Def^n: Let $(\Omega, \mathcal{F}, P)$ be a Probability space. A vector $X = (X_1, X_2, \dots, X_n)$

$X : \Omega \to \mathbb{R}^n$ defined as

$$\underline{X(\omega)} = (\underline{X_1(\omega)}, X_2\underline{(\omega)}, \dots X_n\underline{(\omega)})$$

is called n-dimensional random vector iff

$$\{X_1 \le x_1, \dots X_n \le x_n\} \in \mathcal{F} \quad \forall\ x_1, \dots x_n \in \mathbb{R}$$

$$\|$$

$$\{\omega \mid X_1(\omega) \le x_1, \dots, X_n(\omega) \le x_n\} \in \mathcal{F}$$

$$\Updownarrow$$

$$\{\omega \mid (X_1(\omega), \dots X_n(\omega)) \in B\} \quad \forall\ B \in \mathcal{B}(\mathbb{R}^n)$$

Theorem: Let $(\Omega, \mathcal{F}, P)$ be a Prob. space. $X_1, \dots X_n$ are Random variables on $(\Omega, \mathcal{F}, P)$ iff $(\underline{X_1, X_2, \dots X_n})$

is a random vector.

<u>Proof</u>: Let $X_1, \ldots X_n$ be n random variables

$$\{X_1 \le x_1, \ldots, X_n \le x_n\} = \underline{\bigcap_{i=1}^{n} \{X_i \le x_i\} \in \mathcal{F}}$$

$\because \{X_i \le x_i\} \in \mathcal{F} \;\; \forall i = 1, \ldots n$

Let $(X_1, X_2, \ldots X_n)$ be an <u>RV.</u>

$$\underline{\{X_i \le x_i\}} = \{\omega \mid X_i(\omega) \le x_i, \; X_1(\omega) \in \mathbb{R} \; \ldots, \; X_n(\omega) \in \mathbb{R}\}$$

$$= \{\omega \mid (X_1(\omega), \ldots X_i(\omega), \ldots X_n(\omega)) \in \underbrace{\mathbb{R} \times \ldots (-\infty, x_i] \times \ldots \mathbb{R}}_{\in \mathcal{B}(\mathbb{R}^n)}\}$$

$$\in \mathcal{F}$$

$\because \mathbb{R} \times \ldots \times (-\infty, x_i] \times \ldots \times \mathbb{R}$ is a Borel set of $\mathbb{R}^n$.

For notational simplicity, we focus on two dimensional RV

Def^n: A two dimensional RV (X,Y) is said to be a discrete RV if there exists a countable set

$$E = \{(x_i, y_j) \mid i = 1, \dots \infty,\ j = 1, \dots \infty\}$$

such that $P\{(X,Y) \in E\} = 1$

Define $p_{ij} = P\{X = x_i, Y = y_j\}$

$$p_{ij} \geq 0, \quad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\{X = x_i, Y = y_j\}$$

$$= P\left(\bigcup_{i=1, j=1}^{\infty\ \infty} \{\underline{X = x_i, Y = y_j}\}\right)$$

$$= P\{(X,Y) \in \underline{E}\}$$

$$= 1$$

$$p_{ij} \geq 0, \quad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{ij} = 1$$

$\infty \ \infty$

Collection $\{p_{ij}\}_{i=1, j=1}$ is called joint PMF of $\underline{(X, Y)}$.

Joint CDF of (X, Y) is defined by

$$F_X(x, y) = P\{X \le x, Y \le y\}$$
$$= \sum_{x_i \le x, y_j \le y} P\{X = x_i, Y = y_j\}$$

$$F_X(x, y) = \sum_{\{(i,j) \mid x_i \le x, y_j \le y\}} p_{ij}$$

__Example__: A fair die is rolled and a fair coin is tossed independently

$\Omega = \{(1,T) \ldots (6,T), (1,H), \ldots (6,H)\}$

$X =$ face value of die

$Y = 0$ if tail comes
$= 1$ if head comes.

$$(X, Y) \in \{(1,0), (2,0), \ldots (6,0), (1,1), \ldots (6,1)\}$$

PMF

$$p_{ij} = 1/12 \quad , \quad i = 1, \dots 6, \; j = 0, 1$$

$$p_{10} = P\{X=1, Y=0\} = P\{\underline{(1,T)}\} = 1/12$$

CDF

$$F(x,y) = P\{X \le x, Y \le y\} = \begin{cases} 0 & x<1, -\infty<y<\infty \;;\; y<0, -\infty<x<\infty \\ 1/12 & 1 \le x < 2, \; 0 \le y < 1 \\ 1/6 & 1 \le x < 2, \; y \ge 1 \\ 1/6 & 2 \le x < 3, \; 0 \le y < 1 \\ 1/3 & 2 \le x < 3, \; y \ge 1 \\ \vdots & \vdots \text{ Complete it.} \\ 1 & x \ge 6, \; y \ge 1 \end{cases}$$

Defn : An RV (X,Y) is said to be a continuous RV if there exists a non-negative function $f(x,y)$ for all $x, y \in \mathbb{R}$ such that

$$P\{(X,Y) \in C\} = \iint_C f(x,y)\,dx\,dy$$

for all $C \in \mathcal{B}(\mathbb{R}^2)$

The function $f(x,y)$ is called as joint PDF of (X,Y).

<u>Def^n</u>: A function $f(x,y)$ is said to be joint PDF of (X,Y)

if $f(x,y) \geq 0 \quad \forall\ x,y$

& $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

If $C = \{(x,y) : x \in A, y \in B\}$

$$P\{(X,Y) \in C\} = P\{X \in A, Y \in B\}$$

$$= \iint_{A\,B} f(x,y)\, dy\, dx$$

CDF :-

$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$$
$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dy\, dx$$

Given we have information about joint PMF or joint PDF of (X,Y). How do we find the PMF or PDF of X and Y individually.?

Marginal PMF or PDF :-

Let $(\Omega, \mathcal{F}, P)$ be a Prob. space.

(X,Y) is a discrete random vector with joint PMF $\{p_{ij}\}_{i=1, j=1}^{\infty\ \infty}$

what is PMF of X ? $\left| p_{ij} = P\{X = x_i, Y = y_j\}\right.$

what is PMF of X ?

$p_{ij} = P\{X=x_i, Y=y_j\}$

$$\{X = x_i\} = \{X=x_i\} \cap \Omega$$

$$= \underline{\{X=x_i\}} \cap \bigcup_{j=1}^{\infty} \underline{\{Y=y_j\}}$$

$$\{Y=y_j\} = \{\omega \mid Y(\omega)=y_j\}$$

$$Y : \Omega \to \{y_j\}_{j=1}^{\infty}$$

$$\bigcup_{j=1}^{\infty} \{Y=y_j\} = \Omega$$

$$\{X = x_i\} = \bigcup_{j=1}^{\infty} \{X=x_i, Y=y_j\}$$

$$P\{X=x_i\} = \sum_{j=1}^{\infty} P\{X=x_i, Y=y_j\} = \sum_{j=1}^{\infty} p_{ij}$$

$Y(1,H) = 1$

$Y(2,H) = 1$

$Y(6,H) = 1$

$\{Y=1\}$

$= 3H$

$$\Omega = \{(1,H), \ldots (6,H), (1,T) \ldots (6,T)\}$$

$X(1,H) = 1, \ldots X(6,H) = 6$

$Y(1,H) = 1 \quad Y(1,T) = 0$

$(X,Y) : \Omega \to A$

$(X,Y)(1,H) = (1,1)$

$(X(1,H), Y(1,H)) = (1,1)$

$$P\{X=x_i\} = \sum_{j=1}^{\infty} p_{ij}$$

$$\| \\ p_{i\cdot}$$

$$p_{i\cdot} \geq 0 \quad \forall i \qquad \sum_{i=1}^{\infty} p_{i\cdot} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_{ij} = 1$$

$\{p_{i\cdot}\}_{i=1}^{\infty}$, $\quad p_{i\cdot} = \sum_{j=1}^{\infty} p_{ij}$ is PMF of X.

Marginal PMF of X.