- Existence & Uniqueness of the solⁿ of 2^{nd} order linear ODE.

- General solⁿ

Higher Order Linear ODEs

Lets Consider the homogeneous n^{th} order linear ODE

$$x^{(n)} + a_1(t)\,x^{(n-1)} + a_2(t)\,x^{(n-2)} + \cdots + a_n(t)\,x = 0 \quad \text{--- (1)}$$

We can reduce ODE (1) in linear system of first order ODEs by introducing

$$x_1 = x, \quad x_2 = x_1' = x', \quad x_3 = x_2' = x'', \quad \ldots, \quad x_n = x_{n-1}' = x^{(n-1)}$$

Using (1)

$$x_n' = x^{(n)} = -a_1(t)\,x^{(n-1)} - a_2(t)\,x^{(n-2)} \ldots - a_n(t)\,x$$

$$= -a_1(t)\,x_n - a_2(t)\,x_{n-1} \cdots - a_n(t)\,x_1$$

We have,

$$(2) \quad \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \end{cases}$$

$$x_3' = \ldots$$

$$\vdots$$

$$x_{n-1}' = x_n$$

$$x_n' = -a_1(t)\,x_n - a_2(t)\,x_{n-1} \cdots - a_n(t)\,x_1$$

(3) —
$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ \vdots & & & & & \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

Define

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix}_{n\times n}$$

then (3) can be written as

$$X' = AX$$

Initial condition
$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ \vdots \\ x_n(t_0) \end{bmatrix}_{n\times 1} = \underbrace{X(t_0) = X_0}_{} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ \vdots \\ k_n \end{bmatrix}$$

$k_i \in \mathbb{R}$, are given $1 \le i \le n$.

$$x(t_0) = x_1(t_0) = k_1$$

$$x'(t_0) = x_2(t_0) = k_2$$

$$x''(t_0) = x_3(t_0) = k_3$$

$$\vdots$$

$$x^{(n-1)}(t_0) = x_n(t_0) = k_n$$

Initial conditions for (1) are

$$(4) \left\{ \begin{aligned} x(t_0) &= k_1 \\ x'(t_0) &= k_2 \\ &\vdots \\ x^{(n-1)}(t_0) &= k_n \end{aligned} \right.$$

$$\left.\text{ODE (1)} + \text{Initial conditions (4)}\right\} \underline{\underline{\text{IVP}}}$$

<u>Def^n</u> (Solution)

A solution of an n^{th} order ODE on some open interval I is a function $x = h(t)$ which is n-times differentiable on I & satisfies DE (1).

<u>Thm1 (Existence & Uniqueness thm)</u>

If the coefficient f^n's $a_1(t), a_2(t) \cdots a_n(t)$ of DE (1) are continuous & bounded on some open interval I & $t_0 \in I$ then the IVP

$$\text{IVP (1)} \begin{cases} x^{(n)} + a_1(t)\, x^{(n-1)} + \cdots + a_n(t)\, x = 0 \\ x(t_0) = K_1 \\ x'(t_0) = K_2 \\ \vdots \\ x^{(n-1)}(t_0) = K_n \end{cases}$$

has a unique sol^n on I, where $K_i \in \mathbb{R}, 1 \le i \le n$ (given).

Superposition Principle

(verify)

If $x_1, x_2 \cdots x_n$ are sol^{n}s of DE (1) then their linear combination $c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is also a sol^n of DE (1), here c_i's are constants.

(Linear Dependence & Linear Indendence of f^ns)

Def^n The functions $x_1(t), x_2(t) \cdots x_n(t)$ are

L.D. if there exist constants $c_1, c_2 \cdots c_n$, not all zero, such that

$$c_1 x_1(t) + c_2 x_2 + \cdots + c_n x_n = 0.$$

$x_1(t), x_2(t) \cdots x_n(t)$ are L.I. if

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \Rightarrow c_1 = 0, c_2 = 0 \cdots c_n = 0.$$

<u>Def^n</u> (Wronskian) The Wronskian of $x_1(t), x_2(t) \cdots x_n(t)$ is defined by

$$W(x_1, x_2 \cdots x_n)(t) = \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ x_1' & x_2' & \cdots & x_n' \\ \vdots & & & \\ x_1^{(n-1)} & x_2^{(n-1)} & \cdots & x_n^{(n-1)} \end{vmatrix}.$$

<u>Thm2</u> (Abel's Formula) Let $x_1(t), x_2(t) \cdots x_n(t)$ are solutions of DE (1). Then,

$$W(x_1, x_2 \cdots x_n)(t) = c\, e^{-\int a_1(t)\, dt}, \quad c \to \text{constant}.$$

Proof. $n=2 \rightarrow$ already done

for $n=3$

$$x''' + a_1 x'' + a_2 x' + a_3 x = 0 \quad \text{--- (5)}$$

$x_1, x_2, x_3 \rightarrow$ sol^{n}s of DE (5).

thrice differentiable

We have,

$$W(t) = W(x_1, x_2, x_3)(t) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix}$$

$$\frac{d}{dt} W(t) = \underbrace{\begin{vmatrix} x_1' & x_2' & x_3' \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1'' & x_2'' & x_3'' \\ x_1'' & x_2'' & x_3'' \end{vmatrix}}_{0}$$

$$+ \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1''' & x_2''' & x_3''' \end{vmatrix} \checkmark$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ -a_1x_1'' - a_2x_1' - a_3x_1 & -a_1x_2'' - a_2x_2' - a_3x_2 & -a_1x_3'' - a_2x_3' - a_3x_3 \end{vmatrix}$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \end{vmatrix} \quad \begin{vmatrix} x_1 & x_2 & x_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1' & x_2' & x_3' \\ -a_1 x_1'' & -a_1 x_2'' & -a_1 x_3'' \end{vmatrix} + \underbrace{\begin{vmatrix} x_1' & x_2' & x_3' \\ -a_2 x_1' & -a_2 x_2' & -a_2 x_3' \end{vmatrix}}_{=0}$$

$$+ \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ -a_3 x_1 & -a_3 x_2 & -a_3 x_3 \end{vmatrix} = 0$$

$$= -a_1(t) \underbrace{\begin{vmatrix} x_1 & x_2 & x_2 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix}}_{= W(t)}$$

$$\frac{d}{dt} W(t) + a_1(t) W(t) = 0$$ → first order linear ODE

Sol^n is $\boxed{W(t) = c\, e^{-\int a_1(t)\,dt}}$.

<u>Corollary</u> The Wronskian of $x_1, x_2, \dots x_n$ of sol^n of DE ① is either identically equal to zero or never zero.

$$W(t) = c\, e^{-\int a_1(t)\,dt}$$

Never zero

- if $c = 0$ then $W(t) \equiv 0$

- If $c \neq 0$ then $W(t)$ is never zero.

Thm Let $x_1, x_2 \cdots x_n$ be sol^{n}s of ①.

Then $x_1(t), x_2(t) \cdots x_n(t)$ are L.D.

$\iff W(x_1, x_2 \cdots x_n)(t) = 0$.

Pf. $(\Rightarrow)$ Suppose $x_1(t), x_2(t) \cdots x_n(t)$ are L.D.

To show that $W(x_1, x_2 \cdots x_n)(t) = 0$.

$\exists$ $c_1, c_2 \cdots c_n$ (constants), not all zero s.t.

$$
(*)\quad \begin{cases}
c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) = 0 & \text{— ⑥} \\
c_1 x_1'(t) + c_2 x_2'(t) + \cdots + c_n x_n'(t) = 0 \\
\vdots \\
c_1 x_1^{(n-1)}(t) + c_2 x_2^{(n-1)}(t) + \cdots + c_n x_n^{(n-1)}(t) = 0
\end{cases}
$$

We have, the system of eq^n

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_1' & x_2' & \cdots & x_n' \\ \vdots & & & \\ x_1^{(n-1)} & x_2^{(n-1)} & \cdots & x_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

has a non-trivial solⁿ.

$$\Rightarrow \qquad \underbrace{\det(\;)}_{\substack{\| \\ W(x_1, x_2 \cdots x_n)(t)}} = 0$$

$(\Leftarrow)$ T.S.T.

$W(t) = W(x_1, x_2 \cdots x_n)(t) = 0 \Rightarrow x_1, x_2 \cdots x_n$ are L.D.

It suffices to show that

$W(t_0) = 0$ for some t_0

$\Rightarrow \underline{x_1, x_2} \cdots x_n$ are L.D.

$$W(t_0) = 0$$
$$\|$$
$$\begin{vmatrix} x_1(t_0) & x_2(t_0) & \cdots & x_n(t_0) \\ \vdots & & & \\ x_1^{(n-1)} & x_2^{(n-1)} & \cdots & x_n^{(n-1)}(t_0) \end{vmatrix}$$

$| x_1 (t_0) \quad x_2(t_0) \quad \cdots \quad x_n (t_0) \ldots$

$\Rightarrow$ the following system of eq^{n}s

$$(*) \quad \underbrace{\begin{bmatrix} x_1(t_0) & x_2(t_0) & \cdots & x_n(t_0) \\ \vdots & & & \\ \vdots & & & \\ x_1^{(n-1)}(t_0) & x_2^{(n-1)}(t_0) & \cdots & x_n^{(n-1)}(t_0) \end{bmatrix}}_{} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has a non-trivial sol^n, say

$c_1 = \alpha_1, \; c_2 = \alpha_2 \; \ldots \; c_n = \alpha_n$ (α_i's are not all zero)

consider the f^n

$$z(t) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$
$$= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

Since $x_1, x_2 \ldots x_n$ are sol^n of DE (1)

$\Rightarrow$ $z(t)$ is also a sol^n of DE (1).

From (*)

$$z(t_0) = c_1 x_1(t_0) + \cdots + c_n x_n(t_0) = 0$$

$$z'(t_0) = c_1 x_1'(t_0) + \cdots + c_n x_n'(t_0) = 0$$

$\vdots$

$$z^{(n-1)}(t_0) = c_1 x_1^{(n-1)}(t_0) + \cdots + c_n x_n^{(n-1)}(t_0) = 0.$$

Thus $z(t)$ is a solⁿ of IVP

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0$$

$$x(t_0) = 0, \quad x'(t_0) = 0 \quad \cdots \quad x^{(n-1)}(t_0) = 0.$$

Note that $x(t) = 0$ is also a solⁿ of same IVP & hence by uniqueness of solⁿ we have

$$z(t) \equiv 0$$

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0$$

$\Rightarrow$ x_i's $i = 1, 2 \cdots n$ are L.D.

Corollary Let $x_1, x_2 \cdots x_n$ be solⁿ's of DE (1).

Then

$x_1, x_2 \cdots x_n$ are L.I. $\Leftrightarrow$ $W(x_1, x_2 \ldots x_n)(t) \not\equiv 0$.

(verify).

Thm Let $x_1(t), x_2(t) \cdots x_n(t)$ are L.I. solⁿ of DE (1) in some open interval I & let $y(t)$ be any solⁿ of (1) then there exists constants $c_1, c_2 \cdots c_n$ s.t.

$$y(t) = c_1 x_1(t) + \cdots \cdots + c_n x_n(t),$$

i.e. any sol^n of (1) belongs to the linear span of $x_1(t), x_2(t) \ldots x_n(t)$.

<u>Proof.</u> Let $y(t)$ be any sol^n of DE (1).

Given: $x_1(t), x_2(t) \ldots x_n(t)$ are L.I. sol^n of DE (1).

$x_1(t), x_2(t), \ldots x_n(t)$ are L.I.

$$\Rightarrow \quad W(x_1, x_2 \ldots x_n)(t) \neq 0$$
$$W(t) =$$

i.e.

$$\exists\ t_0 \in I \quad \text{s.t.} \quad W(t_0) \neq 0$$

$$\| $$

$$\begin{vmatrix} x_1(t_0) & x_2(t_0) & \cdots & x_n(t_0) \\ \vdots & & & \\ x_1^{(n-1)}(t_0) & x_2^{(n-1)}(t_0) & \cdots & x_n^{(n-1)}(t_0) \end{vmatrix}$$

$\Rightarrow$ The system of eq^n

$$\begin{bmatrix} x_1(t_0) & x_2(t_0) & \cdots & x_n(t_0) \\ \vdots & & & \\ x_1^{(n-1)} & x_2^{(n-1)} & & x_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \end{bmatrix} = \begin{bmatrix} y(t_0) \\ y'(t_0) \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

$$x_1^{(n-1)}(t_0) \quad x_2^{(n-1)}(t_0) \quad \cdots \quad x_n^{(n-1)}(t_0) \qquad c_n \qquad y^{(n-1)}(t_0)$$

has a unique sol^n, say.

$$c_1 = \beta_1, \; c_2 = \beta_2 \cdot \cdots \cdot c_n = \beta_n \cdot$$

Define

$$z(t) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Note that $z(t)$ is a sol^n of DE ① since it is a linear combination of sol^n of DE ①.

We have,

$$z(t_0) = c_1 x_1(t_0) + c_2 x_2(t_0) + \cdots + c_n x_n(t_0) = y(t_0)$$

$$z'(t_0) = y'(t_0)$$

$$\vdots$$

$$z^{(n-1)}(t_0) = y^{(n-1)}(t_0).$$

Thus $z(t)$ & $y(t)$ solves the same IVP. Therefore by existence & uniqueness theorm, we have

$$\boxed{y(t) \equiv z(t) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n}.$$

General sol^n involves n arbitrary constants of DE ①

Fundamental set of sol^{n}s $\rightarrow \{x_1, x_2 \cdots x_n\}$

Fundamental set of sol^{n}s

Any set $\{x_1(t), x_2(t) \cdots x_n(t)\}$ of n- linearly independent sol^n of DE ① on some interval (open) I is said to be the fundamental set of sol^{n}s on I.

$\{x_1, x_2 \cdots x_n\} \rightarrow$ fundamental set of sol^{n}s of ①

then general sol^n of ① is

$$\boxed{x(t) = c_1 x_1(t) + \cdots + c_n x_n(t)}$$