ELL100: INTRODUCTION TO ELECTRICAL ENGG.

Single Phase AC Power Circuits

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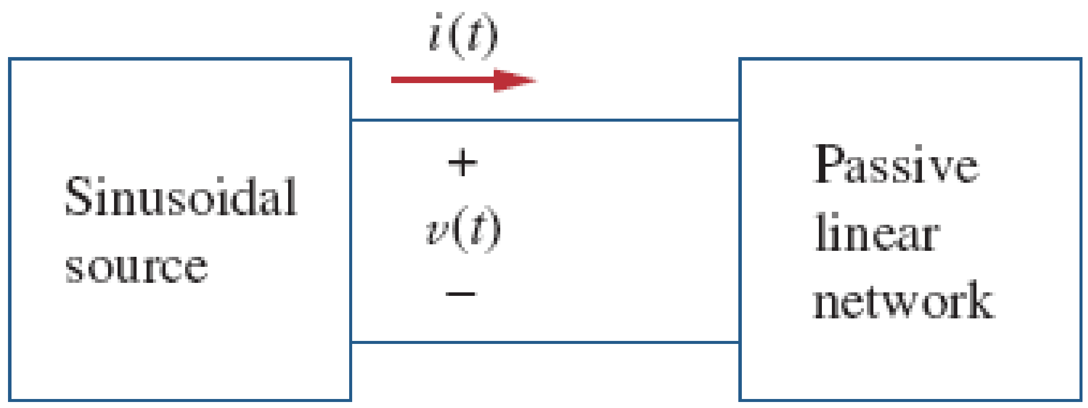
Notations in AC Power

- Instantaneous voltage/current: v(t), i(t)
  - $f(t) = f_m \cos(\omega t + \theta)$
- RMS (Root Mean Square) Value:
  - $f_{rms} = \boldsymbol{F} = \sqrt{\frac{1}{T}\int_0^T (f(t))^2 dt}$
  - In case of sinusoidal signals: $\boldsymbol{F} = \frac{f_m}{\sqrt{2}}$
- Instantaneous power: $p(t) = v(t)i(t)$
- Average Power: $\mathrm{P} = \frac{1}{T}\int_0^T p(t)dt$
- RMS Power: $P_{RMS} = \sqrt{\frac{1}{T}\int_0^T p^2(t)dt}$

Instantaneous Power

- Let the voltage and current be

$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$



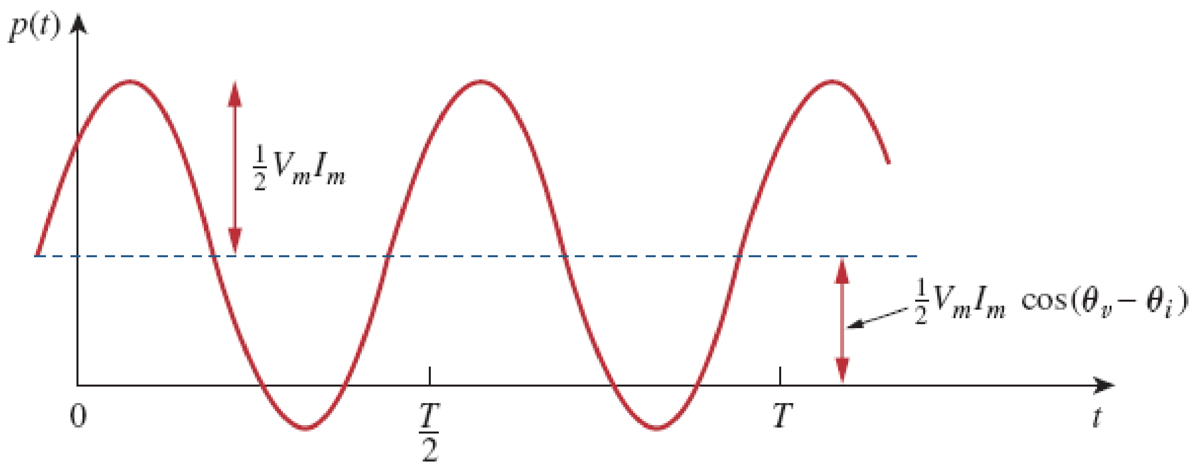
- Instantaneous power is

$$\begin{aligned} p(t) &= v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i) \\ &= \frac{V_m I_m}{2}\left(\cos(2\omega t + \theta_v + \theta_i) + \cos(\theta_v - \theta_i)\right) \end{aligned}$$

$$T = \frac{2\pi}{\omega}$$
$$p(t + \tfrac{T}{2}) = p(t)$$

$$\boxed{\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]}$$

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)$$
$$P = \mathrm{Re}\left[\tfrac{1}{2}\mathbf{V}\mathbf{I}^*\right]$$



Power: Resistive Load

$$p(t) = \frac{V_m I_m}{2}\left(\cos(2\omega t + \theta_v + \theta_i) + \cos(\theta_v - \theta_i)\right)$$

$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$

- Resistor:

$$\theta_v - \theta_i = 0 \implies p(t) = \frac{V_m I_m}{2} + \frac{V_m I_m}{2}\cos(2\omega t + 2\theta_v) \geq 0$$

- Average Power:

$$P = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} p(t)dt = \frac{V_m I_m}{2} = V_{\text{rms}} I_{\text{rms}} = \frac{V_{\text{rms}}^2}{R} = I_{\text{rms}}^2 R$$

where V_{rms} and I_{rms} are the RMS values of the voltage and current signals.

Power: Inductive Load

$$p(t) = \frac{V_m I_m}{2}\left(\cos(2\omega t + \theta_v + \theta_i) + \cos(\theta_v - \theta_i)\right)$$

$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$

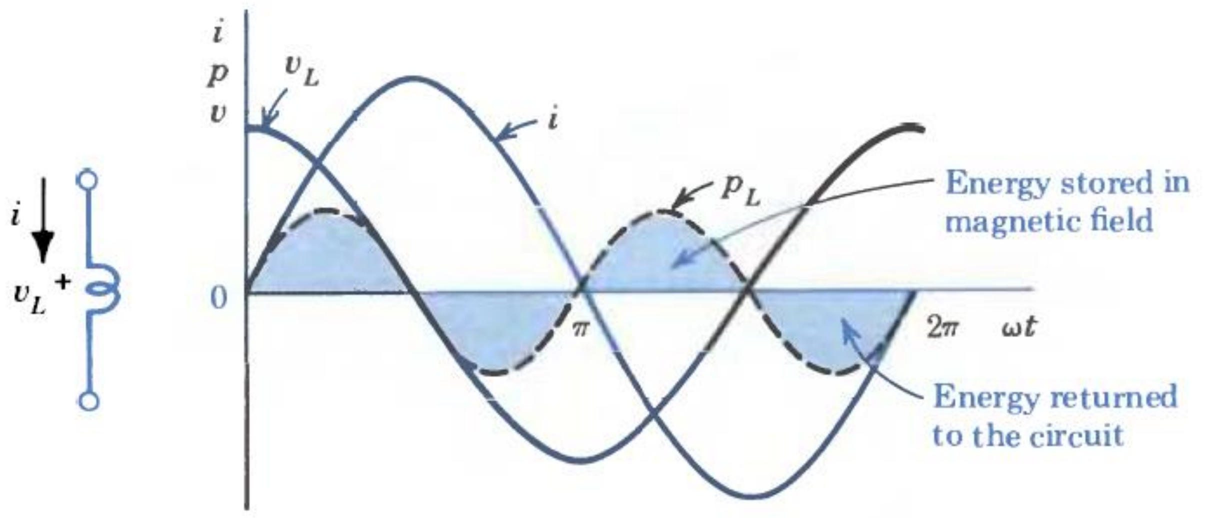
- Inductor:

$$\theta_v - \theta_i = \frac{\pi}{2} \implies p(t) = \frac{V_m I_m}{2}\cos(2\omega t + 2\theta_v - \pi/2) = I_{\text{rms}}^2 X_L \sin(2(\omega t + \theta_v))$$
$$X_L = \omega L$$

- Average Power:

$$P = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} p(t)dt = 0$$

- p(t) < 0 means that energy stored in L or C is being returned to the circuit



Power: Capacitive Load

$$p(t) = \frac{V_m I_m}{2}\left(\cos(2\omega t + \theta_v + \theta_i) + \cos(\theta_v - \theta_i)\right)$$

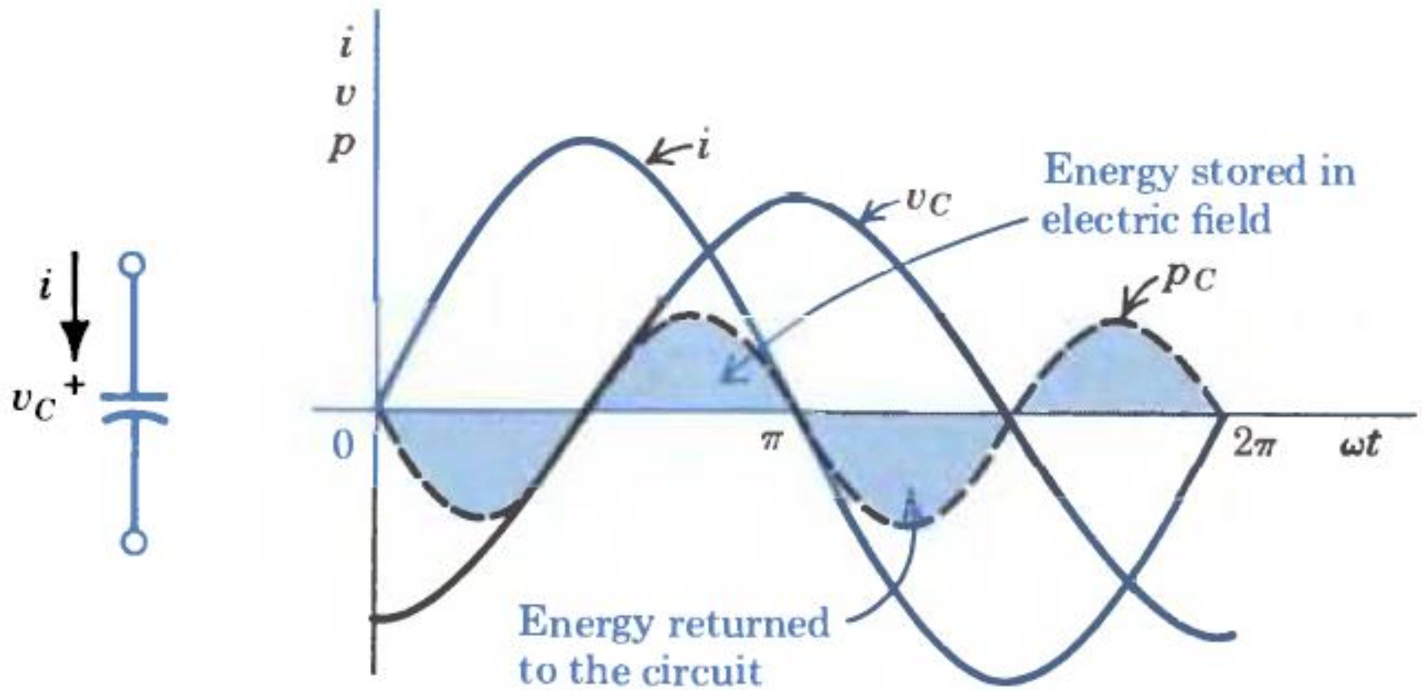
$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$

- Capacitor:

$$\theta_v - \theta_i = -\frac{\pi}{2} \implies p(t) = \frac{V_m I_m}{2}\cos(2\omega t + 2\theta_v + \pi/2) = I_{\mathrm{rms}}^2 X_C \sin(2(\omega t + \theta_v))$$
$$X_C = -1/\omega C$$

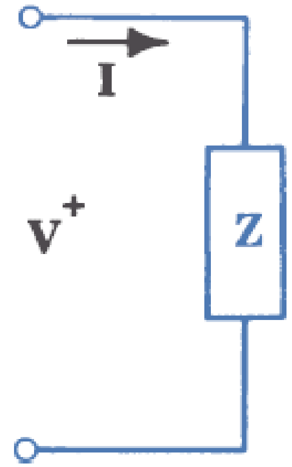
- Average Power:

$$P = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} p(t)dt = 0$$



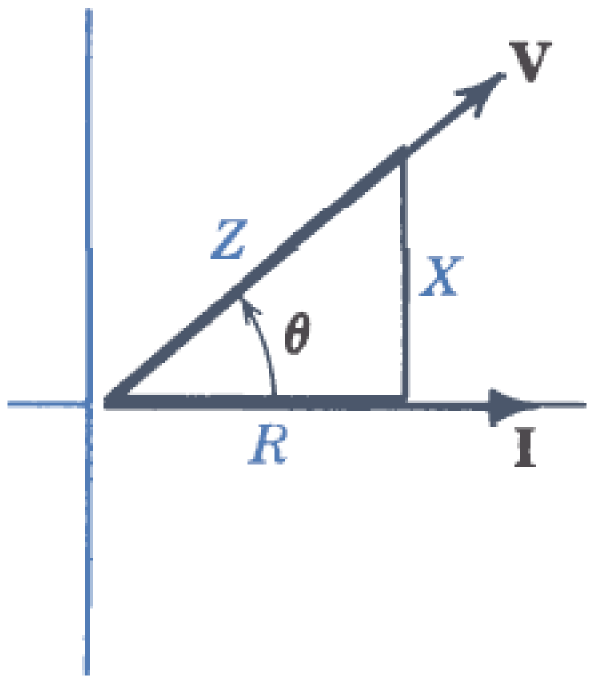
Power: General Impedance Load

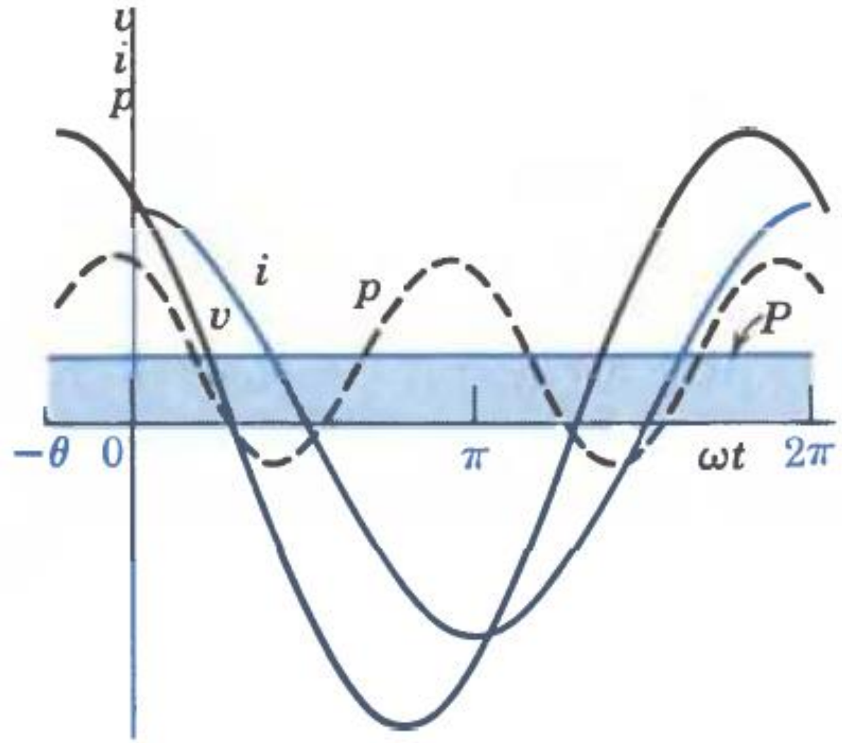
$$p(t) = \frac{V_m I_m}{2}\left(\cos(2\omega t + \theta_v + \theta_i) + \cos(\theta_v - \theta_i)\right)$$



- Average Power:

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta) = I_{\text{rms}}(I_{\text{rms}}|Z|\cos(\theta))$$
$$\theta = \theta_v - \theta_i$$





Power: Active and Reactive Load

- **Active or Real Power/Load**: Power that is dissipated in the resistive elements of the circuit (Watts: W)

$$P = VI\cos(\theta)$$

- **Reactive Power**: Power that is periodically stored in and returned to the circuit from energy storing elements like inductor and capacitor (Volt-Ampere Reactive: VAR)

$$Q = VI\sin(\theta)$$

- **Apparent Power**: The magnitude of power that is apparently used by the circuit (Volt-Ampere: VA)

$$P_A = VI$$

Power Factor and Reactive Factor

Power Factor: It is fraction of the total power that is available for consumption

$$\mathrm{pf} = \cos\theta = \frac{P}{VI}$$

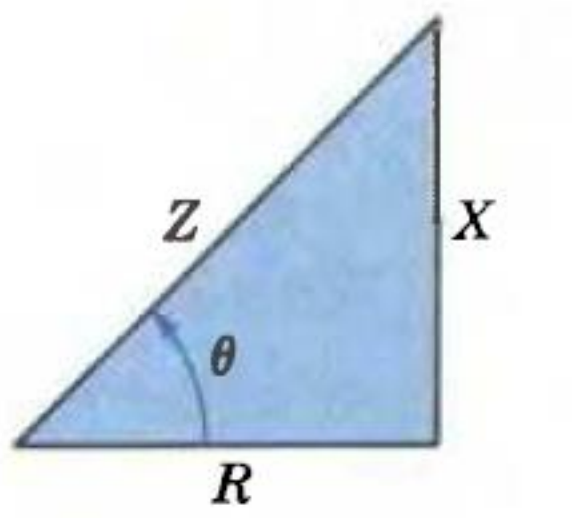
Reactive Factor: The fraction of the total power that 'circulates' in the energy storing elements: sin(θ)

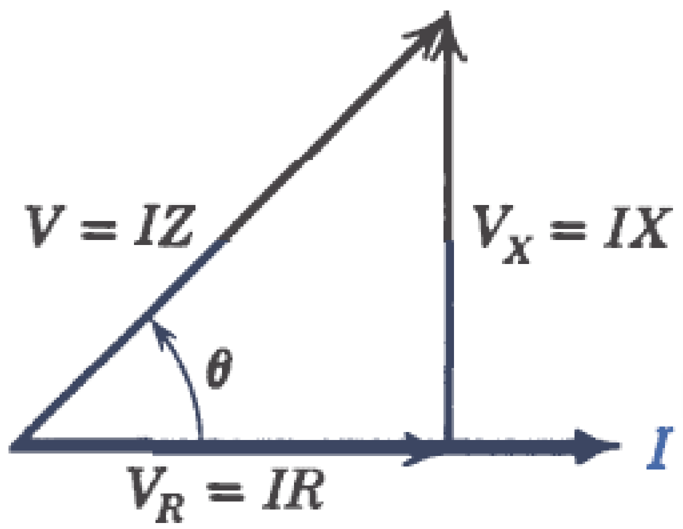
Leading and Lagging Load

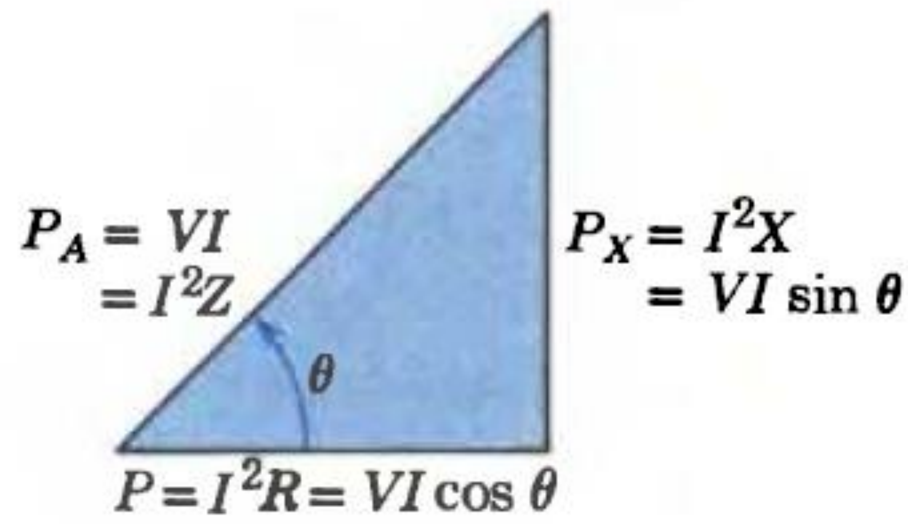
- AC load is always connected in parallel across a common voltage line
- The Voltage Signal is the reference signal while calculating phasors
- **Leading Load/power factor**: Current leads voltage → Load with **capacitive** component ($0>\theta>-\pi/2$)
- **Lagging Load/power factor**: Current lags voltage → Load with **inductive** component ($0<\theta<\pi/2$)

Complex Power

- A load with an inductive component causes current to lag voltage.
- If all components are divided by I, we get impedance triangle.
- If multiplied by I, we get the power triangle.







Complex Power

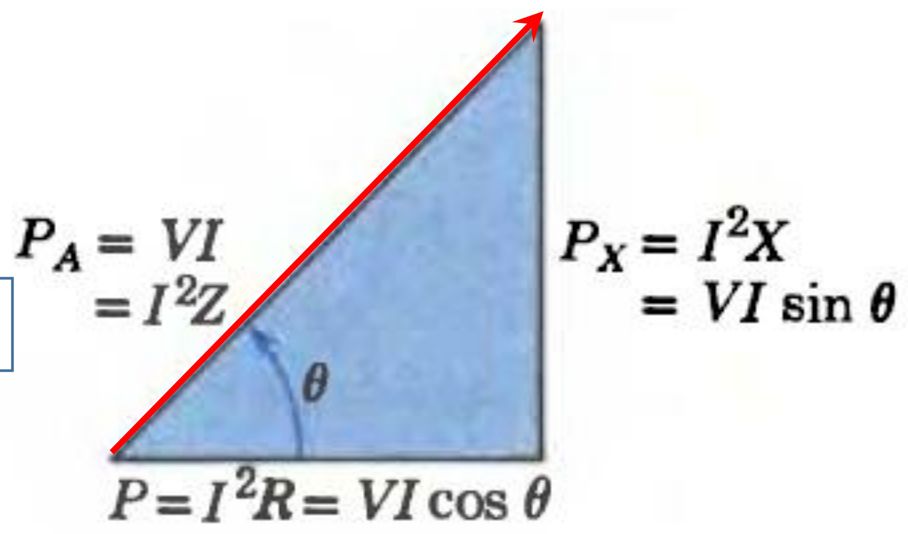
- Two of the sides of the power triangle can be identified as *active* and *reactive* power.
- The hypotenuse has magnitude = *Apparent Power*

$$\overrightarrow{\mathbf{P}_A} = VI\cos(\theta) + jVI\sin(\theta) = VI\angle\theta$$

$$\mathbf{S} = P + jP_X = P + jQ = P_A\angle\theta$$

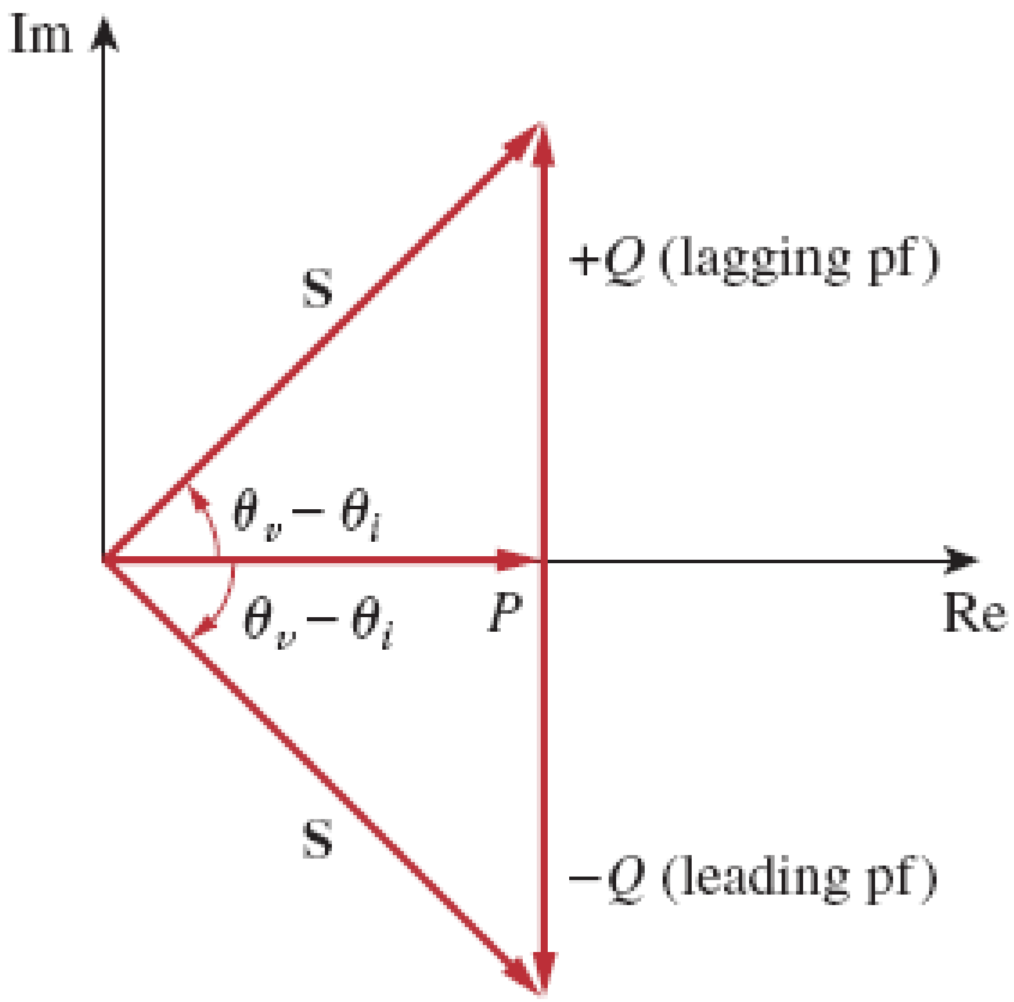
Vector

Magnitude



- **S** is the *complex power*. The source should be able to produce this quantity of complex power.
- Complex power captures all info about power absorbed by the load.

Complex Power



Complex Power

- Note that apparent power has magnitude VI and active power as VI cos θ.
- If both **V** and **I** are represented as phasors

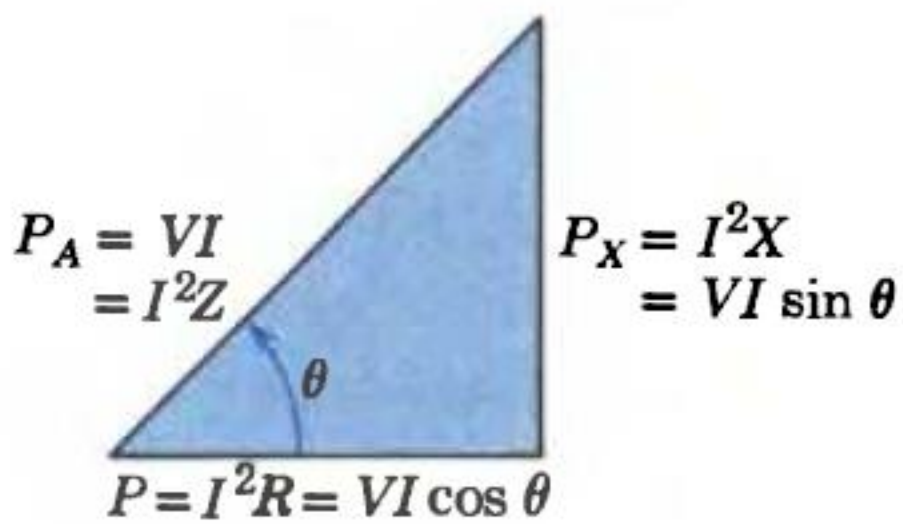
$$P = \mathrm{Re}\,[\mathbf{VI}^*]$$
$$Q = \mathrm{Im}\,[\mathbf{VI}^*]$$

$$\mathbf{VI}^* = (V_{\mathrm{rms}}\angle\theta_v)(I_{\mathrm{rms}}\angle-\theta_i)$$

$$= V_{\mathrm{rms}}I_{\mathrm{rms}}\angle\theta_v - \theta_i$$

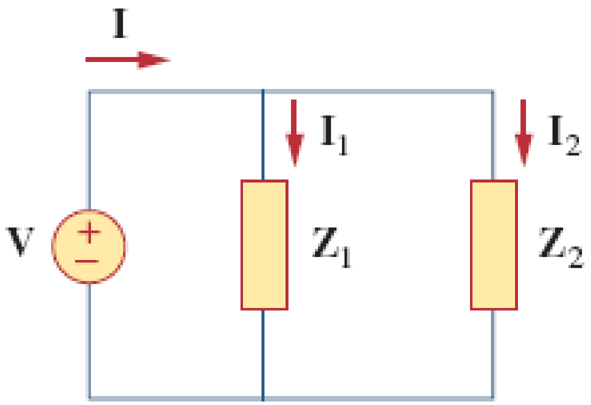
$$= V_{\mathrm{rms}}I_{\mathrm{rms}}\cos\theta + jV_{\mathrm{rms}}I_{\mathrm{rms}}\sin\theta$$

$$= P + jQ$$



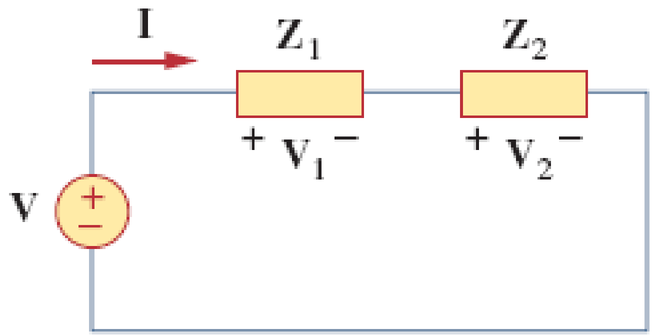
Conservation of AC Power

- Law of conservation of power in DC circuits, is also extendable to power in AC circuits.



$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2$$

$$\begin{aligned} \mathbf{S} &= \mathbf{V}\mathbf{I}^* = \mathbf{V}(\mathbf{I}_1^* + \mathbf{I}_2^*) \\ &= \mathbf{V}\mathbf{I}_1^* + \mathbf{V}\mathbf{I}_2^* = \mathbf{S}_1 + \mathbf{S}_2 \end{aligned}$$



$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$$

$$\begin{aligned} \mathbf{S} &= \mathbf{V}\mathbf{I}^* = (\mathbf{V}_1 + \mathbf{V}_2)\mathbf{I}^* \\ &= \mathbf{V}_1\mathbf{I}^* + \mathbf{V}_2\mathbf{I}^* = \mathbf{S}_1 + \mathbf{S}_2 \end{aligned}$$

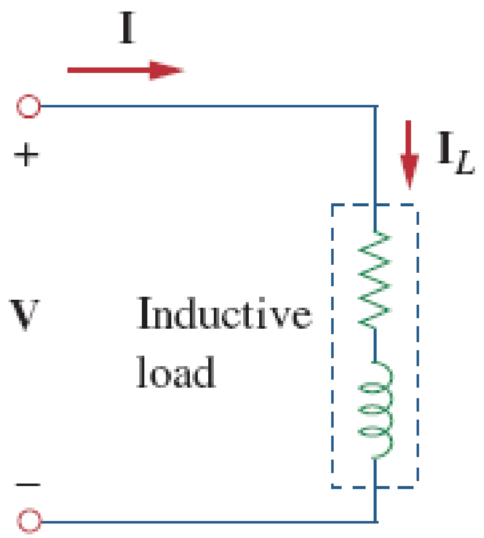
The complex, active and reactive powers of the sources equal the respective sums of the complex active and reactive powers of the individual loads.
(Note that Apparent power is missing in the list)

Power Factor Correction

- In AC circuit, less active power does not mean less power required
- A poor power factor (close to zero) is undesirable, implies larger reactive power for same active power delivery
- More current is needed and larger losses in transmission
- From viewpoint of transmission, pf close to unity is desired (Hence, discoms charge power factor penalty)

Power Factor Correction

- Reactive power minimization or power factor correction is done by consumers. Eg: done by adding capacitors to a lagging load



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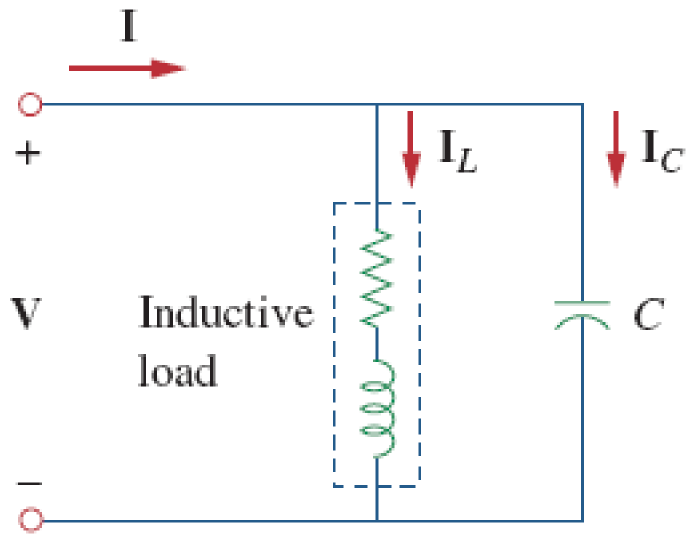
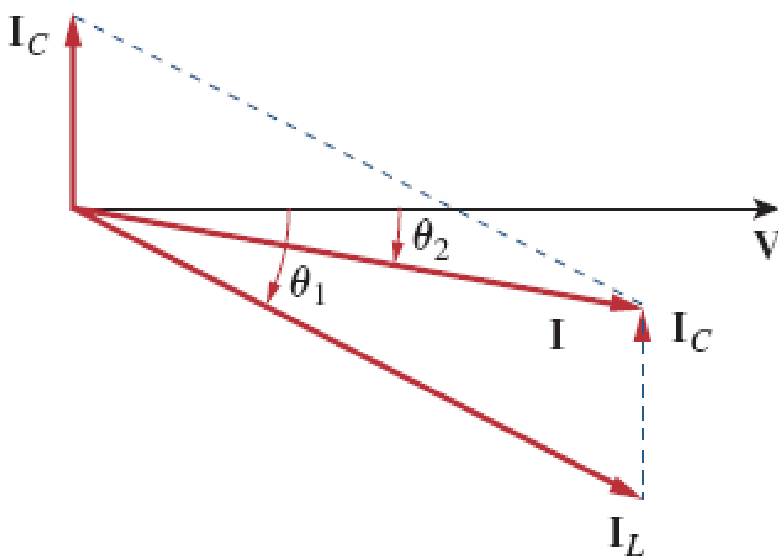
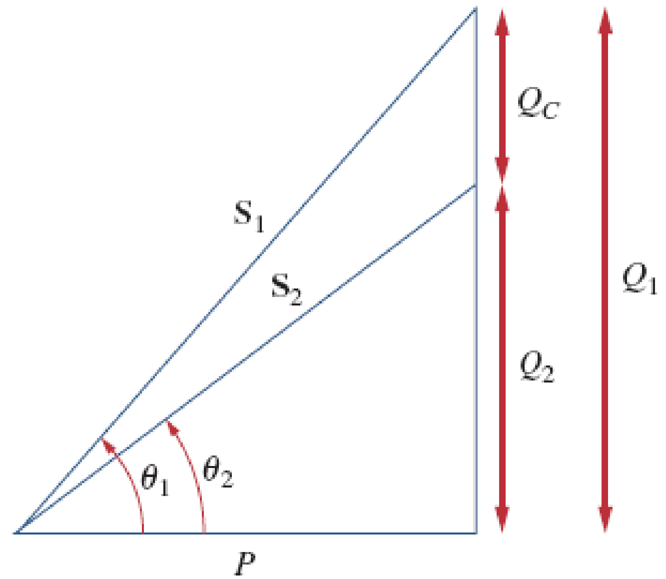


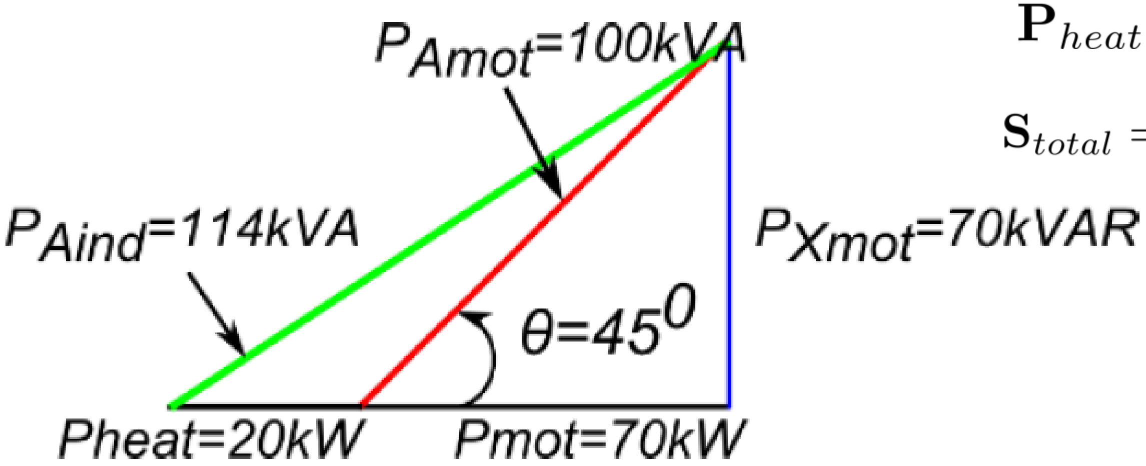




$$Q_1 = P\tan\theta_1$$
$$Q_2 = P\tan\theta_2$$
$$Q_C = Q_1 - Q_2 = P(\tan\theta_1 - \tan\theta_2)$$
$$Q_C = V_{\mathrm{rms}}^2/X_C = \omega C V_{\mathrm{rms}}^2$$

Example 1

- An industrial load consists of: 20 kW of heating and 100 kVA of induction motors operating at 0.707 pf lag.
- Find out the total active and reactive power drawn and the plant power factor



$$\mathbf{S}_{motor} = 1 \times 10^5 \angle 45^\circ = 70.7 \times 10^3 + j70.7 \times 10^3 \ VA$$

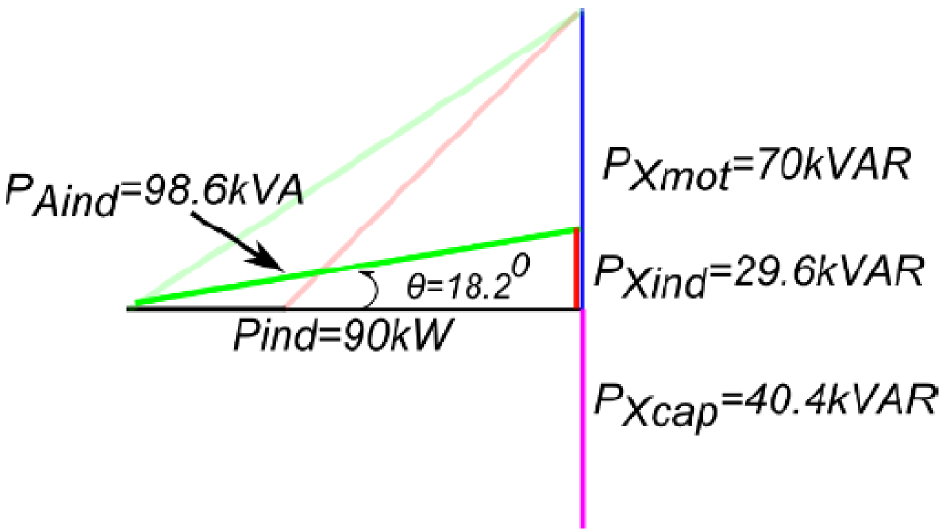
$$\mathbf{P}_{heat} = 20 \times 10^3 \ W$$

$$\mathbf{S}_{total} = 90.7 \times 10^3 + j70.7 \times 10^3 \ VA = 115 \times 10^3 \angle 37.9^\circ \ VA$$

$$\text{p.f.} = \frac{90.7}{115} = 0.789 \text{ lagging}$$

Example 2

- If power factor needs to be improved to 0.95 lag, what should be installed? What should be its power rating?



$$\mathbf{S}_{total} = 90.7 \times 10^3 + j70.7 \times 10^3 \ VA = 115 \times 10^3 \angle 37.9^\circ \ VA$$

Inclusion of capacitive element would improve the power factor by supplying reactive power.

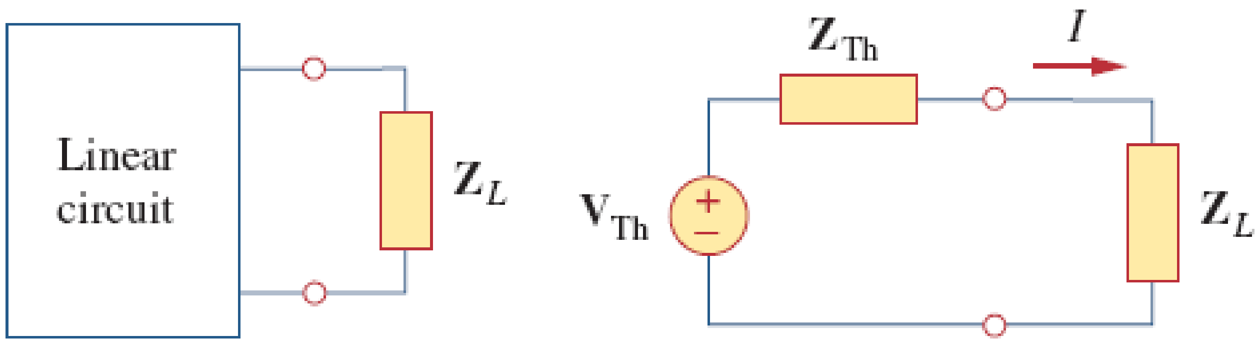
Required reactive power:
$j(90.7 \times 10^3) \tan(\cos^{-1}(0.95)) = j29.8 \ kVAR$

$$P_{X,cap} = j(29.8 - 70.7) kVAR = -j40.9 \ kVAR$$

$P_{app,corr} = (90.7 + j29.8) \ kVA$
Less power drawn than the uncorrected case.

Maximum Average Power Transfer

- We had seen maximum power transfer in DC circuits, with resistors.
- When is maximum average power transferred in an AC circuit?



$$Z_{Th} = R_{Th} + jX_{Th}$$
$$Z_L = R_L + jX_L$$

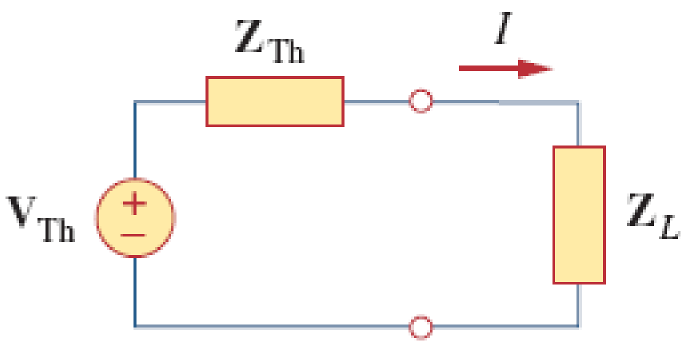
$$I = \frac{V_{Th}}{Z_{Th}+Z_L} = \frac{V_{Th}}{(R_{Th}+R_L)+j(X_{Th}+X_L)}$$

$$P = \mathbf{I}^2 R_L = \frac{\mathbf{V}_{Th}^2 R_L}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}$$

Here, V_{Th}, I represent the peak values of voltage and current. and $\mathbf{V}_{Th}$ and $\mathbf{I}$ are used to represent the RMS values.

Maximum Average Power Transfer

- To get the optimal impedance for maximum power transfer,



$$Z_{Th} = R_{Th} + jX_{Th}$$
$$Z_L = R_L + jX_L$$

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{Z_{Th}+Z_L} = \frac{\mathbf{V}_{Th}}{(R_{Th}+R_L)+j(X_{Th}+X_L)}$$

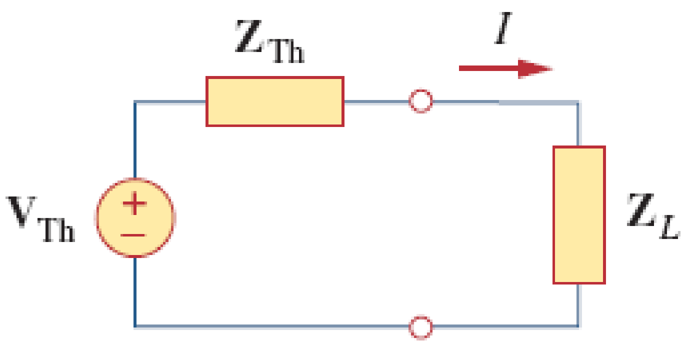
$$P = \mathbf{I}^2 R_L = \frac{\mathbf{V}_{Th}^2 R_L}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}$$

$$\frac{\partial P}{\partial X_L} = -\frac{\mathbf{V}_{Th}^2 R_L}{((R_{Th}+R_L)^2+(X_{Th}+X_L)^2)^2}\frac{\partial\big((R_{Th}+R_L)^2+(X_{Th}+X_L)^2\big)}{\partial X_L} = 0$$

$$\frac{\partial P}{\partial X_L} = -\frac{2\mathbf{V}_{Th}^2 R_L(X_{Th}+X_L)}{((R_{Th}+R_L)^2+(X_{Th}+X_L)^2)^2} = 0 \implies X_L = -X_{Th}$$

Maximum Average Power Transfer

- To get the optimal impedance for maximum power transfer,



$$Z_{Th} = R_{Th} + jX_{Th}$$
$$Z_L = R_L + jX_L$$

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{Z_{Th}+Z_L} = \frac{\mathbf{V}_{Th}}{(R_{Th}+R_L)+j(X_{Th}+X_L)}$$

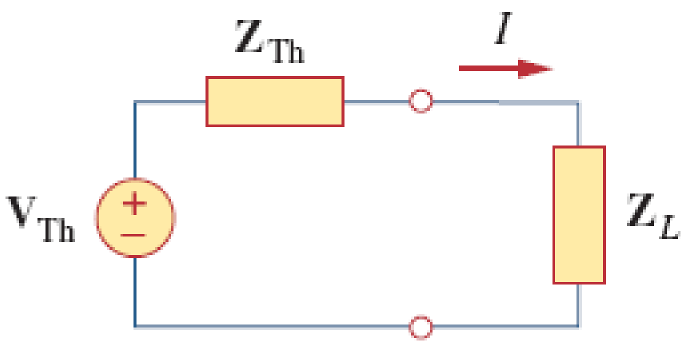
$$P = \mathbf{I}^2 R_L = \frac{\mathbf{V}_{Th}^2 R_L}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}$$

$$\frac{\partial P}{\partial X_L} = 0 \implies X_L = -X_{Th}$$

$$\frac{\partial P}{\partial R_L} = \frac{\mathbf{V}_{Th}^2((R_{Th}+R_L)^2+(X_{Th}+X_L)^2-2R_L(R_L+R_{Th}))}{((R_{Th}+R_L)^2+(X_{Th}+X_L)^2)^2} = 0$$
$$\implies R_L = \sqrt{R_{Th}^2 + (X_L + X_{Th})^2}$$

Maximum Average Power Transfer

- To get the optimal impedance for maximum power transfer,



$$Z_{Th} = R_{Th} + jX_{Th}$$
$$Z_L = R_L + jX_L$$

$$I = \frac{V_{Th}}{Z_{Th}+Z_L} = \frac{V_{Th}}{(R_{Th}+R_L)+j(X_{Th}+X_L)}$$

$$P = \mathbf{I}^2 R_L = \frac{\mathbf{V}_{Th}^2 R_L}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}$$

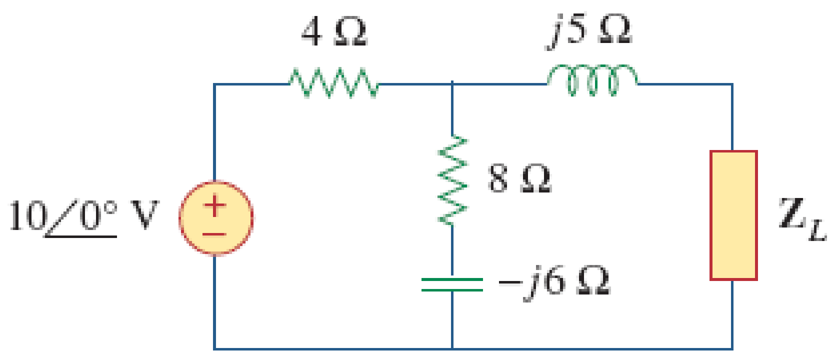
$$\boxed{X_L = -X_{Th} \qquad R_L = \sqrt{R_{Th}^2 + (X_L + X_{Th})^2} \implies R_L = R_{Th}} \longrightarrow Z_L = R_{Th} - jX_{Th} = Z_{Th}^*$$

$$P_{\max} = \frac{|\mathbf{V}_{Th}|^2}{4R_{Th}} = \frac{|V_{Th}|^2}{8R_{Th}}$$

In case of resistive load $X_L = 0$: $R_L = \sqrt{R_{Th}^2 + X_{Th}^2}$

Example 3

- Determine load impedance Z_L which maximizes the average power drawn from the following circuit. What is the maximum average power?



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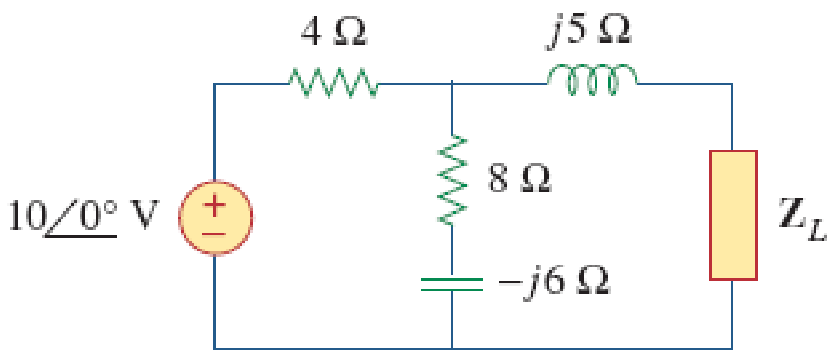
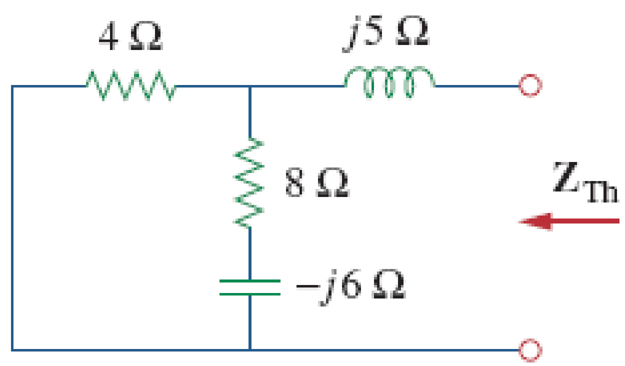


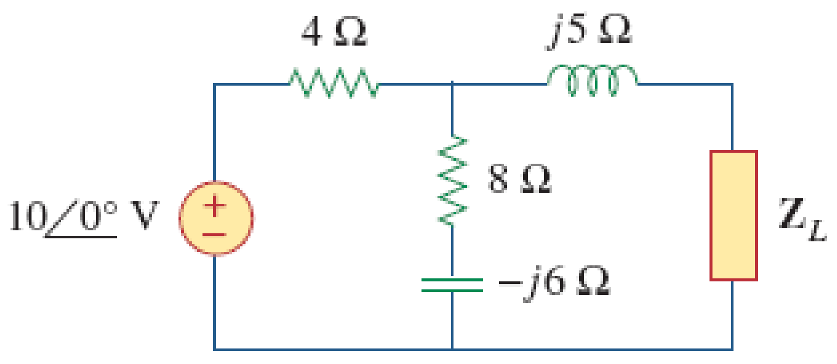
To determine Z_{Th}.



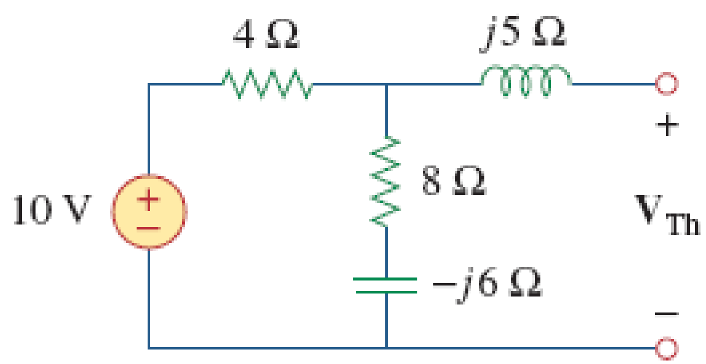
$$Z_{Th} = j5 + 4||(8 - j6) = j5 + \frac{4(8-j6)}{4+8-j6} = 2.933 + j4.467\ \Omega$$

Example 3

- Determine load impedance Z_L which maximizes the average power drawn from the following circuit. What is the maximum average power?



To determine V_{Th}:



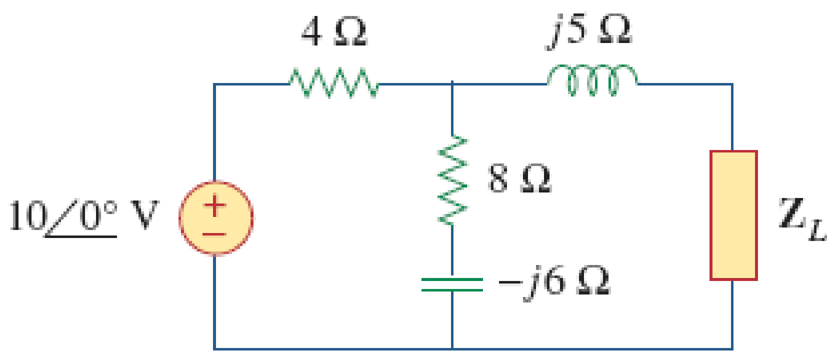
$$Z_{Th} = 2.933 + j4.467\ \Omega$$

$$V_{Th} = \frac{8-j6}{4+8-j6}(10) = \frac{80-j60}{12-j6}$$
$$= \frac{1320-j240}{180} = 7.454\angle(-10.3°)\ V$$

$$\mathbf{V}_{Th,rms} = 7.454\ V$$

Example 3

- Determine load impedance Z_L which maximizes the average power drawn from the following circuit. What is the maximum average power?



$$Z_{Th} = 2.933 + j4.467\ \Omega$$

$$\mathbf{V}_{Th} = 7.454\ V$$

$$Z_L = Z_{Th}^* = 2.933 - j4.467\ \Omega$$

$$P_{max} = \frac{\mathbf{V}_{Th,rms}^2}{4R_{Th}} = \frac{(7.454)^2}{4(2.933)} = 4.736\ W$$