Lecture-1 : Vector spaces & Linear algebra

General outline :

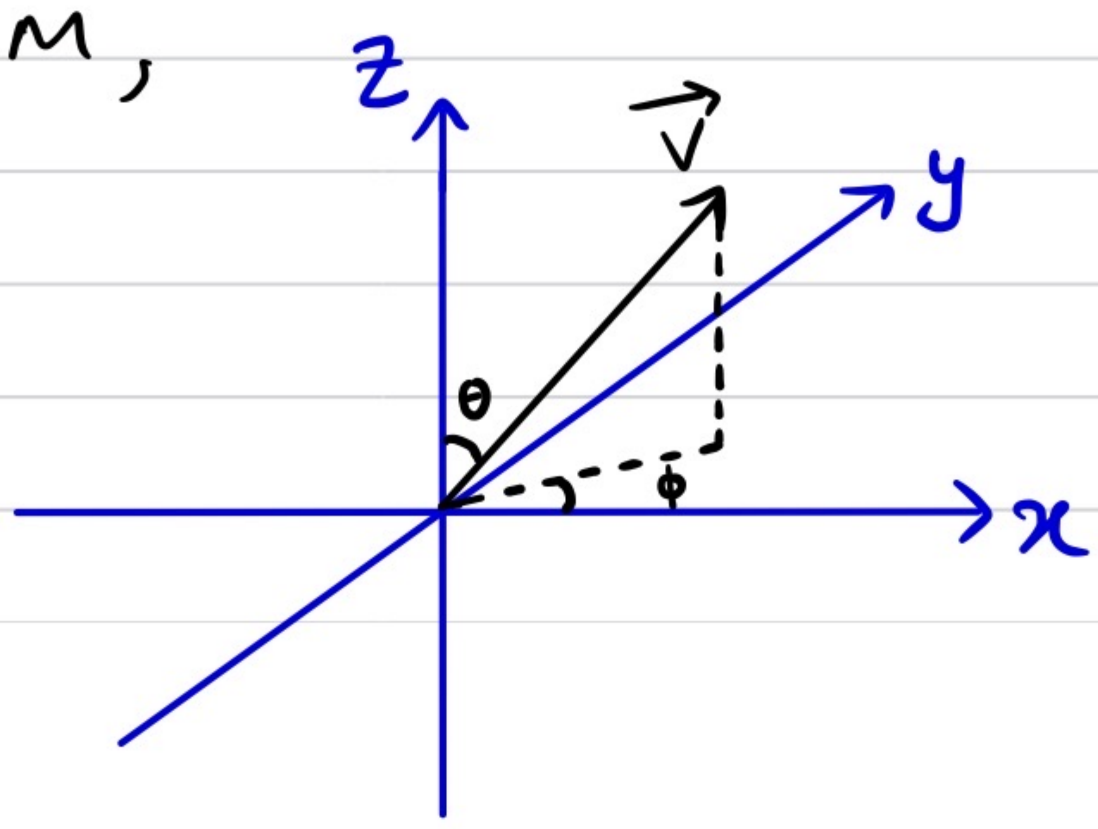
- Generalize the idea of vectors to a more abstract notion of vector spaces & see that many physical systems (most prominantly the space of states in Quantum mechanics) can be described in terms of vector spaces.

- Define linear transforms on vector spaces and their representation as matrices. We will see that many physical operations (e.g. time evolution, symmetry transformations & physically measurable quantities in QM) are represented by such matrices/operators.

- Learn a variety of properties of vector spaces (Linear independence, scalar products, Cauchy-Schwarz inequality, metric, Orthogonalization etc.) & Linear transformations which will be useful later to solve problems of physical interest.

- From our usual familiar notion of 3 dimensional vector, we intuitively think of them as "Arrows".
  - These arrows/vectors have some specific direction & their length represent their magnitude.

  e.g. The velocity, momentum, angular momentum & force e.t.c.



- Given a bunch of vector $\vec{V}_1, \vec{V}_2, \cdots$ we can construct other vector quantities by multiplying these with numbers & adding/subtracting them

i.e. $\vec{V} = a_1\vec{V}_1 + a_2\vec{V}_2 + \cdots$ is another vector while $a_1, a_2, \cdots$ are numbers.

These above notions can be mathematically formalized to abstractly define vectors spaces.

Vector spaces : A vector space V over a field F consists of

1. A set $V \equiv \{|a\rangle, |b\rangle, \cdots\}$ whose elements are called vectors, along with a binary operation called vector addition (+) under which V is closed i.e.

$$\forall \; |a\rangle, |b\rangle \in V \qquad |a\rangle + |b\rangle \in V$$

- The vector addition is commutative & associative

$$|a\rangle + |b\rangle = |b\rangle + |a\rangle$$

$$|a\rangle + (|b\rangle + |c\rangle) = (|a\rangle + |b\rangle) + |c\rangle$$

- Additive identity for vector addition : $\exists$ a unique vector $|0\rangle$ s.t.

$$|a\rangle + |0\rangle = |a\rangle \quad \forall \quad |a\rangle \in V$$

- Additive inverse : $\forall \; |a\rangle \in V$ $\exists$ a unique vector $|-a\rangle$ s.t.

$$|a\rangle + |-a\rangle = |0\rangle$$

2. A set $F \equiv \{\alpha, \beta, \cdots\}$ called field, whose elements are called scalars along with binary operations scalar addition (+) and multiplication (·) satisfying

Commutativity : $\alpha + \beta = \beta + \alpha$; $\alpha \cdot \beta = \beta \cdot \alpha$

Associativity : $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$; $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$

Additive Identity : $\exists$ unique $0 \in F$ s.t.
$$0 + \alpha = \alpha \quad \forall \quad \alpha \in F$$

Additive Inverse : $\forall \ \alpha \in F \ \exists$ unique $(-\alpha) \in F$ s.t.
$$\alpha + (-\alpha) = 0$$

Multiplicative Identity : $\exists$ unique $1 \in F$ s.t.
$$1.\alpha = \alpha \quad \forall \ \alpha \in F$$

Multiplicative Inverse : $\forall \ \alpha \in F$ except 0, $\exists$ unique $\tilde{\alpha} \in F$ s.t. $\alpha . \tilde{\alpha} = 1$

* For our discussion in this course, the field F will always be either $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers) with usual addition & multiplication. In this case the above properties of F are obvious

3. A joint binary operation - Multiplication between a scalar and a vector

{ Even though it is an independent operation, we will denote it by the same symbol as multiplication of scalars (.) }

Satisfying

- $\alpha.(\beta.|a\rangle) = (\alpha.\beta).|a\rangle$
- $(\alpha+\beta).|a\rangle = \alpha.|a\rangle + \beta.|a\rangle$
- $\alpha.(|a\rangle + |b\rangle) = \alpha.|a\rangle + \beta.|b\rangle$
- $1.|a\rangle = |a\rangle$

$$\forall \ |a\rangle, |b\rangle \in V \ \ \& \ \ \alpha, \beta \in F.$$

Lets look at some examples of vector spaces

1. $V = \mathbb{R}^n$ over $F = \mathbb{R}$ with usual addition & multiplication

$$|a\rangle \equiv (a_1, a_2, \cdots a_n) \quad , \quad |b\rangle \equiv (b_1, b_2, \cdots b_n) \quad \in \mathbb{R}^n$$

then $\alpha|a\rangle + \beta|b\rangle = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \cdots \alpha a_n + \beta b_n)$

Additive identity : $|0\rangle = |0, 0, \cdots 0\rangle$

Additive inverse : $|-a\rangle = |-a_1, -a_2, \cdots -a_n\rangle$

- The $n=3$ case is our usual 3 dimensional vector space.
- $n \neq 3$ are generalization to higher/lower dimensions.
- Replacing $\mathbb{R}$ with $\mathbb{C}$ also gives valid vector spaces.

2. $V =$ Real/Complex $m \times n$ matrices over $F = \mathbb{R}/\mathbb{C}$

$$|a\rangle = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \quad ; \quad |b\rangle = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \vdots \\ b_{m1} & \cdots & \cdots & b_{mn} \end{pmatrix}$$

with $\left(\alpha|a\rangle + \beta|b\rangle\right)_{ij} = \alpha a_{ij} + \beta b_{ij} \quad , \quad \alpha, \beta \in \mathbb{R}/\mathbb{C}$

Additive identity : $|0\rangle = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$

Additive inverse : $|-a\rangle_{ij} = -a_{ij}$

It is also straightforward to check that some restricted class of matrices also form vector space.

for example

- Symmetric/Antisymmetric square matrices over $\mathbb{C}$
  matrix M s.t. $M^T = \pm M$

- Complex hermitian/antihermitian matrices over $\mathbb{R}$

M s.t. $M^{\dagger} = \pm M$

Ex : Do ① real orthogonal matrices $(M^T M = 1)$
② Complex unitary matrices $(M^{\dagger} M = 1)$
form vector spaces over $\mathbb{R}$ or $\mathbb{C}$?
- Which property is violated ?

3. V = Function from $\mathbb{R}$ to $\mathbb{R}$ over $F = \mathbb{R}$ $\left\{ \mathbb{R} \text{ can be replaced with } \mathbb{C} \right\}$

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a mapping which takes a real number x as input & gives another real number $f(x)$ as output.

The space of all such maps is a vector space over $\mathbb{R}$ with vector addition & scalar multiplication defined as

$$(f+g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x)$$

Additive identity : $0(x) = 0 \quad \forall\, x \in \mathbb{R}$

Additive inverse : $(-f)(x) = -f(x)$.

As in EXM-2 above, some restricted class of functions also form vector spaces by themselves.

E.g. Real polynomial functions of degree n over $F = \mathbb{R}$

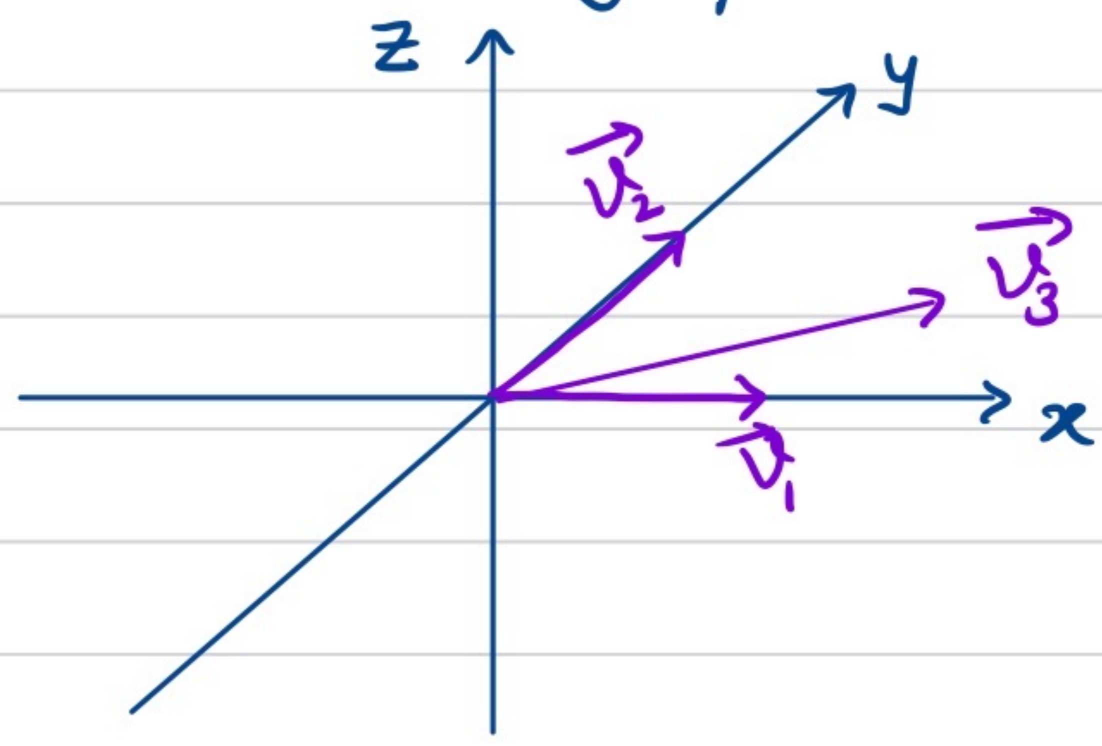
$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Linear independence : Given a set of vectors $(|v_1\rangle, |v_2\rangle, \ldots |v_n\rangle)$ in a vector space V over field F $(\mathbb{R}/\mathbb{C})$, the vectors are said to be linearly independent if $\nexists$ any set of scalars $(\alpha_1, \alpha_2, \ldots \alpha_n)$ not all zero, such that

$$\alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \cdots + \alpha_n|v_n\rangle = |0\rangle$$

Ex: Consider the 3 vectors lying in the x-y plane in the 3-dimensional space

$$\vec{v}_1 = 2\hat{\imath}, \quad \vec{v}_2 = 3\hat{\jmath}, \quad \vec{v}_3 = 3\hat{\imath} + 2\hat{\jmath}$$



$$\Rightarrow \boxed{\frac{3}{2}\vec{v}_1 + \frac{2}{3}\vec{v}_2 - \vec{v}_3 = \vec{0}} \qquad \vec{v}_4 = \hat{k}$$

Thus $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ are not linearly independent, While any pair $(\vec{v}_1, \vec{v}_2)$, $(\vec{v}_2, \vec{v}_3)$ or $(\vec{v}_1, \vec{v}_3)$ is linearly independent.

Span : Given a set of vectors $S = (|v_1\rangle, \ldots |v_n\rangle) \in V$, the span of S is the set of all linear combinations of vectors in S with coefficients in F.

$$\text{Span}(S) = \{ \alpha_1|v_1\rangle + \cdots + \alpha_n|v_n\rangle \;\; \forall \; \alpha_1, \alpha_2, \ldots \alpha_n \in F \}$$

From our previous example the span of each of the set $(\vec{v}_1, \vec{v}_2)$, $(\vec{v}_2, \vec{v}_3)$, $(\vec{v}_1, \vec{v}_3)$ as well as $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is the set of all vectors in x-y plane.

Basis : Given a vector space V over field F, any set $B = (|v_1\rangle\, |v_2\rangle, \ldots)$ of linearly independent vectors such that

$$\text{Span}_F(B) = V$$

is called a Basis of V.

Again, in our canonical example of 3 dimensional space its usual to take $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$ as the Basis set. But any

set of 3 linearly independent vectors
e.g. $(\hat{i}+2\hat{j}, 3\hat{j}+5\hat{k}, \hat{j})$ is an equally good Basis.

Ex : Argue that the number of Elements in any Basis of V is the same and is the maximum number of Linearly independent vectors in V.

Dimension : The number of Elements in any Basis of V is called the dimension of V ($\dim(V)$).

- As the name already suggests, the dimension of our 3 dimensional space is 3.
- The vector space of all polynomial functions over $\mathbb{C}$ has dimension Infinity! $B = \{1, x, x^2, x^3, \ldots\}$

Subspace : Given a vector space V over field F, a subset $W \subset V$ is called a subspace of V if W itself is a vector space over F w.r.t. the scalar multiplication and vector addition of V.

Exm

- V and the empty subset O are subspaces of V.
- The x-y plane in 3 dim'al space forms a 2 dim'al subspace. More generally any plane through origin is a subspace. Simillary any line through origin is a 1-dim'al subspace.

  Notice that the subset constituting any quadrant of say x-y plane or even the half plane is not a subspace!
- The subset symmetric (or anti-symmetric) $n \times n$ matrices is a subspace of the vector space of all $n \times n$ matrias over $F = \mathbb{R}/\mathbb{C}$.
- The subset of Hermitian (or anti-Hermition) $n \times n$ matrices is a subspace of $n \times n$ complex matrices only for $F = \mathbb{R}$ (but not for $F = \mathbb{C}$).

- Let, V = vector space of all $n \times 1$ matrices (column vectors). over $F = \mathbb{R}/\mathbb{C}$
  & A = a given fixed $m \times n$ matrix.
  The subset of column vector X $(n \times 1)$ in V s.t.
  $A \cdot X = 0$ {m linear homogeneous eqns in n variables}
  form a subspace of V.
  i.e. The set of solutions of homogeneous linear equations forms a vector space. → Check & convince yourself.

★ The same is true for homogeneous linear (partial) differential equations as well.

E.g. the Schroedinger equation, whose space of solutions is state space of corresponding Quantum mechanical system is a Vector space over $\mathbb{C}$.

Comments :

- The Span (over F) of any subset of a Vector space is a subspace of V.
- The intersection of any number of subspaces $W_1, W_2, \ldots W_n$ of V is also a subspace of V.
- The union of subspaces is generally not a subspace.

Coordinates on a Vector space :

Let $(V;F)$ be a vector space and $B = \{|\beta_1\rangle, |\beta_2\rangle, \dots |\beta_n\rangle\}$ be a Basis of V. Clearly then $\forall\ |\alpha\rangle \in V\ \exists$ a unique set of scalars $(a_1, a_2, \dots a_n)$ s.t.

$$|\alpha\rangle = a_1|\beta_1\rangle + a_2|\beta_2\rangle + \dots + a_n|\beta_n\rangle$$

This n-tuple of scalars $(a_1, a_2, \dots a_n)$ are reffered to as the <u>coordinates of $|\alpha\rangle$</u> <u>w.r.t the Basis B</u>.

→ The coordinates for any vector are Basis dependent & the same vector $|\alpha\rangle$ will look different in different Basis.

$$\begin{aligned}|\alpha\rangle &= a_1|\beta_1\rangle + a_2|\beta_2\rangle + \dots + a_n|\beta_n\rangle \\ &= a_1'|\beta_1'\rangle + a_2'|\beta_2'\rangle + \dots + a_n'|\beta_n'\rangle\end{aligned}$$

where $B = \{|\beta_1\rangle, |\beta_2\rangle, \dots |\beta_n\rangle\}$ &

$B' = \{|\beta_1'\rangle, |\beta_2'\rangle, \dots |\beta_n'\rangle\}$ are two different Basis of V.

Of course the coordinates $\{a_i\}$ & $\{a_i'\}$ are related to each other since they represent the same vector. Let's see how that works.

Since B & B' are Basis for V, lets decompose $|\beta_i'\rangle$ in B basis.

$$|\beta_1'\rangle = M_{11}|\beta_1\rangle + M_{12}|\beta_2\rangle + \dots + M_{1n}|\beta_n\rangle$$

$$\vdots$$

$$|\beta_n'\rangle = M_{n1}|\beta_1\rangle + M_{n2}|\beta_2\rangle + \dots + M_{nn}|\beta_n\rangle$$

More compactly $|\beta_i'\rangle = \sum_{j=1}^{n} M_{ij}|\beta_j\rangle$

The coefficients $\{M_{ij}\}$ form an $n\times n$ matrix M

$$|\alpha\rangle = \sum_{i=1}^{n} a'_i|\beta'_i\rangle = \sum_{i=1}^{n} a'_i\left(\sum_{j=1}^{n} M_{ij}|\beta_j\rangle\right)$$
$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a'_i M_{ij}\right)|\beta_j\rangle$$
$$= \sum_{j=1}^{n} a_j|\beta_j\rangle$$

Matrix notation

$$\Rightarrow \boxed{a_j = \sum_{i=1}^{n} a'_i M_{ij}} \equiv \boxed{A^T = A'^T \cdot M}$$

where $A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $A' = \begin{pmatrix} a'_1 \\ a'_2 \\ \vdots \\ a'_n \end{pmatrix}$

* The matrix M relating the vectors in two Basis B & B' is always invertible.
  - The rows of M are all linearly independent since they are just the coordinates of linearly independent vectors $|\beta'_i\rangle$'s in B Basis.

Thus
$$\left.\begin{aligned} |\beta'_i\rangle &= M_{ij}|\beta_j\rangle \\ \Rightarrow |\beta_i\rangle &= (M^{-1})_{ij}|\beta'_j\rangle \end{aligned}\right\} \text{ — (1)}$$

Notation: Repeated indicies are summed over

Similarly
$$a_i = a'_j M_{ji}$$
$$\Rightarrow a'_i = a_j (M^{-1})_{ji}$$

$$\left\{\begin{aligned} A &= M^T \cdot A' \\ A' &= (M^T)^{-1} \cdot A \end{aligned}\right\} \text{ — (2)}$$

* Note that once we know the matrix M relating the two Basis B & B' (eqn (1)) we know how to transform the coordinates of any arbitrary vector in V using equation (2).

Comments:

- The matrix relating any two arbitrary Basis of V is always invertible.
- Given any Basis B of an n-dimensional vector space

V and any invertible $n \times n$ matrix M, we can construct a new Basis B' of V.

$$\text{Let } B = \{ |\beta_i\rangle \quad i = 1, 2, \dots n \}$$

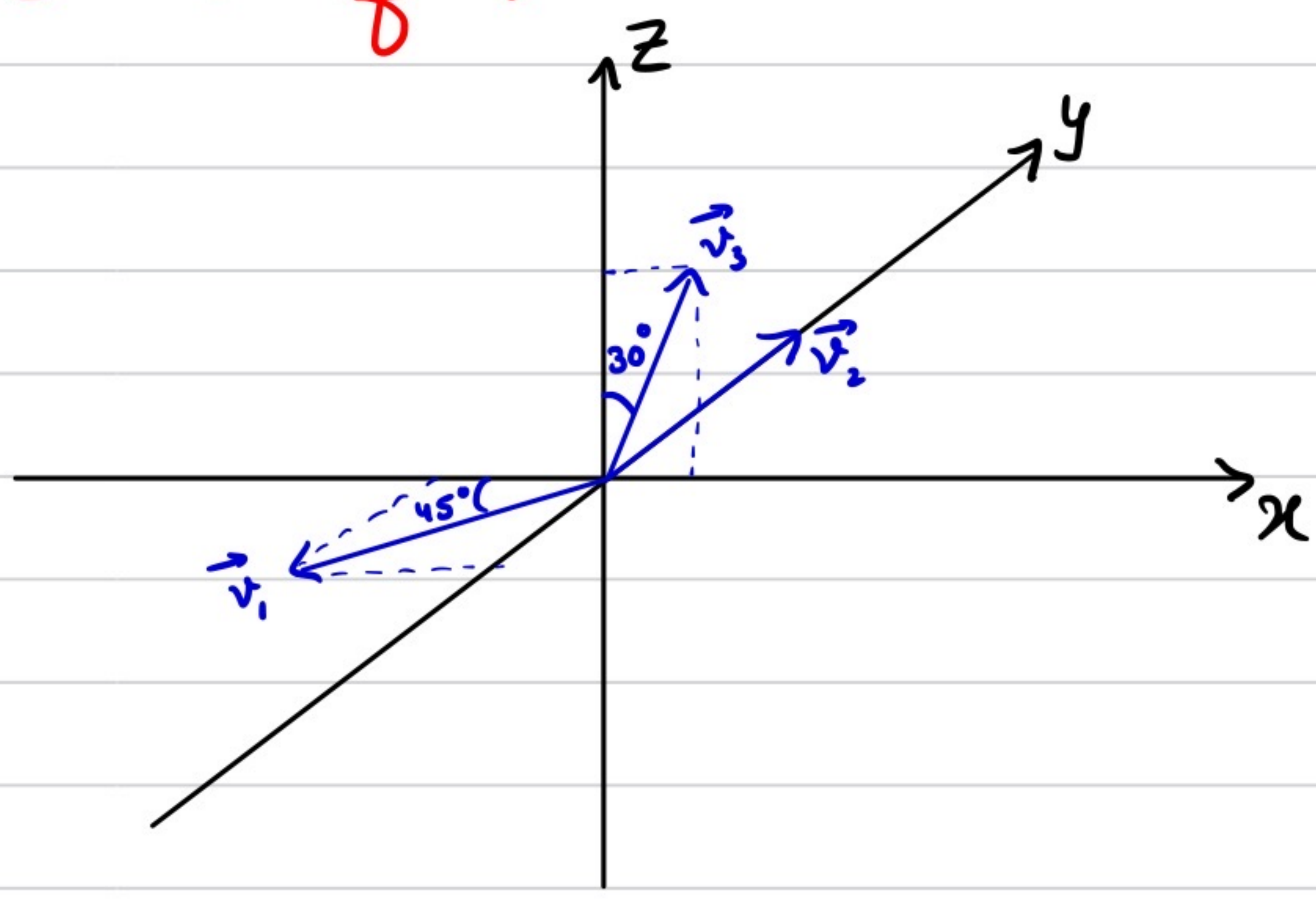
$$\text{Define } B' = \left\{ |\beta_i'\rangle = M_{ij} |\beta_j\rangle \; ; \quad i = 1, 2, \dots n \right\}$$

$\longrightarrow$ Invertibility of M (rows linearly independent), implies that all the n $|\beta_i'\rangle$ are all linearly independent, & hence B' is a Basis for V.

Ex: Consider $V = \mathbb{R}^3$ over $F = \mathbb{R}$, our real 3 dimensional space & consider the following two Basis of V

$$B = \{ \hat{i}, \hat{j}, \hat{k} \}$$

$$B' = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$$



shown in figure.
- $\vec{v}_1$: Unit vector in x-y plane making angle 45° with $-$ve x- axis
- $\vec{v}_2$: Length 2 vector along +ve y-axis
- $\vec{v}_3$: unit vector in x-z plane making 45° angle with +ve z axis.

Construct the matrix M relating the elements of B to B' i.e.

$$\hat{i} = M_{1j} \vec{v}_j$$

$$\hat{j} = M_{2j} \vec{v}_j$$

$$\hat{k} = M_{3j} \vec{v}_j$$

— Check that M is invertible.