INDIAN INSTITUTE OF TECHNOLOGY, KANPUR, CSE

CS698D, Assignment 1

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1 PROBLEM 1

The question is done according to the algorithm described in notes.

Data: A sequence of words and numbers in an array: A
Result: Decompressed sequence of words: in $output$
while $(A[i] \neq \phi)$ **do**
  if $A(i+1).type == word$ **then**
    $output+ = A(i+1)$;
    $arr.first = A(i+1)$;
    $i+ = 2$;
  else
    if $i > arr.length()$ **then**
      $return$ Error
    end
    $word = arr(i)$;
    $output+ = word$;
    $arr.remove(i)$;
    $arr.begin = word$;
    $i++$;
  end
end

2 Problem 2

Expected Length of first run:

$$
\begin{aligned}
E_l(2) &= \lim_{n\to\infty}\left(\sum_{l=1}^{n} p(l)l\right)\\
&= 1\left(\frac{1}{2^2}+\frac{1}{2^2}\right)+2\left(\frac{1}{2^3}+\frac{1}{2^3}\right)+\ldots\infty\\
&= 1.\left(\frac{1}{2}\right)+2\left(\frac{1}{2^2}\right)+\ldots+i\left(\frac{1}{2^i}\right)\ldots\infty\\
\frac{E_1(2)}{2} &= 1.\left(\frac{1}{2^2}\right)+2\left(\frac{1}{2^3}\right)+\ldots i\left(\frac{1}{2^{(i+1)}}\right)\ldots\infty\\
&\text{Subtracting the above 2 equations:}\\
\frac{E_1(2)}{2} &= \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\ldots\\
&= 1\\
E_1(2) &= 2
\end{aligned}
$$

For alphabet os size: s

$$
\begin{aligned}
E_l(s) &= 1\left(\frac{s(s-1)}{s^2}\right)+2\left(\frac{s(s-1)}{s^3}\right)+3\left(\frac{s(s-1)}{s^4}\right)+\ \ldots\infty\\
\frac{E_l(s)}{s} &= 1\left(\frac{s(s-1)}{s^3}\right)+2\left(\frac{s(s-1)}{s^4}\right)+3\left(\frac{s(s-1)}{s^5}\right)+\ \ldots\infty\\
&\text{Subtracting the above 2 equations:}\\
E_l(s)\left(1-\frac{1}{s}\right) &= s(s-1)\left(\frac{1}{s^2}+\frac{1}{s^3}+\ldots\right)\\
&= \frac{s(s-1)}{s^2}\left(\frac{1}{1-\frac{1}{s}}\right)\\
E_l(s) &= \frac{s}{s-1}
\end{aligned}
$$

3 Problem 3

We have a sequence of words: $w_1, w_2, w_3..w_n$
To show:

$$
l_p(w_{i-1}, w_i) \geq l_p(w_i, w_j)\forall j < (i-1) \tag{3.1}
$$

Proof by contradiction:
Suppose, $\exists j < i-1$ s.t. $l_p(w_{i-1}, w_i)[=l_1] \geq l_p(w_i, w_j)[=l_2]$

Then, w_{i-1}, w_j match till l_1 and $w_{i-1}(l_1+1) \neq w_j(l_1+1)$

$$
\begin{aligned}
&w_{i-1}(l_1+1) \leq w_i(l_1+1) \text{ [Lexicographically sorted]}\\
&w_{i-1}(l_1+1) \leq w_j(l_1+1)\\
&w_{i-1}(l_1+1) < w_j(l_1+1)
\end{aligned}
$$

But then $j > i-1$
$\Rightarrow\Leftarrow$
Therefore, $l_p(w_{i-1}, w_i) \geq l_p(w_i, w_j) \forall j < (i-1)$

4 PROBLEM 4

To prove

$$
\begin{aligned}
H(Y|X) = 0 \leftrightarrow\ Y(X) \text{ or}\\
\forall x_0 \in X\ \exists y_0 \in Y \text{ s.t.} P(y_0|x_0) = 1\text{and}\\
P(y_1|x_0) = 0\ \forall y_1 \neq y_0
\end{aligned}
$$

Proof
$\Rightarrow$

$$\sum_y \sum_x P(x, y) \log \frac{1}{P(y|x)} = 0 \tag{4.1}$$

Since, each of the term in the summation is greater than 0, each term has to individually 0.
Now, suppose,

$$
\begin{aligned}
\exists y_1 \text{s.t.} P(y_1|x) \in (0, 1)\\
\exists y_2 \text{s.t.} P(y_2|x) \in (0, 1)
\end{aligned}
$$

Then, $P(x, y_1) \log \frac{1}{P(y_1|x)} \neq 0$.
Also, $P(x, y_2) \log \frac{1}{P(y_2|x)} \neq 0$
$\Rightarrow\Leftarrow$
$\Leftarrow$
Suppose, $\exists y_0$ s.t. $Pr(y_0|x_0) = 1$ and $Pr(y_1|x_0) = 0 \forall y \neq y_0$
Then, it is easy to see that all the terms in the equation 4.2 are 0.
Hence, proved.

5 PROBLEM 5

5.1

Case 1: Sequence is unbounded.
Then trivially limit is ∞ because the sequence is monotonically deceasing.

Case 2: Sequence is bounded.
Let $x = \inf\ x_n$
$\forall \epsilon > 0 \exists x_n$ s.t. $(x+\epsilon) > x_n$ [By inf definition]
$\forall m > n\ x_m < x_n$
$\forall m > n\ x_m < x+\epsilon$
$\forall \epsilon \forall m > N\ |x_m - x| < \epsilon$
So, $x_m \to x$

5.2

Let the sequence be $x_1, x_2, x_3..$

$$lim \sup x = lim_{n\to\infty} \sup_{m>n} x_m$$
$$\sup_{m>i} x_m \geq \sup_{m>(i+1)} x_m$$

So, $\sup_{m>i} x_m, \sup_{m>(i+1)} x_m, ...$ form a decreasing sequence. By analogy of previous result increasing sequence of real numbers has a limit.
Similarly,

$$lim \inf x = lim_{n\to\infty} \inf_{m>n} x_m$$
$$\inf_{m>i} x_m \leq \inf_{m>(i+1)} x_m$$

So, $\inf_{m>i} x_m, \inf_{m>(i+1)x_m}, ...$ form an increasing sequence. By previous result this forms a decreasing sequence of real numbers has a limit.

6 PROBLEM 6

We will show,

$$f = O(g) \leftrightarrow \lim sup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty \tag{6.1}$$

$\Rightarrow$
If $g(n) = 0$ then trivially $f(n) = 0$. If $g(n) \neq 0:$
$\exists C \exists n > N\ |f(n)| < C|g(n)|$

$$\frac{|f(n)|}{|g(n)|} \leq C \tag{6.2}$$

If a sequence: x is bounded above by C then $\lim supx \leq C$
Since, the sequence is bounded above by C. So,

$$\lim_{n\to\infty} sup \frac{|f(n)|}{|g(n)|} \leq C \tag{6.3}$$

$\Leftarrow$
We say that $\alpha = \lim sup_{n\to\infty} f(n)$ if following conditions hold:

$$(\epsilon > 0)(\exists N)(\forall n > N)(f(n) < \alpha + \epsilon)$$
$$(\epsilon > 0)(\forall N)(\exists n > N)(f(n) > \alpha - \epsilon)$$

Putting $\epsilon = 1$ in the above definition.

$$\forall n > N \frac{|f(n)|}{|g(n)|} < C + 1 \tag{6.4}$$

$$\forall n > N \, |f(n)| \leq (C+1)|g(n)| \tag{6.5}$$

So, $f(n) = O(g(n))$. Hence, proved.
Reference: parc.im.pwr.wroc.pl/ cichon/Math/BigO.pdf

7 Problem 7

The function $f(x) = \frac{1}{x}$ is undefined on $x = 0$ so it is discontinuous. [

$$\lim_{x\to 0^-} f(x) = -\infty$$
$$\lim_{x\to 0+} f(x) = +\infty$$

Since both the limits are different we cannot define $f(x)$ at $x = 0$ s.t. the function becomes continuous.]
$f : \mathbb{R} \to \mathbb{R}$ is continuous $\leftrightarrow$ $\forall \epsilon, x \exists \delta \forall y (|x - y| < \delta) \rightarrow (|f(x) - f(y)| < \epsilon)$
Consider some point $x \neq 0$ at which we will show $f(x)$ is continuous. If $y > \frac{x}{2}$
Choose $\delta < \min\left(\frac{|x|}{2}, \frac{\epsilon x^2}{2}\right)$

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| \leq \frac{\delta}{|x|(|x| - \delta)} \leq \frac{2\delta}{x^2} \leq \epsilon$$

If $x \neq 0$ then $\forall \epsilon$ we can find a suitable value of δ s.t. the continuity condition holds.
At $x = 0$ the function is discontinuous.

8 Problem 8

Given:
Continuous function $f : [0,1] \to \mathbb{R}$ Since, f is continuous on $[0,1]$,
$\forall \epsilon, x \exists \delta \forall y \ \ |x - y| < \delta \ \rightarrow \ |f(x) - f(y)| < \epsilon, \ x, y \in [0,1]$
Let the partitions of the interval $[0,1]$ be denoted by $x_1, x_2, ... x_n$ s.t. $|x_{i+1} - x_i| < \delta \ \forall i \in [n]$

For partition P_i

$$
\begin{aligned}
U_i &= f(x_{i+1})(x_{i+1} - x_i) \\
L_i &= f(x_i)(x_{i+1} - x_i) \\
U_i - L_i &= (f(x_{i+1}) - f(x_i))(x_{i+1} - x_i) \\
&\leqslant \epsilon(x_{i+1} - x_i) \\
\sum_{i=1}^{i=n}(U_i - L_i)s = \epsilon(1-0)
\end{aligned}
$$

$\Rightarrow f$ is integrable.

9 Problem 9

$$
\begin{aligned}
L^1 &= 0 \\
L^2 &= \frac{1}{2^3}(0^2 + 1^2) \\
L^3 &= \frac{1}{3^3}(0^2 + 1^2 + 2^2) \\
L^4 &= \frac{1}{4^3}(0^2 + 1^2 + 2^2 + 3^2) \\
L^5 &= \frac{1}{5^3}(0^2 + 1^2 + 2^2 + 3^2 + 4^2) \\
L^n &= \frac{1}{n^3}(0^2 + 1^2 + \ldots + (n-1)^2) \\
&= \frac{(2n-1)n(n-1)}{6n^3} \\
L^n_{n\to\infty} &= \frac{1}{3}
\end{aligned}
$$

Now, checking for upper limit:

$$
\begin{aligned}
U^1 &= 1 \\
U^2 &= \frac{(1^2 + 2^2)}{2^3} \\
U^3 &= \frac{(1^2 + 2^2 + 3^2)}{3^3} \\
U^n &= \frac{(2n+1)(n+1)n}{6n^3} \\
U^n_{n\to\infty} &= \frac{1}{3}
\end{aligned}
$$

Since, $L^n_{n\to\infty} = U^n_{n\to\infty}$ the function is Reinmann integrable and the value of the integral is: $\frac{1}{3}$

10 PROBLEM 10

To prove:

$$\mu(A_1 \triangle A_2) = 0 \rightarrow \mu(A_1) = \mu(A_2) \tag{10.1}$$

Proof:

$$\begin{aligned} \mu(A_1 \triangle A_2) &= 0 \\ \Rightarrow \mu(A_1 \setminus A_2 \cup A_2 \setminus A_1) &= 0 \\ \text{Since both the sets ar disjoint} \\ \Rightarrow \mu(A_1 \setminus A_2) + \mu(A_2 \setminus A_1) &= 0 \end{aligned}$$

Since the range of μ is $\mathbb{R}^+ \cup 0$, both the terms have to be individually 0.

$$\begin{aligned} \Rightarrow \mu(A_1 \setminus A_2) &= 0 \\ \text{Also, } \mu(A_1) &= \mu(A_1 \setminus A_2 \cup A_1 \cap A_2) \\ \mu(A_1) &= \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) \\ \text{Since both the sets ar disjoint} \\ \mu(A_1) &= \mu(A_1 \cap A_2) \\ \text{Similarly, } \mu(A_2) &= \mu(A_1 \cap A_2) \end{aligned}$$

Therefore, $\mu(A_1) = \mu(A_2)$