COL 352 Introduction to Automata and Theory of Computation

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Lecture 18: Limitations of Context-Free Grammars

Chomsky normal form

Definition

A context-free grammar is said to be in Chomsky normal form if every rule is of the form

$$A \to BC$$

$$A \to a$$

where $a \in T$, $A, B, C \in V$, neither B nor C is the start variable, i.e. start variabe does not appear on the right of any rule. Moreover, epsilon does not appear on the right of any rule except as $S \to \epsilon$.

Lemma

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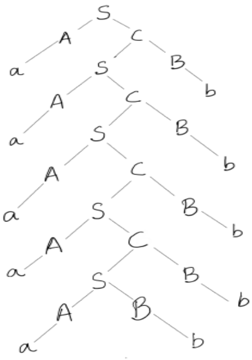

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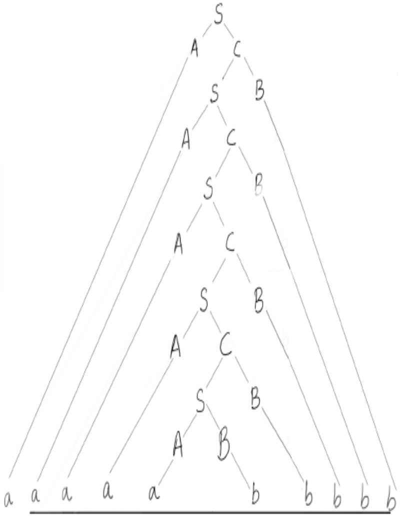
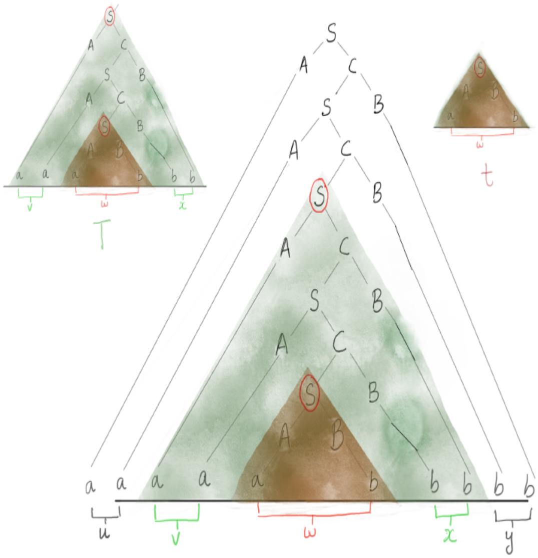


Pumping Lemma for CFLs



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Let L be a CFL. Then there exists a constant k such that if z is a string in L of length at least k, then we can write $z = uvwxy$ such that

- $|vwx| \leq k$
- $vx \neq \epsilon$,
- *For all $i \geq 0$ the string $uv^iwx^iy \in L$.*

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Let G be a CFG accepting L. Let b be an upper bound on the size of the RHS of any production rule of G.

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- Upper bound on the length of strings in L with parse-tree of height $\ell + 1 = b^\ell$
- Let $N = |V|$ be the number of variables in G. What can we say about the strings z in L of size greater than b^N?
- In every parse tree of z, there is a path where a variable repeats.
- Consider a minimum size parse-tree generating z, and consider a path where at least a variable repeats, and consider the last such variable.

Proof of pumping lemma for CFLs

Pumping Lemma for CFLs

Theorem (Pumping Lemma for CFLs)

$L \in \Sigma^*$ is a context-free language $\Longrightarrow$
there exists $k \geq 1$ such that
for all strings $z \in L$ with $|z| \geq k$ we have that
there exists $u, v, w, x, y \in \Sigma^*$ with $z = uvwxy, |vx| > 0$, $|vwx| \leq k$ such that
for all $i \geq 0$ we have that $uv^iwx^iy \in L$.

Theorem (Contrapositive of Pumping Lemma for CFLs)

For all $k \geq 1$ we have that
there exists strings $z \in L$ with $|z| \geq k$ such that
for all $u, v, w, x, y \in \Sigma^*$ with $z = uvwxy$, $|vx| > 0$, $|vwx| \leq k$ we have that
there exists $i \geq 0$ such that $uv^iwx^iy \notin L \Longrightarrow$
$L \in \Sigma^*$ is not a context-free language.

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$L \in \Sigma^*$ is not a context-free language.

- Demon picks $k \geq 0$.
- You pick $z \in L$ of length at least k.
- The demon picks strings u, v, w, x, y such that $z = uvwry$, $|vr| > 0$, and $|vwx| \leq k$.
- You pick $i \geq 0$. If $uv^iwx^iy \notin L$ you win

A few examples

Prove!

Prove that the following languages are not context-free:

1. $L = \{0^n 1^n 2^n : n \geq 0\}$
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- Therefore, $uwy \notin L$. why?

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Exercise: Is $\{0^n1^n2^m3^m \mid n \geq 1\}$ a CFL?

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$$L = \{ww \mid w \in \{0,1\}^*\}$$

- For each n, choose $z = uvwxy = 0^n 1^n 0^n 1^n \in L$.
- consider subwords vwx of $0^n 1^n 0^n 1^n$ such that $|vwx| \leq n$ and $|vx| > 0$.
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Equivalence of NPDAs and CFGs

Theorem

$L = L(G)$ for some context-free grammar G if and only if it is accepted by some NPDA.

CFG to NPDA

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Proof ($\Rightarrow$).

- Assume CFG is in the Chomsky normal form.
- Push S_0 on the stack and make it the current variable.

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- If the the top is a terminal, then match it off with the input bit,
- If the top of the stack is $\perp$ then accept else make that the new current variable.

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Repeat the above procedure. (It will either accept or loop forever.) □

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CFG to NPDA

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Let $G = (V, T, P, S)$ then $A_G = (Q, \Sigma, \Gamma, \delta, q_0, \bot, F)$ where

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- $Q = \{q\} = q_0$ (Single state)

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- $Q = \{q\} = q_0$ (Single state)
- $\Sigma = T$ (Input Alphabet is set of terminals)
- $\Gamma = V \cup T$ (Stack alphabet is terminals and non-terminals)

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- $Q = \{q\} = q_0$ (Single state)
- $\Sigma = T$ (Input Alphabet is set of terminals)
- $\Gamma = V \cup T$ (Stack alphabet is terminals and non-terminals)
- $\bot = S$ (stack bottom is start symbol of CFG)
- $F = \varnothing$ (Acceptance by empty stack)

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Let $G = (V, T, P, S)$ then $A_G = (Q, \Sigma, \Gamma, \delta, q_0, \perp, F)$ where

- $Q = \{q\} = q_0$ (Single state)
- $\Sigma = T$ (Input Alphabet is set of terminals)
- $\Gamma = V \cup T$ (Stack alphabet is terminals and non-terminals)
- $\perp = S$ (stack bottom is start symbol of CFG)
- $F = \varnothing$ (Acceptance by empty stack)
- δ is defined as:

$$\delta(q, \epsilon, B) := \{(q, \beta) \mid B \to \beta \text{ in } P\}$$

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CFG to NPDA

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Guess production rule and push on to the stack and verify guess while popping.

NPDA to CFG

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- Idea: Design a CFG that for each pair of states p, q in P, have a variable $A_{p,q}$ which generates all strings that can take P from p (with empty stack) to q (with empty stack).
- Modify P so that
  - It has single accept state.
  - It empties its stack before accepting.
  - Each transition either pushes a symbol or pops a symbol (not both).

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Given a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, \perp, q_F)$ with restriction that every transition is either pushes a symbol or pops a symbol form the stack, i.e. $\delta(q, a, X)$ contains either (q_0, YX) or (q_0, ϵ).

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Lemma

$L(G_p) = L(P)$.

PDA to CFG

Lemma

If $A_{p,q} \Longrightarrow^* x$ *then* x *can bring the PDA* P *from state* p *on empty stack to state* q *on empty stack.*

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- **Inductive step.** If $A_{p,q} \Longrightarrow^* x$ in $n + 1$ steps. The first step in the derivation must be $A_{p,q} \to A_{p,r} A_{r,q}$ or $A_{p,q} \to a A_{r,s} b$.

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  - If it is $A_{p,q} \rightarrow A_{p,r}A_{r,q}$, then the string x can be broken into two parts x_1x_2 such that $A_{p,r} \Longrightarrow^* x_1$ and $A_{r,q} \Longrightarrow^* x_2$ in at most n steps. The claim easily follows in this case.
  - If it is $A_{p,q} \rightarrow aA_{r,s}b$, then the string x can be broken as ayb such that $A_{r,s} \Longrightarrow^* y$ in n steps. Notice that from p on reading a the PDA pushes a symbol X to stack, while it pops X in state s and goes to q.

□