Tutorial Sheet - 1

1. Let A and B be non-empty bounded subsets of $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Then prove that
   (i) $\inf(A+B) = \inf(A) + \inf(B)$.
   (ii) $\sup(A+B) = \sup(A) + \sup(B)$.

2. Let $r \in \mathbb{R}$. Prove that there exists a sequence $\{x_n\}$ of rational numbers such that $\lim\limits_{n\to\infty} x_n = r$.

3. (a) If $\{a_n\}$ is a sequence of real numbers and if the subsequences $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n-1}\}_{n=1}^{\infty}$ both converge to the same limit, then show that $\{a_n\}$ also converges to the same limit.
   (b) If $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is another sequence which converges to 0, show that the product sequence also converges to 0. What can you say about the product sequence, if $\{b_n\}$ converges, but to a non-zero point?

4. Let $\{a_n\}$ be a sequence of real numbers. Define the sequence $\{s_n\}$ by $s_n = \frac{1}{n}\sum_{i=1}^{n} a_i$.
   (i) If $\{a_n\}$ is monotone and bounded, then show that $\{s_n\}$ is also monotone and bounded.
   (ii) If $\{a_n\}$ converges to a, then show that the sequence $\{s_n\}$ also converges to a.

5. Let $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$ and let $t_n = \max\{a_n, b_n\}$, $s_n = \min\{a_n, b_n\}$. Show that $\{t_n\}$ and $\{s_n\}$ are convergent and
$$\lim_{n\to\infty} t_n = \max\{a,b\}, \quad \lim_{n\to\infty} s_n = \min\{a,b\}.$$

6. If $a_1 > 0$ and for $n \geq 1$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, then show that the sequence $\{a_n\}_{n=2}^{\infty}$ is nonincreasing and bounded. Also, find its limit.

7. For $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = \dfrac{1}{4}(x_n^2 + 3)$ for all $n \geq 1$. Examine the convergence of the sequence $\{x_n\}$ for different values of a. Also, find $\lim\limits_{n\to\infty} x_n$ whenever it exists.

8. Prove or disprove that the sequence $\sum_{k=0}^{n} \frac{1}{(n+k)^2}$ converges to 0.

9. Check if the following sequences are Cauchy sequences or not.
   (a) $a_n = \sum_{k=1}^{n} \frac{1}{k!}$ for $n \in \mathbb{N}$
   (b) $a_1 = 1, a_{n+1} = (1 + \frac{(-1)^n}{2^n})a_n$ for $n \in \mathbb{N}$
   (c) Given $a, b \in \mathbb{R}$, let $x_1 = a$, $x_2 = b$, and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n \geq 3$.

10. Find the limit superior and the limit inferior for the sequence $\left\{(-1)^n(1 + \frac{1}{n})\right\}_{n=1}^{n=\infty}$.