MULTIVARIABLE CALCULUS
LECTURE 26

1. Constrained Extrema

In the previous lecture, we considered the problem of finding the local maximum or minimum of a function $f : D(\subset \mathbb{R}^2) \to \mathbb{R}$. In this lecture we will learn how to determine extrema of a function of two/three variables, subject to the constraint given by the vanishing of another function.

Let us illustrate with the following examples.

Problem : Find the shortest distance from origin to the plane

$$z = 2x + y - 5.$$

Let us formulate the above problem in constraint extrema language. We need to minimize the function

$$f(x, y, z) = (x - 0)^2 + (y - 0)^2 + (z - 0)^2,$$

subject to the constraint $2x + y - 5 - z = 0$. We will solve this problem using *substitution methods*. Let us begin the proof by defining the following function :

$$h(x, y) := f(x, y, 2x + y - 5) := x^2 + y^2 + (2x + y - 5)^2.$$

Then

$$\frac{\partial h}{\partial x} = 2x + 4(2x + y - 5) = 0,$$
$$\frac{\partial h}{\partial y} = 2y + 2(2x + y - 5) = 0.$$

Solving the above equation we obtain

$$x = \frac{5}{3} \quad y = \frac{5}{6}.$$

Substituting back in z variable we obtain $z = 2.\frac{5}{3} + \frac{5}{6} - 5 = -\frac{5}{6}$. Moreover at $x = \frac{5}{3}, y = \frac{5}{6}$,

$$\frac{\partial^2 h}{\partial x^2} = 2 + 8 = 10 > 0$$
$$\frac{\partial^2 h}{\partial y^2} = 2 + 2 = 4 > 0$$
$$\frac{\partial^2 h}{\partial x \partial y2} = 4.$$

Therefore $AC - B^2 = 40 - 16 = 24 > 0$. So the point $(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6})$ **lies in the plane** and it is a point of minima. So the shortest distance is $\frac{\sqrt{150}}{6}$.

Question : Does the above substitution method always work to find constraint extrema. Let us give another example.

Problem : Find the shortest distance from origin to $x^2 - z^2 = 1$.

Let us formulate the above problem in constraint extrema language. We need to minimize the function

$$f(x, y, z) = (x - 0)^2 + (y - 0)^2 + (z - 0)^2,$$

subject to the constraint $x^2 - z^2 = 1$. We will solve this problem using *substitution methods*. Let us begin the proof by defining the following function :

$$h(x, y) := x^2 + y^2 + x^2 - 1 = 2x^2 + y^2 - 1.$$

$$\frac{\partial h}{\partial x} = 4x = 0,$$
$$\frac{\partial h}{\partial y} = 2y = 0.$$

Solving the above equation we obtain $x = 0, \quad y = 0$. This immediately implies $z^2 = -1$!!!! This point does not lies on the (real) hyperbolic cylinder.

To overcome this difficulty we make a different substitution : $x^2 = 1 + z^2$. If we perform the above procedure we obtain $x = \pm 1, y = 0, z = 0$ and lies on the (real) hyperbolic cylinder. Moreover one can easily check $AC - B^2 > 0$, and $A > 0$. Hence a point of minima and the minimum distance is 1.

Remark 1.1. In the substitution method, once we substitute the constraint in the minimizing/maximizing function, then the domain of the function will be the domain of the minimizing/maximizing function. Then the extrema points belongs to this domain which may not be the domain of constraints. This what exactly happened in the above example. The above method of finding point of extrema is quite subtle. It depends on the function and as well as in their domain.

So we need to devise a mathematically elegant way to tackle this problem. Now we will be doing a method known as *Lagrange multiplier method.*

2. Lagrange multiplier method

This is a very well known method to find the extrema subject to some constraint. It has wide varieties of applications from Partial differential equations to Geometry. Let us formulate the problem :

Problem : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable functions with continuous partial derivatives. Find the local maxima or local minima of a function $f(x, y, z)$ on the surface

$$\mathcal{S} := \{(x, y, z) : g(x, y, z) = 0\},$$

where $g : \mathbb{R}^3 \to \mathbb{R}$.

Theorem 2.1. *(Necessary Condition)*

Let $P_0 = (x_0, y_0, z_0)$ be a point on the surface $\mathcal{S} := \{(x, y, z) : g(x, y, z) = 0\}$. Suppose P_0 is a local extremum of f on $\mathcal{S}$. Also assume $\nabla g \neq 0$ at P_0. Then we have the following necessary condition : there exists $\lambda \in \mathbb{R}$ such that there holds

$$\nabla f(P_0) = \lambda \nabla g(P_0), \quad \lambda \in \mathbb{R}.$$

Before proving the theorem let us remark some useful facts from tangents and normals for function of several variables.

Remark 2.1. Let $\mathcal{S}$ be a surface given by $\mathcal{S} := \{(x, y, z) : g(x, y, z) = 0\}$. Find the normal to the surface $\mathcal{S}$ at the point (x_0, y_0, z_0). To do so, let us consider a curve (function) $\gamma : [0, 1] \to \mathcal{S}$. Without loss of generality we may consider $\gamma(t_0) = (x_0, y_0, z_0)$, where $t_0 \in [0, 1]$ and $\gamma(t) = (x(t), y(t), z(t))$. Since γ lies on the surface we have

$$g(\gamma(t)) = 0.$$

Differentiate with respect to t, in the above equation, we obtain

$$\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} = 0,$$

in other words we have at $t = t_0$

$$\begin{aligned} &(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}).(x'(t), y'(t), z'(t)) = 0 \\ &\underbrace{\nabla g(P_0)}_{Normal}. \quad \underbrace{(x'(t), y'(t), z'(t))}_{= \text{Tangent to the curve}} \quad = 0 \quad \text{at } t = t_0. \end{aligned}$$

This implies $\nabla g(P_0)$ is the normal to the surface at (x_0, y_0, z_0).

Now we begin the proof of Theorem 2.1.

Proof of Theorem 2.1 : Let $c(t) = (x(t), y(t), z(t))$ be a curve on the surface $\mathcal{S}$ passing through P_0 on $\mathcal{S}$ and Let $P_0 = c(t_0)$. It is given that f has a local extremum at P_0 on $\mathcal{S}$, this implies f has a local extremum on the curve c as well at the point $c(t_0)$. Therefore defining $g(t) := f(c(t))$, and using one variable calculus we obtain

$$\frac{d}{dt} f(c(t)) = 0, \quad \text{at } t = t_0,$$

i.e., using chain rule we obtain,

$$\nabla f(P_0). \underbrace{\frac{d\,c(t)}{dt}}_{=Tangent} = 0 \quad \text{at } t = t_0.$$

This implies $\nabla f(P_0)$ is perpendicular to the tangent to $\mathcal{S}$ at P_0. This implies $\nabla f(P_0)$ is parallel to Normal to the surface $\mathcal{S}$ at P_0. But we know from the above remark that the normal to the surface is given by $\nabla g(P_0)$. Therefore there exists a $\lambda \in \mathbb{R}$ such that

$$\nabla g(P_0) = \lambda \nabla f(P_0).$$

The above equation is known as Lagrange Multiplier. Also note that $g(P_0) = 0$.

Let us give some examples.

Problem : Find the point on the plane $x + y - 2z = 1$ which is nearest to the origin.

Let us formulate the above problem in constraint extrema language. We need to minimize the function

$$f(x, y, z) = (x - 0)^2 + (y - 0)^2 + (z - 0)^2,$$

subject to the constraint $x + y - 2z = 1$, i.e., $g(x, y, z) = x + y - 2z - 1$. Using Lagrange Multiplier we have

$$\nabla(x^2+y^2+z^2) = \lambda(x+y-2z-1).$$

From above we obtain

$$(2x, 2y, 2z) = \lambda(1, 1, -2).$$

So by equating, $2x = \lambda, 2y = \lambda, z = -\lambda$. Also we have one more equation $x+y-2z-1 = 0$. By substituting these values we obtain
$x = y = \frac{1}{6}$ and $z = -\frac{1}{3}$. So $(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$ is a point of minima.

Problem : Find the maximum and minimum of the function f given by

$$f(x, y, z) = x^2y^2z^2$$

subject to the constraint that (x, y, z) lies on the unit sphere given by

$$\mathcal{S} := \{(x, y, z) : x^2 + y^2 + z^2 = 1\}.$$

So $g(x, y, z) = x^2 + y^2 + z^2 - 1$, and $f(x, y, z) = x^2y^2z^2$. Now using Lagrange multiplier we have $\nabla f = \lambda \nabla g$,

$$(2xy^2z^2, 2x^2yz^2, 2x^2y^2z) = \lambda(2x, 2y, 2z),$$

this implies $2xy^2z^2 = 2\lambda x$, $2x^2yz^2 = 2\lambda y$ and $2x^2y^2z = 2\lambda z$. One immediately see that either a solution of this system of equations will have one of its coordinates is 0. This implies $f = 0$. Or else is

$$x^2 = y^2 = z^2 \quad \lambda = x^4.$$

In addition to this one need to check that these points should satisfy $g(x, y, z) = 0$. Then we have

$$x = \pm\frac{1}{\sqrt{3}} = y = z.$$

Then the value of f at each of the corresponding points is $\frac{1}{27}$ and the minimum value of f is zero,(attained for instance, at $(0, 0, 1)$).