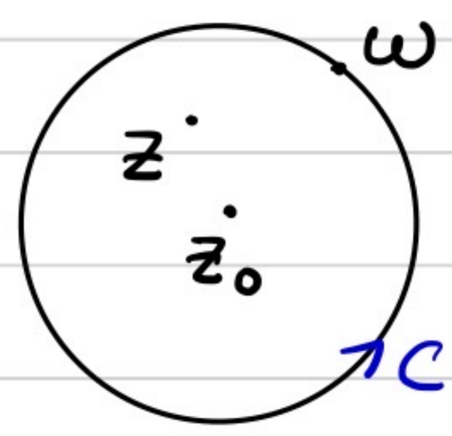
Series expansions

Taylor Series: Let $f(z)$ be analytic in the open disk $|z-z_0| < R$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\,(z-z_0)^n \qquad \forall\ |z-z_0| < R$$

Proof using cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i}\oint_C dw\,\frac{f(w)}{w-z} = \frac{1}{2\pi i}\oint_C \frac{dw\,f(w)}{(w-z_0)-(z-z_0)}$$



$$= \frac{1}{2\pi i}\oint_C \frac{dw\,f(w)}{(w-z_0)}\ \frac{1}{\left(1-\frac{z-z_0}{w-z_0}\right)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{dw}{w-z_0} f(w) \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{n!}{2\pi i}\oint_C \frac{dw\,f(w)}{(w-z_0)^{n+1}}\right)(z-z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\,(z-z_0)^n$$

★ The above proof utilizes the series representation

$$\frac{1}{w-z} = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}$$

which converges $\forall\ |z| < |w|$ since it converges absolutely.

Ex: $f(z) = e^z \rightarrow$ Entire function

$$f^{(n)}(0) = e^0 = 1$$

$$\Rightarrow\ e^z = \sum_{n=0}^{\infty} \frac{1}{n!} f^{n}(0)\, z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Ex: $f(z) = \sinh(z) \equiv \dfrac{e^z - e^{-z}}{2}$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z^n}{n!} - \frac{(-z)^n}{n!}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{n!}\left(1-(-1)^n\right)$$

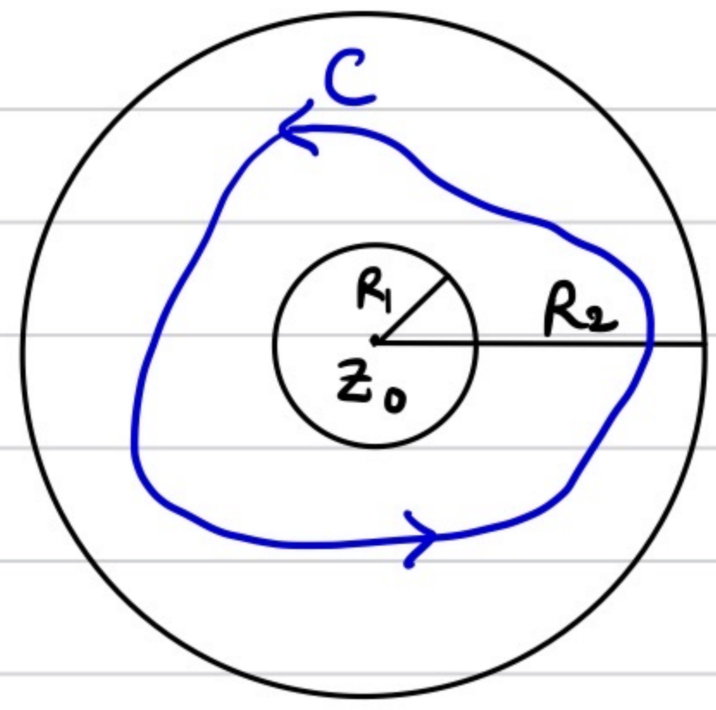
$$= \sum_{n=1,3,5\ldots}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Simillarily $\cosh z = \frac{1}{2}(e^{z} + e^{-z}) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$

Ex: $f(z) = \frac{1}{z}$ around $z = -1$

$$f^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}} \Rightarrow f^{(n)}(z=-1) = -n!$$

$$\Rightarrow \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-n!)}{n!} (z-(-1))^n = -\sum_{n=0}^{\infty} (z+1)^n$$



Laurant series: Let $f(z)$ be analytic in the annulus $R_1 < |z - z_0| < R_2$, then

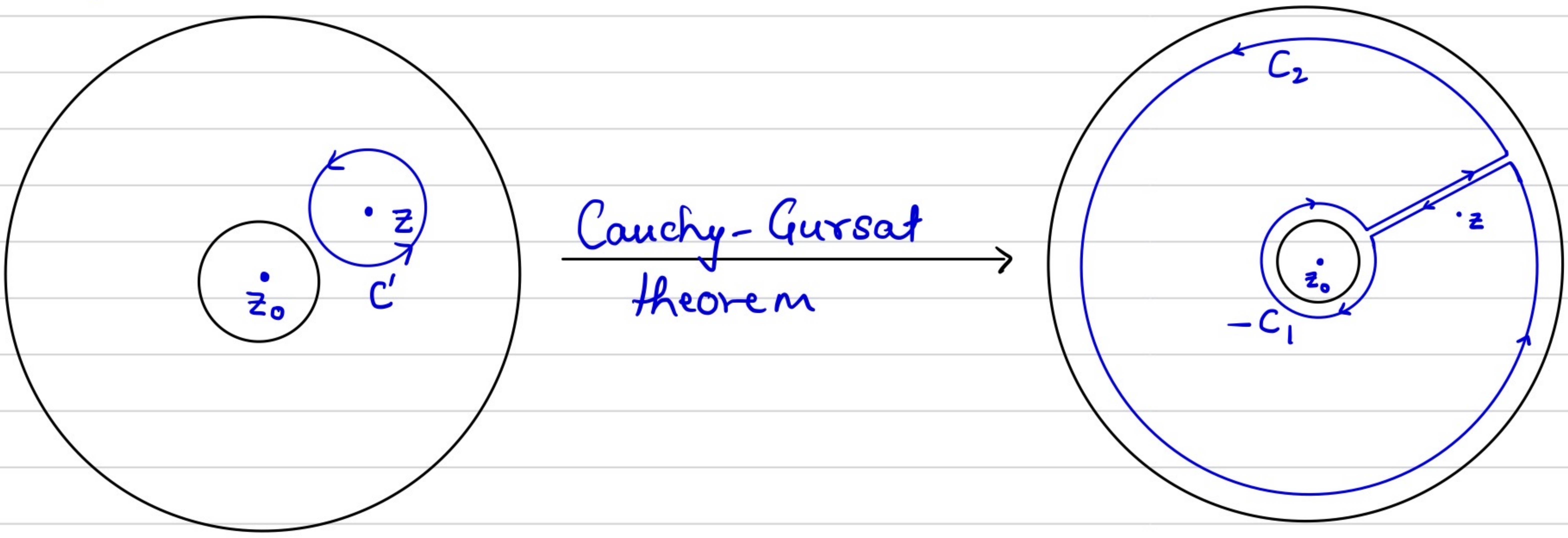
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

w/ $$a_n = \frac{1}{2\pi i} \oint_C \frac{dw\, f(w)}{(w - z_0)^{n+1}}$$

where C is any simple closed clockwise contour in the annulus

$R_1 < |z - z_0| < R_2$

proof:

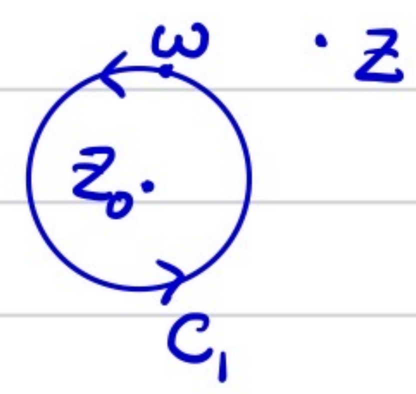


$$f(z) = \frac{1}{2\pi i} \oint_{C'} \frac{dw\, f(w)}{w - z} = \underbrace{-\frac{1}{2\pi i} \oint_{C_1} \frac{dw\, f(w)}{w - z}}_{I_1} + \underbrace{\frac{1}{2\pi i} \oint_{C_2} \frac{dw\, f(w)}{w - z}}_{I_2}$$

* $I_1 = 0$ if $f(w)$ were analytic inside C_1 (Since $\frac{1}{w-z}$ is analytic in C_1) & the above integral would simply reduce to the Taylor series discussed previously.

$$I_1 = -\frac{1}{2\pi i}\oint_{C_1} \frac{dw\; f(w)}{(w-z_0)-(z-z_0)} = \frac{1}{2\pi i}\oint_{C_1}\frac{dw\; f(w)}{(z-z_0)-(w-z_0)}$$

Now on C_1 : $|w-z_0| < |z-z_0|$, thus

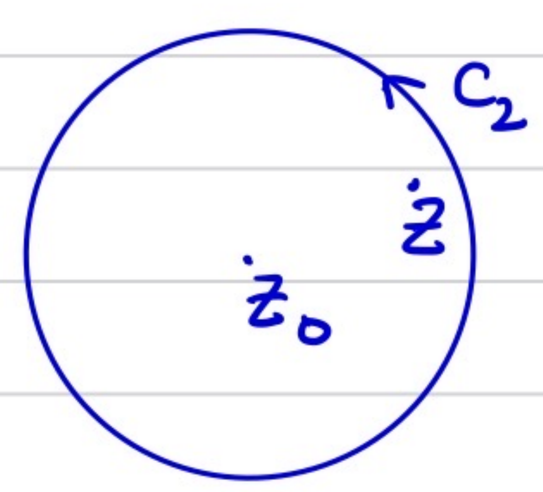


$$= \frac{1}{2\pi i}\oint_{C_1} dw\, f(w) \sum_{n=0}^{\infty} \frac{(w-z_0)^n}{(z-z_0)^{n+1}}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}}\left(\oint_{C_1} dw\; f(w)\;(w-z_0)^n\right)$$

$$= \sum_{n=1}^{\infty}\frac{1}{(z-z_0)^n}\left[\frac{1}{2\pi i}\oint_{C_1}\frac{dw\; f(w)}{(w-z_0)^{-n+1}}\right]$$

$$I_2 = \frac{1}{2\pi i}\oint_{C_2}\frac{dw\, f(w)}{w-z}$$



$$\frac{|z-z_0|}{|w-z_0|} < 1$$

$$= \frac{1}{2\pi i}\oint_{C_2} dw\, f(w) \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint\frac{dw\; f(w)}{(w-z_0)^{n+1}}\right)(z-z_0)^n$$

$$f(z) = I_1 + I_2$$

$$= \sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\oint_{C_1}\frac{dw\; f(w)}{(w-z_0)^{-n+1}}\right)}\frac{1}{(z-z_0)^n} + \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\oint_{C_2}\frac{dw\, f(w)}{(w-z_0)^{n+1}}\right)}(z-z_0)^n$$

Independent of the choice of contour as long as it lies in the annular disk where the integrand is analytic.

$$f(z) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi i}\oint_C\frac{dw\; f(w)}{(w-z_0)^{n+1}}\right)(z-z_0)^n$$