COL 352 Introduction to Automata and Theory of Computation

Nikhil Balaji

Bharti 420
Indian Institute of Technology, Delhi
nbalaji@cse.iitd.ac.in

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Lecture 6: Nondeterminism: Epsilon Transitions

Recap

- Languages, Decision problems.
- Finite State Automata - devices with finite memory.
- Deterministic Finite State Automata (DFA): From one state, reading an action we move to exactly one other state.
- Regular languages: L is regular if there exists some DFA A such that $L = L(A)$.
- Closed under Union, Intersection, Complement.

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- Subset Construction: Every NFA has an equivalent DFA.
- Exponential blowup in state complexity unavoidable! NFAs indeed are very concise.
- Question: Can we always make sure a DFA has exactly one final/accepting state?

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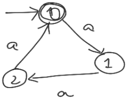

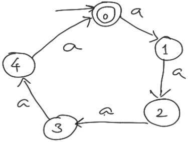

Epsilon Transitions

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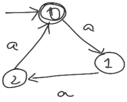

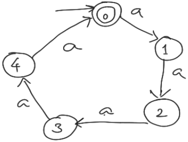

Jump from a state to another without reading any letter.

Such transitions are called ε-transitions.

- How to define them formally?
- Are they more powerful than normal DFA/NFA?
- Usefulness?

Closure under union

Lemma

If L_1 and L_2 are regular then so is $L_1 \cup L_2$

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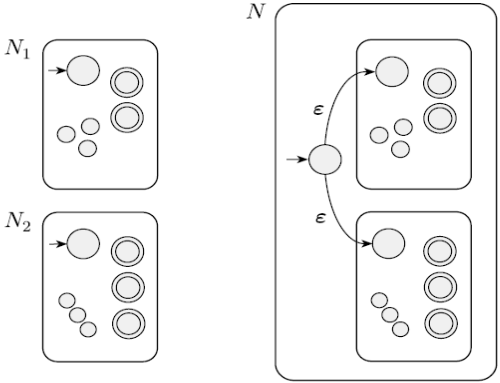

Closure under concatenation

Lemma

If L_1 and L_2 are regular then so is $L_1 \circ L_2$

$$L_1 \circ L_2 = \{w_1 w_2 \in \Sigma^* \mid w_1 \in L1, w_2 \in L_2\}$$

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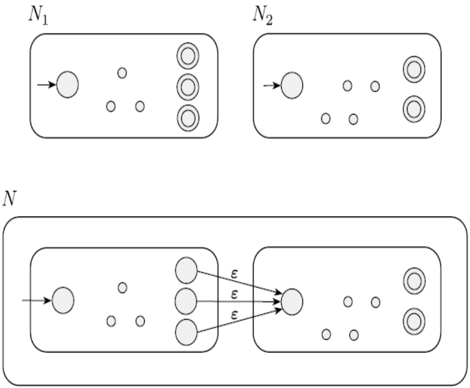

Closure under Kleene star

Lemma

If L is regular then so is L^*.

$$L^* = \{w_1 w_2 \dots w_k \in \Sigma^* \mid k \geq 0 \ \forall i, w_i \in L\}$$

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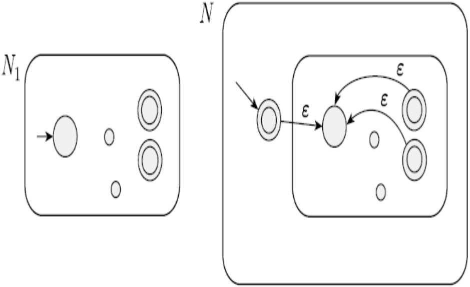

Modelling epsilon transitions

Definition

An ε-nondeterministic finite-state automaton (ε-NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states
- Σ is a finite alphabet, i.e., set of input symbols
- $\delta : Q \times (\Sigma \cup \varepsilon) \to 2^Q$ is a function that takes a state and input symbol and returns the set of possible next states,
- $q_0 \in Q$ is the start/initial state
- $F \subseteq Q$ is the set of final/accepting states.

Epsilon Closure

Definition

Let $(Q, \Sigma, \delta, q_0, F)$ be an ε-NFA. For each set $S \subseteq Q$, $EClose(S)$ is the set of states reachable via ε-transitions from S.

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Extended Transition Relation

Let $\hat{\delta} : Q \times \Sigma^* \to 2^Q$ be defined as:

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Acceptance: An ε-NFA A accepts w iff $\hat{\hat{\delta}}(q_0, w) \cap F \neq \emptyset$.

How Powerful are Epsilon Transitions

Question: Do ε transitions add expressive power to NFAs?

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Answer: No!

Theorem

For any ε-NFA A, there exists an NFA A' (without ε-transitions) such that $L(A) = L(A')$.

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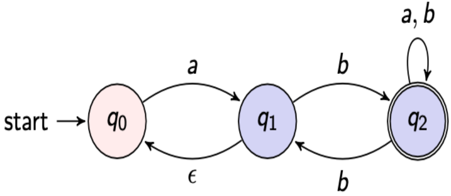

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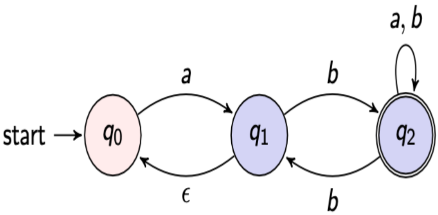

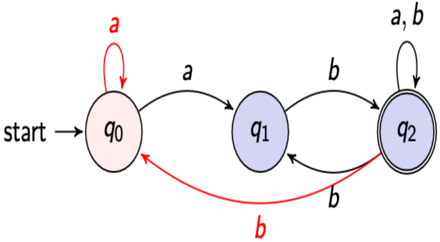

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Proof.

Let $A = (Q, \Sigma, \delta, q_0, F)$ be an ε-NFA. Then, we construct NFA $A = (Q', \Sigma, \delta', q_0', F')$ s.t.,

- $Q' = Q$
- Σ is the same but no -transitions are used now.
- $\delta'(q, a) = EClose(\delta(q, a))$.
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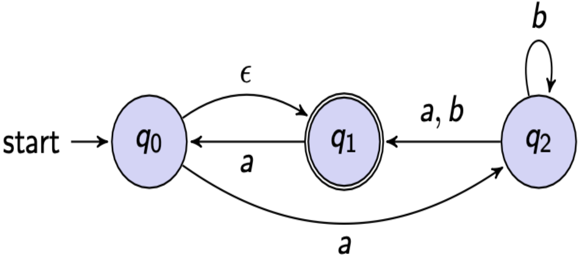
Correctness: $\forall w \in \Sigma^*$, w accepted by A' iff w is accepted by A. Is this always true? What if there are -transitions to start or final state? □

Removing Epsilon Transitions: A Caveat



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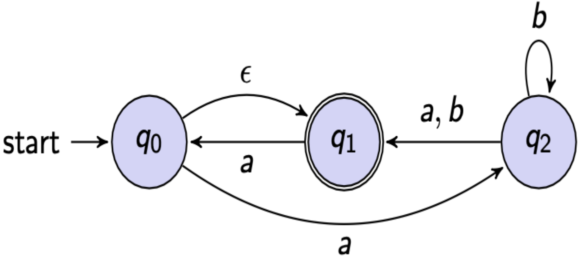

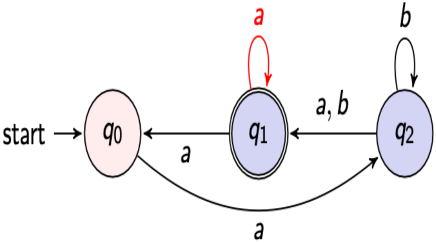

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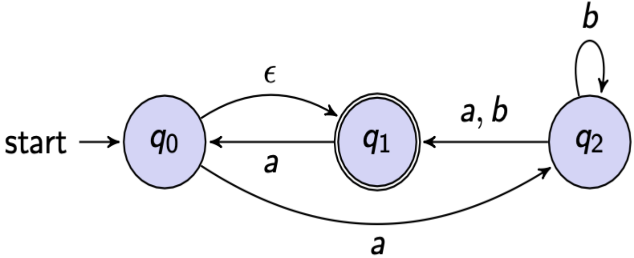

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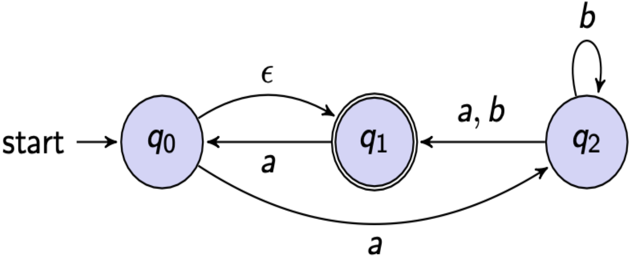

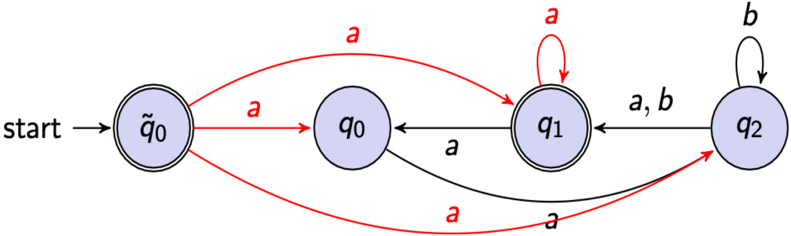

Removing Epsilon Transitions: Fix

- What went wrong?

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- What went wrong?
- Base case was handled incorrectly!
- Need to distinguish between first visit and subsequent visits of q_0.

Proof.

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- $q_0' = \tilde{q}_0$
- $F' = F \cup \{\tilde{q}_0\}$ (if $EClose(\{q_0\}) \cap F \neq \varnothing$) and F (otherwise)
- $\delta'(q, a) = EClose(\delta(EClose(q_0, a)))$ (if $q = \tilde{q}_0$), otherwise $EClose(\delta(q, a))$.

□

Handling Epsilon moves: The Algorithm

Lemma

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Proof Idea.

Construction in 3 steps:

1. **Saturate:** repeatedly add shortcuts that make ϵ-transitions redundant.
2. **Fix final states:** if some state reachable from initial state by ϵ-transitions is final, then make initial state as final!
3. **Remove** ϵ-transitions.