TOC Assignment - 3

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Ques! CFG Construction and Derivation Tree Analysis

Construct a context-free grammar G for the language $L = \{a^n b^m \mid n \geq 1, m \geq 0, \text{ and } n \text{ is even}\}$.

a) Define V, Σ, P, and S

- $\Sigma = \{a, b\}$
- $V = \{S, A\}$
- Start symbol : S
- Productions (P) :

$S \rightarrow aaS \mid aaA$

$A \rightarrow bA \mid \varepsilon$

S generates an even number of a's (at least 2), and A generates any number of b's.

b) Show the leftmost and rightmost derivations for the string "aaaabb".

String : aaaabb

Leftmost Derivation :

$S \rightarrow aaS$

$\rightarrow aa\ aaA$

$\rightarrow aa\ aabA$

$\rightarrow aa\ aa\ b\ bA$

$\rightarrow aa\ aa\ b\ b$

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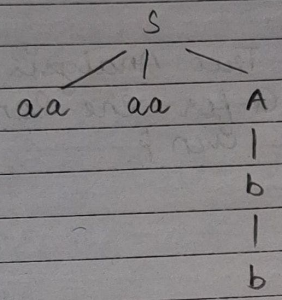
(c) Derivation Tree for the same string.



Leaves: a a a a b b → "aaaabb"

(d) ~~Regularity & Context-Freeness~~ Prove that the language is not regular but is context-free.

Regular expression for L:

$(aa)+ b^*$

Since this RE exists, L is a Regular Language.
And every regular language is also Context-Free.

Ques2 Ambiguity Analysis of a Given Grammar

- Given the grammar: $S \rightarrow aSbS \mid \varepsilon$

a) Generate two different parse trees for the string "aabb".

⇒ Derive aabb

$S \rightarrow aSbS$

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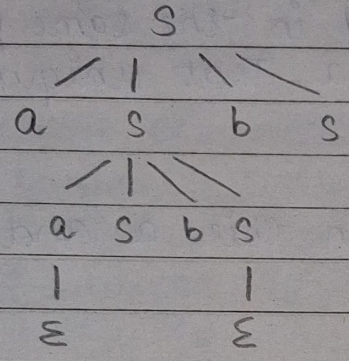
→ a (a S b S) b S
→ a a ∈ b ∈ b ∈ = aabb

→ Parse tree



Leaves (left-to-right): a a b b → aabb

b) Prove that the grammar is ambiguous using formal definitions.

A grammar is ambiguous if there exists at least one string in the language that has two distinct parse trees (equivalently, two different leftmost or rightmost derivations).

Claim : The grammar S → aSbS | ε is unambiguous.

Proof :

- Any nonempty string generated by this grammar must begin with a (because the only production introducing terminals start with a) and therefore must be

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produced by one application of $S \to aS_1bS_2$.

- In any derivation of a nonempty string w, the first a in w must match some b later in w. Let that matching b be at position k. The grammar forces that matching b to be the b produced in the same production aS_1bS_2 that produced the first a. That uniquely splits w into three parts.
  - a (the first terminal)
  - substring generated by S_1 (between this a and its matching b),
  - b (the matching b),
  - substring generated by S_2 (the remainder)

- The position k (first a's matching b) is uniquely determined by the usual balance argument (count a minus b scanning from left) — it is the smallest index where balance returns to the level before the first a. Hence the split into parts for S_1 and S_2 is unique.

- By induction on string length, the derivations (and therefore parse tree) of each of the two substrings are unique.

- Therefore the whole parse tree is unique.

So no string has two different parse trees is unambiguous.

(c) Construct a non-ambiguous grammar that generates the same language.

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The given grammar itself is already non-ambiguous. You can present an equivalent (and maybe clearer) unambiguous grammar:

$S \rightarrow TS \mid \epsilon$
$T \rightarrow a\,S\,b$

T generates one matched pair a....b where the .... is a balanced substring (generated by S), and then $S \rightarrow TS$ allows concatenation of such matched blocks. This grammar is the same as the original $S \rightarrow aSbS \mid \epsilon$ but written to emphasize the unique decomposition $S = (a\,S\,b)\,S$ — hence unambiguous.

Ques3 PDA construction for Balanced Expressions
Construct a PDA that accepts the language $L = \{a^n b^n \mid n \geq 0\}$.

a) Define the PDA as a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$.

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ where:

- $Q = \{q_0, q_1, q_f\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, X\}$ — Z_0 is initial stack symbol, X is marker for each a
- q_0 is start state
- Z_0 is initial stack symbol
- $F = \{q_f\}$ (accept by final state)

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In q_0 push one X for each a. On seeing first b go to q_1 and pop one X per b. If input finished and stack has only Z_0, go to q_f.

b) Provide the complete transition table.

1) $\delta(q0, a, z0) = \{(q0, Xz0)\}$
2) $\delta(q0, a, X) = \{(q0, XX)\}$
3) $\delta(q0, b, X) = \{(q1, \epsilon)\}$
4) $\delta(q1, b, X) = \{(q1, \epsilon)\}$
5) $\delta(q1, \epsilon, z0) = \{(q_f, z0)\}$
6) $\delta(q0, \epsilon, z0) = \{(q_f, z0)\}$

c) Show step-by step instantaneous descriptions (IDs) for the input "aaabbb".

Start: $(q_0, aaabbb, Z_0)$

1. $(q_0, aaabbb, Z_0)$
2. $\overset{a}{\Rightarrow} (q_0, aabbb, XZ_0)$ // used $\delta(q0, a, z0)$
3. $\overset{a}{\Rightarrow} (q_0, abbb, XXZ_0)$ // $\delta(q0, a, X)$
4. $\overset{a}{\Rightarrow} (q_0, bbb, XXXZ_0)$ // $\delta(q0, a, X)$

On reading first b, move to q_1 and pop one X:

5. $\overset{b}{\Rightarrow} (q_1, bb, XXZ_0)$ // $\delta(q0, b, X) \rightarrow (q1, \epsilon)$

Consume remaining remaining two b's in q_1, popping Xs:

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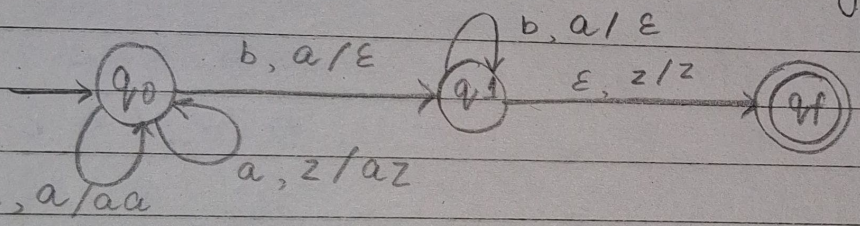
6. $\xrightarrow{b}$ $(q_1, b, XZ_0) \;//\; \delta(q_1, b, X)$
7. $\xrightarrow{b}$ $(q_1, \varepsilon, Z_0) \;//\; \delta(q_1, b, X)$

Now empty and stack has Z0 — take ε-move to final:

8. $\xrightarrow{\varepsilon}$ $(q_f, \varepsilon, Z_0) \;//\; \delta(q_1, \varepsilon, Z_0)$

Since in final state q_f with empty input, string accepted.

d) Draw the state transition diagram.



Ques 4 Grammar Transformation and Normal forms

Given the CFG : $S \to aA \mid bB$; $A \to aA \mid \varepsilon$; $B \to bB \mid \varepsilon$

a) Eliminate ε-productions and unit productions

Step 1 – find nullable non terminals:
A is nullable (since $A \to \varepsilon$).
B is nullable (since $B \to \varepsilon$). S is not nullable.

Step 2 – add productions obtained by omitting nullable occurrences:

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- From $S \to aA$: because A nullable, add $S \to a$.
- From $S \to bB$: because B nullable, add $S \to b$.
- From $A \to aA$: because A nullable, add $A \to a$.
- From $B \to bB$: because B nullable, add $B \to b$.

Step 3 — remove original ε-productions : remove $A \to \varepsilon$ and $B \to \varepsilon$.
No unit productions (of the form $X \to Y$) are present now.

Resulting grammar (no ε, no unit):

$S \to aA \mid a \mid bB \mid b$
$A \to aA \mid a$
$B \to bB \mid b$

b) Convert the resulting grammar into ~~them~~ Chomsky Normal Form (CNF).

CNF requires productions of the form $X \to YZ$ (two nonterminals) or $X \to a$ (single terminal). Also introduce helper nonterminals for terminals when they appear in longer right-hand sides.

Introduce terminal nonterminals :

$X\_a \to a$
$X\_b \to b$

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Replace terminals in mixed productions (length 2 where one symbol terminal):

From $S \to aA$ replace a by X_a : $S \to X\_aA$
From $S \to bB$ replace b by X_b : $S \to X\_bB$
From $A \to aA \to A \to X\_aA$
From $B \to bB \to B \to X\_bB$

Keep terminal-only productions:
$S \to a$ and $S \to b$ and $A \to a$ and $B \to b$ are of allowed form

⇒ Final CNF grammar:

$S \to X\_a\ A$
$S \to X\_b\ B$
$S \to a$
$S \to b$

$A \to X\_a\ A$
$A \to a$

$B \to X\_b\ B$
$B \to b$

$X\_a \to a$
$X\_b \to b$

c) Show derivation of the string "aab" in both the original and CNF forms.

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Given grammar:

$S \rightarrow aA \mid bB$

$A \rightarrow aA \mid \varepsilon$

$B \rightarrow bB \mid \varepsilon$

This grammar generates strings of the form a^+ or b^+ only (like "a", "aa", "bbb" etc). Hence, "dab" cannot be derived because it mixes both a's and b's — no rule allows switching from A(a's) to B(b's).

In both original & CNF, the same limitation exists, so "aab" is not derivable.

d) Discuss how CNF simplifies PDA simulation & parsing.

- Uniform structure: Every production is either $A \rightarrow BC$ or $A \rightarrow a$, making parsing steps systematic.
- Simpler PDA construction: Easier to simulate grammar as stack operations follow fixed patterns.
- Efficient algorithms: CNF allows algorithms like CYK to check string membership efficiently ($O(n^3)$).
- Less ambiguity: Reduces complexity and non determinism during parsing.

Ques 5 CFL Membership via Pumping Lemma

Prove using the Pumping Lemma for CFLs that the

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Language $L = \{ a^n b^n c^n \mid n \geq 0 \}$ is not context-free.

a) State the pumping lemma clearly.

There is a pumping length $p \geq 1$ such that for every string $s \in L$ with $|s| \geq p$, we can write

$$s = uvwxy$$

satisfying

1) $|vwx| \leq p$,
2) $|vx| > 0$ (i.e. at least one of v, x is non empty),
3) for all $i \geq 0$, the string $uv^i wx^i y \in L$.

We will show this property fails for L, so L is not context-free.

b) Choose a string $s \in L$ where $|s| \geq p$ (pumping length).

Let p be the pumping length. Choose

$$s = a^p b^p c^p \in L,$$

so $|s| = 3p \geq p$

Assume $s = uvwxy$ satisfies the pumping-lemma conditions: $|vwx| \leq p$ and $|vx| > 0$.

c) Show some pumped string leaves L

Let $s = a^p b^p c^p$. Since $|vwx| \leq p$, the substring vwx lies fully inside one block (a's / b's / c's) or at most spans two.

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- If vwx is within one block, pumping adds/removes symbols from only that block → unequal a, b, c counts → not in L!

- If vwx spans two blocks, pumping disturbs their balance while the third block stays unchanged → counts mismatch → not in L.

Hence, for some $i \neq 1$, $uv^i wx^i y \notin L$.

a) Conclude why L cannot be accepted by any PDA.

Because the pumping lemma for CFLs is a necessary property of all context-free languages, and L violates it, L is not context-free. By the equivalence of context-free languages and pushdown automata, no PDA can accept L.

Ques6 Suppose, $L(G) = \{a^m b^n \mid m > 0 \text{ and } n \geq 0\}$. Find out the grammar G which produces L(G).

Given: $L(G) = \{a^m b^n \mid m > 0, n \geq 0\}$
To find: Context-Free Grammar G that generates this language.

Grammar:
Let $G = (V, \Sigma, P, S)$ where
- $V = \{S, T\}$
- $\Sigma = \{a, b\}$
- S is the start symbol

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- Productions P:

$S \rightarrow aS \mid aT$
$T \rightarrow bT \mid \varepsilon$

⇒ Explanation:
- S ensures at least one 'a' is produced (since every derivation starts with a).
- T generates zero or more b's

Thus, all strings have one or more a's followed by any number of b's.

⇒ Eg. derivation:

$S \Rightarrow aS \Rightarrow aaT \Rightarrow aa\ bT \Rightarrow aa\ bbT \Rightarrow aa\ bbb$

Generated string: $aabbb \in L(G)$

Hence, the grammar correctly generates
$L(G) = \{a^m b^n \mid m > 0, n \geq 0\}$

Ques I Is this grammar ambiguous? If so, prove it and construct a non-ambiguous grammar that derives the same language.
$S \rightarrow aS \mid aSbS \mid c$

Yes, it is ambiguous. We show two different parse trees for this string.

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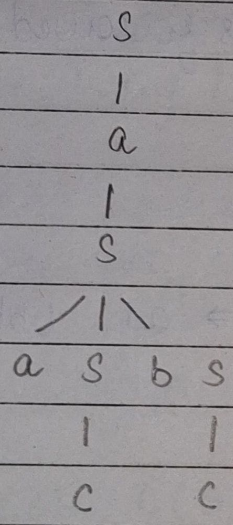
Two different parse trees for aacbc

Derivation A

1. $S \Rightarrow aS$
2. $aS \Rightarrow a(aSbS)$
3. $a(aSbS) \Rightarrow a(acbS)$
4. $a(acbS) \Rightarrow a(acbc) = aacbc$

Parse tree A (shape):



Derivation B

1. $S \Rightarrow aSbS$
2. $aSbS \Rightarrow a(aS)bS$
3. $a(aS)bS \Rightarrow a(ac)bS$
4. $a(ac)bS \Rightarrow a(ac)bc = aacbc$

Parse tree B (different shape):

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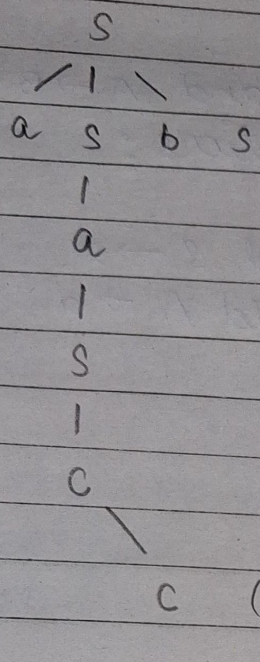

(rightmost $S \to c$)

⇒ The two trees are structurally different (in A the left child of the root is 'a' then a subtree asbS; in B the root expands to asbS immediately). Therefore the grammar is ambiguous (exists a string with two distinct parse trees).

⇒ A grammar is ambiguous iff some string has two different parse trees. We exhibited two distinct parse trees for aacbc. Hence the grammar is ambiguous.

Quiz: Give the CFG $G = (\{S, A, B\}, \{a\}, \{S \to A, A \to B, B \to a\}, S)$. Remove unit productions and rewrite the grammar.

Given CFG:

$G = (\{S, A, B\}, \{a\}, P, S)$ with

$P: S \to A, A \to B, B \to a$

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⇒ Remove unit productions :
We have unit chains $S \to A \to B \to a$ and $A \to B \to a$.
Replace them by direct terminal productions :

- From S follow chain to terminal a ⇒ add $S \to a$.
- From A follow chain to terminal a ⇒ add $A \to a$.
- Keep $B \to a$.

Remove unit rules $S \to A$ and $A \to B$

Resulting grammar (no unit productions):

$S \to a$
$A \to a$
$B \to a$

Note: All non-terminals still generate a. The language is {a}.

Ques Give the CFG $G = (\{S, A, B\}, \{a, b, c\}, \{S \to A, A \to aB, B \to c\}, S)$. Remove useless productions and write the updated grammar.

Given :
$G = (\{S, A, B\}, \{a, b, c\}, P, S)$ with
$P : S \to A,\ A \to aB,\ B \to c.$

Step 1 — Find non terminals that generate terminals (useful for generating):

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- $B \rightarrow C \Rightarrow$ B generates.
- $A \rightarrow aB$ and B generates $\Rightarrow$ A generates.
- $S \rightarrow A$ and A generates $\Rightarrow$ S generates.

So all S, A, B are generating

Step 2 – Find reachable non terminals from start S:

- S is start (reachable).
- From $S \rightarrow A \Rightarrow$ A reachable
- From $A \rightarrow aB \Rightarrow$ B reachable

So, all S, A, B are reachable.

⇒ There are no useless productions (no unreachable or non-generating non-terminals).

* terminal grammar (after removing useless productions):

$S \rightarrow A$
$A \rightarrow aB$
$B \rightarrow C$

Language produced : Strings of the form ac (specifically only "ac").

Ques10 Convert the given CFG to CNF. Consider the given grammar G1:
$S \rightarrow a \mid aA \mid B$

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$A \rightarrow aBB \mid \varepsilon$
$B \rightarrow Aa \mid b$

Step 1 – Remove ε- production:
$A \rightarrow \varepsilon$ is nullable ⇒ update others.

$S \rightarrow a \mid aA \mid B$
$A \rightarrow aBB$
$B \rightarrow Aa \mid a \mid b$

Step 2 – Remove unit productions:
$S \rightarrow B$ ⇒ replace with B's RHS.

$S \rightarrow a \mid b \mid aA \mid Aa$
$A \rightarrow aBB$
$B \rightarrow Aa \mid a \mid b$

Step 3 – Convert to CNF form:
Introduce $X\_a \rightarrow a$, $Y \rightarrow BB$.
Replace terminals in long RHS and make binary:

$S \rightarrow a \mid b \mid X\_aA \mid A\ X\_a$
$A \rightarrow X\_a\ Y$
$Y \rightarrow B\ B$
$B \rightarrow A\ X\_a \mid a \mid b$
$X\_a \rightarrow a$

Final CNF: All rules are either $A \rightarrow BC$ or $A \rightarrow a$.