COL 352 Introduction to Automata and Theory of Computation

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Lecture 4: Closure properties of regular languages, nondeterminism

Recap

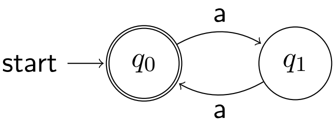
- Languages, Decision problems.
- Finite State Automata - devices with finite memory.
- Deterministic Finite State Automata (DFA)
  - From one state, reading an action we move to exactly one other state.
  - So, for each input word, there is exactly one run!
- Regular languages
  - L is regular if there exists some DFA A such that $L = L(A)$.
  - Closed under Union, Intersection, Complement.
  - Other operations of languages: Concatenation ($L \circ L'$), Kleene star (L^*)

Building complicated DFAs from simple ones

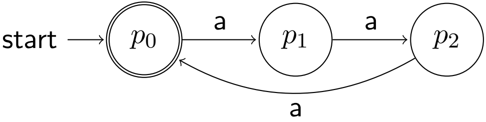
Example

Let $\Sigma = \{a\}$ for this example.

Let $L_1 = \{w \mid |w| \equiv 0 \pmod 2\}$



Let $L_2 = \{w \mid |w| \equiv 0 \pmod 3\}$



What is $L_1 \cap L_2$?

$L_1 \cap L_2 = \{w \mid |w| \equiv 0 \pmod 6\}$

Building complicated DFAs from simple ones

Example

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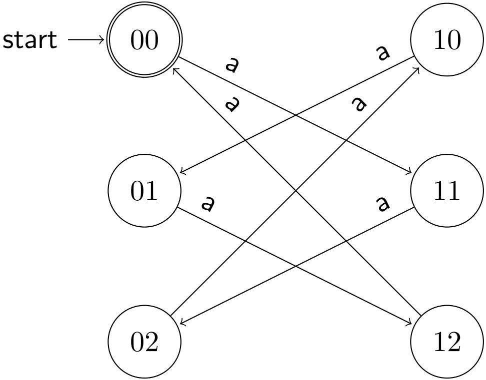

Building complicated DFAs from simple ones

Lemma

Let $L_1, L_2 \subseteq \Sigma^*$ be two regular languages, then $L_1 \cap L_2$ is also a regular language.

Proof.

Product construction

Let $A_1 = (Q_1, \Sigma, q_0^1, F_1, \delta_1)$ and $A_2 = (Q_2, \Sigma, q_0^2, F_2, \delta_2)$ be the automata accepting L_1, L_2, respectively.

Let A be a finite state automaton $(Q, \Sigma, q_0, F, \delta)$ s.t.

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Q & = & \{(q, q') \mid q \in Q_1, q' \in Q_2\} \\
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F & = & \{(q, q') \mid q \in F_1, \text{ and } q' \in F_2\} = F_1 \times F_2 \\
\delta((q, q'), a) & = & (\delta_1(q, a), \delta_2(q', a))
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Correctness

$\forall w \in \Sigma^*$, w is accepted by A iff w is accepted by both A_1 and A_2.

Proof of Correctness

Proof.

Recall, $\delta((q,q'),a) := (\delta_1(q,a), \delta_2(q',a))$

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Inductively, we have

$$\begin{aligned}\hat{\delta}((q,q'),\varepsilon) &= (q,q') \\ \hat{\delta}((q,q'),xa) &= \delta(\hat{\delta}((q,q'),x),a)\end{aligned}$$

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Claim: $L(A) = L(A_1) \cap L(A_2)$

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$$x \in L(A) \iff \hat{\delta}(q_0,x) \in F$$

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□

Building complicated DFAs from simple ones

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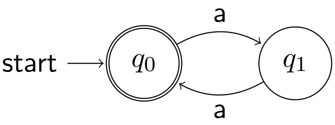
Let $L_1 = \{w \mid |w| \equiv 0 \pmod 2\}$



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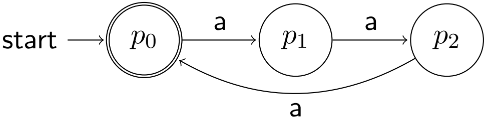

What is $L_1 \cap L_2$?

$L_1 \cup L_2 = \{w \mid |w| \equiv 0 \pmod 2 \text{ or } |w| \equiv 0 \pmod 3\}$

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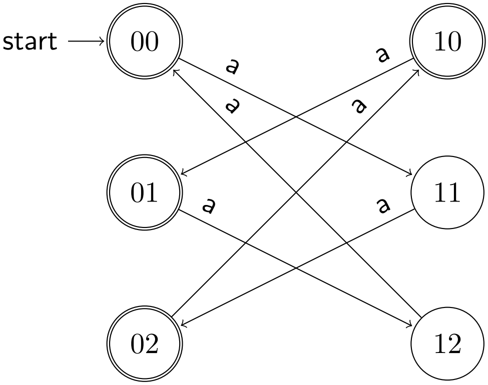

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Correctness

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Complementation

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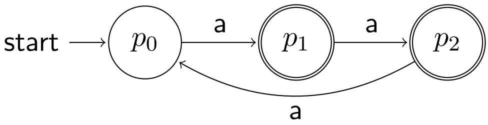


Closure under complement

Lemma

Let $L \subseteq \Sigma^*$ be a regular language, then $\overline{L} = \{w \mid w \notin L\}$ is also a regular language.

Proof.

Let $A = (Q, \Sigma, q_0, F, \delta)$ be the automata accepting L.

Let A' be a finite state automaton $(Q', \Sigma', q_0', F', \delta')$ s.t.

$$\begin{array}{ccc} Q' & = & Q \\ q_0' & = & q_0 \\ F' & = & \{q \in Q \mid q \notin F\} \\ \delta' & = & \delta \end{array}$$

Correctness

$\forall w \in \Sigma^*$, w is accepted by A' iff w is not accepted by A.

$\square$

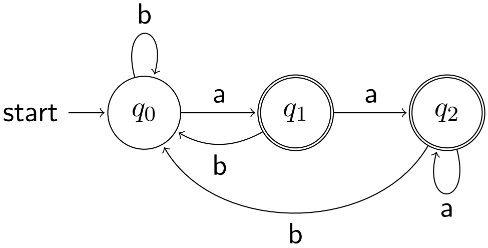
Concatenation and Kleene star

Let $L_1, L_2, L \subseteq \Sigma^*$

$$L_1 \circ L_2 := \{xy \mid x \in L_1, y \in L_2\}$$

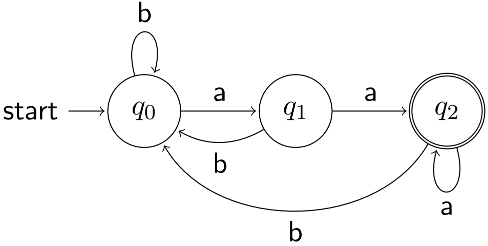
$$\begin{aligned} L^k &:= \{x_1 x_2 \dots x_k \mid x_i \in L\} \\ L^* &:= \bigcup_{k \geq 0} L^k \end{aligned}$$

Example



Non-deterministic finite state automata

Example



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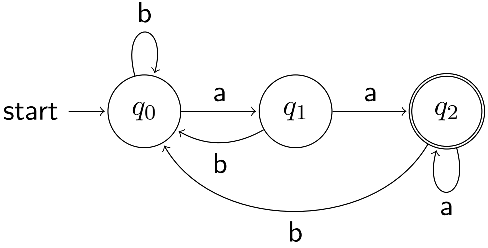
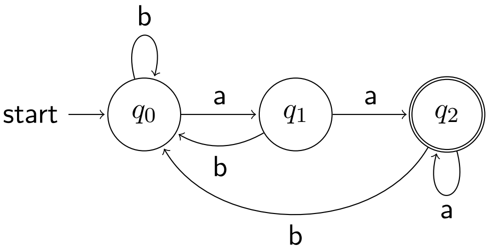


Non-deterministic finite state automata

Example

Input: Text file over the alphabet $\{a, b\}$

Check: does the file end with the string 'aa'



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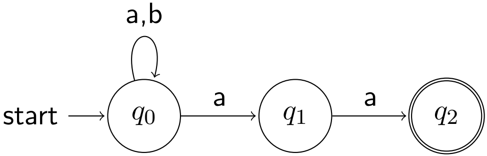

Note that: $\delta(q_0, a) = \{q_0, q_1\}$

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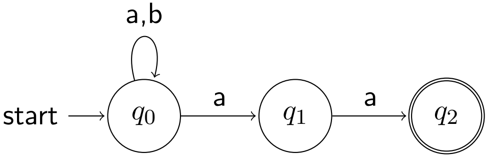

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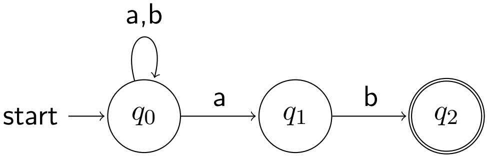
Runs of a NFA



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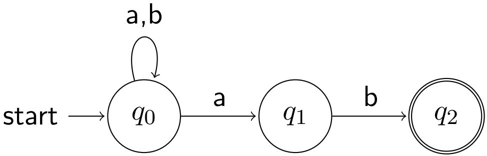

Runs on $b \quad a \quad b \quad a \quad b$

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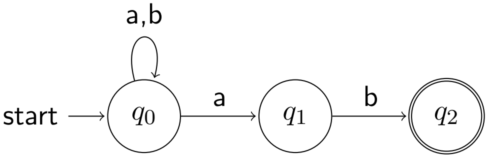

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- $q_0 \xrightarrow{b}$

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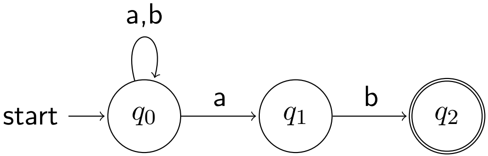

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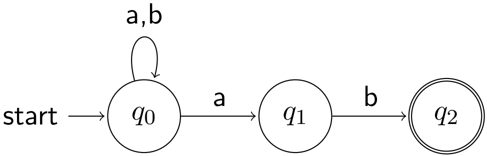

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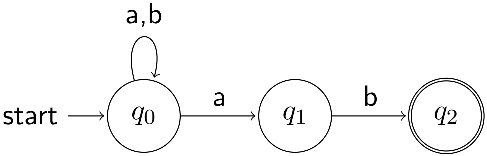

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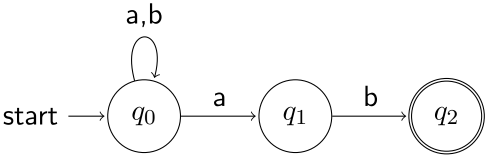

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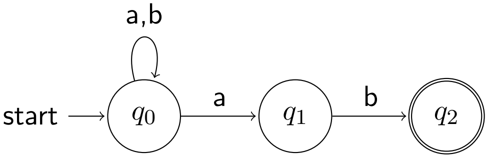

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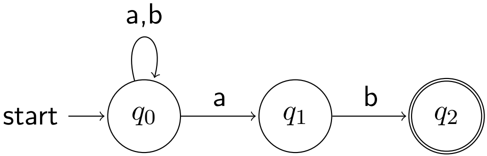

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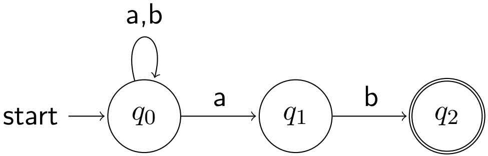

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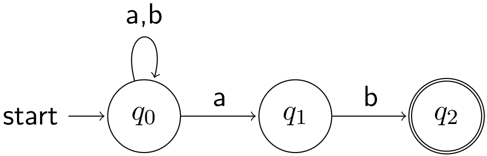

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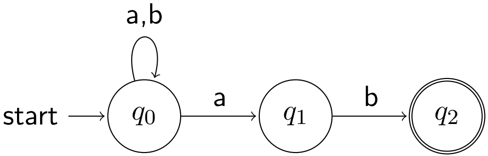

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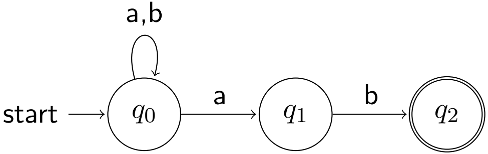

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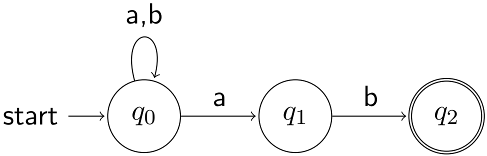

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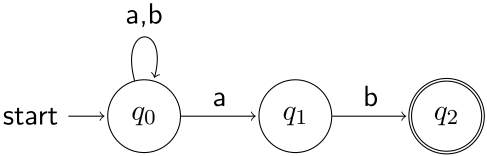

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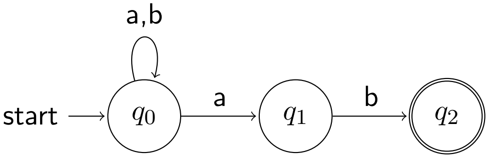

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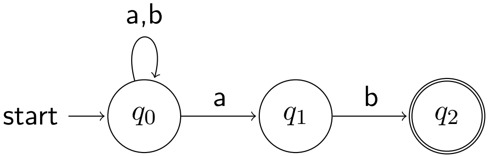

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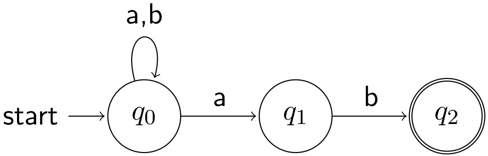

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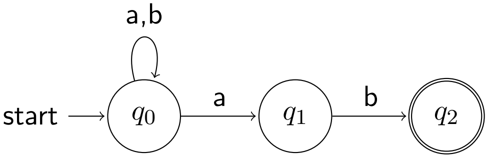

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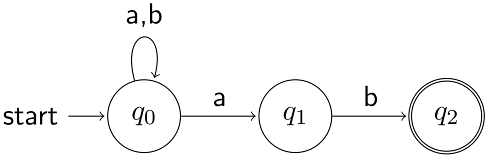

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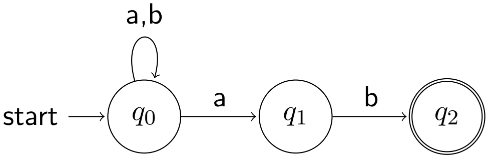

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- Nondeterminism can be thought of as a guess!

Non-deterministic finite state automata

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Input: $w \in \{a, b\}^*$

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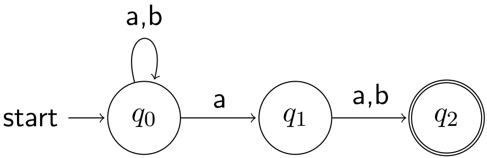

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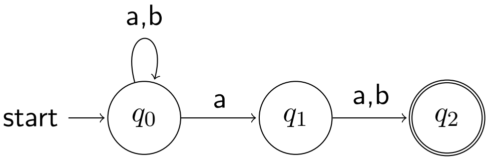

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Informal description: A finite state automaton which can branch out to different states on the same letter.

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$$L(A) = \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in 2^F\}$$

An NFA A is said to accept a language L if $L = \{w \mid A \text{ accepts } w\}$.

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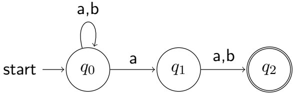

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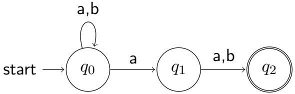

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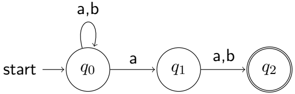

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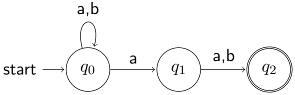

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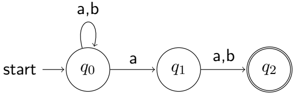

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Equivalent DFA

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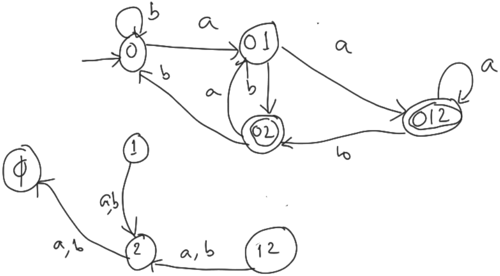

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