Discrete-Time Fourier Series

Lecture 29

DTFS (represented by N complex exponentials)

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_o n}$$

DTFS by Matrix

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} e^{\frac{j2\pi}{N}0.0} & e^{\frac{j2\pi}{N}1.0} & e^{\frac{j2\pi}{N}2.0} & e^{\frac{j2\pi}{N}3.0} \\ e^{\frac{j2\pi}{N}0.1} & e^{\frac{j2\pi}{N}1.1} & e^{\frac{j2\pi}{N}2.1} & e^{\frac{j2\pi}{N}3.1} \\ e^{\frac{j2\pi}{N}0.2} & e^{\frac{j2\pi}{N}1.2} & e^{\frac{j2\pi}{N}2.2} & e^{\frac{j2\pi}{N}3.2} \\ e^{\frac{j2\pi}{N}0.3} & e^{\frac{j2\pi}{N}1.3} & e^{\frac{j2\pi}{N}2.3} & e^{\frac{j2\pi}{N}3.3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

DTFS by Matrix

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Matrices are inverse of each other

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Number of operations increases as N^2

Fast Fourier "Transform"

Divide FS of length 2N into two of length N (divide and conquer)

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^1 & W_4{}^2 & W_4{}^3 \\ W_4{}^0 & W_4{}^2 & W_4{}^0 & W_4{}^2 \\ W_4{}^0 & W_4{}^3 & W_4{}^2 & W_4{}^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

where $W_4^m = e^{-j\frac{2\pi}{4}m}$

Number of Multiplications = 16
Number of Additions = 4×3=12
Total=28

Fast Fourier Transform

Divide it into two 2-point series (divide and conquer)

Even-numbered entries in x[n]:

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} W_2{}^0 & W_2{}^0 \\ W_2{}^0 & W_2{}^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix}$$

Odd-numbered entries in x[n]:

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} W_2{}^0 & W_2{}^0 \\ W_2{}^0 & W_2{}^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix}$$

Number of Multiplications = 2×4

Number of Additions = 2×2=4

Total=12

Fast Fourier Transform

Break the original 4-point DTFS coefficients a_k into two parts:

$a_k=d_k+e_k$

where d_k comes from the even-numbered x[n] and e_k comes form the odd-numbered x[n]

Fast Fourier Transform

The 2-point DTFS coefficients b_k of the even-numbered x[n]

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} {W_2}^0 & {W_2}^0 \\ {W_2}^0 & {W_2}^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix} = \begin{bmatrix} {W_4}^0 & {W_4}^0 \\ {W_4}^0 & {W_4}^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix}$$

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} {W_4}^0 & {W_4}^0 & {W_4}^0 & {W_4}^0 \\ {W_4}^0 & {W_4}^1 & {W_4}^2 & {W_4}^3 \\ {W_4}^0 & {W_4}^2 & {W_4}^0 & {W_4}^2 \\ {W_4}^0 & {W_4}^3 & {W_4}^2 & {W_4}^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Fast Fourier Transform

The 2-point DTFS coefficients b_k of the even-numbered x[n]

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} W_2{}^0 & W_2{}^0 \\ W_2{}^0 & W_2{}^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix}$$

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} W_4{}^0 & & W_4{}^0 & \\ W_4{}^0 & & W_4{}^2 & \\ W_4{}^0 & & W_4{}^0 & \\ W_4{}^0 & & W_4{}^2 & \end{bmatrix} \begin{bmatrix} x[0] \\ \\ x[2] \\ \\ \end{bmatrix}$$

Fast Fourier Transform

The 2-point DTFS coefficients b_k of the even-numbered x[n]

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} W_2{}^0 & W_2{}^0 \\ W_2{}^0 & W_2{}^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix}$$

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \\ W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix}$$

Fast Fourier Transform

The 2-point DTFS coefficients b_k of the even-numbered x[n]

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} W_2{}^0 & W_2{}^0 \\ W_2{}^0 & W_2{}^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix}$$

$$\boxed{\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_0 \\ b_1 \end{bmatrix}} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \\ W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \end{bmatrix}$$

Fast Fourier Transform

The 2-point DTFS coefficients c_k of the odd-numbered x[n]

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} W_2{}^0 & W_2{}^0 \\ W_2{}^0 & W_2{}^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix}$$

$$\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^1 & W_4{}^2 & W_4{}^3 \\ W_4{}^0 & W_4{}^2 & W_4{}^0 & W_4{}^2 \\ W_4{}^0 & W_4{}^3 & W_4{}^2 & W_4{}^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Fast Fourier Transform

The 2-point DTFS coefficients c_k of the odd-numbered x[n]

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} W_2{}^0 & W_2{}^0 \\ W_2{}^0 & W_2{}^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix}$$

$$\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^1 & W_4{}^3 \\ W_4{}^2 & W_4{}^2 \\ W_4{}^3 & W_4{}^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix}$$

Fast Fourier Transform

The 2-point DTFS coefficients c_k of the odd-numbered x[n]

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} W_2{}^0 & W_2{}^0 \\ W_2{}^0 & W_2{}^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix}$$

$$\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} W_4{}^0 c_0 \\ W_4{}^1 c_1 \\ W_4{}^2 c_0 \\ W_4{}^3 c_1 \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^1 & W_4{}^3 \\ W_4{}^2 & W_4{}^2 \\ W_4{}^3 & W_4{}^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix}$$

Fast Fourier Transform

The 2-point DTFS coefficients c_k of the odd-numbered x[n]

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} W_2^{\ 0} & W_2^{\ 0} \\ W_2^{\ 0} & W_2^{\ 1} \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix} = \begin{bmatrix} W_4^{\ 0} & W_4^{\ 0} \\ W_4^{\ 0} & W_4^{\ 2} \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix}$$

$$\boxed{\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} W_4^{\ 0} c_0 \\ W_4^{\ 1} c_1 \\ W_4^{\ 2} c_0 \\ W_4^{\ 3} c_1 \end{bmatrix}} = \begin{bmatrix} W_4^{\ 0} & W_4^{\ 0} \\ W_4^{\ 1} & W_4^{\ 3} \\ W_4^{\ 2} & W_4^{\ 2} \\ W_4^{\ 3} & W_4^{\ 1} \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \end{bmatrix}$$

Fast Fourier Transform

Combine d_k and e_k to get a_k

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} d_0 + e_0 \\ d_1 + e_1 \\ d_2 + e_2 \\ d_3 + e_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_0 \\ b_1 \end{bmatrix} + \begin{bmatrix} W_4{}^0 c_0 \\ W_4{}^1 c_1 \\ W_4{}^2 c_0 \\ W_4{}^3 c_1 \end{bmatrix}$$

FFT procedure:

- Compute b_k and c_k: 2×(2×2) = 8 multiples
- Combine $a_k = b_k + W_4{}^k c_k$: 4 multiples
- Total 12 multiples: fewer than the original 16 multiples
- Total 8 additions: fewer than the original 12 additions

Scaling of the FFT algorithm

Let *M(n)* be the number of multiples to compute n-point FFT

$$M(1) = 0$$

$$M(2) = 2M(1) + 2 = 2$$

$$M(4) = 2M(2) + 4 = 8$$

$$M(8) = 2M(4) + 8 = 24$$

$$M(N) = N \log_2 N$$

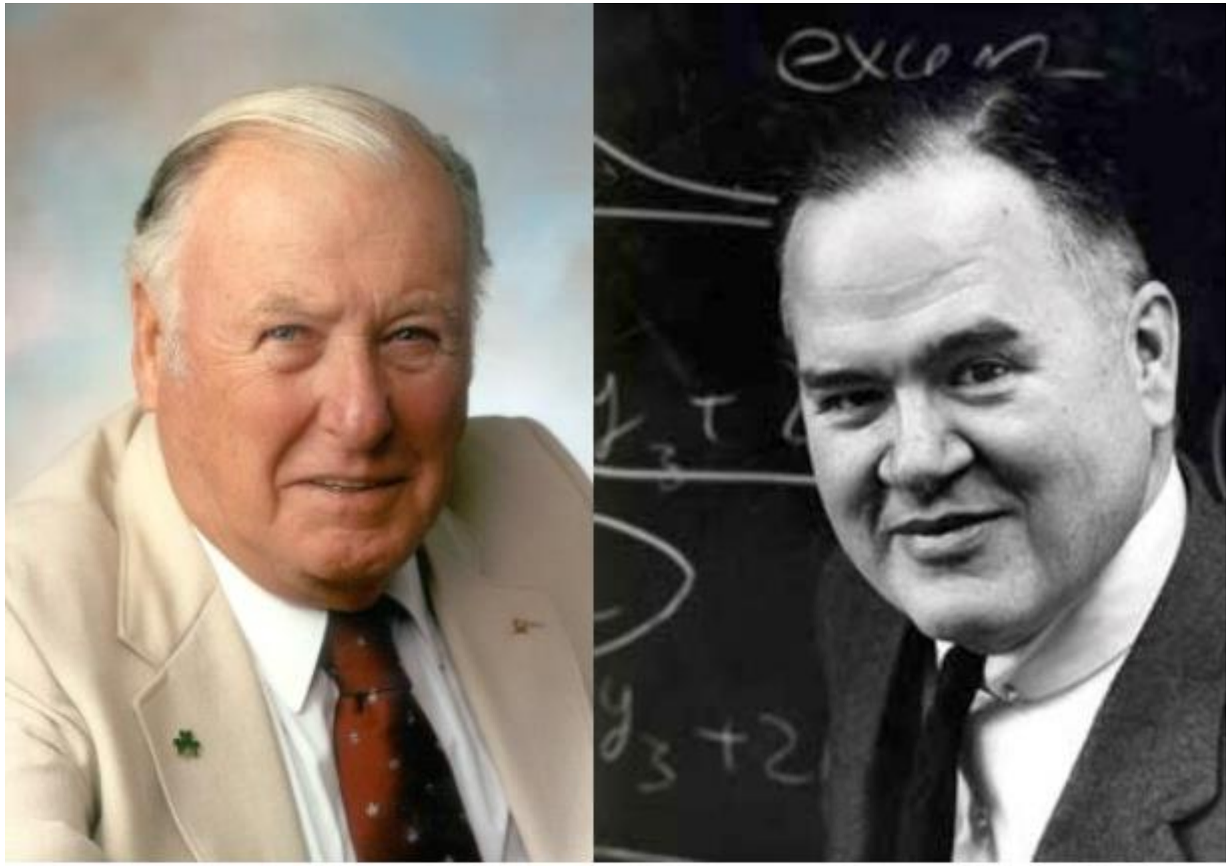
N = 1024

Original = 1048576

FFT = 10240

ONE MILLION OPERATIONS REDUCES TO TEN THOUSAND

Cooley and Tukey

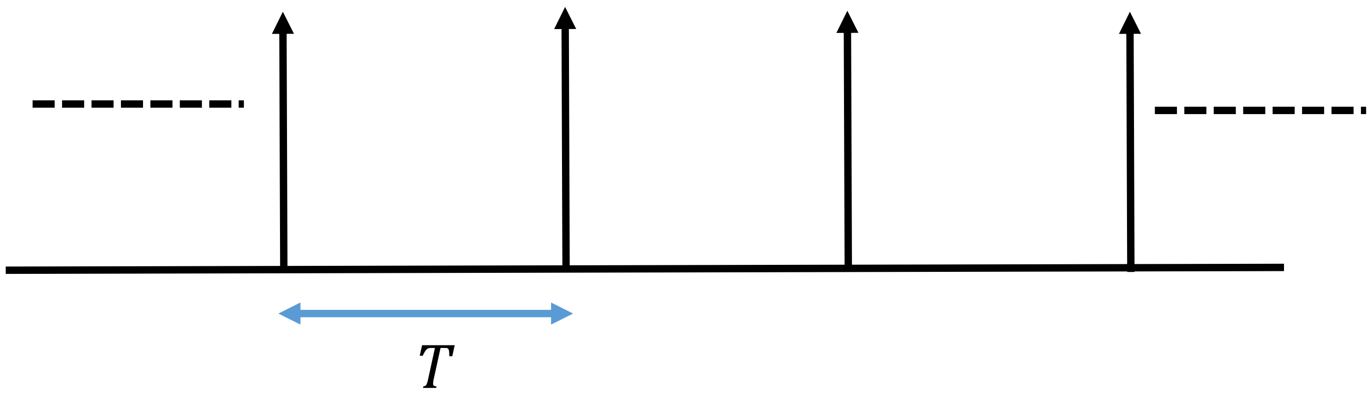


James William Cooley (1926-) John Wilder Tukey (1915-2000)

- James Cooley and John Tukey were research scientists at IBM.
- They published their work in 1965 (read it on Google).
- He developed fast Fourier transforms to understand the data from sensors which were planted to detect nuclear explosions.

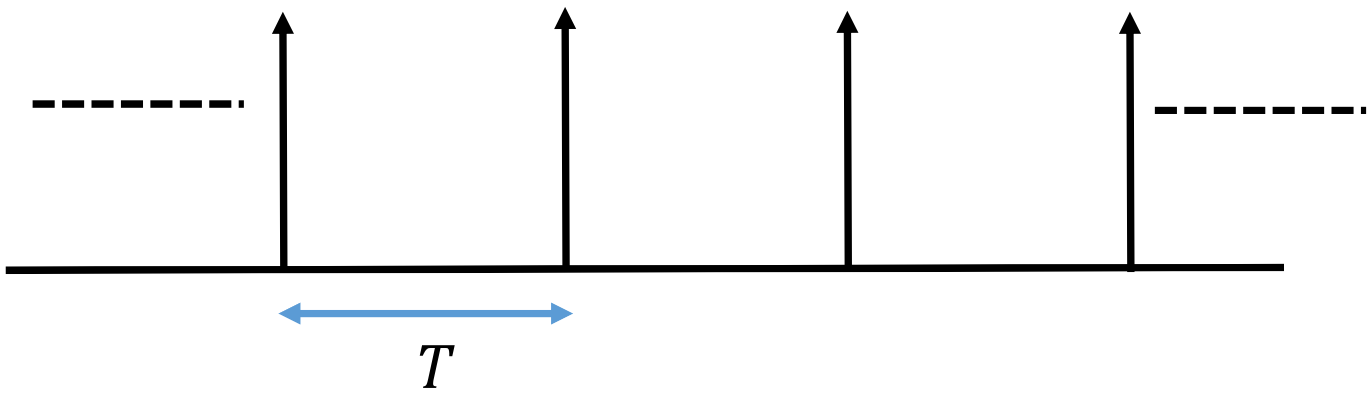
Picket Fence

$$\sum_{k=-\infty}^{\infty} \delta(t - kT)$$

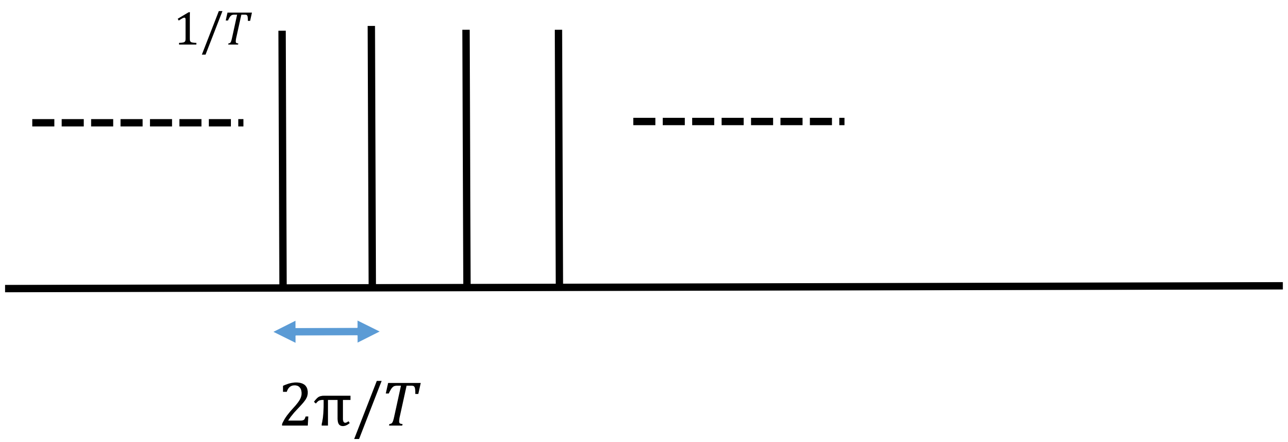


Picket Fence

$$\sum_{k=-\infty}^{\infty} \delta(t-kT)$$

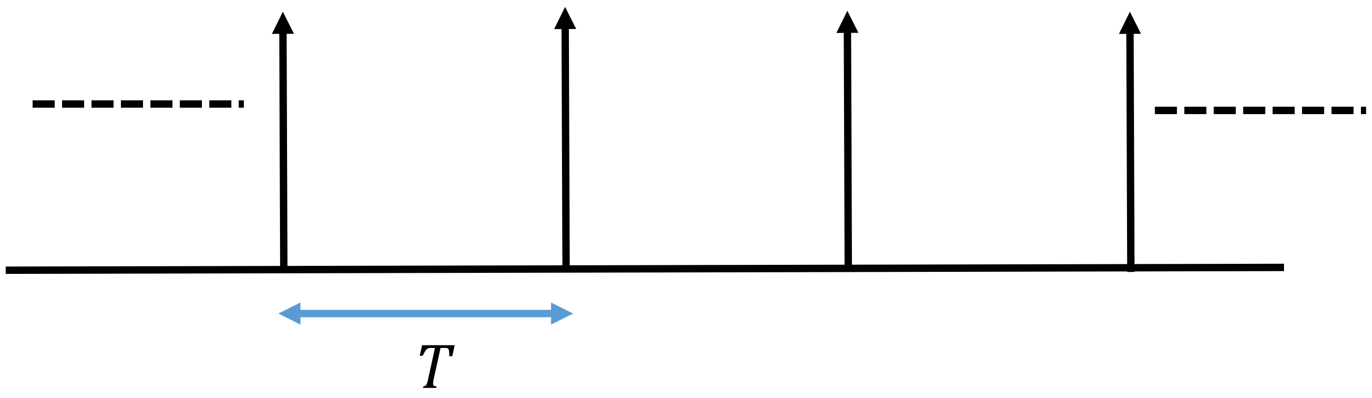


Line Spectrum

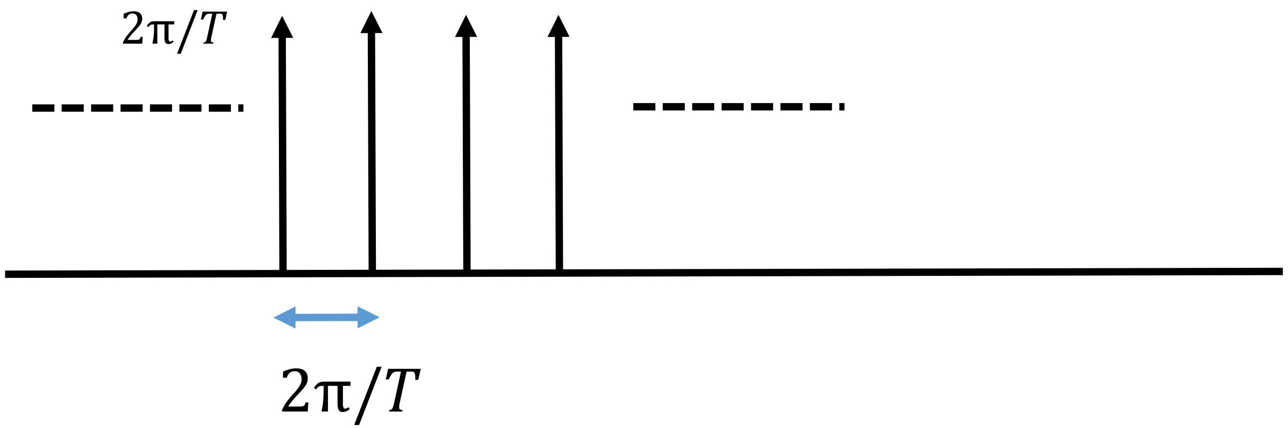


Picket Fence

$$\sum_{k=-\infty}^{\infty} \delta(t - kT)$$

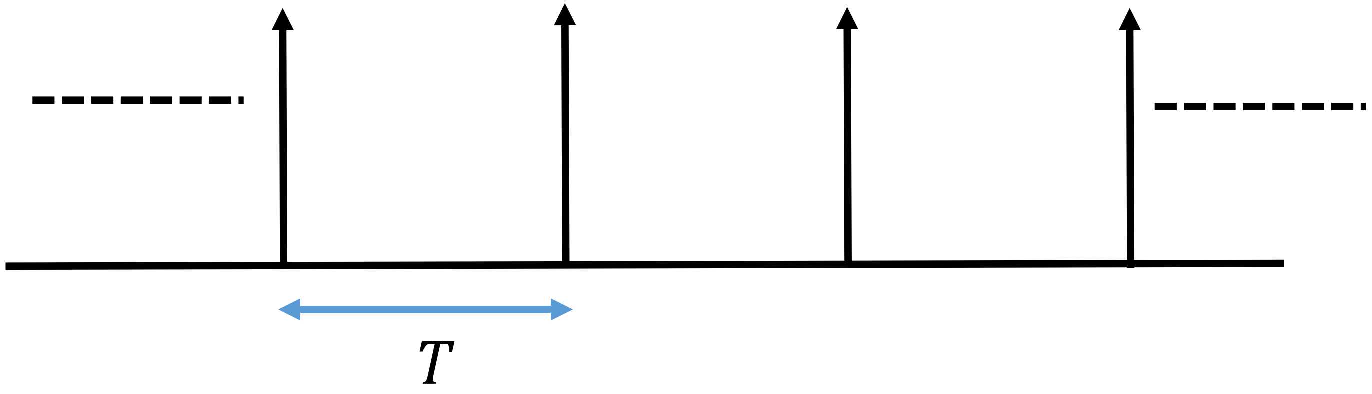


Fourier Spectrum

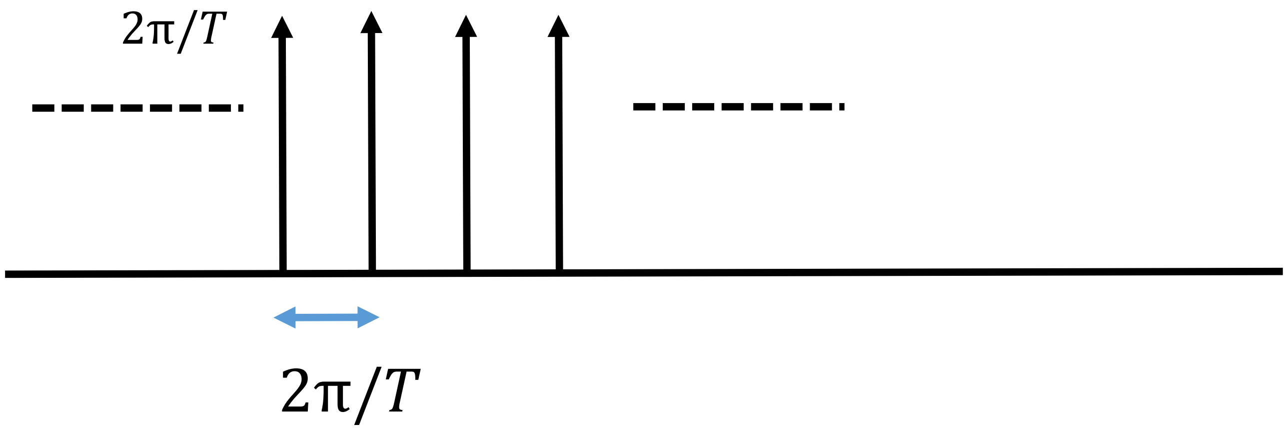


Picket Fence

$$\sum_{k=-\infty}^{\infty} \delta(t - kT)$$



$$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$



Picket Fence

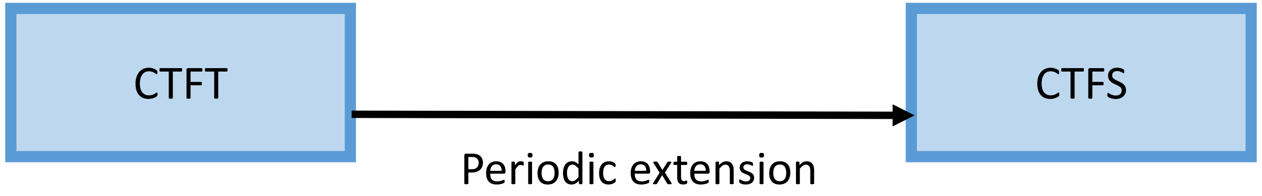
$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \quad \xleftrightarrow{\mathbb{F}} \quad \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

Picket Fence **Picket Fence**

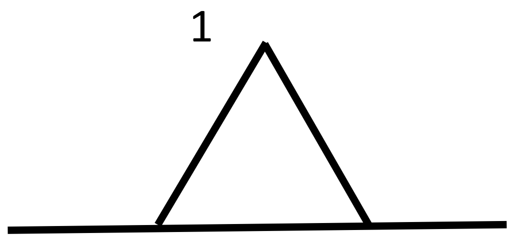
Relations between Fourier Transforms

CTFT

Relations between Fourier Transforms



Periodic extension



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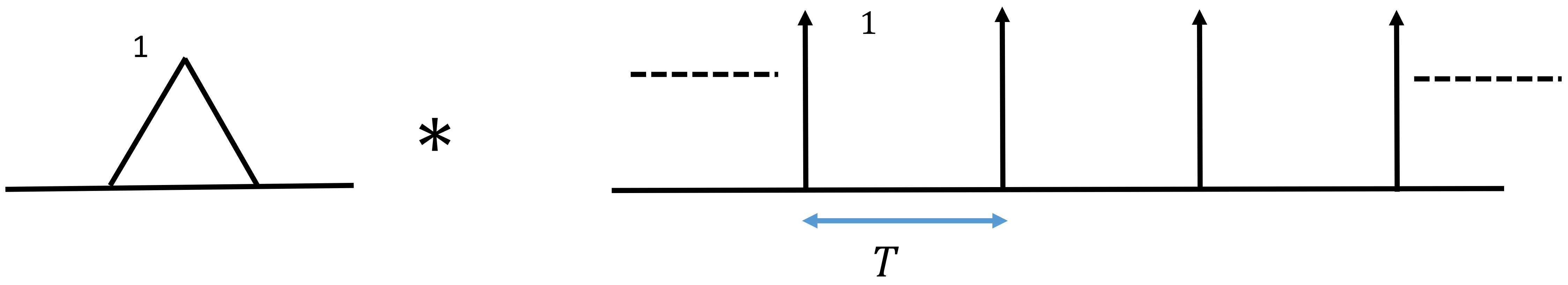
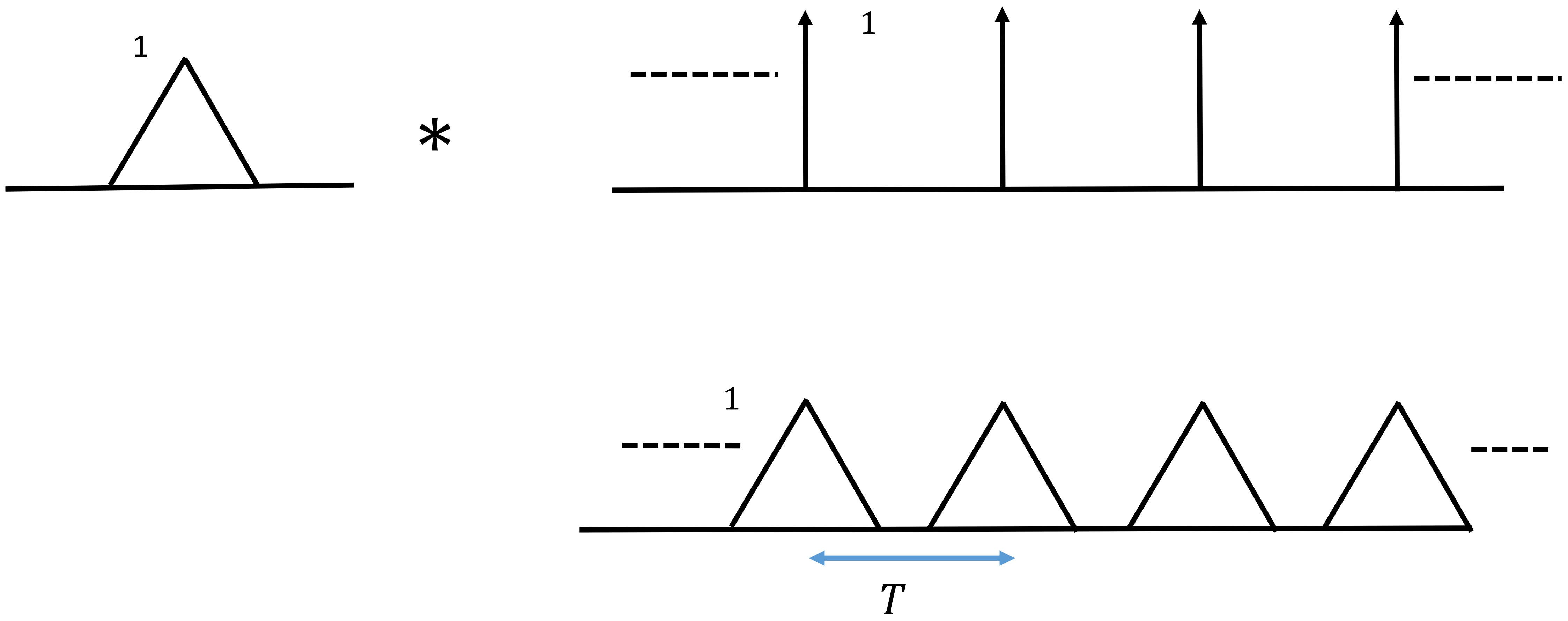
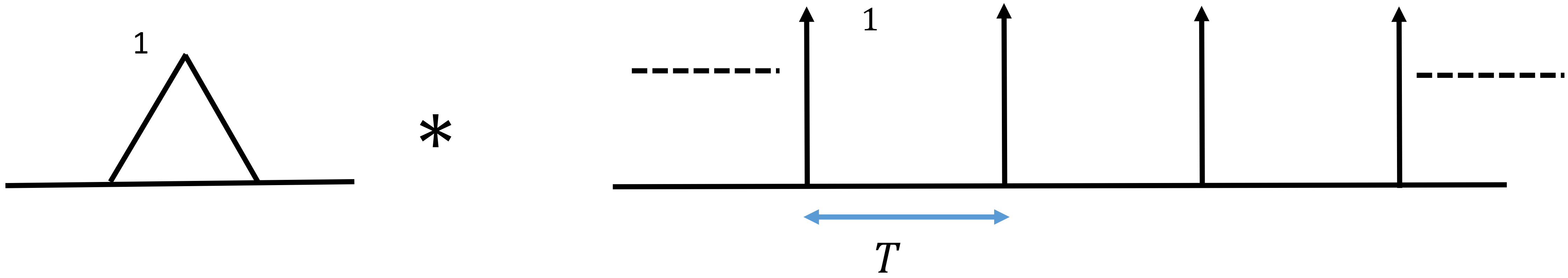


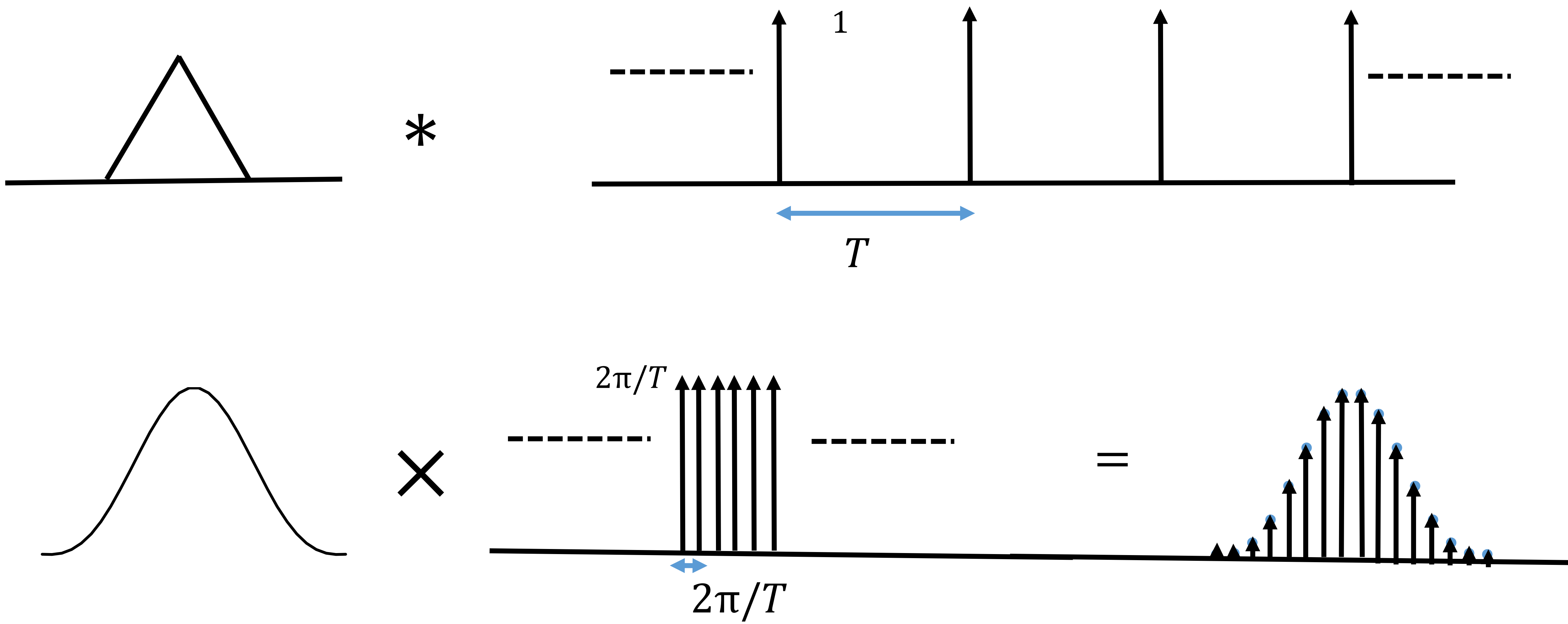
Periodic extension



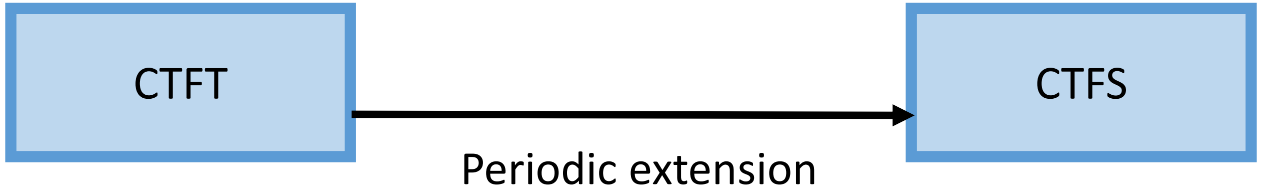
Periodic extension



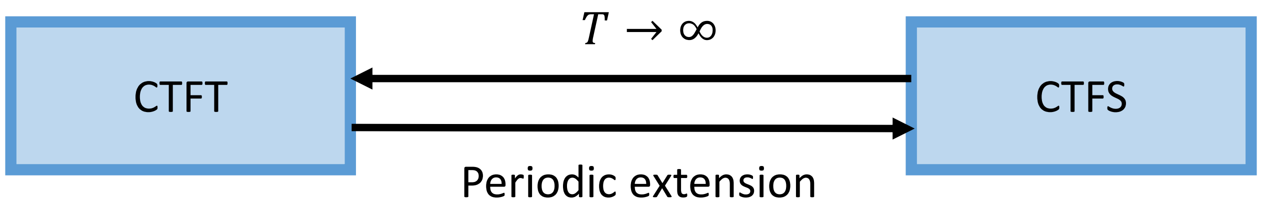
Periodic extension



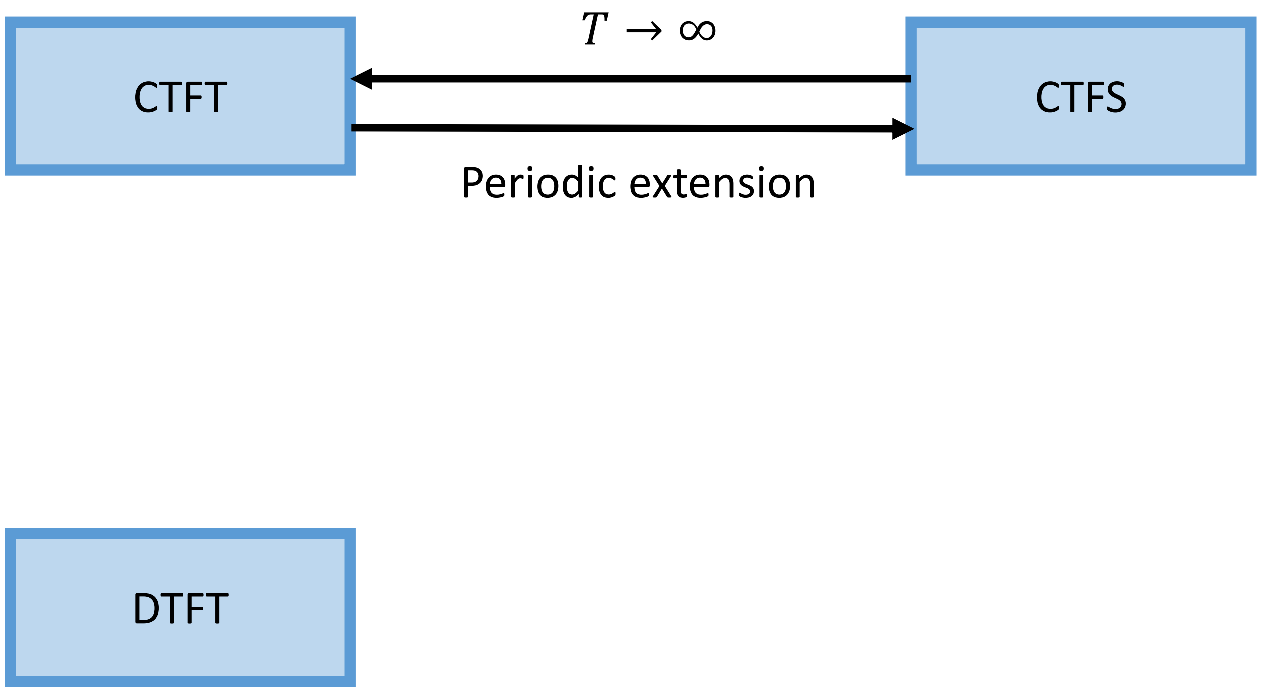
Relations between Fourier Transforms



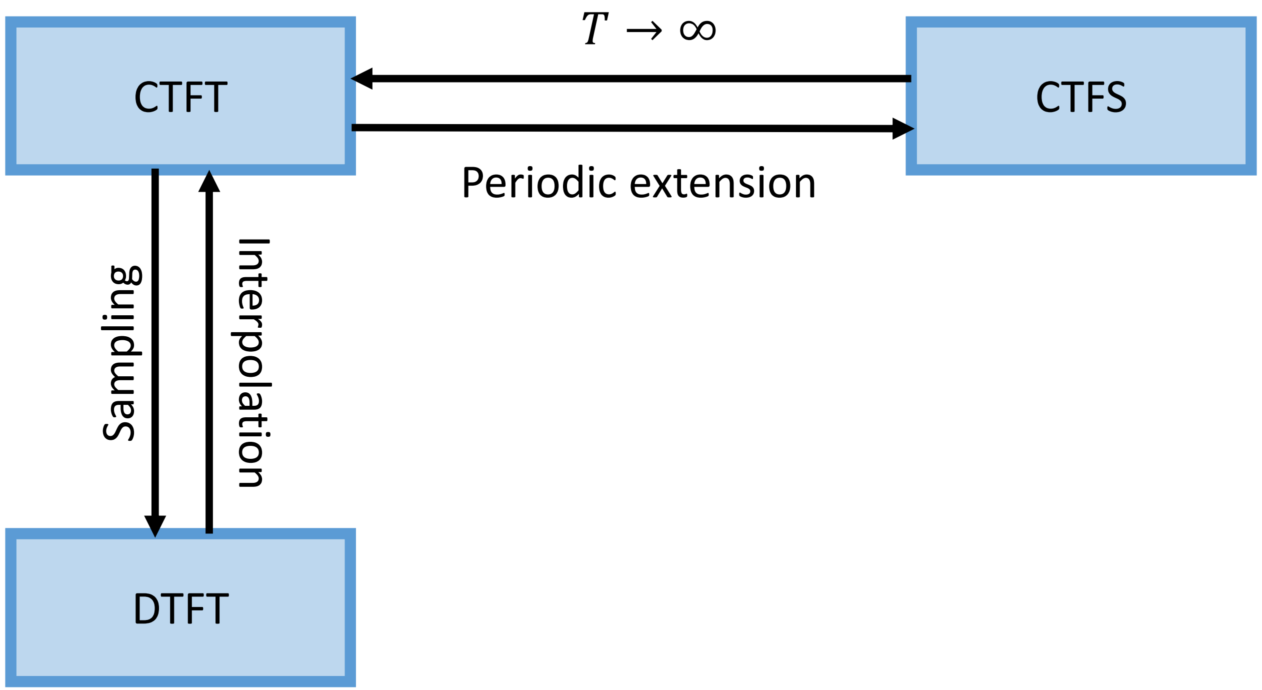
Relations between Fourier Transforms



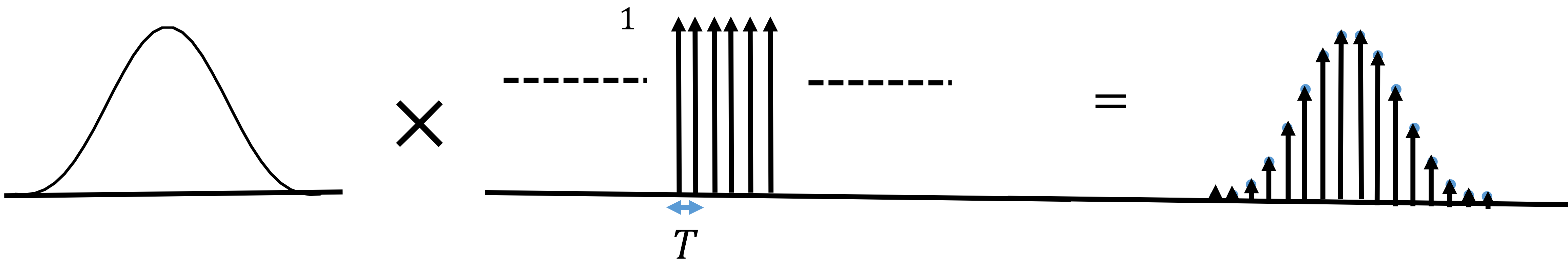
Relations between Fourier Transforms



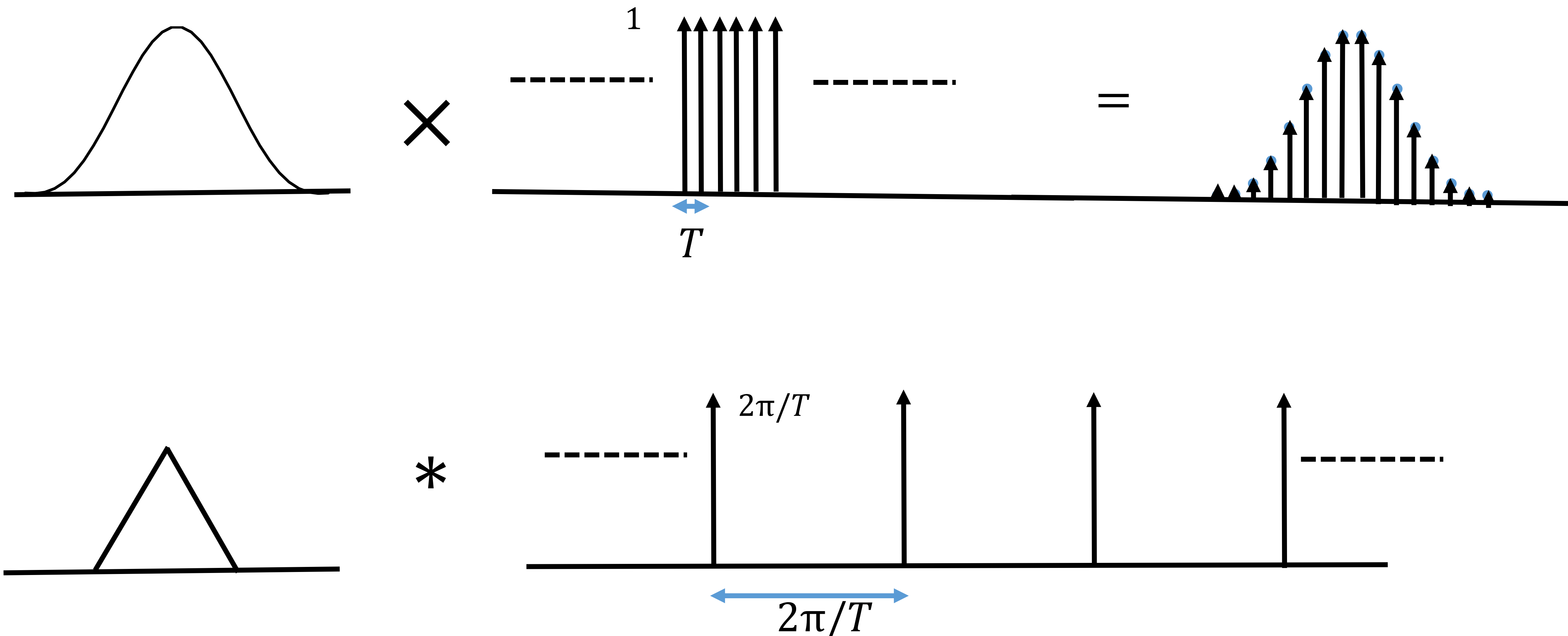
Relations between Fourier Transforms



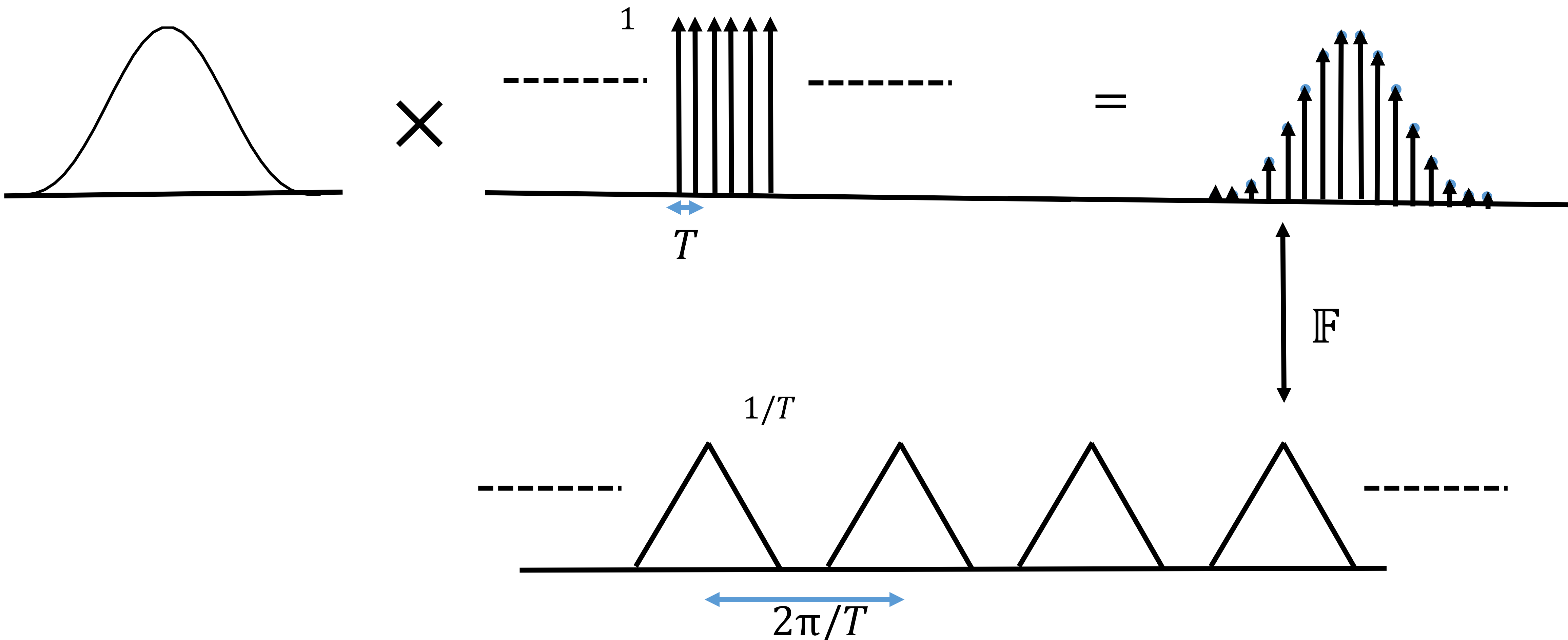
Sampling



Sampling



Sampling

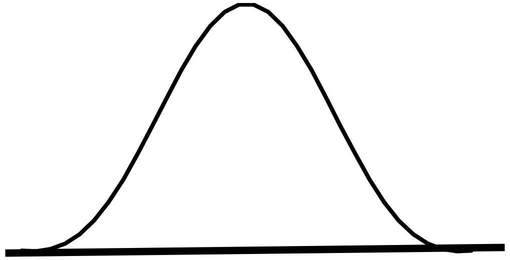


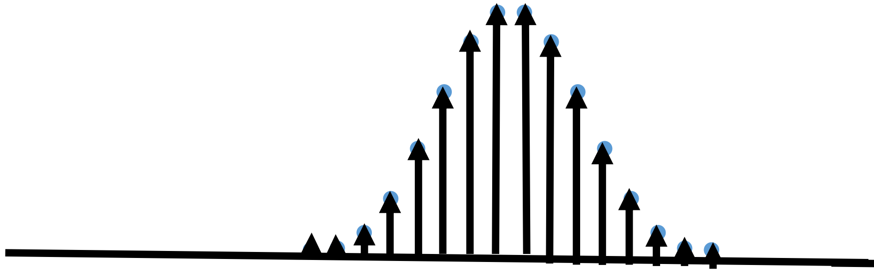
Sampling

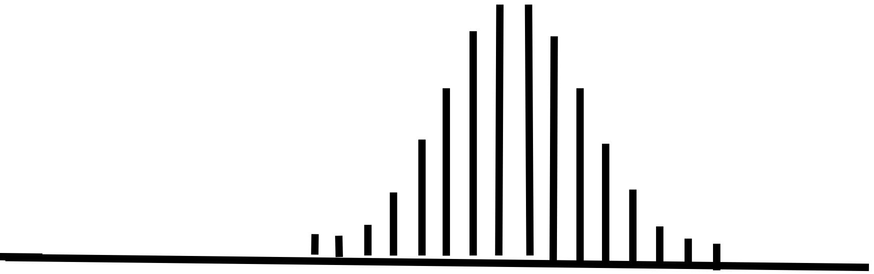
$x(t)$

$\sum_{k} x(kT)\delta(t - kT)$

$x[n] = x(nT)$







$\hat{X}(\omega)$

$$X(\omega) = \sum_{k} \frac{1}{T} \hat{X}\left(\omega - k\frac{2\pi}{T}\right)$$

$X(e^{j\Omega})$

Relationship between Fourier Transforms

$$X(\omega) = \int_{-\infty}^{\infty} x_d(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \sum_{k} x(kT)\delta(t - kT)\, e^{-j\omega t} dt$$

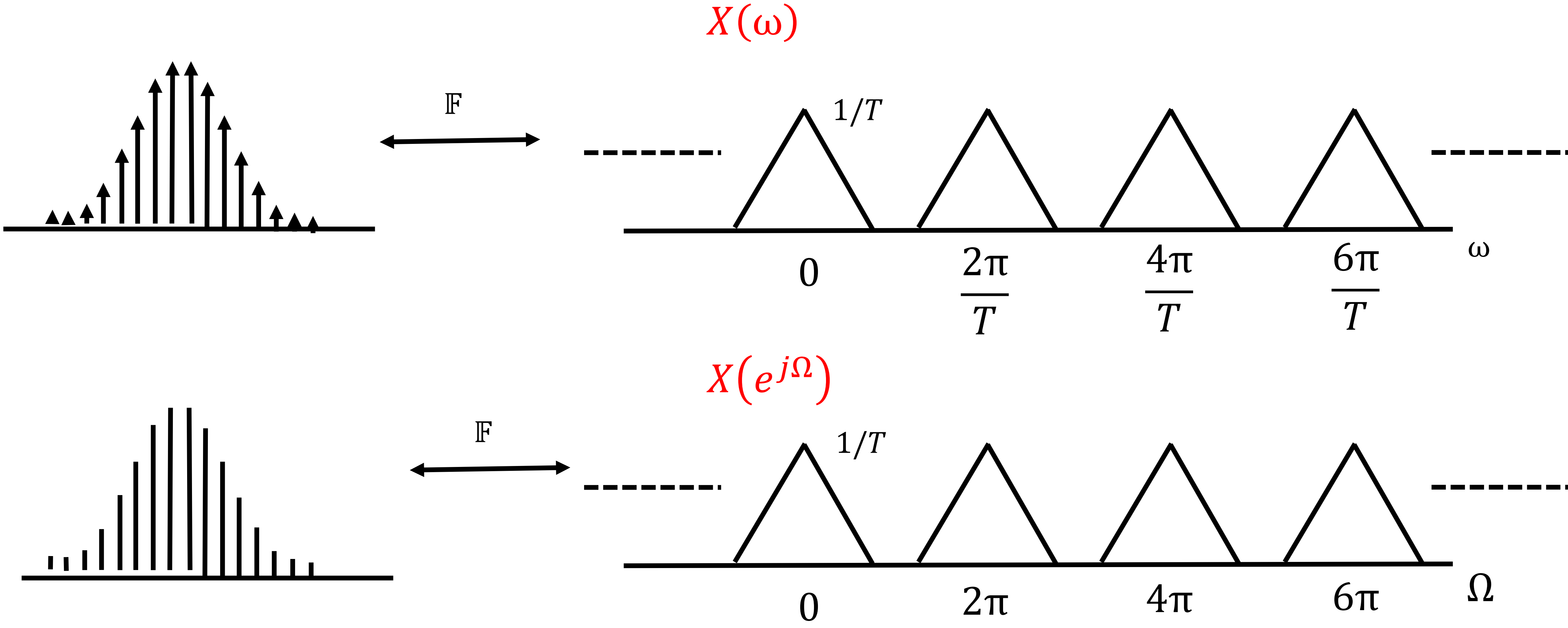
$$X(\omega) = \int_{-\infty}^{\infty} \sum_{n} x[n]\delta(t - nT)\, e^{-j\omega t} dt$$

$$X(\omega) = \sum_{n} x[n] \int_{-\infty}^{\infty} \delta(t - nT) e^{-j\omega t} dt$$

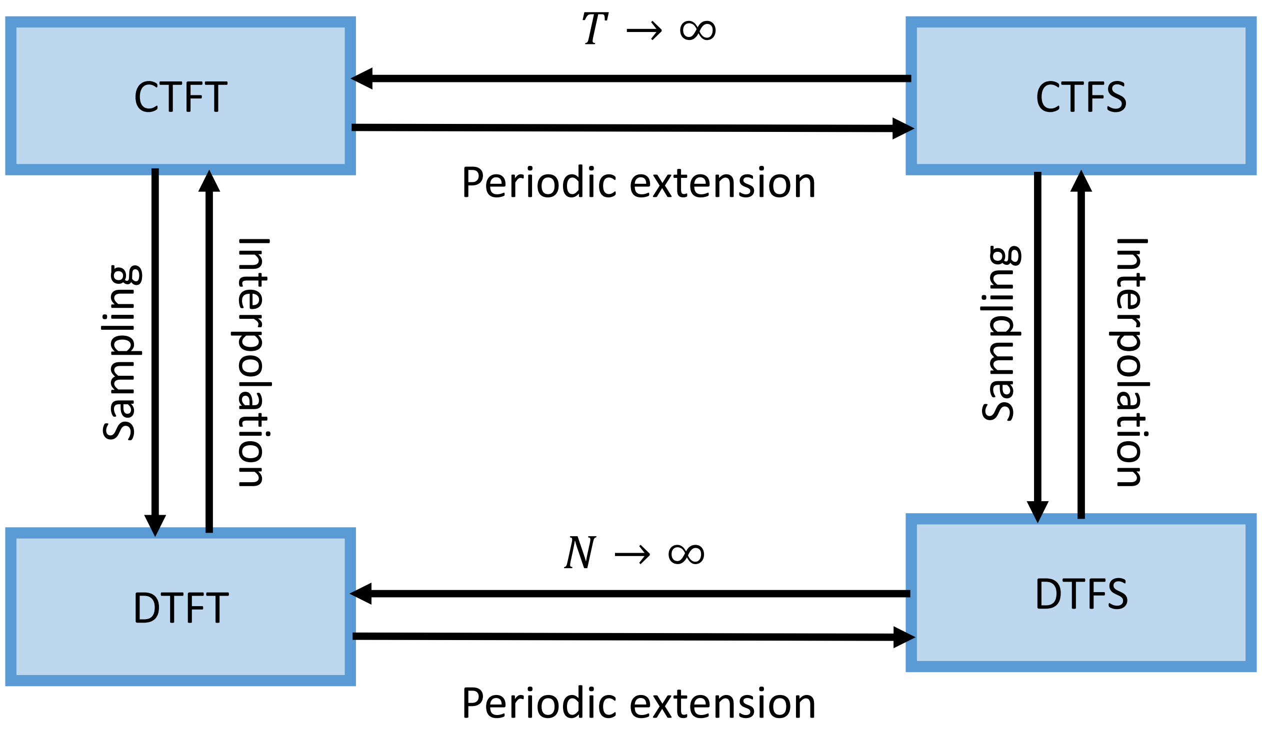
$$X(\omega) = \sum_{n} x[n]\, e^{-j\omega nT} \int_{-\infty}^{\infty} \delta(t - nT) dt$$

$$X(\omega) = \sum_{n} x[n]\, e^{-j\omega nT} = X\left(e^{j\Omega}\right)\Big|_{\Omega = \omega T}$$

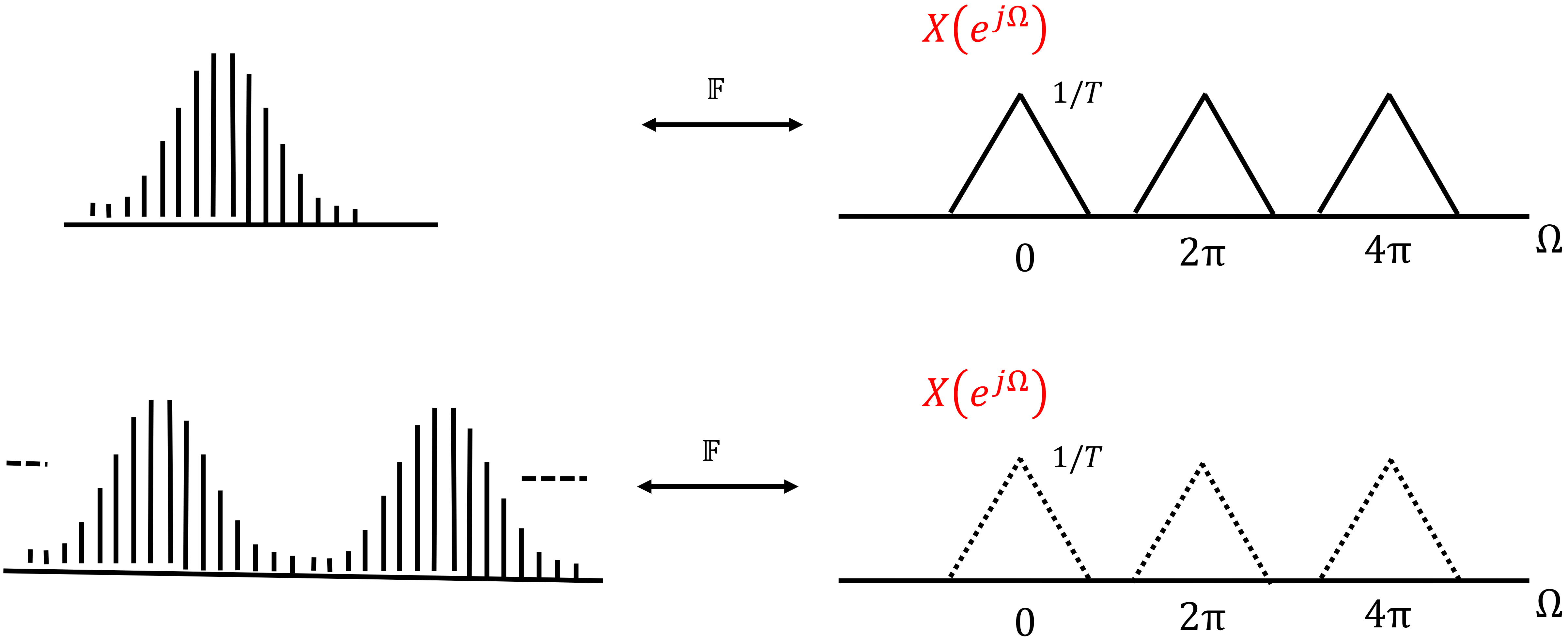
Relationship between Fourier Transforms



Relations between Fourier Transforms



Relationship between Fourier Transforms



Summary

- DFT is most useful
- FFT is the most used computer algorithm