Lecture 3 - Differentiation

Taylor's Theorem:

Suppose f is differentiable upto $n+1$ times on I, i.e, $f', f'', \dots, f^{(n+1)}$ are exists on I and let $a \in I$. For given $x_0 \in I$, $x_0 \neq a$, there exists a ξ between a & x such that

$$f(x_0) = f(a) + \frac{f'(a)}{1!}(x_0-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x_0-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x_0-a)^{n+1}.$$

$$f(x_0) = \underbrace{P_n(x_0)}_{n^{th} \text{ degree Taylor poly. of } f \text{ at } 'a'} + \underbrace{R_n(x_0)}_{n^{th} \text{ reminder term}}.$$

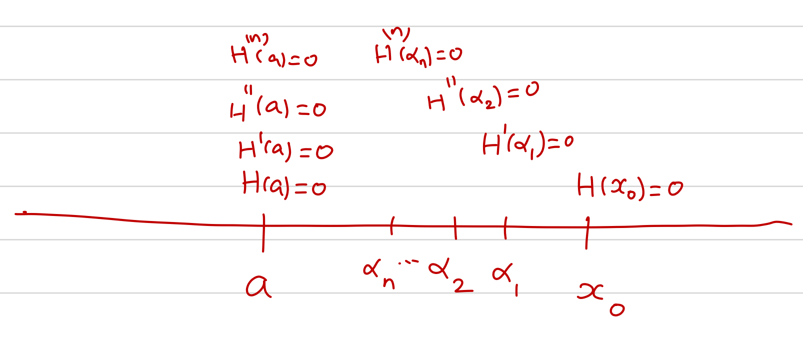
proof: Choose M such that

$$f(x_0) - P_n(x_0) = M(x_0-a)^{n+1}.$$

Claim $M = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x_0-a)^{n+1}$.

Define $H(t) = f(t) - P_n(t) - M(t-a)^{n+1}$.

Observe $H(a) = 0 = H(x_0)$



By Rolle's thm, $\exists\ \alpha_1$ between a & x_0 s.t.

$$H'(\alpha_1) = 0.$$

$\because H'(a) = 0$, again using Rolle's thm, we get α_2 between a & α_1 s.t.

$$H''(\alpha_2) = 0.$$

By repeatedly using Rolle's thm, there exists ξ between a & x_0 s.t.

$$H^{(n+1)}(\xi) = 0.$$

$$f^{(n+1)}(\xi) - M(n+1)! = 0$$

$$M = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\therefore f(x_0) = P_n(x_0) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x_0-a)^{n+1}.$$

— x —

Examples:

1. $f(x) = \sin x$.

Take $a=0$. $\quad f'(x) = \cos x$

$$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right).$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

where ξ is between 0 & x.

$$= x - \frac{x^3}{3!} + \cdots + \frac{\sin\frac{n\pi}{2}}{n!}x^n + \frac{\sin(\xi + \frac{n\pi}{2})}{(n+1)!}x^{n+1}.$$

2. $f(x) = e^x$. Take $a=0$. $\quad f^{(n)}(x) = e^x$

$$f(x) = 1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!} + \frac{e^{\xi}x^{n+1}}{(n+1)!}$$

where ξ is between 0 & x.

Try: $f(x) = \cos x$, $a=0$
$g(x) = \log(1+x)$ "

Error bound:

Thm: Suppose $|f^{(n+1)}(x)| \leq M \quad \forall x \in I$.

Then the approximation error bound between f & P_n is

$$|f(x) - P_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!} \quad \forall x \in I.$$

Because. $\quad |f(x) - P_n(x)| = \left|\frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}\right|$

$$\leq \frac{M|x-a|^{n+1}}{(n+1)!}.$$

Illustration:

$f(x) = \sin x$, $x \in [-1,1]$, $a=0$.

$$|\sin x - P_n(x)| \leq \frac{1 \cdot |x|^{n+1}}{(n+1)!} \leq \frac{1}{(n+1)!}.$$

$n=5$ $\quad \left|\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\right| \leq \frac{1}{6!} \approx 0.0014$

$$< 10^{-2}.$$

(*) Find the n^{th} order Taylor polynomial P_n about $x=0$, for the funct. $\sin x$ with the error bound 10^{-2} in the interval $|x| \leq 1$.

$$|\sin x - P_n(x)| \leq \frac{1}{(n+1)!} < 10^{-2}$$

we get $n = 4$.

$$P_4(x) = x - \frac{x^3}{3!}$$

(*) Find the interval of validity when we approximate $\sin x$ with 3^{rd} order Taylor poly with error bound 10^{-3}.

$$\left|\sin x - x + \frac{x^3}{3!}\right| \leq \frac{1 \cdot |x|^4}{4!} < 10^{-3}$$

$$|x|^4 < 24 \times 10^{-3}$$

$$|x| < (24 \times 10^{-3})^{1/4}.$$

Local Maximum & Minimum

Suppose f', f'' are exists & cts on I & $a \in I$. Suppose $f'(a) = 0$ & $f''(a) < 0$.

Then f has a local maximum at 'a'.

Pf: $\because f''(a) < 0$ & f'' is cts on I, $\exists\ \delta > 0$ s.t. $f''(x) < 0 \quad \forall |x-a| < \delta$.

For $x \in (a-\delta, a+\delta) \setminus \{a\}$,

$$f(x) = f(a) + \cancel{f'(a)(x-a)} + \frac{f''(\xi)}{2!}(x-a)^2$$

where ξ is between a & x

$$\Rightarrow |\xi - a| < \delta.$$

$$\Rightarrow f''(\xi) < 0.$$

$$\Rightarrow f(x) - f(a) = \frac{f''(\xi)}{2!}(x-a)^2 < 0$$

$\therefore$ the pt 'a' is a local max of f.

Revisit L'Hôpital's Rule.

Suppose f & g are diff upto 'n' times on I, $a \in I$. Suppose $f^{(k)}(a) = g^{(k)}(a) = 0$, $k = 0, \dots, n-1$ & $g^{(n)}(a) \neq 0$, $f^{(n)}$ & $g^{(n)}$ are cts on I.

Then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

— x —

By Taylor's thm,

$$f(x) - P_n(x) = R_n(x)$$
$$\to 0 \text{ as } n \to \infty.$$

$$\Rightarrow f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \to ①.$$

① is true iff $R_n(x) \to 0$ as $n \to \infty$.

The Taylor's Series of f around the pt 'a' is given by $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

At $a=0$, it is also known as Maclauren series

Thm Suppose f is infinitely diff on I, $a \in I$, and $|f^{(n)}(x)| \leq M$, $\forall x \in I$ $\forall n \geq 1$.

Then
$$\lim_{n\to\infty} R_n(x) = 0$$

and
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Pf:
$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

$$\to 0 \text{ as } n \to \infty \quad \text{(why?)}$$

— x —

eg:

(1) $f(x) = \sin x$. $\quad |f^{(n)}(x)| \leq 1$.

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty.$$

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

(2) $f(x) = e^x$. $\quad x \in \mathbb{R}$.

$\boxed{a=0}$

$$|R_n(x)| = \left|\frac{e^{\xi}x^{n+1}}{(n+1)!}\right| \quad \text{when } \xi \text{ is betw } 0 \text{ \& } x.$$

$$\Rightarrow e^{\xi} \leq e^{|x|}$$

$$\leq \frac{e^{|x|}|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(3). $f(x) = \log(1+x)$

$$f' = \frac{1}{1+x}$$
$$f''(x) = \frac{(-1)}{(1+x)^2}$$
$$\vdots$$
$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

For $x > 0$,

$$R_n(x) = \frac{f^{(n+1)}(\xi_n)}{(n+1)!} \cdot x^{n+1}, \quad \text{for some } \xi_n \in (0,x).$$

$$|R_n(x)| = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{(1+\xi_n)^{n+1}}$$

$$= \frac{1}{n+1} \cdot \frac{|x|^{n+1}}{(1+\xi_n)^{n+1}}$$

If $x \in (0,1]$, then $|R_n(x)| \leq \frac{1}{n+1}$
$\to 0$ as $n \to \infty$.

$\therefore$ The Maclaurin series for the function

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad 0 < x \leq 1.$$

Think: Can we do a similar argument to derive $\log(1+x) = x - \frac{x^2}{2} + \cdots$ for $x \in (-1, 0)$.

(*)
$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Exc: Verify that f is infinitely diff on $\mathbb{R}$ & $f^{(k)}(0) = 0 \quad \forall k \geq 1$.
(Ross, pa. 238).

The Taylor's Series of f around '0'

$$\sum \frac{f^{(k)}(0)}{k!}x^k = 0.$$

By Taylor's thm, we have

$$e^{-1/x} = \overset{0}{\cancel{P_n(x)}} + R_n(x), \quad x > 0.$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

Qn: Is $R_n(x) \to 0$ as $n \to \infty$, $x > 0$?
Ans: NO.

Suppose $\lim_{n\to\infty} R_n(x) = 0$

$$\Rightarrow f(x) = \sum \frac{f^{(k)}(0)}{k!}x^k, \quad x > 0$$

$$e^{-1/x} = 0$$

$$\Rightarrow \Leftarrow.$$

Rmk: Even though the function f is infinitely differentiable & the Taylor's series about '0' converges but it is not cgs to f for any interval containing '0'.