Functions of complex variable :

A map $f : S \to \mathbb{C}$ from a set $S \subset \mathbb{C}$ to $\mathbb{C}$

S : Domain of definition of f.

$\omega = f(z)$: value of function at point z.

representing $z = x + iy$, we can rewrite

$$f(z) = u(x,y) + i\, v(x,y)$$

where $u(x,y)$ & $v(x,y)$ are real functions of $(x,y) \in \mathbb{R}$.
Equivalently u & v also be regarded as functions of the polar components of $z = re^{i\theta}$

$$u(r,\theta),\ v(r,\theta).$$

- If $f(z)$ gives only one value for every z in its domains then it referred to as Single valued function

  e.g. $f(z) = \frac{1}{z},\ (\bar{z})^2,\ z^9 + 20\bar{z}^5 + 7z\bar{z},\ e^z$

- If $f(z)$ gives more then one values for some z in its domain $\longrightarrow$ Multivalued function

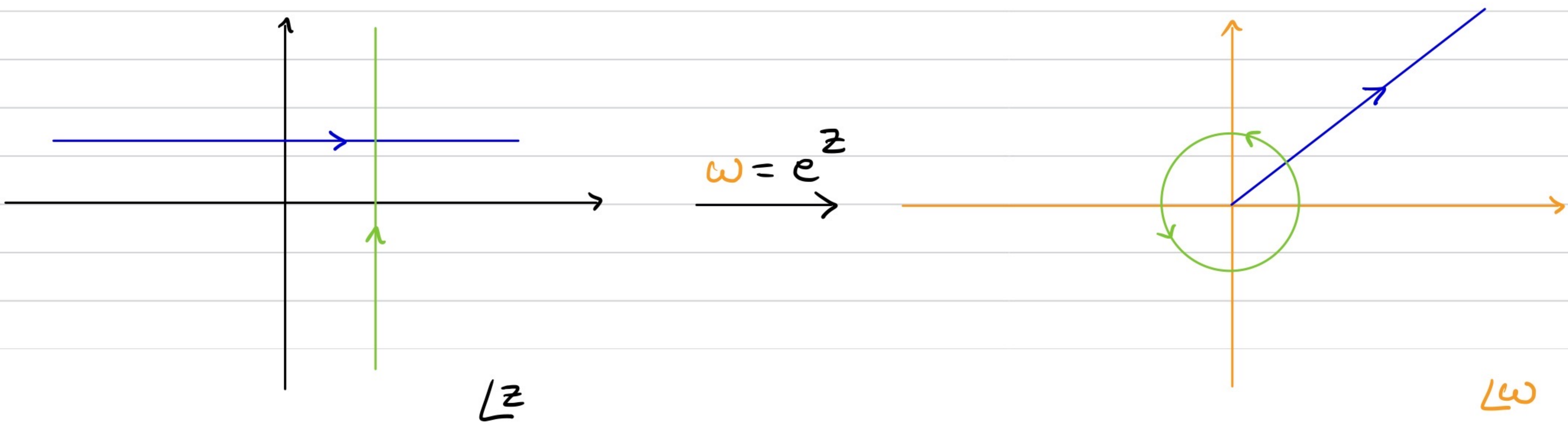
  e.g. $f(z) = z^{1/n},\ \tan^{-1}(z),\ \mathrm{Log}(z)$

Exponential function : $\omega = e^z$

using $z = x + iy$, we have

$$\omega = e^{x+iy} = e^x \cdot e^{iy}$$

i.e. $|\omega| = e^x,\ \arg(\omega) = y$



Limit of a function :

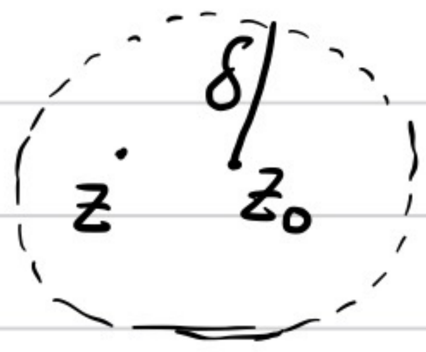
$$\lim_{z \to z_0} f(z) = \omega_0$$ means that the value of the function $f(z)$ get arbitrarily close to ω_0 as z approaches the point z_0.

Equivalently, $\forall\ \epsilon > 0\ \ \exists\ \delta > 0$ such that

$$|f(z) - \omega_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

For the above statement to make sense, $f(z)$ must be defined in the region

$$0 < |z - z_0| < \delta$$



This is refered to as the 'deleted neighbourhood' of z_0.

Ex : $$\lim_{z \to i} \left[\frac{i(z-i)\bar{z}}{z^2+1}\right]$$

In the neighbourhood of i, we can write

$z = i + \epsilon e^{i\theta}$. Then $z \to i \equiv \epsilon \to 0$

$$\lim_{z \to i} \left[\frac{i(z-i)\bar{z}}{z^2+1}\right] = \lim_{\epsilon \to 0} \left[\frac{i\cancel{\epsilon}\cancel{e^{i\theta}}\cdot(-i+\epsilon e^{-i\theta})}{\cancel{\epsilon}\cancel{e^{i\theta}}\cdot(2i+\epsilon e^{i\theta})}\right]$$

using $z^2+1 = (z-i)(z+i)$

$$= \lim_{\epsilon \to 0} \left(\frac{i(-i+\epsilon e^{-i\theta})}{2i+\epsilon e^{i\theta}}\right) = -\frac{i}{2}$$

Ex : $$\lim_{z \to 0} \left(\frac{\bar{z}}{z}\right)$$

Near 0, $z = \epsilon e^{i\theta}$. $\lim_{z \to 0}(\ ) \equiv \lim_{\epsilon \to 0}(\ )$

$$\lim_{z \to 0} \left(\frac{\bar{z}}{z}\right) = \lim_{\epsilon \to 0} \left(\frac{\cancel{\epsilon} e^{-i\theta}}{\cancel{\epsilon} e^{i\theta}}\right) = e^{-2i\theta}$$

→ depends on θ i.e. the direction from which we approach the origin. E.g. it take value $+1$ or -1 when we approach from +ve or -ve x-axis respectively.

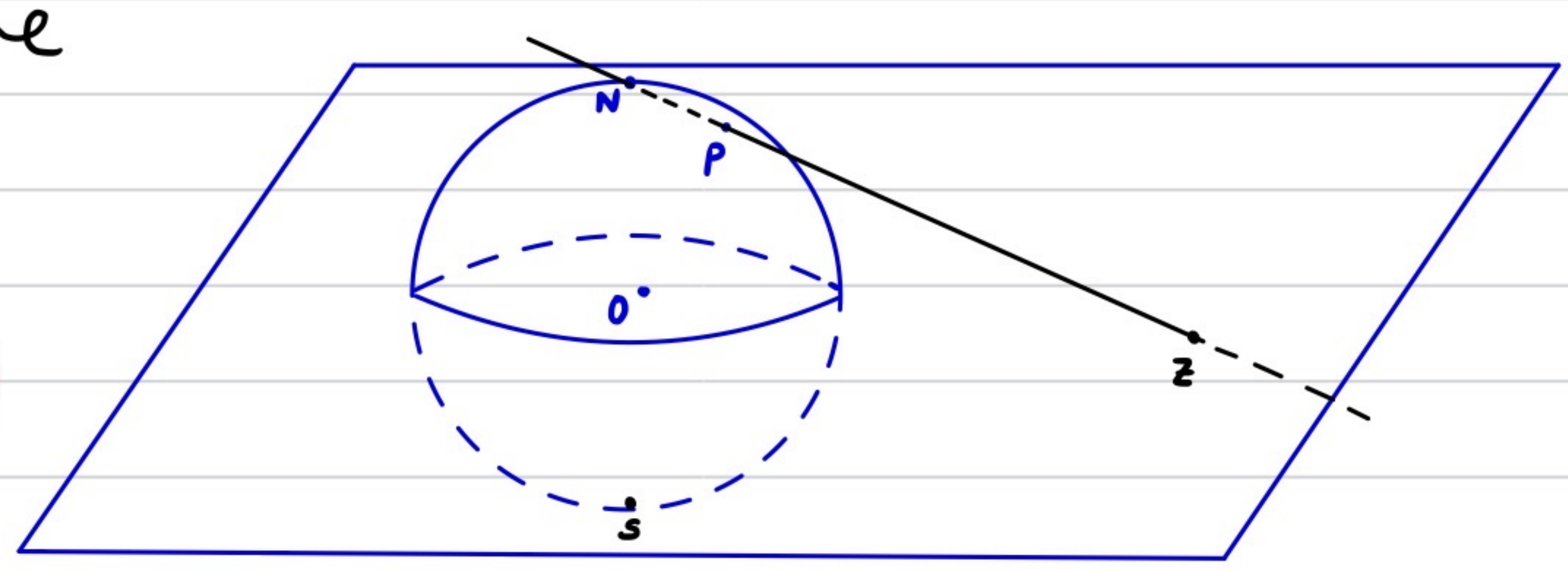
→ The limit is not well defined.

Including the "point at infinity" & the Riemann sphere

Consider the stereographic projection between a 2-sphere and a complex plane

Ex: $(\theta, \phi) \longrightarrow (x, y)$

Find $z(\theta,\phi) = x(\theta,\phi) + i\,y(\theta,\phi)$



- The figure gives a one-to-one mapping between points P on the sphere & the z on the complex plane.
- The south pole maps to the origin. The equator maps to the unit circle.
- As we approach the north pole on sphere the constant ϕ circles map to larger & larger circles on the complex plane.
- The "infinity" on the complex plane maps to a single point on the sphere, namely the North pole.

This representation of the complex plane with the infinity included as a single point is referred to as the Riemann sphere & gives a nice visualization for limits of complex functions when the $z \to \infty$.

Computing Limits involving ∞ : {discuss in tutorial}

- $\lim_{z \to \infty} f(z) \equiv \lim_{z \to 0} f\left(\frac{1}{z}\right)$

- $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$

e.g. $$\lim_{z \to \infty} \frac{iz+1}{5z-i} = \lim_{\epsilon \to 0} \frac{i\left(\frac{1}{\epsilon} e^{-i\theta}\right)+1}{5\left(\frac{1}{\epsilon} e^{-i\theta}\right)-i} = \lim_{\epsilon \to 0} \frac{i \cdot e^{-i\theta} + \epsilon}{5 e^{-i\theta} - i\epsilon}$$

$$= \frac{i \cancel{e^{-i\theta}}}{5 \cancel{e^{-i\theta}}} = \frac{i}{5}$$

$$\lim_{z \to \infty} \frac{iz^3+1}{z^2-1} = \lim_{\epsilon \to 0} \frac{i \frac{e^{-3i\theta}}{\epsilon^3} + 1}{\left(\frac{e^{-2i\theta}}{\epsilon^2} - 1\right)} = \lim_{\epsilon \to 0} \frac{ie^{-3i\theta} + \epsilon^3}{\epsilon\left(e^{-2i\theta} - \epsilon^2\right)}$$

$$= \lim_{\epsilon \to 0} \frac{i e^{-3i\theta}}{\epsilon e^{-2i\theta}} = \lim_{\epsilon \to 0} \frac{i e^{-i\theta}}{\epsilon} = \infty$$

Continuous functions :

- $f(z)$ is continuous at point $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.
- If a function is said to be continuous in region R if it is continuous at every point in R.

— Note that composing continuous functions result in continuous functions

e.g. $f(z) = e^z$ & $g(z) = z^2$ are both continuous & so are $f(g(z)) = e^{z^2}$, $g(f(z)) = e^{2z}$.

— $f(z) = u(x,y) + i\,v(x,y)$ continuous

$\Rightarrow u$ & v are continuous.

Theorem: If $f(z)$ is continuous in a region R which is both closed and bounded then $\exists\ M > 0$ s.t.

$$|f(z)| \leq M \quad \forall\ z \in R$$

$\rightarrow$ follows from the boundedness of real & imaginary part of $f(z)$.

Derivatives of complex functions:

$f(z)$ is differentiable at z_0 if $\boxed{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)}$ exists.

e.g. $f(z) = z^n$ is differentiable for all $n \in \mathbb{Z}^+$ throught the complex plane.

$f(z) = \bar{z}$ is not differentiable!

$$\lim_{z\to z_0} \frac{\bar{z}-\bar{z}_0}{z-z_0} = \lim_{\epsilon\to 0} \frac{(\bar{z}_0+\epsilon e^{-i\theta}) - \bar{z}_0}{\epsilon e^{i\theta}} = e^{-2i\theta}$$

Lets look at $f(z) = |z|^2 = z\bar{z}$

$$\lim_{\epsilon\to 0}\left[\frac{(z_0+\epsilon e^{i\theta})(\bar{z}_0+\epsilon \bar{e}^{i\theta}) - z_0\bar{z}_0}{\epsilon e^{i\theta}}\right]$$

$$= \lim_{\epsilon\to 0}\left[\frac{\cancel{\epsilon}(z_0 e^{-i\theta}+\bar{z}_0 e^{i\theta})}{\cancel{\epsilon} e^{i\theta}}\right] = z_0 e^{-2i\theta} + \bar{z}_0$$

<u>Not differentiable !!</u>

★ $u = \mathrm{Re}(z\bar{z}) = (x^2+y^2)$, $v = \mathrm{Im}(z\bar{z}) = 0$

As real functions both u & v are nice differential function with well defined all order partial derivatives w.r.t. x & y but still the complex function $f(z)$ is not differentiable in the above sense.

★ The complex differentiability condition is stronger then the of real differentiability of real & imaginary part of the function.

What is the extra condition required for complex differentiability on top of real differentiabity of u & v?