Electromagnetic Waves in Linear (Dielectric) Media

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References

- Introduction to Electrodynamics, David J. Griffiths (3rd ed.)
    - Chapter 9, 9.3 Electromagnetic Waves in Linear Media

Linear Dielectric Media

When is a dielectric medium called linear?

- A dielectric medium is called linear if its polarization is linearly proportional to the (weak) electric field

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

- In the nonlinear regime, higher order terms become important and are useful for several optical applications.

$$P = \epsilon_0 \chi_e^{(1)} E + \epsilon_0 \chi_e^{(2)} E^2 + \epsilon_0 \chi_e^{(3)} E^3 + ...$$

- The second order term ($\chi_e^{(2)} E^2$) is responsible for second harmonic generation etc.
- The third order term ($\chi_e^{(3)} E^3$) is responsible for third harmonic generation, intensity dependent refractive index etc.

Linear Dielectric Media

When is a dielectric medium called linear?

- Optical fibers are well known examples where third order nonlinearity is at play.
- For linear dielectrics the electric displacement is also a linear function of electric field.

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi_e)\mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E}$$

  where $\epsilon_r = 1 + \chi_e = \epsilon/\epsilon_0$ is called the relative permittivity or dielectric constant of the material.
- Glass is a very common example of a linear dielectric material, other examples are dilute gases.

Maxwell's Equations in Matter: Recap

- Maxwell's equations in a non-conducting medium read,

$$
\begin{aligned}
\nabla \cdot \mathbf{D} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}
$$

Maxwell's Equations in Matter: Recap

- For a linear medium $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$, so the Maxwell's equations can be written as

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu\epsilon \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

Electromagnetic Wave Equation

- Now if we take curl of the curl equation of E (Faraday's law), and use the Gauss's law

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right)$$

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})$$

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- Similarly, one can derive the wave equation for magnetic field **B** (HW).

Electromagnetic Wave Equation

- On comparison with the standard form of wave equation, we can identify the speed of the propagation of an electromagnetic wave in a linear medium as

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu\epsilon/\mu_0\epsilon_0}} = \frac{c}{n}$$

where $n = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$ is called the refractive index of the medium.

Electromagnetic Wave Equation

- For most materials $\mu \cong \mu_0$ so

$$n \cong \sqrt{\epsilon/\epsilon_0} = \sqrt{\epsilon_r} \Rightarrow v = \frac{c}{\sqrt{\epsilon_r}}$$

- ϵ_r is called the dielectric constant of the medium and is mostly greater than one.
- Therefore EM waves have a speed less than 'c' while traveling in a medium.

Plane Wave Solutions

- The plane wave solutions in the complex notation can be written as for the vacuum case:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 \; e^{i(kz-\omega t)}$$
$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 \; e^{i(kz-\omega t)}$$

where $\tilde{\mathbf{E}}_0 = \mathbf{E}_0 e^{i\delta}$ and $\tilde{\mathbf{B}}_0 = \mathbf{B}_0 e^{i\delta}$.

Plane Wave Solutions

- The solutions must satisfy $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$.
- This condition leads to the following

$$\hat{\mathbf{k}} \cdot \mathbf{E} = 0; \hat{\mathbf{k}} \cdot \mathbf{B} = 0$$

- Electromagnetic waves are transverse waves.
- Now from Faraday's law

$$\nabla \times \tilde{\mathbf{E}} = -\frac{\partial \tilde{\mathbf{B}}}{\partial t}$$

$$\textit{one obtains} \; \boxed{\tilde{\mathbf{B}}_0 = \frac{k}{\omega}\left(\hat{k} \times \tilde{\mathbf{E}}_0\right) = \frac{1}{v}\left(\hat{k} \times \tilde{\mathbf{E}}_0\right)}$$

- Fields **E** and **B** are mutually perpendicular to each other.

Electromagnetic Plane Waves in Linear Media

To summarize the charateristics of EM plane waves in linear media

- They travel in the medium with a speed less than the speed of light.
- They are transverse waves as $\hat{k} \cdot \mathbf{E} = \hat{k} \cdot \mathbf{B} = 0$ for a plane EM wave propagating in arbitrary direction.
- Electric and magnetic fields associated with an EM wave in a linear media are perpendicular to each other.

Energy, Momentum and Intensity of EM Waves

- One can obtain the energy density, momentum density as well as intensity of EM wave in a linear media, by making following substitutions in the vacuum case expressions.

$$\epsilon_0 \Rightarrow \epsilon; \mu_0 \Rightarrow \mu; c \Rightarrow v \tag{1}$$

- The averaged energy density for an EM wave in a linear medium

$$\langle u \rangle = \frac{1}{2}\epsilon E_0^2$$

- Similarly, the Poynting vector will be

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu}(\mathbf{E} \times \mathbf{B}) \\ &= v\epsilon E_0^2 cos^2(kz - \omega t + \delta)\hat{z} \end{aligned}$$

Energy, Momentum and Intensity of EM Waves

- The averaged momentum density

$$\langle p \rangle = \frac{1}{2v} \epsilon E_0^2 \hat{k}$$

- And the intensity of the EM wave

$$I = \langle S \rangle = \frac{1}{2} \epsilon v E_0^2$$

Reflection and Transmission of EM Waves: Boundary Conditions

- The boundary conditions at the interface between two non-conducting linear dielectric media having permittivities ϵ_1 and ϵ_2, and permeabilities μ_1 and μ_2, respectively.

$$(i)\ \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$$

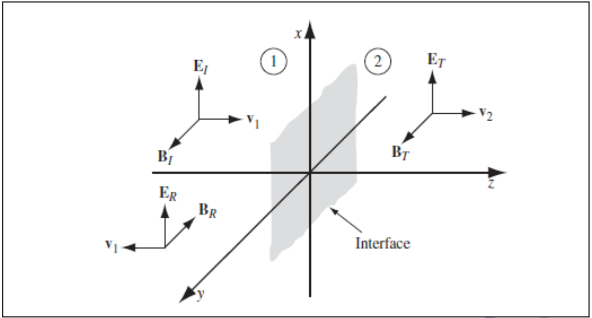
$$(ii)\ B_1^{\perp} = B_2^{\perp}$$

$$(iii)\ \mathbf{E}_1^{||} = \mathbf{E}_2^{||}$$

$$(iv)\ \frac{1}{\mu_1}\mathbf{B}_1^{||} = \frac{1}{\mu_2}\mathbf{B}_2^{||}$$

Reflection+Transmission of EM Waves: Normal Incidence

- Consider a plane wave of frequency ω, having polarization in x-direction, is traveling in z-direction in a linear medium '1'. It encounters an interface (in xy-plane) between medium '1' and another linear medium '2'.
- Depending on the refractive indices of two media, a part of the incident wave energy gets reflected while the remaining part gets transmitted.



Reflection+Transmission of EM Waves: Normal Incidence

- The electric and magnetic fields for incident, reflected and transmitted waves:

Incident Wave

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x}; \qquad \tilde{\mathbf{B}}_I(z,t) = \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{y}$$

Reflected Wave

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x}; \qquad \tilde{\mathbf{B}}_R(z,t) = -\frac{\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)} \hat{y}$$

Transmitted Wave

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{x}; \qquad \tilde{\mathbf{B}}_T(z,t) = \frac{\tilde{E}_{0T}}{v_2} e^{i(k_2 z - \omega t)} \hat{y}$$

Reflection+Transmission of EM Waves: Normal Incidence

- At the interface electric and magnetic fields will satisfy following boundary conditions:

$$E_1^{||} = E_2^{||}$$
$$\frac{B_1^{||}}{\mu_1} = \frac{B_2^{||}}{\mu_2}$$

- The field components normal to the interface plane are zero in case of normal incidence.
- In terms of incident, reflected and transverse components, the above conditions read

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$
$$\frac{1}{\mu_1}(\tilde{B}_{0I} + \tilde{B}_{0R}) = \frac{1}{\mu_2}\tilde{B}_{0T}$$

Reflection+Transmission of EM Waves: Normal Incidence

- Writing B in terms of E

$$\frac{1}{\mu_1}\left(\frac{\tilde{E}_{0I}}{v_1} - \frac{\tilde{E}_{0R}}{v_1}\right) = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}$$

$$\Rightarrow \tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde{E}_{0T} = \beta\tilde{E}_{0T}$$

where

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

- Solving for electric field components one obtains

$$\boxed{\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I} \; and \; \tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I}}$$

Reflection+Transmission of EM Waves: Normal Incidence

- As for most of the linear media $\mu \approx \mu_0$, therefore

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} \approx \frac{\mu_0 v_1}{\mu_0 v_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

- Therefore, we can also write the reflected and transmitted field components in terms of v_1 and v_2 as follows

$$\boxed{\tilde{E}_{0R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{0I} \text{ and } \tilde{E}_{0T} = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{E}_{0I}}$$

- Also in terms of refractive indices of the two media, n_1 and n_2

$$\boxed{\tilde{E}_{0R} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)\tilde{E}_{0I} \text{ and } \tilde{E}_{0T} = \left(\frac{2n_1}{n_1 + n_2}\right)\tilde{E}_{0I}}$$

Reflection+Transmission of EM Waves: Normal Incidence

$$\boxed{\tilde{E}_{0R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{0I} \ \textit{and} \ \tilde{E}_{0T} = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{E}_{0I}}$$

Phases of Reflected and Transmitted Components

- *Transmitted wave always remains in phase with the incident wave.*
- *Reflected wave is in phase with the incident wave if* $v_2 > v_1$ *(or* $n_1 > n_2$*, wave traveling from denser to rarer medium).*
- *Reflected wave is* 180° *out of phase with the incident wave if* $v_1 > v_2$ *(or* $n_2 > n_1$*, wave traveling from rarer to denser medium).*

Reflection+Transmission of EM Waves: Normal Incidence

Reflection and Transmission Coefficients

- Reflection coefficient is the ratio of the reflected intensity to the incident intensity, i.e.

$$\boxed{R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$

- Similarly, transmission coefficient is the ratio of the transmitted intensity to the incident intensity, i.e.

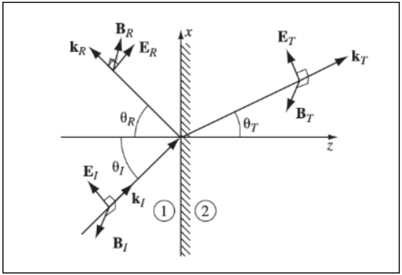
$$\boxed{T = \frac{I_T}{I_I} = \left(\frac{\epsilon_2 v_2}{\epsilon_1 v_1}\right)\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{4 n_1 n_2}{(n_1 + n_2)^2}}$$

- It can be easily verified that as required by conservation of energy

$$\boxed{R + T = 1}$$

Reflection+Transmission of EM Waves: Oblique Incidence

- Consider a plane wave of frequency ω, incident at angle θ_I to the normal of the interface between media'1' and '2'.
- x-z plane is the *plane of incidence* as it contains $\mathbf{k}_I$, $\mathbf{k}_R$, and the normal to the interface ($\hat{z}$).
- A part of the incident wave gets reflected and the remaining part gets refracted (transmitted at an angle different from the angle of incidence).



Reflection+Transmission of EM Waves: Oblique Incidence

- The electric and magnetic fields for incident, reflected and transmitted waves:

Incident Wave

$$\tilde{\mathbf{E}}_I(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}; \qquad \tilde{\mathbf{B}}_I(\mathbf{r}, t) = \frac{1}{v_1}(\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I)$$

Reflected Wave

$$\tilde{\mathbf{E}}_R(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)}; \qquad \tilde{\mathbf{B}}_R(\mathbf{r}, t) = \frac{1}{v_1}(\hat{\mathbf{k}}_R \times \tilde{\mathbf{E}}_R)$$

Transmitted Wave

$$\tilde{\mathbf{E}}_T(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}; \qquad \tilde{\mathbf{B}}_T(\mathbf{r}, t) = \frac{1}{v_2}(\hat{\mathbf{k}}_T \times \tilde{\mathbf{E}}_T)$$

Reflection+Transmission of EM Waves: Oblique Incidence

- Now invariance of the frequency ω leads to,

$$\omega = k_I v_1 = k_R v_1 = k_T v_2$$
$$\Rightarrow k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

- And the electric field and magnetic field must satisfy appropriate boundary conditions at the interface between two media.

Reflection+Transmission of EM Waves: Oblique Incidence

- Writing field components parallel and perpendicular to the interface,

$$
\begin{aligned}
&(i) && \epsilon_1\left(\tilde{E}_I^{\perp}+\tilde{E}_R^{\perp}\right)=\epsilon_2\tilde{E}_T^{\perp}\\
&(ii) && \tilde{B}_I^{\perp}+\tilde{B}_R^{\perp}=\tilde{B}_T^{\perp}\\
&(iii) && \tilde{\mathbf{E}}_I^{||}+\tilde{\mathbf{E}}_R^{||}=\tilde{\mathbf{E}}_T^{||}\\
&(iv) && \frac{1}{\mu_1}\left(\tilde{\mathbf{B}}_I^{||}+\tilde{\mathbf{B}}_R^{||}\right)=\frac{1}{\mu_2}\tilde{\mathbf{B}}_T^{||}
\end{aligned}
$$

Reflection+Transmission of EM Waves: Oblique Incidence

- Using the traveling wave form for incident, reflected, as well as transmitted components

$$\epsilon_1 \left(\tilde{E}^{\perp}_{I0} e^{i(\mathbf{k}_I.\mathbf{r}-\omega t)} + \tilde{E}^{\perp}_{R0} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} \right) = \epsilon_2 \tilde{E}^{\perp}_{T0} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$$

$$\tilde{B}^{\perp}_{I0} e^{i(\mathbf{k}_I.\mathbf{r}-\omega t)} + \tilde{B}^{\perp}_{R0} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} = \tilde{B}^{\perp}_{T0} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{E}}^{||}_{I0} e^{i(\mathbf{k}_I.\mathbf{r}-\omega t)} + \tilde{\mathbf{E}}^{||}_{R0} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} = \tilde{\mathbf{E}}^{||}_{T0} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$$

$$\frac{1}{\mu_1} \left(\tilde{\mathbf{B}}^{||}_{I0} e^{i(\mathbf{k}_I.\mathbf{r}-\omega t)} + \tilde{\mathbf{B}}^{||}_{R0} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} \right) = \frac{1}{\mu_2} \tilde{\mathbf{B}}^{||}_{T0} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$$

- They all seem to have a generic form, at $z = 0$

$$()\, e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} + ()\, e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} = ()\, e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$$

- As boundary conditions must hold for all times at each point on the interface ($z = 0$), the exponents must be equal, i.e.,

$$(\mathbf{k}_I \cdot \mathbf{r} - \omega t) = (\mathbf{k}_R \cdot \mathbf{r} - \omega t) = (\mathbf{k}_T \cdot \mathbf{r} - \omega t)$$
$$\Rightarrow \ \mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}$$

- Incident, reflected and transmitted wave vectors lie in the same plane called the '*plane of incidence*'.

Reflection+Transmission of EM Waves: Oblique Incidence

Law of Reflection

- For incident and reflected wave,

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r}$$
$$k_I sin\theta_I = k_R sin\theta_R$$
$$sin\theta_I = sin\theta_R$$

$$\Rightarrow \boxed{\theta_I = \theta_R}$$

Snell's Law of Refraction

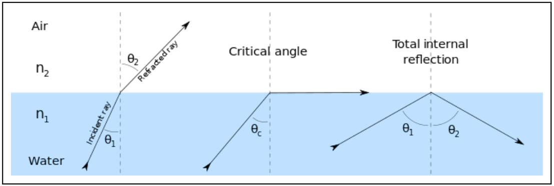
- For incident and transmitted wave,

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}$$
$$k_I sin\theta_I = k_T sin\theta_T$$
$$n_1 sin\theta_I = n_2 sin\theta_T$$

$$\Rightarrow \boxed{\frac{sin\theta_I}{sin\theta_T} = \frac{n_2}{n_1}}$$

Reflection+Transmission of EM Waves: Oblique Incidence

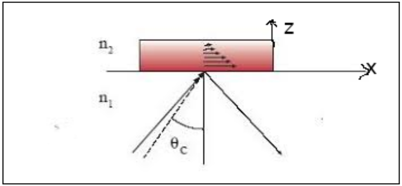
- Now from Snell's law $sin\ \theta_T = \frac{n_1}{n_2} sin\ \theta_I$.
- For $n_1 > n_2$ if $\theta_I = sin^{-1}(n_2/n_1)$ then $sin\ \theta_T = 1$, i.e., $\theta_T = \pi/2$. This is called the critical angle of incidence, θ_c.
- The wave incident at critical angle propagates along the interface between the two media.



Reflection+Transmission of EM Waves: Oblique Incidence

- Also $cos^2\theta_T = 1 - sin^2\theta_T = -\frac{n_1^2}{n_2^2}\left(sin^2\theta_I - sin^2\theta_c\right)$
- For $\theta_I > \theta_c$, $cos\ \theta_T$ will become imaginary. The incident wave completely reflects into medium '1'. This is called 'Total internal reflection (TIR)'.
- An evanescent wave is generated in the rarer medium '2', propagates along the interface and gets attenuated along z-direction [1]

$$\mathbf{E}_T = \mathbf{E}_{T0} e^{-\alpha z} e^{i(k_T x - \omega t)} \ where\ \alpha = \frac{\omega}{c}\sqrt{n_1^2 sin^2\theta_I - n_2^2}$$



[1] See OPTICS by Ajoy Ghatak (7^{th} ed.), sec.23.6,23.9.

Reflection+Transmission of EM Waves: Oblique Incidence

- Now to obtain the amplitudes of reflected and trasmitted (refracted) wave for a given amplitude of the incident wave, let us consider an incident wave polarized in the plane of incidence (x-z plane).
- This means $B^{\perp} = B_z = 0$, $\mathbf{B}^{||} = B_y \hat{y}$ and $\mathbf{E}^{||} = E_x \hat{x}$.
- From boundary condition (i)

$$\epsilon_1 \left(\tilde{E}_{I0} + \tilde{E}_{R0} \right)_z = \epsilon_2 \left(\tilde{E}_{T0} \right)_z$$

$$\epsilon_1 \left(-\tilde{E}_{I0} sin\ \theta_I + \tilde{E}_{R0} sin\ \theta_R \right) = -\epsilon_2 \tilde{E}_{T0} sin\ \theta_T$$

$$\Rightarrow\ \tilde{E}_{I0} - \tilde{E}_{R0} = \beta \tilde{E}_{T0}$$

$$where\ \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

Reflection+Transmission of EM Waves: Oblique Incidence

- Similarly from boundary condition (iii)

$$\left(\tilde{E}_{I0} + \tilde{E}_{R0}\right)_x = \left(\tilde{E}_{T0}\right)_x$$
$$\left(\tilde{E}_{I0} cos\ \theta_I + \tilde{E}_{R0} cos\ \theta_R\right) = \tilde{E}_{T0} cos\ \theta_T$$
$$\Rightarrow\ \tilde{E}_{I0} + \tilde{E}_{R0} = \alpha \tilde{E}_{T0}$$
$$where\ \alpha = \frac{cos\ \theta_T}{cos\ \theta_I}$$

Fresnel's Equations

- Solving for electric field components

$$\boxed{\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0I} \; \textit{and} \; \tilde{E}_{0T} = \left(\frac{2}{\alpha + \beta}\right)\tilde{E}_{0I}}$$

- These are called the Fresnel's equations[a] for the polarization in the plane of incidence.
- Similarly, one can obtain the following Fresnel's equations for the case of polarization perpendicular to the plane of incidence (HW).

$$\boxed{\tilde{E}_{0R} = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)\tilde{E}_{0I} \; \textit{and} \; \tilde{E}_{0T} = \left(\frac{2}{1 + \alpha\beta}\right)\tilde{E}_{0I}}$$

[a] As α is a function of θ_I, reflected and transmitted wave amplitudes also depend on θ_I.

Special Cases

- **Normal Incidence**: For $\theta_I = 0$, $\alpha = 1$

$$\boxed{\tilde{E}_{0_R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0_I} \; and \; \tilde{E}_{0_T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0_I}}$$

- **Grazing Incidence**: For $\theta_I \approx \pi/2$, $\alpha = \infty$, leading to total reflection.

$$\boxed{\tilde{E}_{0_R} = \tilde{E}_{0_I} \; and \; \tilde{E}_{0_T} = 0}$$

- **Incidence at Brewster's Angle**: If p-polarized wave is incident at Brewster's angle, it gets completely transmitted.

$$\boxed{\tilde{E}_{0_T} = \tilde{E}_{0_I} \; and \; \tilde{E}_{0_R} = 0}$$

Reflection+Transmission of EM Waves: Oblique Incidence

$$\boxed{\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0I} \;\; \textit{and} \;\; \tilde{E}_{0T} = \left(\frac{2}{\alpha + \beta}\right)\tilde{E}_{0I}}$$

Phases of Reflected and Transmitted Components

- *Transmitted wave always remains in phase with the incident wave.*
- *Reflected wave is in phase with the incident wave if* $\alpha > \beta$.
- *Reflected wave is* 180° *out of phase with the incident wave if* $\beta > \alpha$.

Brewster's Angle

- For $\tilde{E}_{0R} = 0$, $\alpha = \beta$, i.e.

$$\alpha = \frac{\cos\,\theta_T}{\cos\,\theta_I} = \frac{\sqrt{1 - \sin^2\theta_T}}{\sqrt{1 - \sin^2\theta_I}} = \frac{\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I}}{\sqrt{1 - \sin^2\theta_I}} = \beta$$

$$\Rightarrow\ \sin^2\theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2}$$

- As usually $\mu_1 \simeq \mu_2$, so $\beta \simeq n_2/n_1$, $\sin^2\theta_B \simeq \frac{\beta^2}{1+\beta^2}$ and therefore

$$\boxed{\tan\,\theta_B \simeq \frac{n_2}{n_1}}$$

Reflection+Transmission of EM Waves: Oblique Incidence

Brewster's Angle

- For an EM wave incident on glass from air $\theta_B \approx 56^\circ$.
- Also from $\alpha = \beta$ and using Snell's law,

$$\frac{\cos\,\theta_I}{\cos\,\theta_T} = \frac{n_1}{n_2} = \frac{\sin\,\theta_T}{\sin\,\theta_I}$$
$$\Rightarrow\ \sin 2\theta_I = \sin 2\theta_T$$
$$\Rightarrow\ 2\theta_R = \pi - 2\theta_T \quad as\ \theta_I = \theta_R$$
$$\Rightarrow\ \theta_R + \theta_T = \pi/2$$

- Therefore for an EM wave incident at Brewster's angle, reflected and transmitted waves are perpendicular to each other.

Brewster's Angle

- For a s-polarized EM wave incident at Brewster's angle

$$\boxed{\tilde{E}_{0_R} = \left(\frac{1-\beta^2}{1+\beta^2}\right)\tilde{E}_{0_I} \;\; and \;\; \tilde{E}_{0_T} = \left(\frac{2}{1+\beta^2}\right)\tilde{E}_{0_I}}$$

- Thus for an unpolarized EM wave incident at Brewster's angle, reflected light is linearly polarized with its polarization perpendicular to the plane of incidence (s-polarized) whereas the transmitted light is partially polarized.

Reflection+Transmission of EM Waves: Oblique Incidence

Reflection and Transmission Coefficients

- The incident, reflected and transmitted intensities are

$$I_I = \frac{1}{2}\epsilon_1 v_1 E_{0I}^2 cos\theta_I; \;\; I_R = \frac{1}{2}\epsilon_1 v_1 E_{0R}^2 cos\theta_R; \;\; I_T = \frac{1}{2}\epsilon_2 v_2 E_{0T}^2 cos\theta_T$$

- Therefore, reflection and transmission coefficients for a p-polarized incident wave can be written as

$$\boxed{R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2}$$

$$\boxed{T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0R}}{E_{0I}}\right)^2 \frac{cos\ \theta_T}{cos\ \theta_I} = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2}$$

- It can be easily verified that $R + T = 1$, as required by conservation of energy.

Reflection+Transmission of EM Waves: Oblique Incidence

Reflection and Transmission Coefficients

- Similarly, reflection and transmission coefficients for a s-polarized incident wave can be written as

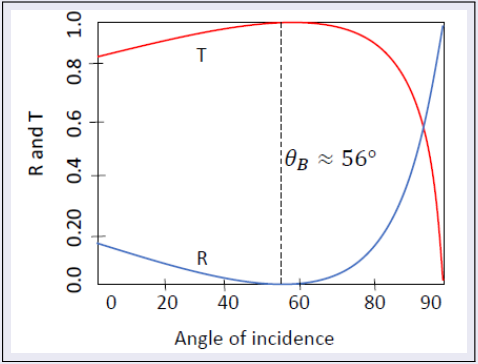
$$\boxed{R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2}$$

$$\boxed{T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0R}}{E_{0I}}\right)^2 \frac{\cos\,\theta_T}{\cos\,\theta_I} = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2}$$

- Here also $R + T = 1$ holds good, as required by conservation of energy.

Reflection and Transmission Coefficients

- As α is a function of the angle of incidence, reflection and transmission coefficients also are and their dependence for p-polarization of incident light look like as shown below



Reflection+Transmission of EM Waves: Oblique Incidence

Reflection and Transmission Coefficients

- For s-polarization of the incident light, the complete transmission does not happen, as is evident from the following plot