Design & Analysis of Algorithms

Assignment-1

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① What do you understand by Asymptotic notations. Define different asymptotic notations with examples.

The main idea of asymptotic analysis is to have a measure of the efficiency of algorithms that don't depend on machine-specific constant and doesn't require algorithms to be implemented and time taken by programs to be compared. Asymptotic notation are mathematical tools to represent the time complexity of algorithms for asymptotic analysis. The f/w^n 3 asymptotic notation are mostly used to represent the time complexity of algo.

(i) **Θ Notation:** The theta notation bounds a function from above & below, so it defines exact asymptotic behaviour. A simple way to get theta notation of an expression is to drop low order terms & ignore leading constant. for example.

$3n^3 + 6n^2 + 6000 = \Theta(n^3)$.

$\Theta(g(n)) = \{f(n)$: there exist +ve constants n_0 & c_1, c_2 that $0 <= c_1 * g(n) <= f(n) <= c_2 * g(n)$ for all $n >= n_0\}$

The above definition means, if f(n) is theta g(n), then the value f(n) is always b/w^n $c_1 * g(n)$ & $c_2 * g(n)$ for large values of n ($n >= n_0$). The definition of theta also requires that f(n) must be non-negative for values of n greater than n_0.

Big O notation :- The Big O notation defines an upper bound of an algo, it bounds a function only from above. for eg., consider the case of Insertion sort. It takes linear time in best case & quadratic time in worst case. We can safely say that the time complexity of Insertion sort is $O(n^2)$, it covers linear time.

If we use Θ notation to represent time complexity of Insertion sort, we have to use 2 statements for best and worst cases:

1. The worst case time complexity of Insertion sort is $\Theta(n^2)$
2. The best case time complexity of Insertion sort is $\Theta(n)$.

$O(g(n))$ = { f(n): there exist +ve constant c & n_0 such that $0 <= f(n) <= c * g(n)$ for all $n >= n_0$ }

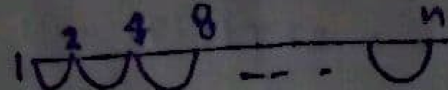
③ **Ω notation** :- Just as Big O notation provides an asymptotic upper bound on a function, Ω notation provides an asymptotic lower bound.

Ω notation can be useful when we have lower bound on time complexity of an algo. The best case performance of an algo. is generally not useful, the omega notation is the least used notation among all three.

$\Omega(g(n))$ = { f(n): there exist +ve constants c & n_0 such that $0 <= c * g(n) <= f(n)$ for all $n >= n_0$ }.

(2) What should be time complexity of -

for(i = 1 to n) (i = i*2)



$i = 1, 2, 4, 8 \ldots n$

$= 2^0, 2^1, 2^2, 2^3 \ldots 2^k$

This is an G.P so,

$a = 1, \quad r = \frac{t_2}{t_1} = \frac{2}{1} = 2$

k th term $\quad t_k = ar^{k-1}$

$n = 1 \cdot 2^{k-1}$

$2^k = 2n$

$k = \log_2(2n)$

$= \log_2(n) + \log_2(2)$

$k = \log n + 1$

Time complexity $= O(\log n+1)$

$= O(\log n)$

(3). →

$T(n) = \{ 3T(n-1)$ if $n > 0$, otherwise $1 \}$

$T(n) = 3T(n-1)$ — (1)

$T(1) = 1$

~~$T(3) = T(2)$~~ .

put $n = n-1$ in eq^n (1)

$T(n-1) = 3T((n-1)-1)$

$T(n-1) = 3T(n-2)$ — (2) .

put eq^n (2) in eq^n (1)

$T(n) = 9T(n-2)$ — (3)

put $n = n-2$ in eq^n (1).

$\cancel{T(n)}$ $T(n-2) = 3T(n-3)$ — (4).

Put (4) in eq^n (3)

$$T(n) = 9[3T(n-3)]$$
$$= 27T(n-3) \quad \text{— (5)}$$

$\cancel{T(n) = 3^k T(n-k) + 3^{k-1} T(n-(k+1)) + \cdots +}$

$$T(n) = 3^k T(n-k)$$

Put $n-k = 1$

$n = k+1 \Rightarrow k = n-1$

$\cancel{T(n) = 3^n T(n-n)}$

$$T(n) = 3^{n-1} T[n-(n-1)]$$
$$T(n) = 3^{n-1} T(1)$$
$$T(n) = 3^{n-1} \cdot 1$$
$$T(n) = \frac{3^n}{3}$$
$$T(n) = O(3^n)$$

This is the time complexity.

(4). $T(n) = \{2T(n-1)-1$ if $n>0$, otherwise $1\}$

$T(1)=1$

$T(n) = 2T(n-1)-1$ — (1)

Put $n = n-1$ in eq^n. (1)

$T(n-1) = 2T((n-2))-1$

$= 2T(n-2)-1$ — (2)

Put (2) in (1)

$T(n) = 2[2T[(n-2)-1]-1$

$= 4T(n-2)-2-1$

$= 4T(n-2)-3$ — (3)

Put $n-2$ in eq^n. (1)

$T(n-2) = 2T(n-2-1)-1$

$T(n-2) = 2T(n-3)-1$ — (4)

Put eq^n (4) in eq^n (3)

$T(n) = 4[2T(n-3)-1]-2-1$

$= 8T(n-3)-4-2-1$

$= 8T(n-3)-7$ — (5)

$T(n) = 2^K T(n-K) - (2^K - 1)$ —⑥

so let $n-K = 1$

$n = K+1$

$K = n-1$

$T(n) = 2^{n-1} T(n-n+1) - (2^{n-1} - 1)$

$= \frac{2^n}{2} T(1) - \left(\frac{2^n}{2} - 1\right)$

$= \frac{2^n}{2} - \left(\frac{2^n}{2} - 1\right)$

$= \frac{2^n}{2}(1-1) - 1$

$T(n) = O(1^n) = O(1)$

⑤ What should be time complexity of –

```
int i=1, s=1;
while (s<=n) {
  i++; s=s+i;
   printf("#");
}.
```

$S(K) = 1 + 2 + 3 + \dots + K$

& stops when $S(K) > n$

$\therefore S(K) \Rightarrow (K * (K+1))/2 \leq n$

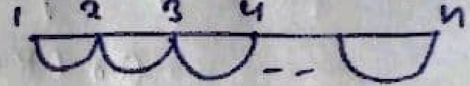
$O(x^2) \leq m$

$x = O$ (root n)

Time complexity $= O(\sqrt{n})$.

⑥ Time complexity of:-

```
void function(int n) {
           1
    int i , count=0;
                  n+1     n
      for( i=1; i*i<=n; i++)
          count++
           n
    }.
```



$$T(n) = 1 + 1 + (n+1) + n + n$$

$$= (3n+3)$$

$$= O(n)$$

7. Time complexity of-

```
void function (int n) {
    int i, j, k, count = 0;
    for ( i = n/2; i <= n, i++)
        for ( j = 1; j <= n; j = j*2)
            for ( k = 1; k <= n; k = k*2 )
                count++;
}.
```

→ for i : Executes O(n) times

for j: Executes O(log n) times

for k: Executes O(log n) times

So. Time. complexity.

$$T(n) = O(n * \log n * \log n)$$

$$= O(n \log^2 n)$$

8. Time complexity of:-

```
Function (int n) {
    if(n==1) return;       //q(n)
    for (i=1 to n) {
        for (j=1 to n) {   // O(1)
            printf("*");
        }
    }
    function (n-3);
}
```

Inner loop. execute only one time due to break statement.

$$T(n) = O(n*1)$$

$$= O(n)$$

(9) Time complexity of -

```
void function( int n) {
    for ( i = 1 to n){                  // O(n)
        for( j=1; j<=n; j=j+i)          // O(n)
            printf("*")
        }
    }
```

for outer loop time complexity = O(n)

for inner loop time complexity = O(n)

So, time complexity

$$T(n) = O(n*n)$$
$$= O(n^2)$$

(6) Time complexity of-

(10). for the func^n, n^k & a^n, what is asymptatic relationship between these functions?

Assume that $K >= 1$ & $a > 1$ are constants - find out the value of c & no for which relation holds

To answer this we need to think about the function, how it grows, & what function binds it together

n^k is a polynomial func^n & c^n is a exponential function. we know that polynomials always grow more slowly than exponential

If we were to say that n^k is $o(c^n)$, then we would be saying that n^k has an asymptatic upper bound of (c^n). As polynomial grow more slowly than exponential.

If we were to say that n^k is $\Omega(c^n)$, then we would be saying that n^k has an asympatic lower bound of $\Omega(c^n)$.- that for a large enough n, n^k always grows faster than c^n Is that true? No, because, polynomial always grow slower than exponential.

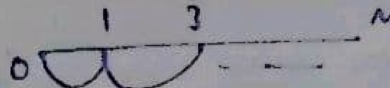
If we were to say that n^k is $\Theta(c^n)$, then we would be saying that n^k is "tightly bound" by $\Theta(c^n)$ - that for large enough n, n^k is always sandwiched b/w^n $K1 * c^n$ & $K2 * c^2$. Is that true? No because polynomial always grow slower than exponentials.

In order for n^k to $\Theta(c^n)$ it would need to be both $o(c^n)$ & $\Omega(c^n)$ which is not possible.

In conclusion, the only true statement here is that n^k is $o(c^n)$.

(11) What is the time complexity of below code & why?

```
void fun (int n) {
  int j = 1, i = 0;
    while (i<n){
      i = i+j;
      j++; } }
```



$= 1 + n + n$

$= 1 + 2n$

$T(n) = O(n)$

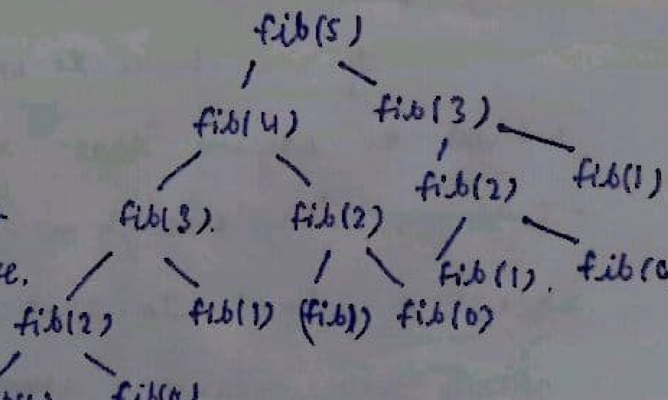
(12). Write recurrence relation for the recursive func^n that prints Fibonacci series. Solve the recurrence relation to get time complexity of the program. & space complexity?

```
int fib( int n)
 {
  if (n<=1)
    return n;
    return fib (n-1) + fib(n-2);
  }
  int main()
    { int n= 9;
      printf ("%d", fib (n));
      getchar();
      return 0;
    }
```

Time complexity $\Rightarrow T(n)$

$T(n) = T(n-1) + T(n-2)$

which is exponential.



Extra space: $O(n)$ if we consider function call stack size, otherwise $O(1)$.

We can observe that implementation does a lot of repeated work. So this is a bad implementation for n^th fibonacci number.

(13). $T(n) = O(n \log n)$

```
int i, j, k = 0,
for ( i = n/2 ; i ≤ n; i++) {
    for (j = 2; j <= n; j = j*2) {
        k = k + n/2
    }
}
```

→ $T(n) = O(n^3)$

```
sum = 0;
for( int i = 1; i <= n; i++)
for( int j = 1; j <= n; j+ = 2)
for( int k = 1; k <= n; k+ = 2)
    sum + = k1
```

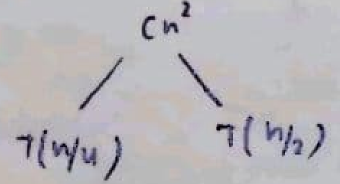
→ $T(n) = O(\log(\log n))$

```
// Here c is a constant greater than 1.
for( int i = 2; i <= n; i = pow (i,c))
{
    // some O(1) expressions.

}
// Here fun is sqrt or cuberoot or any other constant root.

for( int i = n; i > 1; i = fun(i))
{
    //some O(1) expressions
}
```

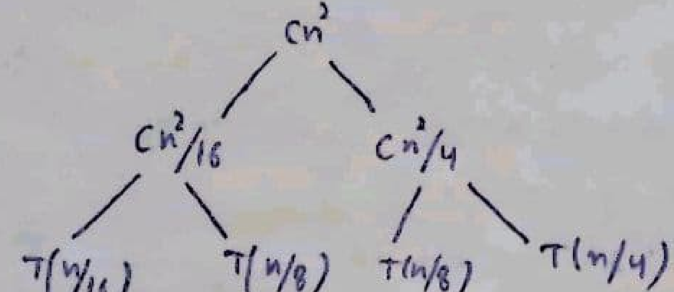
(c) Solve the f/w recurrence relation:

$T(n) = T(n/4) + T(n/2) + cn^2$

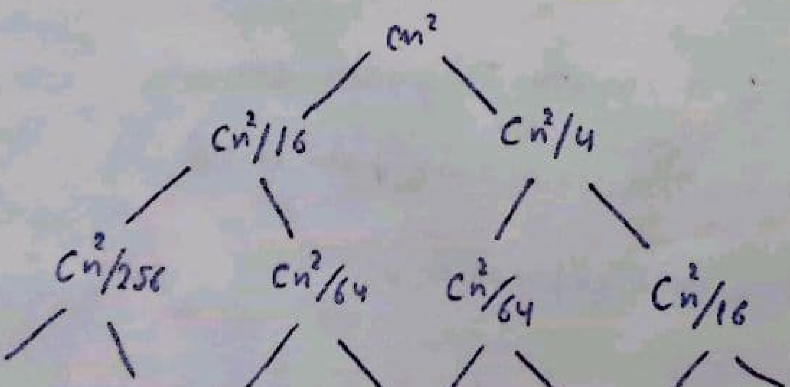
f/w is initial recurrence tree for following recurrence relation



if we break it again, we get f/w recurrence tree



Breaking down further gives us:



To know $T(n)$, we need to calculate sum of tree nodes level by level. If we sum the above tree level by level we get f/w series.

$$T(n) = c(n^2 + 5(n^2)/16 + 25(n^2)/256) + \ldots\ldots$$

Above series is G.P with ratio 5/16.

we can sum above tree for infinite sum.

$$T(n) = n^2/(1-5/16)$$

$$\boxed{T(n) = O(n^2)}$$

(15) what is time complexity of following function fun()?

```
int fun(int n) {
    for(int i=1; i<=n; i++){          // O(n)
    for(int j=1; j<n; j+=i)
      {
        // some O(1) task
      }
    }
}
```

for outer loop

$= 1+1+(n+1)+n$

$= 2n+3$

$T(n) = O(n)$

for inner loop

$T(n) = O(n)$

So, time complexity of fun()

$T(n) = O(n*n)$

$= O(n^2)$

(16) what should be the time complexity of:

```
for(int i=2; i<=n; i=pow(i,k))
  {
    //some O(1) expressions or statements.
  }
```

where k is a constant.

Time complexity of loop is considered as O(log log n) if the loop variables is increased / decreased exponentially by a constant amount.

$T(n) = O(\log \log n)$.

17. Write a recurrence relation when quick sort repeatedly divides the array into 2 parts. Derive time complexity.

Quick sort's worst case is when the choosen pivot is either the largest (19%) or smallest element in the list. When this happens, one of the two sublists will be empty. so quick sort is only called on one list during the sort step.

$$T(n) = T(n-1) + n + 1 \quad \text{(Recurrence relation)}$$

$$T(n) = T(n-2) + n-1 + n-2$$

$$T(n) = T(n-3) + 3n - 1 - 2 - 3$$

$$T(n) = T(1) + \sum_{i=0}^{n-1} (n-i)$$

$$T(n) = \frac{n(n-1)}{2}$$

Now time complexity $T(n) = O(n^2)$

(18). Arrange the f/w^n. in increasing order of rate of growth.

(a) n, $n!$, $\log n$, $\log\log n$, root(n), $\log(n!)$, $n\log n$, 2^n, 2^{2n}, 4^n, n^2, 100.

→ 100, $\log n$, $\log\log n$, $\log(n!)$, ~~$n\log n$~~, n, $n!$, root(n), $n\log n$, n^2, 2^n, 2^{2n}, 4^n,

(b) $2(2^n)$, $4n$, $2n$, 1, $\log(n)$, $\log\log(n)$, $\sqrt{\log(n)}$, $\log 2n$, $2\log(n)$, n, $\log(n!)$, $n!$, n^2, $n\log n$.

→ 1, $\log n$, $\log 2n$, $\sqrt{\log n}$, $2\log n$, $\log(n!)$, $2n$, $4n$, n, $n!$, $n\log n$, n^2, $2(2^n)$.

(c) 8^{2n}, $\log_2 n$, $n\log_6 n$, $n\log_2 n$, $\log(n!)$, $n!$, $\log_8(n)$, 96, $8n^2$, $7n^3$, $5n$

→ 96, $\log_8(n)$, $\log_2(n)$, $\log(n!)$, $5n$, $n!$, $n\log_2 n$, $n\log_6 n$, $8n^2$, $7n^3$, 8^{2n}.

(19). Write linear search code to search an element in a sorted array with minimum comparisons

```
int search (int arr[], int n, int x)
{ int i;
  for(i=0; i<n; i++)
     if (arr[i]==x)
        return i;
  return -1;
}

int main (void)
{ int arr[]= {2,3,4,10,40};
  int x = 10;
  int n = size of (arr) / ~~(arr, n, x)~~;
                     size of (arr[0]);

  //function call
  int result = search (arr, n, x);
  (result == -1)
   ? printf ("Element is not present in array");
   : printf ("Element is present at index %d",
                                 result );
  return 0;
}
```

The time complexity of above algo. is O(n).

(20). Write pseudocode for iterative & recursive insertion sort. Insertion sort is called online sorting. why?

<u>Recursive Insertion sort Alg.</u>

```
//sort an arr[] of size n.
insertion sort (arr, n)
    loop from i=1 to n-1
    (a) Pick element arr [i] & insert
        it into sorted sequence arr[0 -- i-1]
```

Iterative insertion sort

To sort an array of size n in ascending order.

1. Iterate from arr[1] to arr[n] over the array.
2. Compare the current element (key) to its predecessor.
3. If the key element is smaller than its predecessor, compare it to the elements before. Move the greater elements one position up to make space for the swapped element.

An online algo. is one that can process its input piece-by-piece in a serial fashion i.e. in the order that the input is fed to the algo. without having entire input available from the begining.

Insertion sort considers one input element per iteration & produces a partial solution without considering future elements. Thus insertion sort is an online algo.

(21). Complexity of all the sorting algo. that has been discussed.

-) Selection sort: It is sound and easy to understand. It's also very slow & has a time complexity of $O(n^2)$ for both its worst & best case inputs.

-) Insertion sort: Insertion sort has $T(n) = O(n)$ when the input is a sorted list. for an arbitary sorted list $T(n) = O(n^2)$

-) Merge sort: worst case complexity. $T(n) = O(n \log n)$

-) Quick sort: worst case complexity $T(n) = O(n^2)$
best case complexity $T(n) = O(n)$.

(22) Divide all the sorting algo. into in place / stable / online sorting.

In place / outplace technique - A sorting technique is inplace if it does not use any extra memory to sort the array. Among all techniques merge sort is outplaced technique. as it requires an extra array to merge the sorted sub array.

online / offline technique - only insertion sort is online technique because of the underlying algo. it uses.

stable / unstable technique - A sorting technique is stable if it does not change the order of elements with the same value.

Bubble sort, insertion sort & merge sort are stable techniques. while selection sort is unstable. as it may change the order of elements with the same value

(23). Write pseudocode for binary search. What is time & space complexity of linear & binary search.

1. compare x with the middle element.
2. If x matches with the middle element, we return the mid index.
3. Else if x is greater than the middle element, then x can only lie in the right half subarray after the mid element. so we recur for the right half
4. Else (x is smaller) recur for left half.

linear search:- Time complexity : $T(n) = O(n)$.

Space complexity : $O(1)$

we don't need any extra space. to store anything

Binary search: Time complexity : $T(n) = O(\log n)$

Space complexity : $O(1)$ in case of iterative implementation & in case of recursive implement-tation, $O(\log n)$ recursion call stack space.

(24). Write reccurrence relation for binary recursive search

$\rightarrow T(n)$ size of array

```
bool binary search ( int * arr, int l, int r, int key)
{
  if (l>r) return false;   //1
    int mid = (l+r)/2 ;   //1
  if (arr[mid] = = key ) return true;  //1
T(n/2) ->  else if (arr[mid]< key) return binary search
                         (arr, mid+1, r, key);
T(n/2) -> else return binary search (arr, l, mid-1, key);
}.
```

So, recurrence relation.

$$T(n) = T(n/2) + 1$$

$T(1) = 1$ // Base case