ODE Lecture 5

Linear Equations and Some more concepts.

Linear Equations.

A first order linear Equation is of the form

$$y' + p(x)\, y = r(x) \quad \text{---} \quad (*)$$

Note:

1. There are no products of the unknown function and its derivative

2. Neither y nor its derivative y' occur to any power other than the first power.

3. Neither the unknown fn nor its derivatives are cnsider another fn such as e^{y}, $\sqrt{y'}$

Note

II order linear de is of the form

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

(*) can be re written as follows:

$$\frac{dy}{dx} + p(x)\, y = r(x)$$

$$\left(-p(x)\, y + r(x)\right) dx = dy$$

$$\left(p(x)\, y - r(x)\right) dx + dy = 0.$$

$$M = p(x)\, y - r(x)$$

$M = $

$N = 1$

$$\frac{\partial M}{\partial y} = p(x) \qquad \frac{\partial N}{\partial x} = 0.$$

Now

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$= 1\,(p(x) - 0)$$

$= p(x)$, which is a fn of x - alone.

$$I.F = \exp \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx$$

$$= \exp \int p(x)\, dx.$$

Multiplying with IF

$$e^{\int p(x)\,dx}\left(y' + p(x)\,y\right)$$

$$= e^{\int p(x)\,dx} \cdot r(x)$$

$$\frac{d}{dx}\left(e^{\int p(x)\,dx} \cdot y \right) = e^{\int p(x)\,dx} \cdot r(x)$$

Integrating we get

$$e^{\int p(x)\,dx} \cdot y = \int r(x)\, e^{\int p(x)\,dx}\, dx + c$$

$$y(x) = e^{-\int p(x)\,dx} \left[\int e^{\int p(x)\,dx}\, r(x)\, dx + c \right]$$

Let $h(x) = \int p(x)\, dx$.

$$y(x) = e^{-h} \left[\int e^{h} r\, dx + c \right]$$

$y(x) = \ldots$

Example

Solve $y' + 2xy = 2x.$

This is a first order linear differential equation.

Here $p(x) = 2x$

$r(x) = 2x.$

$$I.F = e^{\int p(x)\,dx}$$

$$= e^{\int 2x\,dx}$$

$$= e^{x^2}$$

Given eqn can be written as

$$e^{x^2} y' + 2xy\, e^{x^2} = 2x \cdot e^{x^2}$$

$$\frac{d}{dx}\left(e^{x^2} y\right) = 2x \cdot e^{x^2}$$

$$e^{x^2} y = \int e^{x^2} 2x \, dx + c$$

$$y = e^{-x^2}\left[\int e^{x^2} 2x \, dx + c\right]$$

put $x^2 = u$ for

$$\int e^{x^2} 2x \, dx.$$

we get

$$\int e^{x^2} 2x \, dx$$

u

$$= \int e^{u}\, du = e$$

$$= e^{x^2}$$

$$\therefore\ y = e^{-x^2}\left[e^{x^2} + c\right]$$

$$\Rightarrow y(x) = 1 + c\, e^{-x^2}$$

Comment

The notation $\frac{dy}{dx}$ implies x is independent variable and y is dependent variable.

In solving first order ODE, sometimes it is helpful to reverse the role of x and y and work on the resulting equation.

Example

Solve

$$(4y^3 - 2xy)\, y' = y^2$$

Given

$$4y^3 \frac{dy}{dx} - 2xy \frac{dy}{dx} = y^2$$

— Not linear

$$\frac{dy}{dx}(4y^3 - 2xy) = y^2$$

$$\frac{dy}{dx} = \frac{y^2}{4y^3 - 2xy}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{4y^3 - 2xy}{y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{2}{y}x = 4y$$

This is linear with

x - dependent variable

y - independent variable

Solution is

$$x = e^{-h}\left[\int e^{h} r \, dy + c\right]$$

where $h = \int p(y) \, dy$

Here

$$h = \int \frac{2}{y} dy$$

$$= 2 \ln y = \ln y^2$$

$$x = e^{-\ln y^2}\left[\int e^{\ln y^2} \cdot 4y \, dy + c\right]$$

$$x = \frac{1}{y^2}\left[\int y^2\, 4y\, dy + c\right]$$

$$\Rightarrow \quad x y^2 = \underline{\underline{y^4 + c}}$$

Bernoulli's Equation

This is of the form

$$y' + p(x)\, y = r(x)\, y^{\lambda},$$

where λ is a real number.

$\underline{\lambda = 0}$ $\qquad y' + p\, y = r,$

which is linear.

$\underline{\lambda = 1}$ $\qquad y' + p\, y = r\, y$

$$y' + (p - r)\, y = 0,$$

which is linear

for $\lambda \in \mathbb{R}$ $\lambda \neq 0, 1$

Eqn is nonlinear.

Substituting

$$z = y^{1-\lambda}$$

$$z' = (1-\lambda)\, y^{-\lambda}\, y'$$

$$\Rightarrow z' = (1-\lambda)\, y^{-\lambda} \cdot \left\{ r(x)\, y^{\lambda} - p(x)\, y \right\}$$

$$\Rightarrow z'(x) = (1-\lambda)\, r(x) - p(x)\,(1-\lambda)\, y^{1-\lambda}$$

$$\Rightarrow z'(x) = (1-\lambda)\, r(x) - p(x)\,(1-\lambda)\, z$$

$$\Rightarrow z'(x) + p(x)(1-\lambda) z$$

$$= (1-\lambda) r(x),$$

which is linear in

z and x.

Example

Solve $\quad y' - \frac{y}{x} = y^3$

Given eqn is a Bernoulli's eqn with $\lambda = 3$

Let $\quad z = y^{1-\lambda}$

$$z = y^{-2}$$

$$z' = -2 y^{-3} y'$$

$$\Rightarrow z' = -2 y^{-3} \left[y^3 + \frac{y}{x} \right]$$

$$\Rightarrow \quad z' = -2 - \frac{2}{x}\, y^{-2}$$

$$\Rightarrow \quad z' = -2 - \frac{2}{x}\, z$$

$$\Rightarrow \quad z' + \frac{2}{x}\, z = -2,$$

which is linear in z and x.

Solution is given by

$$z(x) = e^{-h}\left[\int e^{h}\, r\, dx + c\right],$$

where

$$h = \int p(x)\, dx.$$

$$= \int \frac{2}{x}\, dx$$

$$= 2 \ln x$$

$$= \ln x^{2}$$

$= \ln x$

$$Z(x) = e^{-2\ln x}\left[\int e^{\ln x^2}\cdot(-2)\,dx + c\right]$$

$$\Rightarrow Z(x) = \frac{1}{x^2}\left[\int -2x^2 + c\right]$$

$$\Rightarrow Z(x) = \frac{1}{x^2}\left[-2\,\frac{x^3}{3} + c\right]$$

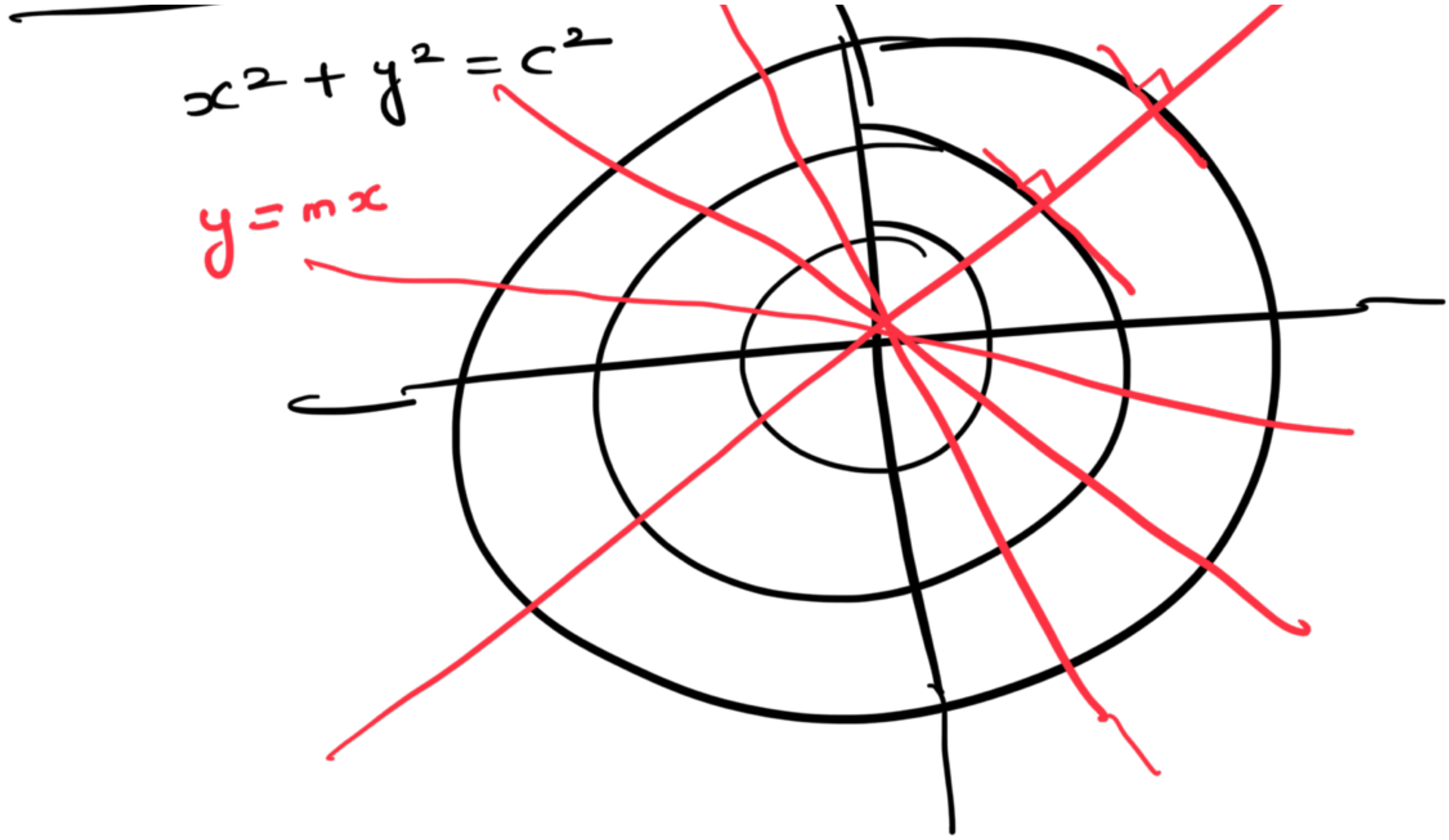
$$\Rightarrow x^2 Z(x) = -\frac{2}{3}x^3 + c$$

Replacing Z we get

$$x^2 y^{-2} = -\frac{2}{3}x^3 + c$$

or $$3\frac{x^2}{y^2} = -2x^3 + c'$$

Orthogonal trajectories



Two families of curves such that each curve in one family is orthogonal to every curve in the other family (whenever they intersect) is orthogonal trajectories of each other.

Examples

1. Curves of equal temperature (isotherms) and curves of heat flow

- curves of equal voltage (equipotential curves) and curves of electric force

How to find Orthogonal trajectories

Suppose the first family is

$$① - \quad F(x, y, c) = 0$$

First, differentiate ① w.r.to x

$$② \quad G(x, y, y', c) = 0.$$

② = (

Eliminate c b/w ① and ②

and find de

③ $H(x, y, y') = 0$.

corresponding to first family

Hence de for orthogonal trajectories is

④ $H\left(x, y, -\frac{1}{y'}\right) = 0$

General Solution of ④ provides required orthogonal trajectories

Example

Find the orthogonal trajectories of straight lines through the origin through the origin

The family of straight lines passing th' origin is given by

$$y = m x.$$

Differentiating $y' = m$

The ODE for this family is

$$y = y' x$$

or $$x y' - y = 0.$$

Now de for orthogonal trajectories is given by

$$x \left(-\frac{1}{y'}\right) - y = 0$$

ie $$x + y y' = 0.$$

$$x = -y\, y' = -y \frac{dy}{dx}$$

$$x \, dx = -y \, dy$$

Integrating we find the solution as

$$x^2 + y^2 = C,$$

which is a family of circles with centre at origin.