# 1.1.11

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**Q (1.1.11)**: Prove rank$A$ = rank$A^\top$.

**Solution.** Let $A$ be $m \times n$ and $r$ = rank$A$. There exist invertible $P \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$ with

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Transpose (1):

$$Q^\top A^\top P^\top = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

Since left/right multiplication by invertible matrices preserves rank,

$$\text{rank}(A^\top) = \text{rank}(Q^\top A^\top P^\top) = \text{rank}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = r.$$

Thus rank$A^\top$ = rank$A$. ■