COL 352 Introduction to Automata and Theory of Computation

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Lecture 10: Pumping Lemma: Examples

Pumping Lemma for regular languages

Pumping Lemma

$L \subseteq \Sigma^*$ is a regular language $\implies$
there exists $n \geq 1$ such that
for all strings $w \in L$ with $|w| \geq n$ we have that
there exists $x, y, z \in \Sigma^*$ with $w = xyz$, $|y| > 0$, $|xy| \leq n$ such that
for all $i \geq 0$ we have that
$xy^i z \in L$.

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Contrapositive

If for all $n \geq 1$ such that
there exists a string $w \in L$ with $|w| \geq n$ such that
for all breakups $x, y, z \in \Sigma^*$ with $w = xyz$, $|y| > 0$, $|xy| \leq n$ we have that
there exists $i \geq 0$ such that
$xy^iz \in L$.
$\implies L \subseteq \Sigma^*$ is not a regular language

Applying Pumping Lemma: Example 1

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$$L_{a,b} = \{a^n b^n \mid n \geq 0\}$$

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- Choose $k = 0$ for each i, j. The corresponding word is

$$a^i (a^j)^0 a^{n-i-j} b^n = a^{n-j} b^n \notin L_{a,b}$$

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Exercise: What is $L \cap a^*b^*$?

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- Choose $k = 2$ for each i, j. The corresponding word is

$$a^i (a^j)^2 a^{n-i-j} bba^n = a^{n+j} bba^n \notin PAL$$

Food for thought

- If $L_1 \cup L_2$ is regular, are L_1 and L_2 regular?
- Is this regular: $\{a^n b^m \mid n \neq m\}$?
- Is this regular: $\{a^n b^m \mid n \geq m\}$?
- Is this regular: $\{a^n b^{n+1} \mid n \geq 0\}$?

Need for infinite memory

Feels like all non-regular languages needed to remember infinite memory.

In $\{a^n b^n \mid n \geq 0\}$ we need to remember the number of seen a's and count the b's to match.

Finite number of states cannot count unboundedly increasing number.

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- Consider $xy^kz = 1^r 1^{ks} 1^t = 1^{p+(k-1)s}$
- We need $p + (k-1)s$ to be prime for all $k \geq 0$ to satisfy pumping condition.

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- Choose $k = p + 1$, we have $p + (k-1)s = p + ps = (s+1)p$.

Not a sufficient condition

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Exercise: Find a language that is not regular, but which satisfies the pumping conditions.

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- However, let $k = 0$ (pumping down). Then, $xy^k z = a^i a^{n-j-i} b^n \notin L$.

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Homomorphisms : Examples

- Given $\forall a \in \Sigma, h(a)$, we can uniquely determine $h(w)$ for every $w \in \Sigma^*$
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$$L_{a,b} = \{a^n b^n \mid n \in \mathbb{N}\}$$
- What is $h(L_{a,b})$ under $h : \Sigma \to \Sigma$ given by $h(a) = h(b) = a$?

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