Lecture 18
Signals and Systems (ELL205)

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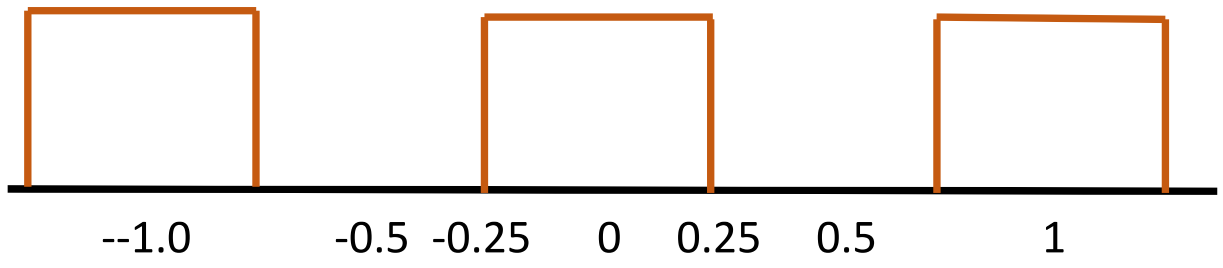
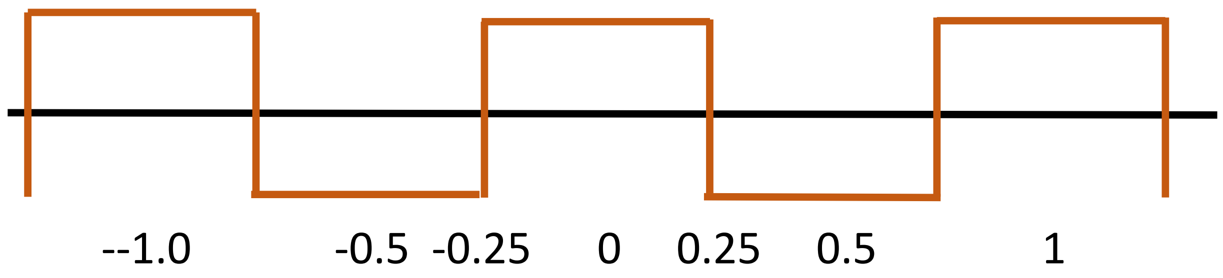
Use properties

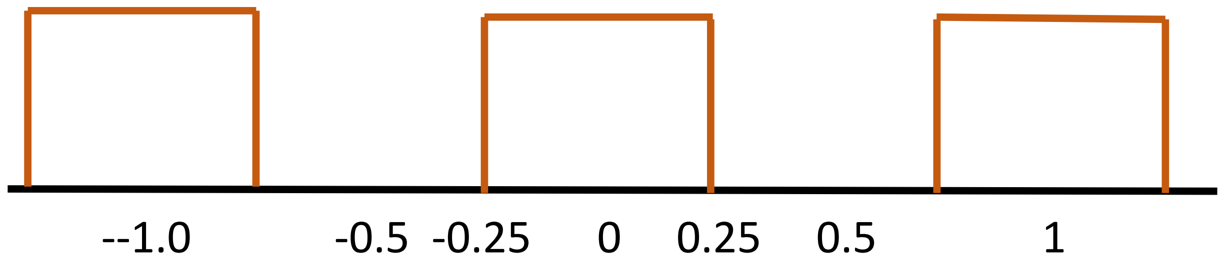


a_k



$b_k = ?$

Use properties



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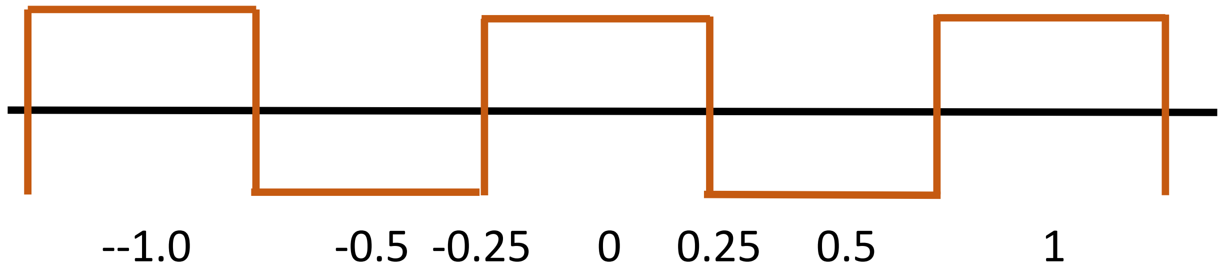

$b_k = ?$

$b_k = a_k$ for $k \neq 0$

$b_o = 0$

Use properties

$$x(t) = m + \widetilde{x(t)}$$

$\widetilde{x(t)}$ has mean value 0

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{T}\int_T m dt + \frac{1}{T}\int_T \widetilde{x(t)}dt = m$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{m}{T}\int_T e^{-jk\omega_0 t}dt + \frac{1}{T}\int_T \widetilde{x(t)}e^{-jk\omega_0 t}dt = 0 + b_k$$

For $k \neq 0$

Signals as Vectors

Outline

- Introduction to vectors
- Introduction to inner product and projection theorem
- Introduction to signals as vectors

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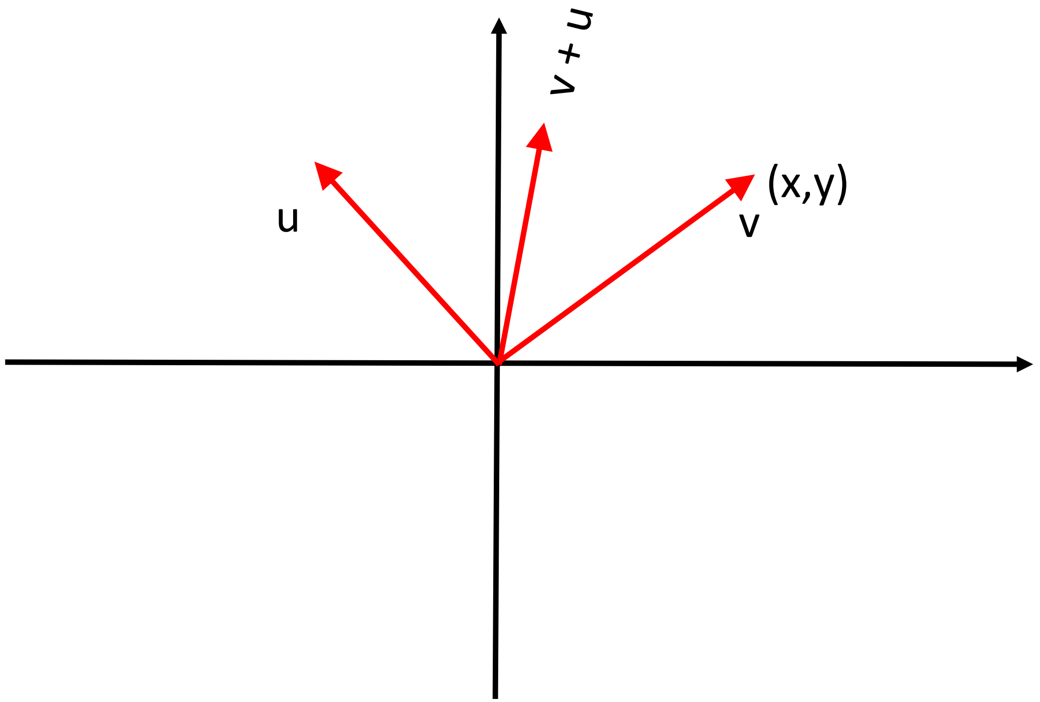
Vector Spaces

2D vector



Vector Spaces

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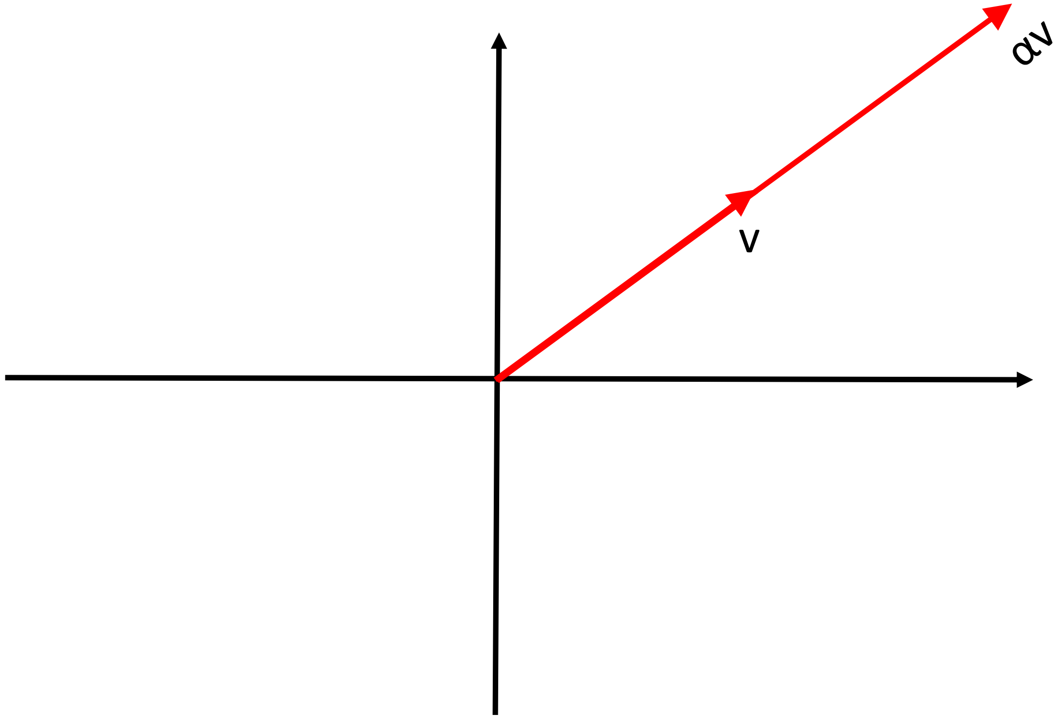
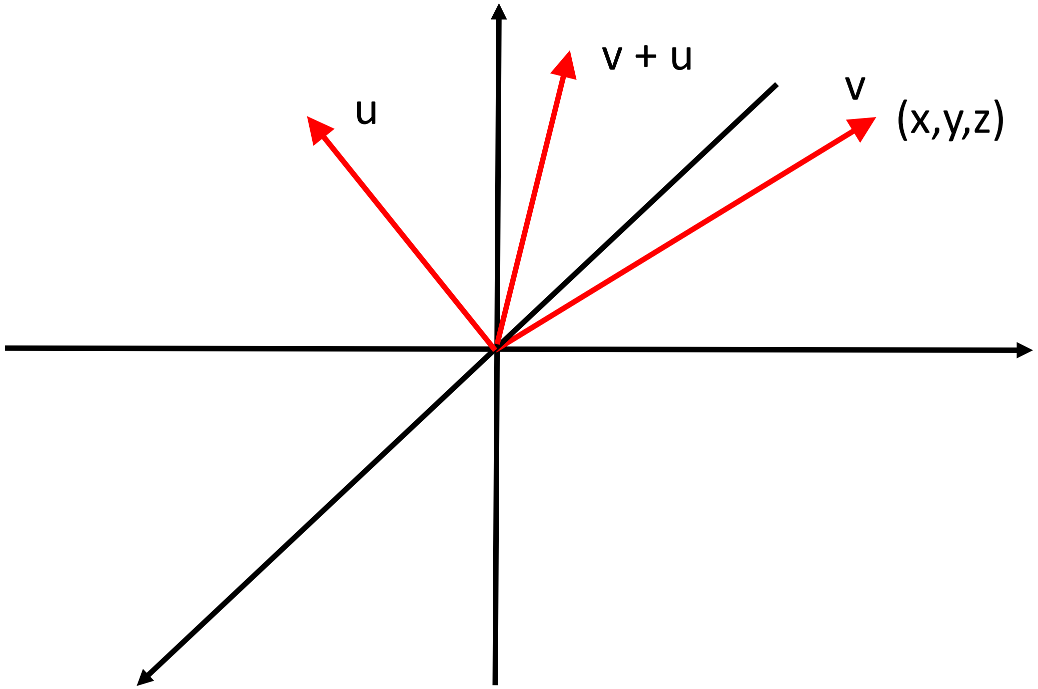


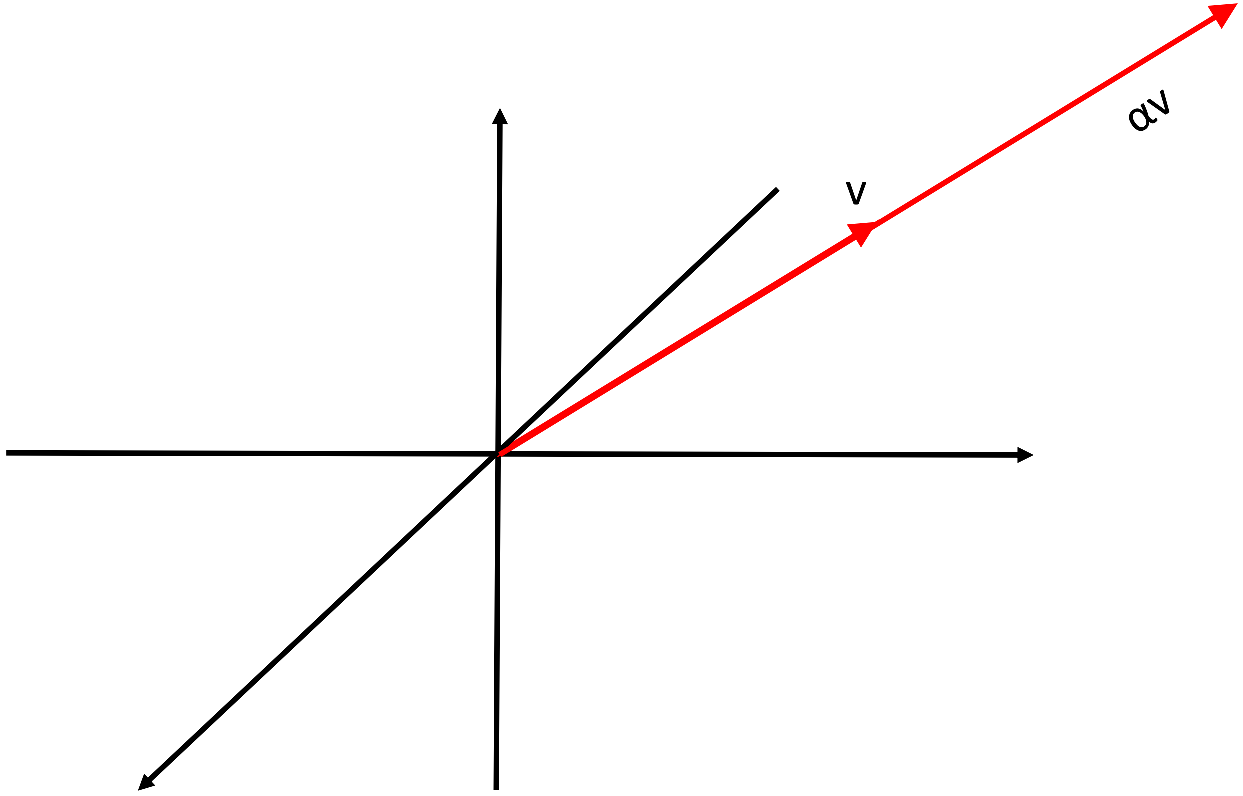
Vector Spaces

3D vector



Vector Spaces

3D vector



Vector Spaces

A set V is called a vector space over the field $\mathbb{k}$ ($\mathbb{R}$ or $\mathbb{C}$) (or equivalently a $\mathbb{K}$-vector space), if following conditions are satisfied:

A. $\exists$ two maps:

$$+ : \mathrm{V} \times V \to V$$
$$(v_1, v_2) \to v_1 + v_2$$

called vector addition

$$. : \mathbb{K} \times V \to V$$
$$(\alpha, v) \to \alpha. v$$

called scalar multiplication

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B. $(V, +)$ is a commutative group (addition is commutative, associative, and identity (0 vector) and inverse vector exist)

C. Other usual rules:

1. $\alpha.(v_1 + v_2) = \alpha.v_1 + \alpha.v_2$
2. $(\alpha_1 + \alpha_2).v = \alpha_1.v + \alpha_2.v$
3. $(\alpha_1.\alpha_2).v = \alpha_1.(\alpha_2.v) = \alpha_2.(\alpha_1.v)$
4. $1.v = v$

Subspace of Vector Spaces

A set W is called a subspace of vector space V if

A. $W \subseteq V$

B. Under same addition and scalar multiplication, W is a vector space in its own right.

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Span

Let $\{v_1, v_2, \dots, v_p\}$ be a set of vectors

Span $\left\{v_1, v_2, \dots, v_p\right\} = \left\langle v_1, v_2, \dots, v_p \right\rangle = \sum_{l=1}^{p} \alpha_l v_l \mid \alpha_l \epsilon \mathbb{K}$

Span is the smallest subspace containing $\{v_1, v_2, \dots, v_p\}$.

Set of vectors is known as the spanning set.

Linear independence of vectors

Linearly dependent vectors

$$\text{If } \sum_{i=0}^{n} a_i v_i = 0$$

for some a_i not all of them zero, then the set of vectors is referred to as linearly dependent.

E.g.,

$$a v_1 + b v_2 + c v_3 = 0$$

$$v_3 = -\frac{(a v_1 + b v_2)}{c}$$

Linear independence of vectors

$$v_3 = -\frac{(av_1 + bv_2)}{c}$$

$$v = a_1v_1 + a_2v_2 + a_3v_3$$

$$v = a_1v_1 + a_2v_2 - a_3\frac{(av_1 + bv_2)}{c}$$

$$v = \left(a_1 - \frac{a_3a}{c}\right)v_1 + \left(a_2 - \frac{a_3b}{c}\right)v_2$$

Linear independence of vectors

Take spanning set, kill all linearly dependent vectors, and get basis set

Linearly independent vectors

If $\sum_{i=0}^{n} a_i v_i = 0$ only for all a_i being zero

Basis

The set is referred to as basis for V if the set is linearly independent and spans V.

Every vector space has a basis (Zorn's lemma).

Dimension of the vector space

If the number of vectors in the basis set is finite, then V is a **finite dimensional space**

otherwise it is **infinite dimensional space** ($\mathcal{L}_2$ is a infinite dimensional space).

Outline

- Introduction to vectors
- Introduction to inner product and projection theorem
- Introduction to signals as vectors

Inner Product Space

Introduces the notion of length and direction.

Symbol: $\langle v, u \rangle$

Hermitian symmetry: $\langle v, u \rangle = \langle u, v \rangle^*$

Hermitian bilinearity: $\langle \alpha v + \beta u, w \rangle = \alpha \langle v, w \rangle + \beta \langle u, w \rangle$

$$\langle v, \alpha u + \beta w \rangle = \alpha^* \langle v, u \rangle + \beta^* \langle v, w \rangle$$

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$\langle v, \alpha u + \beta w \rangle = \alpha^* \langle v, u \rangle + \beta^* \langle v, w \rangle$

Strict positivity: $\langle v, v \rangle \geq 0$ with equality iff $v = 0$

Inner Product Space

$v = \{v_1, v_2, \ldots. v_n\}$ and
$u = \{u_1, u_2, \ldots. u_n\}$

$$\langle v, u \rangle \stackrel{\text{def}}{=} \sum_{i=1}^{n} v_i {u_i}^*$$

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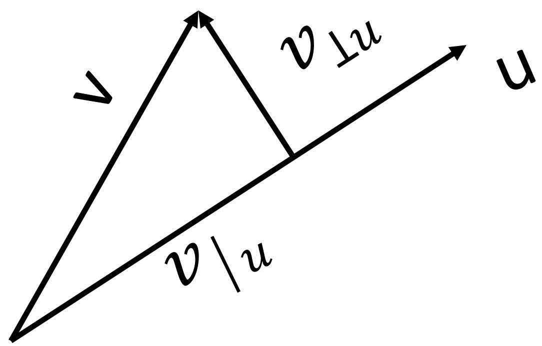
$$\langle v, u \rangle^* \stackrel{\text{def}}{=} \sum_{i=1}^{n} {v_i}^* u_i = \langle u, v \rangle$$

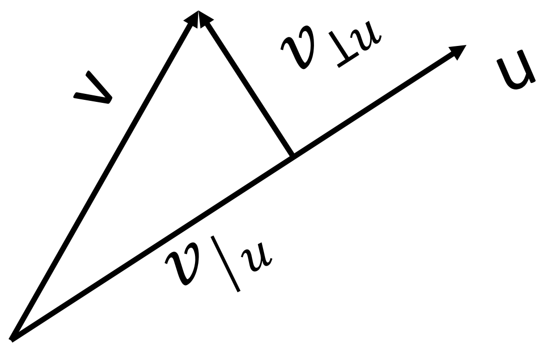
$\|v\| \stackrel{\text{def}}{=} \sqrt{\langle v, v \rangle}$ (norm of a vector, length of a vector)

$\|v\| \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} v_i {v_i}^*} = \sqrt{\sum_{i=1}^{n} |v_i|^2}$

One dimension projection theorem



One dimension projection theorem



$$v_{|u} = \alpha u$$

$$\langle v - v_{|u}, u \rangle = 0$$

$$\langle v - \alpha u\,, u \rangle = \langle v\,, u \rangle - \alpha \langle u\,, u \rangle = 0$$

$$\alpha = \frac{\langle v\,, u \rangle}{\|u\|^2}$$

$$v_{|u} = \frac{\langle v, u \rangle}{\|u\|^2} u$$

$$v_{\perp u} = v - v_{|u} = v - \frac{\langle v, u \rangle}{\|u\|^2} u$$

Direction

- $\cos\left(\angle(v, u)\right) = \frac{\langle v,u \rangle}{\|v\| \|u\|}$ (If v and u are real)
- Two vectors are orthogonal if $\langle v, u \rangle = 0$
- Cauchy-Schwarz inequality

$$|\langle v, u \rangle| \leq \|v\| \|u\|$$

Equality is satisfied when $v = \alpha u$