NUMERICAL ANALYSIS - I

Introduction:

Most of the Engineering, Physical and economical sciences can be formulated in terms of system of linear or non-linear equations, ordinary or partial differential equations. In majority of the cases, the solutions to these problems in analytical form are non-existent or difficult or not amenable for direct interpretation. In all such problems, numerical analysis provides approximate solutions, practical and amenable for analysis. Numerical analysis does not strive for exactness. Instead, it yields approximations with specified degree of accuracy. The early disadvantage of the several numbers of computations involved has been removed through high speed computation using computers, giving results which are accurate, reliable and fast. Numerical analysis is not only a science but also an art because the choice of appropriate procedure which best suits to a given problem yields good solutions.

Interpolation

Introduction:

The statement

$$y = f(x), \qquad x_0 \le x \le x_n,$$

means: corresponding to every value of x in the range $x_0 \le x \le x_n$ there exists one or more values of $f(x)$. Assuming that $f(x)$ is single-valued, continuous and that it is known explicitly, then the values of $f(x)$ corresponding to certain given values of x, say $x_0, x_1, \ldots, x_n$ can easily be computed and tabulated. The central problem of numerical analysis is the converse of this: Given the set of tabular values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ satisfying the relation $\mathrm{y} = f(x)$ where the explicit nature of $f(x)$ is not known, it is required to find a simpler function, say $\phi(x)$, which approximates f(x), such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. Such a process of approximation of an unknown function by a known function within the range where it is defined, suchthat both functions assume same values at the given set of tabulated points is called interpolation. The extrapolation is the process of approximating the unknown function by the known function at a point outside the range of definition. If $\phi(x)$ is a polynomial, then the process is called polynomial interpolation and $\phi(x)$ is called the interpolating polynomial. The following principle called Weierstrass approximation theorem justifies the polynomial approximation. This principle states that 'If f(x) is continuous in $x_0 \le x \le x_n$, then given any ε > 0, there exists a polynomial P(x) such that $|f(x) - P(x)| < \varepsilon$ for all x in (x_0, x_n).'

In this chapter, we discuss the problem of approximating a given function by polynomials. There are two main uses of these approximating polynomials. The first use is to reconstruct the function $f(x)$ when it is not given explicitly and only values of $f(x)$ and/ or its certain order derivatives are given at a set of distinct points called *nodes* or *tabular points*. The second use is to perform the required operations which were intended for $f(x)$, like determination of roots, differentiation and integration etc. can be carried out using the approximating polynomial $P(x)$. The approximating polynomial $P(x)$ can be used to predict the value of $f(x)$ at a non-tabular point. The deviation of $P(x)$ from $f(x)$, that is $f(x) - P(x)$, is called the *error of approximation*.

Let $f(x)$ be a continuous function defined on some interval $[a, b]$, and be prescribed at $n+1$ distinct tabular points $x_0, x_1, ..., x_n$ such that $a = x_0 < x_1 < ... < x_n = b$. The distinct tabular points $x_0, x_1, ..., x_n$ may be nonequidistant or equidistant, that is $x_{k+1} - x_k = h,\ k = 0,1,...,n.$ The problem of polynomial approximation is to find a polynomial $P(x)$ of degree $\leq n,$ which fits the given data exactly, that is, $P(x_i) = f(x_i), i = 0,1,...,n.$ These conditions are called the *interpolating conditions*.

Remark: Through two distinct points, we can construct a unique polynomial of degree 1(straight line). Through three distinct points, we can construct a unique polynomial of degree atmost two (a parabola or a straight line). In general, through $n+1$ distinct points, we can construct a unique polynomial of degree $\leq n$. *The interpolation polynomial fitting a given data is unique*. We may express it in various forms but are otherwise the same polynomial. For example, $f(x) = x^2 - 2x - 1$ can be written as $x^2 - 2x - 1 = -2 + (x-1) + (x-1)(x-2)$.

To construct the interpolating polynomials we use finite differences.

Finite differences

Let y= f(x) be defined at a tabular set of equidistant points $x_{0,} x_1, x_2, ..., x_n$ as $y_k = f(x_k)\ for\ k = 0,1,2,......., n.$ Since $x_{0,} x_1, x_2, ..., x_n$ are equidistant, we have $x_k = x_{k-1} + h = x_0 + kh,\ for\ k = 1,2,......, n.$ Hence $y_k = f(x_k) = f(x_{k-1} + h) = f(x_{k+1} - h).$

Now we define the finite differences as follows.

Forward difference:

Consider a function, $y = f(x)$ of an independent variable x. Let $y_0, y_1, \dots, y_r$ be the values of y corresponding to the equidistant values $x_0, x_1, \dots, x_r$ of x respectively.

The differences $y_1 - y_0 y_2 - y_1$, ..., in general $y_{r+1} - y_r,\ r = 0,1,2, \dots$

are called the first order forward differences of y = f(x) and are denoted by $\Delta y_0, \Delta y_1, \Delta y_2 ....\Delta y_r$ where Δ is called the *forward difference operator*.

The second order forward differences are $\Delta^2 y_0, \Delta^2 y_1, \dots, \Delta^2 y_r$ are defined recursively as follows

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta y_1 - \Delta y_0 = y_2 - 2y_1 + y_0$$

In general $\Delta^2 y_r = y_{r+2} - 2y_{r+1} + y_r$.

Similarly, one can find the n^{th} forward differences recursively as ,

$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \;\; r = 0,1,2, \dots$

These forward differences can be tabulated and the table so obtained is called forward difference table:

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	
x_2	y_2	Δy_2	$\Delta^2 y_2$		
x_3	y_3	Δy_3			
x_4	y_4				

Backward difference:

The differences of the form, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \dots$, in general $\nabla y_r = y_r - y_{r-1}, r = 1,2,3$ ..., are called first backward differences and ∇ is called the *backward difference operator*.

The second backward differences $\nabla^2 y_2, \nabla^2 y_3, \dots, \nabla^2 y_r$ are similarly defined.

We have $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1} = y_r - 2y_{r-1} + y_{r-2}$

Similarly one can define the n^{th} backward differences,

$\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}$.

The relation between Δ and ∇ is given by $\nabla y_r = \Delta y_{r-1}$.

Backward difference table:

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0	∇y_1	$\nabla^2 y_1$	$\nabla^3 y_1$	$\nabla^4 y_1$
x_1	y_1	∇y_2	$\nabla^2 y_2$	$\nabla^3 y_2$	
x_2	y_2	∇y_3	$\nabla^2 y_3$		
x_3	y_3	∇y_4			
x_4	y_4				

Alternate notations for $y = f(x)$:

1. $\Delta y_x = f(x+h) - f(x)$
2. $\nabla y_x = f(x) - f(x-h)$

Where h is the difference between two consecutive x values.

The shift operator E:

The first forward difference of a function $f(x)$ for equally spaced values of x, with step length h is,

$$\Delta f(x) = f(x+h) - f(x)$$
$$f(x+h) = \Delta f(x) + f(x)$$
$$f(x+h) = (\Delta + 1) f(x)$$

The operator $1 + \Delta$ operating on $f(x)$ shifts $f(x)$ forward to its immediately succeeding value $f(x+h)$.

$\therefore E = 1 + \Delta$is called the first order *shift operator* i.e. $Ef(x) = f(x+h)$

$E^2 f(x) = E(E(f(x)) = E\big(f(x+h)\big) = f(x+2h)$

In general, $E^n f(x) = f(x+nh), n = 1,2, \dots$

An alternate form of the above equation is $E^n y_m = y_{m+n}$

Inverse shift operator:

We know that, $\nabla y_r = \Delta y_{r-1}$ or $\nabla f(x) = \Delta f(x-h)$

i.e.,$\nabla f(x) = f(x) - f(x-h)$

$f(x-h) = (1-\nabla) f(x)$

The operator $1-\nabla$ operating on $f(x)$ shifts $f(x)$ backwards to it's immediately preceding value, $f(x-h)$.

Here $1 - \nabla = E^{-1}$ is called first order *inverse shift operator*.

$E^{-1} f(x) = f(x-h)$
$E^{-2} f(x) = f(x-2h)$
...
$E^{-n} f(x) = f(x-nh), n = 1,2, \dots$

The alternate form of the above equation is $E^{-n} y_m = y_{m-n}$

Properties of finite differences:

1. Linearity property: All finite differences are linear. i.e.,
   For any two constants a, b and for any two functions $f(x), g(x)$, we have,
   $\Delta\big(af(x) \pm bg(x)\big) = a\,\Delta f(x) \pm b\,\Delta g(x)$

2. Index law:
   If m, n are positive integers then, $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$.
3. The first order difference of a polynomial of degree n is a polynomial of degree n-1, then^{th} order difference is a constant and $(n+1)^{th}$ order difference of the polynomial of n^{th} degree is zero.

Proof:

Let $y = f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l$ be a polynomial of degree.

Giving an increment h to x, we get

$y + \Delta y = a(x+h)^n + b(x+h)^{n-1} + c(x+h)^{n-2} + \cdots + k(x+h) + l$

where$\Delta x = h$

$\Delta y = a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \cdots + kh$

Expanding $(x+h)^n, (x+h)^{n-1}$ and so on by binomial expansion

$$\Delta y = a\left[x^n + nhx^{n-1} + \frac{n(n-1)}{2}h^2x^{n-2} + \cdots - x^n\right] + b\left[x^{n-1} + (n-1)hx^{n-2} + \frac{(n-1)(n-2)}{2}h^2x^{n-3} + \quad \ldots - x^{n-1}\right] + kh$$

$$\Delta y = anhx^{n-1} + \left[ah^2\frac{n(n-1)}{2} + hb(n-1)\right]x^{n-2} + \cdots$$

Here the coefficients of x^{n-2}, x^{n-3}, ... are all constants.

$\therefore\ \Delta \mathrm{y} = \mathrm{anhx}^{\mathrm{n}-1} + \mathrm{b}'\mathrm{x}^{\mathrm{n}-2} + \mathrm{c}'\mathrm{x}^{\mathrm{n}-3} + \cdots + \mathrm{k}'\mathrm{x} + \mathrm{l}$

The first difference of a polynomial of n^{th} degree is another polynomial of degree $n-1$.

To find the second difference, we give x an increment, $\Delta x = h$

$$\begin{aligned}\Delta(y + \Delta y) &= \Delta y + \Delta(\Delta y)\\ &= anh(x+h)^{n-1} + b'(x+h)^{n-2} + c(x+h)^{n-3} + \cdots + k'(x+h) + l'\end{aligned}$$

$$\Delta^2 y = anh[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - x^{n-2}] + \cdots + k'h$$
$$\Delta^2 y = an(n-1)h^2x^{n-2} + b''x^{n-3} + c''x^{n-4} + \cdots + k''x + l''$$

The second difference is thus a polynomial of degree $n-2$.

By continuing in this manner, we arrive at a polynomial of zero degree.

$$\begin{aligned}\Delta^n y &= a(n(n-1)(n-2)\ldots 2.1)h^n x^{n-1}\\ &= an!\,h^n, \text{ which is a constant.}\end{aligned}$$

Since n^{th} difference is therefore constant and all higher differences are zero.

Note: The converse of the above property is also true. i.e., if for a function f(x) the n^{th} order finite difference is a constant and $(n+1)^{th}$ ordered finite difference is zero then f(x) is a polynomial of degree n.

4. $\Delta = E\nabla = \nabla E$.
5. $\Delta^k y_r = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} y_{k+i}$

   Proof: Since
   $E = 1 + \Delta,\ we\, have\, \Delta = 1 - E$
   $\Rightarrow \Delta^k = (1-E)^k = \sum_{i=1}^{k} (-1)^i \binom{k}{i} E^i$
   $\therefore \Delta^k y_r = \left(\sum_{i=1}^{k} (-1)^i \binom{k}{i} E^i \right) y_r = \sum_{i=1}^{k} (-1)^i \binom{k}{i} E^i y_r = \sum_{i=1}^{k} (-1)^i \binom{k}{i} y_{k+i}$
6. $\Delta^k y_r = \nabla^k y_{k+r}$

Problems

1. Find the suitable assumptions find the missing terms in the following table:

x	1	2	3	4	5	6	7
y	103.4	97.6	122.9	?	179.0	?	195.8

   Here 5 values of y are known. Hence there exists a polynomial of degree atmost4 approximating y = f(x) for which the fifth order difference vanishes.
   $\Delta^5 y_0 = 0$ i.e.,$(E-1)^5 y_0 = 0$
   $(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_0 = 0$
   $y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$
   Also $(E-1)^5 y_1 = 0$ i.e.$y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$
   Substituting for y_0, y_1, y_2, y_4, y_6
   $y_5 + 10y_3 = 1739.4$
   $5y_5 + 10y_3 = 2502.7$
   $y_5 = 190.825, y_3 = 154.8575$

2. Establish the following identity $E = e^{hD}$

Proof:

a) $Ef(x) = f(x+h)$

By Taylor series expansion,

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots \\ &= f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \cdots \\ &= \left(1 + hD + \frac{h^2}{2!} + \cdots\right) f(x) \\ &= e^{hD} f(x) \end{aligned}$$

$$\therefore Ef(x) = e^{hD} f(x)$$

1.1 Interpolation with evenly spaced points-Newton's formulae for interpolation:

Let the data $(x_i, f(x_i)), i = 0,1,...,n$ be given with uniform spacing, that is, the nodal points are given by $x_i = x_0 + ih, i = 1,2,...,n.$

1.1.1 Newton's Forward Difference Interpolation Formula

Let the interpolating polynomial be given by

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + ... + a_n(x - x_0)(x - x_1)...(x - x_{n-1}).$$

Imposing the interpolating conditions, we obtain

$$a_0 = y_0; a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}; a_2 = \frac{\Delta^2 y_0}{h^2 2!}; ...; a_n = \frac{\Delta^n y_0}{h^n n!};$$

Setting $x = x_0 + ph$, we obtain

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + ... + \frac{p(p-1)...(p-n+1)}{n!}\Delta^n y_0,$$ which is called Newton- Gregory forward difference interpolation formula and is useful for interpolating near the beginning of a set of tabular values.

Remark: It can be shown that the error committed in replacing the function $f(x)$ by means of the polynomial $y_n(x)$ is

$$y(x) - y_n(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!} y^{(n+1)}(\xi)$$

$$= \frac{p(p-1)(p-2)\ldots(p-n)}{(n+1)!} \Delta^{(n+1)} y(\xi), x_0 < \xi < x_n.$$

Example 1 *Derive the Newton's forward difference formula using the operator relations.*

Solution:

$$y(x) = y(x_0 + ph) = E^p(y_0) = (1+\Delta)^p(y_0)$$

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \ldots + \frac{p(p-1)\ldots(p-n+1)}{n!}\Delta^n\right\} y_0.$$

Example 2 *Find the cubic polynomial which takes the following values:*

$y(1) = 24, y(3) = 120, y(5) = 336,$ and $y(7) = 720.$ *Hence, or otherwise, obtain the value of* $y(8)$.

Solution: We form the difference table:

x	y	Δ	Δ^2	Δ^3
1	24			
3	120	96		
5	336	216	120	
7	720	384	168	48

Here h = 2 and p = (x-1)/2.

Therefore,

$$y(x) = 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48)$$

$$= x^3 + 6x^2 + 11x + 6.$$

To determine y (8), we observe that p = 7/2. Hence y (8) = 990.

Example 3 *Using Newton's forward difference formula, find the sum* $S_n = 1^3 + 2^3 + 3^3 + ... + n^3$.

Solution: $\Delta S_n = (n+1)^3, \Delta^2 S_n = 3n^2 + 9n + 7, \Delta^3 S_n = 6n + 12, \Delta^4 S_n = 6.$

Thus S_n is a polynomial of degree 4 in 'n'.

Further, $S_1 = 1, \Delta S_1 = 8, \Delta^2 S_1 = 19, \Delta^3 S_1 = 18, \Delta^4 S_1 = 6.$

Hence,

$$S_n = 1 + (n-1)(8) + ... + \frac{(n-1)(n-2)(n-3)(n-4)}{24}(6)$$
$$= \left[\frac{n(n+1)}{2}\right]^2.$$

1.1.2 Newton's Backward Difference Interpolation Formula

If we choose $y_n(x)$ in the form

$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + ... + a_n(x - x_n)(x - x_{n-1})...(x - x_1)$ and impose the interpolating conditions, we obtain(after some simplification)

$$y_n(x) = y_n(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + ... + \frac{p(p+1)...(p+n-1)}{n!}\nabla^n y_n,$$

where $x = x_n + ph.$

This is Newton-Gregory backward difference interpolation formula and it uses tabular values to the left of y_n. This formula is therefore useful for interpolation near the end of the tabular values.

Remark: It can be shown that the error in this formula may be written as

$$y(x) - y_n(x) = \frac{p(p+1)(p+2)...(p+n)}{(n+1)!}\nabla^{(n+1)} y(\xi), x_0 < \xi < x_n.$$

Example 1 *Derive the Newton's backward difference formula using the operator relations.*

Solution:

$$y(x) = y(x_n - ph) = E^{-p}(y_n) = (1-\nabla)^{-p}(y_n)$$

$$= \left\{1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \ldots + \frac{p(p+1)\ldots(p+n-1)}{n!}\nabla^n\right\} y_n.$$

Example 2*Values of x (in degrees) and sin x are given in the following table:*

x	*15*	*20*	*25*	*30*	*35*	*40*
Sin x	*0.2588190*	*0.3420201*	*0.4226183*	*0.5*	*0.5735764*	*0.6427876*

Determine the value of sin 38^0.

Solution: The difference table is

x	Sin x	∇	∇^2	∇^3	∇^4	∇^5
15	0.2588190	0.0832011	-0.0026029	-0.0006136	0.0000248	0.0000041
20	0.3420201	0.0805982	-0.0032165	-0.0005838	0.00002879	
25	0.4226183	0.0773817	-0.0038053	-0.0005599		
30	0.5	0.0735764	-0.0043652			
35	0.5735764	0.0692112				
40	0.6427876					

Here x = 38 and h = 5. Therefore p = -0.4.
Hence, y (38) = 0.6427876-0.02768448+0.00052382+0.00003583-0.00000120
= 0.6156614.

Example 3*Find the missing term in the following table:*

x	*0*	*1*	*2*	*3*	*4*
y	*1*	*3*	*9*	-	*81*

Explain why the result differs from $3^3 = 27$?

Solution: Since four points are given, the given data can be approximated by a third degree polynomial in x. Hence $\Delta^4 y_0 = 0$.

$$0 = (E-1)^4 y_0 = (E^4 - 4E^3 + 6E^2 - 4E + 1)y_0 \Rightarrow y_3 = 31.$$

The tabulated function is 3^x and the exact value of y (3) is 27. The error is due to the fact that the exponential function $3^x$is approximated by means of a polynomial in x of degree 3.

Example 4*The table below gives the values of tan x for* $0.10 \le x \le 0.30$:

x	*0.10*	*0.15*	*0.20*	*0.25*	*0.30*

$y=\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Find: (a) tan 0.12 (b) tan 0.26 (c) tan 0.40 and (d) tan 0.50.

Solution: The difference table is

x	tan x	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
0.15	0.1511	0.0508			
0.20	0.2027	0.0516	0.0008		
0.25	0.2553	0.0526	0.0010	0.0002	
0.30	0.3093	0.0540	0.0014	0.0004	0.0002

(a) p = 0.4. Hence tan (0.12) = 0.1205.
(b) p = -0.8. Hence tan (0.26) = 0.2662.
(c) p = 2. Hence tan (0.40) = 0.4241. (Extrapolation).
(d) p = 4. Hence tan (0.50) = 0.5543. (Extrapolation).

Remark: Comparison of the computed and actual values shows that in the first two cases (i.e., of interpolation) the results obtained are fairly accurate whereas in the last-two cases (i.e., of extrapolation) the errors are quite considerable. The example therefore demonstrates the important result that if a tabulated function is other than a polynomial, then extrapolation very far from the table limits would be dangerous-although interpolation can be carried out very accurately.

Exercise:

1. *The population in decennial census were as under. Estimate the population for the year 1955:*

year	*1921*	*1931*	*1941*	*1951*	*1961*
population	*46*	*66*	*81*	*93*	*101*

2. *Find the missing term in the table:*

X	*0*	*1*	*2*	*3*	*4*
Y	*1*	*?*	*9*	*27*	*81*

Explain why the result differs from $3^1=3$*?*

3. *The probability integral* $p=\sqrt{\frac{2}{\pi}}\int_0^x \exp\left(-\frac{1}{2}t^2\right)dt$ *has the following values:*

X	1.00	1.05	1.10	1.15	1.20	1.25
p	0.682689	0.706282	0.728668	0.749856	0.769861	0.788700

Calculate p for x = 1.235 .

4. *The values of the elliptic integral* $K(m)=\int_0^{\frac{\pi}{2}}\left(1-m\sin^2\theta\right)^{-1/2}d\theta$ *for certain equidistant values of m are given below. Determine K(0.25).*

m	0.20	0.22	0.24	0.26	0.28	0.30
K(m)	1.659624	1.669850	1.680373	1.691208	1.702374	1.713889

5. *From the following table, find y when x = 1.45.*

X	1.0	1.2	1.4	1.6	1.8	2.0
Y	0.0	-0.112	-0.016	0.336	0.992	2.0

6. *Evaluate sin (0.197) from the following table:*

X	0.15	0.17	0.19	0.21	0.23
Sin x	0.14944	0.16918	0.18886	0.20846	0.22798

7. *Given the table of values:*

X	150	152	154	156
$Y=\sqrt{x}$	12.247	12.329	12.410	12.490

Evaluate $\sqrt{155}$.

8. *From the following table find the number of students who obtained less than 45 marks.*

Marks :	< 40	40 - 50	50 - 60	60 – 70	70 - 80
No. of students :	31	42	51	35	31

[Hint: Apply NFIF for cumulative frequency]

9. *Given that u(0)=-1, u(1)=1, u(3)=17 and u(5)=89 find u(2) and u(4). State the result used.*

1.2 Interpolation with unevenly spaced points:

Let the data $(x_i, y = f(x_i)), i = 0,1,...,n$ be given at distinct unevenly spaced points or non-uniform points $x = x_i, i = 0,1,...,n.$.
This data may also be given at evenly spaced points.
For this data, we can fit a unique polynomial of degree $\leq n$.

1.2.1 Lagrange Interpolation

Let y(x) be continuous and differentiable (n+1) times in its domain. We wish to find a polynomial of degree n, say $L_n(x)$, such that $L_n(x_i) = y_i, i = 0,1,...,n.$
Let

$$L_n(x) = a_0(x-x_1)(x-x_2)...(x-x_n) + a_1(x-x_0)(x-x_2)...(x-x_n) + ...$$
$$+ a_{n-1}(x-x_0)(x-x_1)...(x-x_{n-1})$$

be the desired polynomial of the nth degree such that the interpolating conditions are satisfied.
Substituting these conditions, we obtain,

$$L_n(x) = \frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} y_1 + ...$$
$$+ \frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})} y_n.$$

Remark The linear interpolating polynomial is given by

$$L_1(x) = \sum_{i=0}^{1} l_i(x) y_i; l_0(x) = \frac{x-x_1}{x_0-x_1}, l_1(x) = \frac{x-x_0}{x_1-x_0} \text{ and}$$
$$l_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Example 1 *Certain corresponding values of x and* $\log_{10} x$ *are (300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871). Find* $\log_{10} 301$.

Solution:

$$\log_{10} 301 = \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)}(2.4771) + \frac{(1)(-4)(-6)}{(4)(-1)(-3)}(2.4829)$$
$$+\frac{(1)(-3)(-6)}{(5)(1)(-2)}(2.4843) + \frac{(1)(-3)(-4)}{(7)(3)(2)}(2.4871)$$
$$= 2.4786.$$

Example 2 *Find the Lagrange interpolating polynomial of degree 2 approximating the function* $y = \ln x$ *defined by the following table of values. Hence determine the value of* $\ln 2.7$.

x	*2*	*2.5*	*3.0*
ln x	*0.69315*	*0.91629*	*1.09861*

Solution:

$$L_2(x) = (2x^2 - 11x + 15)(0.69315) - (4x^2 - 20x + 24)(0.91629)$$
$$+(2x^2 - 9x + 10)(1.09861) = -0.08164x^2 + 0.81366x - 0.60761,$$

which is the required quadratic polynomial.

Putting x = 2.7, we get $\ln 2.7 \approx L_2(2.7) = 0.9932518$.

Example 3 *The function* $y = \sin x$ *is tabulated below*

x	*0*	$\pi/4$	$\pi/2$
y	*0*	*0.70711*	*1.0*

Using Lagrange's interpolation formula, find the value of $\sin(\pi/6)$.

Solution:

$$\sin(\pi/6) \approx \frac{(\pi/6-0)(\pi/6-\pi/2)}{(\pi/4-0)(\pi/4-\pi/2)}(0.70711) + \frac{(\pi/6-0)(\pi/6-\pi/4)}{(\pi/2-0)(\pi/2-\pi/4)}(1)$$
$$= 0.51743.$$

Example 4 *Using Lagrange's interpolation formula, find the form of the function y (x) from the following table*

X	*0*	*1*	*3*	*4*
Y	*-12*	*0*	*12*	*24*

Solution: Since y = 0 when x = 1, it follows that (x-1) is a factor.

Let y (x) = (x-1)R(x). Then R(x)=y(x)/(x-1). We now tabulate the values of x and R(x).

x	0	3	4

R(x)	12	6	8

Applying Lagrange's formula, we get

$R(x) = x^2 - 5x + 12.$

Hence the required polynomial approximation toy(x) is given by

$y(x) = (x-1)(x^2 - 5x + 12).$

Inverse interpolation

Suppose that a data $(x_i, f(x_i)), i = 0,1,...,n,$ is given. In interpolation, we predict thevalue of the ordinate $f(x')$ at a non-tabular point $x = x'$. In many applications, we require thevalue of the abscissa x' for a given value of the ordinate $f(x')$. For this problem, we consider thegiven data as $(f(x_i), x_i), i = 0,1,...,n$ and construct the interpolation polynomial. That is, weconsider $f(x)$ as the independent variable and x as the dependent variable. This procedure iscalled inverse interpolation.

Example 1 If $y_1 = 4, y_3 = 12, y_4 = 19$ and $y_x = 7,$ find x.

Solution:

$$x = \frac{(-5)(-12)}{(-8)(-15)}(1) + \frac{(3)(-12)}{(8)(-7)}(3) + \frac{(3)(-5)}{(15)(7)}(4)$$

$= 1.86.$

Exercise:

1. *Applying Lagrange's formula, find a cubic polynomial which approximates the following data:*

x	*-2*	*-1*	*2*	*3*
y(x)	*-12*	*-8*	*3*	*5*

2. *Using Lagrange's formula, express the rational function*
$\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$ *as a sum of partial functions.*
[Hint: Let $f(x) = 3x^2 + x + 1$. Form a table of values of f(x) for x =1,2,3. Obtain the second-order Lagrange polynomial $L_2(x)$][Stanton].

3. *Given the data points (1,-3),(3,9),(4,30) and (6,132) satisfying the function y = f(x), compute f(5) .*

4. *Given the table values*

x	50	52	54	56
$\sqrt[3]{x}$	3.684	3.732	3.779	3.825

Use Lagrange's formula to find x when $\sqrt[3]{x} = 3.756$.

5. *Find a real root of* $f(t) = 0$, *if* $f(-1) = 2, f(2) = -2, f(5) = 4$ and $f(7) = 8$, *using interpolation.*

Remark: For a given data, it is possible to construct the Lagrange interpolation polynomial.However, it is very difficult and time consuming to collect and simplify the coefficients of $x = x_i, i = 0,1,...,n$.

Now, assume that we have determined the Lagrange interpolation polynomialof degree n based on the data values $(x_i, y = f(x_i)), i = 0,1,...,n$ at the $(n+1)$ distinct points.Suppose that to this given data, a new value $(x_{n+1}, y = f(x_{n+1}))$ at the distinct point x_{n+1} is added atthe end of the table. If we require the Lagrange interpolating polynomial for this new data,then we need to compute all the Lagrange fundamental polynomials again. The nth degreeLagrange polynomial obtained earlier is of no use. This is the disadvantage of the Lagrangeinterpolation.

However, Lagrange interpolation is a fundamental result and is used in provingmany theoretical results of interpolation.

DIVIDED DIFFERENCES:

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ be the given points with unequal intervals. Then the first divided difference for the arguments x_0, x_1 is defined by the relation

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}, \text{ Similarly } [x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}, [x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2}, \dots.$$

$$[x_{n-1}, x_n] = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

The second divided differences for x_0, x_1, x_2 is defined as

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}, \text{ Similarly } [x_1, x_2, x_3] = \frac{[x_2, x_3] - [x_1, x_2]}{x_3 - x_1}, \dots,$$

$$[x_{n-2}, x_{n-1}, x_n] = \frac{[x_{n-1}, x_n] - [x_{n-2}, x_{n-1}]}{x_n - x_{n-2}}.$$

The third divided differences for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}, [x_1, x_2, x_3, x_4]$$
$$= \frac{[x_2, x_3, x_4] - [x_1, x_2, x_3]}{x_4 - x_1}, \dots,$$

$$[x_{n-3}, x_{n-2}, x_{n-1}, x_n] = \frac{[x_{n-2}, x_{n-1}, x_n] - [x_{n-3}, x_{n-2}, x_{n-1}]}{x_n - x_{n-3}}.$$

Similarly the other higher order divided differences are also defined.

Divided difference table:

x	y	$I\ DD$	$II\ DD$		$n^{th} DD$
x_0	y_0				
		$[x_0, x_1]$			
x_1	y_1				
			$[x_0, x_1, x_2]$		
		$[x_1, x_2]$			
x_2	y_2				
			$[x_1, x_2, x_3]$		
		$[x_2, x_3]$			$[x_0, x_1, \dots, x_n]$
x_3	y_3		.		
.	.	.	.	$\dots\ \dots$	
.	.	.	.		
.	.	.	$[x_{n-3}, x_{n-2}, x_{n-1}]$		
x_{n-2}	y_{n-2}				
		$[x_{n-2}, x_{n-1}]$			
x_{n-1}	y_{n-1}				
			$[x_{n-2}, x_{n-1}, x_n]$		
x_n	y_n	$[x_{n-1}, x_n]$			

Newton's divided difference interpolation formula:

Let $y_0, y_1, \dots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \dots, x_n$. Then from the definition of divided differences, we have

$$[x_0,\ x_1] = \frac{y-y_0}{x-x_0} \Rightarrow y = y_0 + (x-x_0)[x,\ x_0] \text{ ----------- (1)}$$

Again $[x,\ x_0,\ x_1] = \frac{[x,x_0]-[x_0,\ x_1]}{x-x_1} \implies [x,x_0] = [x_0,\ x_1] + (x-x_1)[x,\ x_0,\ x_1]$.

On using this in (1)

$$y = y_0 + (x-x_0)[x_0,\ x_1] + (x-x_0)(x-x_1)[x,\ x_0,\ x_1] \text{---------(2)}$$

Also $[x,\ x_0,\ x_1,\ x_2] = \frac{[x,\ x_0,\ x_1]-[x_0,\ x_1,\ x_2]}{x-x_2}$

$$\implies [x,\ x_0,\ x_1] = [x_0,\ x_1,\ x_2] + (x-x_2)[x,\ x_0,\ x_1,\ x_2]$$

Using this in (2);

$$y = y_0 + (x-x_0)[x_0,\ x_1] + (x-x_0)(x-x_1)[x_0,\ x_1,\ x_2]$$
$$+(x-x_0)(x-x_1)(x-x_2)[x,x_0,x_1,\ x_2].$$

Proceeding in this manner, we arrive at

$$y = y_0 + (x-x_0)[x_0,\ x_1] + (x-x_0)(x-x_1)[x_0,\ x_1,\ x_2]$$
$$+(x-x_0)(x-x_1)(x-x_2)[x,x_0,x_1,\ x_2] + \cdots$$
$$+(x-x_0)(x-x_1)(x-x_2)\dots(x-x_n)[x_0,x_1,\dots,x_n].$$

is called Newton's general interpolation formula with divided difference.

Examples:

1. Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Evaluate $f(9)$ using Newton's divided difference formula.

x	$f(x)$	I DD	II DD	III DD	IV DD
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457			
13	2366			1	
			42		
		709			
17	5202				

On using divided difference formula, we have

$$f(9) = 150 + 121\,(9-5) + 24\,(9-5)(9-7) + 1(9-5)\,(9-7)\,(9-11) = 810.$$

2. Use Newton's divided difference formula to find $f(4)$ from the given data

x	0	2	3	6
$f(x)$	-4	2	14	158

x	y	I DD	II DD	III DD
0	-4			
		3		
2	2		3	
		12		1
3	14		9	
		48		
6	158			

On using divided difference formula, we have

$$y = f(4) = -4 + (4-0)3 + (4-0)(4-2)3 + (4-0)(4-2)(4-3)1$$
$$= 40.$$

3. Find the interpolating polynomial using Newton's divided difference formula for the following data and also find y when x=8.

x	0	1	2	5
y	2	3	12	147

x	y	I DD	II DD	III DD
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
5	147			

$$y = 2 + (x-0)(1) + (x-0)(x-1)(4) + (x-0)(x-1)(x-2)1$$
$$= x^3 + x^2 - x + 2$$

And also $y(8) = 570.$

4. Find the Newton's divided differences polynomials for the data and also find $f(2.5)$.

x	-3	-1	0	3	5
$f(x)$	-30	-22	-12	330	3458

x	$f(x)$	I DD	II DD	III DD	IV DD
-3	-30				
		4			
-1	-22		2		
		10		4	
0	-12		26		5
		114			
3	330			44	
			290		
		1564			
5	3458				

On using Newton's divided difference polynomial, we have

$$
\begin{aligned}
y = f(x)& \\
&= -30 + (x+3)4 + (x+3)(x+1)2 \\
&\quad + (x+3)(x+1)(x-0)4 \\
&\quad +(x+3)(x+1)x(x-3).
\end{aligned}
$$

$$y = f(x) = 5x^4 + 9x^3 - 27x^2 - 21x - 12.$$

When $x = 2.5$, $y = 102.6785$.

Exercise:

5. Fit an interpolating polynomial for the data $u_{10} = 355, u_0 = -5, u_8 = -21, u_1 = -14, u_4 = -125$ by using Newton's interpolation formula and hence evaluate u_2.

6. Construct the interpolation polynomial for the data given below using Newton's general interpolation formula for the divided differences

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

7. Find $f(4.5)$ by using suitable interpolation

x	-1	0	2	5	10
$f(x)$	-2	-1	7	124	999

8. Fit a polynomial the data $(-4, 1245), (-1, 33), (0, 5), (2, 9), (5, 1335)$. Hence find $f(1)$ and $f(7)$.

Numerical Differentiation:

Let $y_0, y_1, \ldots, y_n$ are the values of a function $y = f(x)$ corresponding to $x_0, x_1, \ldots, x_n$, the process of computing $f'(x), f''(x)$ at some particular value of independent variable x is known as numerical differentiation.

The approximate values of these derivatives are obtained by differentiating an appropriate interpolation formula. If x is nearer to x_0 or nearer to x_n we use Newton's
forward or backward interpolation formula provided the values $x_0, x_1, \ldots, x_n$ are equidistant. If the values $x_0, x_1, \ldots, x_n$ are at unequal intervals we use Newton's general interpolation formula.

Case(i):
The given values of x are equidistant and the given x is near to x_0, we prepare the forward difference table and consider the Newton's forward interpolation formula.

$$f(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

Diffn.w. r. t.r, we obtain

$$f'(x_0 + rh) = \frac{1}{h}\left\{\Delta y_0 + \frac{2r-1}{2}\Delta^2 y_0 + \frac{3r^2-6r+2}{3!}\Delta^3 y_0 + \frac{4r^3-18r^2+22r-6}{24}\Delta^4 y_0 + \cdots\right\} \quad (3)$$

Again on diffn.w r. t. r, we have

$$f''(x_0 + rh) = \frac{1}{h^2}\left\{\Delta^2 y_0 + (r-1)\Delta^3 y_0 + \frac{6r^2-18r+11}{12}\Delta^4 y_0 \ldots\right\}. \quad (4)$$

When the derivative is required at a basic tabulated point x_i,
then choose $x = x_{0,}\ so\ r = 0$. Thus

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left\{\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \frac{1}{4}\Delta^4 y_0 + \cdots\right\}$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right\}$$

Similarly the other higher order derivatives can be computed.

Case(ii):

The given values of x are equidistant and the given x is near to x_n, we prepare the backward difference table and consider the Newton's backward interpolation formula.

$$f(x_n + rh) = y_n + r\nabla y_n + \frac{r(r+1)}{2}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

Diffn.w. r. t.r, we obtain

$$f'(x_n + rh) = \frac{1}{h}\left\{\nabla y_n + \frac{2r+1}{2}\nabla^2 y_n + \frac{3r^2+6r+2}{3!}\nabla^3 y_n + \cdots\right\} \qquad (5)$$

Again on diffn.w r. t. r, we have

$$f''(x_n + rh) = \frac{1}{h^2}\{\nabla^2 y_n + (r+1)\nabla^3 y_n + \cdots\} \qquad (6)$$

When $x = x_{n,}\ so\ r = 0$. Thus

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left\{\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right\}$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left\{\nabla^2 y_n - \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n - \frac{5}{6}\nabla^5 y_n + \cdots\right\}$$

Similarly the other higher order derivatives can be computed.

Examples:

1. Given that

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.75	10.031

Find $\frac{dy}{dx}$ at $x = 1.1$ and $x = 1.6$.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		−0.036				
		0.378		0.006			
1.2	8.781		−0.030		−0.002		
		0.348		0.004		0.001	
1.3	9.129		−0.026		−0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		−0.023		0.002		
		0.299		0.005			
1.5	9.75		−0.018				
		0.281					
1.6	10.031						

We have $r = \frac{x - x_0}{h} = \frac{1.1 - 1.0}{0.1} = 1.$

On using (3),

$$\frac{dy}{dx} = \frac{1}{0.1}\left\{0.414 + \frac{1}{2}(-0.036) - \frac{1}{6}(0.006)\right\} = 3.95.$$

Similarly, from (4), we have

$$\frac{dy}{dx} = \frac{1}{0.1}\left\{0.281 + 0.5\,(-0.018) + \frac{2}{6}(0.005) + \cdots\right\} = 2.7366.$$

2. Given that

x	-2	-1	0	1	2	3
y	0	0	6	24	60	120

Compute $\left(\frac{dy}{dx}\right)_{x=4.5}$ and $\left(\frac{d^2y}{dx^2}\right)_{x=4.5}$.

x	y	Δ	Δ^2	Δ^3	Δ^4
-2	0				
		0			
-1	0		6		
		6		6	
0	6		12		0
		18		6	
1	24				
			18		0
		36			
2	60			6	
			24		
		60			
3	120				

Here $r = \frac{x-x_n}{n} = \frac{4.5-3}{1} = 1.5.$

$$\frac{dy}{dx} = \frac{1}{1}\left\{60 + \frac{3+1}{2}\,24 + \frac{3(1.5)^2+9+2}{6}\,6\right\} = 125.75.$$

$$\frac{d^2y}{dx^2} = \frac{1}{1^2}\{24 + (1.5+1)6\} = 39.$$

Exercise:

3. A rod is rotating in a plane. The following table gives the angle θ in radians through which the rod has turned for various values of the time t seconds.

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.2	4.67

Calculate the angular velocity and angular acceleration of the rod when $t = 0.6\ sec.$ and $t = 0.9\ sec.$

4. The following data gives corresponding values of pressure and specific volume of a superheated steam

v	2	4	6	8	10
p	105	42.7	25.3	16.7	13

Find the rate of change of

i. Pressure with respect to volume when $v = 2$.

ii. Volume with respect to pressure when $p = 105$.

5. Given the following table of values of x and y

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y	1.0000	1.0247	1.0488	1.0723	1.0954	1.1180	1.1401

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$at $= 1.00$.

NUMERICAL INTEGRATION

Consider the definite integral $I=\int_a^b ydx$ where y is known to be a function of x. In many problems of mathematics and applied sciences we need to evaluate this integral. If the function y = f(x) is not known explicitly or the function cannot be integrated by the analytical methods then we use numerical integration. The process of evaluating a definite integral from a set of tabulated values of the integrand f(x) is called numerical integration. Given a set of data points $(x_0,y_0),(x_1,y_1),...,(x_n,y_n)$ of a function y = f(x), where f(x) is not known explicitly or cannot be integrated by analytic methods to compute the value of the definite integral $I=\int_a^b ydx$, we replace f(x) by an interpolating polynomial $\varphi(x)$ and on integration we obtain an approximate value of the definite integral. Thus we can obtain different integration formulae depending upon the type of interpolation formula used.

Newton-Cotes quadrature formula:

This formula is obtained by using Newton's forward difference formula.

Let the interval [a,b] be divided into n equal subintervals such that $a=x_0<x_1<...<x_n=b.$ Clearly $x_n=x_0+nh.$ Hence the integral becomes $I=\int_{x_0}^{x_n} ydx.$

Approximating y by Newton's difference formula, we obtain

$$I=\int_{x_0}^{x_n}\left[y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{6}\Delta^3 y_0+...\right]dx$$

Since $x_n=x_0+ph,\ dx=h\,dp$ and hence the above integral becomes

$$I=h\int_0^n\left[y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{6}\Delta^3 y_0+...\right]dp$$

Which gives on simplification

$$\int_{x_0}^{x_n} ydx=nh\left[y_0+\frac{n}{2}\Delta y_0+\frac{n(2n-3)}{12}\Delta^2 y_0+\frac{n(n-2)^2}{24}\Delta^3 y_0+...\right] \qquad (1)$$

This is known as Newton-Cotes quadrature formula. From this general formula we deduce the following important quadrature rules by taking n = 1, 2, 3.

Trapezoidal rule

Put n = 1 in (1). Then all differences higher than the first will become zero and we obtain

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0+\frac{1}{2}\Delta y_0\right]=h\left[y_0+\frac{1}{2}(y_1-y_0)\right]=\frac{h}{2}[y_0+y_1]$$

We deduce similarly for the interval $[x_1, x_2]$,

$$\int_{x_1}^{x_2} y\,dx=\frac{h}{2}[y_1+y_2] \text{ and so on.}$$

For the last interval $[x_{n-1}, x_n]$ we have

$$\int_{x_{n-1}}^{x_n} y\,dx=\frac{h}{2}[y_{n-1}+y_n]$$

Combining all these expressions, we obtain

$\int_{x_0}^{x_n} y\,dx=\frac{h}{2}\left[y_0+2(y_1+y_2+...+y_{n-1})+y_n\right]$ which is known as the trapezoidal rule.

Geometrical significance

The curve y = f(x) is replaced by n straight lines joining the points (x_0,y_0) and (x_1,y_1); (x_1,y_1) and (x_2,y_2);...;(x_{n-1},y_{n-1}) and (x_n,y_n). Then the area bounded by the curve y = f (x), the ordinates $x = x_0$ and $x = x_n$ and the X-axis is approximately equivalent to the sum of the areas of the n trapeziums obtained.

Note: - The error in the trapezoidal formula is

$E=-\frac{(b-a)}{12}h^2 y''(\bar{x})$ where $y''(\bar{x})$ is the largest value of the second derivatives.

Simpson's 1/3 – rule:

Put n =2 in (1), we get

$$\int_{x_0}^{x_2} ydx=2h\left[y_0+\Delta y_0+\frac{1}{6}\Delta^2 y_0\right]=\frac{h}{3}[y_0+4y_1+y_2]$$

Similarly, $\int_{x_2}^{x_4} ydx=\frac{h}{3}[y_2+4y_3+y_4]$

……………………………………………

and finally $\int_{x_{n-2}}^{x_n} ydx=\frac{h}{3}[y_{n-2}+4y_{n-1}+y_n]$

Summing up we obtain

$\int_{x_0}^{x_n} ydx=\frac{h}{3}\left[y_0+4(y_1+y_3+...+y_{n-1})+2(y_2+y_4+...+y_{n-2})+y_n\right]$ which is Simpson's 1/3 – rule.

Note:-While applying Simpson's 1/3 – rule, the given interval must be divided into even number of equal subintervals.

The error in Simpson's rule is

$E=-\frac{(b-a)}{180}h^4 y^{iv}(\bar{x})$ where $y^{iv}(\bar{x})$ is the largest value of the fourth derivatives.

Simpson's 3/8 – rule:

Put n = 3 in (1), we obtain

$$\int_{x_0}^{x_3} ydx=3h\left[y_0+\frac{3}{2}\Delta y_0+\frac{3}{4}\Delta^2 y_0+\frac{1}{8}\Delta^3 y_0\right]=\frac{3h}{8}[y_0+3y_1+3y_2+y_3]$$

Similarly, $\int_{x_3}^{x_6} ydx=\frac{3h}{8}[y_3+3y_4+3y_5+y_6]$

and finally, $\int_{x_{n-3}}^{x_n} ydx=\frac{3h}{8}[y_{n-3}+3y_{n-2}+3y_{n-1}+y_n]$

summing up all these, we obtain

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[y_0+3y_1+3y_2+2y_3+3y_4+3y_5+2y_6+\ldots+2y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right]$$

Note:-While applying Simpson's 3/8 – rule the number of sub-intervals should be taken as multiple of 3.

The error in Simpson's 3/8 – rule is $E=-\frac{3}{80}h^5 y^{iv}(\bar{x})$.

Examples

1. Evaluate $y=\int_0^6 \frac{dx}{1+x^2}$ by using

   (i) Trapezoidal rule (ii) Simpson's rule (iii) Simpson's 3/8 – rule.

Solution.

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Trapezoidal rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$=\frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)\right]=1.4108$$

(ii) By Simpson's 1/3 – rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$=\frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right]=1.3662$$

(iii) By Simpson's 3/8 – rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$=\frac{3}{8}\left[(1+0.027)+3(0.5+0.2+0.0588+0.0385)+2(0.1)\right]=1.3571$$

2. A solid of revolution is formed by rotating about the X-axis the area between X-axis, the blines x=0 and x = 1 and a curve through the points with the following coordinates.

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed.
Solution:

Volume V= $\pi\int_0^1 y^2 dx$

x	0.00	0.25	0.50	0.75	1.00
y^2	1.0000	0.9793	0.9195	0.8261	0.7081

h = 0.25.

Using Simpson's rule,

$$V = \frac{\pi\times 0.25}{3}\left[1+4(0.9793+0.8261)+2(0.9195)+0.7081\right]=2.8192.$$

3. Compute the value of $\int_{0.2}^{1.4}\left(\sin x-\log x+e^x\right)dx$ using Simpson's 3/8th rule.

Solution:
Let y=sinx-logx+e^x and h=0.2,n=6
The values of y are as given below:

x	0.2	0.4	0.6	0.8	1.00	1.2	1.4
y	3.0295	2.7975	2.8976	3.1660	3.5597	4.0698	4.4042

By Simpson's 3/8th rule, we have

$$\int_{0.2}^{1.4}\left(\sin x-\log x+e^x\right)dx=\frac{3h}{8}\left[(y_0+y_6)+3(y_{`1}+y_2+y_4+y_5)+2y_3\right]$$

$$=\frac{3\times 0.2}{8}\left[(3.0295+4.4042)+3(2.7975+2.8976+3.5597+4.0698)\right.$$
$$\left.+2(3.1660)\right]$$
$$=4.0304$$

EXERCISES:

1. Given that

x	4.0	4.2	4.4	4.6	4.8	5.00	5.2
Y=logx	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} y\,dx$ by

(i) Trapezoidal rule (ii) Simpson's rule (iii) Simpson's 3/8 – rule

2. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,d\theta$ using Simpson's rule taking 9 ordinates.
3. A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below:

h(ft)	10	11	12	13	14
A(sq.ft)	950	1070	1200	1350	1530

If t denotes time t in minutes, the rate of flow of the surface is given by $\frac{dh}{dt} = -48\sqrt{\frac{h}{A}}$. Estimate the time taken for the water level to fall from 14 to 10ft. above the sluices.

4. The velocities of a car (running on a straight road) at intervals of 2 minutes are given below.

Time in minutes	0	2	4	6	8	10	12
Velocity in km/hr	0	22	30	27	18	7	0

Apply Simpson's rule to find the distance covered by the car.

5. A curve is drawn to pass through the points given by the following table:

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve, X-axis and the lines x = 1, x = 4.