Electromagnetic Waves in Vacuum

PYL101: Electromagnetics and Quantum Mechanics

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References

- Introduction to Electrodynamics, David J. Griffiths (3rd ed.)
  - Chapter 9, 9.1 Waves in One Dimension
  - Chapter 9, 9.2 Electromagnetic Waves in Vacuum

Notion of a Wave

What are waves?

Wave: A propagating disturbance that transfers energy through a medium (or vacuum) without transfering any significant amount of mass.

Familiar examples

- Waves on a string/rope
- Sound waves
- Water waves
- Sun light
- Radio waves
- Microwaves

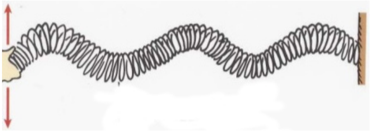
Longitudinal vs Transverse Waves

Longitudinal waves

Displacement of particles happens along the propagation direction.
Ex.: Sound waves, Compression rarefaction waves in a spring, etc.

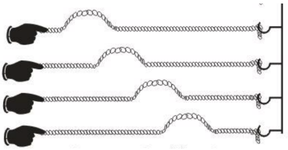
Transverse waves

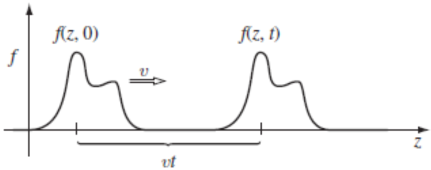
Displacement of particles happens perpendicular to the propagation direction.
Ex.: Waves on a string/rope, Water waves, Radio waves, etc.



Mathematical Representation of a Wave

In the simplest case, a wave represents a disturbance propagating in a medium with fixed shape and constant velocity [1].





Since the shape remains fixed as the wave propagates we have:

displacement at point z, at time t = displacement at point $(z - vt)$, at time 0:

$$f(z,t) \;=\; f(z - vt, 0)$$

[1] In an absorbing medium the amplitude will diminish whereas in a dispersive medium wave will spread with a reduction in amplitude

Mathematical Representation of a Wave

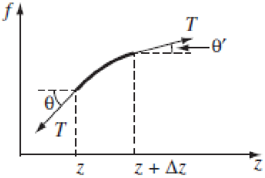
- If function f(z,t) depends on z and t only in a special combination $z - vt$ (or $z + vt$), it represents a wave of fixed shape traveling in the $+z$ direction ($-z$ direction) with speed v.
- Which of the following functions represent waves travelling with speed v ?

$$f(z,t) = Ae^{-b(z-vt)^2}, f(z,t) = Asin[b(z-vt)]$$
$$f(z,t) = Asech[b(x-vt^2)], f(z,t) = Ae^{-b(bz^2+vt)}$$
$$f(z,t) = Asin(bz)cos(bvt), f(z,t) = Atanh[b(z+vt)]$$

- Consider a long horizontal string under tension 'T(z,t)' as it is slightly displaced from its equilibrium with vertical displacement f(z,t).

- The net transverse/vertical force on the segment shown below

$$\Delta F \; = \; T(z+\Delta z) sin\ \theta' \; - \; T(z) sin\ \theta$$



One Dimensional Wave Equation

- For small displacements, θ (and θ') remains also small such that $cos\ \theta \cong 1$ and $sin\ \theta \cong tan\ \theta$, therefore from the balance of horizontal force components,

$$\begin{aligned} T(z+\Delta z)cos\ \theta' &\cong T(z)cos\ \theta \\ \rightarrow T(z+\Delta z) &\cong T(z) \cong T \end{aligned}$$

- The net transverse force acting upwards is therefore,

$$\Delta F \cong T(tan\ \theta' - tan\ \theta)$$

One Dimensional Wave Equation

- If μ is the mass density of the string, mass of the string segment between z and $z + dz$

$$m = \mu \Delta z$$

- Now from Newton's second law

$$\begin{aligned} \Delta F &= m \frac{\partial^2 f}{\partial t^2} \\ &= \mu \Delta z \frac{\partial^2 f}{\partial t^2} \end{aligned}$$

- On the other hand, *tan* θ and tan θ' are the slopes of the string at points z and $z + \Delta z$.
- Therefore one can write

$$\mu \Delta z \frac{\partial^2 f}{\partial t^2} = T \left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z} \right) \cong T \frac{\partial^2 f}{\partial z^2} \Delta z$$

OR

$$\boxed{\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial z^2}}$$

where $v = \sqrt{T/\mu}$ is the speed of the wave propagation. This equation is called the *wave equation*.

Solution of the Wave Equation

- As name suggests, wave equation admits traveling wave solutions having a form

$$f(z,t) \;=\; g(z-vt)$$

- As the wave equation involves square of v, functions of the form $h(z+vt)$ are also solutions of the wave equation and represent waves traveling in the $-z$ direction.

- HW: Verify that these two functions satisfy the one dimensional wave equation.

Solution of the Wave Equation

- As the wave equation is a linear differential equation, its general solution will be a linear combination of the possible solutions,

$$f(z,t) \;=\; A\, g(z-vt) \;+\; B\; h(z+vt)$$

- Standing waves belong to one such class of solutions of the wave equation.

Amplitude, Wavelength, Phase and Frequency

- The most familiar wave solution is the sinusoidal one, given by:

$$f(z,t) \;=\; A\, cos[k(z - vt) + \delta] \;=\; A\, cos[kz - \omega t + \delta]$$

  Here A is the amplitude, k is the wave number[2] and is related to the wavelength as $k = 2\pi/\lambda$, δ is the phase constant, and $\omega = kv = 2\pi v/\lambda$ is the frequency of the wave.

- The entire expression inside the square brackets, i.e., $(kz - \omega t + \delta)$ is called the phase of the wave.

[2] Number of wavelengths within 2π units of length

The Sinusoidal Wave Form : Complex Notation

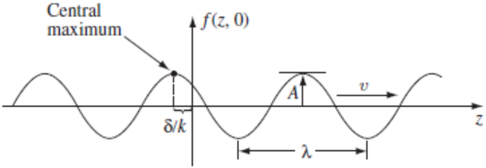
- In complex notation, one can write:

$$f(z,t) = Re\left[A\ e^{i(kz-\omega t+\delta)}\right] = Re\left[\tilde{A}\ e^{i(kz-\omega t)}\right]$$

  where $\tilde{A} = A\ e^{i\delta}$ is the complex amplitude.
- Physical wave function is the real part of the above complex function.
- Complex notation just facilitates mathematical manipulations.

The Sinusoidal Wave Form

- The wave form of the sinusoidal solution discussed above looks like



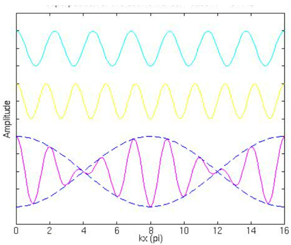
- For $z = vt - \delta/k = \omega t/k - \delta/k$, the phase is zero and can be termed as central maximum.

- δ/k is the distance by which the central maximum is delayed as is evident from the above figure [3].

[3] For a waveform $A\ cos[kz - \omega t - \delta]$, the central maximum is ahead by a distance δ/k

Concept of a Wave Packet

As the wave equation is linear, a linear superposition of two or more sinusoidal solutions is also a solution of the wave equation. Such a coexisting group of waves is called a *wave packet*.



Imagine a case of two waves with same amplitudes but different wavelengths and frequencies,

$$f_1(z,t) = A\ cos[(k+\Delta k)z - (\omega + \Delta\omega)t]$$
$$f_2(z,t) = A\ cos[(k-\Delta k)z - (\omega - \Delta\omega)t]$$

The resultant wave packet is given by (HW)

$$\boxed{f(z,t)\ =\ 2\ A\ cos(kz - \omega t)\ cos[(\Delta k)z - (\Delta\omega)t]}$$

Phase Velocity vs Group Velocity

- Any given wave packet consists of several sinusoidal waves of different wavelengths and frequencies [4].
- The speed of each constituent sinusoidal wave is known as its phase velocity and is given by $v_{ph} = \omega/k$.
- The speed with which the resultant wave packet moves is known as its group velocity and is given by $v_{gr} = \Delta\omega/\Delta k = d\omega/dk$.

[4] the constituent frequencies can be identified by looking at the fourier spectrum (transform) of the wavepacket

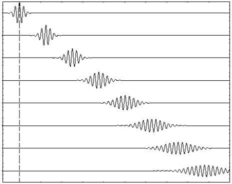
Phase Velocity vs Group Velocity

- Energy/information is transferred with group velocity and not with phase velocity.
- Therefore group velocity of a wave packet can not be larger than the velocity of light else it will violate causality.
- On the other hand, phase velocity of a sinusoidal wave can be larger than the velocity of light without violating causality.

Dispersion of a Wave Packet

- When the phase velocities of the constituent waves of a wave packet depend on their respective wavelengths, the wave packet can not retain its shape as it propagates through the medium [5].

$$v_{ph} = \frac{\omega}{k} = F(k)$$



- This usually happens due to wavelength dependent medium response (refractive index etc).

[5] for example, different wavelengths of VIBGYOR travel at different v_{ph} across the prism resulting in their splitting

Light as Electromagnetic Waves

- Initial contribution (1665-1850) to wave theory of light notably from Christiaan Huygens, Thomas Young and Augustin-Jean Fresnel among others, although based on a hypothetical medium called aether.
- Around 1850, Faraday proposed that light is an electromagnetic vibration that could propagate without a need of any medium (as the ether prosed by Huygens).
- James-Clerk Maxwell inspired by Faraday's work proposed the electromagnetic theory of light around 1864.
- Heinrich Hertz confirmed Maxwell's theory by his laboratory experiments on Radio waves around 1888.

Maxwell's Equations: Recap

Maxwell's equations describing EM Waves in integral form,

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{enc}}{\epsilon_0} \qquad \textit{Gauss's law}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \qquad \textit{Absence of magneic charge}$$

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \phi_B}{\partial t} \qquad \textit{Faraday's law}$$

$$\oint_L \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{\partial \phi_E}{\partial t} \qquad \textit{Mod.Ampere's law}$$

Maxwell's Equations: Recap

Maxwell's equations in differential form read,

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \text{Gauss's law}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{Absence of magneic charge}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's law}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{Mod.Ampere's law}$$

Maxwell's Equations in vacuum: Recap

Maxwell's equations in vacuum read,

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 && \mathit{Gauss's\ law} \\
\nabla \cdot \mathbf{B} &= 0 && \mathit{Absence\ of\ magneic\ charge} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \mathit{Faraday's\ law} \\
\nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} && \mathit{Ampere's\ law\ with\ displacement\ current} \\
& && \mathit{and\ without\ conduction\ current}
\end{aligned}
$$

Electromagnetic Wave Equation

- Now if we take curl of the curl equation of E (Faraday's law), and use the Gauss's law

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

  which has the form of a three dimensional wave equation. Similarly, one can derive the wave equation for magnetic field **B** (HW).

- If you separate the components of these vector equations, each cartesian component of **E** and **B** fields satisfy the three dimensional wave equation.

Electromagnetic Wave Equation

- On comparison with the standard form of wave equation, we can identify the speed of the propagation of an electromagnetic wave as

$$
\begin{aligned}
v^2 = \frac{1}{\mu_0\epsilon_0} &= \frac{4\pi}{\mu_0}\frac{1}{4\pi\epsilon_0} \\
&= \frac{9\times 10^9}{10^{-7}} \\
&= 9\times 10^{16} m^2/s^2
\end{aligned}
$$

$$
v = \frac{1}{\sqrt{\mu_0\epsilon_0}} \;=\; 3.00\times 10^8 \;\; m/s
$$

which is the speed of light.

- Light is an electromagnetic wave !

Plane Wave Solutions

- An EM wave with frequency ω travelling in z-direction and having no dependence on x and y co-ordinates are called monochromatic plane waves. The EM wave equations for such a wave are written as,

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\frac{\partial^2 \mathbf{B}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

- The plane wave solutions in the complex notation can be written as:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 \; e^{i(kz-\omega t)}$$
$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 \; e^{i(kz-\omega t)}$$

where $\tilde{\mathbf{E}}_0 = \mathbf{E}_0 e^{i\delta}$ and $\tilde{\mathbf{B}}_0 = \mathbf{B}_0 e^{i\delta}$.

Plane Wave Solutions

- Now all the solutions of Maxwell's equations must obey the EM wave equations but the converse is not true. To represent an electromagnetic wave, these solutions must satify the Maxwell's equations.
- The solutions must satisfy $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$.

Plane Wave Solutions

- Starting from Gauss's law

$$\nabla \cdot \tilde{\mathbf{E}}(z,t) = 0$$

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\hat{x}\tilde{E}_{0x} + \hat{y}\tilde{E}_{0y} + \hat{z}\tilde{E}_{0z}\right) e^{i(kz-\omega t)} = 0$$

$$\rightarrow ik\tilde{E}_{0z}e^{i(kz-\omega t)} = 0$$

$$\rightarrow \boxed{\tilde{E}_{0z} = E_{0z} = 0}$$

- Similarly, it can be shown that $\nabla \cdot \mathbf{B} = 0$ leads to $\boxed{\tilde{B}_{0z} = B_{0z} = 0}$.

Plane Wave Solutions

- As the wave is propagating in z-direction, $E_{0z} = B_{0z} = 0$ means that electric and magnetic fields are perpendicular to the direction of propagation.
- For a plane wave traveling in arbitrary direction we would have

$$\hat{\mathbf{k}} \cdot \mathbf{E} = 0; \hat{\mathbf{k}} \cdot \mathbf{B} = 0$$

- Therefore electromagnetic waves are transverse waves.

Plane Wave Solutions

- Now from Faraday's law

$$\nabla \times \tilde{\mathbf{E}} = -\frac{\partial \tilde{\mathbf{B}}}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E_{0x}} & \tilde{E_{0y}} & \tilde{E_{0z}} \end{vmatrix} e^{i(kz-\omega t)} = -\frac{\partial}{\partial t}\left(\hat{x}\tilde{B}_{0x} + \hat{y}\tilde{B}_{0y} + \hat{z}\tilde{B}_{0z}\right) e^{i(kz-\omega t)}$$

$$(-\hat{x}\tilde{E}_{0y} + \hat{y}\tilde{E}_{0x})\frac{\partial}{\partial z}e^{i(kz-\omega t)} = -\left(\hat{x}\tilde{B}_{0x} + \hat{y}\tilde{B}_{0y}\right)\frac{\partial}{\partial t}e^{i(kz-\omega t)}$$

$$ik\left(-\hat{x}\tilde{E}_{0y} + \hat{y}\tilde{E}_{0x}\right)e^{i(kz-\omega t)} = i\omega\left(\hat{x}\tilde{B}_{0x} + \hat{y}\tilde{B}_{0y}\right)e^{i(kz-\omega t)}$$

Plane Wave Solutions

- Separating the vector components,

$$-ik\ \hat{x}\tilde{E}_{0y} = i\omega\ \hat{x}\tilde{B}_{0x}$$
$$ik\ \hat{y}\tilde{E}_{0x} = i\omega\ \hat{y}\tilde{B}_{0y}$$

- This leads to the following relations between the E and B components

$$\tilde{B}_{0x} = -\frac{k}{\omega}\tilde{E}_{0y};\ \tilde{B}_{0y} = \frac{k}{\omega}\tilde{E}_{0x};\ \ whereas\ \ \tilde{B}_{0z} = 0$$

- This can be compactly written in the vector form as

$$\boxed{\tilde{\mathbf{B}}_0 = \frac{k}{\omega}\left(\hat{z} \times \tilde{\mathbf{E}}_0\right)}$$

- Fields **E** and **B** are mutually perpendicular to each other.

Plane Wave Solutions

- Similarly from the last Maxwell's equation

$$\nabla \times \tilde{\mathbf{B}} = \mu_0 \epsilon_0 \frac{\partial \tilde{\mathbf{E}}}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{B_{0x}} & \tilde{B_{0y}} & \tilde{B_{0z}} \end{vmatrix} e^{i(kz-\omega t)} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\hat{x} \tilde{E}_{0x} + \hat{y} \tilde{E}_{0y} + \hat{z} \tilde{E}_{0z} \right) e^{i(kz-\omega t)}$$

$$(-\hat{x} \tilde{B}_{0y} + \hat{y} \tilde{B}_{0x}) \frac{\partial}{\partial z} e^{i(kz-\omega t)} = \mu_0 \epsilon_0 \left(\hat{x} \tilde{E}_{0x} + \hat{y} \tilde{E}_{0y} \right) \frac{\partial}{\partial t} e^{i(kz-\omega t)}$$

$$ik \left(-\hat{x} \tilde{B}_{0y} + \hat{y} \tilde{B}_{0x} \right) e^{i(kz-\omega t)} = -i\omega \mu_0 \epsilon_0 \left(\hat{x} \tilde{E}_{0x} + \hat{y} \tilde{E}_{0y} \right) e^{i(kz-\omega t)}$$

Plane Wave Solutions

- Separating the vector components,

$$-ik\ \hat{x}\tilde{B}_{0y} = -i\omega\mu_0\epsilon_0\ \hat{x}\tilde{E}_{0x}$$
$$ik\ \hat{y}\tilde{B}_{0x} = -i\omega\mu_0\epsilon_0\ \hat{y}\tilde{E}_{0y}$$

- This leads to the following relations between the E and B components

$$\tilde{E}_{0x} = \frac{k}{\omega\mu_0\epsilon_0}\tilde{B}_{0y};\ \tilde{E}_{0y} = -\frac{k}{\omega\mu_0\epsilon_0}\tilde{B}_{0x};\ \ whereas\ \ \tilde{E}_{0z} = 0$$

- This can be compactly written in the vector form as

$$\boxed{\tilde{\mathbf{E}}_0 = -\frac{k}{\omega\mu_0\epsilon_0}\left(\hat{z}\times\tilde{\mathbf{B}}_0\right) = -\frac{kc^2}{\omega}\left(\hat{z}\times\tilde{\mathbf{B}}_0\right)}$$

- So we have two relations between E_0 and B_0

$$\tilde{\mathbf{B}}_0 = \frac{k}{\omega}\left(\hat{z} \times \tilde{\mathbf{E}}_0\right)$$
$$\tilde{\mathbf{E}}_0 = -\frac{kc^2}{\omega}\left(\hat{z} \times \tilde{\mathbf{B}}_0\right)$$

- Combining these two we can write

$$\begin{aligned}\tilde{\mathbf{B}}_0 &= -\frac{k}{\omega}\frac{kc^2}{\omega}\left(\hat{z} \times \left(\hat{z} \times \tilde{\mathbf{B}}_0\right)\right)\\ &= -\frac{k^2c^2}{\omega^2}\left(\left(\hat{z} \cdot \tilde{\mathbf{B}}_0\right)\hat{z} - (\hat{z} \cdot \hat{z})\,\tilde{\mathbf{B}}_0\right)\\ &= +\frac{k^2c^2}{\omega^2}\tilde{\mathbf{B}}_0\end{aligned}$$

$$\Rightarrow \boxed{\frac{\omega}{k} = c}$$

Plane Wave Solutions

- The plane wave solutions for a monochromatic EM wave traveling in an arbitrary direction

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \tilde{E}_0 \; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}}$$

$$\tilde{\mathbf{B}}(\mathbf{r}, t) = \frac{\tilde{E}_0}{c} \; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\left(\hat{\mathbf{k}} \times \hat{\mathbf{n}}\right) = \frac{1}{c}\left(\hat{\mathbf{k}} \times \tilde{\mathbf{E}}\right)$$

- The physical solution is the real part of these complex fields

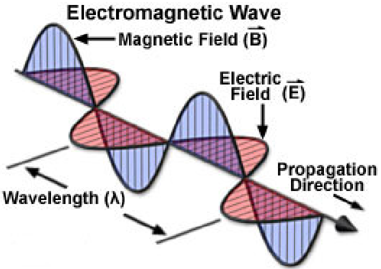
$$\mathbf{E}(\mathbf{r}, t) = E_0 \; cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)\hat{\mathbf{n}}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{E_0}{c} \; cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)\left(\hat{\mathbf{k}} \times \hat{\mathbf{n}}\right) = \frac{1}{c}\left(\hat{\mathbf{k}} \times \mathbf{E}\right)$$

Electromagnetic Plane Waves in Vacuum

To summarize the charateristics of EM plane waves in vacuum

- They travel with the speed of light.
- They are transverse waves as $E_{0z} = B_{0z} = 0$ for a plane EM wave propagating in z-direction.
- Electric and magnetic fields associated with an EM wave are perpendicular to each other.



Electromagnetic Spectrum

The image below summarizes the whole spectrum of electromagnetic waves.

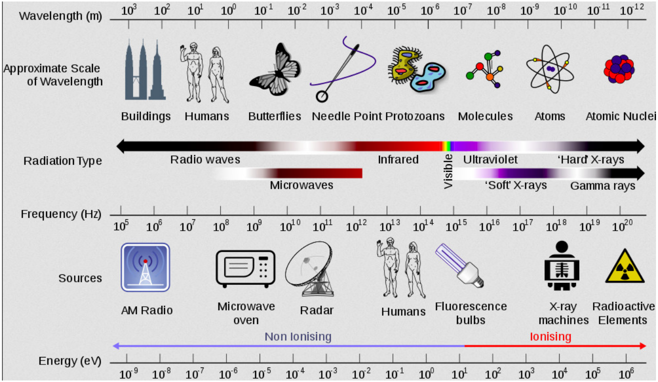


Image Source: Wikimedia Commons

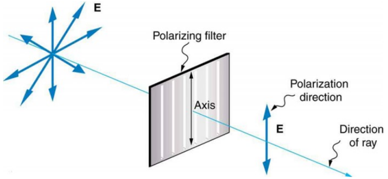
Polarization of Electromagnetic Waves

- An EM wave being a transverse wave, its electric field and magnetic field vectors are allowed to oscillate only in a plane perpendicular to the propagation direction.

- An EM wave is called polarized when the oscillations of the associated electric field vector are further restricted, say to a line (linearly polarized light) or to a circle (circularly polarized light).

Polarization of Electromagnetic Waves

- Most of the light sources, including the Sun, emit unpolarized light.
- The sunlight while getting scattered from the atmosphere gets partially polarized.
- Reflected light also becomes partially polarized[6].
- Unpolarized light can also be made polarized using an optical filter called polarizer.



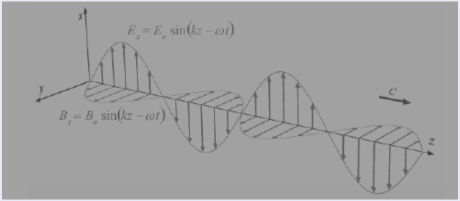
[6] completely polarized if incident at Brewster's angle (will be discussed in later lectures)

Polarization of Electromagnetic Waves

Linear Polarization

- In a linearly polarized EM wave associated electric field vector oscillates along a straight line (direction of polarization).
- An EM wave polarized in x-direction and propagating in z-direction is expressed as

$$\mathbf{E}(z,t) = E_0 cos(kz - \omega t + \delta)\hat{x}$$
$$\mathbf{B}(z,t) = \frac{E_0}{c} cos(kz - \omega t + \delta)\hat{y}$$



Circular Polarization

- In a circularly polarized EM wave the associated electric field vector revolves in a circle in the transverse plane.
- A circularly polarized EM wave is composed of two linearly polarized EM waves with equal amplitudes[a], perpendicular polarizations and a phase difference of $\pi/2$.
- Mathematically, the electric field for a circularly polarized EM wave is given by [b]

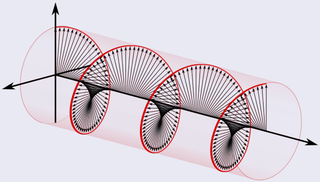
$$\mathbf{E}(z,t) = E_0[cos(kz - \omega t + \delta)\hat{x} - sin(kz - \omega t + \delta)\hat{y}]$$

[a] If amplitudes are different it is called an elliptically polarized EM wave.

[b] Draw for yourself the electric field vectors for phase difference of $+\pi/2$ and $-\pi/2$ to get an idea of handedness (left or right) in circular polarization.

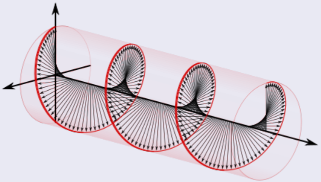
Polarization of Electromagnetic Waves

Left Handed Circularly Polarized



$$\mathbf{E}(z,t) = E_0[cos(kz - \omega t + \delta)\hat{x} - sin(kz - \omega t + \delta)\hat{y}]$$

Right Handed Circularly Polarized



$$\mathbf{E}(z,t) = E_0[cos(kz - \omega t + \delta)\hat{x} + sin(kz - \omega t + \delta)\hat{y}]$$

Energy, Momentum and Intensity of EM Waves

- Electrostatics: the energy per unit volume stored in an electric field:

$$u_e = \frac{1}{2}\epsilon_0 E^2$$

- Magnetostatics: the energy per unit volume stored in a magnetic field:

$$\begin{aligned} u_m &= \frac{1}{2\mu_0} B^2 \\ &= \frac{1}{2\mu_0}(E^2/c^2) = \frac{1}{2\mu_0}\mu_0\epsilon_0 E^2 = \frac{1}{2}\epsilon_0 E^2 \end{aligned}$$

- Therefore for a monochromatic plane wave both electric and magnetic fields contribute equally to the energy density of the wave.

Energy, Momentum and Intensity of EM Waves

- The energy density stored in EM wave in free space is therefore
$$u_{em} = u_e + u_m = \epsilon_0 E^2$$
- Using the electric field expression from the plane wave solution
$$u_{em} = \epsilon_0 E_0^2 cos^2(kz - \omega t + \delta)$$
which is an oscillatory function of time.
- The average energy density associated with the electromagnetic plane wave traveling along z-direction in vacuum [7]
$$\langle u_{em} \rangle = \frac{1}{2} \epsilon_0 E_0^2$$

[7] Averaged over a period of the wave which is 2π.

Energy, Momentum and Intensity of EM Waves

- Poynting vector corresponds to the energy flux density (energy per unit area per unit time) transported by the EM waves and is given by (HW: See 8.1.2 of Griffiths for the derivation)

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

- For a monochromatic wave traveling in vacuum,

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\left(\frac{1}{c}\hat{\mathbf{k}}\times\mathbf{E}\right)\\ &= \frac{1}{\mu_0 c}\mathbf{E}\times\left(\hat{\mathbf{k}}\times\mathbf{E}\right)\\ &= c\epsilon_0\left[\hat{\mathbf{k}}(\mathbf{E}\cdot\mathbf{E}) - \mathbf{E}(\mathbf{E}\cdot\hat{\mathbf{k}})\right]\\ &= c\epsilon_0 E^2\hat{\mathbf{k}}\end{aligned}$$

Energy, Momentum and Intensity of EM Waves

- For a wave propagating along z-direction

$$\mathbf{S} = c\epsilon_0 E_0^2 cos^2(kz - \omega t + \delta)\hat{\mathbf{z}} = cu_{em}\hat{z}$$

- The average energy density transported by a monochromatic EM wave traveling in vacuum shall be

$$\langle \mathbf{S} \rangle = \frac{1}{2} c\epsilon_0 E_0^2 \hat{\mathbf{z}}$$

Energy, Momentum and Intensity of EM Waves

- The momentum density stored in the EM fields (HW: see 8.2.3 of Griffiths)

$$\mathbf{p} = \frac{1}{c^2}\mathbf{S} = \frac{1}{c^2}c\epsilon_0 E_0^2 cos^2(kz - \omega t + \delta)\hat{\mathbf{z}}$$
$$= \frac{1}{c}u_{em}\hat{\mathbf{z}}$$

- The average momentum stored

$$\langle \mathbf{p} \rangle = \frac{1}{2c}\epsilon_0 E_0^2 \hat{\mathbf{z}}$$

- And the intensity (average power per unit area transported) of the EM wave is

$$I = \langle S \rangle = \frac{1}{2}c\epsilon_0 E_0^2$$