ESL300

26/7/23.

* Introduction to Dynamics
  - Ian Percival & Derek Richards
* Cambridge University Press
2) Regular & ~~chaotic~~ Stochastic Motion
  - Lieberman & A J Lichtenbarg
3) Order within Chaos
  - P Berge, Y Pomean & C Vidal. James Gleick

Non linear dynamics

$$\ddot{x} + \omega^2 x = f\cos\omega t$$
$$+ \Gamma\dot{x} + ax^2$$

1) I order Dynamical System
2) II order, n^{th} order
-1 stable, unstable equillibrium
-2 Fixed Points, stability, phase flow diagram
3) Population Dynamics - logistic equation :- fixed pt, stability, phase flow dia.
4) Autonomous System
Non autonomous System - Concept of Propagators, Properties of Propagators
↓
Period Propagators

Logistic Maps - attractors, fixed points, bifurcations to chaotic state and then self organised state.

Models for features of attractors in logistic Maps, no. of attractors, value of attractors in attractors.

Feigenbaum Numbers

Fractals - Self-similar Objects

2/8/23.

$$\frac{dx^2}{dt^2} + \omega^2 x = 0 \quad \Big\} \text{ II order system, Harmonic oscillator}$$

Put $\frac{dx}{dt} = y$ $\quad \frac{dy}{dt} + \omega^2 x = 0$ - 2 coupled 1^{st} order system.

III order

n^{th} order dynamical system → n coupled I order systems

Damped oscillator

$\Gamma \frac{dx}{dt} \propto v$ $\quad \frac{d^2x}{dt^2} + \underbrace{\Gamma}_{\text{damping coeff.}} \frac{dx}{dt} + \omega^2 x = 0 \quad \Big\}$ ex of a body is moving through viscous fluid.

Applica^n of self dynamic fusion,

for large μ, ignore $\frac{d^2x}{dt^2}$ eq^n becomes $\mu\frac{dx}{dt} + \omega^2 x = 0$

Higher Order $\rightarrow$ Lower Order

→ Population Dynamics: Birth Rate, Death Rate

x - Population.

$\frac{dx}{dt} \propto b(x)\,x$

↳ birth rate

$\propto d(x)\,x$

↳ death rate

$$\frac{dx}{dt} = b(x)\,x - d(x)\,x$$

$$= cx(1-x) \rightarrow \text{Logistic eq}^n$$

→ Phase Space

1D motion — 2D

3D Motion — 6D

nD motion — $2n$D

→ 1st order autonomous System:

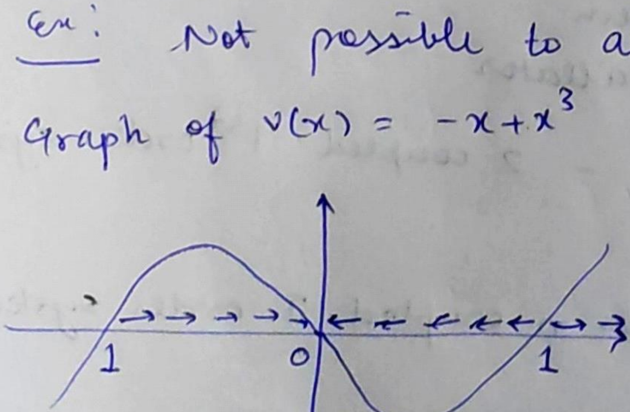
Eq^n of motion $\frac{dx}{dt} = v(x,t) = x$ $\quad v =$ vel. f^n $=$ phase vel.

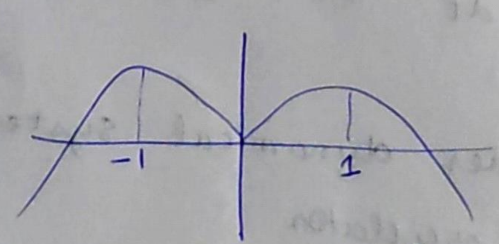
$\dot{x} = v(x)$] auto. sys.

$$\frac{dx}{dt} = v(x) \Rightarrow \int_{x_0}^{x} \frac{dx}{v(x)} = t - t_0$$

Ex: Not possible to always integrate, we'll do graphically.

Graph of $v(x) = -x + x^3$





$V_x = \frac{1}{2}x^2 - \frac{1}{4}x^4$

$v(x) = -\frac{dU}{dx}$ $\quad U =$ pot.

or

$$v(x) = v_0 - \int_0^x dx'\, v(x')$$

Phase flow, stab. of fixed pts.

$v(x) = -x + x^3$

$x_k = 0, \pm 1$

→ Phase Portrait (steps to draw)

1) Take suitable set of pts x s.t. $x = x_s$
2) For each $x = x_s$, draw an arrow whose length $\propto v(x_s)$
3) Draw an arrow near or on x-axis with centre of x & pointing in dirn of ↑ing or ↓ x depending on sign of $v(x_s)$.

For calc. max

$v(x) = -x + x^3 = 0$

$\frac{dv}{dx} = 0 \Rightarrow x^2 = 1/3 \Rightarrow x = \frac{1}{\sqrt{3}} = 0.577$

Arrow represent vel. of liq. flowing.

8/8/23.

Phase
Fixed pts of 1st order autonomous sys. & stability

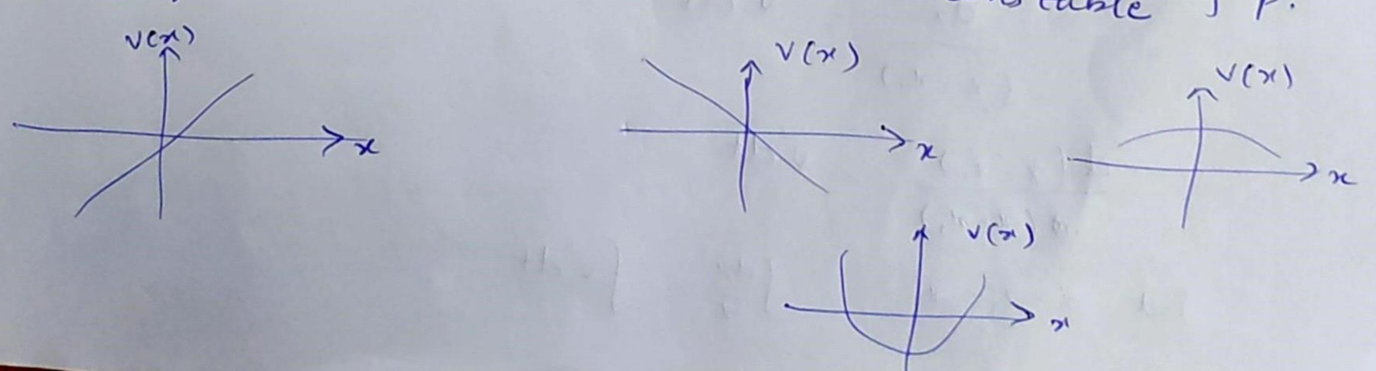
$\frac{dx}{dt} = v(x) \qquad v(x) = \text{vel-}f^n$

$v(x_k) = 0$

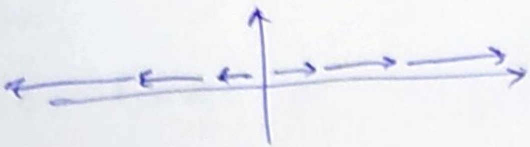
→ Stab. of fixed pt:

If there are pts $x = x_k$ around which $v(x)$ is ↓ f^n of x, so that neighbourhood states approx x_k. Then this is stable fixed pt.

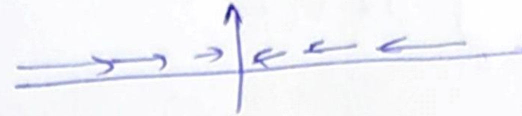
If there are pts $x = x_k$ around which $v(x)$ is ↑ f^n of x, so that neighbourhood states leaves x_k, unstable f.p.



unstable



stable

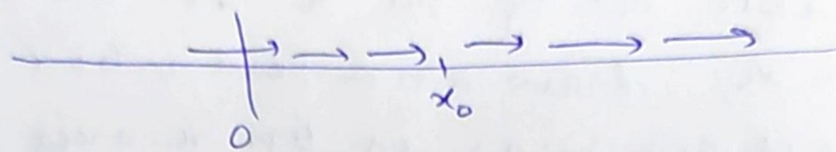


Ex 1) $v(x) = 0$ $\quad \frac{dx}{dt} = 0$ $\quad$ ~~sys~~ everywhere at rest & always

$x = x_0$

2) $v(x) = a$ $\quad a \neq 0$ $\quad x = x_0$ at $t = 0$

$x = x_0 + at$ $\quad a > 0$



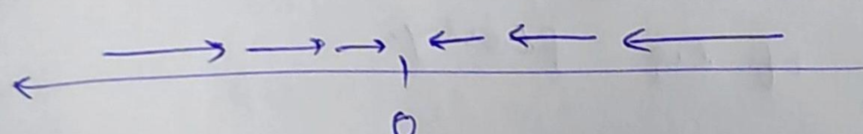
~~mag~~ mag of arrow = constant

3) $v(x) = bx$ $\quad b \neq 0$

$x = x_0 e^{bt}$

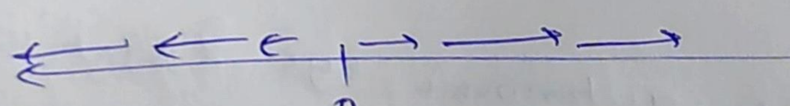
$\frac{dx}{dt} = bx$ $\qquad \int \frac{dx}{x} = \int b\,dt$

$b < 0$



stable

$b > 0$



unstable

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$$\frac{dx}{dt} = v(x)$$

$v(x) = 0$ at $x = x_1$ F.P.

$$v(x) = v(x_1) + (x - x_1)\left.\frac{\partial v}{\partial x}\right|_{x = x_1} + \ldots \text{ higher order terms}$$

$$\frac{dx}{dt} = (x - x_1)\, v' \quad ; \; v(x_1) = 0$$

$$\frac{dx}{x - x_1} = v' dt$$

$$\frac{d(x - x_1)}{x - x_1} = v' dt$$

Now, put $x - x_1 = x'$ $\Rightarrow \frac{dx'}{x'} = v' dt$

Integrating both sides, $\Rightarrow \int \frac{dx'}{x'} = \int v' dt \Rightarrow \ln x' - \ln x_0 = v' t$

[where $x_0 = x(t=0)$] $\Rightarrow \frac{x'}{x_0} = e^{v't}$

$$\frac{dx'}{dt} = x_0 e^{v't} v' \qquad \Rightarrow x' = x_0 e^{v't}$$

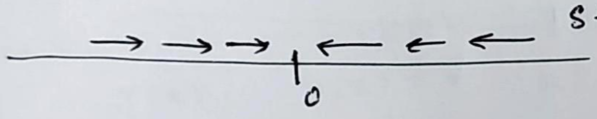
$\Rightarrow v = x' v'$

$\Rightarrow v = (x - x_1)\, v'$

$\Rightarrow v(x) = -x + x^3 \quad ; \; x_1 = 0, +1, -1$

1) $x_1 = 0$

$v' < 0$



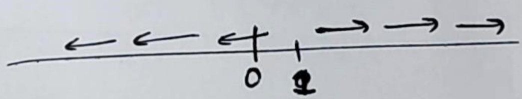
Stable

2) $x_1 = 1$

$v = (x - 1)\, v'$

$v' = -1 + 3 = 2$

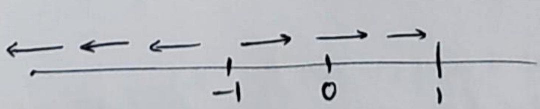
$v' > 0$



(Diverging) unstable

3) $x_1 = -1$

$v = (x + 1)\, 2$



unstable

→ Percival chap-10 ; Linear Systems

- Non-Autonomous systems

$$\frac{dx}{dt} = \dot{x} = v(x, t) = a(t)\, x + b(t)$$

$$\frac{dx}{dt} = ax + b$$ (linear system and non-auto-nomous in x, time)

→ General Solution

$$x(t) = e^{at}\left[x(t_0)\,e^{-at_0} + \int_{t_0}^{t} b(t')\,e^{-at'}\,dt'\right]$$

consider two cases : (i) $a<0$ (ii) $a>0$

$a<0$; x decreasing

$a>0$; x increasing

Put $b=0$ $\quad \dfrac{dx}{dt} = a(t)x$ ① or $\cancel{x(t) = x(t_0)\int}$ $\quad x(t) = x(t_0) \times e^{\int_{t_0}^{t} a(t)\,dt}$

Integrate

1) Fixed Point is origin

→ Propagators :

Suppose $x = x_1(t)$ is a solution of ① then $cx_1(t)$ is also a solution

Let $x_1(t_0) = 1$

$x_1(t_1) = k$

$\Rightarrow x(t_0) = c\,x_1(t_0)$ $\quad$ Similarly $\quad x(t_1) = c\,x_1(t_1) = ck$

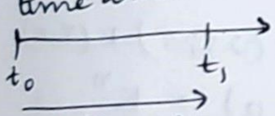
$= c$

$\therefore x(t_1) = k\,x(t_0)$

k depends on t_0 and t_1 (mapping from start to end).

General, $\quad x(t_1) = \underbrace{k(t_1, t_0)}\, x(t_0)$

↳ Propagator from t_0 to t_1



→ Properties of Propagators :

(1) $k(t_0, t_0) = 1$

Identity

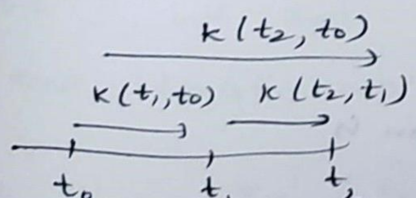
(2) Inverse : $k(t_1, t_0) \times k(t_0, t_1) = 1$

Proof : $x(t_1) = k(t_1, t_0)\,x(t_0)$

$x(t_0) = k(t_0, t_1)\,x(t_1)$

$x(t_0) = \dfrac{x(t_1)}{k(t_1, t_0)}$ $\Rightarrow k(t_1, t_0) \times k(t_0, t_1) = 1$

(3)



$k(t_2, t_0) = k(t_1, t_0)\,k(t_2, t_1)$

- multiplicative

$x_2 = x(t_2) = k(t_2, t_0)\,x(t_0)$

$x_1 = x(t_1) = k(t_1, t_0)\,x(t_0)$

$x_3 = k(t_2, t_1)\,x(t_1)$

$= k(t_2, t_1)\,x_1$

$= k(t_2, t_1)\,k(t_1, t_0)\,x_0$

$= k(t_2, t_1)\,k(t_1, t_0)\,\dfrac{x_2}{k(t_2, t_0)}$

Pg 166

11/8/23.

$$\frac{dx(t)}{dt} = a(t)\,x(t)$$

periodic $\quad a(t) = a(t+T) = a(t+nT)$

Solution at $(t+T)$ is same as $x(t)$

$$\frac{dx(t+T)}{dt} = a(t+T)\,x(t+T)$$
$$= a(t)\,x(t+T) \quad ; \ a(t) \text{ is periodic.}$$

T - constant

If $x(t)$ is a solution, $x(t+T)$ is also a solution.

By induction, $x(t+nT)$ is also a solution for $n \in I^+$, $n \in I^-$

If $x(t_1) = c\,x(t_0)$

$x(t_1 + nT) = c\,x(t_0 + nT)$

By defn of Propagators.

$x(t_1) = K(t_1, t_0)\,x(t_0)$

$x(t_1 + nT) = K(t_1 + nT, t_0 + nT)\,x(t_0 + nT)$

$x(t_1 + nT, t_0 + nT) = K(t_1, t_0)$

In particular, $t_1 = T$, $t_0 = 0$

$K((n+1)T, nT) = K(T, 0) = K$

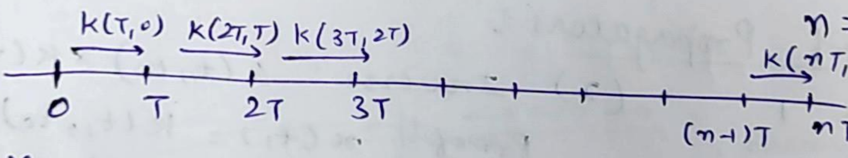
By multiplication rule,

$K((n-1)T, (n-2)T) \ldots$

$K(nT, (n-1)T) \ldots K(3T, 2T)\, K(2T, T)\, K(T, 0)$

$= K(nT, 0) = K^n$

$n = 0, \pm 1, \pm 2, \ldots$

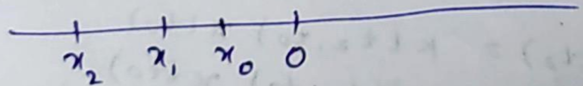


$x_n = K^n x_0$

If $|K| < 1$ $\lim_{n\to\infty} x_n = 0$

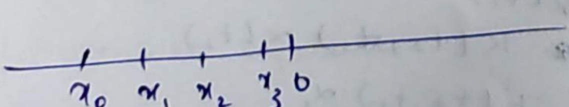
If $|K| > 1$, then $\lim x_n$ diverges.

$K > 1$



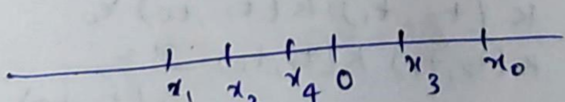
F.P. origin is unstable

$0 < K < 1$



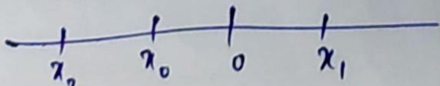
as it is approaching 0, so stable

$-1 < K < 0$



Stable but oscillatory

$K < -1$



unstable but oscillatory

(10.2, 10.3) Read ch-10, 2 Ex on Pg 167

Population Dynamics

16/8/23.

$\dot{x} = bx - cx^2$ - Logistic equation

F.P. ① $x = 0 \Rightarrow x = 0$ Ignore

② $x = b/c$

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$ } use this into map form or difference equation

$x_0 \to x_1$

$x_{n-1} \to x_n$

$x_{n+1} = F_b(x_n) \to$ continuous, differentiable & have single maximum.

x_n = population from n^{th} sample.

$x_{n+1} = F_b(x_n)$

$x_{n+1} = bx(1-x)$ - Logistic map

$$\frac{dx}{dt} = \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}$$

$$= bx(t_n) - cx^2(t_n)$$

$\Rightarrow x_{n+1} - x_n = (\Delta t_n)(bx_n - cx_n^2)$

$x_{n+1} = x_n + (\Delta t_n) x_n (b - cx_n)$

$= x_n[1 + (\Delta t_n)(b - cx_n)]$

$x_{n+1} = x_n(1 + b\Delta t_n) - c\Delta t_n x_n^2$ ①

Put $\Delta t_n = 1$, without loss of generality

$x_{n+1} = x_n(1 + b - cx_n)$

Let $1 + b = b_1$

$\Rightarrow x_{n+1} = x_n b_1 \left(1 - \frac{c}{b_1} x_n\right)$

From ①, we get $x_{n+1} = x_n(1 + b\Delta t_n)\left(1 - \left(\frac{c\Delta t_n}{1 + b\Delta t_n}\right) x_n\right)$

~~and~~ $1 + b\Delta t_n = b'$ and $x_n' = \frac{x_n(c\Delta t_n)}{1 + b\Delta t_n}$

$\Rightarrow x_{n+1} = x_n'(1 - b' x_n)$

Now, $\frac{dx}{dt} = cx - bx^2$

$\frac{x_{t+1} - x_t}{\Delta t} = cx - bx^2 \Rightarrow x_{t+1} = x_t + c\Delta t x_t - b\Delta t x_t^2$

$x_{t+1} = x_t(1 + c\Delta t) - b\Delta t x_t^2$

(x_m is constant) $\frac{x_{t+1}}{x_m} = \frac{x_t}{x_m}(1 + c\Delta t) - \frac{b\Delta t x_t^2 x_m}{x_m^2}$

$x'_{t+1} = x'_t(1+c\Delta t) - b\Delta t\, x'^2_t x_m$

$1+c\Delta t = b_1$ and $b\Delta t\, x_m = b'_1$

$\frac{1+c\Delta t}{b\Delta t\, x_m} = b_1 \Rightarrow \frac{1+c\Delta t}{b\Delta t} = b_1 x_m$

$\frac{x_m b\Delta t}{1+c\Delta t} = 1, \quad x_m = \frac{1+c\Delta t}{b\Delta t}$

$\rightarrow x_{n+1} = b x_n(1-x_n)$

1) $0 < x_n < 1$ 2) $1 < b \leq 4$ } interesting, extraordinary subtle and varied behaviour.

1) • If $x_n < 0$; decreasing population ; $x_n \rightarrow 0$

~~2)~~ • If $x_n < 1$; iterates will be positive

2) Diff. RHS w.r.t. x $bx(1-x)$ and put equal to 0.

$= b(1-x) + bx(-1) = 0$

$\Rightarrow x = 1/2$

$x_{max} = b/4$ (Max. value of iterator)

If $b > 1$, $x_n > 1$ (-ve population)

If $b < 1$, x_n (population decreases)

If $b > 1$, (population increases)

Ex: 1) $b = 2$, $x_0 = 0.2$, find $x_1, x_2, \dots$

$x_1 = 2(0.2)(1-0.2) = 0.32$

$x_2 = 2(0.32)(1-0.32) = 0.4354$

$x_3 = 2(0.4354)(1-0.4354) = 0.5$

2) $x_0 = 0.7$; $b = 2$

$x_1, x_2, \dots$ converge to 0.5

3) $x_0 = 0.6$, $b = 2$

$x_1, x_2, x_3, \dots$ converge to 0.5 (no matter what is seed value).

The system is having attractor 0.5.
(it is fixed pt. and stable)

18 | Aug | 2023.

→ Logistic Map

$x_{n+1} = b x_n(1-x_n) \quad n = 0, \pm 1, \pm 2, \dots$

$0 < x_n < 1$

$1 < b < 4$

→ for $b < 1$, population decreases and goes to zero and we don't want that case.

Q for $b = 2$, $x_0 = 0.2$. Find $x_1, x_2, \dots$

Sol: $x_1 = 2(0.2)(1-0.2)$; $x_2 = 2(0.32)(1-0.32)$

$x_1 = 0.32$ $x_2 = 0.4352$

Similarly, $x_3 \simeq 0.5$

Now, for $b = 2$

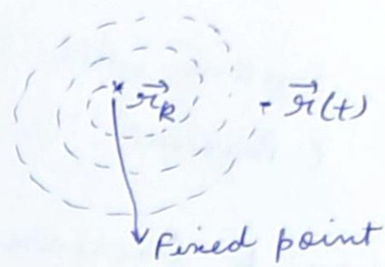
I $\quad v(\vec{r}) \quad \vec{r}$ = position vector in n-dimensional space.

$v(r_k) = 0$

$\Rightarrow r = r_k$ (F.P.)

Fixed point maybe attractor or not.

$\lim_{t \to \infty} \vec{r}(t) = \vec{r}_k$



• Fixed point ($\vec{r}_k$) is attractor if trajectory $\lim_{t \to \infty} (\vec{r}(t)) = \vec{r}_k$

II

Strongly stable F.P. but not attractor. Neighbouring trajectory converges near F.P.

F.Ps can be stable/unstable.

→ Attractor should be stable F.P.

Problem:

If b changes, no. of attractors also changes. Find upper limit of b so that system is a single attractor system. check this is 3.

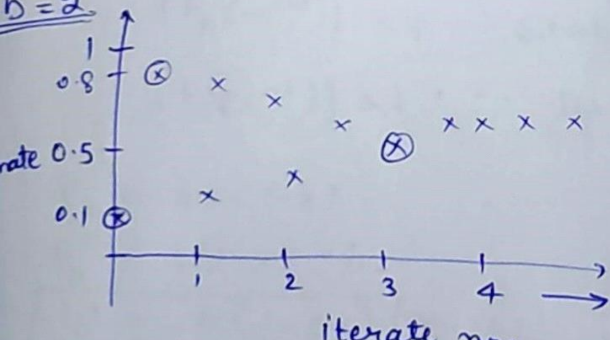
$b = 2$ single attractor

suppose $b = 3.1$. let $x_0 = 0.1$ then find attractor i.e. $bx(1-x)$.

(Home exercise)

There are two attractors of the system : 0.557, 0.765

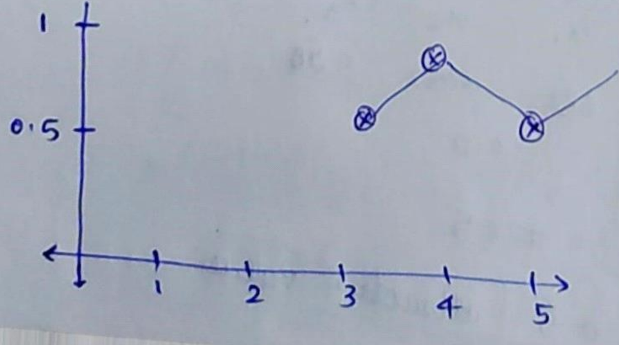
b = 2



(Period 1)
(constant)

-- converges to a single value --

b = 3.1



Period 2 attractor

2 steps

-- converges to two values --

Problem! Find upper limit of b so that system is having tw[o] attractors system. In particular, upper limit of $(b = 1+\sqrt{6}$

→ when $b > 1+\sqrt{6}$, we have 2^2 attractors
4 periods
System will return to its original value after four steps.
(Percival, Ch-11, Pg 201)

When b increases further, then we have 2^3 attractors
(8 period.)

it will continue like this till 2^n attractors.
when $b_c = 3.57$ critical value, system never returns to its initial value. System becomes chaotic.
(Chaos or Irregular State)

when $b = b_c$, no periodic behaviour, system is irregular, and sys. never settles to a periodic or regular system. This kind of transition from regular to chaotic or irregular state is a common feature of complex systems which are inherently non linear. Example!- population dynamics, finance, chemical reactions.

→ when $b > b_{cr}$ (3.57 to 4)
Surprisingly, the system will return to periodic system.
$3^1, 3^2, \ldots$

From chaotic ⟶ self organised state

$b > b_{c_2}$ → changes to $5^1, 5^2, \ldots$

19/08/23.

$x_{n+1} = b x_n (1 - x_n)$ Logistic Map

$\frac{dx}{dt} = cx - bx^2$ Logistic D.E.

$0 < x < 1$

$1 < b \leq 4$

b	x_0	x_1	x_2	x_3
2	0.2	0.32	0.432	0.50
2	0.4	$\simeq 0.5$		
2	0.6	$\simeq 0.5$		

0.5 = attractor value

$$x_{n+1} = b x_n (1 - x_n)$$

F.P. $\quad x_{n+1} = x_n$

$$b \cancel{x_n}(1-x_n) = \cancel{x_n}$$

$$b(1-x_n) = 1$$

$$1 - x_n = \frac{1}{b}$$

$$\boxed{x_n = 1 - 1/b}$$

$b = 2 \; ; \; x_n = 0.5$ F.P.

→ Stability of F.P.

~~forgoted~~ Propagator

period

$x = x_k$ f.p. $\qquad x = x^{(1)}$ F.P.

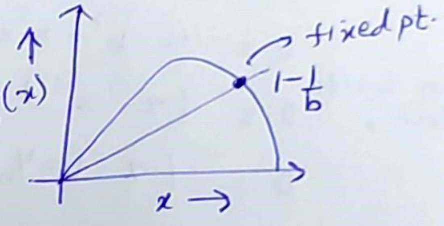
$$x_{n+1} = F_b^{(1)}(x_n)$$

$$x_{n+2} = F_b(F_b(x_n))$$

$$= F_b^{(2)}(x_n)$$

$$F_b(x) = F_b(x^{(1)}) + F_b'(x^{(1)})(x - x^{(1)}) \quad \text{(Taylor's Expansion)}$$

$$+ (x - x^{(1)})^2(--) + \dots$$



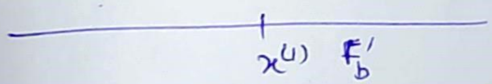
$$F_b(x) = bx(1-x)$$

$$F_b(x^{(1)}) = b\left(1 - \frac{1}{b}\right)\left(1 - 1 + \frac{1}{b}\right)$$

$$= b\left(1 - \frac{1}{b}\right)\frac{1}{b} = \left(1 - \frac{1}{b}\right)$$

$$F_b(x^{(1)}) = x^{(1)}$$

$$y = x$$



$|F_b'(x^{(1)})| < 1 \quad$ Stable Fixed Pt.

$|F_b'(x^{(1)})| > 1 \quad$ unstable

$$F_b = bx(1-x)$$

$$F_b' = b(1-x) - bx$$

$$\boxed{F_b' = b(1-2x)}$$

$$F_b'(x^{(1)}) = b\left(1 - 2\left(1 - \frac{1}{b}\right)\right)$$

$$= b\left(1 - 2 + 2/b\right)$$

$$= b\left(-1 + 2/b\right) = |2 - b| < 1$$

$$b - 2 < 1 \Rightarrow b < 3$$

$\Rightarrow \boxed{1 < b < 3}$ For Stable F.P.

Period 1 attractor

22/Aug/23. (Tuesday)

When $b = 3.1$, 2 attractor $F_b' > 1$

If $x = x^{(1)}$ is fixed point for Ist generation map, $x_{n+1} = F_b(x$

Comment on it for second generation. $\frac{1}{I}$ Gen.

$$F_b^{(2)}(x) = F_b(F_b(x))$$

At $x = x^{(1)}$ $\quad = F_b(x^{(1)})$

$$F_b^{(2)}(x^{(1)}) = x^{(1)}$$

$x = x^{(1)}$ is f.p. for second gen. Then $x = x^{(1)}$ is f.p. fo

n^{th} generation.

$\rightarrow$ $x = F_b^{(2)}(x) = F_b(F_b(x))$

$$x = b \cdot b x \cdot (1-x)(1 - bx(1-x))$$

$$1 = b^2(1-x)(1-bx(1-x))$$

$$x = 0 \quad ; \quad x = x^{(1)} = 1 - 1/b$$

$$0 = 1 - b^2(1-x) + b^3 x(1-x)^2$$

$$0 = 1 - b^2 + b^2 x + b^3 x + b^3 x^3 - 2x^2 b^3 \quad (1)$$

Also, we can write like this, $\quad 0 = (x - x^{(1)})(b^2x^2 + \beta x + \gamma)$

$$0 = (x - 1 + 1/b)(b^2x^2 + \beta x + \gamma)$$

$$0 = (bx - b + 1)(b^2x^2 + \beta x + \gamma)$$

$$0 = (1 - b + bx)(b^2x^2 + \beta x + \gamma)$$

$$0 = x^2(b^2 - b^3 + \beta b) + (\beta - \beta b + \gamma b)x + \gamma - \gamma b \quad (2)$$

comparing equations (1) and (2)

$$b^2 - b^3 + \beta b = -2b^3$$

$$\beta - \beta b + \gamma b = b^2 + b^3$$

$$\gamma - \gamma b = (1+b)(1-b)$$

$$\Rightarrow \boxed{\gamma = (1+b)}$$

$$\beta(1-b) + (1+b)b = b^2(1+b)$$

$$\beta(1-b) = b^2(1+b) - b(1+b)$$

$$\beta(1-b) = b(1+b)(b-1)$$

$$\Rightarrow \boxed{\beta = -b(1+b)}$$

This should hold in (2) also

$x^{(1)} = 1 - 1/b \qquad b - 1 - bx = 0$

$$b^2x^2 - b(1+b)x + (1+b) = 0$$

$$x = \frac{1}{2b^2}\left\{ b(1+b) \pm \left[b^2(1+b)^2 - 4(1+b)b^2\right]^{1/2} \right\}$$

$x = (1+b \pm (1+b)^{1/2}(b-3)^{1/2})/2b$.

If $b<3$, roots are imaginary. So, the system will converge to a single attractor

$b=3$, upper limit of single attractor

$b=3$, $x = 1+b/2b$

Suppose $b=3.1$, we will get two roots $\underbrace{x^{(2)} = 0.558,\ x^{(3)} = 0.764}$ FP

23/Aug/23. How to prove stability criterian?

→ modulus of I derivative of mapping function of II Generation i.e

$$\left|\frac{d}{dx} F_b^{(2)}(x)\right|\Big|_{x^{(2)}} < 1 \qquad \text{for stability}$$

$$\left|\frac{d}{dx} F_b^{(2)}(x)\right|\Big|_{x^{(3)}} < 1$$

I Gen F.P. $\begin{cases} 0 & \text{- not of interest} \\ x^{(1)} & \text{- ? stability analysis - Prove it is unstable f.p.} \\ x^{(2)} & \\ x^{(3)} & \end{cases}$

$\left(\text{show } \left|\frac{d}{dx} F_b^{(2)}(x)\right|\Big|_{x^{(1)}} > 1\right)$

- consider

$$\frac{d}{dx} F_b^{(2)}(x) = \frac{d}{dx} F_b(\underbrace{F_b(x)}_{\to y})$$

$$= F_b'(y)\frac{dy}{dx}$$

$$= F_b'(y)\, y'$$

$$= F_b'(F_b(x))\, F_b'(x)\Big|_{x=x^{(1)}}$$

$$= F_b'(F_b(x^{(1)}))\, F_b'(x^{(1)})$$

$$= \left(F_b'(x^{(1)})\right)^2$$

for $b>3$, $x^{(1)}$ is u.f.p. and not 2-attractor.

$$\frac{d}{dx} F_b^{(2)}(x)\Big|_{x^{(2)}} = ?$$

$F_b(x)\Big|_{x^{(2)}} = x^{(3)} \qquad F_b(x)\Big|_{x^{(3)}} = x^{(2)}$?

$$b^2x^2 - b(1+b)x + 1+b = 0$$

$$x^{(2)} + x^{(3)} = (b+1)/b$$

$$x^{(2)}x^{(3)} = 1+b/b^2$$

$$\Rightarrow \frac{d}{dx} F_b(F_b(x))\Big|_{x=x^{(2)}}$$

$$\frac{d}{dx} F_b(F_b(x))\Big|_{x=x^{(2)}}$$

$$= F_b'(F_b(x))\Big|_{x=x^{(2)}}$$

$$= F_b'(x^{(3)})\, F_b'(x^{(2)})$$

Similarly, $\frac{d}{dx} F_b^{(2)}(x)\Big|_{x=x^{(3)}}$

$$\frac{d}{dx} F_b(F_b(x^{(3)}))\Big|_{x=x^{(3)}}$$

$$= \frac{d}{dx} F_b'(F_b(x))\Big|_{x=x^{(3)}}$$

$$= F_b'(F_b(x^{(3)}))\, F_b'(x^{(3)})$$

So, Stability criterion for $x^{(2)}$ & $x^{(3)}$ are same.

$$= F_b'(x^{(2)})\, F_b'(x^{(3)})$$

As $F_b'(x) = b(1-2x)$

$\Rightarrow F_b'(x^{(3)}) = b(1-2x^{(3)})$ and $F_b'(x^{(2)}) = b(1-2x^{(2)})$

$$\Rightarrow F_b'(x^{(3)})\, F_b'(x^{(2)}) = b^2\left(1 - 2(x^{(2)} + x^{(3)}) + 4x^{(2)}x^{(3)}\right)$$

$$= b^2\left(1 - 2\left(\frac{1+b}{b}\right) + 4\left(\frac{1+b}{b^2}\right)\right)$$

$$= b^2\left(1 - 2/b - 2 + 4/b^2 + 4/b\right)$$

$$= b^2\left(-1 + 2/b + 4/b^2\right)$$

$$= -b^2 + 2b + 4$$

$$= -(b-1)^2 + 5$$

For stability, $|5-(b-1)^2| < 1$

$$(b-1)^2 - 5 < 1$$
$$(b-1)^2 < 6$$
$$b-1 < \pm\sqrt{6}$$

$$5-(b-1)^2 < 1$$
$$4 < (b-1)^2$$
$$\pm 2 < (b-1)$$
$$b > 3$$
$$b > 1$$

-ve values of b are not allowed.

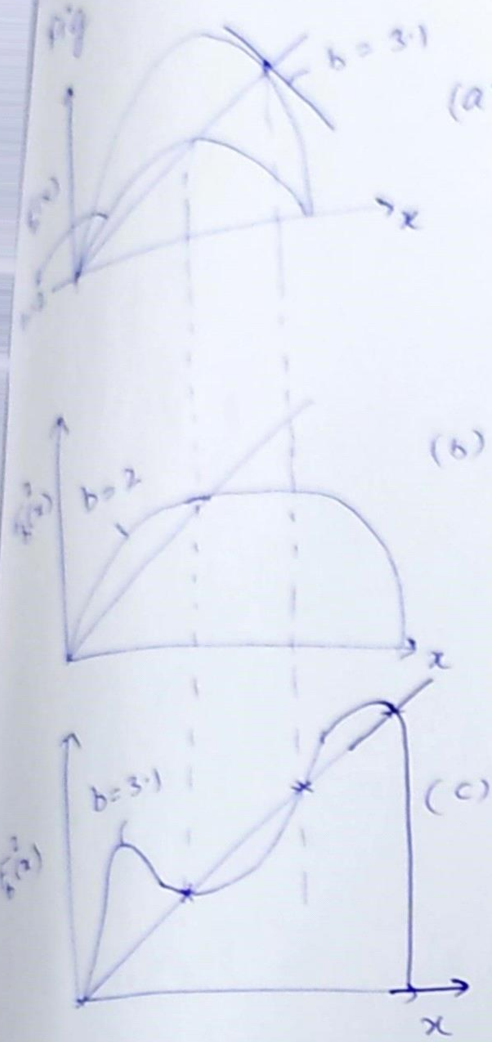
$\boxed{3 < b < 1+\sqrt{6}}$ (stability criterion for two attractors)

Q Find upper limit of b such that it is 4 attractor system or cycle-4.

Ch-11 Percival (203) (204, 205)

26/8/23.

Fig



Observations:-

1) Value of b for which F_b becomes unstable at $x = x^{(1)}$ is a value for which $F_b^{(2)}$ has unit slope at $x = x^{(1)}$

$$\frac{d}{dx} F_b^{(2)}(x)\Big|_{x = x^{(1)}} = F_b'(x^{(1)})^2$$

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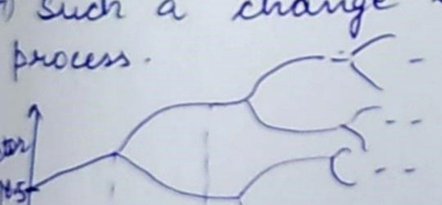
4) The two new solutions $x^{(2)}$ & $x^{(3)}$ are fixed points of $F_b^{(2)}(x)$ but not of $F_b(x)$. They belong to cycle of period 2^1.

Ex: For given b, there is only 1 period 2 cycle ($x^{(2)}$ & $x^{(3)}$ are unique).

5) If slope of ~~$x^{(2)}$~~ $F_b^{(2)}(x)$ at $x^{(2)}$ & $x^{(3)}$ are <1, then 2 cycle system is stable cycle.

6) At the value of b where fixed point $x^{(1)}$ becomes unstable, a stable cycle of period 2^1 is generated.

7) Such a change in nature of stable motion is bifurcation process.



Pitchfork Bifurcation

If value of b is further increased beyond $1+\sqrt{6}$, the slope of $F_b^{(2)}(x)$ at $x^{(2)}$ & $x^{(3)} > 1$, $x^{(2)}$ & $x^{(3)}$ will become unstable and at a location of $x^{(2)}$ & $x^{(3)}$ a new cycle of 2^2 will be generated which will remain stable a certain value.

If we continue this process, we can generate 2^n cycle, where n is +ve integer, n can become so large that system will never return to itself and hence, the system is chaotic.

$b_c = 3.57$

1/9/22.

Fractal crystals, Fractal Dimensions

Ex: Lorentz attractors, trees, clouds, coastline.
↓
Phase space

Fractal set

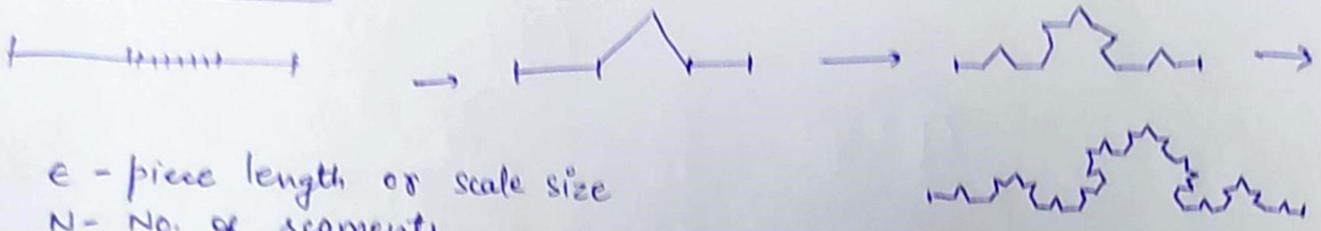
1. Cantor set

Take a st. line. Divide into 3 parts. Remove middle part. .... Repeat.

If this kind of set is examined under high magnification, its structure is indistinguishable from unmagnified version. This property of invariance under a change of scale.
- self similarity property

This is common feature of fractal objects.

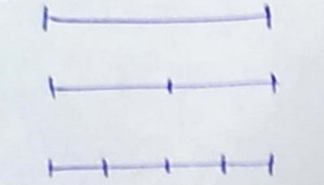
→ Koch curve :-



ϵ - piece length or scale size
N - No. of segments
d - dimension

$N(\epsilon) \propto\sim k\epsilon^{-d}$

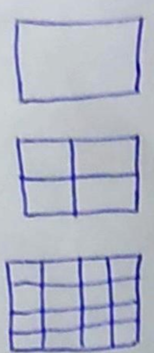
①



N	ϵ
1	L
2	L/2
4 (2^2)	L/4 / $L/2^2$
2^n	$L/2^n$

$N(\epsilon) = L/\epsilon$

②



N	ϵ
1	L
4	L/2
$4^2 = 16$	L/4
⋮	
.... 4^n	$L/2^n$

$$N(\epsilon) = (L/\epsilon)^2$$

Generalise

$$N(\epsilon) = (L/\epsilon)^d$$

$$\log N(\epsilon) = d(\log L - \log \epsilon)$$

$$d = \frac{\log N(\epsilon)}{\log L + \log(1/\epsilon)}$$

Capacity dimension $\lim_{\epsilon \to 0}$ $= \dfrac{\log N(\epsilon)}{\log(1/\epsilon)} = d_c$

I $\quad d_c = \dfrac{\log 2^n}{\log(1/2^n)} = 1$

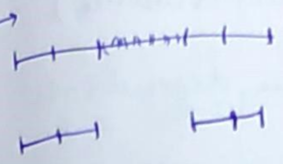
II $\quad d_c = \dfrac{\log 4^n}{\log(1/1/2^n)} = \dfrac{\log 2^{2n}}{\log 2^n} = 2$

→

N	ϵ
1	1
2	$1/3$
2^2	$1/3^2$
⋮	
2^n	$1/3^n$

$$d_c = \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3}$$

→



N	ϵ
1	1
2	$2/5$

$$d_c = \frac{\log 2}{\log 5/2}$$

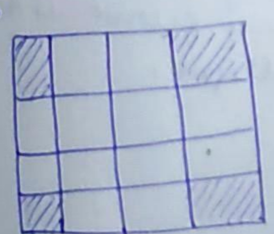
Koch curve - $d_c = \dfrac{2\log 2}{\log 3}$

→

N	ϵ
1	1
10	$1/5$
10^2	$1/5^2$

$$d_c = \frac{2\log 10}{2\log 5}$$

→



N	ϵ
1	1
12	$1/4$
12^2	$1/4^2$
⋮	⋮

$$d_c = \frac{\log 12^n}{\log 4^n} = \frac{\log 12}{\log 4} = 1.79$$

5/9/23.

$\dfrac{dx}{dt} = v(x,t)$

II order systems $\quad \ddot{x} + \omega^2 \sin x = f$

Fixed points : Parabolic
Hyperbolic
Stars
Spiral
Elliptic , Circular etc.

→ Eigen values of characteristic Matrix
→ Transformation of Matrices.
→ Anharmonic Oscillator
→ Pendulum with varying length (periodic / random)
→ Hamiltonian System (Generalized concept of energy)
  • H eq^{n}s
→ Lagrangian formulation
→ Variational Principle
  • Euler-Lagrange eq^{n}s.
⇒ Non linear Schrodinger wave eq^n - Signal Propagation
⇒ Protein Folding Problem.

⇒ **II order Dyn. Sys.**

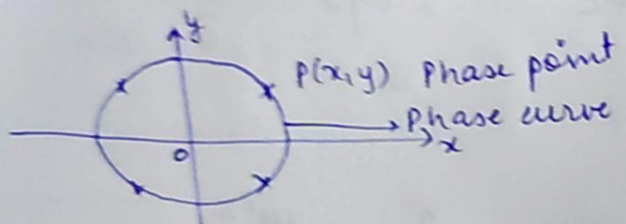
$$\frac{d\bar{x}}{dt} = \dot{\bar{x}} = \bar{v}(\bar{x}) \equiv \text{Autonomous (Implicit time dependance)}$$

$$\bar{v}(\bar{x}, t) \equiv \text{Non-Autonomous (Explicit time dependance)}$$

$$\bar{x} = (x(t), y(t))$$

$$\frac{dx}{dt} = v_x(x, y) \quad ; \quad \frac{dy}{dt} = v_y(x, y)$$

$$\vec{x} = \hat{x}\, x(t) + \hat{y}\, y(t)$$



$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \rightarrow \text{second order sys.}$$

$$\underbrace{\frac{dx}{dt} = y(t) \Rightarrow \frac{dy}{dt} = -\omega^2 x}_{\boxed{\text{coupled 1st order sys}}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\omega^2 x}{y} \rightarrow \text{Integrate it.}$$

Phase curve is an ellipse.

force,

$$F(x) = m\frac{d^2x}{dt^2} = m\ddot{x}$$

if $\dot{x} = y$, then $\dot{y} = F(x)/m$

$$\bar{x} = (x, y) \rightarrow \text{Phase pt.}$$

$$\left.\begin{aligned} \dot{\bar{x}} &= (\dot{x}, \dot{y}) \\ &= (y, F/m) \end{aligned}\right\} \text{Phase velocity}$$

→ Method to obtain Phase Portrait:

① Take suitable set of pts in phase plane & through each pt. P (take arbitrary x, y and locate P). draw a short arrow with centre at P to denote vel vector.

② Length of arrow ∝ magnitude of $\bar{v}$ at $P(x, y)$. Dir^n is $\tan^{-1}\frac{dy}{dx}$

Ex :- Particle falling under Gravity

$$\frac{dx}{dt} = v_x(x,y) = \lambda_1 x = v_x(x)$$
$$\frac{dy}{dt} = v_y(x,y) = \lambda_2 x = v_y(y)$$

if λ_1 and λ_2 are constants then they are separable.
o/w λ_1 & λ_2 are f^n of x & y.

→ $\ddot{x} + \alpha(\dot{x})^2 + \omega^2 x + \beta x^2 + \gamma x^4 = 0$ [anharmonic oscillator with damping.

$$\frac{dx}{dt} = y = v_x(y)$$
$$\frac{dy}{dt} = -\alpha y^2 - \omega^2 x - \beta x^2 - \gamma x^4 = v_y(x,y)$$

f.p. : $v_x = 0$, $v_y = 0$

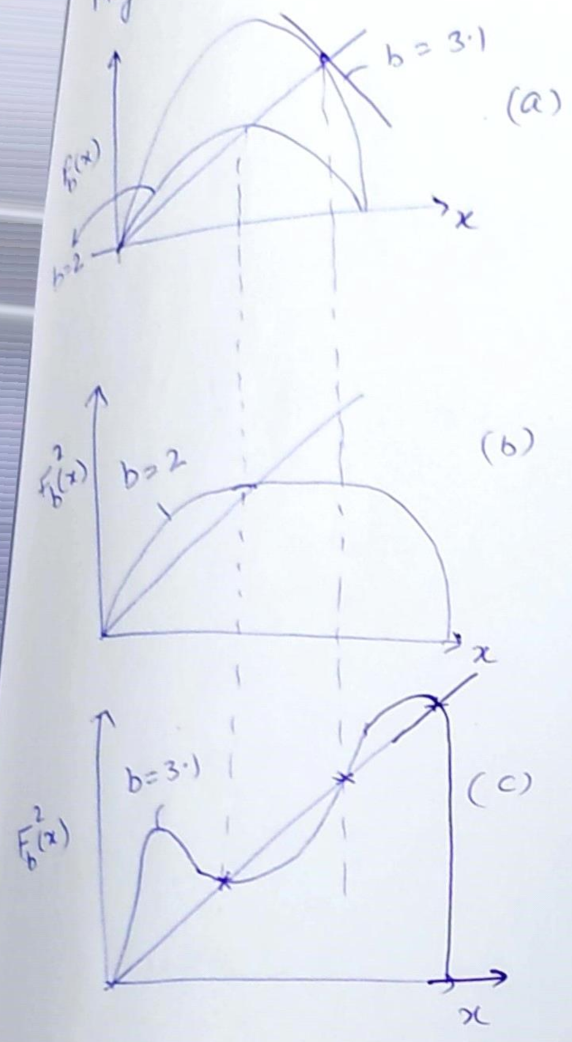
If $x = a$ is f.p. of one I order sys.,
$x = b$ is f.p of second I order sys.,
$\bar{r} = (a, b)$ is f.p. of combined II order sys. consisting of 2 coupled I order systems.

for I order sys, we know $(0,0)$ is fixed pt. & $\frac{dy}{dx} = \frac{\lambda_2 y}{\lambda_1 x}$

$$\Rightarrow \left(\frac{x}{x_0}\right)^{\lambda_2} = \left(\frac{y}{y_0}\right)^{\lambda_1}$$

25/8/23

Fig



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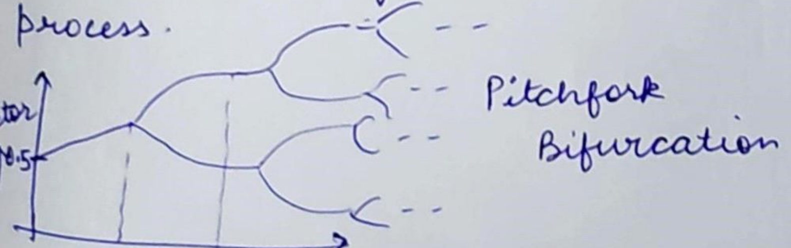

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