1) Bernoulli Distribution

2) Binomial Distribution

3) Poisson Distribution.

Let $X_1, X_2, \ldots, X_k$ be independent RVs where $X_i \sim b(n_i, p)$. Then

$$S_n = \sum_{k=1}^{n} X_k \sim b\left(\sum_{k=1}^{n} n_k, p\right)$$

$X \sim b(n, p)$
$M_X(t) = \underline{(1-p+pe^t)^n}$

$$M_{S_n}(t) = E[e^{tS_n}]$$

$$= E[e^{t(X_1+X_2+\cdots+X_n)}]$$

$$= E\Big[\prod_{k=1}^{n} \underline{e^{tX_k}}\Big]$$

$$= \prod_{k=1}^{n} E[e^{tX_k}]$$

$$= \prod_{k=1}^{n} M_{X_k}(t)$$

$$= \prod (1-p+pe^t)^{n_k} \quad \because X_k \sim b(n_k, p)$$

$$= \prod_{k=1}^{n} (1-p+pe^t)^{n_k} \quad \because X_k \sim b(n_k, p)$$

$$= \underline{(1-p+pe^t)^{\sum_{k=1}^{n} n_k}} \longrightarrow \text{MGF of } b\left(\sum_{k=1}^{n} n_k, p\right)$$

$$\Rightarrow \quad S_n \sim b\left(\sum_{k=1}^{n} n_k, p\right)$$

Let $X_1, X_2, \dots X_k$ be independent Poisson RVs with $X_k \sim P(\lambda_k)$, $k=1, \dots n$

$$\sum_{k=1}^{n} X_k \sim P\left(\sum_{k=1}^{n} \lambda_k\right)$$

Use MGF to prove this

$M_X(t) = e^{\lambda(e^t-1)}$ if $X \sim P(\lambda)$

(exercise)

<u>Exercise</u>: Let X and Y be independent Poisson RVs with Parameters $\lambda_1 > 0$, $\lambda_2 > 0$, respectively. Show that

the conditional distribution of X given X+Y is a binomial distribution.

Exercise: If $X \sim P(\lambda)$ and the Conditional distribution of Y given $X = x$ is $b(x, p)$, then $Y \sim P(\lambda p)$.

Negative Binomial distribution:

Consider a succession of trials of an experiments with success probability p.

Let us compute the probability of observing exactly r successes, where $r \geq p$ is fixed integer.

We do this by counting the number

of failures before r^{th} success.

$X \to$ no. of failures before r^{th} success.

To make this happen $X+r$ trials are required.

$(X+r)^{th}$ trial is success.

and there are exactly X failures in $X+r-1$ trials.

$$P\{X=k\} = \binom{k+r-1}{k} p^r (1-p)^k \qquad k=0,1,2,\ldots \quad \text{--- (+)}$$

A RV X with PMF given by (+) is said to follow negative binomial dist.

$$\binom{k+r-1}{k} = \frac{(k+r-1)!}{k!\,(r-1)!}$$

$$= \frac{(k+r-1)\,(k+r-2)\cdots(r+1)\,(r)\,\cancel{(r-1)}!}{k!\;\cancel{(r-1)}!}$$

$$= (-1)^k \cdot \frac{(-r)\,(-r-1)\cdots(-r-k+1)}{k!}$$

$$\| \quad \binom{-r}{k}$$

$$\binom{k+r-1}{k} = (-1)^k \binom{-r}{k}$$

Let $q = (1-p) < 1$

$$\underline{(1-q)^{-r}} = \sum_{k=0}^{\infty} \binom{-r}{k} (-q)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{-r}{k} q^k$$

$$\boxed{1 = \sum_{k=0}^{\infty} \binom{k+r-1}{k} p^r (1-p)^k}$$

$$M_X(t) = E[e^{tX}]$$

$$= \sum^{\infty} \binom{k+r-1}{\ldots} p^r \ldots^k \; e^{tk}$$

$$= \sum_{k=0}^{\infty} \binom{k+r-1}{k} p^r (1-p)^k e^{tk}$$

$$= p^r \sum_{k=0}^{\infty} \binom{k+r-1}{k} \left((1-p)e^t\right)^k$$

$$M_X(t) = p^r \left(1 - (1-p)e^t\right)^{-r}$$

if $e^t(1-p) < 1$

$e^t < \frac{1}{1-p}$

$\boxed{t < \log\left(\frac{1}{1-p}\right)}$

$$\boxed{\begin{aligned} E[X] &= \frac{r(1-p)}{p} \\ Var(X) &= \frac{r(1-p)}{p^2} \end{aligned}}$$

exercise.

Special case : $r = 1$

$X \longrightarrow$ number of failures before first success

$$P\{X = k\} = p \cdot (1-p)^k \qquad k = 0, 1, \ldots \infty$$

Here X is said to follow a geometric distribution.

$$E[X] = \frac{1-p}{p} \qquad Var(X) = \frac{1-p}{p^2}$$

Example : An oil company conducts a geological study that indicates that an exploratory oil well should have 20% chance of striking oil.

i) what is the probability that third strike comes in seventh drill.

ii) what is the probability that first strike comes on third well drilled.

(Exercise) i) $r = 3$, $P\{X=4\}$

$X \sim NB(p = 0.2)$

ii) $X \sim$ geometric distribution

$P\{X=2\}$, $p=0.2$

Theorem : If X has a geometric distribution, then for any two non-negative integers m, n,

$$\underline{P\{X>m+n \mid X>m\}} = \underline{P\{X \geq n\}}$$

Proof :

$$P\{X>m+n \mid X>m\} = \frac{P\{X>m+n,\ X>m\}}{P\{X>m\}}$$

$$= \frac{P\{X>m+n\}}{P\{X>m\}} = \frac{(1-p)^{m+n+1}}{(1-p)^{m}}$$

$$= (1-p)^{n+1} = P\{X \geq n\}$$

Theorem : Let $X_1, X_2, \ldots X_n$ be

independent geometric RVs with Parameters $p_1, p_2, \ldots p_n$, respectively. Then

$X_{(1)} = \min\{X_1, \ldots X_n\}$ is also a geometric RV with parameter

$$p^* = 1 - \prod_{i=1}^{n} (1 - p_i)$$

<u>Proof</u> :- $X \simeq$ geometric RV with Parameter p

$$P\{X \le k\} = 1 - P\{X > k\}$$

$$F(k) = 1 - (1-p)^{k+1} \quad \forall\ k.$$

$$\begin{aligned} P\{X_{(1)} \le k\} &= 1 - P\{X_{(1)} > k\} \\ &= 1 - P\{\mathrm{Min}\{X_1, \ldots X_n\} > k\} \\ &= 1 - P\{X_1 > k, X_2 > k, \ldots X_n > k\} \\ &= 1 - \prod_{i=1}^{n} P\{X_i > k\} \end{aligned}$$

$$\prod_{i=1} P\{X_i > k\}$$

$$= 1 - \prod_{i=1}^{n} (1-p_i)^{k+1}$$

$$= 1 - \left(\prod_{i=1}^{n} (1-p_i)\right)^{k+1}$$

$$= 1 - (1-p^*)^{k+1}$$

$$p^* = 1 - \prod_{i=1}^{n} (1-p_i)$$

$$\Rightarrow \prod_{i=1}^{n} (1-p_i) = 1-p^*$$

$$F_{X_{(1)}}(k) = 1 - (1-p^*)^{k+1}$$

$\Rightarrow$ $X_{(1)}$ is geometric RV with

Parameter p^*.

$\square$