Lecture 5 Tue 1 Aug 2023

* Direction (recession direction)

$X \rightarrow$ convex set in $\mathbb{R}^n$.

$x_0 \in X \qquad d \in \mathbb{R}^n, \ d \neq 0$

$x_0 + \lambda d \in X \quad \forall \lambda \geq 0$

$D_X(x_0) = \{ d \in \mathbb{R}^n : x_0 + \lambda d \in X \ \forall \lambda \geq 0 \}$

X: bounded $\Leftrightarrow$ $\boxed{D_X(x_0) = \phi}$

* Extreme Dir^n

A vector $d \neq 0$, $d \in D_X(x_0)$ is called an extreme dir^n if,

$\nexists \ d^1, d^2 \in D_X(x_0), \ d^1 \neq \mu d^2, \ \mu \geq 0$ s.t.

$d = \lambda_1 d^1 + \lambda_2 d^2$ for some $\lambda_1, \lambda_2 \geq 0$.

$S = \{ x \in \mathbb{R}^n : Ax \leq b, \ x \geq 0 \} \qquad x_0 \in S$

$d \in D_X(x_0), \ d \neq 0, \ x_0 + \lambda d \in S, \ \forall \lambda \geq 0$.

$Ax_0 + \lambda Ad \leq b, \quad x_0 + \lambda d \geq 0, \ \forall \lambda \geq 0$.

$\lambda Ad \leq \underbrace{b - Ax_0}_{\geq 0} \quad \forall \lambda \geq 0. \quad \text{—— } (*)$

$\lambda \geq 0 \Leftarrow$

If $Ad \leq 0$ then $(*)$ is always true. $\forall \lambda \geq 0$.

Also, if any component of Ad is > 0, then $(*)$ does not hold $\forall \lambda > 0$.

We know, $x_0 + \lambda d \geq 0$,

if any component of d is $-ve$, then we can make that component of $x_0 + \lambda d < 0$

$\therefore d \geq 0$

Note: $x_0 + \lambda d \geq 0 \quad \forall \lambda \geq 0$ iff $d \geq 0$.

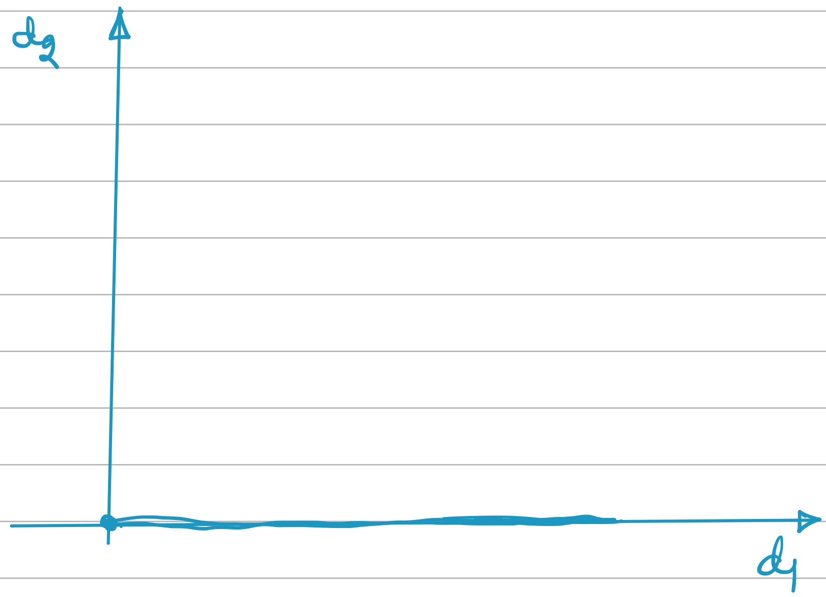
$S = \{(x, y) : x + y \geq 2,\ y \leq 4,\ x, y \geq 0\}$.

$$A = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \qquad A\begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

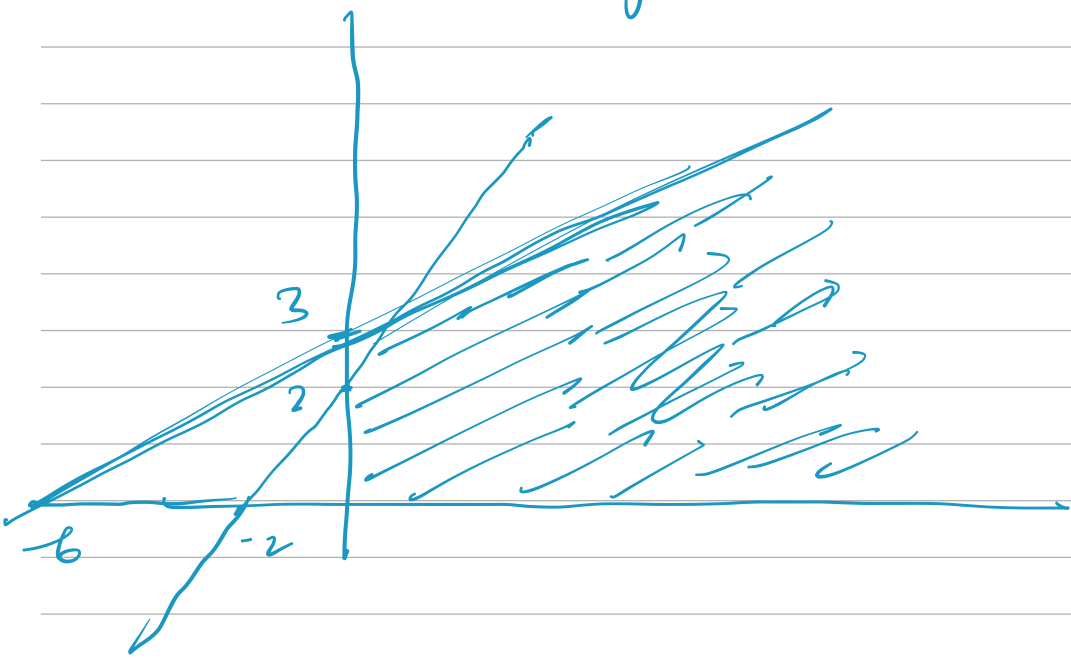
$$Ad \leq 0 \implies \left.\begin{array}{l} -d_1 - d_2 \leq 0 \\ d_2 \leq 0 \end{array}\right\} \quad \begin{array}{l} d_1 + d_2 \geq 0 \\ d_2 \leq 0. \end{array}$$

$$D_S = \left\{ d \in \mathbb{R}^2,\ d \neq 0 : \begin{array}{l} d_1 + d_2 \geq 0 \\ d_2 \leq 0 \\ d \geq 0 \end{array} \right\}$$

$$= \{ d \in \mathbb{R}^2,\ d \neq 0,\ d_1 \geq 0,\ d_2 = 0 \}$$



$$S = \left\{ (x, y) : \begin{array}{l} -x + 2y \le 6 \\ -x + y \le 2 \\ x, y \ge 0 \end{array} \right\}$$



$$D_S = \left\{ d \in \mathbb{R}^n,\ d \neq 0,\ \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} d \le 0,\ d \ge 0 \right\}$$

$$-d_1 + 2d_2 \le 0 \qquad\qquad d_1, d_2 \ge 0$$

$$-d_1 + d_2 \le 0$$

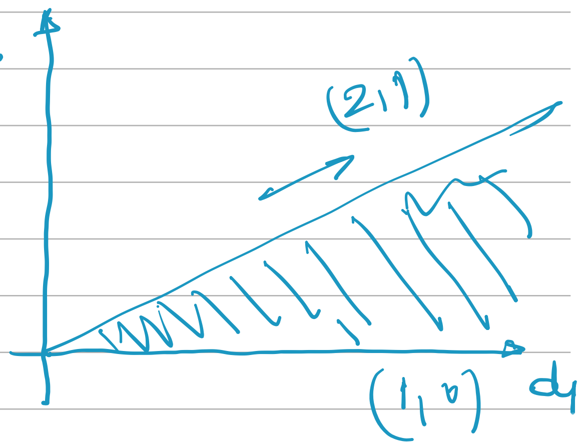
$\Rightarrow$ $d_1 \ge 2d_2$

$d_1 \ge d_2$, $d_1, d_2 \ge 0$.

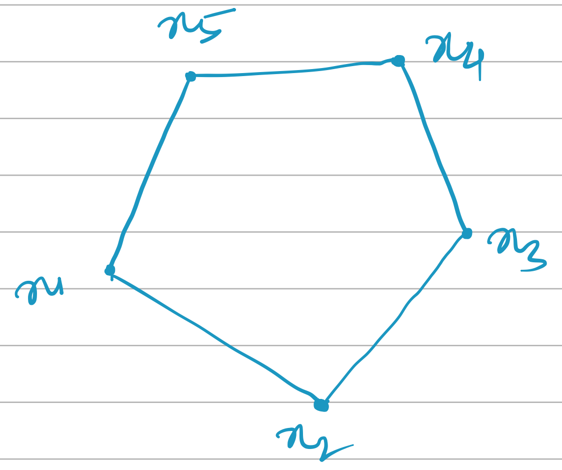
$$D_S = \{ d \in \mathbb{R}^2 : d \neq 0,\ d_1 \geq 2d_2 \ge 0 \}$$

Two extreme dir^n

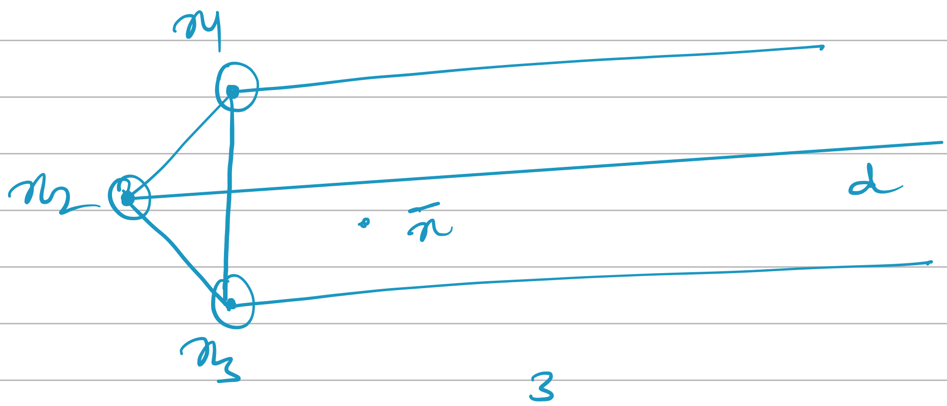
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



* Representation theorem of polyhedron.



$$\bar{x} = \sum_{i=1}^{5} \lambda_i x_i \,, \quad \sum_{i=1}^{5} \lambda_i = 1 \,, \quad \lambda_i \geq 0 \ \forall i$$



$$\bar{x} = \sum_{i=1}^{3} \lambda_i x_i + \mu d$$

Let X be a polyhedron in $\mathbb{R}^n$.
Then, any point $\bar{x} \in X$ can be represented by,

$$\bar{x} = \sum_{i=1}^{p} \lambda_i x^i + \sum_{j=1}^{s} \mu_j d^j$$

$$\lambda_i \geq 0 \,, \ \forall i$$

$$\mu_j \geq 0 \,, \ \forall j$$

$$\sum_{i=1}^{p} \lambda_i = 1$$

Where, $\{x^1, x^2, \ldots, x^p\}$ are extremes pts of X.
and, $\{d^1, d^2, \ldots, d^s\}$ are extreme dir^n of X.

(LP) $\max \; z = c^T x$.

s.t. $x \in S = \{x \in \mathbb{R}^n : Ax \le b, \; x \ge 0\}$

$$\max. \; c^T \bar{x} = \sum_{i=1}^{p} \lambda_i (c^T x^i) + \sum_{j=1}^{s} \mu_j \underbrace{c^T}_{1\times n} \underbrace{d^j}_{n\times 1}$$

subject to $\lambda_i \ge 0$
$\mu_j \ge 0$
$\sum \lambda_i = 1$

Suppose $c^T d^j > 0$ for atleast one j.
$j = 1, 2, \ldots, s$.

We can choose corresponding μ_j's $\to \infty$
and all other μ_k's $= 0$.
$\lambda_1 = 1, \ldots, \lambda_i = 0, \; \forall i \ne 1$.
$\Rightarrow z \to \infty \; \therefore$ (LP) is unbounded.

Suppose $c^T d^j \le 0 \; \forall j = 1, \ldots, s$.
$\downarrow$
Take all μ_j's $= 0$.

$$\max c^T \bar{x} = \max \sum_{i=1}^{p} \lambda_i (c^T x^i),$$

$\lambda_i \ge 0, \; \sum \lambda_i = 1$

$$\Rightarrow \max c^T \bar{x} = \max_{i=1,\ldots,p} \{c^T x^i\}$$

Hence, if answer exist, it is an extreme pt

Take $\lambda_k = 1$ for which $c^T x^k$ was largest and all other λ_i's $= 0$, $i \neq k$,