Lecture 15
Signals and Systems (ELL205)

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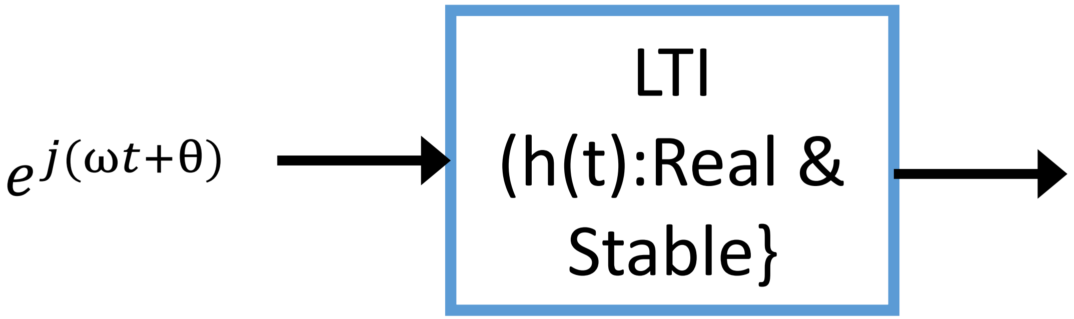
Outline of the lecture

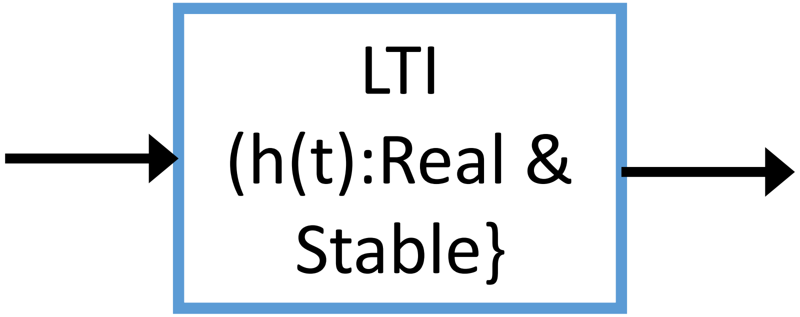
- How an LTI system react to sinusoids?
- How do we break signal as sums of sinusoids?

Output of an LTI system to Complex Exponentials



Output of an LTI system to Complex Exponentials

$e^{j(\omega t+\theta)}$



$$y(t) = e^{j(\omega t+\theta)}H(\omega)$$

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$$

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

Frequency response

Output of an LTI system to Complex Exponentials

$e^{-j(\omega t+\theta)}$ → LTI (h(t):Real & Stable} →

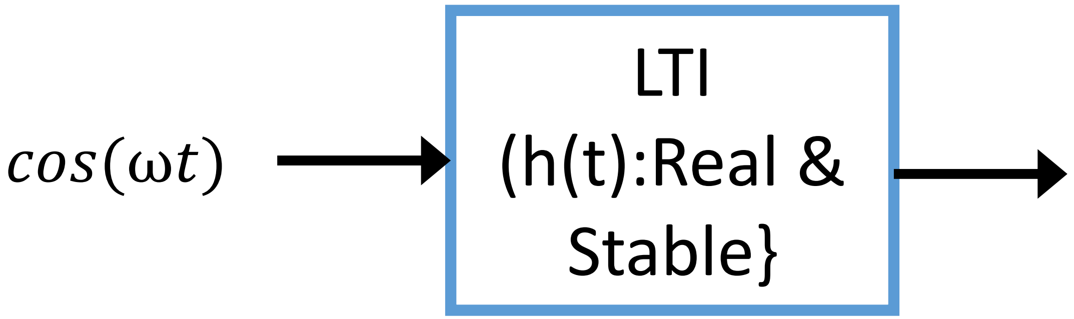
$$y(t) = e^{-j(\omega t+\theta)}H(-\omega)$$

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau < \infty$$

$$H(-\omega) = \int_{-\infty}^{\infty}h(\tau)e^{j\omega\tau}d\tau$$

Frequency response

Output of an LTI system to Complex Exponentials



$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

Output of an LTI system to Complex Exponentials

$$\left\{\frac{e^{j(\omega t)}+e^{-j(\omega t)}}{2}\right\} \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{(h(t):Real \&}\\ \text{Stable\}}\end{array}} \longrightarrow \frac{1}{2}\{e^{j\omega t}H(\omega)+e^{-j\omega t}H(-\omega)\}$$

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty$$

$$H(-\omega)=\int_{-\infty}^{\infty}h(\tau)e^{j\omega\tau}d\tau$$

$$=\overline{\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau}$$

$$=\overline{H(\omega)}$$

Output of an LTI system to Complex Exponentials

$$\{\frac{e^{j(\omega t)}+e^{-j(\omega t)}}{2}\} \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{(h(t):Real \&}\\ \text{Stable\}}\end{array}} \longrightarrow \frac{1}{2}\{e^{j\omega t}H(\omega)+e^{-j\omega t}H(-\omega)\}$$

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau < \infty$$

$$H(-\omega) = \overline{H(\omega)}$$

{Conjugate symmetry}

Output of an LTI system to Complex Exponentials

$$\{\frac{e^{j(\omega t)}+e^{-j(\omega t)}}{2}\}$$

LTI
(h(t):Real &
Stable}

$$\frac{1}{2}\{e^{j\omega t}H(\omega)+e^{-j\omega t}\overline{H(\omega)}\}$$

$$H(-\omega)=\overline{H(\omega)}$$

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty$$

{Conjugate symmetry}

Output of an LTI system to Complex Exponentials

$$\{\frac{e^{j(\omega t)}+e^{-j(\omega t)}}{2}\} \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{(h(t):Real \&}\\ \text{Stable\}}\end{array}} \longrightarrow R_e\{e^{j\omega t}H(\omega)\}$$

$$H(-\omega) = \overline{H(\omega)}$$

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$$

{Conjugate symmetry}

Output of an LTI system to Complex Exponentials

$$\left\{\frac{e^{j(\omega t)}+e^{-j(\omega t)}}{2}\right\} \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{(h(t):Real \&}\\ \text{Stable\}}\end{array}} \longrightarrow R_e\{e^{j\omega t}|H(\omega)|e^{\angle H(\omega)}\}$$

$$H(-\omega) = \overline{H(\omega)}$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

{Conjugate symmetry}

Output of an LTI system to Complex Exponentials

$$\left\{\frac{e^{j(\omega t)}+e^{-j(\omega t)}}{2}\right\}$$

LTI
(h(t):Real &
Stable}

$$|H(\omega)|R_e\{e^{j\omega t+\angle H(\omega)}\}$$

$$H(-\omega) = \overline{H(\omega)}$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

{Conjugate symmetry}

Output of an LTI system to Complex Exponentials

$$cos\omega t \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{(h(t):Real \&}\\ \text{Stable\}}\end{array}} \longrightarrow |H(\omega)|\cos(\omega t + \angle H(\omega))$$

$$H(-\omega) = \overline{H(\omega)}$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

{Conjugate symmetry}

Bounds on $H(\omega)$

When is $H(\omega)$ guaranteed to be bounded?

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$|H(\omega)| = \left| \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right|$$

$$|H(\omega)| \leq \int_{-\infty}^{\infty} \left| h(\tau) e^{-j\omega\tau} \right| d\tau$$

Schwartz inequality

Bounds on $H(\omega)$

When is $H(\omega)$ guaranteed to be bounded?

$$|H(\omega)| \leq \int_{-\infty}^{\infty} \left|h(\tau)e^{-j\omega\tau}\right| d\tau$$

Schwartz inequality

$$|H(\omega)| \leq \int_{-\infty}^{\infty} |h(\tau)|\left|e^{-j\omega\tau}\right| d\tau$$

$$|H(\omega)| \leq \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

Bounds on $H(\omega)$

When is $H(\omega)$ guaranteed to be bounded?

$$|H(\omega)| \leq \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

If system is stable,

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

Thus for a stable system

$$|H(\omega)| < \infty$$

Bounds on $H(\omega)$

Is stability a necessary condition for $H(\omega)$ to be bounded?

No !!.

Outline of the lecture

- Why sinusoids?
- How an LTI system react to sinusoids?
- How do we break signal as sums of sinusoids?

Towards Fourier Series

Jean Baptiste Joseph Fourier

Brief Biography

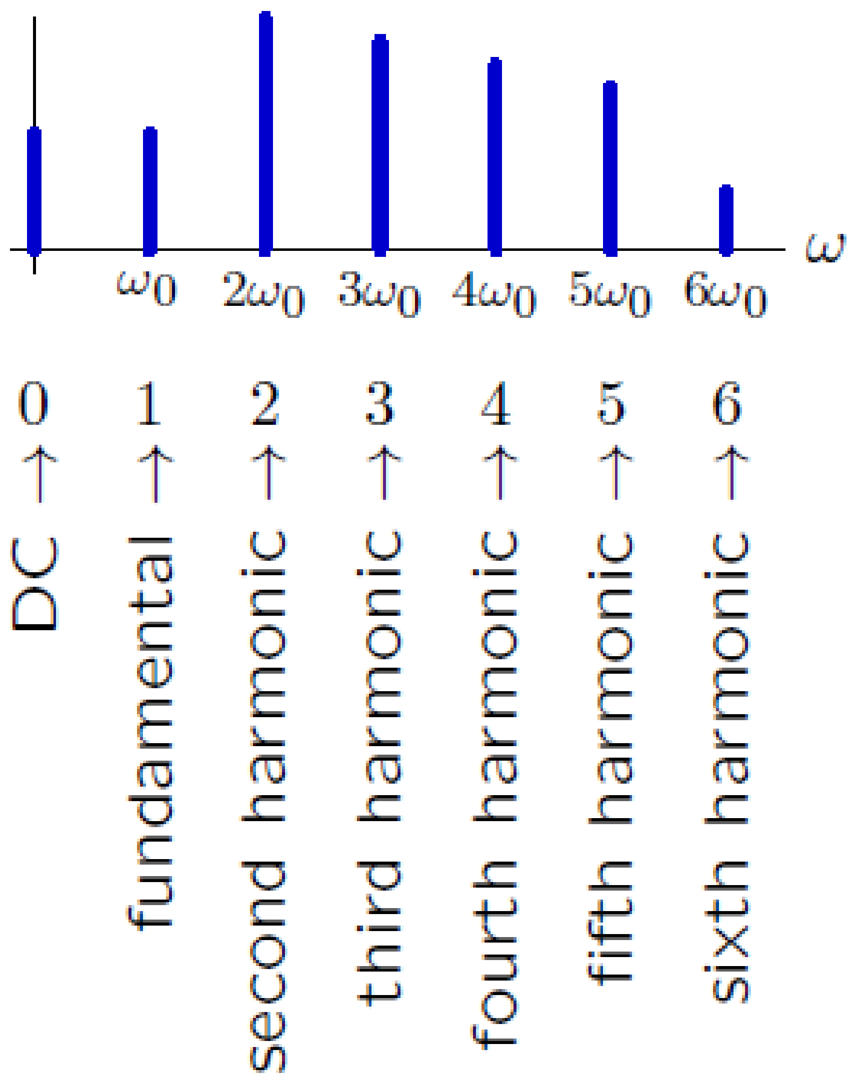
Jean-Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) was a French mathematician and physicist born in Auxerre.
He lived during the time of French revolution and even escaped guillotine.
Fouier was an Egyptologist and accompanied Napolean Bonaparte on his Egyptian expedition.
He is buried in the Père Lachaise Cemetery in Paris.
His name is one of the 72 names inscribed on the Eiffel Tower.
Worked at *École Polytechnique*.
He is best known for Fourier series. Fourier Series is described by many as the most beautiful mathematical poem.

Criticized heavily

S. F. Lacroix
G. Monge
P. S. De Laplace
J. L. Lagrange

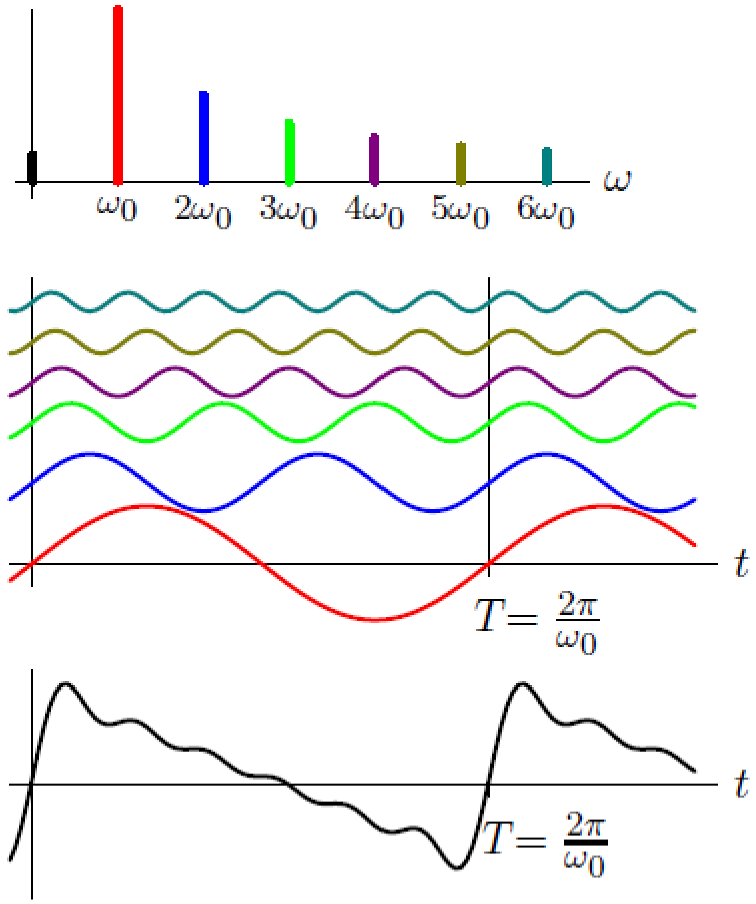
Further reviewed by
Poisson
and Legendre

Harmonics



ω
ω_0
$2\omega_0$
$3\omega_0$
$4\omega_0$
$5\omega_0$
$6\omega_0$
0
1
2
3
4
5
6
DC
fundamental
second harmonic
third harmonic
fourth harmonic
fifth harmonic
sixth harmonic

What signals can be represented by the sum of fundamentals and its harmonics?



Fourier Series

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$\mathrm{y}(t) = y(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} H(k\omega_o)$$

a_k are called Fourier series spectral coefficients

Check it ?

If $x(t)$ is real, what is true about a_k

1)$a_k = \overline{a_{-k}}$	**2)**$a_k = \overline{a_k}$
3)$a_k = a_{-k}$	4)$a_k = -a_{-k}$

Check it ?

If $x(t)$ is real, what is true about a_k

1)$a_k = \overline{a_{-k}}$	**2)**$a_k = \overline{a_k}$
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Check it ?

If $x(t)$ is real, what is true about a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$\overline{x(t)} = \overline{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}} = \sum_{k=-\infty}^{\infty} \overline{a_k} e^{-jk\omega_o t} = \sum_{m=-\infty}^{\infty} \overline{a_{-m}} e^{jm\omega_o t} = \sum_{k=-\infty}^{\infty} \overline{a_{-k}} e^{jk\omega_o t}$$

If $x(t) = \overline{x(t)}$, then $a_k = \overline{a_{-k}}$

Fourier Spectral Coefficients

$$x(t) = x(t+T) = \sum_k a_k e^{jk\omega_o t}$$

Multiplying with $e^{-jl\omega_o t}$ and integrating over one time period

$$\int_T x(t)e^{-jl\omega_o t}dt = \int_T \sum_k a_k e^{jk\omega_o t}\, e^{-jl\omega_o t}dt$$

Changing order of integration and summation and using $\int_T e^{j(k-l)\omega_o t}dt = T\delta[k-l]$ we get

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_o t}dt$$

Analysis and Synthesis equation

Synthesis $$x(t) = x(t+T) = \sum_{k} a_k e^{jk\omega_o t}$$

Analysis $$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\omega_o t}dt$$