Name: Harshit Jain
User ID: hmj5262

Problem 1

1. Let's suppose $\sqrt{13}$ is a rational number. Then we can write it $\sqrt{13} = \frac{a}{b}$ where a, b are whole numbers, b not zero.

   We additionally assume that this $\frac{a}{b}$ is simplified to lowest terms, since that can obviously be done with any fraction. Notice that in order for $\frac{a}{b}$ to be in simplest terms, both of a and b cannot be even. One or both must be odd. Otherwise, we could simplify $\frac{a}{b}$ further.

   From the equality $\sqrt{13} = \frac{a}{b}$ it follows that $13 = \frac{a^2}{b^2}$, or $a^2 = 13b^2$. Since 13 is prime and a^2 is a multiple of 13, then a is multiple of 13.

   If we substitute $a = 13k$ into the original equation $\sqrt{13} = \frac{a}{b}$, we get:

   $\Rightarrow (13k)^2 = 13b^2$

   $\Rightarrow b^2 = 13k^2$

   Since 13 is prime and b^2 is a multiple of 13 then b is multiple of 13.

   We now have a contradiction since a and b must have no common factors (except 1) but we have proved that if $\frac{a}{b}$ exits then a and b must have common factor 13.

   So $\frac{a}{b}$ can not exist and the square root of 13 is irrational.

2. Yes, we can prove that square root of any prime number is irrational.

   Let's suppose $\sqrt{p}$ is a rational number, where p is any prime number. Let $\sqrt{p} = \frac{m}{n}$ where $m, n \in N$. and m and n have no factors in common.

   Now $p = \frac{m^2}{n^2}$, or $m^2 = pn^2$.

   Since p is prime and m^2 is a multiple of p then m is multiple of p.

   If we substitute $m = pk$ into the original equation $\sqrt{p} = \frac{m}{n}$, we get:

   $\Rightarrow (pk)^2 = pn^2$

   $\Rightarrow n^2 = pk^2$

   Since p is prime and n^2 is a multiple of p then n is multiple of p.

   We now have a contradiction since m and n must have no common factors (except 1) but we have proved that if $\frac{m}{n}$ exits then m and n must have common factor p.

   So $\frac{m}{n}$ can not exist and the square root of any prime is irrational.

Problem 2

$Q = \{q_0, q_1, q_2, q_3\}$

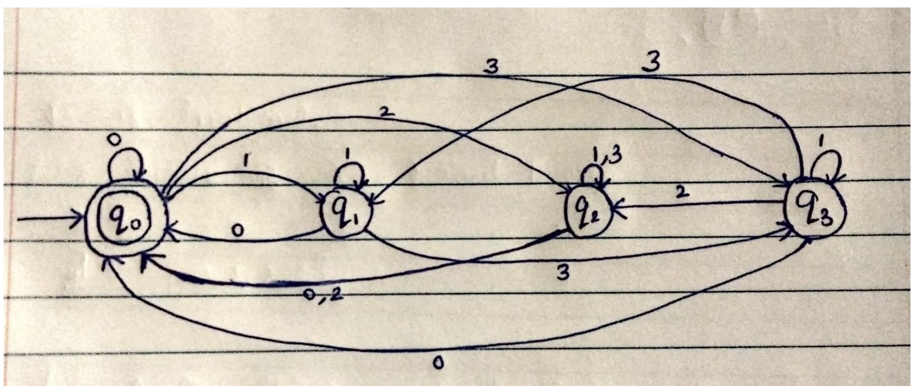
q_0 : Initial state, state after reading a digit that leaves a remainder of 0 when divided by 4
q_1 : State after reading a digit that leaves a remainder of 1 when divided by 4
q_2 : State after reading a digit that leaves a remainder of 2 when divided by 4
q_3 : State after reading a digit that leaves a remainder of 3 when divided by 4

$\Sigma = \{0, 1, 2, 3\}$

$q_0 = \{q_0\}$ (Start State)

$F = \{q_0\}$ (Accept State)

$$\delta = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline q_0 & q_0 & q_1 & q_2 & q_3 \\ q_1 & q_0 & q_1 & q_2 & q_3 \\ q_2 & q_0 & q_2 & q_0 & q_2 \\ q_3 & q_0 & q_3 & q_2 & q_1 \end{array}$$



Problem 3

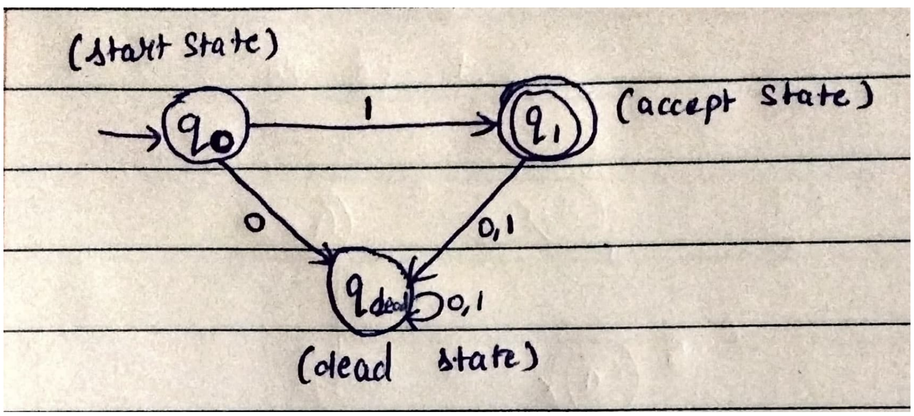
1. Assume there exists a DFA over the alphabet $\{0,1\}$ that recognizes the language $L = \{1\}$ with less than three states.

   A DFA has states, transitions, an initial state, and accepting states. Let's analyze the possibilities:

   - We must have at least one start state. Let the start state be q_0.
   - For the string 1, there must be an accepting state. Let this state be q_1.
   - Let's introduce a dead state to q_{dead} handle strings other than 1. This state will be a non-accepting state, and we can transition to it from q_0 to q_1 on input 0 or 1.

   Now, this DFA has three states: q_0, q_1, and q_{dead}. It ensures that only the string "1" is accepted, and all other strings lead to the dead state.

   Therefore, we have shown that it is impossible to create a DFA with less than three states that recognizes the language $L = \{1\}$, and we need at least three states, including a dead state, to handle strings other than 1.

2. Yes, the DFA over the alphabet $\{1\}$ that recognizes the language $L = \{1\}$ needs to have at least three states.

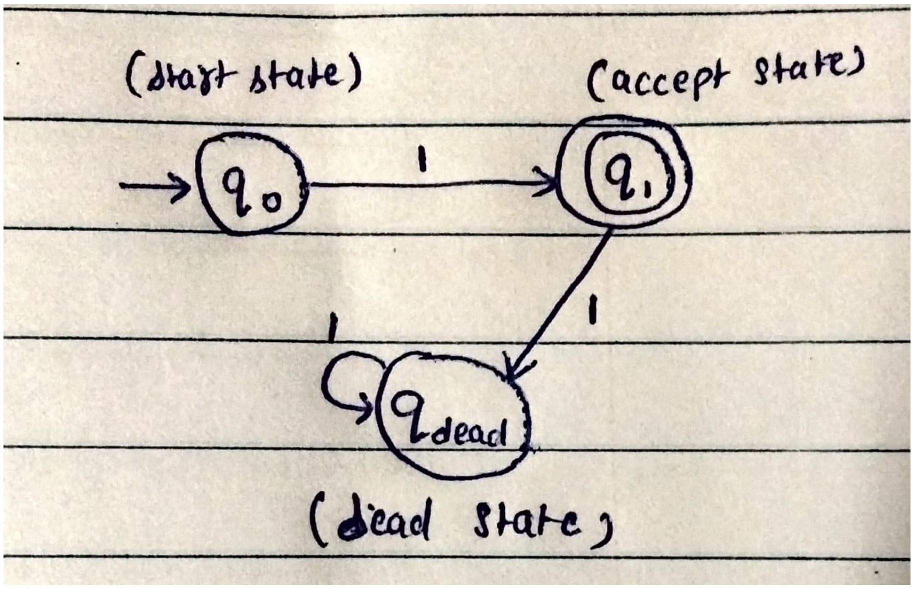
   <u>States:</u>

   - q_0 (Start state)
   - q_1 (Accept state for the valid string 1)
   - q_{dead} (Dead state for any other invalid string)

   <u>Transitions:</u>

   - From q_0, on input 1, transition to q_1
   - From q_1, on input 1, transition to q_{dead}

This DFA ensures that only the string 1 is accepted (transitioning from q_0 to q_1), and any additional 1s are rejected by transitioning to the dead state q_{dead}. Thus, a minimum of three states is required to handle both acceptance and rejection of strings in the language $L = \{1\}$ over the alphabet $\{1\}$.



Problem 4

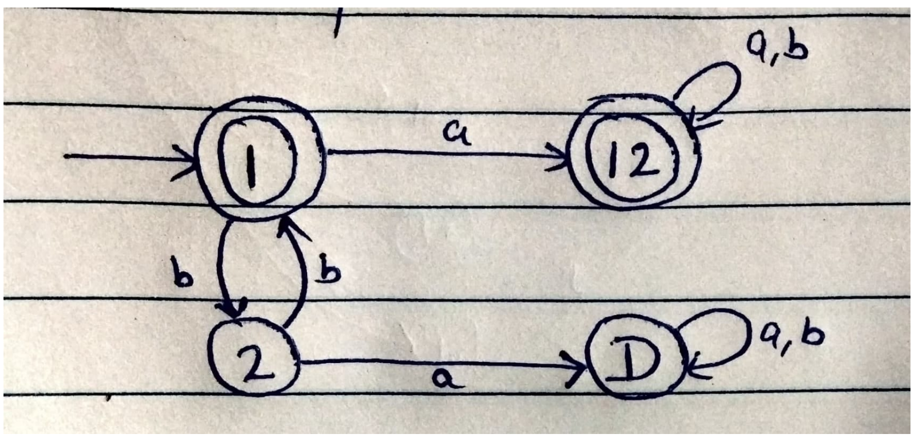
From the given NFA, we will make the transition table, $\delta_{NFA} = \begin{array}{c|cc} & a & b \\ \hline 1 & \{1,2\} & 2 \\ 2 & \phi & 1 \end{array}$

From here, we can make an equivalent DFA transition table as follows, $\delta_{DFA} = \begin{array}{c|cc} & a & b \\ \hline 1 & 12 & 2 \\ 12 & 12 & 12 \\ 2 & D & 1 \\ D & D & D \end{array}$

<u>States:</u>

- 1 (Start state and Accept State)
- 12 (Accept state)
- D (Dead state for any other invalid string)

So, the converted DFA is as follows:



Problem 5

1. Every regular language can be recognized by a DFA. Since a DFA is a special case of an NFA, we can convert any DFA recognizing a regular language into a 4NFA by replicating the accept states.

   Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$, we can construct a 4NFA, $M' = (Q', \Sigma, \delta', q_0, F')$ as follows:

   - Let $Q' = Q \times \{1, 2, 3, 4\}$, where each state $q \in Q$ is replicated four times with different indices.
   - Define δ' such that for each transition $\delta(q, a) = p$, where $q, p \in Q$ and $a \in \Sigma$, we have $\delta'((q, i), a) = (p, i+1)$ for $i = 1, 2, 3$. Additionally, $\delta'((q, 4), a) = (p, 1)$.
   - Let $F' = \{(p, i) \mid p \in F, i = 4\}$, i.e., the accept states are the ones in the last replication.

   This construction ensures that in the 4NFA M', at least four branches must have reached an accept state.

2. To establish that every language recognized by a 4NFA is regular, we need to demonstrate that we can convert any 4NFA into an equivalent DFA.

   Given a 4NFA, $M = (Q, \Sigma, \delta, q_0, F)$, we can construct an equivalent DFA $D = (Q', \Sigma, \delta', q_0', F')$ as follows:

   - The states of D are subsets of the states of M. Formally, $Q' = 2^Q$ (the power set of Q).
   - For each state $S \in Q'$ and input symbol $a \in \Sigma$:
     - Let $S' = \bigcup_{q \in S} \delta(q, a)$.
     - Define $\delta'(S, a) = S'$.
   - $q_0' = \{q_0\}$ (the set containing the initial state of M).
   - $F' = \{S \in Q' \mid |S \cap F| \geq 4\}$ (the sets with at least four states from F in them).

   This construction guarantees that every language recognized by a 4NFA can also be recognized by a DFA. Thus, the language recognized by a 4NFA is regular.

3. The number 4 is somewhat arbitrary in this context. The concept of k-NFAs (for $k > 1$) implies that more than one branch must reach an accept state for acceptance. The specific choice of 4 in the question might be used to emphasize that more than a single branch must reach an accept state, but other values of k (e.g., 3NFA or 5NFA) would still capture the idea that multiple branches are required for acceptance. The key is that k-NFAs allow for multiple paths to acceptance, making them more expressive than traditional NFAs or DFAs.