Lecture 10 Mon 14 Aug 2023

EP of $S \longleftrightarrow$ BFS

$$S = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

Conversely: Let $x \in S$ be an EP of S.

$$x = (x_1, \ldots, x_n)^T, \quad x_i \geq 0, \ \forall\ 1 \leq i \leq n.$$

Let k be the index s.t. (w.l. of generality)
$x_1, x_2, \ldots, x_k > 0$ and $x_{k+1}, \ldots, x_n = 0$
where, $0 \leq k \leq n$

$$x = (x_1, \ldots, x_k, 0, \ldots 0)^T$$

Let the corresponding columns of A
(with $x_i > 0$) be $\{a_1, a_2, \ldots, a_k\}$.

$\downarrow$

These are LI vectors in $\mathbb{R}^n$.

Suppose $\{a_1, \ldots, a_k\}$ be LD.

$\Rightarrow \exists$ some scalars λ_i's, not all zeros,

s.t. $\sum_{i=1}^{k} \lambda_i a_i = 0$

Set $\mu = \min \left\{ \frac{x_i}{|\lambda_i|}, 1 \leq i \leq k, \lambda_i \neq 0 \right\}$

$\Rightarrow \quad \mu > 0$

Take any number ε, $0 < \varepsilon < \mu$

$x_i' = x_i + \varepsilon \lambda_i$, $1 \le i \le k$ & $x_i' = x_i = 0$ $\forall\, k+1 \le i \le n$

$x_i'' = x_i - \varepsilon \lambda_i$, $1 \le i \le k$ & $x_i'' = x_i = 0$ $\forall\, k+1 \le i \le n$.

Note: (1) $x' \neq x''$ $\because$ atleast one of the $\lambda_i \neq 0$, $1 \le i \le k$

(2) $x = \frac{1}{2}(x' + x'')$.

(3) $x', x'' \geq 0$ by choice of $\varepsilon > 0$.

As $0 < \varepsilon < \mu = \min\left\{ \dfrac{x_i}{|\lambda_i|}, \lambda_i \neq 0 \right\}$

$\Rightarrow 0 < \varepsilon < \dfrac{x_i}{|\lambda_i|}$, $\forall\, i$, $1 \le i \le k$ for which $\lambda_i \neq 0$.

$\lambda_i > 0$

$0 < \varepsilon < x_i / \lambda_i$

$\Downarrow$

$x_i - \varepsilon \lambda_i > 0$

$\Downarrow$

$\lambda_i < 0$

$0 < \varepsilon < \dfrac{x_i}{-\lambda_i}$

$\Downarrow$

$x_i + \varepsilon \lambda_i > 0$

$\Downarrow$

$x_i' > 0$, $x_i'' \geq 0$

(4) $Ax' = \sum a_i x_i' = \sum_{i=1}^{k} a_i x_i'$

$= \sum_{i=1}^{k} a_i (x_i + \varepsilon \lambda_i) = \sum_{i=1}^{k} a_i x_i + \varepsilon \underbrace{\sum_{i=1}^{k} \lambda_i a_i}_{=0}$

$$= \sum_{i=1}^{n} a_i x_i = Ax = b. \quad (\because x \in S).$$

Similarly, $Ax'' = b$.

By (3) & (4), $x', x'' \in S$

By (1) $x' \neq x''$

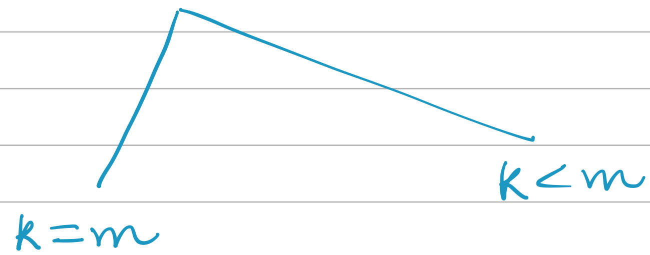
By (2) $x = \frac{1}{2}(x' + x'')$

which contradicts that x is an EP of S.

$\therefore$ (1) must implies all $\alpha_i = 0$.

$\Rightarrow \{a_1, \ldots, a_k\}$ are LI in $\mathbb{R}^n$.

$\Rightarrow k \leq m$.



Case $k = m$

$B = [a_1, a_2, \ldots, a_m]_{m \times m}$

is invertible

$\neq Ax = b$

$\Rightarrow \sum_{i=1}^{n} a_i x_i = b$

$\Rightarrow \sum_{i=1}^{n} a_i x_i = b \Rightarrow Bx = b$

$x \geq 0$

x is a BFS of the system

$Ax = b$, $x \geq 0$

+ it is non degenerate
as $x_i > 0$ $\forall$ $1 \leq i \leq k = m$.

Case $k < m$

$\{a_1, a_2, -, a_k\}$ in LI in $\mathbb{R}^n$, $k < m$.

$$B = [\underset{m\times 1}{a_1}, -, \underset{m\times k}{a_k}, \underbrace{-, -, -}_{(m-k)}]_{m\times m}$$

This set can be extended to form a basis of $\mathbb{R}^m$.

Also, rank (A) = column rank (A)
= row rank (A) = m

$\therefore$ We can find $m-k$ columns in A which together with $\{a_1, -, a_k\}$ forms a basis of $\mathbb{R}^m$.

$\Rightarrow$ x is a BFS of the system
(degenerate BFS).