

DEFINITE INTEGRATION

Let $f(x)$ be a function defined on $[a, b]$. If $\int f(x)dx = F(x)$, then $F(b) - F(a)$ is called the definite integral of $f(x)$ over $[a, b]$. It is denoted by $\int_a^b f(x)dx$. The real number ‘ a ’ is called **the lower limit** and the real number ‘ b ’ is called **the upper limit**.

This is known as fundamental theorem of integral calculus.

Geometrical Interpretation of Definite Integral: If $f(x) > 0$ for all x in $[a, b]$ then $\int_a^b f(x)dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the x -axis and the lines $x=a$ and $x=b$ i.e., $\int_a^b f(x)dx$.

Properties of Definite Integrals:

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$ i.e., definite integral is independent of its variable.

2). $\int_a^b f(x)dx = - \int_b^a f(x)dx$.

3. If $a < c < b$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

3. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

4. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

5. $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$, if $f(x)$ is an even function

$= 0$, if $f(x)$ is an odd function.

6. $\int_0^{2a} f(x)dx = 2 \int_0^a f(x)dx$ if $f(2a-x) = f(x)$
 $= 0$ if $f(2a-x) = -f(x)$.

Theorem:

If $f(x)$ is an integrable function on $[a, b]$ and $g(x)$ is derivable on $[a, b]$ then

$$\int_a^b (f \circ g)(x) g'(x) dx = \int_{g(a)}^{g(b)} f(x) dx \int_a^b (f \circ g)(x) g'(x) dx = \int_{g(a)}^{g(b)} f(x) dx .$$

PROBLEMS

Evaluate the Following as Limit of Sum

1. Evaluate $\int_0^5 (x+1)dx$.

Sol: We use the following formula for $p = 5$ and $f(x) = x^2 + 1$, $x \in [0, 5]$ and f is continuous over $[0, 5]$.

$$\begin{aligned} \therefore \int_0^5 (x+1)dx &= \int_0^5 f(x)dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{np} f\left(\frac{i}{n}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{5n} \left(\frac{i}{n} + 1 \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{n} + 1 + \frac{2}{n} + 1 + \frac{3}{n} + 1 + \frac{4}{n} + 1 + \left(\frac{5n}{n} + 1 \right) \right] \end{aligned}$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{5n}{n} \right) + 5n \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2} [1 + 2 + 3 + \dots + 5n \text{ terms}] + 5 \end{aligned}$$

$$\left(\because \Sigma n = \frac{n(n+1)}{2} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^2} \frac{5n(5n+1)}{2} + 5$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^2} \frac{n^2 \cdot 5 \left(5 + \frac{1}{n} \right)}{2} + 5$$

$$= \frac{25}{2} + 5 = \frac{35}{2} \quad \left(\because \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) = 0 \right)$$

2. Evaluate $\int_0^4 x^2 dx$.

Sol: $\int_0^4 x^2 dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{np} f\left(\frac{i}{n}\right)$

Here $p = 4$ and $f(x) = x^2$

$$\therefore \int_0^4 x^2 dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{4n} \left(\frac{i}{n} \right)^2$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(\frac{1}{n} \right)^2 + \left(\frac{2}{n} \right)^2 + \left(\frac{3}{n} \right)^2 + \dots + \left(\frac{4n}{n} \right)^2 \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1^2 + 2^2 + 3^2 + \dots + (4n)^2}{n^2} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\frac{4n(4n+1)(8n+1)}{6} \right]$$

$$\left(\because \sum n^2 = \frac{n(n+1)(2n+1)}{6} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{6n^3} \left[n^3 \left(4 \left(4 + \frac{1}{n} \right) \left(8 + \frac{1}{n} \right) \right) \right]$$

$$= \frac{1}{6} 4(4)(8) = \frac{64}{3} \quad \left(\because \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \right)$$

3. Evaluate $\int_0^4 (x + e^{2x}) dx$.

Sol: Here $p = 4$, and $f(x) = x + e^{2x}$

$$\therefore \int_0^4 (x + e^{2x}) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{np} f\left(\frac{i}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{4n} \left(\frac{i}{n} + e^{2i/n} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(\frac{1}{n} + e^{2/n} \right) + \left(\frac{2}{n} + e^{4/n} \right) + \left(\frac{3}{n} + e^{6/n} \right) + \dots + \left(\frac{4n}{n} + e^{8n/n} \right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{4n}{n} \right] + \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^{\frac{2}{n}} + e^{\frac{4}{n}} + \dots + e^{\frac{8n}{n}} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^2} [1 + 2 + 3 + \dots + 4n \text{ terms}] + \lim_{n \rightarrow \infty} \frac{1}{n} e^{\frac{2}{n}} [1 + e^{2/n} + \dots + e^{4n/n}]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^2} \left(\frac{4n(4n+1)}{2} \right) + \lim_{n \rightarrow \infty} \frac{1}{n} e^{2/n} \left[\frac{(e^{2/n})^{4n} - 1}{e^{2/n} - 1} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{n^2}{n^2} \left(\frac{4 \left(4 + \frac{1}{n} \right)}{2} \right) + \lim_{n \rightarrow \infty} \left\{ \frac{e^{2/n} (e^8 - 1)}{\left(\frac{e^{2/n} - 1}{2/n} \right)} \left(\frac{1}{2} \right) \right\}$$

$$= 8 + \left(\frac{e^8 - 1}{2} \right) = \frac{16 + e^8 - 1}{2}$$

$$\left(\because \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) = 0 \text{ and } \lim_{2/n \rightarrow 0} \left(\frac{e^{2/n} - 1}{2/n} \right) = 1 \right)$$

$$= \frac{15 + e^8}{2}$$

$$\left(\because \lim_{n \rightarrow \infty} \frac{2}{n} = 0 \text{ and } \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = 1 \right)$$

4. $\int_0^1 (x - x^2) dx$

Sol: Here $p = 1$ and $f(x) = x - x^2$

$$\therefore \int_0^1 (x - x^2) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n} - \left(\frac{i}{n} \right)^2 \right)$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{n} + \frac{2}{n} + \dots + \frac{n}{n} \right] - \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(\frac{1}{n} \right)^2 + \left(\frac{2}{n} \right)^2 + \dots + \left(\frac{n}{n} \right)^2 \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n^2} [1+2+3+\dots+n] - \lim_{n \rightarrow \infty} \frac{1}{n^3} [1^2+2^2+3^2+\dots+n^2] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \frac{n(n+1)}{2} \right] - \lim_{n \rightarrow \infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \\
 &= \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \frac{n^2 \left(1 + \frac{1}{n}\right)}{2} \right] - \lim_{n \rightarrow \infty} \frac{1}{n^3} \frac{n^3 \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)}{6} \\
 &= \frac{1}{2} - \frac{2}{6} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad \left(\because \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) = 0 \right).
 \end{aligned}$$

Very Short Answer Questions

I. Evaluate the following definite integrals.

1. $\int_0^a (a^2x - x^3) dx$

Sol. $\int_0^a (a^2x - x^3) dx = \left[\frac{a^2x^2}{2} - \frac{x^4}{4} \right]_0^a = \frac{a^4}{2} - \frac{a^4}{4} = \frac{a^4}{4}$

2. $\int_2^3 \frac{2x dx}{1+x^2}$

Sol. $\int_2^3 \frac{2x dx}{1+x^2} = \left[\ln|1+x^2| \right]_2^3 = \ln 10 - \ln 5 = \ln(10/5) = \ln 2$

3. $\int_0^\pi \sqrt{2+2\cos\theta} d\theta$

Sol. $\int_0^\pi \sqrt{2+2\cos\theta} d\theta = \int_0^\pi \sqrt{2 \cdot 2 \sqrt{\cos^2 \frac{\theta}{2}}} d\theta = \int_0^\pi 2\cos(\theta/2) d\theta$

$$= \left[4\sin \frac{\theta}{2} \right]_0^\pi = 4 \left(\sin \frac{\pi}{2} - \sin 0 \right) = 4$$

4. $\int_0^\pi \sin^3 x \cdot \cos^3 x dx$

Sol. $\int_0^\pi \sin^3 x \cdot \cos^3 x dx = \int_0^\pi \sin^3(\pi-x) \cos^3(\pi-x) dx$

$$= - \int_0^\pi \sin^3 x \cos^3 x dx = -I$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0 = -I \Rightarrow 2I = 0 \Rightarrow I = 0$$

5. $\int_0^2 |1-x| dx$

Sol. $\int_0^2 |1-x| dx = \int_0^1 -(x-1) dx + \int_1^2 (x-1) dx$

$$= \int_0^1 (-x+1) dx + \int_1^2 (x-1) dx$$

$$= \left[\frac{-x^2}{2} + x \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^2$$

$$= -\frac{1}{2} + 1 + \left(\frac{4}{2} - 2 \right) - \left(\frac{1}{2} - 1 \right) = \frac{1}{2} + \frac{1}{2} = 1$$

$$6. \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$$

Sol. Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx \dots (i)$

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos(\pi/2 - \pi/2 - x) dx}{1+e^{-x}} \left(\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right) \text{ Adding (1) and (2),}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{e^x \cos x dx}{1+e^x} \dots (2)$$

$$2I = \int_{-\pi/2}^{\pi/2} \frac{\cos x(1+e^x)}{1+e^x} dx = \int_{-\pi/2}^{\pi/2} \cos x dx$$

$$2I = 2 \int_0^{\pi/2} \cos x dx (\because \cos x \text{ is even function})$$

$$\Rightarrow I = [\sin x]_0^{\pi/2} \Rightarrow I = 1$$

$$7. \int_0^1 \frac{dx}{\sqrt{3-2x}}$$

Sol. $\int_0^1 \frac{dx}{\sqrt{3-2x}} = \left(\frac{2\sqrt{3-2x}}{-2} \right)_0^1 = -(\sqrt{3-2 \cdot 1} - \sqrt{3-2 \cdot 0}) = -(1 - \sqrt{3}) = (\sqrt{3} - 1)$

$$8. \int_0^a (\sqrt{a} - \sqrt{x})^2 dx$$

Sol. $\int_0^a (\sqrt{a} - \sqrt{x})^2 dx = \int_0^a (a + x + 2\sqrt{a}\sqrt{x}) dx$

$$= \left[ax + \frac{x^2}{2} - 2\sqrt{a} \cdot x^{3/2} \cdot \frac{2}{3} \right]_0^a$$

$$a^2 + \frac{a^2}{2} - \frac{4}{3}a^2 = \frac{6a^2 + 3a^2 - 8a^2}{6} = \frac{1}{6}a^2$$

$$9. \int_0^{\pi/4} \sec^4 \theta d\theta$$

$$\int_0^{\pi/4} \sec^4 \theta d\theta = \int_0^{\pi/4} \sec^2 \theta \cdot \sec^2 \theta d\theta = \int_0^{\pi/4} \sec^2 \theta (1 + \tan^2 \theta) d\theta$$

$$\text{Sol. Let } I = \int_0^{\pi/4} (\sec^2 \theta + \sec^2 \theta \tan^2 \theta) d\theta = \int_0^{\pi/4} \sec^2 \theta d\theta + \int_0^{\pi/4} \tan^2 \theta \sec^2 \theta d\theta$$

$$= \tan \theta \Big|_0^{\pi/4} + \left(\frac{\tan^3 \theta}{3} \right) \Big|_0^{\pi/4} = 1 - 0 + \frac{1}{3}(1 - 0) = \frac{4}{3}$$

$$10. I = \int_0^3 \frac{x}{\sqrt{x^2 + 16}} dx$$

$$\text{Sol. } x^2 + 16 = t^2 \quad 9 + 16 = t^2$$

$$2x dx = 2t dt \quad 0 + 16 = t^2$$

$$x dx = t dt$$

$$I = \int_4^5 \frac{tdt}{t} = \int_4^5 dt = [t]_4^5 = 5 - 4 = 1$$

$$11. \int_0^1 x \cdot e^{-x^2} dx$$

$$\text{Sol. } \int_0^1 x \cdot e^{-x^2} dx = \frac{1}{2} \int_0^1 2x e^{-x^2} dx, \quad \text{put } -x^2 = t$$

$$\Rightarrow -2x dx = dt \Rightarrow 2x dx = -dt$$

$$x = 1 \Rightarrow t = 1, x = 0 \Rightarrow t = 0$$

$$I = \frac{1}{2} \int_0^{-1} -e^t dt = \frac{1}{2} \left[-e^t \right]_0^{-1}$$

$$= \frac{1}{2} \left[e^0 - e^{-1} \right] = \frac{1}{2} \left(1 - \frac{1}{e} \right)$$

$$12. I = \int_1^5 \frac{dx}{\sqrt{2x-1}}$$

Sol. Let $2x - 1 = t^2$ UL : $t = 3$

$$2 dx = 2t dt \quad LL : t = 1$$

$$dx = t dt$$

$$= \int_1^3 \frac{tdt}{t} = \int_1^3 dt = [t]_1^3 = 3 - 1 = 2$$

Short Answer Questions

$$1. \int_0^4 \frac{x^2}{1+x} dx$$

$$\text{Sol. } \int_0^4 \frac{x^2}{1+x} dx = \int_0^4 \frac{x^2 - 1 + 1}{1+x} dx \Rightarrow I = \int_0^4 (x-1) dx + \int_0^4 \frac{dx}{1+x}$$

$$\begin{aligned} &= \left[\frac{x^2}{2} - x \right]_0^4 + [\log(1+x)]_0^4 \\ &= \frac{4^2}{2} - 4 + \log 5 - \log 1 = 4 + \log 5 \end{aligned}$$

$$2. \int_{-1}^2 \frac{x^2}{x^2 + 2} dx$$

$$\text{Sol. } \int_{-1}^2 \frac{x^2 + 2 - 1}{x^2 + 2} dx = \int_{-1}^2 \left(1 - \frac{2}{x^2 + 2} \right) dx$$

$$= \int_{-1}^2 dx - 2 \int_{-1}^2 \frac{dx}{x^2 + (\sqrt{2})^2}$$

$$= [x]_{-1}^2 - 2 \cdot \frac{1}{\sqrt{2}} \left[\tan^{-1} \left(\frac{x}{\sqrt{2}} \right) \right]^2$$

$$= [2 - (-1)] - \sqrt{2} \left[\tan^{-1} \left(\frac{2}{\sqrt{2}} \right) - \tan^{-1} \left(-\frac{1}{\sqrt{2}} \right) \right]$$

$$\begin{aligned}
 &= 3 - \sqrt{2} \left[\tan^{-1}(\sqrt{2}) - \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right) \right] \\
 &= 3 + \sqrt{2} \left[\tan^{-1}\left(-\frac{1}{\sqrt{2}}\right) - \tan^{-1}(\sqrt{2}) \right]
 \end{aligned}$$

3. $\int_0^1 \frac{x^2}{x^2 + 1} dx$

$$\text{Sol. } \int_0^1 \frac{x^2}{x^2 + 1} dx = \int_0^1 \frac{x^2 + 1 - 1}{x^2 + 1} dx = \int_0^1 dx - \int_0^1 \frac{dx}{x^2 + 1}$$

$$[x]_0^1 - [\tan^{-1} x]_0^1 = 1 - \tan^{-1} 1 = 1 - \frac{\pi}{4}$$

4. $\int_0^{\pi/2} x^2 \sin x dx$

$$\text{Sol. } \int_0^{\pi/2} x^2 \sin x dx = \left[x^2 (-\cos x) \right]_0^{\pi/2} - \int_0^{\pi/2} (2x)(-\cos x) dx = (0 - 0) + 2 \int_0^{\pi/2} x \cos x dx$$

$$= 2 \left[x \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} (2x)(-\cos x) dx$$

$$= 2 \left[\frac{\pi}{2} \times 1 \right] + 2 [\cos x]_0^{\pi/2}$$

$$= \pi + 2(0 - 1) = \pi - 2$$

5. $\int_0^4 |2-x| dx$

$$\text{Sol. } \int_0^2 |2-x| dx + \int_2^4 |2-x| dx$$

$$= \int_0^2 (2-x) dx + \int_2^4 (x-2) dx$$

$$= \left[2x - \frac{x^2}{2} \right]_0^2 + \left[\frac{x^2}{2} - 2x \right]_2^4$$

$$\begin{aligned}
 &= \left(4 - \frac{4}{2} \right) - \left[(8 - 8) - \left(4 - \frac{4}{2} \right) \right] \\
 &= 2 - 0 + 2 = 4
 \end{aligned}$$

6. $\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx \dots (1)$

$$\begin{aligned}
 I &= \int_0^{\pi/2} \frac{\sin^5(\pi/2 - x) dx}{\sin^5(\pi/2 - x) + \cos^5(\pi/2 - x)} \\
 &\left(\because \int_0^a f(a - x) dx = \int_0^a f(x) dx \right) \\
 &= \int_0^{\pi/2} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} \dots (2)
 \end{aligned}$$

Adding (1) and (2),

$$2I = \int_0^{\pi/2} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx = \int_0^{\pi/2} 1 \cdot dx$$

$$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

7. $\int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx \dots (1)$

$$I = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x) - \cos^2(\pi/2 - x)}{\sin^3(\pi/2 - x) + \cos^3(\pi/2 - x)} dx$$

$$\left(\because \int_0^a f(a - x) dx = \int_0^a f(x) dx \right)$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x - \sin^2 x}{\cos^3 x + \sin^3 x} dx \quad \dots (2)$$

Adding (1) and (2),

$$2I = \int_0^{\pi/2} \frac{0dx}{\cos^3 x + \sin^3 x} \Rightarrow I = 0$$

8. Evaluate $\lim_{n \rightarrow \infty} \frac{\sqrt{n+1} + \sqrt{n+2} + \dots + \sqrt{n+n}}{n\sqrt{n}}$

Sol: For determining the limit we use the result that if f is continuous on $[0, 1]$ and

$$P = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1 \right\} \text{ is a partition then } \int_0^1 f(x)dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right)$$

$$\text{Given } \lim_{n \rightarrow \infty} \left(\frac{\sqrt{n+1} + \sqrt{n+2} + \dots + \sqrt{n+n}}{n\sqrt{n}} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{\sum_{i=1}^n \sqrt{n} \sqrt{\left(1 + \frac{1}{n}\right)} + \sqrt{n} \sqrt{\left(1 + \frac{2}{n}\right)} + \dots + \sqrt{n} \sqrt{\left(1 + \frac{n}{n}\right)}}{\sqrt{n}} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \sqrt{1 + \frac{i}{n}}$$

$$= \int_0^1 \sqrt{1+x} dx = \frac{2}{3} \left[(1+x)^{3/2} \right]_0^1$$

$$= \frac{2}{3} [2^{3/2} - 1] = \frac{2}{3} [2\sqrt{2} - 1]$$

9. $\lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+6n} \right]$

Sol: $\lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+5n} \right]$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \dots + \frac{1}{1+\frac{5n}{n}} \right]$$

$$= \int_0^5 \frac{1}{1+x} dx = [\log(1+x)]_0^5 = \log 6$$

Here $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 5\right\}$ is a partition of $[0, 5]$ and $\int_0^5 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right)$

10. $\lim_{n \rightarrow \infty} \frac{1}{n} \left[\tan \frac{\pi}{4n} + \tan \frac{2\pi}{4n} + \dots + \tan \frac{n\pi}{4n} \right]$

Sol: $\lim_{n \rightarrow \infty} \frac{1}{n} \left[\tan \frac{\pi}{4n} + \tan \frac{2\pi}{4n} + \dots + \tan \frac{n\pi}{4n} \right]$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum \tan \frac{i\pi}{4n}$$

$$= \int_0^1 \tan \frac{x\pi}{4} dx$$

$$= \frac{4}{\pi} \left[\log \sec \left(\frac{x\pi}{4} \right) \right]_0^1$$

$$= \frac{4}{\pi} \left[\log \sec \frac{\pi}{4} - \log \sec 0 \right]$$

$$= \frac{4}{\pi} \left[\log \sqrt{2} \right] - 0$$

$$= \frac{4}{\pi} \cdot \frac{1}{2} \log 2 = \frac{2 \log 2}{\pi}$$

11. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^3}{i^4 + n^4}$

Sol: $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^3}{i^4 + n^4}$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\frac{i^3}{n^3} \cdot \frac{1}{1 + \left(\frac{i}{n}\right)^4}}{\left(\frac{i}{n}\right)^4 + 1} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum \frac{\left(\frac{i}{n}\right)^3}{\left(\frac{i}{n}\right)^4 + 1} = \int_0^1 \frac{x^3}{x^4 + 1} dx$$

$$x^4 + 1 = 5 \text{ then } x^3 dx = \frac{1}{4} dt.$$

Upper limit when $x = 1$ is $t = 2$.

Lower limit when $x = 0$ is $t = 1$.

$$\begin{aligned}\therefore \int_0^1 \frac{x^3}{x^4 + 1} dx &= \frac{1}{4} \int_1^2 \frac{dt}{t} \\ &= \frac{1}{4} [\log t]_1^2 = \frac{1}{4} (\log 2)\end{aligned}$$

12. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^2 + i^2}$

Sol: $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^2 + i^2}$

Dividing numerator and denominator by n^2 we get

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{\frac{1}{n}}{1 + \left(\frac{1}{n}\right)^2}$$

$$= \int_0^1 \frac{x dx}{1+x^2}$$

Let $1 + x^2 = t$ then $x dx = \frac{1}{2} dt$

Upper limit when $x = 1$ is $t = 2$.

Lower limit when $x = 0$ is $t = 1$.

$$\therefore \int_0^1 \frac{x dx}{1+x^2} = \frac{1}{2} \int_1^2 \frac{dt}{t} = \frac{1}{2} [\log t]_1^2$$

$$= \frac{1}{2} \log 2 = \log \sqrt{2}.$$

$$13. \lim_{n \rightarrow \infty} \left(\frac{1+2^4+3^4+\dots+n^4}{n^5} \right)$$

$$\text{Sol: } \lim_{n \rightarrow \infty} \left(\frac{1+2^4+3^4+\dots+n^4}{n^5} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{i^4}{n^4}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n} \right)^4$$

$$= \int_0^1 x^4 dx = \left[\frac{x^5}{5} \right]_0^1 = \frac{1}{5}.$$

$$14. \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n^2} \right) \left(1 + \frac{2^2}{n^2} \right) \dots \left(1 + \frac{n^2}{n^2} \right) \right]^{1/n}$$

Sol: Let

$$y = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n^2} \right) \left(1 + \frac{2^2}{n^2} \right) \dots \left(1 + \frac{n^2}{n^2} \right) \right]^{1/n}$$

$$= \lim_{n \rightarrow \infty} \left[\left(\frac{n^2 + 1^2}{n^2} \right) \left(\frac{n^2 + 2^2}{n^2} \right) \dots \left(\frac{n^2 + n^2}{n^2} \right) \right]^{1/n}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n^2 + i^2}{n^2} \right)^{1/n}$$

$$\therefore y = \lim_{n \rightarrow \infty} \left(\frac{n^2 + i^2}{n^2} \right)^{1/n}$$

$$\therefore \log y = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{n^2 + i^2}{n^2} \right)^{1/n}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left[1 + \left(\frac{i}{n} \right)^2 \right]$$

$$= \int_0^1 \log(1+x^2) dx$$

(Using integration by parts)

$$= \left[x \log(1+x^2) \right]_0^1 - \int_0^1 x \frac{2x}{1+x^2} dx$$

$$= \log 2 - 2 \int_0^1 \left(\frac{x^2 + 1 - 1}{x^2 + 1} \right) dx$$

$$= \log 2 - 2[x]_0^1 + 2 \left[\tan^{-1} x \right]_0^1$$

$$= \log 2 - 2 + 2(\tan^{-1} 1)$$

$$= \log 2 - 2 + 2 \frac{\pi}{4}$$

$$\therefore \log_e y = \log 2 - 2 + \frac{\pi}{2}$$

$$y = e^{\log 2 - 2 + \pi/2}$$

$$= e^{\log_e 2 - e^{\frac{\pi}{2}}}$$

$$= 2e^{\frac{\pi-4}{2}}$$

$$\therefore \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n^2} \right) \left(1 + \frac{2^2}{n^2} \right) \dots \left(1 + \frac{n^2}{n^2} \right) \right]^{1/n} = 2e^{\frac{\pi-4}{2}}.$$

15. $\lim_{n \rightarrow \infty} \left[\frac{(n!)^{1/n}}{n} \right]$

Sol: $\lim_{n \rightarrow \infty} \left[\frac{(n!)^{1/n}}{n} \right]$

$$= \lim_{n \rightarrow \infty} \left[\frac{(\underline{n})^{1/n}}{n} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{(\underline{n})}{n^n} \right]^{1/n}$$

$$\text{Let } y = \lim_{n \rightarrow \infty} \left[\frac{(\underline{n})}{n^n} \right]^{1/n}$$

$$\log_e y = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left[\left(\frac{1}{n} \right) \left(\frac{2}{n} \right) \dots \left(\frac{i}{n} \right) \dots \left(\frac{n}{n} \right) \right]$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \log \frac{r}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum \log \left(\frac{r}{n} \right) \\
 &= \int_0^1 \log_e x = [x \log x]_0^1 - \int_0^1 dx \\
 &= [x \log x - x]_0^1 \\
 &= [x(\log x - 1)]_0^1 = -1 \\
 \therefore y &= e^{-1} = \frac{1}{e}.
 \end{aligned}$$

Long Answer Questions

1. $\int_0^{\pi/2} \frac{dx}{4+5\cos x}$

$$\begin{aligned}
 \text{Sol. } \int_0^{\pi/2} \frac{dx}{4+5\cos x} &= \int_0^{\pi/2} \frac{dx}{4+5 \left[\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} \right]} \\
 &= \int_0^{\pi/2} \frac{dx}{4 \left[\tan^2 \frac{x}{2} + 1 \right] + 5 \left[1 - \tan^2 \frac{x}{2} \right]} \\
 &\quad \left[\tan^2 \frac{x}{2} + 1 \right]
 \end{aligned}$$

put $\tan \frac{x}{2} = t \Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dt \Rightarrow dx = \frac{2dt}{1+t^2}$

$x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{2} \Rightarrow t = 1$

$$\begin{aligned}
 &= \int_0^1 \frac{(1+t^2)}{4t^2 + 4 + 5 - 5t^2} \frac{2dt}{1+t^2} \\
 &= \int_0^1 \frac{2}{9-t^2} dt = \frac{2}{2 \cdot 3} \ln \left[\left| \frac{3+t}{3-t} \right| \right]_0^1 \\
 &= \frac{1}{3} \left[\ln \frac{4}{2} \right] = \frac{1}{3} \ln 2
 \end{aligned}$$

2. $\int_a^b \sqrt{(x-a)(b-x)} dx$

Sol. $\int_a^b \sqrt{(x-a)(b-x)} dx = \int_a^b \sqrt{-x^2 + (a+b)x - ab} dx$

$$= \int_a^b \sqrt{\left(\frac{b-a}{2}\right)^2 - \left[x - \left(\frac{a+b}{2}\right)\right]^2} dx$$

$$\left(\because -x^2 + (a+b)x - ab = -(x^2 - (a+b)x + ab) = -\left(x - \frac{a+b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 + ab \right)$$

$$= \left[\frac{1}{2} \left(x - \left(\frac{a+b}{2} \right) \right) \sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{4 \cdot 2} \sin^{-1} \frac{\left(x - \left(\frac{a+b}{2} \right) \right)}{\left(\frac{b-a}{2} \right)} \right]_a^b$$

$$= 0 + \frac{(b-a)^2}{8} \left[\sin^{-1}(1) - \sin^{-1}(-1) \right]$$

$$= \frac{(b-a)^2}{8} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{\pi}{8} (b-a)^2$$

3. $\int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

Sol. and $x = \sin t$

$$x=0 \Rightarrow t=0 \text{ and } x=\frac{1}{2} \Rightarrow t=\frac{\pi}{6}$$

$$\int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/6} t \sin t dt = \left(t \int \sin t dt \right)_{0}^{\pi/6} - \int_0^{\pi/6} 1 \cdot (-\cos t) dt$$

$$= t(-\cos t) \Big|_0^{\pi/6} + (\sin t) \Big|_0^{\pi/6} = \frac{\pi}{6} \left(-\frac{\sqrt{3}}{2} \right) - 0 + \frac{1}{2} - 0$$

$$= \frac{1}{2} - \frac{\pi \sqrt{3}}{12}$$

4. $\int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x} dx$

Sol. $\int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x} dx = \int_0^{\pi/4} \frac{\sin x + \cos x}{9+16[1-(\sin x - \cos x)^2]} dx$

put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

$$x=0 \Rightarrow t=-1 \text{ and } x=\frac{\pi}{4} \Rightarrow t=0$$

$$= \int_{-1}^0 \frac{dt}{25-16t^2} = \frac{1}{16} \int_{-1}^0 \frac{dt}{\frac{25}{16}-t^2}$$

$$= \frac{1}{16} \times \frac{1}{2 \times \frac{5}{4}} \left[\ln \left| \frac{\frac{5}{4}+t}{\frac{5}{4}-t} \right| \right]_{-1}^0$$

$$= -\frac{1}{40} \ln \left[\frac{1/4}{9/4} \right] = \frac{1}{40} \cdot 2 \ln 3 = \frac{1}{20} \ln 3$$

5. $\int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx$

Sol. let $I = \int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx ----- (1)$

$$= \int_0^{\pi/2} \frac{a \sin \left(\frac{\pi}{2} - x \right) + b \cos \left(\frac{\pi}{2} - x \right)}{\sin \left(\frac{\pi}{2} - x \right) + \cos \left(\frac{\pi}{2} - x \right)} \left(\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right) I = \int_0^{\pi/2} \frac{a \cos x + b \sin x}{\sin x + \cos x} dx ----- (2)$$

$$(1)+(2) \Rightarrow 2I = \int_0^{\pi/2} \frac{a(\sin x + \cos x) + b(\sin x + \cos x)}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} (a+b)dx = (a+b) \frac{\pi}{2} \Rightarrow I = (a+b) \frac{\pi}{4}$$

6. $\int_0^a x(a-x)^n dx$

Sol. let $I = \int_0^a x(a-x)^n dx \dots\dots (1)$

$$I = \int_0^a (a-x)(x)^n dx \dots\dots (2)$$

$$\begin{aligned} I &= \int_0^a ax^n dx - x^{n+1} dx \\ &= \left[\frac{ax^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^a = \frac{a^{n+2}}{n+1} - \frac{a^{n+2}}{n+2} \end{aligned}$$

$$I = \frac{a^{n+2}}{(n+1)(n+2)}$$

7. $\int_0^2 x\sqrt{2-x} dx$

Sol. $I = \int_0^a x\sqrt{2-x} dx$

$$\begin{aligned} \int_0^a f(x)dx &= \int_0^a f(a-x)dx \\ &= \int_0^2 (2-x)\sqrt{x} dx = \int_0^2 (2\sqrt{x} - x\sqrt{x}) dx \\ &= \int_0^2 [2(x)^{1/2} - x^{3/2}] dx = \left[\frac{2x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} \right]_0^2 \\ &= \frac{4}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2} = \sqrt{2} \left[\frac{8}{3} - \frac{8}{5} \right] = \frac{16\sqrt{2}}{15} \end{aligned}$$

8. $\int_0^\pi x \sin^3 x dx$

Sol. $I = \int_0^\pi x \sin^3 x dx = \int_0^\pi (\pi - x) \sin^3 (\pi - x) dx \quad \left(\because \int_0^a f(x) dx = \int_0^a f(a - x) dx \right)$

$$I = \int_0^\pi (\pi - x) \sin^3 x dx = \int_0^\pi \pi \sin^3 x dx - \int_0^\pi x \sin^3 x dx$$

$$= \int_0^\pi \pi \sin^3 x dx = I$$

$$\Rightarrow 2I = \int_0^\pi \pi \sin^3 x dx = \pi \int_0^\pi \frac{3 \sin x - \sin 3x}{4} dx$$

$$= \frac{\pi}{4} \left(-3 \cos x + \frac{\cos 3x}{3} \right)_0^\pi = \frac{\pi}{4} \left(-3(-1) - \frac{1}{3} + 3 - \frac{1}{3} \right)$$

$$= \frac{\pi}{4} (6 - 2/3) = \frac{\pi}{4} \cdot 16/3$$

$$\therefore I = \frac{\pi}{2.4} \frac{16}{3} = \frac{2\pi}{3}$$

9. $\int_0^\pi \frac{x}{1 + \sin x} dx$

Sol. $I = \int_0^\pi \frac{x}{1 + \sin x} dx \quad \dots (i)$

$$I = \int_0^\pi \frac{(\pi - x) dx}{1 + \sin(\pi - x)} = \int_0^\pi \frac{\pi dx}{1 + \sin x} - \int_0^\pi \frac{x dx}{1 + \sin x}$$

$$= \int_0^\pi \frac{\pi dx}{1 + \sin x} - I \quad \dots (ii)$$

$$2I = \int_0^\pi \frac{\pi dx}{1 + \sin x} \Rightarrow I = \frac{\pi}{2} \int_0^\pi \frac{dx}{1 + \sin x}$$

$$= \frac{\pi}{2} \int_0^\pi \frac{(1 - \sin x)}{1 - \sin^2 x} dx = \frac{\pi}{2} \int_0^\pi \left(\frac{1 - \sin x}{\cos^2 x} \right) dx$$

$$= \frac{\pi}{2} \left(\int_0^\pi \frac{1}{\cos^2 x} dx - \int_0^\pi \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx \right)$$

$$= \frac{\pi}{2} \int_0^{\pi} \sec^2 x dx - \int_0^{\pi} \sec x \cdot \tan x dx$$

$$\begin{aligned} &= \frac{\pi}{2} \left([\tan x]_0^{\pi} - [\sec x]_0^{\pi} \right) \\ &= \frac{\pi}{2} [(0-0) - (-1-1)] = \frac{\pi}{2} \cdot 2 = \pi \end{aligned}$$

10. $\int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx$

$$\begin{aligned} \text{Sol. } I &= \int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx \\ &= \int_0^{\pi} \frac{(\pi-x) \sin^3(\pi-x)}{1 + \cos^2(\pi-x)} dx = \int_0^{\pi} \frac{(\pi-x) \sin^3 x}{1 + \cos^2 x} dx \\ &= \pi \int_0^{\pi} \frac{\sin^3 x}{1 + \cos^2 x} dx - \int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx \quad \dots (1) \\ &= \pi \int_0^{\pi} \frac{\sin^3 x}{1 + \cos^2 x} dx - I \end{aligned}$$

$$2I = \int_0^{\pi} \frac{\sin^3 x dx}{1 + \cos^2 x}$$

Put $t = \cos x \Rightarrow dt = -\sin x dx$

$$\begin{aligned} 2I &= \int_1^{-1} -\frac{(1-t^2)}{1+t^2} dt = \int_{-1}^1 \frac{1-t^2}{1+t^2} dt \\ &= \int_{-1}^1 \left(-1 + \frac{2}{1+t^2} \right) dt = \left[-t + 2 \tan^{-1} t \right]_{-1}^1 \\ &= \left[-1 + 2 \tan^{-1} 1 \right] - \left[-1 + 2 \tan^{-1}(-1) \right] \\ &= -1 + 2 \cdot \frac{\pi}{4} + 1 - 2 \left(-\frac{\pi}{4} \right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi \end{aligned}$$

11. $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Sol. $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Put $x = \tan \theta$

$$dx = \sec^2 \theta d\theta$$

$$\text{l.l : } x = 0 \Rightarrow \theta = 0$$

$$\text{u.l: } x = 1 \Rightarrow \theta = \pi/4$$

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx = \int_0^{\pi/4} \frac{\log(1+\tan \theta) \sec^2 \theta d\theta}{(1+\tan^2 \theta)}$$

$$= \int_0^{\pi/4} \log(1+\tan \theta) d\theta$$

$$\text{let } I = \int_0^{\pi/4} \log(1+\tan \theta) d\theta$$

$$\begin{aligned} &= \int_0^{\pi/4} \log \left[1 + \tan \left(\frac{\pi}{4} - \theta \right) \right] d\theta = \int_0^{\pi/4} \log \left[1 + \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \right] d\theta \\ &= \int_0^{\pi/4} \log \left[1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right] d\theta = \int_0^{\pi/4} \log \left[\frac{1 + \tan \theta + 1 - \tan \theta}{1 + \tan \theta} \right] d\theta = \int_0^{\pi/4} \log \frac{2}{1 + \tan \theta} d\theta = \int_0^{\pi/4} [\log 2 - \log(1 + \tan \theta)] d\theta \\ &= \log 2 \int_0^{\pi/4} d\theta - \int_0^{\pi/4} \log(1 + \tan \theta) d\theta = \log 2 \int_0^{\pi/4} d\theta - I \quad 2I = \log 2(\theta)_0^{\pi/4} = (\log 2) \frac{\pi}{4} \end{aligned}$$

$$\therefore I = \frac{\pi}{8} \log 2$$

12. $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$

Sol. $I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{(\pi - x) \sin(\pi - x) dx}{1 + \cos^2(\pi - x)}$

$$= \int_0^\pi \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{\sin x dx}{1 + \cos^2 x} - \int_0^\pi \frac{x \sin x dx}{1 + \cos^2 x}$$

$$= \pi \left\{ \tan^{-1}(-\cos x) \right\}_0^\pi - I$$

$$2I = \pi \left\{ \tan^{-1} 1 - \tan^{-1}(-1) \right\} = \pi \left(\frac{\pi}{4} + \frac{\pi}{4} \right) = 2 \frac{\pi^2}{4}$$

$$I = \frac{\pi^2}{4} \Rightarrow \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$$

13. $\int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx$

Sol., $I = \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx \dots\dots 1.$

$$= \int_0^{\pi/2} \frac{\sin^2 \left(\frac{\pi}{2} - x \right)}{\cos \left(\frac{\pi}{2} - x \right) + \sin \left(\frac{\pi}{2} - x \right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x dx}{\sin x + \cos x} \dots\dots 2.$$

Adding 1. and 2.

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

Consider $\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

Put $\tan(x/2) = t$

$$dx = \frac{2dt}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}$$

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x} &= \int_0^1 \frac{2tdt}{2t + (1-t^2)} \\ &= 2 \int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2} = 2 \cdot \frac{1}{2\sqrt{2}} \left[\log \frac{\sqrt{2}+t-1}{\sqrt{2}-t+1} \right]_0^1 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2}} \left(\log 1 - \log \frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \\
 &= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} \\
 &= \frac{1}{\sqrt{2}} \log (\sqrt{2}+1)^2 = \frac{2}{\sqrt{2}} \log (\sqrt{2}+1)
 \end{aligned}$$

$$I = \frac{1}{\sqrt{2}} \log (\sqrt{2}+1)$$

14. Suppose that $f: \mathbf{R} \rightarrow \mathbf{R}$ is a continuous periodic function and T is the period of it. Let

$a \in \mathbf{R}$. Then prove that for any positive integer n , $\int_0^{a+nT} f(x)dx = n \int_0^{a+T} f(x)dx$.

$$\text{Sol. } \int_0^{a+nT} f(x)dx = \int_0^{a+T} f(x)dx + \int_{a+T}^{a+2T} f(x)dx + \dots + \int_{a+rT}^{a+(r+1)T} f(x)dx + \dots + \int_{a+(x-1)T}^{a+xT} f(x)dx \dots (1)$$

Consider $(r+1)$ th integral of RHS

$$\int_{a+rT}^{a+(r+1)T} f(x)dx$$

$$\text{Let } x = y + rT \Rightarrow dx = dy$$

$$x = a + rT \Rightarrow y = a$$

$$x = a + (r+1)T \Rightarrow y = a + T$$

$$\begin{aligned}
 \int_{a+rT}^{a+(r+1)T} f(x)dx &= \int_a^{a+T} f(y+rT)dy \\
 &= \int_a^{a+T} f(y)dy \quad (f \text{ is periodic}) = \int_a^{a+T} f(x)dx
 \end{aligned}$$

Similarly we can show that each integral of (1) is equal to $\int_a^{a+T} f(x)dx$.

$$\begin{aligned}
 \therefore \int_0^{a+nT} f(x)dx &= \int_0^{a+T} f(x)dx + \int_0^{a+T} f(x)dx \dots n \text{ terms} \\
 &= n \int_0^{a+T} f(x)dx
 \end{aligned}$$

15. $\int_0^\pi \frac{1}{3+2\cos x} dx$

Sol: Let $\tan \frac{x}{2} = t$ then

$$dx = \frac{2dt}{1+t^2} \text{ And } \cos x = \frac{1-t^2}{1+t^2}$$

Upper limit when $x = \pi$ is $t = \infty$

Lower limit when $x = 0$ is $t = 0$.

$$\begin{aligned} \therefore \int_0^\pi \frac{dx}{3+2\cos x} &= \int_0^\infty \frac{\frac{2dt}{1+t^2}}{3+2\left(\frac{1-t^2}{1+t^2}\right)} = \int_0^\infty \frac{2dt}{5+t^2} \\ &= 2 \int_0^\infty \frac{dt}{(\sqrt{5})^2 + t^2} = \frac{2}{\sqrt{5}} \left[\tan^{-1} \frac{t}{\sqrt{5}} \right]_0^\infty \\ &= \frac{2}{\sqrt{5}} \left[\tan^{-1} \infty - \tan^{-1} 0 \right] \\ &= \frac{2}{\sqrt{5}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{5}}. \end{aligned}$$

16. $\int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$

Sol: Let $x = \tan \theta$ then $dx = \sec^2 \theta d\theta$

$$\therefore \text{Upper limit when } x = 1 \text{ is } \theta = \frac{\pi}{4}$$

And lower limit when $x = 0$ is $\theta = 0$.

$$\begin{aligned} \therefore \int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx \\ &= \int_0^{\pi/4} \sin^{-1} \left(\frac{2 \tan \theta}{1+\tan^2 \theta} \right) \sec^2 \theta d\theta \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{\pi/4} \sin^{-1}(\sin 2\theta) \sec^2 \theta d\theta \\
 &= \int_0^{\pi/4} 2\theta \cdot \sec^2 \theta d\theta \\
 &= [2\theta \tan \theta]_0^{\pi/4} - \int_0^{\pi/4} 2 \cdot \tan \theta d\theta \\
 &= [2\theta \tan \theta]_0^{\pi/4} - 2[\log(\sec \theta)]_0^{\pi/4} \\
 &= \frac{2\pi}{4} - 2 \left[\log \sec \left(\frac{\pi}{4} \right) - 0 \right] \\
 &= \frac{\pi}{2} - 2 \log \sqrt{2} \\
 &= \frac{\pi}{2} - 2 \left(\frac{1}{2} \right) \log 2 \\
 &= \frac{\pi}{2} - \log 2
 \end{aligned}$$

17. $\int_0^1 x \tan^{-1} x dx$

Sol: Let $x = \tan \theta$ then $dx = \sec^2 \theta d\theta$

$$\therefore \text{Upper limit when } x = 1 \text{ is } \theta = \frac{\pi}{4}$$

And lower limit when $x = 0$ is $\theta = 0$.

$$\therefore \int_0^1 x \tan^{-1} x dx = \int_0^{\pi/4} \theta \tan \theta \sec^2 \theta d\theta$$

Using integration by parts by taking $u = \theta$ and $v = \tan \theta \sec^2 \theta$ we get

$$\int_0^{\pi/4} \theta \tan \theta \sec^2 \theta d\theta = \left[\theta \cdot \frac{\tan^2 \theta}{2} \right]_0^{\pi/4} - \frac{1}{2} \int_0^{\pi/4} \tan^2 \theta d\theta$$

$$\begin{aligned}
 &= \frac{\pi}{4} \left(\frac{1}{2} \right) - \frac{1}{2} \int_0^{\pi/4} (\sec^2 \theta - 1) d\theta \\
 &= \frac{\pi}{8} - \frac{1}{2} \int_0^{\pi/4} \sec^2 \theta d\theta + \frac{1}{2} \int_0^{\pi/4} d\theta = \frac{\pi}{8} - \frac{1}{2} [\tan \theta]_0^{\pi/4} + \frac{1}{2} [\theta]_0^{\pi/4} \\
 &= \frac{\pi}{8} - \frac{1}{2}[1] + \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8} + \frac{\pi}{8} - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}.
 \end{aligned}$$

18. $\int_{-1}^{3/2} |x \sin \pi x| dx$

Sol. We know that $|x \cdot \sin \pi x| = x \cdot |\sin \pi x|$

Where $-1 \leq x \leq 1$

And $|x \cdot \sin \pi x| = -x \sin \pi x$ where $1 < x \leq 3/2$

$$\begin{aligned}
 \therefore \int_{-1}^{3/2} |x \sin \pi x| dx &= \int_{-1}^1 |x \sin \pi x| dx + \int_1^{3/2} |x \sin \pi x| dx \\
 &= \int_{-1}^1 x \sin \pi x dx - \int_1^{3/2} x \sin \pi x dx \\
 &= \left(-\frac{x \cdot \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} \right)_{-1}^1 - \left(-x \cdot \frac{\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} \right)_1^{3/2} \\
 &= \frac{1}{\pi} + \frac{(-1)(-1)}{\pi} - \left[-\frac{1}{\pi^2} - \frac{1}{\pi} \right] \\
 &= \frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi} = \frac{3}{\pi} + \frac{1}{\pi^2}
 \end{aligned}$$

19. $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

Sol. $= \int_0^{\pi} \frac{(\pi - x) \cdot \sin(\pi - x) dx}{1 + \cos^2(\pi - x)}$ $\left(\int_0^a f(x) dx = \int_0^a f(a - x) dx \right)$

$$= \pi \int_0^{\pi} \frac{(\pi - x) \sin x dx}{1 + \cos^2 x} = \pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x} - I$$

$$2I = \pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x} \Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x}$$

$$t = \cos x \, dx = -\sin x \, dx$$

$$\int_0^{\pi} \frac{\sin x \, dx}{1 + \cos^2 x} = \int_{-1}^1 \frac{-dt}{1 + t^2} = \int_{-1}^1 \frac{dt}{1 + t^2} = 2 \int_0^1 \frac{dt}{1 + t^2}$$

$f(x)$ is even

$$= 2 \left(\tan^{-1} t \right)_0^1 = 2(\tan^{-1} 1 - \tan^{-1} 0)$$

$$= 2 \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{2}$$

$$I = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

20. $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$

Sol. $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$

$$\text{Put } x = a \sin \theta \Rightarrow dx = a \cos \theta \, d\theta$$

$$x = 0 \Rightarrow \theta = 0$$

$$x = a \Rightarrow \theta = \pi/2$$

$$I = \int_0^{\pi/2} \frac{a \cos \theta \, d\theta}{a \sin \theta + \sqrt{a^2 - a^2 \sin^2 \theta}}$$

$$= \int_0^{\pi/2} \frac{a \cos \theta \, d\theta}{a(\sin \theta + \cos \theta)}$$

$$= \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)} = \int_0^{\pi/2} \frac{\sin \theta \, d\theta}{\cos \theta + \sin \theta}$$

$$I = \int_0^{\pi/2} \frac{\sin \theta \, d\theta}{\sin \theta + \cos \theta}$$

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - \theta\right) \, d\theta}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)}$$

$$= \int_0^{\pi/2} \frac{\cos \theta d\theta}{\cos \theta + \sin \theta} = \frac{1}{2} \int_0^{\pi/2} \frac{(\sin \theta + \cos \theta) d\theta}{\sin \theta - \cos \theta}$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{1}{2} (\theta)_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

21. $\int_0^{\pi/4} \log(1 + \tan x) dx$

$$\begin{aligned} \text{Sol. I} &= \int_0^{\pi/4} \log \left[1 + \tan \left(\frac{\pi}{4} - x \right) \right] dx \\ &= \int_0^{\pi/4} \log \left[1 + \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x} \right] dx \\ &= \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan x}{1 + \tan x} \right) dx \\ &= \int_0^{\pi/4} \log \left(\frac{1 + \tan x + 1 - \tan x}{1 + \tan x} \right) dx \\ &= \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)] dx \\ &= \int_0^{\pi/4} \log 2 dx - \int_0^{\pi/4} \log(1 + \tan x) dx \\ &= \log 2(x)_0^{\pi/4} - I \\ 2I &= \frac{\pi}{4} \log 2 \Rightarrow I = \frac{\pi}{8} \log 2 \end{aligned}$$

Reduction Formulae:

Theorem-1:

If $I_n = \int_0^{\pi/2} \sin^n x dx$ then $I_n = \frac{n-1}{n} I_{n-2}$.

Proof:

$$\begin{aligned}
 I_n &= \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x dx \\
 &= \left[-\sin^{n-1} x \cos x \right]_0^{\pi/2} + \int_0^{\pi/2} (n-1) \sin^{n-2} x \cdot \cos^2 x dx \\
 &= (n-1) \int_0^{\pi/2} \sin^{n-2} x (1 - \sin^2 x) dx \\
 &= (n-1) \left[\int_0^{\pi/2} \sin^{n-2} x dx - \int_0^{\pi/2} \sin^n x dx \right] \\
 &= (n-1) I_{n-2} - (n-1) I_n
 \end{aligned}$$

$$I_n (1+n-1) = (n-1) I_{n-2} \Rightarrow I_n n = (n-1) I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2} \quad \text{---(1)}$$

Note:

In (1), replace n by n-2, n-3, then $I_n = \frac{n-1}{n} I_{n-2} \Rightarrow I_{n-2} = \frac{n-3}{n-2} I_{n-4} \Rightarrow I_{n-4} = \frac{n-5}{n-4} I_{n-6}$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot I_0 \quad \text{Or } I_1 \text{ according as n is even or odd.}$$

$$\text{But } I_0 = \int_0^{\pi/2} \sin^0 x dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$I_1 = \int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2}$$

$$= -\cos \frac{\pi}{2} + \cos 0 = -0 + 1 = 1$$

$$\therefore \int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{If n is even.}$$

$$\therefore \int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1 \text{ If } n \text{ is odd.}$$

Theorem- 2:

$$\text{If } I_n = \int_0^{\pi/2} \cos^n x dx \text{ then } I_n = \frac{n-1}{n} I_{n-2}.$$

$$I_n = \int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \cos^n \left(\frac{\pi}{2} - x \right) dx = \int_0^{\pi/2} \sin^n x dx$$

Theorem- 3:

$$\text{If } I_n = \int_0^{\pi/4} \tan^n x dx \text{ then } I_n + I_{n-2} = \frac{1}{n-1}.$$

Proof:

$$\begin{aligned} I_n &= \int_0^{\pi/4} \tan^n x dx = \int_0^{\pi/4} \tan^{n-2} x \tan^2 x dx \\ &= \int_0^{\pi/4} \tan^{n-2} x (\sec^2 x - 1) dx = \int_0^{\pi/4} \tan^{n-2} x \sec^2 x dx - \int_0^{\pi/4} \tan^{n-2} x dx \\ &= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2} \\ \therefore I_n + I_{n-2} &= \frac{1}{n-1} \end{aligned}$$

Theorem-4:

$$\text{If } I_n = \int_0^{\pi/4} \sec^n x dx \text{ then } I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

Proof:

$$\begin{aligned} I_n &= \int_0^{\pi/4} \sec^n x dx = \int_0^{\pi/4} \sec^{n-2} x \sec^2 x dx \\ &= \left[\sec^{n-2} x \tan x \right]_0^{\pi/4} - \int_0^{\pi/4} (n-2) \sec^{n-2} x \sec x \tan^2 x dx \end{aligned}$$

$$\begin{aligned}
 &= (\sqrt{2})^{n-2} - (n-2) \int_0^{\pi/4} \sec^{n-2} x (\sec^2 x - 1) dx \\
 &= (\sqrt{2})^{n-2} - (n-2) \left[\int_0^{\pi/4} \sec^n x dx - \int_0^{\pi/4} \sec^{n-2} x dx \right] \\
 &= (\sqrt{2})^{n-2} - (n-2) I_n + (n-2) I_{n-2}
 \end{aligned}$$

$$I_n(1+n-2) = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$$

$$\Rightarrow I_n(n-1) = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$$

$$\therefore I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Theorem 5: If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx$ then $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx$.

Proof:

$$\begin{aligned}
 I_{m,n} &= \int_0^{\pi/2} \sin^m x \cos^n x dx = \int_0^{\pi/2} \sin^{m-1} (\sin x \cos^n x) dx \\
 &= \left[\frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos^{n+1} x}{n+1} (m-1) \sin^{m-2} x \cos x dx \\
 &= \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \cos^n x \cos^2 x dx \\
 &= \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \cos^n x (1 - \sin^2 x) dx \\
 &= \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \cos^n x - \sin^m x \cos^n x dx \\
 &= \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n} \\
 \Rightarrow I_{m,n} \left(1 + \frac{m-1}{n+1} \right) &= \frac{m-1}{n+1} I_{m-2,n} \\
 \Rightarrow I_{m,n} \left(\frac{n+m}{n+1} \right) &= \frac{m-1}{n+1} I_{m-2,n}
 \end{aligned}$$

$$\therefore I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \quad \text{-----(1)}$$

Note: replacing m by m-2,m-4,----

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} I_{m-4,n} = \frac{m-1}{m+n} \frac{m-3}{m+n-2} \frac{m-5}{m+n-4} \dots I_{0,n}$$

Or $I_{1,n}$ according as n is even or odd.

$$\text{But } I_{0,1} = \int_0^{\pi/2} \sin^0 x \cos^n x dx = \int_0^{\pi/2} \cos^n x dx$$

$$I_{1,n} = \int_0^{\pi/2} \sin x \cos^n x dx = \left[-\frac{\cos^{n+1} x}{n+1} \right]_0^{\pi/2} = \frac{1}{n+1}$$

$$\begin{aligned} \therefore I_{m,n} &= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots \frac{1}{n+1} \text{ if m is odd} \\ &= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots \int_0^{\pi/2} \cos^n x dx \text{ if m is even} \end{aligned}$$

Corollary 2: If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx$ then $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$.

Very Short Answer Questions

I. Find the values of the following integrals.

1. $\int_0^{\pi/2} \sin^{10} x dx$

Sol. n=10 even

$$\therefore \int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^{10} x dx = \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{63\pi}{512}$$

2. $\int_0^{\pi/2} \cos^{11} x dx$

Sol. n=11 is odd

$$\therefore \int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1$$

$$\int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3}$$

$$= \frac{11-1}{11} \cdot \frac{11-3}{9} \cdot \frac{11-5}{7} \cdots \frac{2}{3} = \frac{256}{693}$$

3. $\int_0^{\pi/2} \cos^7 x \cdot \sin^2 x dx$

Sol. $I = \int_0^{\pi/2} \cos^7 x \cdot \sin^2 x dx$,

m=2, n=7

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x dx \quad \text{Here m even,}$$

n odd

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

$$= \frac{7-1}{9} \times \frac{7-3}{7} \times \frac{7-5}{5} \times \frac{1}{2+1}$$

$$= \frac{6}{9} \cdot \frac{4}{7} \cdot \frac{2}{5} \cdot \frac{1}{3} = \frac{16}{315}$$

4. $\int_0^{\pi/2} \sin^4 x \cdot \cos^4 x \cdot dx$

Sol. $\int_0^{\pi/2} \sin^m x \cdot \cos^n x dx . m=n=4$

Here m, n even

$$= \frac{(n-1)(n-3) \dots 1}{(m+n)(m+n-2) \dots 2} \cdot \frac{\pi}{2} = \frac{(4-1)(4-3)(3\pi)}{8 \cdot 6 \cdot 4 \cdot 2 \cdot 2} = \frac{3\pi}{256}$$

5. $\int_0^{\pi} \sin^3 x \cos^6 x dx$

Sol. let $f(x) = \sin^3 x \cos^6 x \Rightarrow f(\pi-x) = \sin^3(\pi-x) \cos^6(\pi-x)$
 $= \sin^3 x \cos^6 x = f(x)$

$$\begin{aligned} \therefore \int_0^{\pi} \sin^3 x \cos^6 x dx &= 2 \int_0^{\frac{\pi}{2}} \sin^3 x \cos^6 x dx \\ &= 2 \cdot \frac{(n-1)}{m+n} \cdot \frac{(n-3)}{m+n-2} \cdots \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{2}{3} \cdot 1 = 2 \cdot \frac{5}{9} \cdot \frac{3}{7} \cdot \frac{1}{5} \cdot \frac{2}{3} = \frac{4}{63} \end{aligned}$$

6. $\int_0^{2\pi} \sin^2 x \cos^4 x dx$

Sol. $f(x) = \sin^2 x \cos^4 x$
 $\Rightarrow f(2\pi-x) = \sin^2(2\pi-x) \cos^4(2\pi-x)$

$$\sin^2 x \cos^4 x = f(x)$$

$$\text{and } f(\pi-x) = \sin^2(\pi-x) \cos^4(\pi-x) = f(x)$$

$$I = 4 \int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x dx$$

$$= \frac{4(4-1)(4-3)\frac{\pi}{2}}{6 \cdot 4 \cdot 2} = \frac{4 \cdot 3 \pi}{6 \cdot 4 \cdot 2 \cdot 2} = \frac{\pi}{8}$$

7. $\int_{-\pi/2}^{\pi/2} \sin^2 \theta \cdot \cos^7 \theta d\theta$

Sol. $\sin^2 \theta \cos^7 \theta$ is even function

$$f(\theta) = \sin^2 \theta \cos^7 \theta d\theta$$

$$f(-\theta) = \sin^2 (-\theta) \cos^7 (-\theta) = f(\theta)$$

$$= 2 \int_0^{\pi/2} \sin^2 \theta \cdot \cos^7 \theta d\theta$$

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x dx$$

n is odd, n = 7

$$= 2 \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

$$= 2 \cdot \frac{7-1}{9} \cdot \frac{7-3}{9-2} \cdot \frac{7-5}{9-4} \cdot \frac{1}{3}$$

$$= 2 \cdot \frac{6}{9} \cdot \frac{4}{7} \cdot \frac{2}{5} \cdot \frac{1}{3} = \frac{32}{315}$$

8. $\int_{-\pi/2}^{\pi/2} \sin^3 \theta \cos^3 \theta d\theta$

Sol. $f(\theta) = \sin^3 \theta \cos^3 \theta$

$$f(-\theta) = \sin^3 (-\theta) \cos^3 (-\theta)$$

$$= -\sin^3 \theta \cos^3 \theta = -f(\theta)$$

$f(\theta)$ is odd

$$\therefore \int_{-\pi/2}^{\pi/2} \sin^3 \theta \cos^3 \theta d\theta = 0$$

$$9. \int_0^a x(a^2 - x^2)^{7/2} dx$$

$$\text{Sol. } x = a \sin \theta \quad a = a \sin \theta$$

$$dx = a \cos \theta d\theta \quad \theta = \pi/2$$

$$\begin{aligned} &= \int_0^{\pi/2} a \sin \theta (a^2 - a^2 \sin^2 \theta)^{7/2} a \cos \theta d\theta \\ &= \int_0^{\pi/2} a^9 \cos^8 \theta \sin \theta d\theta = a^9 \int_0^{\pi/2} \cos^8 \theta \sin \theta \cdot d\theta \\ &= a^9 \left(\frac{-\cos^9 \theta}{9} \right)_0^{\pi/2} = a^9 \left(-0 + \frac{1}{9} \right) = \frac{a^9}{9} \end{aligned}$$

$$10. \int_0^2 x^{3/2} \sqrt{2-x} dx$$

$$\text{Sol. } x = 2 \cos^2 \theta, dx = 4 \cos \theta \sin \theta d\theta$$

$$\begin{aligned} I &= \int_{\pi/2}^0 (2)^{3/2} \cos^3 \theta \\ &\quad \sqrt{2-2\cos^2 \theta} (-4 \cos \theta \sin \theta) d\theta \\ &= 4 \int_0^{\pi/2} 2^{3/2} \cdot 2^{1/2} \cdot \cos^4 \theta \cdot \sin^2 \theta d\theta \\ &\quad \left[16 \cdot \int_0^{\pi/2} \cos^4 \theta \sin^2 \theta d\theta \right] \\ n &= \text{even}, m = \text{even} \\ &= 16 \left[\frac{4-1}{6} \cdot \frac{4-3}{5-2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right] = \frac{\pi}{2}. \end{aligned}$$

Short Answer Questions

1. $\int_0^1 x^5(1-x)^{5/2} dx$

Sol. given integral is $I = \int_0^1 x^5(1-x)^{5/2} dx$

Put $x = \sin^2 \theta$

$$dx = 2\sin \theta \cos \theta d\theta$$

$$U.L.X = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$L.L X=O \Rightarrow \theta=0$$

$$I = \int_0^{\pi/2} \sin^{10} \theta (1-\sin^2 \theta)^{5/2} 2\cos \theta \sin \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{11} \theta \cos^6 \theta d\theta$$

$m=11$ odd and $n=6$ even

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \frac{m-1}{m} \cdots \frac{2}{3}$$

$$I = 2 \cdot \frac{5}{17} \cdot \frac{3}{15} \cdot \frac{1}{13} \cdot \frac{10}{11} \cdot \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{512}{153153}$$

2. $\int_0^4 (16-x^2)^{5/2} dx$

Sol. $I = \int_0^4 (16-x^2)^{5/2} dx$

Put $x = 4 \sin \theta$

$$dx = 4 \cos \theta d\theta$$

$$U.L. x=4 \Rightarrow \theta = \pi/2$$

$$L.L X=0 \Rightarrow \theta=0.$$

$$I = \int_0^{\pi/2} (16-16\sin^2 \theta)^{5/2} \cdot 4 \cos \theta \cdot d\theta$$

$$\begin{aligned}
 &= \int_0^{\pi/2} (4)^5 \cdot \cos^5 \theta \cdot d\theta = (4)^6 \int_0^{\pi/2} \cos^6 \theta \cdot d\theta \\
 &= (4)^6 \cdot \frac{6-1}{6} \cdot \frac{6-3}{6-2} \cdot \frac{6-5}{6-4} \cdot \frac{\pi}{2} \\
 &= (4)^6 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = (4)^4 \cdot \frac{5}{2} \cdot \pi = 640\pi
 \end{aligned}$$

3. $\int_{-3}^3 (9-x^2)^{3/2} dx$

Sol: Let $x = 3 \sin \theta$ then $dx = 3 \cos \theta d\theta$

Upper limit when $x = 3$ is $\theta = \frac{\pi}{2}$

And lower limit when $x = -3$ is $\theta = -\frac{\pi}{2}$.

$$\begin{aligned}
 &\therefore \int_{-3}^3 (9-x^2)^{3/2} dx \\
 &= \int_{-\pi/2}^{\pi/2} (9-9\sin^2 \theta)^{3/2} 3\sin \theta 3\cos \theta d\theta
 \end{aligned}$$

$$= 3^5 \int_{-\pi/2}^{\pi/2} \cos^4 \theta \sin \theta d\theta$$

$$\begin{aligned}
 f(\theta) &= \cos^4 \theta \sin \theta \text{ and } f(-\theta) \\
 &= \cos^4 (-\theta) \sin (-\theta) \\
 &= -\cos^4 \theta \sin \theta \\
 &= -f(\theta)
 \end{aligned}$$

Hence f is an odd function of θ .

$$\therefore \int_{-\pi/2}^{\pi/2} f(\theta) d\theta = 0.$$

4. $\int_0^5 x^3(25-x^2)^{7/2} dx$

Sol: Let $x = 5 \sin \theta$ then

Upper limit when $x = 5$ is $\theta = \frac{\pi}{2}$

And lower limit when $x = 0$ is $\theta = 0$.

$$\begin{aligned}\therefore \int_0^5 x^3(25-x^2)^{7/2} dx \\ &= \int_0^{\pi/2} 5^3 \sin^3 \theta (25 - 25 \sin^2 \theta)^{7/2} 5 \cos \theta d\theta\end{aligned}$$

$$\begin{aligned}&= 5^{11} \int_0^{\pi/2} \sin^3 \theta \cos^7 \theta \cos \theta d\theta = 5^{11} \int_0^{\pi/2} \sin^3 \theta \cos^8 \theta d\theta \\ &= 5^{11} \left(\frac{8-1}{8+3} \cdot \frac{8-3}{8+3-2} \cdot \frac{8-5}{8+3-4} \cdot \frac{8-7}{8+3-6} \right) \left(\frac{3-1}{3} \cdot \frac{\pi}{2} \right) \\ &= 5^{11} \frac{7}{11} \cdot \frac{5}{9} \cdot \frac{3}{7} \cdot \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{\pi}{2} = 5^{11} \times \frac{1}{11} \times \frac{1}{9} \pi = \frac{5^{11} \times \pi}{99}\end{aligned}$$

[Using the formula $\int_0^{\pi/2} \sin^m x \cos^n x dx$ if m, n are even $= \left(\frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \right)$]

5. $\int_{-\pi}^{\pi} \sin^8 x \cos^7 x dx$

Sol: Take $f(x) = \sin^8 x \cos^7 x$

Then $f(-x) = \sin^8(-x) \cos^7(-x)$

$$\begin{aligned}&= \sin^8 x \cos^7 x \\ &= f(x)\end{aligned}$$

f is an even function of x .

$$\therefore \int_{-\pi}^{\pi} \sin^8 x \cos^7 x dx = 2 \int_0^{\pi} \sin^8 x \cos^7 x dx$$

Now $f(x) = \sin^8 x \cos^7 x$

And $f(\pi - x) = \sin^8(\pi - x) \cos^7(\pi - x)$

$$= -\sin^8 x \cos^7 x = -f(x)$$

Hence $\int_0^\pi \sin^8 x \cos^7 x dx = 0$

$$\therefore \int_{-\pi}^{\pi} \sin^8 x \cos^7 x dx = 0$$

[By the result that $f = [0, 2a] \rightarrow \mathbb{R}$ is integrable on $[0, a]$ and if $f(2a - x) = -f(x) \forall x \in [a, 2a]$

then $\int_0^{2a} f(x) dx = 0$].

6., $\int_3^7 \sqrt[7]{\frac{7-x}{x-3}} dx$

Sol: Let $x = 3 \cos^2 \theta + 7 \sin^2 \theta$

Then $dx = -6 \cos \theta \sin \theta + 14 \sin \theta \cos \theta$

$$= 8 \cos \theta \sin \theta$$

Upper limit when $x = 7$ is

$$7 = 3 \cos^2 \theta + 7 \sin^2 \theta$$

$$\Rightarrow 7(1 - \sin^2 \theta) = 3 \cos^2 \theta$$

$$\Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Lower limit when $x = 3$ is

$$3 = 3 \cos^2 \theta + 7 \sin^2 \theta$$

$$\Rightarrow 3 \sin^2 \theta = 7 \sin^2 \theta$$

$$\Rightarrow 4 \sin^2 \theta = 0 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0$$

Also $\sqrt{\frac{7-x}{x-3}} = \sqrt{\frac{7-3\cos^2 \theta - 7\sin^2 \theta}{3\cos^2 \theta + 7\sin^2 \theta - 3}} = \sqrt{\frac{4\cos^2 \theta}{4\sin^2 \theta}} = \cot \theta$

$$\therefore \int_3^7 \sqrt{\frac{7-x}{x-3}} dx = \int_0^{\pi/2} \cot \theta (8 \cos \theta \sin \theta) d\theta$$

$$= 8 \int_0^{\pi/2} \cos^2 \theta d\theta$$

$$= 8 \left(\frac{2-1}{2} \right) \frac{\pi}{2} = 8 \left(\frac{1}{2} \right) \frac{\pi}{2} = 2\pi$$

$$\left(\because \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{2} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \right)$$

7. $\int_2^6 \sqrt{(6-x)(x-2)} dx$

Sol: Put $x = 2 \cos^2 \theta + 6 \sin^2 \theta$

$$\text{Then } dx = (-4 \cos \theta \sin \theta + 12 \sin \theta \cos \theta) d\theta$$

$$= 8 \sin \theta \cos \theta d\theta$$

Upper limit when $x = 6$ is

$$6 = 2 \cos^2 \theta + 6 \sin^2 \theta$$

$$\Rightarrow 6 \cos^2 \theta = 2 \cos^2 \theta$$

$$\Rightarrow 4 \cos^2 \theta = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Lower limit when $x = 2$ is

$$2 = 2 \cos^2 \theta + 6 \sin^2 \theta$$

$$\Rightarrow 2 \sin^2 \theta = 6 \sin^2 \theta$$

$$\Rightarrow 4 \sin^2 \theta = 0 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0$$

$$\therefore \int_2^6 \sqrt{(6-x)(x-2)} dx$$

$$= \int_0^{\pi/2} \sqrt{\frac{(6-2 \cos^2 \theta - 6 \sin^2 \theta)}{(2 \cos^2 \theta + 6 \sin^2 \theta - 2)}} \cdot 8 \sin \theta \cos \theta d\theta$$

$$\begin{aligned}
 &= \int_0^{\pi/2} (2\cos\theta)(2\sin\theta)(8\sin\theta\cos\theta)d\theta \\
 &= 32 \int_0^{\pi/2} \cos^2\theta \sin^2\theta d\theta = \frac{32}{4} \int_0^{\pi/4} \sin^2 2\theta d\theta \\
 &= -8 \int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2} \right) d\theta \\
 &= 4 \left[(\theta)_0^{\pi/2} - \frac{1}{4} (\sin 4\theta)_0^{\pi/2} \right] \\
 &= 4 \left(\frac{\pi}{2} \right) - 0 = 2\pi.
 \end{aligned}$$

8. $\int_0^{\pi/2} \tan^5 x \cos^8 x dx$

Sol: $\int_0^{\pi/2} \tan^5 x \cos^8 x dx$

$$= \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x} \cos^8 x dx$$

$$= \int_0^{\pi/2} \sin^5 x \cos^3 x dx$$

$$= \int_0^{\pi/2} \sin^5 x \cos^2 x \cos x dx$$

$$= \int_0^{\pi/2} \sin^5 x (1 - \sin^2 x) \cos x dx$$

Let $\sin x = t$ then $\cos x dx = dt$

Upper limit when $x = \frac{\pi}{2}$ is $t = 1$.

Lower limit when $x = 0$ is $t = 0$.

$$\begin{aligned}
 &= \int_0^1 t^5 (1 - t^2) dt \\
 &= \left(\frac{t^6}{6} - \frac{t^8}{8} \right)_0^1 = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.
 \end{aligned}$$

Long Answer Questions

1. $\int_0^1 x^{7/2} (1-x)^{5/2} dx$

Sol: Let $x = \sin^2 \theta$ then $dx = 2 \sin \theta \cos \theta d\theta$

Upper limit when $x = 1$ is $\sin^2 \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$.

Lower limit when $x = 0$ is $\sin^2 \theta = 0 \Rightarrow \theta = 0$.

$$\begin{aligned} & \therefore \int_0^1 x^{7/2} (1-x)^{5/2} dx \\ &= \int_0^{\pi/2} (\sin^2 \theta)^{7/2} (1-\sin^2 \theta)^{5/2} 2 \sin \theta \cos \theta d\theta \\ &= 2 \int_0^{\pi/2} \sin^7 \theta \cos^5 \theta \sin \theta \cos \theta d\theta \\ &= 2 \int_0^{\pi/2} \sin^8 \theta \cos^6 \theta d\theta \end{aligned}$$

When m, n are even, m = 8, n = 6

$$\begin{aligned} I_{m,n} &= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{\pi}{2} \\ &= \frac{6-1}{8+6} \cdot \frac{6-3}{8+6-2} \cdot \frac{6-5}{8+6-4} \\ &\quad \frac{8-1}{8} \cdot \frac{8-3}{8-2} \cdot \frac{8-5}{8-4} \cdot \frac{8-7}{8-6} \cdot \frac{\pi}{2} \\ &= \frac{5}{14} \cdot \frac{3}{12} \cdot \frac{1}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\ &= \frac{5\pi}{2048}. \end{aligned}$$

2. $\int_0^\pi (1 + \cos x)^3 dx$

Sol: $\int_0^\pi (1 + \cos x)^3 dx$

$$= \int_0^\pi \left(1 + 2\cos^2 \frac{x}{2} - 1 \right)^3 dx$$

$$= \int_0^\pi \left(2\cos^2 \frac{x}{2} \right)^3 dx$$

$$= 8 \int_0^\pi \cos^6 \frac{x}{2} dx$$

Let $\frac{x}{2} = t$ then $dx = 2 dt$

Upper limit is $t = \frac{\pi}{2}$

Lower limit is $t = 0$

$$= 16 \int_0^{\pi/2} \cos^6 t dt$$

$$= 16 \left(\frac{6-1}{6} \cdot \frac{6-3}{6-2} \cdot \frac{6-5}{6-4} \cdot \frac{\pi}{2} \right)$$

$$= 16 \left(\frac{5}{6} \right) \left(\frac{3}{4} \right) \left(\frac{1}{2} \right) \left(\frac{\pi}{2} \right)$$

$$= \frac{5\pi}{2}$$

$$\left[\because I_n = \int_0^{\pi/2} \cos^n x dx \text{ when } n \text{ is even} \right]$$

$$= \left(\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{\pi}{2} \right)$$

$$3. \int_4^9 \frac{dx}{\sqrt{(9-x)(x-4)}}$$

Sol: Take $x = 4 \cos^2 \theta + 9 \sin^2 \theta$

$$\text{Then } dx = (-8 \cos \theta \sin \theta + 18 \sin \theta \cos \theta)d\theta$$

$$= 10 \cos \theta \sin \theta$$

Upper limit when $x = 9$ is

$$\begin{aligned} 9 &= 4 \cos^2 \theta + 9 \sin^2 \theta \\ \Rightarrow 9(1 - \sin^2 \theta) &= 4 \cos^2 \theta \\ \Rightarrow 5 \cos^2 \theta &= 0 \\ \Rightarrow \cos \theta &= 0 \Rightarrow \theta = \frac{\pi}{2} \end{aligned}$$

Lower limit when $x = 4$ is

$$\begin{aligned} 4 &= 4 \cos^2 \theta + 9 \sin^2 \theta \\ \Rightarrow 4(1 - \cos^2 \theta) &= 9 \sin^2 \theta \\ \Rightarrow 5 \sin^2 \theta &= 0 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0 \end{aligned}$$

Also $\sqrt{(9-x)(x-4)}$

$$\begin{aligned} &= \sqrt{\frac{[9 - (4 \cos^2 \theta + 9 \sin^2 \theta)]}{(4 \cos^2 \theta + 9 \sin^2 \theta - 4)}} \\ &= \sqrt{5 \cos^2 \theta \sin^2 \theta} = 5 \cos \theta \sin \theta \\ \therefore \int_2^4 \frac{dx}{\sqrt{(9-x)(x-4)}} &= \int_0^{\pi/2} \frac{10 \cos \theta \sin \theta}{5 \cos \theta \sin \theta} d\theta \\ &= 2[\theta]_0^{\pi/2} = 2\left(\frac{\pi}{2}\right) = \pi. \end{aligned}$$

$$4. \int_0^5 x^2 (\sqrt{5-x})^7 dx$$

Sol: Let $x = 5 \sin^2 \theta$ then $dx = 10 \sin \theta \cos \theta d\theta$

$$\text{Upper limit when } x = 5 \text{ is } \sin^2 \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\text{Lower limit when } x = 0 \text{ is } \sin^2 \theta = 0 \Rightarrow \theta = 0.$$

$$\therefore \int_0^5 x^2 (\sqrt{5-x})^7 dx$$

$$= \int_0^{\pi/2} 25 \sin^4 \theta (\sqrt{5-5 \sin^2 \theta}) 10 \sin \theta \cos \theta d\theta$$

$$= 250.5^{7/2} \int_0^{\pi/2} \sin^5 \theta \cos^8 \theta d\theta$$

Then value of $I_{m,n}$ when m is even and n is odd is

$$I_{m,n} = \left(\frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdot \frac{n-7}{m+n-6} \right) \left(\frac{m-1}{m} \cdot \frac{m-3}{m-2} \right)$$

$$\therefore \int_0^5 x^2 (\sqrt{5-x})^7 dx$$

$$= 250.5^{7/2} \left[\frac{7}{13} \cdot \frac{5}{11} \cdot \frac{3}{9} \cdot \frac{1}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \right]$$

$$= \frac{250.5^{7/2} \cdot 8}{99 \times 13} = \frac{250.(5^{7/2})}{1287}$$

$$= \frac{5^3(2)(8)(5)^{7/2}}{1287}$$

$$= \frac{5^{13/2} \cdot 16}{1287},$$

5. $\int_0^{2\pi} (1+\cos x)^5 (1-\cos x)^3 dx$

Sol: $\int_0^{2\pi} (1+\cos x)^5 (1-\cos x)^3 dx$

$$= \int_0^{2\pi} \left(2\cos^2 \frac{x}{2}\right)^5 \left(2\sin^2 \frac{x}{2}\right)^3 dx$$

$$= 2^8 \int_0^{2\pi} \cos^{10} \frac{x}{2} \sin^6 \frac{x}{2} dx$$

Let $\frac{x}{2} = t \Rightarrow dx = 2dt$ then

Upper limit when $x = 2\pi$ is $t = \pi$

And lower limit when $x = 0$ is $t = 0$.

$$= 2^8 \times 2 \times \int_0^\pi \cos^{10} t \sin^6 t dt$$

$$= 2^8 \times 2 \times 2 \times \int_0^{\pi/2} \cos^{10} t \sin^6 t dt$$

$$= 2^{10} \int_0^{\pi/2} \cos^{10} t \sin^6 t dt$$

$\left(\because \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(2a-x) = f(x) \right)$ Use the form $I_{m,n}$ when m and n are even.

$$I_{m,n} = \left(\frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \dots \right) \left(\frac{m-1}{m} \cdot \frac{m-3}{m-2} \dots \frac{\pi}{2} \right)$$

$$\therefore \int_0^{2\pi} (1+\cos x)^5 (1-\cos x)^3 dx$$

$$= 2^{10} \left[\frac{5}{16} \cdot \frac{3}{14} \cdot \frac{1}{12} \cdot \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$= 2^{10} \times \frac{45\pi}{32 \times 32 \times 64}$$

$$= \frac{32 \times 32 \times 45\pi}{32 \times 32 \times 64} = \frac{45\pi}{64}$$

6. Show that

$$\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$$

Sol. Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{\sin x + \cos x} dx$$

$$2I = \int_0^{\pi/2} \left(\frac{x}{\sin x + \cos x} + \frac{(\pi/2 - x)}{\sin x + \cos x} \right) dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$\text{Put } t = \tan \frac{x}{2} \Rightarrow dx = \frac{2dt}{1+t^2}$$

$$I = \frac{\pi}{4} \int_0^1 \frac{2 \frac{dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \frac{\pi}{2} \int_0^1 \frac{dt}{2t+1-t^2}$$

$$= \frac{\pi}{2} \int_0^1 \frac{dt}{(\sqrt{2})^2 + (t-1)^2} = \frac{\pi}{2} \left(\frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}+t-1}{\sqrt{2}-t+1} \right)_0^1$$

$$= \frac{-\pi}{4\sqrt{2}} \left(\log \frac{\sqrt{2}-1}{\sqrt{2}+1} \right) = \frac{\pi}{4\sqrt{2}} \log(\sqrt{2}+1)^2$$

$$= \frac{\pi}{4\sqrt{2}} 2 \log(\sqrt{2}+1) = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)$$

7. Find $\int_0^\pi \frac{x \cdot \sin x}{1 + \sin x} dx$

$$\text{Sol. I} = \int_0^\pi \frac{x \cdot \sin x}{1 + \sin x} dx = \int_0^\pi \frac{(\pi - x) \sin(\pi - x)}{1 + \sin(\pi - x)} dx$$

$$\left(\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$= \int_0^\pi \frac{(\pi - x) \sin x}{1 + \sin x} dx$$

$$I = \int_0^\pi \frac{\pi \sin x}{1 + \sin x} dx - \int_0^\pi \frac{x \sin x}{1 + \sin x} dx$$

$$= \int_0^\pi \frac{\pi \sin x}{1 + \sin x} dx - I \Rightarrow 2I = \pi \int_0^\pi \frac{\sin x}{1 + \sin x} dx$$

$$= \pi \int_0^\pi \left(1 - \frac{1}{1 + \sin x} \right) dx$$

$$= \pi \int_0^\pi dx - \pi \int_0^\pi \frac{1}{1 + \sin x} dx \quad \dots (1)$$

$$\text{Consider } \int_0^\pi \frac{1}{1 + \sin x} dx = 2 \int_0^{\pi/2} \frac{1}{1 + \sin x} dx$$

$$= 2 \int_0^{\pi/2} \frac{1}{1 + \sin\left(\frac{\pi}{2} - x\right)} dx = 2 \int_0^{\pi/2} \frac{dx}{1 + \cos x}$$

$$= 2 \int_0^{\pi/2} \frac{dx}{2 \cos^2(x/2)} = \int_0^{\pi/2} \sec^2 \frac{x}{2} dx$$

$$= \left(2 \tan \frac{x}{2} \right)_0^{\pi/2} = 2 \cdot \tan \frac{\pi}{2} - 2 \cdot 0 = 2 - 0 = 2$$

$$2I = \pi(x)_0^\pi - 2\pi = \pi(\pi) - 2 = \pi^2 - 2\pi$$

$$I = \frac{\pi^2}{2} - \pi$$