ELL100: INTRODUCTION TO ELECTRICAL ENGG.

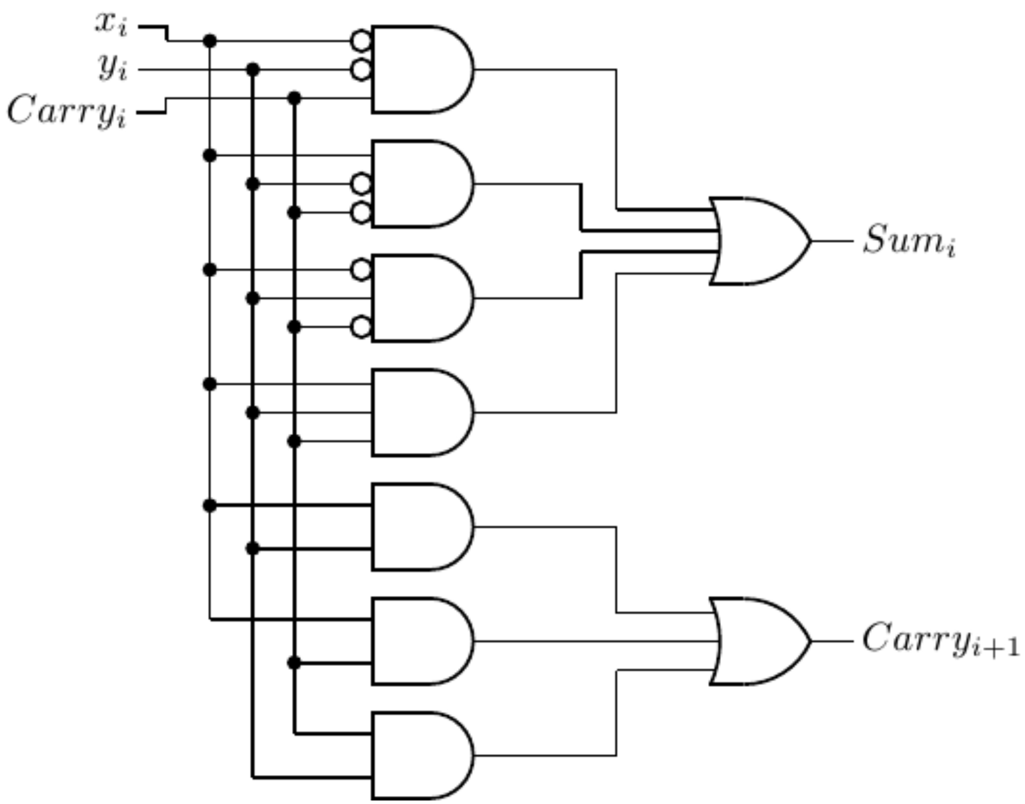
Minimization through Karnaugh maps

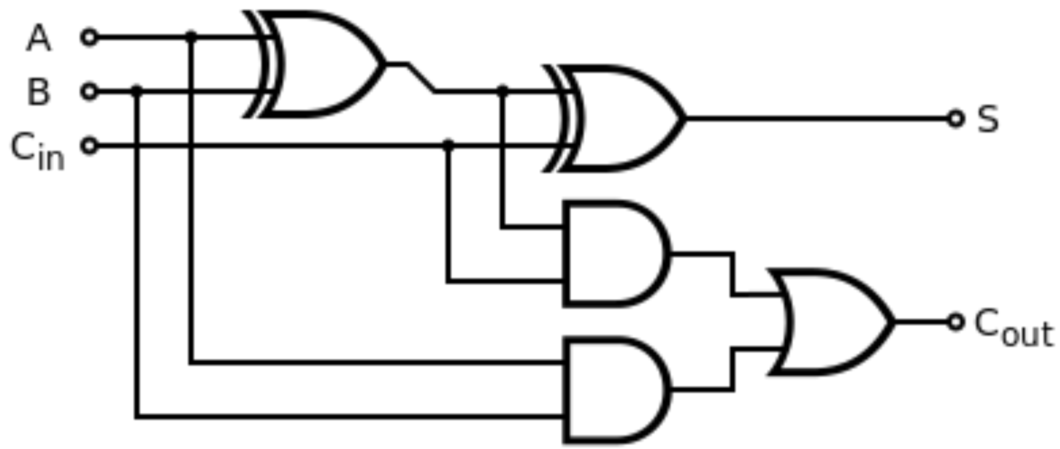
Instructor: Debanjan Bhowmik

Textbook: Moris Mano's 'Digital Design'

Chapter 3 (Gate-Level Minimization)

Multiple Expressions for One Truth Table





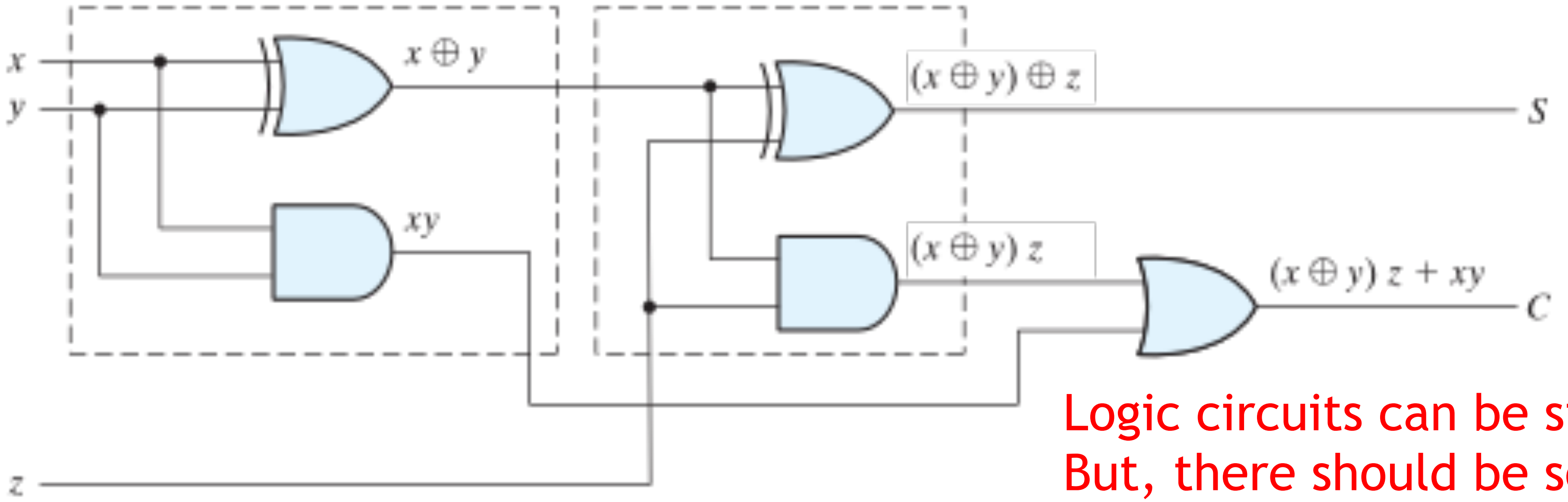
It is useful if the same logic can be implemented using lesser number of logic gates and logic levels.

Note: There is more than one way to implement the same logic

Simplification is not Simple

$$S = (x \oplus y) \oplus z$$

$$\begin{aligned} C &= xy + yz + xz = xy + (x+y)z \\ &= xy + (x(y+y') + (x+x')y)z = xy + (xy + x \oplus y)z \\ &= xy + xyz + (x \oplus y)z = xy + (x \oplus y)z \end{aligned}$$



Logic circuits can be simplified.
But, there should be some
systematic approach.

Karnaugh Map

- **Karnaugh Map**: A graphical and systematic way to obtain simplest expression for a truth table

Definitions

Some definitions:

$$C(x,y,z) = \boxed{\begin{array}{l} x'yz + xy'z \\ +xyz' + xyz \end{array}} = \boxed{xy + yz + xz}$$

$$C(x,y,z) = \boxed{\begin{array}{l} (x'+y+z)(x+y'+z)\ldots \\ (x+y+z')(x+y+z) \end{array}} = \boxed{(x+y)(y+z)(x+z)}$$

Sum of Products (SOP) representation
Derived from entries where the Output is 1 (TRUE)

Product of Sums (POS) representation
Derived from entries where the Output is 0 (FALSE)

Definitions

Some definitions:

$$
\begin{aligned}
C(x,y,z) &= \boxed{\begin{array}{l} x'yz + xy'z \\ +xyz' + xyz \end{array}} \\
&= xy + yz + xz
\end{aligned}
$$

Canonical Sum of Products (SOP) representation. All inputs present in all terms

$$
\begin{aligned}
C(x,y,z) &= \boxed{\begin{array}{l} (x'+y+z)(x+y'+z)\ldots \\ (x+y+z')(x+y+z) \end{array}} \\
&= (x+y)(y+z)(x+z)
\end{aligned}
$$

Canonical Product of Sums (POS) representation. All inputs present in all terms

Definitions

Some definitions:

$$C(x, y, z) = x'yz + xy'z + xyz' + xyz = \boxed{xy + yz + xz}$$

Minimal Sum of Products (SOP) representation. Cannot have a shorter SOP form.

$$C(x, y, z) = (x' + y + z)(x + y' + z)\ldots (x + y + z')(x + y + z) = \boxed{(x + y)(y + z)(x + z)}$$

Minimal Product of Sums (POS) representation. Cannot have a shorter POS form.

Definitions

Some definitions:

$$
\begin{aligned}
C(x,y,z) &= \boxed{x'yz} + \boxed{xy'z} \\
&\quad + \boxed{xyz'} + \boxed{xyz} \\
&= xy + yz + xz
\end{aligned}
$$

$$
\begin{aligned}
C(x,y,z) &= \boxed{(x' + y + z)}\boxed{(x + y' + z)} \ldots \\
&\quad \boxed{(x + y + z')}\boxed{(x + y + z)} \\
&= (x + y)(y + z)(x + z)
\end{aligned}
$$

Minterm: Each of the terms in a canonical SOP form

Maxterm: Each of the terms in a canonical POS form

Definitions

Some definitions:

$$
\begin{aligned}
C(x,y,z) &= x'yz + xy'z \\
&\quad +xyz' + xyz \\
&= xy + yz + xz \\
&= m_3 + m_5 + m_6 + m_7
\end{aligned}
$$

If the inputs are ordered, then minterms can be indexed.

$$xy'z \rightarrow m_{101_2} \rightarrow m_5$$

$$(x=1, y=0, z=1) \implies (C=1)$$

$$
\begin{aligned}
C(x,y,z) &= (x'+y+z)(x+y'+z)\ldots \\
&\quad (x+y+z')(x+y+z) \\
&= (x+y)(y+z)(x+z) \\
&= M_4 M_2 M_1 M_0
\end{aligned}
$$

If inputs are ordered, then maxterms can be indexed.

$$(x+y'+z) \rightarrow M_{010_2} \rightarrow M_2$$

$$(x=0, y=1, z=0) \implies (C=0)$$

Definitions

Some definitions:

$$
\begin{aligned}
C(x,y,z) &= x'yz + xy'z \\
&\quad + xyz' + xyz \\
&= xy + yz + xz \\
&= m_3 + m_5 + m_6 + m_7
\end{aligned}
$$

If the inputs are ordered, then minterms can be indexed.

$$C(x,y,z) = \sum m(3,5,6,7)$$

$$
\begin{aligned}
C(x,y,z) &= (x'+y+z)(x+y'+z)... \\
&\quad (x+y+z')(x+y+z) \\
&= (x+y)(y+z)(x+z) \\
&= M_4 M_2 M_1 M_0
\end{aligned}
$$

If inputs are ordered, then maxterms can be indexed.

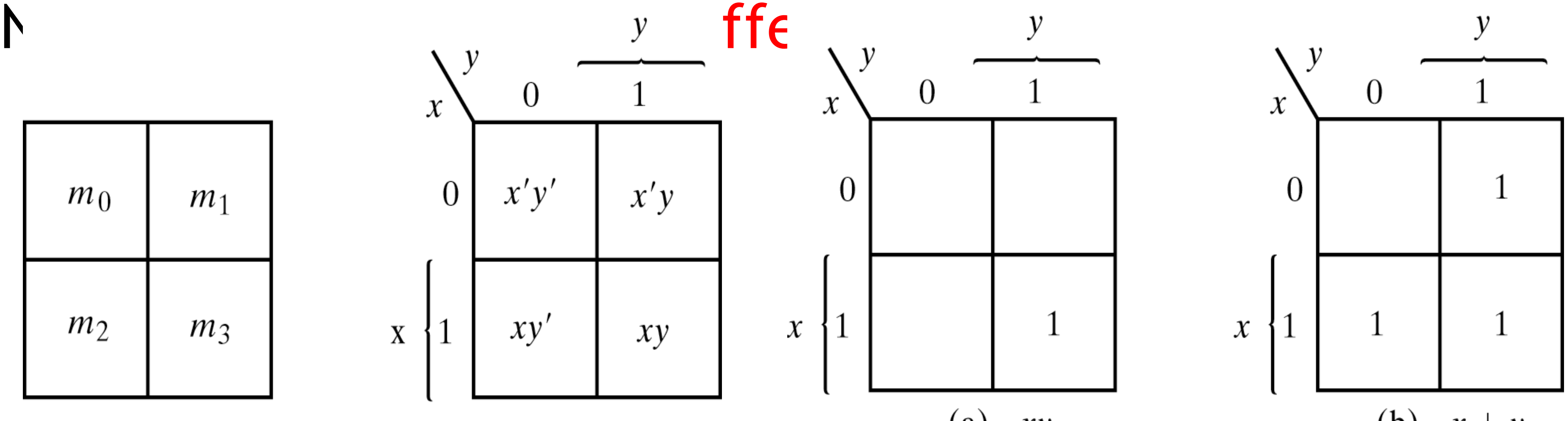
$$C(x,y,z) = \prod M(0,1,2,4)$$

Karnaugh Map

- **Karnaugh Map**: A graphical and systematic way to obtain simplest expression for a truth table (K-Map)
- The map is made up of squares, with each square representing one minterm of the function
- This produces a circuit diagram with a minimum number of gates and the minimum number of inputs to the gate
- It is sometimes possible to find two or more expressions that satisfy the minimization criteria

Two Variable Map

- Two-variable has four minterms, and consists of four squares.
- $m_1 + m_2 + m_3 = x'y + xy' + xy = x + y$
- N ffe



(a) xy

(b) $x + y$

Three Variable Map

- Note that the minterms are not arranged in a binary sequence.
- For simplifying Boolean functions, we must recognize the basic property possessed by adjacent squares.
- $m_5+m_7= xy'z + xyz = xz(y' + y) = xz$

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6

cancel

$x \backslash yz$	00	01	11	10
0	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
x 1	$xy'z'$	$xy'z$	xyz	xyz'

y (columns 11, 10); z (columns 01, 11)

y

Simplification

- A larger number of adjacent squares are combined, we obtain a product term with fewer literals.

  1 square = 1 term with three literals

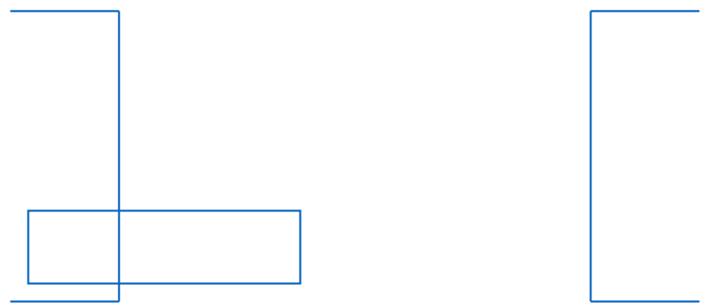
  2 adjacent squares = 1 term with two literals

  4 adjacent squares = 1 term with one literal

  8 adjacent squares encompass the entire map and produce a function that is always equal to 1
- The number of adjacent squares is combined in a power of two such as 1, 2, 4, and 8.

Example 1

$F(x, y, z) = \Sigma(0, 2, 4, 5, 6)$



Example 2

Design the minimal product-of-sums expression for the function

$$f(x_1, x_2, x_3) = \Sigma\, m(0, 2, 4, 5, 6, 7)$$

Minterms and Maxterms

$$f(x_1, x_2, x_3) = \sum m(0, 2, 4, 5, 6, 7)$$

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x}_1\overline{x}_2\overline{x}_3$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \overline{x}_1\overline{x}_2 x_3$	$M_1 = x_1 + x_2 + \overline{x}_3$
2	0	1	0	$m_2 = \overline{x}_1 x_2\overline{x}_3$	$M_2 = x_1 + \overline{x}_2 + x_3$
3	0	1	1	$m_3 = \overline{x}_1 x_2 x_3$	$M_3 = x_1 + \overline{x}_2 + \overline{x}_3$
4	1	0	0	$m_4 = x_1\overline{x}_2\overline{x}_3$	$M_4 = \overline{x}_1 + x_2 + x_3$
5	1	0	1	$m_5 = x_1\overline{x}_2 x_3$	$M_5 = \overline{x}_1 + x_2 + \overline{x}_3$
6	1	1	0	$m_6 = x_1 x_2\overline{x}_3$	$M_6 = \overline{x}_1 + \overline{x}_2 + x_3$
7	1	1	1	$m_7 = x_1 x_2 x_3$	$M_7 = \overline{x}_1 + \overline{x}_2 + \overline{x}_3$

Minterms and Maxterms

$$f(x_1, x_2, x_3) = \sum m(0, 2, 4, 5, 6, 7)$$

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x}_1\overline{x}_2\overline{x}_3$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \overline{x}_1\overline{x}_2x_3$	$M_1 = x_1 + x_2 + \overline{x}_3$
2	0	1	0	$m_2 = \overline{x}_1x_2\overline{x}_3$	$M_2 = x_1 + \overline{x}_2 + x_3$
3	0	1	1	$m_3 = \overline{x}_1x_2x_3$	$M_3 = x_1 + \overline{x}_2 + \overline{x}_3$
4	1	0	0	$m_4 = x_1\overline{x}_2\overline{x}_3$	$M_4 = \overline{x}_1 + x_2 + x_3$
5	1	0	1	$m_5 = x_1\overline{x}_2x_3$	$M_5 = \overline{x}_1 + x_2 + \overline{x}_3$
6	1	1	0	$m_6 = x_1x_2\overline{x}_3$	$M_6 = \overline{x}_1 + \overline{x}_2 + x_3$
7	1	1	1	$m_7 = x_1x_2x_3$	$M_7 = \overline{x}_1 + \overline{x}_2 + \overline{x}_3$

The function is 1 for these rows

Minterms and Maxterms

$$f(x_1, x_2, x_3) = \sum m(0, 2, 4, 5, 6, 7) = \prod M(1, 3)$$

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x}_1\overline{x}_2\overline{x}_3$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \overline{x}_1\overline{x}_2 x_3$	$M_1 = x_1 + x_2 + \overline{x}_3$
2	0	1	0	$m_2 = \overline{x}_1 x_2\overline{x}_3$	$M_2 = x_1 + \overline{x}_2 + x_3$
3	0	1	1	$m_3 = \overline{x}_1 x_2 x_3$	$M_3 = x_1 + \overline{x}_2 + \overline{x}_3$
4	1	0	0	$m_4 = x_1\overline{x}_2\overline{x}_3$	$M_4 = \overline{x}_1 + x_2 + x_3$
5	1	0	1	$m_5 = x_1\overline{x}_2 x_3$	$M_5 = \overline{x}_1 + x_2 + \overline{x}_3$
6	1	1	0	$m_6 = x_1 x_2\overline{x}_3$	$M_6 = \overline{x}_1 + \overline{x}_2 + x_3$
7	1	1	1	$m_7 = x_1 x_2 x_3$	$M_7 = \overline{x}_1 + \overline{x}_2 + \overline{x}_3$

The function is 1 for these rows

The function is 0 for these rows

Minterms and Maxterms

$$f(x_1, x_2, x_3) = \sum m(0, 2, 4, 5, 6, 7) = \prod M(1, 3)$$

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x}_1\overline{x}_2\overline{x}_3$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \overline{x}_1\overline{x}_2 x_3$	$M_1 = x_1 + x_2 + \overline{x}_3$
2	0	1	0	$m_2 = \overline{x}_1 x_2\overline{x}_3$	$M_2 = x_1 + \overline{x}_2 + x_3$
3	0	1	1	$m_3 = \overline{x}_1 x_2 x_3$	$M_3 = x_1 + \overline{x}_2 + \overline{x}_3$
4	1	0	0	$m_4 = x_1\overline{x}_2\overline{x}_3$	$M_4 = \overline{x}_1 + x_2 + x_3$
5	1	0	1	$m_5 = x_1\overline{x}_2 x_3$	$M_5 = \overline{x}_1 + x_2 + \overline{x}_3$
6	1	1	0	$m_6 = x_1 x_2\overline{x}_3$	$M_6 = \overline{x}_1 + \overline{x}_2 + x_3$
7	1	1	1	$m_7 = x_1 x_2 x_3$	$M_7 = \overline{x}_1 + \overline{x}_2 + \overline{x}_3$

$$f(x_1, x_2, x_3) = \prod M(1, 3) = (x_1 + x_2 + \bar{x}_3).(x_1 + \bar{x}_2 + \bar{x}_3)$$

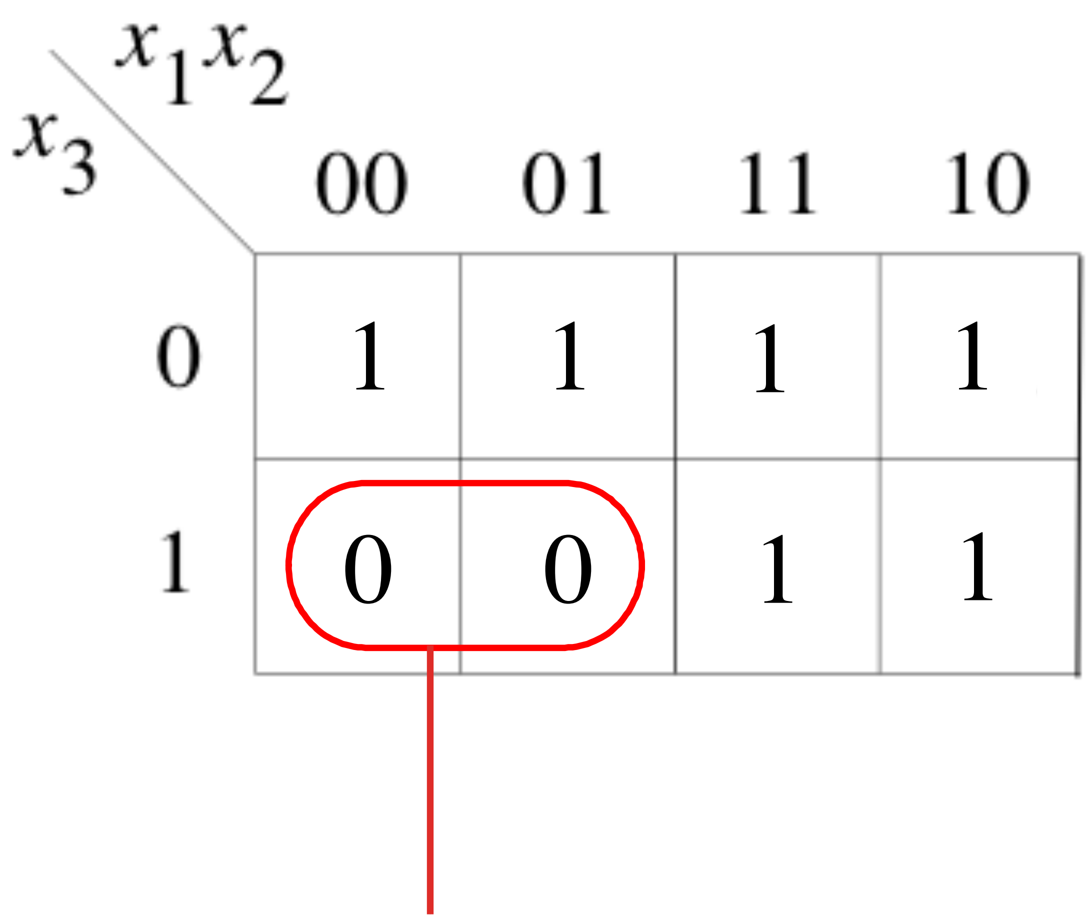
The K-Map

$x_3 \backslash x_1 x_2$	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

The K-Map

$x_3 \backslash x_1 x_2$	00	01	11	10
0	1	1	1	1
1	0	0	1	1

The K-Map - Solution



$(x_1 + \overline{x_3})$

F = 0 when x1=0 AND x3=1

F = 1 when x1=1 OR x3=0
(De Morgan’s theorems)

4 Variable Map

1 square = 1 term with 4 literals

2 adjacent squares = 1 term with 3 literals

4 adjacent squares = 1 term with 2 literals

8 adjacent squares = 1 term with 1 literal

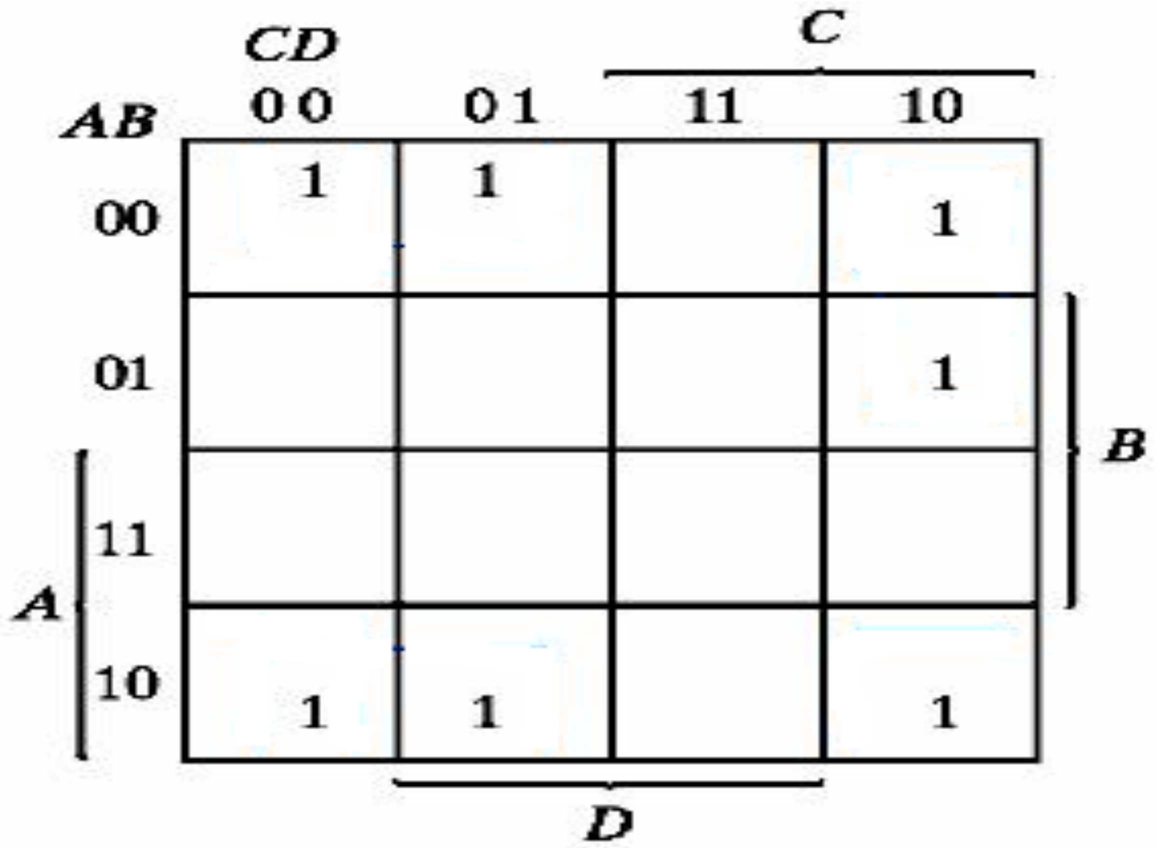
16 adjacent squares = 1

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6
m_{12}	m_{13}	m_{15}	m_{14}
m_8	m_9	m_{11}	m_{10}

				y		
	wx \ yz	00	01	11	10	
	00	$w'x'y'z'$	$w'x'y'z$	$w'x'yz$	$w'x'yz'$	
	01	$w'xy'z'$	$w'xy'z$	$w'xyz$	$w'xyz'$	x
w	11	$wxy'z'$	$wxy'z$	$wxyz$	$wxyz'$	
	10	$wx'y'z'$	$wx'y'z$	$wx'yz$	$wx'yz'$	
				z		

4 Variable Map Example

$$F = \sum m(0, 1, 2, 6, 8, 9, 10)$$

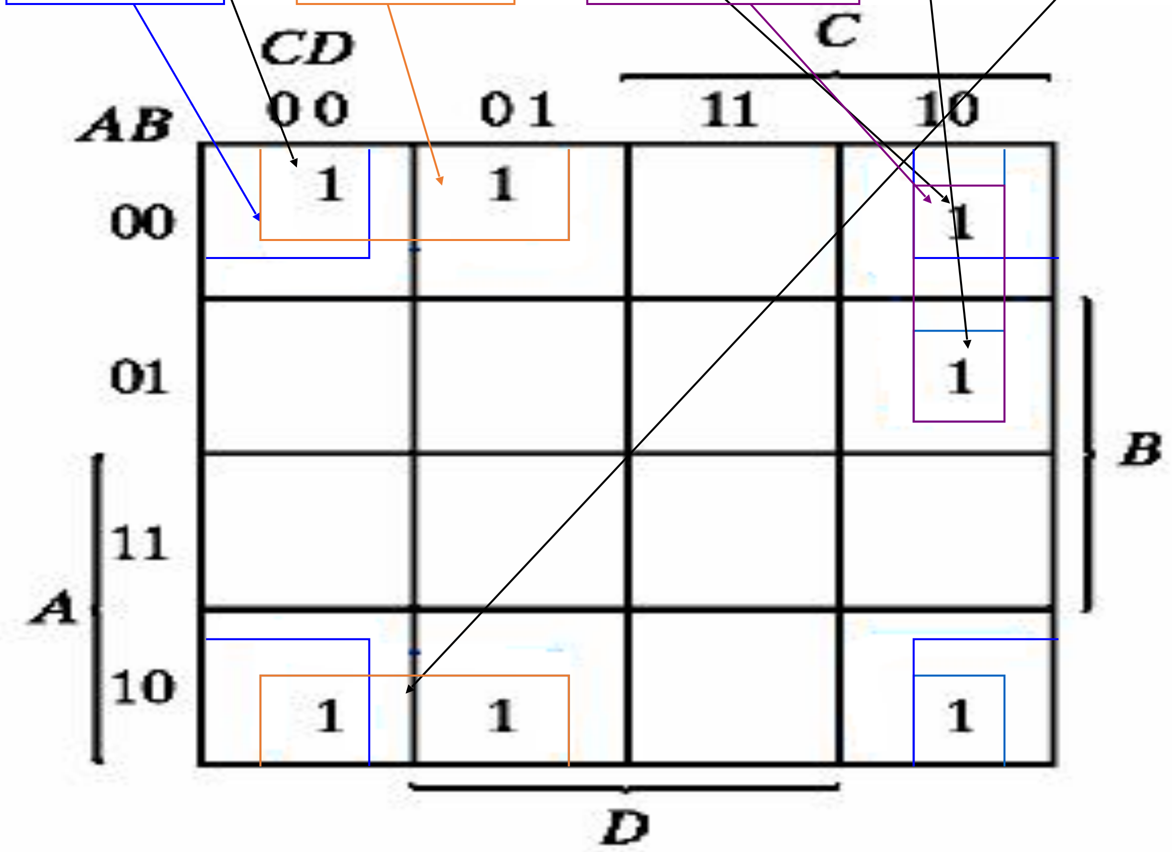


4 Variable Map Example

$$F = \sum m(0, 1, 2, 6, 8, 9, 10)$$

F = A'B'C' + B'CD' + A'BCD' + AB'C'

= B'D' + B'C' + A'CD'



Prime Implicant

- A prime implicant is a product term obtained by combining the maximum possible number of adjacent squares in the map.

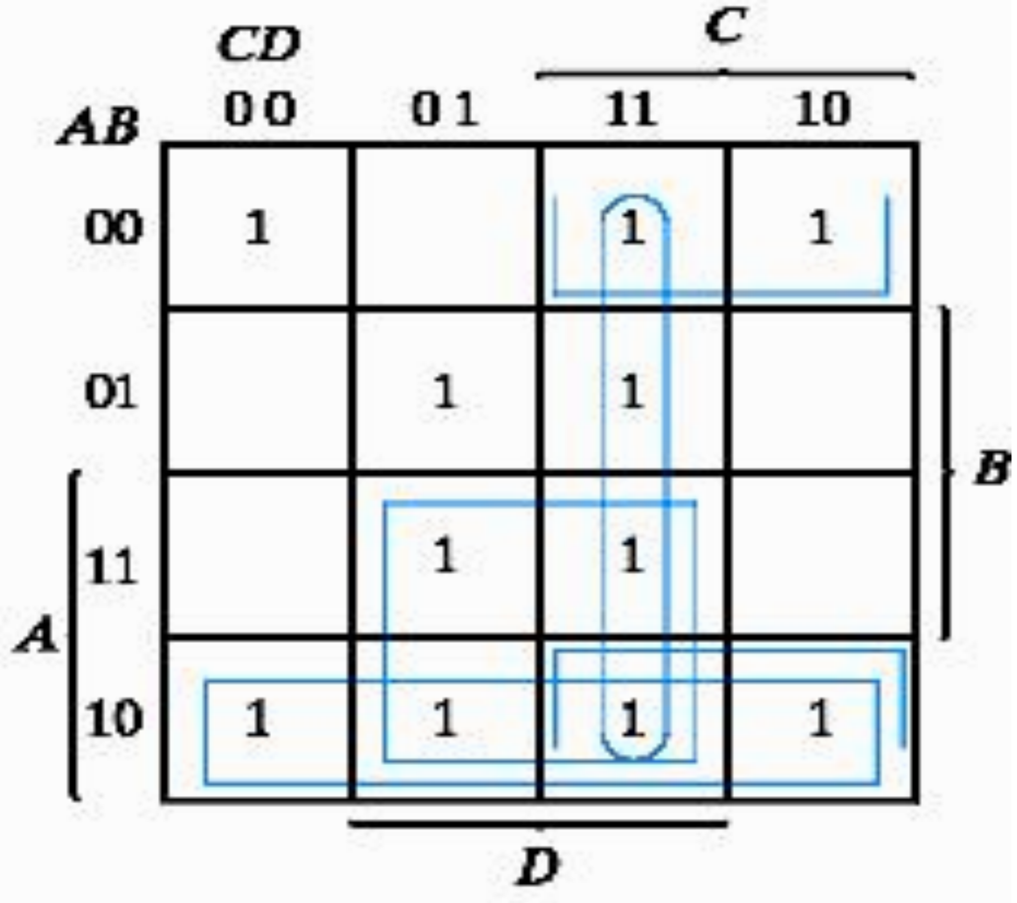
- This shows all possible ways that the three minterms (m_3,m_9,m_{11}) can be covered with prime implicants.

F = BD+B'D'+CD+AD

= BD+B'D'+CD+AB'

= BD+B'D'+B'C+AD

= BD+B'D'+B'C+AB'



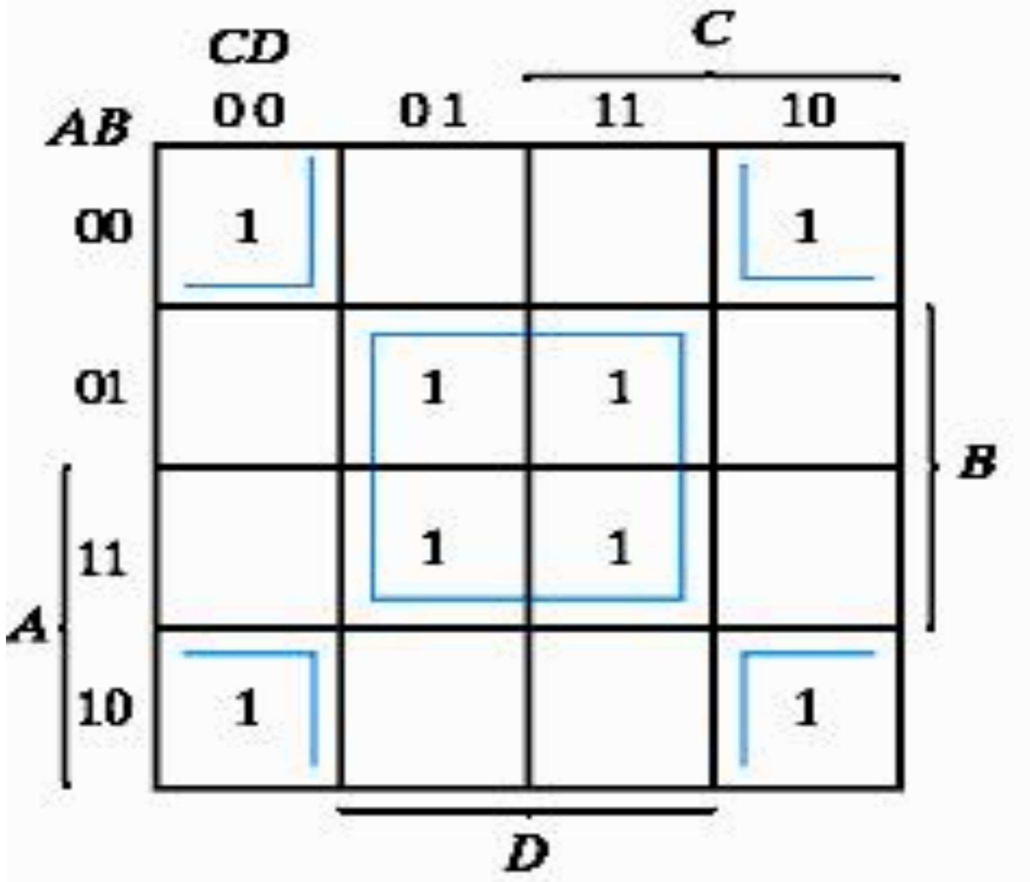
(b) Prime implicants CD, B'C AD, and AB'

Note: A Prime implicant is never completely covered by another prime implicant.

Essential Prime Implicant

- If a minterm in a square is covered by only one prime implicant, then the prime implicant is said to be essential.

- F = BD+B'D'



(a) Essential prime implicants BD and B'D'