MULTIVARIABLE CALCULUS
LECTURE 21

1. Continuity of a function of two variables

1.1. **Definition and Examples.** Let us recall the definition of continuous function in one variable. We define the continuity of a function of one variable and saw how it relied on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x = a$ if
1) $f(a)$ exists.
2) $\lim_{x\to a} f(x)$ exists and
3) $\lim_{x\to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well. In a similar note we define

Definition 1.1. A function $f(x, y)$ is said to be continuous at a point (a, b) in its domain if the following conditions are satisfied:
1) $f(a, b)$ exists.
2) $\lim_{(x,y)\to(a,b)} f(x, y)$ exists and
3) $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example 1.1. *Let D be any subset of $\mathbb{R}^2$. So the map $f : D \to \mathbb{R}$ such that $f(x, y) = c$ $\forall (x, y) \in D$ is continuous.*

Example 1.2. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Check the continuity of f at $(0, 0)$. Using AM-GM inequality we have

$$|f(x,y) - f(0,0)| = |f(x,y) - 0| = |f(x,y)| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{1}{2}\sqrt{x^2+y^2} < \varepsilon,$$

whenever $\sqrt{x^2+y^2} < 2\varepsilon := \delta$.

Example 1.3. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Check the continuity of f at $(0, 0)$. Consider the path $(x, y) \sim (x, x)$, along this path the limit,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2}(\neq f(0,0) = 0)$$

So f is NOT continuous at $(0,0)$.

Example 1.4. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \dfrac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Check the continuity of f at $(0,0)$. We required to prove $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

Consider the $(x,y) \sim (0,y)$ is approaching to $(0,0)$ along the line $y-$axis, Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} \frac{0.y}{0+y^2} = 0.$$

If we consider (x,y) is approaching to $(0,0)$ along the line $x-$axis, Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \frac{x^2.0}{x^4+0} = 0.$$

If we consider (x,y) is approaching to $(0,0)$ along the line $y = mx$, Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{mx^3}{x^4+(mx)^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0.$$

But if we consider (x,y) is approaching to $(0,0)$ along the parabola $y = x^2$, Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,x^2)\to(0,0)} \frac{x^3}{x^4+x^4} = \frac{1}{2} \neq 0.$$

f is NOT continuous at $(x,y) = (0,0)$.

Next we shall introduce concept of partial derivatives.

2. Partial Derivatives

Now that we have examined limits and continuity of functions of two variables, now we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this lecture. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

Derivatives of a Function of Two Variables when studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibnitz notation for the derivative is $\frac{df}{dx}$ which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x,y)$ two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition 2.1. Let $z = f(x, y)$ a function of two variables.
Then the partial derivative of f with respect to x, written as $\frac{\partial f}{\partial x}$ or f_x is defined as

$$\frac{\partial f}{\partial x}(a,b) := f_x(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}, \quad \text{provided limit exists.} \tag{1}$$

The partial derivative of f with respect to y, written as $\frac{\partial f}{\partial y}$ or f_y is defined as

$$\frac{\partial f}{\partial y}(a,b) := f_y(a,b) = \lim_{h\to 0}\frac{f(a,b+k)-f(a,b)}{k}, \quad \text{provided limit exists.} \tag{2}$$

Remark 2.1. This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "∂" is usually called "partial," so $\frac{\partial f}{\partial x}$ spoken as the "partial of f with respect to x") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 2.1. *Use the definition of the partial derivative as a limit to calculate* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *for the function*

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$$

Now $f(x+h, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12 + 2xh + h^2 - 3yh - 4h$. *Hence*

$$\begin{aligned} f(x+h,y) - f(x,y) &= h(2x+h-3y-4) \\ \frac{f(x+h,y)-f(x,y)}{h} &= 2x+h-3y-4. \\ \frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} &= 2x-3y-4. \end{aligned}$$

Similarly one can calculate for $\dfrac{\partial f}{\partial y}$.

Example 2.2. Partial Derivatives may exist without being Continuous *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x,y) := \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Now,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h.0}{h^2+1} - 0}{h} = 0$$

Similarly $\frac{\partial f}{\partial x}(0,0) = 0$. *But* f *is NOT continuous at* $(0,0)$ *(Example 1. 3). Although partial derivatives exist.*

Example 2.3. Do partial derivatives always exist? *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = |x| + |y|$. *Now*

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h},$$

hence the limit does not exist. Similarly $\frac{\partial f}{\partial y}(0,0)$ *does not exist. Although the function is continuous at* $(0,0)$. *Indeed*

$$|f(x,y) - f(0,0)| = |f(x,y)| = |x| + |y| = \sqrt{(|x| + |y|)^2} \leq \sqrt{2}\sqrt{(x-0)^2 + (y-0)^2}.$$

Therefore, $|f(x,y) - f(0,0)| < \varepsilon$, *whenever* $\sqrt{(x-0)^2 + (y-0)^2} < \dfrac{\varepsilon}{\sqrt{2}} := \delta.$

Remark 2.2. For functions two variables : Continuity $\not\Longrightarrow$ Existence of partial derivatives $\not\Longrightarrow$ Continuity at some point on the domain of function. The existence of both partial derivatives at a point need not imply continuity of the function at that point. The reason being that the partial derivatives only exhibit the RATE of change of f only along two paths (x and y-axis).

Remark 2.3. From one variable calculus one can easily see that

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h},$$

provided exists. Equivalently, one could write as : there exists $\varepsilon > 0$ such that

$$f(a+h, b) - f(a,b) = h\frac{\partial f}{\partial x} + \varepsilon_1 \, h,$$

where $\varepsilon \to 0$, as $h \to 0$. Similarly for $\dfrac{\partial f}{\partial y}$.

Theorem 2.1. Sufficient condition for continuity: *Suppose one of the partial derivatives exist at* (a, b) *and the other partial derivative exists and is bounded in a neighborhood of* (a, b). *Then* $f(x, y)$ *is continuous at* (a, b).

Proof. Let f_y exists at (a, b). Then using Remark 2.3,

$$f(a, b+k) - f(a,b) = kf_y(a,b) + k\varepsilon_1,$$

where $\varepsilon_1 \to 0$ as $k \to 0$. Since f_x exists and bounded in a neighborhood of at (a, b), So there exists a real number M such that $|f_x(u, v)| \leq M$ for all $(u, v) \in N_\delta$. Hence for all $(a + h, b + k) \in N_\delta$, we have

$$\begin{aligned} f(a+h, b+k) - f(a,b) &= f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a,b) \\ &= hf_x(a + h\theta, b+k) + kf_y(a,b) + k\varepsilon_1, \\ &\leq |h|M + |k||f_y(a,b)| + |k||\varepsilon_1| \ \to 0 \text{ as } (h,k) \to (0,0). \end{aligned}$$

So $\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$. Hence $f(x, y)$ is continuous at (a, b). $\square$

3. Directional derivatives

Definition 3.1. Let $\vec{p} = p_1 i + p_2 j$ be any unit vector, i.e., $|\vec{p}| = 1$. Then the directional derivative of $f(x, y)$ at (a, b) in the direction of $\vec{p}$ is

$$D_{\vec{p}} f(a,b) = \lim_{s\to 0} \frac{f(a+sp_1, b+sp_2) - f(a,b)}{s}$$

Example 3.1. *$f(x, y) = x^2 + xy$ at $(1, 2)$ in the direction of unit vector $\vec{p} = \frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.*

$$\begin{aligned} D_{\vec{p}}(1,2) &= \lim_{s\to 0} \frac{f(1+\frac{s}{\sqrt{2}}, b+\frac{s}{\sqrt{2}}) - f(1,2))}{s} \\ &= \lim_{s\to 0} \frac{\left(s^2 + s(2\sqrt{2}+\frac{1}{\sqrt{2}})\right)}{s} = 2\sqrt{2} + \frac{1}{\sqrt{2}} \end{aligned}$$

QUESTION : Does existence of partial derivatives implies existence of directional derivatives.

ANSWER : The existence of partial derivatives does not guarantee the existence of directional derivatives in all directions. For example

Example 3.2. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Let $\vec{p} = p_1 i + p_2 j$ be any unit vector. The directional derivative of f along $\vec{p}$ at $(0, 0)$ is

$$D_{\vec{p}} f(0,0) = \lim_{s\to 0} \frac{f(sp_1, sp_2) - f(0,0)}{s} = \lim_{s\to 0} \frac{p_1 p_2}{s(p_1^2 + p_1^2)},$$

exists iff either $p_1 = 0$ or $p_2 = 0$.

Example 3.3. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \sqrt{|xy|}$. Then*

$$\frac{\partial f}{\partial x}(0,0) := \frac{f(h,0) - f(0,0)}{h} = 0.$$

Similarly, $\dfrac{\partial f}{\partial y}(0, 0) = 0$. Let $\vec{p} = p_1 i + p_2 j$ be any unit vector. The directional derivative of f along $\vec{p}$ at $(0, 0)$ is

$$\begin{aligned} D_{\vec{p}} f(0,0) &= \lim_{h\to 0} \frac{f(0+hp_1, 0+hp_2) - \ f(0,0)}{h} \\ &\lim_{h\to 0} \frac{f(hp_1, hp_2)}{h} = \lim_{h\to 0} \frac{|h|}{h} \sqrt{p_1 p_2}, \end{aligned}$$

does not exist, provided $p_1 p_2 = 0$. Moreover, f is continuous at $(0, 0)$. Indeed,

$$|f(x,y) - f(0,0)| - |f(x,y) - 0| = \sqrt{|xy|} \leq \frac{1}{\sqrt{2}} \sqrt{(x-0)^2 + (y-0)^2} < \varepsilon,$$

whenever $\sqrt{(x-0)^2 + (y-0)^2} < \sqrt{2}\varepsilon := \delta$.

Example 3.4. *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x,y) := \begin{cases} \dfrac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Let $\vec{p} = p_1 i + p_2 j$ *be any unit vector. The directional derivative of* f *along* $\vec{p}$ *at* $(0,0)$ *is*

$$\begin{aligned} D_{\vec{p}} f(0,0) &= \lim_{h\to 0} \frac{f(0+hp_1, 0+hp_2) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{p_1^2 p_2}{h^2p_1^2 + p_2^2} = \frac{p_1^2 p_2}{p_2^2}, \end{aligned}$$

provided $p_2 \neq 0$. *If* $p_2 = 0$, *then* $D_{\vec{p}} f(0,0) = 0$. *All directional derivatives exist. Moreover* f *is not continuous (see Example 1.4).*

Remark 3.1. In the above example, all directional derivatives exist but f is NOT continuous.