MTL-100 Lec-5

Theorem: Let $(a_n)_{n=1}^{\infty}$ be a sequence such that $a_n > 0$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \in \mathbb{R}$.

Then $\lim_{n\to\infty} a_n^{1/n} = L$.

Proof: Let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, $\exists N \in \mathbb{N}$ s.t.

$$L - \frac{\varepsilon}{2} < \frac{a_{n+1}}{a_n} < L + \frac{\varepsilon}{2} \quad \forall n \geq N \quad \text{—(i)}$$

Let $m \geq N+1$. Taking $n = N, N+1, \dots, m-1$ in (i) and multiplying, we get

$$\left(L-\frac{\varepsilon}{2}\right)^{m-N} < \frac{\cancel{a_{N+1}}}{a_N} \cdot \frac{\cancel{a_{N+2}}}{\cancel{a_{N+1}}} \cdots \frac{a_m}{\cancel{a_{m-1}}} < \left(L+\frac{\varepsilon}{2}\right)^{m-N}$$

$$\Rightarrow \left(L-\frac{\varepsilon}{2}\right)^{m-N} < \frac{a_m}{a_N} < \left(L+\frac{\varepsilon}{2}\right)^{m-N} \quad \forall m \geq N+1$$

$$\Rightarrow \underbrace{a_N^{1/m}\left(L-\frac{\varepsilon}{2}\right)^{1-\frac{N}{m}}}_{\downarrow \text{ as } m\to\infty \;\; \left(L-\frac{\varepsilon}{2}\right)} < a_m^{1/m} < \underbrace{a_N^{1/m}\left(L+\frac{\varepsilon}{2}\right)^{1-\frac{N}{m}}}_{\downarrow \text{ as } m\to\infty \;\; \left(L+\frac{\varepsilon}{2}\right)} \quad \text{—(ii)} \quad \forall m \geq N+1$$

$\therefore \exists M \in \mathbb{N}$ s.t. $a_N^{1/m}\left(L+\frac{\varepsilon}{2}\right)^{1-N/m} < L+\frac{\varepsilon}{2}+\frac{\varepsilon}{2} = L+\varepsilon$

$$a_N^{1/m}\left(L-\frac{\varepsilon}{2}\right)^{1-N/m} > L-\frac{\varepsilon}{2}-\frac{\varepsilon}{2} = L-\varepsilon \quad \forall m \geq M \quad \text{—(iii)}$$

So, if $m \geq \max\{N+1, M\}$, then by (ii) & (iii),

$$L-\varepsilon < a_m^{1/m} < L+\varepsilon$$

$$\Rightarrow \lim_{m\to\infty} a_m^{1/m} = L.$$

Corollary: ① If $a_n > 0 \ \forall n$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then $\lim_{n\to\infty} a_n = 0$

② If $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$, then $\lim_{n\to\infty} a_n = \infty$.

Proof: ① Since $L < 1$, we can choose a real number l s.t. $L < l < 1$.

Now by the previous thm,

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

So, if we choose $\varepsilon = l - L > 0$, we can find $N \in \mathbb{N}$ s.t.

$$n \geq N \implies a_n^{1/n} < L + \varepsilon = l$$

$$\implies 0 < a_n < l^n \quad \forall n \geq N.$$

Since $l < 1$, $\lim_{n\to\infty} l^n = 0$.

By the Sandwich thm., $\lim_{n\to\infty} a_n = 0$.

② Left as an exercise.

Examples: ① $\lim_{n\to\infty} n^{1/n} = 1$.

② If $\alpha \in \mathbb{R}$ and $|x| < 1$, then

$$\lim_{n\to\infty} n^{\alpha} x^{n} = 0.$$

Solution: ① Let $a_n = n > 0 \ \forall n$.

Then $\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1$ as $n\to\infty$

$\therefore$ By the prev. thm, $\lim_{n\to\infty} a_n^{1/n} = 1$

i.e. $\lim n^{1/n} = 1$.

② Let $a_n = n^{\alpha} x^{n}$.

To show: $\lim_{n\to\infty} a_n = 0$

$|a_n| = n^{\alpha} |x|^{n}$. Assume $x \neq 0$

Then $|a_n| > 0$.

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Also, $$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^\alpha |x|^{n+1}}{n^\alpha |x|^n} = \left(1+\frac{1}{n}\right)^\alpha |x|$$

$$\Rightarrow \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| < 1$$

By Corollary 1, $\lim_{n\to\infty} |a_n| = 0$

$$\Rightarrow \lim_{n\to\infty} a_n = 0 .$$

Limit superior and limit inferior

Definition: Let (a_n) be a bounded sequence.
The limit superior of the sequence (a_n), denoted by $\limsup\limits_{n\to\infty} a_n$, is defined as

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$

Note that if we write $\alpha_n = \sup\{a_k : k \geq n\}$, then $(\alpha_n)_{n=1}^{\infty}$ is a non-increasing seq.

$$\alpha_1 = \sup\{a_1, a_2, a_3, \cdots\}$$
$$\alpha_2 = \sup\{a_2, a_3, a_4, \cdots\}$$
$$\alpha_3 = \sup\{a_3, a_4, \cdots\}$$
$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$$

$\Downarrow$

$$\lim_{n\to\infty} \alpha_n = \inf_{n\geq 1} \alpha_n$$

$\therefore \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \alpha_n \quad , \alpha_n = \sup\{a_k : k \geq n\}$

$= \inf\{\alpha_n : n \geq 1\}$

$\therefore \limsup_{n\to\infty} a_n = \inf_{n\geq 1} \sup\{a_k : k \geq n\}$

Similarly, $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \underbrace{\inf\{a_k : k \geq n\}}_{\beta_n}$

$= \lim_{n\to\infty} \beta_n$

(β_n) is a nondecreasing seq.

$\therefore \lim_{n\to\infty} \beta_n = \sup_{n\geq 1} \beta_n$

$\therefore \liminf_{n\to\infty} a_n = \sup_{n\geq 1} \inf\{a_k : k \geq n\}.$

Examples:

① $(a_n) = (0,1,0,1,0,1,\dots)$

$\alpha_n = \sup\{a_k : k \geq n\} = 1 \quad \forall n$

$\beta_n = \inf\{a_k : k \geq n\} = 0 \quad \forall n.$

$\therefore \quad \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \alpha_n = 1$

$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \beta_n = 0.$

② $(a_n) = \left(\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \frac{1}{4}, \frac{4}{5}, \dots\right)$

$a_{2n-1} = \frac{1}{n+1}, \quad n = 1,2,3,\dots$

$a_{2n} = \frac{n+1}{n+2}, \quad n = 1,2,3,\dots$

Then $\alpha_n = \sup\{a_k : k \geq n\}$

$$\geq a_{2n} = \frac{n+1}{n+2}$$

Also, since $a_n < 1 \ \forall n$, $\alpha_n \leq 1$

$\therefore \quad \frac{n+1}{n+2} \leq \alpha_n \leq 1 \quad \forall n \in \mathbb{N}$.

By sandwich thm, $\lim_{n\to\infty} \alpha_n = 1$

i.e. $\boxed{\limsup_{n\to\infty} a_n = 1}$

Now, $\beta_n = \inf\{a_k : k \geq n\} \leq a_{2n-1} = \frac{1}{n+1}$

Also, since $a_k > 0 \ \forall n$, $\beta_n \geq 0$

$\therefore \quad 0 \leq \beta_n \leq \frac{1}{n+1} \quad \forall n$.

$\Rightarrow \quad \lim_{n\to\infty} \beta_n = 0$ i.e. $\boxed{\liminf_{n\to\infty} a_n = 0}$

Theorem: Let (a_n) & (b_n) be bounded sequences. Then

① $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$.

② If $a_n \leq b_n \; \forall n \in \mathbb{N}$, then

$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.

and $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

③ $\limsup_{n\to\infty} (a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

and $\liminf_{n\to\infty} (a_n+b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.

The proof is left as an exercise.

Remark: The inequalities in (3) may be strict.

Consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$ for $n \in \mathbb{N}$.

$$\liminf_{n\to\infty} a_n = -1 = \liminf_{n\to\infty} b_n$$

$$\limsup_{n\to\infty} a_n = 1 = \limsup_{n\to\infty} b_n$$

But, $a_n + b_n = 0 \quad \forall n.$

$\therefore$

$$\limsup_{n\to\infty} (a_n + b_n) = 0 < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

$$\liminf_{n\to\infty} (a_n + b_n) = 0 > \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

Theorem: Suppose (a_n) is a convergent sequence and $\lim_{n\to\infty} a_n = L$. Then $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$.

Proof: Let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty} a_n = L$, $\exists N \in \mathbb{N}$ st.

$$L - \frac{\varepsilon}{2} < a_n < L + \frac{\varepsilon}{2} \quad \forall n \geq N.$$

Now, $\alpha_n = \sup\{a_k : k \geq n\}$

Then if $n \geq N$, $\quad a_k < L + \frac{\varepsilon}{2} \quad \forall k \geq n.$

$$\Rightarrow \quad \sup\{a_k : k \geq n\} \leq L + \frac{\varepsilon}{2}.$$

ie. $\quad \alpha_n \leq L + \frac{\varepsilon}{2} < L + \varepsilon \quad \forall n \geq N.$

Also, $\quad \alpha_n \geq a_n > L - \frac{\varepsilon}{2} \quad \forall n \geq N.$

$$\therefore \quad L - \varepsilon < \alpha_n < L + \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \alpha_n = L.$$

Similarly, we can prove that $\liminf_{n\to\infty} a_n = L$.

(Exercise).

Theorem: If $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L \in \mathbb{R}$,

then $\lim_{n\to\infty} a_n = L$.

Proof: Let $\varepsilon > 0$ be given.

Since $L = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \alpha_n$,

where $\alpha_n = \sup\{a_k : k \geq n\}$.

$\exists\ N_1 \in \mathbb{N}$ s.t. $L-\varepsilon < \alpha_n < L+\varepsilon \quad \forall n \geq N_1$.

Now, $a_n \leq \alpha_n < L+\varepsilon \quad \forall n \geq N_1$ —(i).

$\beta_n = \inf\{a_k : k \geq n\} \leq a_n$.

$\liminf_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} \beta_n = L$.

$\therefore \exists N_2 \in \mathbb{N}$ s.t $L-\varepsilon < \beta_n < L+\varepsilon \quad \forall n \geq N_2$.

$\therefore \quad a_n \geq \beta_n > L-\varepsilon \quad \forall n \geq N_2$ — (ii)

By (i) & (ii), $\quad L-\varepsilon < a_n < L+\varepsilon \quad \forall n \geq \max\{N_1, N_2\}$

$\Rightarrow \quad \lim_{n\to\infty} a_n = L$.

Example: Show that $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{1}{k!} = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n$.

Solution: Let $a_n = \sum\limits_{k=0}^{n} \frac{1}{k!}$ & $b_n = \left(1+\frac{1}{n}\right)^n$.

$$b_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= 2 + \sum_{k=2}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}\cdot\frac{1}{n^k}$$

$$= 2 + \sum_{k=2}^{n} \frac{1}{k!}\, 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$\leq \sum_{k=0}^{n} \frac{1}{k!} = a_n$$

$$\Rightarrow \quad \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty} a_n \qquad \text{——} (i)$$

Now fix $m \in \mathbb{N}$. Then for $n \geq m$

$$b_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{1}{n^k}$$

$$\geq \sum_{k=0}^{m} \binom{n}{k} \frac{1}{n^k} = 2 + \sum_{k=2}^{m} \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$= 2 + \sum_{k=2}^{m} \frac{1}{k!} \underbrace{\left(1-\frac{1}{n}\right)}_{\downarrow \, 1} \cdots \underbrace{\left(1-\frac{k-1}{n}\right)}_{\downarrow \, 1} .$$

as $n \to \infty$.

As $n \to \infty$, the R.H.S. of the above $\to 2 + \sum_{k=2}^{m} \frac{1}{k!}$

$$\therefore \quad \liminf_{n\to\infty} b_n \geq 2 + \sum_{k=2}^{m} \frac{1}{k!} = \sum_{k=0}^{m} \frac{1}{k!} = a_m$$

This is true for every $m \in \mathbb{N}$.

$$\therefore \quad \liminf_{n\to\infty} b_n \geq a_n \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow \quad \liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n \quad \text{— (ii)}$$

By (i) & (ii), we get

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n \le \limsup_{n\to\infty} a_n \quad \text{— (iii)}$$

Now, clearly $a_n = \sum_{k=0}^{n} \frac{1}{k!}$ is an increasing seq. and $a_n < 3 \quad \forall n \in \mathbb{N}$.

$\therefore \quad \lim_{n\to\infty} a_n$ exists.

$\therefore$ We must have equalities in (iii).

Hence, $\lim_{n\to\infty} b_n = \lim a_n$.