Lecture - 4

# Riemann Integration

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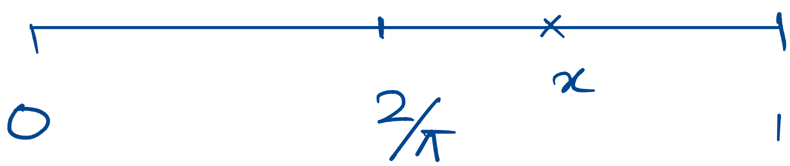
# Non Integrable functions.

**Example:**

$$f : [0,1] \to \mathbb{R}$$

$$f(x) = \begin{cases} 0 , & x \in \mathbb{Q} \\ \sin \frac{1}{x} , & x \notin \mathbb{Q} . \end{cases}$$

**Soln:**




We will show '$f$' is not R-int in $[\frac{2}{\pi}, 1]$.

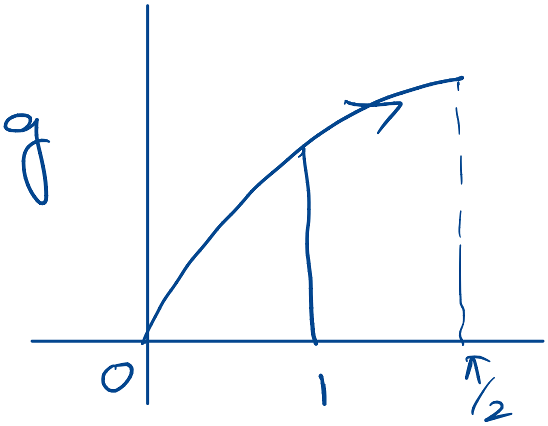
We know for any $x \in [\frac{2}{\pi}, 1]$

$$\frac{2}{\pi} \le x \le 1$$

$$\Rightarrow \quad 1 \le \frac{1}{x} \le \frac{\pi}{2}$$

Observe!

sine is an increasing fn in $[1, \pi/2]$




So $\sin 1 \leq \sin \frac{1}{x} \leq 1$.

ie, $\sin 1 \leq \sin 1/x$, $x \in \left[\frac{2}{\pi}, 1\right]$

Let 'P' be any arbitrary partition of $\left[\frac{2}{\pi}, 1\right]$.

We know,

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i ,$$

$$M_i = \sup_{[x_{i-1}, x_i]} f(x)$$

From the above fact we get

$$M_i \geq \sin 1$$

$$\Rightarrow \quad U(P, f) \geq \sin 1 \left(1 - \frac{2}{\pi}\right).$$

Consider,

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i ,$$

$$m_i = \inf_{[x_{i-1}, x_i]} f(x)$$

We know, $f(x) \geq 0$ in $[\frac{2}{\pi}, 1]$

Then, $L(P,f) = 0$.

Calculate,

$$U(P,f) - L(P,f) \geq \left(1 - \frac{2}{\pi}\right) \sin 1.$$

So if $\epsilon < \left(1 - \frac{2}{\pi}\right) \sin 1$

we cannot find a partition such that $U(P,f) - L(P,f) < \epsilon$.

$\Rightarrow$ $f$ is not $\boxed{\text{R- integrable.}}$

# Fundamental Theorem

Let a function

$f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then for each $x \in [a,b]$, $f$ is integrable on $[a,x]$.

Hence $\int_a^x f(t)\,dt$ exists and it depends on $x$.

Then we can define a function, $F$ on $[a,b]$ by,

$$\boxed{F(x) = \int_a^x f(t)\,dt}$$

Theorem: Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ then the function $F$ defined by

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a,b]$$

is continuous on $[a,b]$.

If $f$ is continuous at a point '$x_0$' then $F$ is differentiable at $x_0$ and

$$\boxed{F'(x_0) = f(x_0).}$$

Proof: We will assume $f$ is a bounded function on $[a,b]$.

Define

$$M = \sup\{|f(x)|, x \in [a,b]\}$$

Now then,

$$|F(x) - F(y)|$$

$$= \left| \int_a^x f(t)\,dt - \int_a^y f(t)\,dt \right|$$ [from defn].

$$= \left| \int_y^x f(t)\,dt \right|$$ [Using properties of R-int. fns]

$$\leq \int_y^x |f(t)|\,dt$$

$$\leq M|x-y|$$

So it follows.

$$|F(x) - F(y)| \leq M|x-y|,$$

$$\forall\, x, y \in [a,b].$$

This implies,

"$F$ is actually uniformly continuous".

Now assume,

$f$ is continuous at '$x_0$'.

Claim: $F$ is differentiable at $x_0$.

By continuity we can say,

given $\epsilon > 0$, $\exists\, \delta > 0$ s.t

$$|f(x) - f(x_0)| < \epsilon, \quad |x - x_0| < \delta.$$

For some $x \in [a, b]$, such that $|x - x_0| < \delta$, we look at,

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right|$$

$$\Rightarrow \left| \frac{1}{x - x_0} \int_{x_0}^{x} f(t)\,dt - f(x_0) \right|$$

$$= \left| \frac{1}{x - x_0} \int_{x_0}^{x} f(t)\,dt - \underline{\frac{1}{x - x_0} \int_{x_0}^{x} \underline{f(x_0)}dt} \right|$$

$$= \left| \frac{1}{x - x_0} \int_{x_0}^{x} \big( f(t) - f(x_0) \big)\,dt \right|$$

$$\leq \frac{1}{|x - x_0|} \int_{x_0}^{x} | f(t) - f(x_0) |\,dt$$

$$\leq \frac{\epsilon}{|x - x_0|} |x - x_0| = \epsilon .$$

[$\because$ $t$ varies in between $x_0$ & $x$ and by continuity criteria, $|f(t) - f(x_0)| < \epsilon$ $\forall$ such $t$ ]

So we get

$$\boxed{\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta .}$$

$$\Rightarrow \quad F'(x_0) = f(x_0) .$$

[Proved]

## Example

1) Let

$$f(x) = 0, \quad -1 \le x \le 0$$
$$= 1, \quad 0 < x \le 1$$

Since $f$ is bdd on $[-1,1]$ & it is continuous except at only '0', we know $f$ is - Riemann int.

For, $-1 \le x \le 0$,

$$F(x) = \int_{-1}^{x} f(t)\,dt = 0.$$

$0 < x \le 1$,

$$F(x) = x \quad (?)$$

We got, $F(x) = 0, \quad -1 \le x \le 0$
$\quad\quad = x, \quad 0 < x \le 1.$ } $F$ is cont.

Note: Here $f$ is not continuous on $[-1,1]$, but $F$ is continuous on $[-1,1]$

# Second Fundamental Theorem.

## Theorem

Suppose $f$ be a Riemann Int. function. If there exists a differentiable function $g$ such that,

$$g'(x) = f(x) \quad \forall x \in [a,b]$$

then

$$\int_a^b f(x) = g(b) - g(a).$$

[Another way of writing.

$$\int_a^b g'(x) = g(b) - g(a)$$ ].

"We skip the proof of this one".

The function 'g' is then called the 'antiderivative' of f.

Note: The evaluation of $\int_a^b f$ in terms of the anti-derivative is possible if f satisfies the conditions of the theorem.

These conditions are independent of each other.

Ex:

1) $f: [-1,1] \to \mathbb{R}$,

$$f(x) = 0, \quad -1 \leq x < 0$$
$$= 1, \quad 0 \leq x \leq 1.$$

Check: f has no antiderivatives on $[-1,1]$ although f is integrable on $[-1,1]$.

2) $f: [-1,1] \to \mathbb{R}$

$$f(x) = 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}, \quad x \neq 0$$

$$= 0, \quad x = 0$$

Check:

$f$ has an antiderivative although $f$ is not integrable.

## Change of variable

Theo: Suppose $u(t)$, $u'(t)$ be two continuous fn on $[a,b]$. Let $f$ be a cont. fn on the interval $u([a,b])$.

Then,
$$\int_a^b f(u(x))\,u'(x)\,dx$$
$$= \int_{u(a)}^{u(b)} f(y)\,dy.$$

Exercise: Check.

Example:
$$I = \int_0^1 x\sqrt{1+x^2}\,dx.$$

Soln: Lets take

$$y = u(x) = 1+x^2$$

Then $u'(x) = 2x$

$$I = \frac{1}{2}\int_0^1 f(u(x))\, u'(x),$$

where $f(u(x)) = \sqrt{1+x^2}$

$= \sqrt{u}$

$$= \frac{1}{2}\int_{u(0)}^{u(1)} \underline{f(y)\,dy}.$$

Calculate,

$u(0) = 1, \quad u(1) = 2$

$$I = \frac{1}{2}\int_1^2 f(y)\,dy.$$

$$= \frac{1}{2}\int_1^2 \sqrt{y}\;dy = \frac{1}{3}\left[2^{3/2}-1\right].$$

# Improper Integrals

1. The function $f(x)$ defined on unbounded interval $[a, \infty)$ and $f \in R[a,b]$, for all $b > a$.

2. The function is not defined at some points on $[a, b]$.

Example

1. $\displaystyle\int_0^\infty e^{-x}\,dx$, $\displaystyle\int_0^\infty \frac{dx}{1+x}$

2. $\displaystyle\int_0^1 \frac{dx}{x}$, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

# Improper Integral of first kind

Suppose $f$ is a bdd function defined on $[a,\infty)$ and $f \in \mathcal{R}[a,b]$ for all $b>a$.

## Definition:

The improper integral of $f$ on $[a,\infty)$ is defined as

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

If the limit exists and is finite, we say that the improper integral converges. If the limit goes to infinity or does not exist then we say that the improper integral diverges.

## Examples:

1. $$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\left(1-\frac{1}{b}\right)$$
$$= 1$$

2. $$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$
$$= \lim_{b\to\infty} \left[\arctan x\right]_0^b = \frac{\pi}{2}$$

3. $$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$
$$= \lim_{b\to\infty} \left(1 - e^{-b}\right) = 1$$

4. $$\int_0^\infty \frac{dx}{1+x} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x}$$
$$= \lim_{b\to\infty} \left[\log(1+b)\right] = \infty$$
$\longrightarrow$ divergent.

# Tests for Convergence

# Comparison test

## Theorem:

Suppose $0 \leq f(x) \leq \phi(x)$ for all $x \geq a$, then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty \varphi(x)\,dx$ converges.

2. $\int_a^\infty \varphi(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

**Proof:** Define,

$$F(x) = \int_a^x f(t)\,dt \quad \&$$

$$G(x) = \int_a^x \varphi(t)\,dt.$$

By the property of R-int,
we have,

$$0 \leq F(x) \leq G(x)$$

Also it given,

$$\lim_{x \to \infty} \int_a^x \varphi(t)\,dt \quad \text{exists},$$

i.e, $\lim_{x \to \infty} G(x)$ exists.

So then $G(x)$ is bounded.
Now, then, $F$ is monotonically increasing and also bounded above (Why?)

So then, $\lim_{x \to \infty} F(x)$ exists.

[Proved]

# Limit Comparison test

**Theorem:** Let $f(x) > 0$ and $g(x) > 0$ for all $x \geq a$, be defined and

$$\boxed{\lim_{x \to \infty} \frac{f(x)}{g(x)} = L}$$

1. If

$$0 < L < \infty,$$

then

$$\int_a^{\infty} \underline{f(x)}\,dx \text{ converge (diverge)} \Longleftrightarrow \int_a^{\infty} \underline{g(x)}\,dx \text{ converge. (diverge)}$$

2. If

$$L = 0$$

then

$\int_a^\infty g(x)\,dx$ converges

$$\Longrightarrow \int_a^\infty f(x)\,dx \text{ converges}$$

3. If

$$L = \infty$$

then

$\int_a^\infty g(x)\,dx$ diverges

$$\Longrightarrow \int_a^\infty f(x)\,dx \text{ diverges.}$$

## Examples

1.) $$\int_1^\infty \frac{dx}{x^2(1+e^x)}$$

Note that,

$$\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}$$

& (easy to check) $\int_1^\infty \frac{dx}{x^2}$ converges.

So by comparison test,

$$\int_1^\infty \frac{dx}{x^2(1+e^x)} \quad \text{converges.}$$

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2.) $$\int_1^\infty \frac{x^3}{x+1}\,dx$$

Note that $\frac{x^3}{x+1} > \frac{x^3}{2x}$ on $[1,\infty)$

$$= \frac{x^2}{2}$$

& $\int_1^\infty x^2 dx$ diverges.

Hence $\int_1^\infty \frac{x^3}{x+1} dx$ diverges.

3.) $\int_1^\infty \frac{dx}{\sqrt{x+1}}$

Let $f(x) = \frac{1}{\sqrt{x+1}}$ and

consider $g(x) = \frac{1}{\sqrt{x}}$.

Then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$ &

using the fact $\int_1^\infty \frac{dx}{\sqrt{x}}$ converges

we can conclude that

$\int_1^\infty \frac{dx}{\sqrt{x+1}}$ converges (By LCT).

4) $$\int_1^\infty \frac{dx}{x(1+x^2)}$$

Let $f(x) = \dfrac{1}{x(1+x^2)}$

& consider $g(x) = \dfrac{1}{x^3}$

Then $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

& we know. $\displaystyle\int_1^\infty \frac{dx}{x^3}$

converges and hence

by LCT,

$$\int_1^\infty \frac{dx}{x(1+x^2)} \text{ converges.}$$

# Dirichlet's test

**Theorem:** Let

i) $\phi$ be a monotonic and bounded function on $[a,\infty)$ and $\lim_{x\to\infty} \phi(x) = 0$

ii) the integral,

$$\int_a^b f(x)\,dx$$ be bounded on $[a,b]$ for all $b>a$.

Then $$\int_a^\infty f(x)\,\phi(x)\,dx$$ is convergent.

Example.

1) Consider, $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx.$

Since

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1,$$

so '0' is not a point of infinite discontinuity of the integrand.

Therefore $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ is convergent. — 1

Now Let us consider,

$$\int_1^\infty \frac{\sin x}{x}\,dx.$$

Let $f(x) = \sin x$, $x \geq 1$

and $g(x) = \frac{1}{x}$,

Then

i) $g$ is bounded and monotone decreasing function on $[1, \infty)$.

Moreover,

$$\lim_{x \to \infty} g(x) = 0$$

Now,

$$\left| \int_1^a f(x)\,dx \right| = |-\cos a + \cos 1| < 2$$

$\therefore$ $\int_1^a f(x)\,dx$ is bounded in $[1, a]$, $\forall a > 1$.

So by Dirichlet's test

$$\int_1^\infty \frac{\sin x}{x}\,dx \quad \text{converges.} \quad \text{—} (2)$$

By (1) & (2)

$$\int_0^\infty \frac{\sin x}{x} \quad \text{converges.}$$