Lecture -12

Linear transformation & rank-nullity thm.

Definition: Let V and W be vector spaces over the same field $\mathbb{F}$.
A map $T : V \longrightarrow W$ is called a **linear transformation** if

$$T(au + bv) = a\,T(u) + b\,T(v)$$

for all $a, b \in \mathbb{F}$ and for all $u, v \in V$.

Observation: If $T : V \longrightarrow W$ is a lin. trans then $T(0) = 0$.

Proof:

$$T(0) = T(0+0) \quad (\because 0 = 0+0)$$

$$\Rightarrow T(0) = T(0) + T(0)$$

$$\Rightarrow T(0) - T(0) = T(0) + T(0) - T(0)$$

$$\Rightarrow 0 = T(0) + 0$$

$$\Rightarrow T(0) = 0$$

Remark: $0 \in V$ $\quad$ $0 \in W$.

Example:

(1) Let V, W be vectors spaces over $\mathbb{F}$.

Then $T: V \to W$ defined by

$$T(v) = 0 \quad \forall v \in V$$

is a linear transformation and it is called the zero map or the zero linear transformation.

(2) $Id_V : V \longrightarrow V$ the identity map, i.e. $Id_V(v) = v \quad \forall v \in V$ is a linear transformation, called the identity map on V or the identity operator.

(3) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$T(x,y) = (x-y+1, x+y)$ is NOT a linear transformation, since $T(0,0) = (1,0) \neq (0,0)$.

(4) Fix $a, b, c, d \in \mathbb{R}$.

The map $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (for $F = \mathbb{R}$) given by

$$T(x, y) = (ax + by, cx + dy)$$

is a linear transformation. (Check!)

Exercise: Prove that any linear trans. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by

$T(x, y) = (ax + by, cx + dy)$ for some choice of $a, b, c, d \in \mathbb{R}$.

(5) For $1 \leq i \leq n$, let

$p_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined

by $p_i(a_1, \ldots, a_n) = a_i$.

Then p_i is a linear transformation and it is called i-th projection.

(6) Let $V = C([0,1])$ be the space of all $\mathbb{R}$-valued continuous maps on $[0,1]$.

Define $T : C([0,1]) \longrightarrow \mathbb{R}$ by

$$T(f) = f\left(\tfrac{1}{2}\right),$$

called the evaluation map at $1/2$, is a linear transformation.

Let $f_1, f_2 \in C([0,1])$ and $\alpha, \beta \in \mathbb{R}$

Then
$$
\begin{aligned}
T(\alpha f_1 + \beta f_2) &= (\alpha f_1 + \beta f_2)\left(\tfrac{1}{2}\right) \\
&= (\alpha f_1)\left(\tfrac{1}{2}\right) + (\beta f_2)\left(\tfrac{1}{2}\right) \\
&= \alpha f_1\left(\tfrac{1}{2}\right) + \beta f_2\left(\tfrac{1}{2}\right) \\
&= \alpha T(f_1) + \beta T(f_2)
\end{aligned}
$$

Remark! In fact, you can fix any $x_0 \in [0,1]$ and define the evaluation map at x_0, by

$T(f) = f(x_0)$ and it is a linear transformation from $V \to \mathbb{R}$.

(7) Take $V = \mathbb{C}[x] = W$

Define $D : \mathbb{C}[x] \longrightarrow \mathbb{C}[x]$ by

$$D(f) = \frac{d}{dx}(f)$$ the formal differentiation (i.e. $\frac{d}{dx}(x^n) = n x^{n-1}$).

Then D is a linear transformation.

Remark: T is onto but not 1-1.

(8) Take $V = \mathbb{C}[x] = W$.

Define $M : \mathbb{C}[x] \longrightarrow \mathbb{C}[x]$ by

$$M(f(x)) = x f(x).$$

Then M is a linear transformation.
This map is called multiplication by x.

Remark: You can fix a poly, say $P(x)$ and define the multiplication by $P(x)$ and it is a lin. trans.

Remark: The map M is injective but NOT surjective.

Remarks / definitions:

1. If $T : U \to V$ and $S : V \to W$ are linear transformations then the composition $S \circ T : U \longrightarrow W$ is also a linear transformation.

2. A linear trans. $T : V \to V$ is also called a linear operator.

3. If a linear trans. $T : V \to W$ is bijective then the inverse map $T^{-1} : W \to V$ is also a linear trans. If this is the case, we say that T is an isomorphism.

4. Vector space V and W are called isomorphic if $\exists$ an isomorphism $T : V \to W$.

5. A linear operator $T : V \to V$ is called invertible if T^{-1} exists. (i.e. T is bijective).

Definition: Let V, W be vector spaces over $\mathbb{F}$ and $T : V \longrightarrow W$ a lin. trans.

Define $\ker(T) := \{v \in V : T(v) = 0\}$

and $T(V) := \{T(v) : v \in V\}$.

Observation: $\ker(T)$ and $T(V)$ are subspaces of V and W respectively.

Proof: (1) Let $u, v \in \ker(T)$ & $a, b \in \mathbb{F}$.

Then $T(au + bv) = aT(u) + bT(v)$

$= a \cdot 0 + b \cdot 0$ $\quad (u, v \in \ker(T))$

$= 0$

$\Rightarrow au + bv \in \ker(T)$

i.e. $\ker(T)$ is a subspace of V.

(2) Let $T(u), T(v) \in T(V)$ & $a, b \in \mathbb{F}$.

Then $aT(u) + bT(v)$

$= T(au + bv)$ $\quad$ (T is linear)

$\in T(V)$ $\quad$ since $au + bv \in V$.

i.e. $T(V)$ is a subspace.

Definition: Let $T : V \to W$ be a linear transformation.

(1) The space $\ker(T)$ is called the kernel or the null space of T and the dimension of $\ker T$ is called the nullity of T. (written as $\text{nullity}(T)$)

(2) The space $T(V)$ is called the image space or the range space of T and the dimension of $T(V)$ is called the rank of T. (written as $\text{rank}(T)$).

Lemma: Let T be a linear trans. Then T is injective iff $\ker(T) = \{0\}$.

Proof: ($\Rightarrow$) Assume T is injective.

If $u \in \ker(T)$ then

$$T(u) = 0 = T(0)$$

$\Rightarrow$ $u = 0$ since T is injective.

i.e. $\ker(T) = \{0\}$.

($\Leftarrow$) Assume $\ker(T) = \{0\}$.

For $u, v \in V$, assume $T(u) = T(v)$

$$\Rightarrow T(u-v) = 0$$
$$\Rightarrow u - v \in \ker(T) = \{0\}$$
$$\Rightarrow u - v = 0$$
$$\Rightarrow u = v$$

i.e. T is injective.

Theorem: (rank-nullity theorem)

Let $T : V \longrightarrow W$ be a linear trans. Suppose that V is finite dimensional. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

Proof: Since V is finite dimensional, $\ker(T) \subseteq V$ is also finite dimensional.

Let $\{u_1, \dots, u_m\}$ be a basis of the $\ker(T)$.

Extend the lin. ind. set $\{u_1, \ldots, u_m\}$ to a basis $\{u_1, \ldots, u_m, v_1, \ldots, v_r\}$ of V, so that $\dim V = m + r$.

<u>Claim:</u> $B := \{T(v_1), \ldots, T(v_r)\}$ is a basis of $T(V)$.

- If this claim is true then

$$\begin{aligned} \dim V &= m + r \\ &= \dim(\ker(T)) + \dim(T(V)) \\ &= \text{nullity}(T) + \text{rank}(T). \end{aligned}$$

<u>Proof of the claim:</u>

B is lin. ind.: For $a_1, \ldots, a_r \in \mathbb{F}$

suppose $\sum_{i=1}^{r} a_i T(v_i) = 0$

$$\Rightarrow \quad T\left(\sum_{i=1}^{r} a_i v_i\right) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{r} a_i v_i \in \ker(T)$$

$$\Rightarrow \sum_{i=1}^{r} a_i v_i = \sum_{j=1}^{m} b_j u_j$$

for some $b_1, \ldots, b_m \in F$.

$$\Rightarrow \sum_{j=1}^{m} b_j u_j + \sum_{i=1}^{r} (-a_i) v_i = 0$$

$$\Rightarrow b_j = 0 = a_i \quad \forall\ j = 1, \ldots, m, \quad i = 1, \ldots, r$$

since $\{u_1, \ldots, u_m, v_1, \ldots, v_r\}$ is a basis of V hence lin. ind..

In particular, $a_i = 0 \ \forall\ i = 1, \ldots, r$ proving that B is lin. ind.

$T(V) = L(B)$ the span of B:

Let $w \in T(V)$. Then $\exists\, v \in V$ s.t. $w = T(v)$.
Since $\{u_1, \ldots, u_m, v_1, \ldots, v_r\}$ is a basis of V, $v = \sum_{i=1}^{m} \alpha_i u_i + \sum_{j=1}^{r} \beta_j v_j$

Then $w = T(v) = \sum_{i=1}^{m} \alpha_i T(u_i) + \sum_{j=1}^{r} \beta_j T(v_j)$

$$= \sum_{j=1}^{r} \beta_j T(v_j)$$

$$\in L(B).$$

Que: How to compute rank and nullity of a lin. trans. ?

Example: Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be given by

$$T(x, y, z) = (x+y-z,\ x-y+z,\ y-z).$$

$$\ker(T) = \{(x,y,z) \in \mathbb{R}^3 : T(x,y,z) = (0,0,0)\}$$

$$= \left\{(x,y,z) \in \mathbb{R}^3 : \begin{aligned} x+y-z &= 0 \\ x-y+z &= 0 \\ y-z &= 0 \end{aligned}\right\}$$

Solve this system of lin. equations.

$$\ker(T) = \{(x,y,z) : x = 0,\ y = z\}$$

$$= \{(0,t,t) : t \in \mathbb{R}\}$$

and $\{(0,1,1)\}$ is a basis of $\ker(T)$.

We get $\text{nullity}(T) = \dim \ker(T) = 1.$

& $\text{rank}(T) = \dim \mathbb{R}^3 - \text{nullity}(T)$

$$= 3 - 1$$

$$= 2.$$

On the other hand, one can compute rank(T) directly without calculating nullity(T).

Take a basis of V, say $\{e_1, e_2, e_3\}$ the standard bases.

Then $T(V)$ is generated by $\{T(e_1), T(e_2), T(e_3)\}$

$$T(e_1) = (1, 1, 0)$$
$$T(e_2) = (1, -1, 1)$$
$$T(e_3) = (-1, 1, -1)$$

Consider $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$ with rows $T(e_1), T(e_2), T(e_3)$.

Then the row space is the range space. Then a basis of row space is a bases of the range space.

To find a bases of the row space of A we find the row-reduced echelon form of A, which is $\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$

Hence $\{(1, 0, \frac{1}{2}), (0, 1, -\frac{1}{2})\}$ is a basis of $T(V)$.

Thus $\text{rank}(T) = 2$.

Applications of rank-nullity theorem:

Let $T: V \to W$ be a lin. trans.
Suppose dimension of V is finite.

Note that $\text{rank}(T) \geq 0$ & $\text{nullity}(T) \geq 0$.

We have $\text{rank}(T) + \text{nullity}(T) = \dim V$.

So $\text{rank}(T) \leq \dim V$
$\text{nullity}(T) \leq \dim V$.

Recall: T is injective $\iff \ker(T) = \{0\}$
$\iff \text{nullity}(T) = 0$.

T is surjective $\iff \text{rank}(T) = \dim W$.

Then rank-nullity theorem gives following:

Theorem:

(a) $\nexists$ injective lin. trans. $T: \mathbb{R}^m \to \mathbb{R}^n$ if $m > n$.

(b) $\nexists$ surjective lin. trans. $T: \mathbb{R}^m \to \mathbb{R}^n$ if $m < n$.

(c) $\exists$ isomorphism (a bijective lin. trans.) $T: \mathbb{R}^m \to \mathbb{R}^n$ if and only if $m = n$.

Proof: $T : \mathbb{R}^m \to \mathbb{R}^n$ lin. trans.

(a) T is injective $\Rightarrow$ nullity$(T) = 0$

then $\text{rank}(T) + 0 = \dim \mathbb{R}^m = m$

Since $T(V) \subseteq \mathbb{R}^n$, $\text{rank}(T) \leq n$

i.e. $m \leq n$

So if $m > n$ then $\nexists$ injective lin. trans $T : \mathbb{R}^m \to \mathbb{R}^n$

(b) T is surjective $\Rightarrow T(V) = \mathbb{R}^n$

$\Rightarrow \text{rank}(T) = n$

$n + \text{nullity}(T) = m$

$\Rightarrow m - n = \text{nullity}(T) \geq 0 \Rightarrow m \geq n$

So if $m < n$ then $\nexists$ surjective lin. trans $T : \mathbb{R}^m \to \mathbb{R}^n$

(c) T is an isomorphism

$\Rightarrow T$ is injective and surjective

$\Rightarrow m \leq n$ and $m \geq n$ (by (a) & (b)).

$\Rightarrow m = n$

If $m = n$, $Id : \mathbb{R}^m \to \mathbb{R}^m$ is an isomorphism

Theorem: Suppose V is fin. dim. space over $\mathbb{F}$, and $T: V \longrightarrow V$ a linear operator. Then T is injective $\iff$ T is surjective.

Proof: Since V is finite dimensional,

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

Then T is injective.

$$\begin{aligned}
&\iff \ker(T) = \{0\} \\
&\iff \text{nullity}(T) = 0 \\
&\iff \text{rank}(T) = \dim V \\
&\iff T(V) = V \\
&\iff T \text{ is surjective.}
\end{aligned}$$