NUMERICAL ANALYSIS - II

Solution of Algebraic and Transcendental Equations

Preliminaries :

A problem of great importance in science and engineering is that of determining the roots/zeros of an equation of the form f(x) = 0.

A polynomial equation of the form

$$f(x) = a_0x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots\ldots\ldots\ldots a_{n-1} x + a_n = 0$$

is called an algebraic equation. An equation which contains polynomials, exponential functions, trigonometric functions and logarithmic functions etc. is called transcendental equation.

Initial approximation for an iterative procedure :

We use the following theorem of calculus to determine an initial approximation. It is also called intermediate value theorem.

Theorem : If f(x) is continuous on some interval [a,b] and f(a) f(b) < 0, then the equation f(x) = 0 has at least one real root or an odd number of real roots in the interval (a, b).

BISECTION METHOD

For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the root lies between a and b and let its approximate value be given by

$x_0 = \dfrac{a+b}{2}$. If $f(x_0) = 0$, we conclude that x_0 is a root of the equation $f(x) = 0$, Otherwise the root lies between x_0 and b orbetween x_0 and a depending on whether $f(x_0)$ is positive or negative. We designate this new interval as $[a_1, b_1]$ whose length is $\dfrac{|b-a|}{2}$, as before this is bisected at x_1 and the new interval will be exactly half the length of the previous one. We repeat this process until the latest interval (which contains the root) is the small as desired, say ε. It is clear that the interval width will reduced by a factor of one-half at each step and at the end of the nth step, the new interval will be $[a_n, b_n]$ of length $\frac{|b-a|}{2^n}$, we then have

$$\frac{|b-a|}{2^n} \leq \varepsilon \text{ which gives on simplification } n \geq \frac{\log\left(\frac{|b-a|}{\varepsilon}\right)}{\log_e 2} \quad \ldots\ldots\ldots(1)$$

Inequality (1) gives the number of iterations required to achieve accuracy ε. For example, if $|b-a|<1$ and $\varepsilon=0.001$, then it can be seen that $n \geq 10$. …….. (2)

The method is shown graphically as below

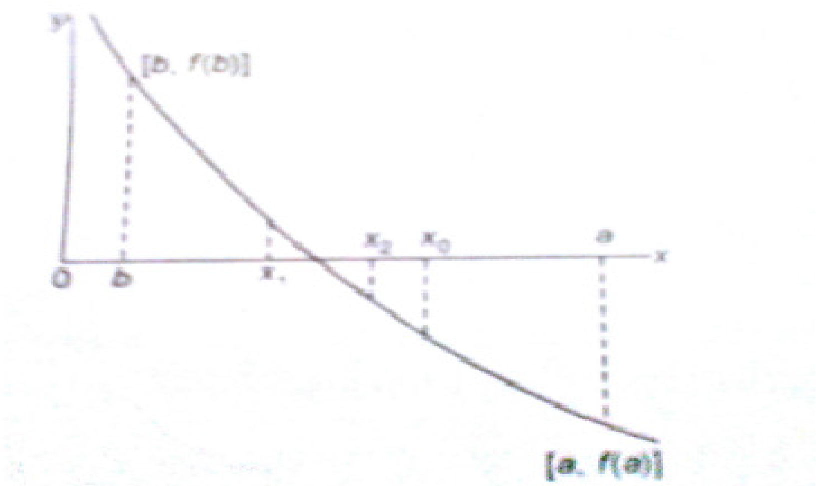


Fig 1.1 Graphical representation of the bisection method

It should be noted that this method always succeeds : If there are more roots than one in the interval, bisection method finds one of the roots. It can be easily programmed using the following computational steps.

1. Choose two real numbers a and b such that f(a) f(b) < 0.

2. Set $x_r = \frac{(a+b)}{2}$.

3. (a) If $f(a)f(x_r)<0$, the root lies in the interval (a, x_r). Then, set b = x_r,

   and go to step 2

   (b) If $f(a)f(x_r)<0$, the root lies in the interval (x_r, b). Then, set a =x_r,

   and go to step2

   (c) If $f(a)f(x_r)=0$, it means that $x_r$is a root of the equation f(x) = 0 and

   the computation may be terminated.

In practical problems, the roots may not be exact so that the condition (c) above is never satisfied. In such a case we need to adapt a criterion for deciding when to terminate the computations. A convenient criterion is to compute the percentage error ε_r defined by

$$\varepsilon_r = \left|\frac{x_r^1 - x_r}{x_r^1}\right| \times 100\% \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Where x_r^1 is the new value of x_r. The computations can be terminated when ε_r becomes less than a prescribed tolerance ε_p. In addition the maximum number of iterations may also be specified in advance.

Problems:

Example 1. Find the real root of the equation $f(x) = x^3 - 2x - 5$ by using bisection method

Solution: Let $f(x) = x^3 - 2x - 5$

$f(2) = -1$ and $f(3) = 16$

Hence the root lies between 2 and 3 and we take $x_0 = \frac{2+3}{2} = 2.5$

Since $f(x_0) = 5.6250$, we choose [2, 2.5] as the new interval. Then

$$x_1 = \frac{2+2.5}{2} = 2.25$$

And $f(x_1) = 1.890625$ proceeding in this way, the following table is obtained

n	a	b	x	$f(x)$
1	2	3	2.5	5.6250
2	2	2.5	2.25	1.8906
3	2	2.25	2.125	0.3457
4	2	2.125	2.0625	-0.3513
5	2.0625	2.125	2.09375	-0.0089
6	2.09375	2.125	2.10938	0.1668
7	2.09375	2.10398	2.10156	0.07856
8	2.09375	2.10156	2.09766	0.03471
9	2.09375	2.09766	2.09570	0.01286
10	2.09375	2.09570	2.09473	0.00195
11	2.09375	2.09473	2.09424	-0.0035
12	2.09424	2.09473		

At n =12, it is seen that the difference between two successive iterates is 0.0005, which is less than 0.001. Thus this result agrees with condition (2)

Example 2. Find the positive real root of the equation $xe^x = 1$, which lies between 0 and 1

Solution: Let $f(x) = xe^x - 1$ since $f(0) = -1$ and $f(1) = 1.718$

It follows that the root lies between 0 and 1 and we take $x_0 = \frac{0+1}{2} = 0.5$

Since f(0.5) is negative, it follows that the root lies between 0.5 and 1. Hence the new root is 0.75, $x_1 = 0.75$. using the values of x_0 and x_1, we calculate €

$$\varepsilon_1 = \left|\frac{x_1 - x}{x_1}\right| \times 100 = 33.33\%$$

Again we find that $f(0.75)$, is positive and hence the root lies between 0.5 and 0.75 $ie\ x_2 = 0.625$

Now the new error is

$$\varepsilon_1 = \left|\frac{0.625 - 0.75}{0.625}\right| \times 100 = 20\%$$

Proceeding in this way, the following table is constructed where only the sign of the function value is indicated. The prescribed tolerance is 0.05%

n	a	b	x	$sign\ of\ \ f(x)$	Er (%)
1	0	1	0.5	negative	--
2	0.5	1	0.75	positive	33.33
3	0.5	0.75	0.625	positive	20.00
4	0.5	0.625	0.5625	negative	11.11
5	0.5625	0.625	0.5938	positive	5.263
6	0.5625	0.5938	0.5781	positive	2.707
7	0.5625	0.5781	0.5703	positive	1.368
8	0.5625	0.5703	0.5664	negative	0.688
9	0.5664	0.5703	0.5684	positive	0.352
10	0.5664	0.5684	0.5674	positive	0.176
11	0.5664	0.5674	0.5669	negative	0.088
12	0.5669	0.5674	0.5671	negative	0.035

After 12 iterates the error ε_r finally satisfies the prescribed tolerance, viz., 0.05%. Hence the required root is 0.567 and it is easily seen that this value is correct to three decimal places.

Exercises

Using bisection method, find the approximate roots of the following equations in the specified intervals.

(1) $x^3 - 9x + 1 = 0\ \ in\ (2, 3)\quad carryout\ 5\ steps$

(2) $\cos x - 1.3x = 0\ \ in\ (0, 1)\quad carryout\ 5\ steps,\ 'x'\, is\, in\, radians$

(3) $x^4 - x^3 - 2x^2 - 6x - 4 = 0\ \ in\ (2, 3)\quad carryout\ 5\ steps$

REGULA FALSI METHOD / METOD OF FALSE POSITION :

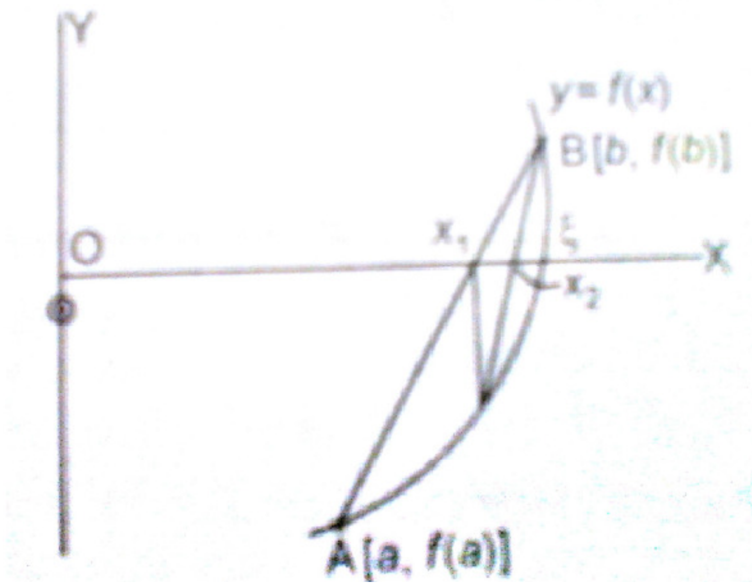
The method is also called linear interpolation or chord method. This is the oldest method for finding the real root of the non linear equation f(x) = 0 and closely resembles the bisection method. This method is also known as method of chords, we choose two points a and b such that f(a) and f(b) are of opposite signs. Hence the root must lie in the interval [a, b]. We know the equation of chord joining the two points $[a, f(a)]$ and $[b, f(b)]$is given by

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \qquad \text{...}(1)$$

The method consists in replacing the part of the curve between the points $[a, f(a)]$ and $[b, f(b)]$ by means of the chord joining the points, and taking the point of intersection of the chord with the x- axis as an approximation to the root. The point of intersection in the present case is obtained by using $y = 0$ in equation (1) thus we obtain

$$x_1 = \frac{a\,f(b) - b\,f(a)}{b - a} \qquad \text{..............}(2)$$

Which is the first approximate root of the equation f(x) = 0. If now $f(x_1)$ and f(a) are of opposite signs, then the root lies between a and x_1, and we replace b by x_1 in (2) and obtain the next approximation. Otherwise we replace a by x_1 and generate the next approximations. The procedure is repeated till the root is obtained to the desired accuracy. The following figure gives a graphical representation of the method.



1.Compute the real root of the equation $x \log x_{10} - 1.2 = 0$ by the method of false position. Carry out three iteration

Solution: let $f(x) = \log x_{10} - 1.2$

$$f(2) = -0.6 < 0, \quad f(3) = 0.23 > 0$$

The real root lies in the interval (2, 3) and from the values of f(x) at x = 2, 3 and we expect the root in the neighbourhood of 3 and let us find (a, b) for applying the method such that (b - a) is small enough.

$f(2.7) = -0.0353 < 0, \quad f(2.8) = 0.052$

The root lies between (2.7, 2.8) the successive approximations are obtained as follows

I iteration:

$a = 2.7, \quad f(2.7) = -0.0353$

$b = 2.8, \quad f(2.8) = 0.052$

$$x_1 = \frac{a\ f(b) - b\ f(a)}{f(b) - f(a)} = 2.7404$$

II iteration:

$a = 2.7404, \quad f(2.7404) = -0.00021 < 0$

$b = 2.8, \quad f(2.8) = 0.052 > 0$

$$x_2 = \frac{a\ f(b) - b\ f(a)}{f(b) - f(a)} = 2.7406$$

III iteration:

$a = 2.7406, \quad f(2.7406) = -0.00004$

$b = 2.8, \quad f(2.8) = 0.052$

$$x_3 = \frac{a\ f(b) - b\ f(a)}{f(b) - f(a)} = 2.7406$$

Comparing x_2 and x_3 we have the same value up to the places of fourth decimal.

Thus the required approximate root is **2.7406**

2. Find the real root of the equation $f(x) = cosx + 1 - 3x$ by Regula false method

correct to four decimal places.

Solution: let $f(x) = cosx + 1 - 3x$

$$f(0) = 2 > 0, \quad f(1) = -1.46 < 0$$

The real root lies in the interval (0, 1) and we expect the root in the neighbourhood of 1 $f(0.6) = 0.0253 > 0, \quad f(0.7) = -0.3352 < 0$

The root lies between (0.6, 0.7)

I iteration:

$a = 0.6, \quad f(0.6) = 0.0253$

$b = 0.7 \quad f(0.7) = -0.3352$

$$x_1 = \frac{a\ f(b) - b\ f(a)}{f(b) - f(a)} = 0.607$$

II iteration:

$a = 0.607, \quad f(0.607) = 0.00036 > 0$

$b = 0.7, \quad f(0.7) = -0.3352 < 0$

$$x_2 = \frac{a\ f(b) - b\ f(a)}{f(b) - f(a)} = 0.607$$

Comparing x_1 and x_2, we have the same value up to third decimal places

Hence the real root correct to three decimal places is **0.607**

Exercises:

1. Using Regulafalsi method, find the approximate roots of the following equations $x^3 - 2x - 5 = 0$ *correct to four decimal places*
2. Show that a real root of the equation $\tan x + \tanh x = 0$ lies between 2 and 3 by using Regulafalsi method, by taking 5 approximations.
3. Find the real root of the equation $\cos x = 3x - 1$ *correct to three decimal places* by using Regulafalsi method
4. Find the real root of the equation $x^4 - x^3 - 2x^2 - 6x - 4 = 0$ *in* $(2, 3)$ *carryout 5 steps* by using Regulafalsi method

ITERATIVE METHODS

We have so for discussed root finding methods, which require the interval in which the root lies. We now describe methods which require one or more starting values of x. These values need not necessarily bracket the root. The first is the iteration method, which requires one starting value of x.

To describe this method for finding the roots of the equation $f(x) = 0$ ....(1)

We rewrite this equation in the form $x = \phi(x)$ ....(2)

There are many ways of doing this. For example, the equation

$x^3 + x^2 - 1 = 0$ can be expressed as either of the form

$$x = (1+x)^{-1/2}, \qquad x = \left(1-x^3\right)^{1/2}, \qquad x = \left(1-x^2\right)^{1/3} \; ....$$

Let x_0 be an approximate value of the desired root ξ. Substituting it for x on the right side of (2), we obtain the first approximation

$$x_1 = \phi(x_0)$$

The successive approximations are then given by

$$x_2 = \phi(x_1), \quad x_3 = \phi(x_2), \; ...... \quad x_n = \phi(x_{n-1}),$$

A number of questions now arise;

(i) Does the sequence of approximations

$x_0, \quad x_1, \quad x_2 \, ...x_n$ always converge to some number ξ

(ii) If it does, will ξ be a root of the equation

(iii) How should we choose ϕ in order that the sequence $x_o, x_1, ..., x_n$ converges to the root?

The answer to the first question is negative. As an example, we consider the equation

$$x = 10^x + 1$$

If we take $x_0 = 0,\ x_1 = 2,\ x_2 = 101,\ x_3 = 10^{101} + 1,$ etc and as n increases, x_n increases without limit. Hence, the sequence $x_0, x_1, x_2, ..., x_n$ does not always converge and, in Theorem 2.1 below, we state the conditions which are sufficient for the convergence of the sequence.

The second question is easy to answer, for consider the equation $x_{n+1} = \phi(x_n)$ ………..(3)

Which gives the relation between the approximations at the nth and (n+1)th stages. As n increases, the left side tends to the root ξ , and if ϕ is continuous the right side tends to $\phi(\xi)$. Hence, in the limit, we have $\phi(\xi)$ which shows that is a root of the equation $x = \phi(x)$

The answer to the third question is contained in the following theorem:

Theorem 2.1

Let $x = \xi$ be a root of f(x) =0 and let I be an interval containing the point $x = \xi$. Let $\phi(x)$ *and* $\phi^1(x)$ be continuous in I, where $\phi(x)$ is defined by the equation

$x = \phi(x)$ Which is equivalent to f(x) = 0. Then if $|\phi^1(x)| < 1$ for all x in I, the sequence of approximations $x_0, x_1, x_2, ..., x_n$

Defined by (3) converges to the root ξ, provided that the initial approximation x_0 is chosen in I.

Proof

Since x_0 is a root of the equation $x = \phi(x)$, we have

$$\xi = \phi(\xi) \text{ ....} \qquad (4)$$

From (3)

$$x_1 = \phi(x_0) \text{ ....(5)}$$

Subtraction gives

$$\xi - x_1 = \phi(\xi) - \phi(x_0)$$

By using the mean value theorem, the right-hand side can be written as

$(\xi - x_0)\phi^1(\xi_0), \qquad x_0 < \xi_0 < \xi$

Hence we obtain

$$\xi - x_1 = (\xi - x_0)\ \phi^1(\xi_0),\ \ x_0 < \xi_0 < \xi \text{ ....(6)}$$

Similarly we obtain

$\xi - x_2 = (\xi - x_0)\ \phi^1(\xi_1),\ \ x_1 < \xi_1 < \xi$ ....(7)

$\xi - x_3 = (\xi - x_2)\ \phi^1(\xi_2),\ \ x_2 < \xi_2 < \xi$ ....(8)

....

$\xi - x_{n+1} = (\xi - x_n)\ \phi^1(\xi_n),\ \ x_n < \xi_n < \xi$ ....(9)

If we let

$|\phi^1(\xi_i)| \le k < 1 \quad for\ all\ i$ ....(10)

Then equation (6) to (10) give

$|\xi - x_1| \le |\xi - x_0|, \qquad |\xi - x_2| \le |\xi - x_1|$

Which show that each successive approximation remain in I provided that the initial approximation is chosen in I. now multiply equation (6) to (10) and simplifying, we obtain

$$\xi - x_{n+1} = (\xi - x_0)\, \phi^1(\xi_0),\, \phi^1(\xi_1)\phi^1(\xi_2)......\phi^1(\xi_n) \text{.................(11)}$$

As n tends to infinity the right hand side of (11) tends to zero, and it follows that the sequence of approximations x_0, x_1, …… converges to the root ξ.

Problems:

1. Find the real root of the equation $f(x) = x^3 + x^2 - 1$ by iterative method on the interval

[0, 1] with an accuracy of 10^{-4}

Solution: To find this root, we rewrite the given equation in the form

$$x = \frac{1}{\sqrt{x+1}} \text{-------------------------------------(1)}$$

Thus $\phi(x) = \dfrac{1}{\sqrt{x+1}}$, $\phi^1(x) = \dfrac{-1}{2(x+1)^{3/2}}$

and $\max_{[0,1]}\left|\phi^1(x)\right| = \dfrac{1}{2\sqrt{8}} = k = 0.17678 < 0.2$

Using $|x_n - x_{n-1}| \le \dfrac{1-k}{k}$

$$|x_n - x_{n-1}| \le \frac{1-0.2}{0.2} \times 0.0001 = 0.0004$$

Hence the absolute value of the difference does not exceed 0.0004, the required accuracy will be achieved and then the iteration can be terminated.

Starting with $x_0 = 0.75$, we obtain the following table

n	x_n	$\sqrt{x_n+1}$	$x_{n+1} = 1/\sqrt{x_n+1}$

0	0.75	1.3228756	0.7559289
1	0.7559289	1.3251146	0.7546517
2	0.7546517	1.3246326	0.5749263

At this stage we find that $|x_{n+1} - x_n| = 0.7549263 - 0.7546517 = 0.0002746$
Which is less than 0.0004. The iteration is therefore terminated and the root to the required accuracy is 0.7549.

Example 2. Find the root of the equation 2x =cosx +3 correct to 3 decimal places.

Solution:we rewrite the equation in the form

$$x = \frac{1}{2}(\cos x + 3)$$

$$\textit{so that} \quad \phi(x) = \frac{1}{2}(\cos x + 3)$$

$$\textit{and} \quad |\phi^1(x)| = \left|\frac{\sin x}{2}\right| < 1$$

Hence the iteration method can be applied to the equation (1) and we start with $x_0 = \frac{\pi}{2}$

The successive iterations are

$x_1 = 1.5,$

$x_2 = 1.535,$

$x_3 = 1.518$

$x_4 = 1.526$

$x_5 = 1.522$

$x_6 = 1.524$

$x_7 = 1.523$

$x_8 = 1.524$

Hence we take the solution as 1.524 correct to three decimal places.

Exercises:

1. Using iteration method, find the approximate roots of the following equations $x^4 - x - 13 = 0$ *correct to four decimal places*
2. Show that a real root of the equation

   $\log x - x + 3 = 0$ lies between 0.1 and 0.23 by using iterationv method, by taking 5 approximations.
3. Find the real root of the equation

   $\log x - \cos x = 0$ *correct to three decimal places* root near to 1 by iteration method
4. By iteration method find the real root of the equation

   $x^4 - x^3 - 2x^2 - 6x - 4 = 0$ *in* $(2, 3)$ *carryout* 5 *steps*
5.

The Newton- RaphsonMethod

Introduction

Around 1669, Newton originated the idea of solving the non-linear equations numerically. A systematic and simple method was introduced by Raphson in 1690. So the iteration method is called ***Newton - Raphson Method***.

It is a powerful technique for solving algebraic, transcendental equations numerically. It is based on the simple idea of linear approximation. Geometrically, it is described as tangent method or also a chord method in which we approximate the curve near a root by a straight line. This method is also called Newton's Method

Consider the equation $f(x) = 0.$
Let x_0 *be an approximation to the root of* $f(x) = 0.$
If $x_1 = x_0 + h$ *be the exact root then* $f(x_1) = 0$

Now

$$f(x_1)=0 \quad \Rightarrow \quad f(x_0+h)=0$$

$$f(x_0)+h\,f^{'}(x_0)+\frac{h^2}{2!}f^{''}(x_0)+........=0 \qquad \text{by Taylor's theorem}$$

neglecting h^2 *and higher powers of* h *we get*

$$f(x_0)+h\,f^{'}(x_0)=0 \quad \Rightarrow \quad h=-\frac{f(x_0)}{f^{'}(x_0)}$$

Thus $x_1=x_0-\dfrac{f(x_0)}{f^{'}(x_0)}$ *is close to the root of* $f(x)=0$

Starting with x_1 *still closer value of the root of* $f(x)=0$ *is given by*

$$x_2=x_1-\frac{f(x_1)}{f^{'}(x_1)} \quad \text{continuing this process}$$

we get values which are closer and closer to the actual root. and these steps are called iterations.

Thus $(n+1)^{th}$ *iteration is*

$$x_{n+1}=x_n-\frac{f(x_n)}{f^{'}(x_n)}, \qquad n=0,1,2,3,......... \quad ---- \ (1)$$

equation (1) is called Newton's iteration formula.

Example :1 : Find the root of $x^3-5x+3=0$ by Newton – Raphsonmethod.

Solution : The given equation is $f(x)=x^3-5x+3=0$.

Here f(1) = -1 < 0 and f(2) = 1 >0 so root lies between 1 and 2

Let $x_0 = 1.5$ and $f'(x)=3x^2-5$

Using Newton's formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$=x_n-\frac{x_n^3-5x_n+3}{3x_n^2-5}$$

$$=\frac{2x_n^3-3}{3x_n^2-5} \qquad n=0,1,2,3.....$$

Which gives $x_1 = 2.1429$, $x_2 = 1.9007$ $x_3 = 1.8385$ $x_4 = 1.834$
$x_5 = 1.8342$

Thus x = 1.834 is the root correct to three decimal places.

Example : 2 : Use Newton- Raphsonmethod to find the real root of 3x =cosx + 1

Solution : The given equation is $f(x) = 3x - \cos x - 1 = 0$

f(0) = -2 < 0 and f(1) = 1.4597 > 0

clearly root lies between 0 and 1. We take $x_0 = 0.6$ as the root close to unity.

Also $f'(x) = 3 + \sin x$

By Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}, \qquad n = 0, 1, 2, 3.....$$

For n = 0, $x_1 = 0.6071$, $x_2 = 0.6071$ since $x_1 = x_2$

So 0.6071 is the root correct to four decimal places.

Exercises :

Use Newton - Raphson method to solve the following equations correct to three decimal places.

(i) $x + \log x = 2$ (ii) $\cos x = x e^x$

(iii) $x^4 - x - 13 = 0$ (iv) $e^x \sin x = 1$

Initial Value Problems for Ordinary Differential Equations

Introduction :

The analytic methods of solving differential equations are applicable only to limited class of equations. The differential equations appearing in physical problems do not fall into the category of familiar types. Therefore one has to study numerical method to solve such equations. Basically a first order differential equation of the form

$$y' = f(x,y), \quad y(x_0) = y_0$$

……………………………………..(1)

is to be solved numerically by different methods. The methods for the solution of the initial value problem (1) can be classified mainly into two types. They are single step method and multi step methods. In single step methods, the solution at any point x_{i+1} is obtained using the solution at only the previous point x_i where as in multistep method the solution is obtained using the solution at a number of previous points.

Taylor Series Method :

Here, we describe the method to solve the initial value problems using Taylor series method. This method is the fundamental numerical method for the solution of initial value problems (1).

Consider the differential equation

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0 \text{ ....(1)}$$

Let y = y(x) be a continuously differentialbe function satisfying the equation (1).Expanding y interms of Taylor's series around the point $x = x_0$, we get

$$y = y_0 + (x - x_0) y_0^{'} + \frac{(x-x_0)^2}{2!} y_0^{''} + \frac{(x-x_0)^3}{3!} y_0^{'''} + ... \quad (2)$$

the value of the differential coefficients

$$y_0^{'}\ ,\ y_0^{''},\quad y_0^{'''},\ ....\ at\ \ x = x_0\ \ can\ be\ computed\ \ from\ the\ equation\ \ y' = f(x, y)$$

Example : 1 : Use Taylor's series method to find y(0.1) and y (0.2) from the equation

$y' = y^2 + x, \qquad y(0) = 1$

Solution : Given

$$y' = y^2 + x, \qquad y(0) = 1$$

$$\Rightarrow \qquad f(x, y) = y^2 + x, \qquad x_0 = 0, \quad y_0 = 1$$

$$y_0^{'} = 1^2 + 0 = 1,$$

$$y'' = 2yy' + 1 \qquad \Rightarrow \qquad y_0^{''} = 2.1.1 + 1 = 3$$

$$y''' = 2(y')^2 + 2y\,y'' \quad \Rightarrow \quad y_0^{'''} = 2(1)^2 + 2.1.3 = 8$$

$$y^{iv} = 4y'y'' + 2yy' = 6y'y'' + 2yy''' \quad \Rightarrow \quad y_0^{iv} = 6.1.3 + 2.1.8 = 34$$

Thus the Taylor's series solution is

$$y = y_0 + \left(xy_0^{'} + \frac{x^2}{2!} y_0^{''} + \frac{x^3}{3!} y_0^{'''} + \frac{x^4}{4!} y^{iv} + ... \right)$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{17}{12}x^4 + \frac{31}{20}x^5 + ...$$

Therefore $\quad y\,(0.1) = 1.11647 \quad$ *and* $\quad y(0.2) = 1.27296$

Example : 2 : Find by Taylor's series method, the value of y at x = 0.1 and x = 0.2 to five decimal places from

$$\frac{dy}{dx} = x^2 y - 1, \qquad y(0) = 1$$

Solution : Here

$$y' = x^2 y - 1, \qquad y_0 = 1, \quad x_0 = 0$$

differenting successively and substituting, we get

$$y' = x^2 y - 1 \qquad \Rightarrow \qquad y_0' = -1$$
$$y'' = 2xy + x^2 y' \qquad \Rightarrow \qquad y_0'' = 0$$
$$y''' = 2y + 4x\,y' + x^2 y'' \qquad \Rightarrow \qquad y_0''' = 2$$
$$y^{iv} = 6y' + 6x\,y''' + x^2 y''' \qquad \Rightarrow \qquad y_0^{iv} = -6 \quad \text{and so on.}$$

Thus by Taylor's series method

$$y = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots\ldots\ldots\ldots$$
$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence $y(0.1) = 0.90033$ *and* $y(0.2) = 0.80227.$

Exercises : Use Taylor series method to solve the differential equations numerically correct to three decimal places.

(i) $\dfrac{dy}{dx} = 1 + xy, \qquad y(0)=1 \;\; at \;\; x = 0.1$

(ii) $\dfrac{dy}{dx} = \dfrac{1}{x^2 + y}, \qquad y(4)=4. \qquad Find \;\; y \;\; at \;\; x = 4.1 \;\; and \;\; 4.2$

Euler's Method

In Taylor series method, we express a series for y in terms of powers of x, from which the value of y can be obtained by direct substitution. As the approximation is poor, we derive Euler method by using Taylor series method, where the values of y are computed by short steps ahead for equal intervals h of the independent variable.

We consider the differential equation

$\dfrac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$.

If $y(x)$ is the exact solution of the above equation, then the Taylor's series for y(x) around $x = x_0$ is given by

$y(x) = y_0 + (x - x_0)y_0^{1} + \dfrac{(x - x_0)^2}{2!} y_0^{11} + \ldots$ Neglecting second and higher order terms,

we get $y(x) = y_0 + (x - x_0)y_0'$ .Denoting $x - x_0 = h$,

we have $y(x_0 + h) = y_0 + h\, y(x_0)'$ where $y(x_0)' = f(x_0, y_0)$.

Taking $x_1 = x_0 + h$, we get $y_1 = y_0 + h\, f(x_0, y_0)$.

Similarly $y_2 = y_1 + h\, f(x_1, y_1)$.

Proceeding in this way , for $x_{n+1} = x_n + h$, we obtain the general formula

$$y_{n+1} = y_n + h\, f(x_n, y_n) .$$

Remark : The process is very slow and to obtain reasonable accuracy with Euler's method, we need to take a smaller value for h. Because of this restriction on h, the method is unsuitable for practical use a modification of it, known as modified Euler's method, which gives more accurate results.

Example 1. Using Euler's method, find an approximate value of y corresponding to x=1, given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.

Solution. We take n=10 and h=0.1 . The various calculations are arranged as follows.

X	Y	$x + y = dy/dx$	old $y + 0.1(dy/dx) =$ new y
0	1	1	1+0.1(1.00)=1.10
0.1	1.1	1.20	1.10+0.1(1.20)=1.22
0.2	1.22	1.42	1.22+0.1(1.42)=1.36
0.3	1.36	1.66	1.36+0.1(1.66)=1.53
0.4	1.53	1.93	1.53+0.1(1.93)=1.72
0.5	1.72	2.22	1.72+0.1(2.22)=1.94
0.6	1.94	2.54	1.94+0.1(2.54)=2.19
0.7	2.19	2.89	2.19+0.1(2.89)=2.48
0.8	2.48	3.89	2.48+0.1(3.89)=2.81
0.9	2.81	3.71	2.81+0.1(3.71)=3.18
1.0	3.18		

Thus the required approximate value of y=3.18.

Example 2. Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with initial condition $y = 1$ at $x = 0$: find y for $x = 1$ By Euler's method.

Solution. We divide the interval (0, 0.1) into five steps. The various calculations are arranged as follows:

x	Y	dy/dx=(y-x)/(y+x)	Old y+0.02(dy/dx)=new y
0	1	1	1.02
0.02	1.02	0.9615	1.0392
0.04	1.0392	.926	1.0577
0.06	1.0577	0.893	1.0756
0.08	1.0756	0.862	1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

Modified Euler's method:

Instead of approximating $f(x,y)$ by $f(x_0,y_0)$, we approximate the integral by means of Trapezoidal rule to obtain $y_1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(0)})\right]$.

We thus obtain the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(n)})\right], \quad n = 0,1,2,\ldots$$

Where $y_1^{(n)}$ is the nth approximation to y_1, which can be found using Euler's formula $y_1^{(0)} = y_0 + hf(x_0, y_0)$.

Example 3. Using modified Euler's method, Solve $\frac{dy}{dx} = x^2 + y, \quad y(0) = 1$.

Choose h= 0.05, Compute y(0.1). Modify the solution twice in each step.

Solution: $x_0 = 0, \quad y_0 = 1$ and $f(x_0, y_0) = x_0^2 + y_0 = 1$

$$\therefore y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 1.05.$$

For $x_1 = 0.05, \; f(x_1, y_1^{(0)}) = 1.0525$

$$\therefore y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(0)})\right] = 1.0513$$

Again modifying, $\therefore y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(1)})\right] = 1.0513$.

Taking $x_1 = 0.05, \quad f(x_1,y_1) = 1.0538$,

$$\therefore y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 1.10399.$$

Then $y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2,y_2^{(0)})\right] = 1.10549$

Again modifying $y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] = 1.1055$.

Hence the required approximate value of y(0.1)=1.1055.

Example 2. Using Modified Euler's method, find an approximate value of y when x=0.3, given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$, taking h=0.1.

Solution. Given $x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3$ and $y_0 = 1$.

Using Euler's method

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0) = 1.1.$$

Modifying using modified Euler's method,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 1.11.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1.1105$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 1.1105.$$

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1) = 1.2316$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] = 1.2426$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] = 1.2432$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] = 1.2432.$$

$$y_3^{(0)} = y_2 + h\, f(x_2, y_2) = 1.3875$$

$$y_3^{(1)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(0)})\right] = 1.3997$$

$$y_3^{(2)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(1)})\right] = 1.4003$$

$$y_3^{(3)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(2)})\right] = 1.4004.$$

$\therefore$ The required approximate value of y(0.3)=1.4004.

Example 3. Solve the following differential equation $\frac{dy}{dx} = \log_{10}(x+y)$, $y(0) = 2$ at $x = 0.6$ and 0.8 with $h = 0.2$.

Solution. Given $x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8$ and $y_0 = 2$.

Using Euler's method

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0) = 2.0602.$$

Modifying using modified Euler's method,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 2.0655.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 2.0656$$

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1) = 2.1366$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] = 2.1455$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] = 2.1416$$

$$y_3^{(0)} = y_2 + h\, f(x_2, y_2) = 2.2226$$

$$y_3^{(1)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(0)})\right] = 2.2272$$

$$y_3^{(2)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(1)})\right] = 2.2272$$

$$y_4^{(0)} = y_3 + h\, f(x_3, y_3) = 2.3175$$

$$y_4^{(1)} = y_3 + \frac{h}{2}\left[f(x_3, y_3) + f(x_4, y_4^{(0)})\right] = 2.3217$$

$$y_4^{(2)} = y_3 + \frac{h}{2}\left[f(x_3, y_3) + f(x_4, y_4^{(1)})\right] = 2.3217.$$

$\therefore$ $y(0.6) = 2.2272$ and $y(0.8) = 2.3217$ approximately.

Note: In Euler method, the interval length h should be kept small and hence these methods can be applied for tabulating y over a limited range only.

Runge-Kutta Methods

Introduction:

Euler's method is less efficient in practical problems since it requires '***h***' to be small for obtaining reasonable accuracy. The Runge-Kutta methods are designed to give greater accuracy and they possess the advantage of requiring only the function value at some selected points on the subinterval.The basic idea of R-K methods is to approximate the integral by a weighted average of slopes and approximate slopes at a number of points in the interval [x_i, x_{i+1}]

I. Runge – Kutta Method of Second Order:

Consider the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

The ***second order R-K*** method formula is given by,

That is the value of y at x = $x_i$is : $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$.

Where, $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h, y_0 + k_1)$.

Example 1 : Given $\frac{dy}{dx} = y - x$where $y(0) = 2$, find $y(0.1)$ and $y(0.2)$correct to four decimal places.

Solution: With **h=0.1**, $f(x, y) = y - x, x_0 = 0, y_0 = 2$

We have R-K method order two formula $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$.

Where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h, y_0 + k_1)$.

We get $k_1 = 0.2$ and $k_2 = 0.21$

$y_1 = y(0.1) = 2 + \frac{1}{2}(0.2 + 0.21) = 2.2050.$

To determine $y_2 = y(0.2)$ we note that $x_0 = 0.1$ and $y_0 = 2.2050$

Hence $y_2 = y(0.2) = 2.2050 + \frac{1}{2}(0.2105 + 0.22155) = 2.4210.$

Example 2 : Solve $\mathbf{y(1.2)}$ given $\mathbf{y' = x + y + xy}$ with $y(\mathbf{1}) = \mathbf{1}$ take $\mathbf{h = 0.1}$

Solution: Given $\mathbf{f(x, y) = x + y + xy, h = 0.1, x_0 = 1, y_0 = 1}$

We have R-K method order two formula $\mathbf{y_1 = y_0 + \frac{1}{2}(k_1 + k_2)}$.

Where $\mathbf{k_1 = hf(x_0, y_0)}$

$\mathbf{k_2 = hf(x_0 + h, y_0 + k_1)}$.

We get $\mathbf{k_1 = 0.3}$ and $\mathbf{k_2 = 0.383}$

$\mathbf{y_1 = y(1.1) = 1 + \frac{1}{2}(0.3 + 0.383) = 1.3415}$

To determine $\mathbf{y_2 = y(1.2)}$ we note that $\mathbf{x_0 = 1.1}$ and $\mathbf{y_0 = 1.3415}$

Hence $\mathbf{y_2 = y(1.2) = 1.3415 + \frac{1}{2}(0.3917 + 0.5013) = 1.78801}$.

II. Runge-Kutta Method of fourth order:

The most commonly used RK method is a method which uses four slopes and is called RK method of second order. The method is given by :

Consider the Ordinary Differential Equation $\mathbf{\frac{dy}{dx} = f(x, y), y(x_0) = y_0}$.

We need to find **'y'** at $\mathbf{x_n = x_0 + nh}$. The **fourth-order Runge-Kutta** method formula is given by

$$\mathbf{y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

Where

$$\mathbf{k_1 = hf(x_0, y_0)}$$
$$\mathbf{k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})}$$
$$\mathbf{k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})}$$
$$\mathbf{k_4 = hf(x_0 + h, y_0 + k_3)}.$$

Example1 : Given $\mathbf{\frac{dy}{dx} = 3x + \frac{y}{2},\ y(0) = 1}$ compute **y (0.2)** by taking **h = 0.2** using Runge-Kutta method of fourth order.

Solution:

Given $f(x, y) = 3x + \frac{y}{2}, x_0 = 0, y_0 = 1, h = 0.2$

$k_1 = hf(x_0, y_0) = 0.2f(0,1) = (0.2)(3 \times 0 + \frac{1}{2}) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 1.05) = 0.165$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f(0.1, 1.0825) = 0.16825$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.16825) = 0.236825$

Therefore $y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.1672208$

Example 2 :Use fourth order R-K method to solve $(x + y)\frac{dy}{dx} = 1$, $y(0.4) = 1$ at x = 0.5, 0.6, 0.7 correct to four decimal places.

Solution:

By data we have $f(x, y) = \frac{1}{x+y}, x_0 = 0.4,\ \ y_0 = 1,\ \ h = 0.1$

$k_1 = hf(x_0, y_0) = 0.1f(0.4, 1) = (0.1)\frac{1}{0.4+1} = 0.0714$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.45, 1.0357) = 0.0673$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.45, 1.03365) = 0.0674$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.5, 1.0674) = 0.0638$

Therefore $y(0.5) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.0674.$

To compute $y(0.6)$ we use $x_0 = 0.5,\ \ y_0 = 1.0674.$

$k_1 = hf(x_0, y_0) = 0.1f(0.5, 1.0674) = 0.0651$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.55, 1.0999) = 0.0606$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.55, 1.0977) = 0.0607$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.6, 1.0674) = 0.05786$

Therefore $y(0.6) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.1283.$

Example 2 :Use fourth order Runge-Kutta method to solve $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ atx = 0.2 taking h = 0.2.

Solution:

$By\ data\ we\ have\ f(x,y) = \frac{y-x}{y+x}, x_0 = 0,\ \ y_0 = 1,\ \ h = 0.2$

$$k_1 = hf(x_0, y_0) = 0.2f(0,1) = (0.2)\left[\frac{1-0}{1+0}\right] = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)f(0.1, 1.1) = 0.1667$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f(0.1, 1.0835) =$$

0.1662

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)f(0.1, 1.1662) = 0.1414$$

$Therefore\ y(0.5) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.1679$

Exercise:

1) Use fourth order Runge-Kutta method to find y(1.1) given that $\frac{dy}{dx} = xy^{\frac{1}{3}},\ \ y(1) = 1$
2) Use fourth order Runge-Kutta method to solve $\frac{dy}{dx} + y = 2x\ \ at\ \ x = 1.1$ given that y=3 at x = 1 initially.

Predictor and Corrector methods(Multi Step Methods) :

Introduction:

Now we define the predictor and corrector methods or also it is know as P- C methods These methods are methods which require function values at $\boldsymbol{x_n, x_{n-1}, x_{n-2}}$for the computation of the function value at x_{n+1}. A predictor formula is used to predict the value of y at x_{n+1} and then a corrector formula is used to improve the value of y_{n+1}. In this part we are discussing the two important predictor and corrector formulae, namely, Milne's method and Adams-Bashforth method.

I. Milne's method:

Consider the ordinary differential equation

$$y' = \frac{dy}{dx} = f(x,y), with\ y(x_0) = y_0.$$

The Milne's predictor & corrector formulae is given by,

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$
$$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

Where $f_i = y'(x_i, y_i)$ and these values are obtained by means of Taylor's series or Euler's method or R-K method.

Example 1. Use Milne's method to find y(0.3) from $\frac{dy}{dx} = x^2 + y^2$, y(0)=1 after computing y(-0.1), y(0.0), y(0.1) and y(0.2) by Taylor's series method correct to four decimal places.

Solution:

First let us calculate y(-0.1),y(0.0), y(0.1) and y(0.2) by Taylor's series method and we get values as,

$$y(-0.1) = 0.9087,\ y(0) = 1,\ y(0.1) = 1.1114,\ y(0.2) = 1.2529.$$

x	y	$f(x,y) = x^2 + y^2$
$x_0 = -0.1$	$y_0 = 0.9087$	f_0 =0.8357
$\mathbf{x_1 = 0.0}$	$\mathbf{y_1 = 1}$	$\mathbf{f_1 = 1}$
$x_2 = 0.1$	$y_2 = 1.1114$	$f_2 = 1.2452$
$x_3 = 0.2$	$y_3 = 1.2529$	$f_3 = 1.6097$
$x_4 = 0.3$	y_4 =?	

We have Milne's predictor formula $y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$y_4^p = 0.9087 + \frac{4(0.1)}{3}\big(2(1) - 1.2452 + 2(1.6097)\big) = 1.4385.$$

Milne's Corrector formula is given by $y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$y_4^c = 1.1114 + \frac{0.1}{3}(1.2452 + 4(1.6097) + 2.1592) = 1.4395$$

Again applying the corrector formula repeatedly we get,

$$y_4^c = 1.1114 + \frac{0.1}{3}(1.2452 + 4(1.6097) + 2.1621) = 1.4396$$
$$y_4^c = 1.1114 + \frac{0.1}{3}(1.2452 + 4(1.6097) + 2.16212) = 1.4396$$

Hence y(0.3)=1.4396

Example 2. Solve the initial value problem $\frac{dy}{dx} = 1 + xy^2, y(0) = 1$, for x=0.4 by Milne's predictor and corrector method correct to three decimal places.

Solution:

First let us calculate y(0.1) ,y(0.2), y(0.3) by Euler's method and we get values as,

$$y(0.1) = 1.105,\ y(0.2) = 1.223,\ y(0.3) = 1.355$$

x	y	$f(x,y) = 1 + xy^2$
$x_0 = 0.0$	$y_0 = 1$	$f_0 = 1$
$x_1 = 0.1$	$y_1 = 1.105$	$f_1 = 1.122$
$x_2 = 0.2$	$y_2 = 1.223$	$f_2 = 1.299$
$x_3 = 0.3$	$y_3 = 1.355$	$f_3 = 1.550$
$x_4 = 0.4$	$y_4 = ?$	

We have Milne's predictor formula $y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$y_4^p = 1 + \frac{4(0.1)}{3}\big(2(1.122) - 1.299 + 2(1.550)\big) = 1.526.$$

Milne's Corrector formula is given by $y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$y_4^c = 1.223 + \frac{0.1}{3}(1.299 + 4(1.550) + 1.931) = 1.537$$

Again applying the corrector formula repeatedly we get,

$$y_4^c = 1.223 + \frac{0.1}{3}(1.299 + 4(1.550) + 1.944) = 1.537$$

Hence y(0.4)=1.537

Example 3. Given $\frac{dy}{dx} = \frac{1}{2}(1 + x^2)y^2$, $y(0) = 1, y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$. Evaluate y at x=0.4 by Milne's predictor and corrector method correct to two decimal places.

Solution:

First let us calculate y(0.1) ,y(0.2), y(0.3) by Euler's method and we get values as,

x	y	$f(x,y) = \frac{1}{2}(1 + x^2)y^2$
$x_0 = 0.0$	$y_0 = 1$	$f_0 = 0.5$
$x_1 = 0.1$	$y_1 = 1.06$	$f_1 = 0.5674$
$x_2 = 0.2$	$y_2 = 1.12$	$f_2 = 0.6522$
$x_3 = 0.3$	$y_3 = 1.21$	$f_3 = 0.7979$
$x_4 = 0.4$	$y_4 = ?$	

We have Milne's predictor formula $y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$y_4^p = 1 + \frac{4(0.1)}{3}\big(2(0.5674) - 0.6522 + 2(0.7979)\big) = 1.2771.$$

Milne's Corrector formula is given by $y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$y_4^c = 1.12 + \frac{0.1}{3}(0.6522 + 4(0.7979) + 0.9459) = 1.2796$$

Again applying the corrector formula repeatedly we get,

$$y_4^c = 1.12 + \frac{0.1}{3}(0.6522 + 4(0.7979) + 0.9459) = 1.2797$$

Again applying the corrector formula repeatedly we get,

$$y_4^c = 1.12 + \frac{0.1}{3}(0.6522 + 4(0.7979) + 0.9459) = 1.2797$$

Since last two corrector values are same up to four decimal places,

∴y(0.4)=1.2796.

Exercises:

1. By using the Milne's predictor-corrector method find an approximate solution of the equation $y' = \frac{2y}{x}, x \neq 0$ at the point x=2, given that y(1)=2.
2. Determine the value of y(0.4) using Milne's predictor and corrector method correct to four decimal places given that $y' = xy + y^2, y(0) = 1$.

II. Adams-bashforthMethod :

Consider the ordinary differential equation

$$\mathbf{y' = \frac{dy}{dx} = f(x, y)} \text{with } \mathbf{y(x_0) = y_0}.$$

The Adams-Bashforth predictor & corrector formulae is given by,

$$y_1^p = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$y_1^c = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

Where $f_i = y'(x_i, y_i)$ and these values are obtained by means of Taylor's series or Euler's method or R-K method.

Example1. Use Adams-Bashforth method to find y(2) from $\frac{dy}{dx} = \frac{x+y}{2}$, y(0)=**2**.

Solution: Let h=0.5 First let us calculate y(0.5),y(1) and y(1.5) by Euler's method and we get values as, y(0.5)=2.636, y(1)=3.595 and y(1.5)=4.968

X	y	$f(x,y) = \frac{x+y}{2}$
$x_{-3} = 0.0$	$y_{-3} = 2$	$f_{-3} = 1$
$x_{-2} = 0.5$	$y_{-2} = 2.636$	$f_{-2} = 1.568$
$x_{-1} = 1.0$	$y_{-1} = 3.595$	$f_{-1} = 2.2975$
$x_0 = 1.5$	$y_0 = 4.968$	$f_0 = 3.234$
$x_1 = 2.0$	$y_1 = ?$	

We have Adams-Bashforth predictor formula

$$y_1^p = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$y_1^p = 4.968 + \frac{0.5}{24}\big(55(3.234) - 59(2.2975) + 37(1.568) - 9(1)\big) =$$
6.8707.

Adam-Bashforth Corrector formula is given by

$y_1^c = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$

$y_1^c = 4.968 + \frac{0.5}{24}(9(4.4353) + 19(3.234) - 5(2.2975) + 1.568) = 6.8730.$

Again applying Adam-Bashforth Corrector formula we get

$y_1^c = 4.968 + \frac{0.5}{24}(9(4.4365) + 19(3.234) - 5(2.2975) + 1.568) = 6.8733.$

Again applying Adam-Bashforth Corrector formula we get

$y_1^c = 4.968 + \frac{0.5}{24}(9(4.4366) + 19(3.234) - 5(2.2975) + 1.568) = 6.8733.$

Since the last two steps have the values same up to four decimal places

Hence y(2)=6.8733.

Example 2. Obtain the solution of the initial value problem $\frac{dy}{dx} - x^2y = x^2, y(1) = 1$ at x=1.4 using Adams-Bashforth method correct to four decimal places.

Solution: Let h=0.1 First let us calculate y(1.1),y(1.2) and y(1.3) by Taylor's series method and we get values as, y(1.1)=1.2332, y(1.2)=1.5475 and y(1.3)=1.9785

x	y	$f(x,y) = x^2(1+y)$

$x_{-3} = 1.0$	$y_{-3} = 1$	f_{-3} =2
$x_{-2} = 1.1$	$y_{-2} = 1.2332$	$f_{-2} = 2.7021$
$x_{-1} = 1.2$	$y_{-1} = 1.5475$	$f_{-1} = 3.6684$
$x_0 = 1.3$	$y_0 = 1.9785$	$f_0 = 5.0336$
$x_1 = 1.4$	y_1 =?	

We have Adams-Bashforth predictor formula

$$y_1^p = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$y_1^p = 1.9785 + \frac{0.1}{24}\big(55(5.0336) - 59(3.6684) + 37(2.7021) - 9(2)\big)$$

$= 2.5717.$

Adam-Bashforth Corrector formula is given by

$$y_1^c = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$y_1^c = 1.9785 + \frac{0.1}{24}(9(7.0005) + 19(5.0336) - 5(3.6684) + 2.7021)$=2.5743.

Again applying Adam-Bashforth Corrector formula we get

$y_1^c = 1.9785 + \frac{0.1}{24}(9(7.0056) + 19(5.0336) - 5(3.6684) + 2.7021)$=2.5745.

$y_1^c = 1.9785 + \frac{0.1}{24}(9(7.006) + 19(5.0336) - 5(3.6684) + 2.7021)$=2.5745.

Since the last two steps have the values same up to four decimal places

Hence y(1.4)=2.5745.

Exercise:

1. Solve for y(0.4) given that $\frac{dy}{dx} = x - y^2, y(0) = 1$, by Adam-Bashforth Predictor Corrector method correct to four decimal places.
2. Solve $\frac{dy}{dx} = \frac{xy}{2}$ for x=0.4 using Adam-Bashforth Predictor Corrector method correct to fourdecimal places. Given that y(0)=1.