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1. An IPL tournament is played between n cricket teams, where each team plays exactly one match with every other team. How many matches are played? (This is easy.) Assume that no match ends in a tie. We say that a subset S of teams is *consistent* if it is possible to order teams in S as $T_1, \ldots, T_{|S|}$ (think of this as the strongest to weakest ordering) such that for every i, j with $1 \leq i < j \leq |S|$, T_i beats T_j. Prove that irrespective of the outcomes of the matches, there always exists a consistent subset S with $|S| \geq \log_2 n$.

2. Call a non-empty subset S of integers *nice* if, for every $x, y \in S$ and every two integers a, b, we have $ax + by \in S$. Observe that the set of multiples of any integer is a nice set. Prove that, in fact, these are the only nice sets. In other words, prove that for every nice set S there exists an integer x such that $S = \{ax \mid a \in \mathbb{Z}\}$. (You might find one of the results from the tutorial useful.)

3. Prove "Claim 2" from the proof of Schröder-Bernstein Theorem discussed in Lecture 5. Here is the statement of the claim. Let A and B be infinite sets, f be an injection from A to B, and g be an injection from B to A. Let $B' = \{b \in B \mid \exists b^* \in B \setminus \mathrm{Im}(f)\ \exists k \in \mathbb{N} \cup \{0\} :\ b = (f \circ g)^k(b^*)\}$, and $A' = \{g(b) \mid b \in B'\}$. Then for every $b \in B$, the following statements are equivalent.

    1. $b \in B'$.
    2. If $f^{-1}(b)$ exists, then it is in A'.
    3. $g(b) \in A'$.

4. Given a set A, the set of finite length strings over A is denoted by A^*. Prove that if A is a finite set, then A^* is necessarily countable. What can you say about the cardinality of A^* if A is countably infinite instead?

5. Prove by mathematical induction that every graph has at least two vertices having equal degree.