Lecture - 11

Sum and direct sum of subspaces

Recall $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

Subspace: Let V be a vector space over $\mathbb{F}$. A non-empty subset $W \subseteq V$ is called a subspace if W is a vector space under the restricted binary operation & scalar multiplication of the vector space V.

$$\Updownarrow$$

A non-empty set $W \subseteq V$ is a subspace of V if

$\forall\, u, v \in W$ and $\forall\, \alpha, \beta \in \mathbb{F}$

we have $\alpha u + \beta v \in W$.

Definition: Suppose W_1, W_2 are subspaces of a vector space V over $\mathbb{F}$. Define the sum of W_1 and W_2 as follows:

$$W_1 + W_2 := \{\omega_1 + \omega_2 : \omega_1 \in W_1, \omega_2 \in W_2\}.$$

Clearly $W_1 + W_2 \subseteq V$.

Lemma: If W_1, W_2 be subspaces of a vector space V over F then $W_1 + W_2$ is also a subspace of V.

Proof: Use the criterion mentioned above.

Let $u = \omega_1 + \omega_2$, $v = \omega_1' + \omega_2' \in W_1 + W_2$

where $\omega_1, \omega_1' \in W_1$ and $\omega_2, \omega_2' \in W_2$

and $\alpha, \beta \in F$.

Then
$$\begin{aligned}\alpha u + \beta v &= \alpha(\omega_1 + \omega_2) + \beta(\omega_1' + \omega_2') \\ &= (\alpha\omega_1 + \beta\omega_1') + (\alpha\omega_2 + \beta\omega_2') \\ &\in W_1 + W_2\end{aligned}$$

Since $\alpha\omega_1 + \beta\omega_1' \in W_1$, $\alpha\omega_2 + \beta\omega_2' \in W_2$ as W_1 and W_2 are subspaces.

Example:

Consider $V = \mathbb{R}[x]$ the space of all polynomials over $\mathbb{R}$.

Define $W_1 = \{ f(x) \in \mathbb{R}[x] : f(-x) = f(x) \}$

& $W_2 = \{ f(x) \in \mathbb{R}[x] : f(-x) = -f(x) \}$.

W_1 is called the space of even poly. and W_2 is called the space of odd poly.

Then $V = W_1 + W_2$.

Proof: Clearly, $W_1 + W_2 \subseteq V$.

For $f(x) \in V = \mathbb{R}[x]$, we have

$$f(x) = \frac{1}{2}\big(f(x) + f(-x)\big) + \frac{1}{2}\big(f(x) - f(-x)\big)$$

with $\frac{1}{2}\big(f(x) + f(-x)\big) \in W_1$

and $\frac{1}{2}\big(f(x) - f(-x)\big) \in W_2$.

i.e. $f(x) \in W_1 + W_2$

and hence $V = W_1 + W_2$.

Example

Let $V = \mathbb{R}^2$ over $\mathbb{F} = \mathbb{R}$.

Let $W_1 = \{(x, x) : x \in \mathbb{R}\}$

and $W_2 = \{(x, -x) : x \in \mathbb{R}\}$.

Proof:

Then $W_1 + W_2 \subseteq \mathbb{R}^2$.

For any $(a, b) \in \mathbb{R}^2$, we have

$$(a, b) = \left(\frac{a+b}{2}, \frac{a+b}{2}\right) + \left(\frac{a-b}{2}, -\frac{a-b}{2}\right)$$

$$\in W_1 + W_2$$

i.e. $\mathbb{R}^2 \subseteq W_1 + W_2$

We get $W_1 + W_2 = \mathbb{R}^2$.

Exercise: Let W_1, W_2 be two subspaces of a vector space V over $\mathbb{F}$. Let S_1 and S_2 be spanning sets of W_1 and W_2 respectively, i.e. $L(S_1) = W_1$ and $L(S_2) = W_2$.

Then $L(S_1 \cup S_2) = W_1 + W_2$.

Theorem: If W_1, W_2 are finite dimensional subspaces of a vector space V over $\mathbb{F}$, then $W_1 \cap W_2$ and $W_1 + W_2$ are finite dimensional and

$$\boxed{\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).}$$

Proof:

$W_1 \cap W_2$ is finite dimensional:

A basis, say B, of $W_1 \cap W_2$ is a linearly independent subset of the space W_1, which is a finite dimensional space therefore B will be a finite set. Therefore $W_1 \cap W_2$ is finite dimensional.

$W_1 + W_2$ is finite dimensional:

From the exercise above, the union of a basis of W_1 and a basis of W_2 is a spanning set of W_1+W_2.

Therefore $\exists$ a finite subset of the subspace W_1+W_2 which spans W_1+W_2 and hence it is finite dimensional.

Proof of the formula:

Step 1: Since $W_1 \cap W_2$ is finite dimensional, let $S=\{u_1, ---, u_r\}$ be a basis of $W_1 \cap W_2$.

Then $\dim(W_1 \cap W_2) = r$.

Warning: If $W_1 \cap W_2 = \{0\}$ then $S=\phi$ and $\dim(W_1 \cap W_2)=0$.

Step 2: Since S is a linearly independent subset of the space W_1 and W_2, it can be extended to a basis B_1 of W_1 and a basis B_2 of W_2.

Let $B_1 = \{u_1, ---, u_r, v_1, ---, v_s\}$ and
$B_2 = \{u_1, ---, u_r, w_1, ---, w_t\}$ be
bases of W_1 and W_2 respectively.
Then $\dim W_1 = r+s$, $\dim W_2 = r+t$.

Step 3: Consider $B = B_1 \cup B_2$ i.e.
$B = \{u_1, ---, u_r, v_1, ---, v_s, w_1, ---, w_t\}$

It is enough to prove that B is
a basis for $W_1 + W_2$.
(Why?) Because, then

$$\begin{aligned}\dim(W_1+W_2) &= r+s+t \\ &= (r+s)+(r+t)-r \\ &= \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).\end{aligned}$$

Step 4: B is a basis of W_1+W_2.

From above exercise, B spans W_1+W_2
i.e. $L(B) = W_1+W_2$.

Now we prove that B is linearly indep.

Suppose for some

$$a_1, \dots, a_r, b_1, \dots, b_s, c_1, \dots, c_t \in \mathbb{F}$$

we have

$$\circledast - \quad \sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j + \sum_{k=1}^{t} c_k w_k = 0$$

$$\Rightarrow \quad \sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j = -\left(\sum_{k=1}^{t} c_k w_k\right)$$

then L.H.S. $\in W_1$ & RHS $\in W_2$

therefore LHS and RHS both belong to $W_1 \cap W_2 = \operatorname{span}(\{u_1, \dots, u_r\})$.

$\Rightarrow \quad \exists \; d_1, \dots, d_r \in \mathbb{F}$ s.t.

$$-\left(\sum_{k=1}^{t} c_k w_k\right) = \sum_{i=1}^{r} d_i u_i$$

$$\Rightarrow \quad \sum_{k=1}^{t} c_k w_k + \sum_{i=1}^{r} d_i u_i = 0$$

Since B_2 is a bases of W_2, we get

$c_k = 0$ and $d_i = 0 \quad \forall\, k = 1, \dots, t$
$\quad \& \; i = 1, \dots, r.$

Since all c_k's are 0, we get

$$\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j = 0$$

Now, since B_1 is a bases, we get

$a_i = 0, \; b_j = 0 \quad \forall\, i = 1, \dots, r$
$\quad j = 1, \dots, s.$

Outcome: all of a_i, b_j, c_k appearing in $(*)$ are zero, proving the linear independence of B.

Definition: Let W_1, W_2 be two subspaces of a vector space V over $\mathbb{F}$. The sum $W_1 + W_2$ of W_1 and W_2 is called **direct** if $W_1 \cap W_2 = \{0\}$.

In particular, a vector space V is said to be the direct sum of two subspaces W_1 and W_2 if

(i) $V = W_1 + W_2$

(ii) $W_1 \cap W_2 = \{0\}$.

If such is the case, we write

$$V = W_1 \oplus W_2.$$

Theorem: Let W_1, W_2 be subspaces of a vector space V over $\mathbb{F}$ such that $V = W_1 + W_2$. Then $V = W_1 \oplus W_2$ if and only if every $v \in V$ can be written as $v = \omega_1 + \omega_2$ for a unique $\omega_1 \in W_1$ and a unique $\omega_2 \in W_2$.

Proof: ($\Rightarrow$) Suppose $V = W_1 \oplus W_2$.

Then by definition, $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$

If, for $v \in V$ we have

$$v = w_1 + w_2 = w_1' + w_2'$$

for some $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$

Then $\quad w_1 - w_1' = w_2' - w_2$

$\Rightarrow$ LHS $\in W_1$, RHS $\in W_2$ therefore both LHS and RHS belong to $W_1 \cap W_2 = \{0\}$

i.e. $\quad w_1 - w_1' = 0 = w_2' - w_2$

$\Rightarrow \quad w_1 = w_1'$ and $w_2 = w_2'$

($\Leftarrow$) We have $V = W_1 + W_2$ and every vector in V is a sum $w_1 + w_2$ for a unique $w_1 \in W_1$ and a unique $w_2 \in W_2$.

If $W_1 \cap W_2 \neq \{0\}$ take $u \neq 0$ in $W_1 \cap W_2$.

Then $\quad u = u + 0 = 0 + u$

$u \in W_1, 0 \in W_2$ & $0 \in W_1, u \in W_2$

contradicting the uniqueness assumption.

Remark: Let W_1, W_2 be two subspaces of a vector space V over $\mathbb{F}$. Then

1. $W_1 \subseteq W_1 + W_2$ & $W_2 \subseteq W_1 + W_2$.

2. $$W_1 + W_2 = W_2 + W_1.$$

Remark: Let $W_1, \ldots, W_n$ be finitely many subspaces of a vector space V over $\mathbb{F}$. Then one can also define the sum $W_1 + \cdots + W_n$ written as $\sum_{i=1}^{n} W_i$.

Examples:

(1) $V = \mathbb{R}^2$,

$$W_1 = \{(x,x) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

$$W_2 = \{(x,2x) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

Then $V = W_1 \oplus W_2$.

Proof: For $(x,y) \in \mathbb{R}^2$ write

$$(x,y) = (2x-y,\ 2x-y) + (y-x,\ 2(y-x))$$

$\Rightarrow$ $\mathbb{R}^2 \subseteq W_1 + W_2$.

i.e. $V = W_1 + W_2$

For $(x,y) \in W_1 \cap W_2 \Rightarrow x = y$ and $y = 2x$

$\Rightarrow x = y = 0$

i.e. $W_1 \cap W_2 = \{(0,0)\}$

Therefore $V = W_1 \oplus W_2$.

(2) $V = M_n(\mathbb{R})$

W_1 = the subspace of symmetric $n \times n$ matrices over $\mathbb{R}$.

W_2 = the subspace of skew-symmetric $n \times n$ matrices over $\mathbb{R}$.

Then $V = W_1 \oplus W_2$.

Proof: For $A \in M_n(\mathbb{R})$, we have

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$$

with $\frac{1}{2}(A + A^t) \in W_1$ and $\frac{1}{2}(A - A^t) \in W_2$

$$\Rightarrow \quad A \in W_1 + W_2$$

$$\Rightarrow \quad V = W_1 + W_2$$

For $A \in W_1 \cap W_2$

$$\Rightarrow A = A^t \text{ and } A = -A^t$$

$$\Rightarrow A = -A$$

$$\Rightarrow A = 0$$

$$\Rightarrow W_1 \cap W_2 = \{0\}$$

Therefore, $V = W_1 \oplus W_2$.

(3) $V = M_n(\mathbb{R})$

W_1 = the subspace of $n\times n$ upper triangular matrices over $\mathbb{R}$.

W_2 = the subspace of $n\times n$ lower triangular matrices over $\mathbb{R}$.

Then $V = W_1 + W_2$

but $V \neq W_1 \oplus W_2$.

Proof: For $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{R})$

we have

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{pmatrix}$$

i.e. $A \in W_1 + W_2$ i.e. $V = W_1 + W_2$.

Moreover $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = Id \neq 0$ & $Id \in W_1 \cap W_2$

Therefore, $V \neq W_1 \oplus W_2$.

Exercise: Let W_1, W_2 be two subspaces of a vectorspace V over $\mathbb{F}$. Let S_1 and S_2 be spanning sets of W_1 and W_2 respectively, i.e. $L(S_1) = W_1$ and $L(S_2) = W_2$.

Then $L(S_1 \cup S_2) = W_1 + W_2$.

Proof: Clearly, $S_1 \subseteq W_1$, $S_2 \subseteq W_2$

$\Rightarrow$ $S_1 \cup S_2 \subseteq W_1 \cup W_2 \subseteq W_1 + W_2$

$\Rightarrow$ $L(S_1 \cup S_2) \subseteq W_1 + W_2$

(since $W_1 + W_2$ is a subspace).

Let $\omega \in W_1 + W_2$. Then $\exists\, \omega_1 \in W_1$ & $\omega_2 \in W_2$ s.t. $\omega = \omega_1 + \omega_2$.

Since $\omega_1 \in W_1 = L(S_1)$, $\exists\, u_1, \ldots, u_r \in S_1$

s.t. $\omega_1 = \sum_{i=1}^{r} a_i u_i$ for some $a_i \in \mathbb{F}$ & $i = 1, \ldots, r$

Since $\omega_2 \in W_2 = L(S_2)$, $\exists\, v_1, \ldots, v_s \in S_2$

s.t. $\omega_2 = \sum_{j=1}^{s} b_j v_j$ for some $b_j \in \mathbb{F}$ & $j = 1, \ldots, s$.

Then $\omega = \sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j \in L(S_1 \cup S_2)$

since $u_i, v_j \in S_1 \cup S_2$

$\Rightarrow$ $W_1 + W_2 \subseteq L(S_1 \cup S_2)$.