1.1.11

AI25BTECH11009
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Question

Show that $\text{rank}A = \text{rank}A^{\top}$ for any matrix A.

Proof (concise)

Let A be $m \times n$ and set $r = \text{rank}A$. There exist invertible $P \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$ such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Transpose (1):

$$Q^\top A^\top P^\top = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

Left/right multiplication by invertible matrices preserves rank, hence

$$\text{rank}(A^\top) = \text{rank}(Q^\top A^\top P^\top) = \text{rank}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = r = \text{rank}A.$$

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