ELL100: INTRODUCTION TO ELECTRICAL ENGG.

Natural Response : First Order Circuits

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- **State :** Collection of all energy defining quantities.
  - **Current** through **Inductor**
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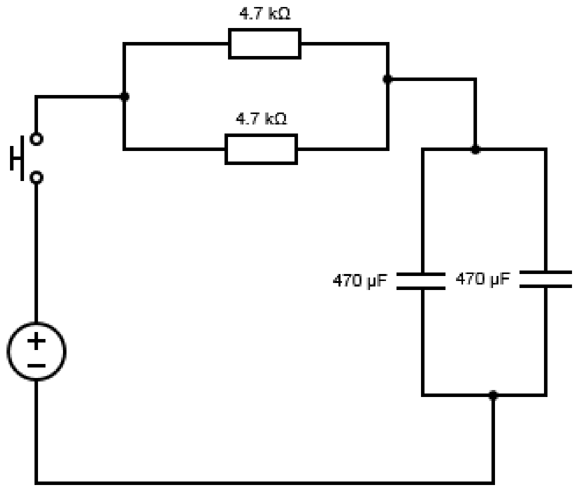
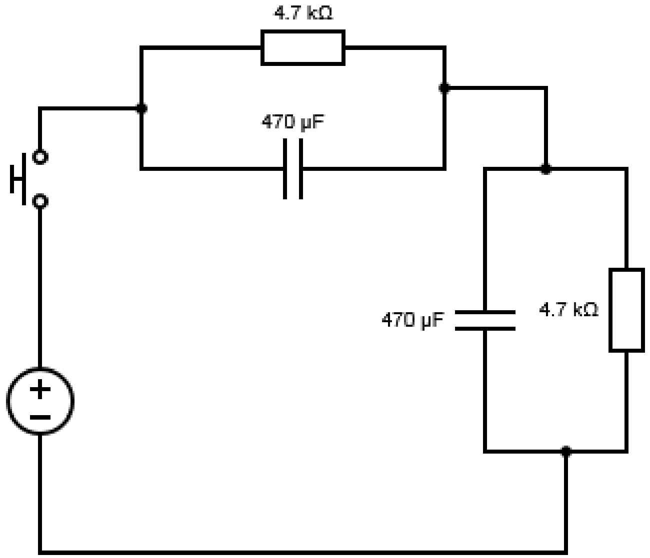
- Typically, the circuit is energized for some time and then let go to observe how it settles **naturally**.
- It is also called **free/unforced response**
- **Forced response** : Part of response solely due to external input.

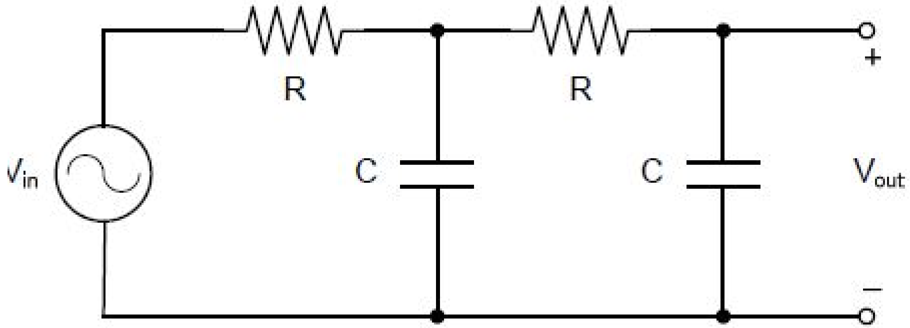
First order Circuits

- If the circuit can be reduced to have **one** energy storing element, it is possible to represent the response with a first order differential equation.
- Eg : Circuits which can be reduced to an equivalent circuit with a single inductor **OR** capacitor connected to a resistor (with/without a source)

Interconnects matter …







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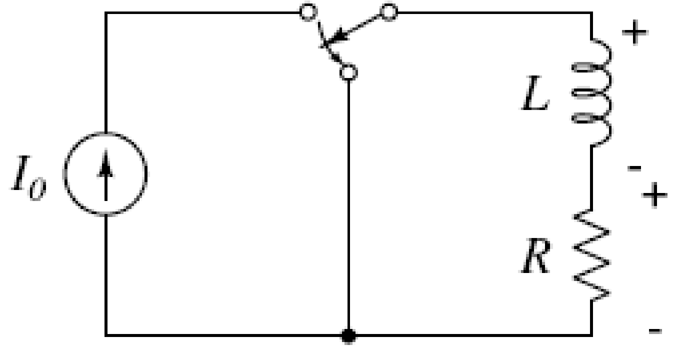
- Eg : Circuits which can be reduced to an equivalent circuit with a single inductor **OR** capacitor connected to a resistor (with/without a source)
- The **state** can be represented using a first order differential equation.

L-R Circuit



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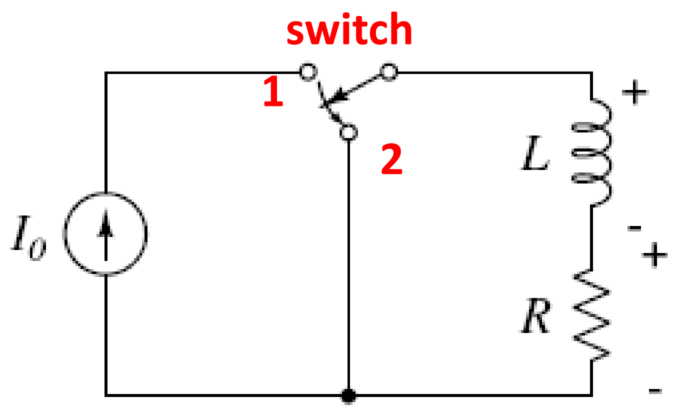

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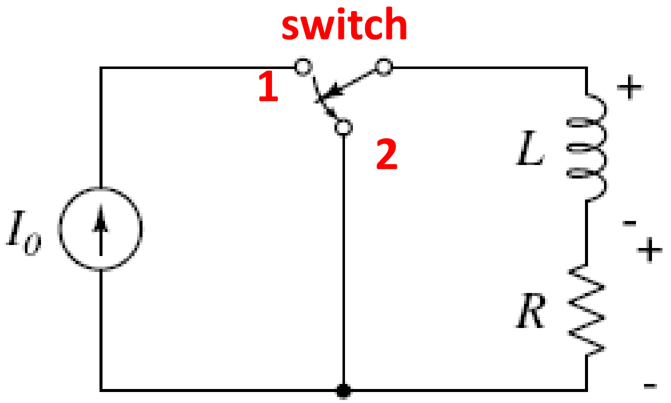

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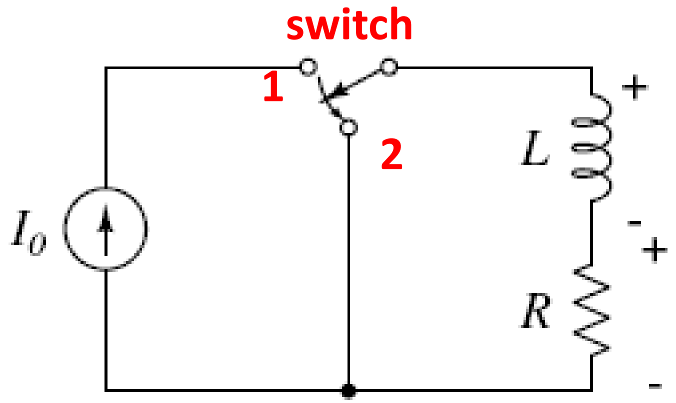

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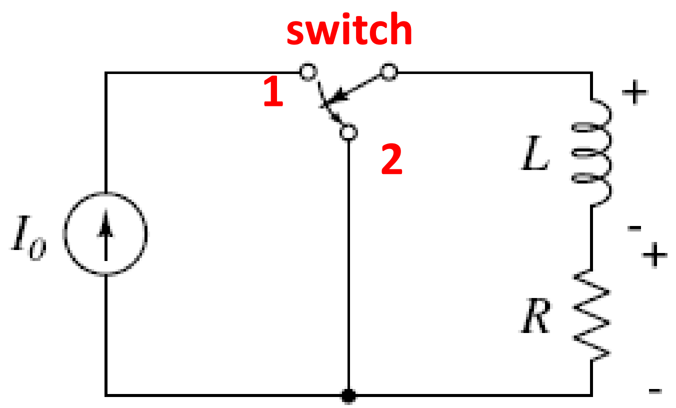
$$L\frac{di}{dt} + Ri = 0$$

L-R Circuit - The Solution



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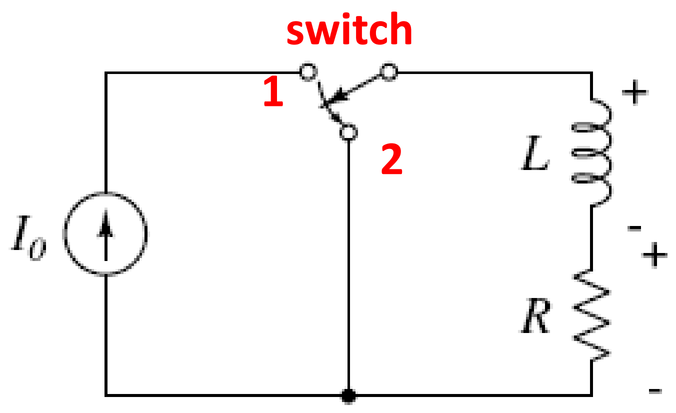

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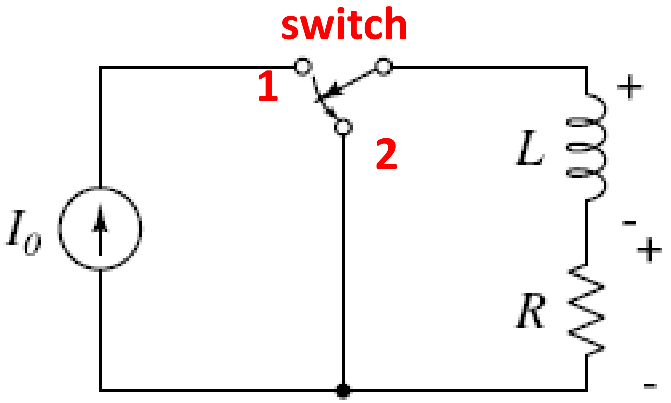

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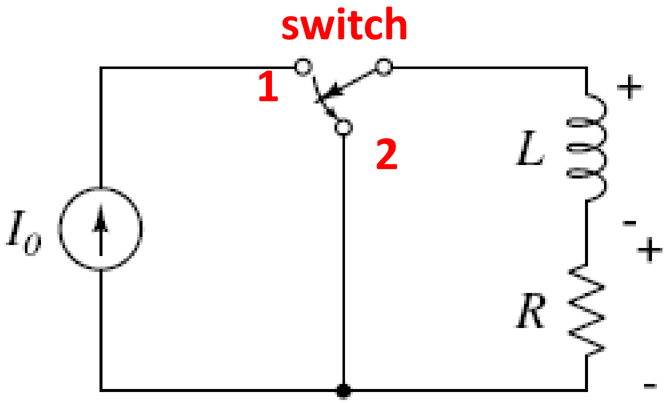
  where A and s are to be found.

L-R Circuit - Decay Rate



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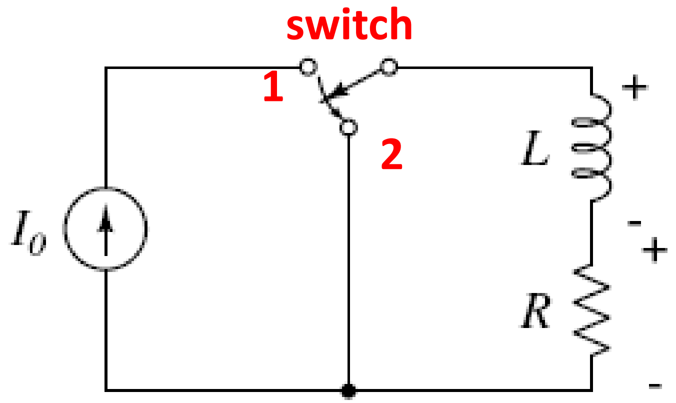

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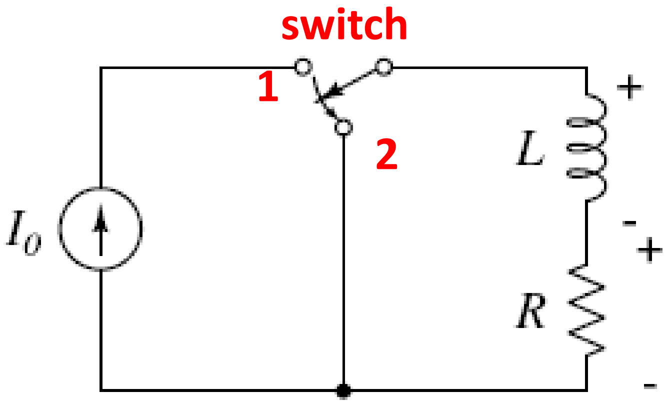

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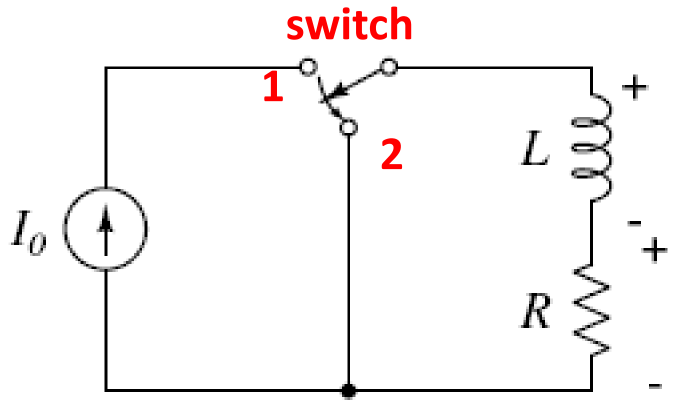

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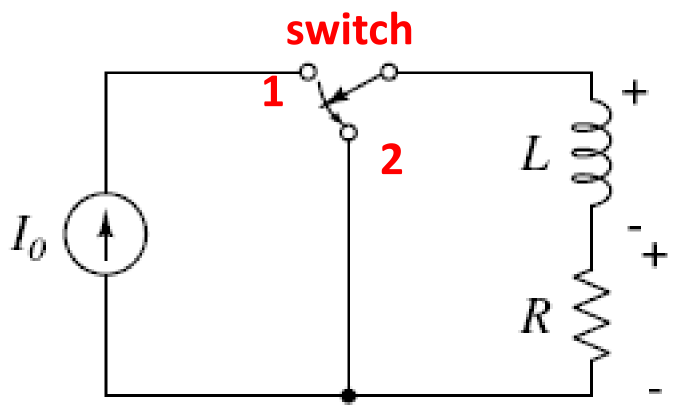

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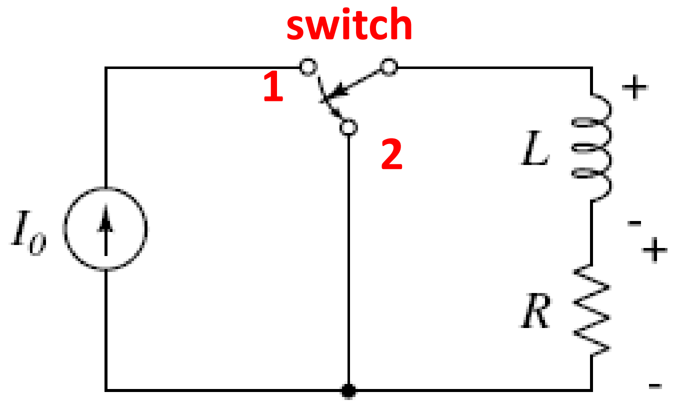

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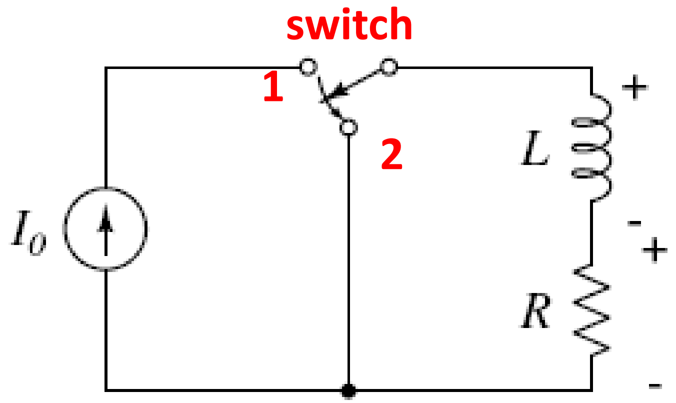

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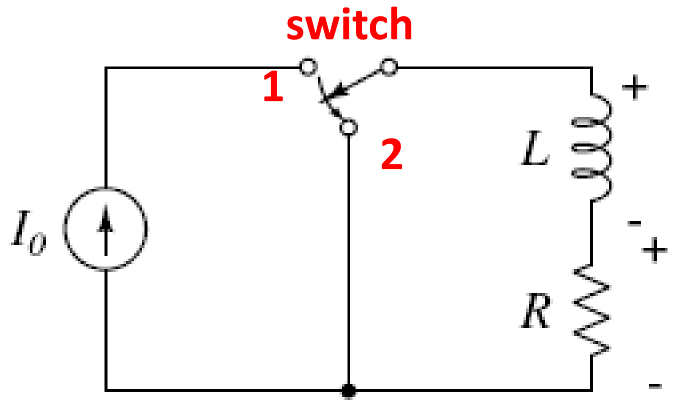

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$$Ae^{-\frac{R}{L}t}\Big|_{t=0^+} = I_0$$

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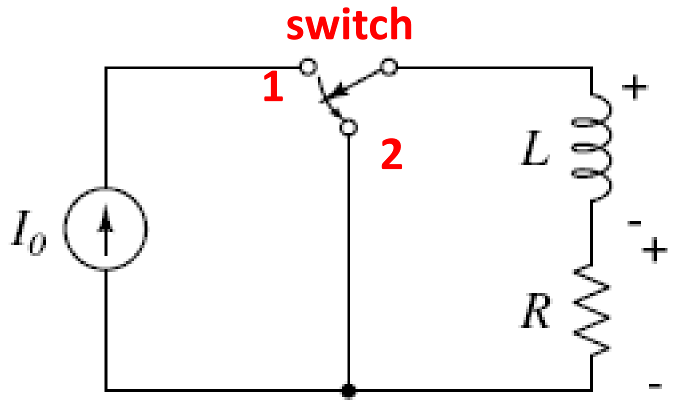

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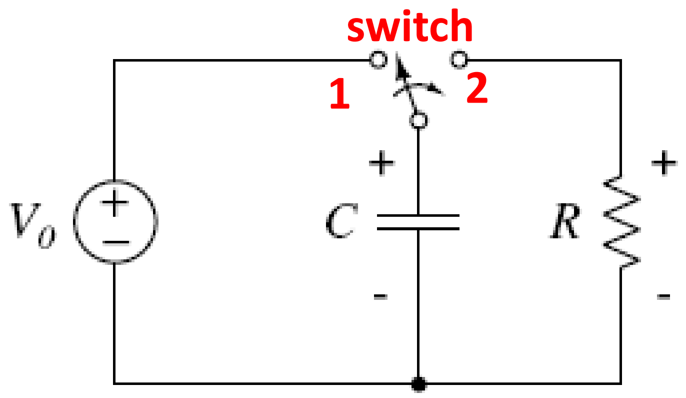
R-C Circuit



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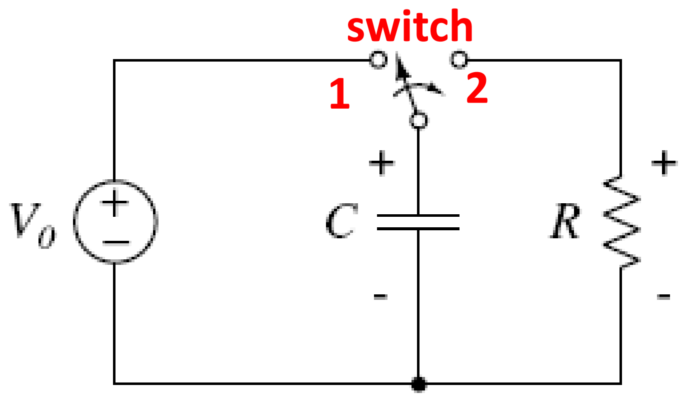
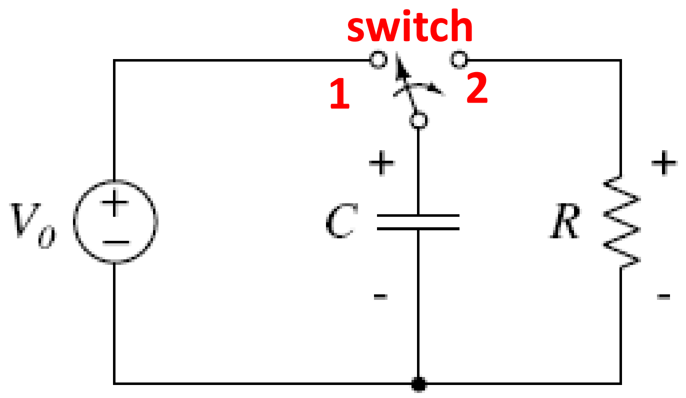


- Initially the switch is in posn. 1 for a long time, then moved to posn. 2. Find response of the circuit after the switch is thrown to posn. 2
- In the R-C circuit, KCL gives

$$\frac{v}{R} + C\frac{dv}{dt} = 0$$

R-C Circuit - Solution



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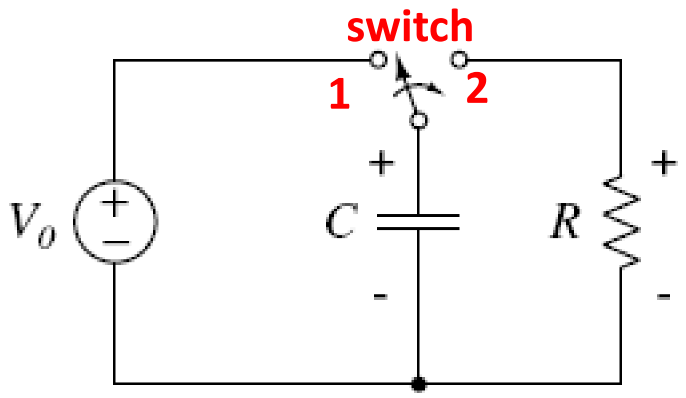

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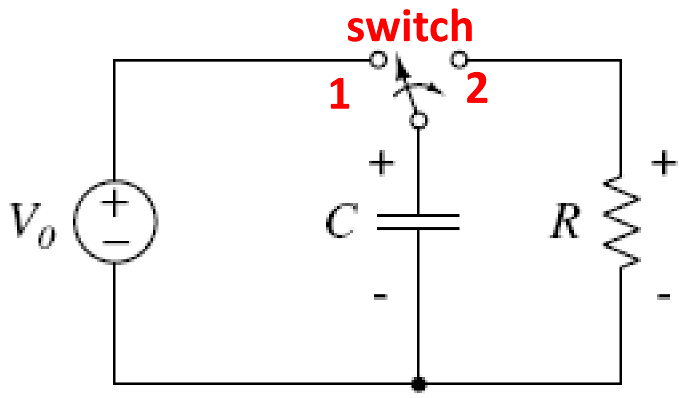

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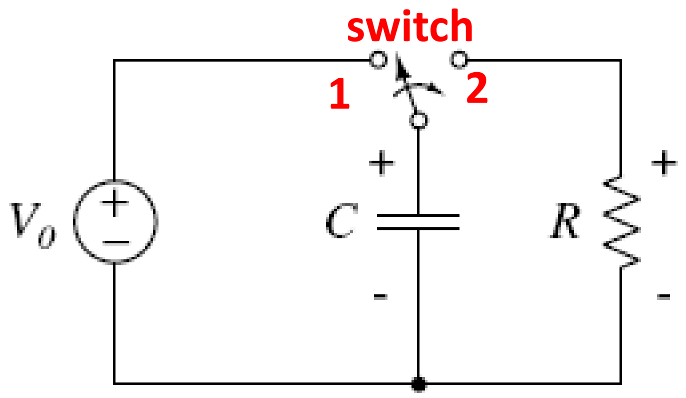

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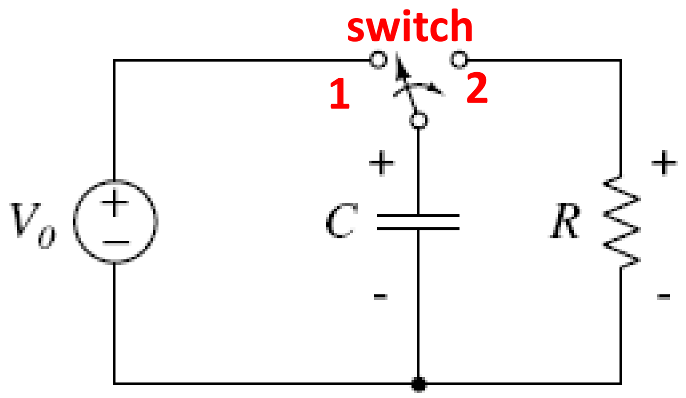
R-C Circuit – Initial Conditions



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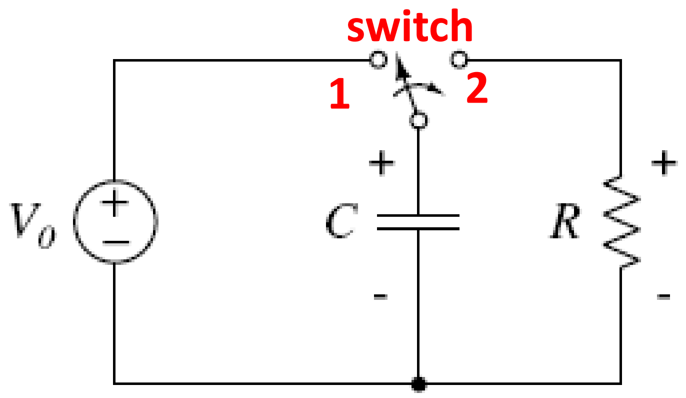

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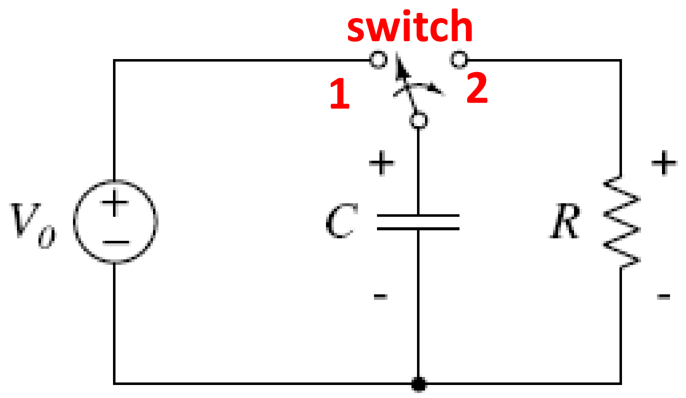
- So, at time $t = 0^+$,

$$v(0^+) = Ae^{-\frac{t}{RC}}\bigg|_{t=0^+} = V_0$$

R-C Circuit – Solution



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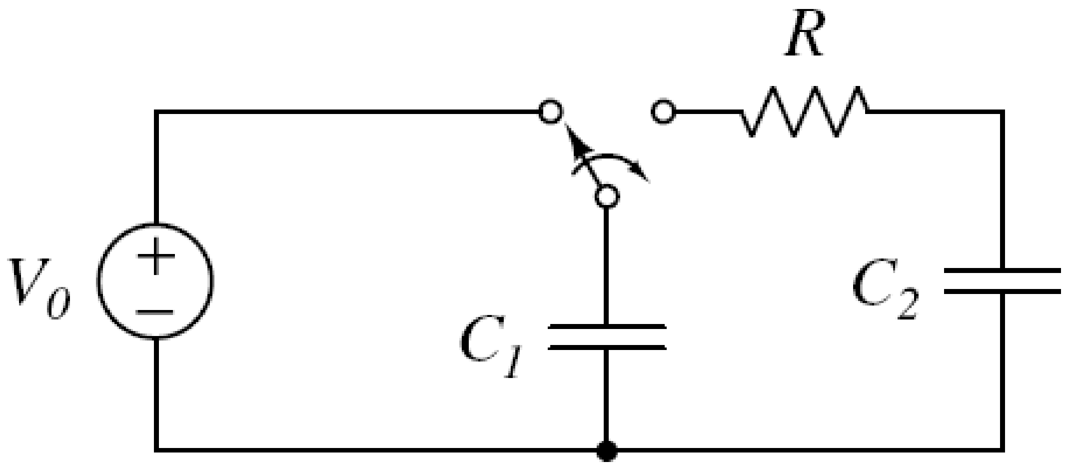
Example 2



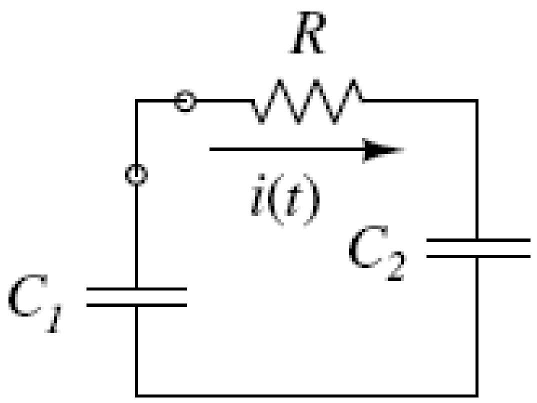
Initially, capacitor C_1 is charged till $t=0^-$ from the voltage source V_0

C_2 is uncharged

Switch is toggled at t=0.

Determine the voltage across and current through C_2.

Solution



- Applying KCL to the loop

$$iR + v_2 - v_1 = 0$$

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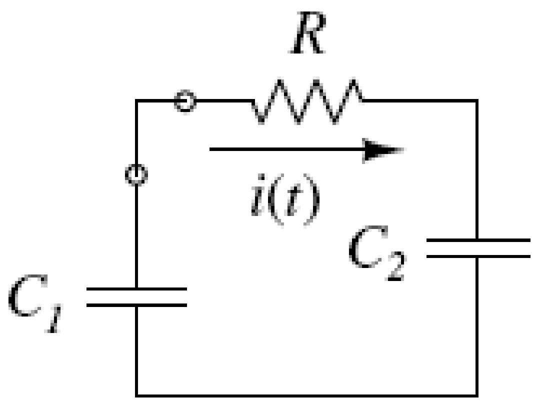

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$$R\frac{di}{dt} + \frac{dv_2}{dt} - \frac{dv_1}{dt} = 0$$

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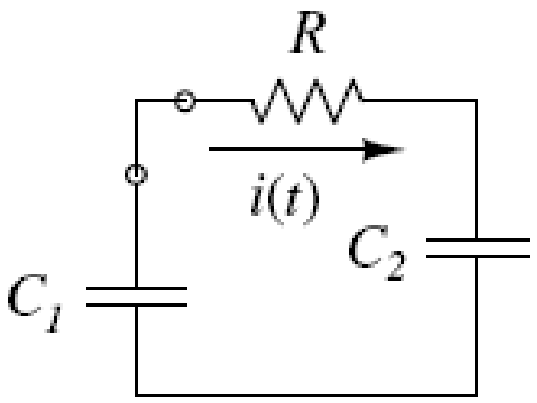

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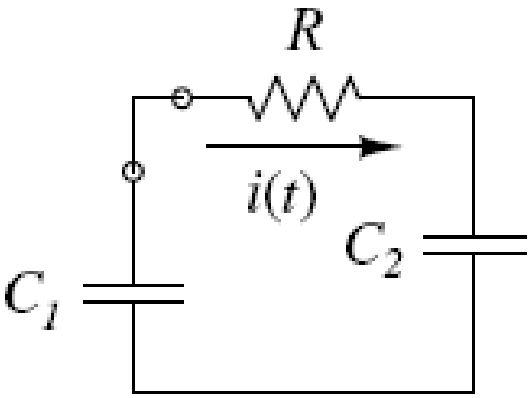

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- Note that $i = C_2\frac{dv_2}{dt} = -C_1\frac{dv_1}{dt}$

$$\implies \quad R\frac{di}{dt} + i\left(\frac{1}{C_2} + \frac{1}{C_1}\right) = 0$$

$$R\frac{di}{dt} + i\frac{C_1+C_2}{C_1C_2} = 0$$

Solution

- Plugging in the template equation $i(t) = Ae^{st}$

$$Ae^{st}\frac{C_1+C_2}{C_1C_2} + ARse^{st} = 0$$
$$s = -\frac{C_1+C_2}{RC_1C_2}$$

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- **Initial Conditions :** At $t=0^+$, $V_{C1}=V_0$, $V_{C2}=0$, so current through resistor is V_0/R

$$Ae^{st}\big|_{t=0^+} = A = \frac{V_0}{R}$$
$$i(t) = \frac{V_0}{R}e^{-\frac{C_1+C_2}{RC_1C_2}t}$$

Solution

- Integrating i(t), to obtain v_2(t)

$$
\begin{aligned}
v_2(t) &= \frac{1}{C_2}\int i(t)dt = \frac{1}{C_2}\int Ae^{st}dt \\
&= \frac{1}{C_2}\frac{A}{s}e^{st} + K \\
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Noting that $v_2(0)=0$

$$
\begin{aligned}
&K = V_0\tfrac{C_1}{C_1+C_2} \\
&v_2(t) = V_0\tfrac{C_1}{C_1+C_2}\left(1 - e^{-t\frac{C_1+C_2}{RC_1C_2}}\right)
\end{aligned}
$$

General Procedure for First order Circuits

- **General procedure to solution :**
  - Write governing equations with KVL/KCL
  - Reduce to a homogenous differential equation
  - Assume solution as Ae^{st}
  - Plug in to the homogenous differential equation to get s.
  - Plug in initial condition in the solution to get A.