LIMIT & CONTINUITY

LECTURE 1 : Limit of functions

LECTURE 2 : Limit of functions

LECTURE 3 : Continuity

LECTURE 4 : Continuity

LECTURE 5 : Uniform Continuity.

Intermediate Value theorem: Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $f(a)<f(b)$. Then for every y s.t. $f(a)<y<f(b)$ $\exists$ $c\in(a,b)$ s.t. $f(c)=y$.

Pf:- Let $g(x)=f(x)-y$

Therefore, g is continuous on $[a,b]$

$$g(a)=f(a)-y<0$$

$$g(b)=f(b)-y>0$$

i.e, $g(a)g(b)<0$.

Thus by our previous theorem, $\exists$ $c\in(a,b)$ s.t.

$g(c)=0$, i.e, $f(c)=y$.

Theorem :- Let $f:[a,b] \to \mathbb{R}$ be a continuous function. Let $M = \sup f([a,b])$ & $m = \inf f([a,b])$. Let $m \le \gamma \le M$. Then $\exists\ c \in [a,b]$ s.t. $f(c) = \gamma$.

Proof :- If $\gamma = m$ or M then we have already proved that, $\exists\ c$ & $d \in [a,b]$ s.t. $f(c) = m$ & $f(d) = M$.

If $m \neq M$ & $m < \gamma < M$

Consider a new function $g:[a,b] \to \mathbb{R}$ defined by

$$g(x) = f(x) - \gamma.$$

Then g is continuous on $[a,b]$.

$$\left.\begin{aligned} g(c) &= f(c) - \gamma = m - \gamma < 0 \\ g(d) &= f(d) - \gamma = M - \gamma > 0 \end{aligned}\right\} \text{ Thus } g(c)\,g(d) < 0$$

Thus, $\exists$ a point $p \in (c,d)$ or (d,c) [depending on $c < d$ or $d < c$] s.t. $g(p) = 0$, i.e., $f(p) = \gamma$.

Theorem:- Let $f : [a,b] \to [a,b]$ be a continuous function. Then $\exists$ $c \in [a,b]$ s.t. $f(c) = c$.

Proof:- Let $g(x) = f(x) - x$

Then $g : [a,b] \to \mathbb{R}$ is a continuous function.

If $f(a) = a$ or $f(b) = b$ then we are done.

If $f(a) \neq a$ & $f(b) \neq b$ then $f(a) > a$ & $f(b) < b$

Then $g(a) = f(a) - a > 0$ & $g(b) = f(b) - b < 0$

thus, $g(a)\,g(b) < 0$

Thus, $\exists$ $c \in (a,b)$ s.t. $g(c) = 0$

$$\Rightarrow f(c) - c = 0$$

$$\therefore f(c) = c.$$

Eg 1:- $f : [0,1) \longrightarrow [0,1)$

$$f(x) = \frac{x}{2} + \frac{1}{2}$$

Eg 2:- $f : (0,1] \longrightarrow (0,1]$

$$f(x) = \frac{x}{2}$$

Eg 3:- $f : (0,1) \longrightarrow (0,1)$

$$f(x) = x^2 \text{ or } \frac{x}{2}$$

Eg:- $f : [0,\infty) \longrightarrow [0,\infty)$

$$f(x) = x+1$$

Check that, the above functions do not have any fixed point.

Eg:- Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x - 3x^2$ has at least 3 real roots.

Solution :- Note that $2 < e < 3$.

$$f(-1) = \frac{1}{e} - 3 < 0$$

$$f(0) = 1 > 0$$

$$f(1) = e - 3 < 0$$

$$f(8) = e^8 - 3 \cdot 8^2 > e^8 - 2^8 > 0$$

Thus by IVT, there are roots in the intervals $(-1, 0)$, $(0, 1)$ and $(1, 8)$.

Thus f has at least 3 real roots.

Uniform Continuity

Definition:- Let $f : A \to \mathbb{R}$ be a function. Then f is said to be uniformly continuous on A if for every $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$x, y \in A$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Theorem:- Let $f : A \to \mathbb{R}$ be a function. Then f is uniformly continuous on A if and only if for any two sequences (x_n) & (y_n) in A such that $|x_n - y_n| \to 0$, we have $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$.

Eg! :- $f(x) = x^2$ on $(0,7)$. Discuss uniform continuity of f.

Sol^n :- $|f(x)-f(y)| = |x^2-y^2| = |(x+y)(x-y)|$

$= |x+y|\,|x-y|$

$\leq (|x|+|y|)\,|x-y|$

$\leq 14\,|x-y|$

$\therefore\ |f(x)-f(y)| \leq 14\,|x-y|$

Therefore, if $|x_n - y_n| \to 0$ then $|f(x_n)-f(y_n)| \to 0$ as $n\to\infty$ for any two arbitrary sequences (x_n) & (y_n) in $(0,7)$.

Therefore, f is uniformly continuous on $(0,7)$.

Eg 2 :- $f(x) = x^2$ in $(0, \infty)$.

Sol^n :- Let $x_n = n + \frac{1}{n}$ & $y_n = n$

$n \to \infty$.

Then $|x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$.

$$|f(x_n) - f(y_n)| = \left|\left(n+\frac{1}{n}\right)^2 - n^2\right|$$

$$= \left|n^2 + 2 + \frac{1}{n^2} - n^2\right|$$

$$= 2 + \frac{1}{n^2} \to 2 \text{ as } n \to \infty$$

Thus f is not uniformly continuous on $(0, \infty)$

Eg 3:- $f(x) = \frac{1}{x}$ in $(0,1)$

Soln: Let $x_n = \frac{1}{n+1}$ and $y_n = \frac{1}{n}$

$$|x_n - y_n| = \left|\frac{1}{n+1} - \frac{1}{n}\right| = \frac{1}{n(n+1)} \to 0 \text{ as } n \to \infty$$

$$|f(x_n) - f(y_n)| = |n+1-n| = 1 \not\to 0 \quad n \to \infty$$

Therefore f is not uniformly continuous on $(0,1)$.

Eg 4:- $f(x) = \log(x)$ in $(0,1)$

Let $x_n = \frac{1}{n}$ & $y_n = \frac{1}{2n}$.

Then $|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{2n}\right| = \frac{1}{2n} \to 0$ as $n \to \infty$

$$|f(x_n) - f(y_n)| = \left|\log\left(\tfrac{1}{n}\right) - \log\left(\tfrac{1}{2n}\right)\right| = \left|\log\left(\frac{2n}{n}\right)\right|$$

$$= \log 2 \not\to 0 \text{ as } n \to \infty.$$

Therefore f is not uniformly continuous on $(0,1)$.

Properties :-

(i) Let $f : A \to \mathbb{R}$ be a uniformly continuous function. Then f is continuous on A.

(ii) Let $f : A \to \mathbb{R}$ be a uniformly continuous function & $B \subseteq A$. Then f is uniformly continuous on B.

(iii) Let f be a real valued function such that f is uniformly continuous on two intervals A & B, and $A \cap B \neq \phi$. Then f is uniformly continuous on $A \cup B$.

Proof :- (i) Let $a \in A$. Let $y_n = a \ \forall n$.

Let (x_n) be an arbitrary sequence in A s.t. $x_n \to a$ as $n \to \infty$

$\Rightarrow |x_n - y_n| \to 0$ as $n \to \infty$.

Since f is uniformly continuous on A, $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$

$\Rightarrow |f(x_n) - f(a)| \to 0 \Rightarrow f(x_n) \to f(a)$ as $n \to \infty$.

$\therefore$ f is continuous at 'a'.

Thus, f is continuous on A.

(ii) Trivial.

(iii) First choose an $\varepsilon > 0$. We need to find a $\delta > 0$ such that $|x-y| < \delta$ and $x, y \in A \cup B \Rightarrow |f(x) - f(y)| < \varepsilon$.

Since f is uniformly continuous on A, for $\frac{\varepsilon}{2} > 0$, $\exists$ a $\delta_1 > 0$ such that $|x-y| < \delta_1$ & $x, y \in A \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$.

Since f is uniformly continuous on B, for $\frac{\varepsilon}{2} > 0$, $\exists$ a $\delta_2 > 0$ such that $|x-y| < \delta_2$ & $x, y \in B \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$.

Now we choose $\delta = \min\{\delta_1, \delta_2\}$.

Let $x, y \in A \cup B$ such that $|x-y| < \delta$. Now we have three cases.

(i) if $x, y \in A$ then $|x-y| < \delta \Rightarrow |x-y| < \delta_1$. Thus, $|f(x) - f(y)| < \frac{\varepsilon}{2} < \varepsilon$

(ii) if $x, y \in B$ then $|x-y| < \delta \Rightarrow |x-y| < \delta_2$. Thus, $|f(x) - f(y)| < \frac{\varepsilon}{2} < \varepsilon$

(iii) if $x \in A$ & $y \in B$ then we choose $z \in A \cap B$ such that $x \leq z \leq y$ or $y \leq z \leq x$ (depending on $x \leq y$ or $y \leq x$).

Since $|x-y| < \delta \Rightarrow |x-z| < \delta \leq \delta_1$ & $|y-z| < \delta \leq \delta_2$

$\Rightarrow |f(x)-f(z)| < \frac{\varepsilon}{2}$ & $|f(z)-f(y)| < \frac{\varepsilon}{2}$

Thus $|f(x)-f(y)| \leq |f(x)-f(z)| + |f(z)-f(y)|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus f is uniformly continuous on $A \cup B$.

Eg:- $f : (0,1) \cup (1,2) \longrightarrow \mathbb{R}$

$$f(x) = 1 \quad , x \in (0,1)$$
$$= 2 \quad , x \in (1,2)$$

Here f is continuous on $(0,1) \cup (1,2)$.

Let $x_n = 1 - \frac{1}{n}$ & $y_n = 1 + \frac{1}{n}$. Then $|x_n - y_n| = \frac{2}{n} \to 0$ as $n \to \infty$

but $|f(x_n) - f(y_n)| = |1-2| = 1 \not\to 0$ as $n \to \infty$.

Therefore, f is not uniformly continuous. But f is uniformly continuous on $(0,1)$ & $(1,2)$.

Theorem:- Let f be a real valued function continuous on $[a,b]$. Then f is uniformly continuous.

Proof:- Assume, f is not uniformly continuous. Then $\exists\ \varepsilon > 0$ and two sequence (x_n) & (y_n) in $[a,b]$ such that $|x_n - y_n| \to 0$ as $n \to \infty$ but $|f(x_n) - f(y_n)| \geq \varepsilon$ $\forall\ n$.

Now (x_n) is bounded. By Bolzano-Weierstrass theorem (x_n) has a convergent subsequence (x_{r_n}).

Let $x_{r_n} \to c \in [a,b]$

$a \leq x_{r_n} \leq b$

$\downarrow$

c

$\Rightarrow a \leq c \leq b.$

Since $x_{r_n} \to c$ and

$|x_{r_n} - y_{r_n}| \to 0$ as $n \to \infty$

$\Rightarrow y_{r_n} \to c$ as $n \to \infty$.

Given f is continuous at 'c'. Thus $f(x_{r_n}) \to f(c)$, $f(y_{r_n}) \to f(c)$ as $n \to \infty$.

$$|f(x_{r_n}) - f(y_{r_n})| \to 0 \text{ as } n \to \infty.$$

But by our construction, $|f(x_{r_n}) - f(y_{r_n})| \geq \varepsilon \quad \forall n.$

This is a contradiction.

Thus, f is uniformly continuous on $[a, b]$.

Eg:- (i) $f(x) = \dfrac{\sin x}{x}$ on $(0, 1)$

Here $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} \dfrac{\sin x}{x} = 1$

$\lim\limits_{x \to 1-} f(x) = \lim\limits_{x \to 1-} \dfrac{\sin x}{x} = \sin 1$

Thus f can be extended continuously on $[0, 1]$.

Since f is continuous on $[0, 1]$, f is uniformly continu.. on $[0, 1]$. Thus f is uniformly continuous on $(0, 1)$ as $(0, 1) \subset [0, 1]$.

(ii) $f(x) = \sin\frac{1}{x}$ on $(0,1)$

Let $x_n = \frac{1}{2n\pi}$ & $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$

Then $x_n, y_n \longrightarrow 0$ as $n \to \infty$

$\Rightarrow |x_n - y_n| \longrightarrow 0$ as $n \to \infty$

But $|f(x_n) - f(y_n)| = |\sin(2n\pi) - \sin(2n\pi + \frac{\pi}{2})|$

$= |0-1| = 1 \not\to 0$ as $n \to \infty$.

Thus f is not uniformly continuous on $(0,1)$.

Lipschitz Continuous :— A function $f: D \to \mathbb{R}$ is said to be Lipschitz continuous on D if there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x-y| \ \forall \ x, y \in D$

Note :— Lipschitz continuous $\Rightarrow$ uniformly continuous.

(iii) $f(x) = \sqrt{x}$ on $(0, \infty)$

Here f can be extended continuously on $[0, \infty)$ as $\lim_{x \to 0+} f(x) = 0$.

Now f is uniformly continuous on $[0, 1]$ as f is continuous on $[0, 1]$. We claim that f is uniformly continuous on $[1, \infty)$ as well.

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}|$$

$$= \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \qquad \left[\begin{array}{l} 1 \leq \sqrt{x} \\ \frac{1}{\sqrt{x}} \leq 1 \end{array} \right.$$

$$\leq \frac{|x - y|}{|\sqrt{x}| + |\sqrt{y}|} \leq \frac{1}{2} |x - y|. \qquad [\text{on } [1, \infty).$$

Thus f is uniformly continuous on $[1, \infty)$. Thus f is uniformly continuous on $[0, \infty)$, and hence on $(0, \infty)$.

(iv) $f(x) = \log(x)$ on (a, ∞), $a > 0$.

$$\frac{|f(x)-f(y)|}{|x-y|} = |f'(c)| \text{ for some } c \in (a,\infty)$$

$$= \left|\frac{1}{c}\right| < \frac{1}{a}$$

$$\therefore \quad |f(x)-f(y)| < \frac{1}{a}|x-y|$$

Thus f is uniformly continuous on (a,∞).

<u>Theorem</u> :- If (x_n) is a Cauchy sequence in A and $f: A \longrightarrow \mathbb{R}$ is a uniformly continuous function then $(f(x_n))$ is also a Cauchy sequence.