Lecture 2 (Infinite Series)

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \longrightarrow \text{convergent}$$

"Alternating series"

Alternating Series A sequence $(x_n)_{n\geq 1}$ of non-zero real numbers is called alternating if any two consecutive terms are of different signs (one +ve, one -ve).

In other words we say, $(x_n)_{n\geq 1}$ is alternating if the terms $(-1)^n x_n$ is either +ve or -ve $\forall\ n\geq 1$.

For example, $((-1)^n)_{n\geq 1}$ alternating sequence.

A series $\sum_{n\geq 1} x_n$ is called alternating series if $(x_n)_{n\geq 1}$ is alternating.

For example, $\sum_{n\geq 1} (-1)^n$, $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$

<u>Observation</u> If $(x_n)_{n\geq 1}$ is alternating, then $(x_n)_{n\geq 1}$ can be written as either $((-1)^n y_n)_{n\geq 1}$ or $((-1)^{n+1} y_n)_{n\geq 1}$ where $(y_n)_{n\geq 1}$ is a sequence of +ve real numbers.

Theorem

Let $(y_n)_{n\geq 1}$ be a nonincreasing sequence of +ve real numbers such that $(y_n) \to 0$ as $n \to \infty$. Then $\sum_{n\geq 1} (-1)^{n+1} y_n$ is convergent.

Proof Exercise.

Follow the proof of convergence of $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$.

1) Let $\sum_{n\geq 1} a_n$, $\sum_{n\geq 1} b_n$ be two convergent series. Then $\sum_{n\geq 1} (a_n \pm b_n)$ is a convergent series. If $\sum_{n\geq 1} a_n = a$ and $\sum_{n\geq 1} b_n = b$, then $\sum_{n\geq 1} (a_n \pm b_n) = a \pm b$.

Proof Denote the seq. of partial sums $\sum_{n\geq 1} a_n$ by S_n

and the seq. of partial sums $\sum_{n\geq 1} b_n$ by t_n

Note that, the seq. of partial sums of $\sum_{n\geq 1}(a_n+b_n)$ is s_n+t_n

and of $\sum_{n\geq 1}(a_n-b_n)$ is s_n-t_n.

(s_n), (t_n) both are convergent sequences, we get $(s_n+t_n)_{n\geq 1}$ and $(s_n-t_n)_{n\geq 1}$ are convergent.

If $s_n \to a$, $t_n \to b$, then we know

$$s_n \pm t_n \to a \pm b.$$

Remark If $\sum_{n\geq 1}(a_n+b_n)$ is convergent, it need not imply $\sum_{n\geq 1} a_n$, $\sum_{n\geq 1} b_n$ are convergent.

For example, take $a_n = 1 \ \forall\, n \geq 1$
$b_n = -1 \ \forall\, n \geq 1$.

Then $\sum_{n\geq 1}(a_n+b_n)$ is convergent but $\sum_{n\geq 1} a_n$, $\sum_{n\geq 1} b_n$ are not convergent.

Exercise If both $\sum_{n\geq 1}(a_n+b_n)$ and $\sum_{n\geq 1}(a_n-b_n)$ are convergent, then $\sum_{n\geq 1} a_n$, $\sum_{n\geq 1} b_n$ are convergent.

2) Let $c \in \mathbb{R}$ and $\sum_{n \geq 1} a_n$ be a convergent series. Then $\sum_{n \geq 1} (c a_n)$ is convergent.

If $\sum_{n \geq 1} a_n = a$, then $\sum_{n \geq 1} (c a_n) = ca$.

<u>Proof</u> Exercise.

3) If $\sum_{n \geq 1} a_n$, $\sum_{n \geq 1} b_n$ are convergent, it <u>DOES NOT</u> gurantee the convergence of $\sum_{n \geq 1} (a_n b_n)$.

For example,

$$a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n}}$$

We can conclude, $\sum\limits_{n\geq 1} a_n$, $\sum\limits_{n\geq 1} b_n$ are convergent.

But $\sum\limits_{n\geq 1}(a_n b_n) = \sum\limits_{n\geq 1} \frac{1}{n}$, not convergent.

So termwise product of two convergent series need not be a convergent series.

Theorem (n-th term test)

Let $\sum_{n\geq 1} a_n$ be a convergent series. Then $(a_n) \to 0$ as $n \to \infty$.

$\sum_{n\geq 1} (-1)^n$ is not convergent as $(-1)^n \not\to 0$ as $n \to \infty$.

Proof Let (S_n) be the seq. of partial sums of $\sum_{n\geq 1} a_n$.

Note, $a_n = S_n - S_{n-1}$ $\forall$ $n \geq 2$.

$$\sum_{n\geq 1} a_n = a \text{ (say)}$$

We get, $(S_n) \to a$ as $n \to \infty$.

$\therefore (S_{n-1}) \to a$ as $n \to \infty$.

$\therefore a_n = S_n - S_{n-1} \to 0$ as $n \to \infty$.

Remark The converse of the above theorem is not true. For example consider $a_n = \frac{1}{n}$. Note that $\frac{1}{n} \to 0$ as $n \to \infty$ but $\sum_{n\geq 1} \frac{1}{n}$ is not convergent.

Defn (K-tail of a series)

Let $\sum_{n\geq 1} a_n$ be an infinite series and K be a +ve integer. Then the infinite series $\sum_{n\geq K} a_n$ is called the K-tail of the series $\sum_{n\geq 1} a_n$.

Remark The series $\sum_{n\geq 1} a_n$ is convergent

$\Rightarrow \forall\ K \geq 1$, the K-tail $\sum_{n\geq K} a_n$ is convergent.

If for some +ve integer K, the K-tail $\sum_{n\geq K} a_n$ is convergent, then $\sum_{n\geq 1} a_n$ is convergent.

For $n\geq 1$, $S_n :=$ the seq. of partial sums of $\sum_{n\geq 1} a_n$

For $n\geq K$, $t_n :=$ the seq. of partial sums of $\sum_{n\geq K} a_n$

finite quantity

$$\left.\begin{aligned} S_n &= a_1 + \cdots + a_n \\ t_n &= a_K + \cdots + a_n \end{aligned}\right\} n\geq K \Rightarrow \boxed{S_n = S_{K-1} + t_n}$$

for any $n\geq K$

Proposition

Let $\sum_{n \geq 1} a_n$ be convergent. Denote, $A_k := \sum_{n \geq k} a_n$. The sequence of k-tails $(A_k) \to 0$ as $k \to \infty$.

Proof Recall, $S_n = S_{k-1} + t_n \ \forall \ n \geq k$.

Letting $n \to \infty$,

$$A_1 = S_{k-1} + A_k.$$

Next let $K \to \infty$,

$$A_1 = A_1 + \lim_{K\to\infty} A_K.$$

$$\therefore \lim_{K\to\infty} A_K = 0.$$

<u>Theorem</u> (Cauchy condensation test)

Let $(a_n)_{n\geq 1}$ be a non-increasing sequence of +ve real numbers. Then $\sum_{n\geq 1} a_n$ converges iff $\sum_{n\geq 0} 2^n a_{2^n}$ converges.

<u>Proof</u> $s_n :=$ seq. of partial sums of $\sum_{n\geq 1} a_n$

$t_n :=$ seq of partial sums of $\sum_{n\geq 0} 2^n a_{2^n}$.

Note, $(s_n), (t_n)$ are increasing sequences as a_n's are +ve real numbers.

<u>Enough to show</u> $(S_n)_{n\geq 1}$ is bounded

$\iff (t_n)_{n\geq 1}$ is bounded.

First let, $(S_n)_{n\geq 1}$ be bounded.

<u>To show</u>: $(t_n)_{n\geq 1}$ is bounded.

Write
$$
\begin{aligned}
S_{2^n} &= a_1 + a_2 + \cdots + a_{2^n} \\
&= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) \\
&\quad + \cdots + (a_{2^{n-1}+1} + \cdots + a_{2^n}) \\
&\geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1} a_{2^n}
\end{aligned}
$$

$$S_{2^n} \geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1} a_{2^n}$$

$$= \frac{1}{2} a_1 + \frac{1}{2} \underbrace{(a_1 + 2a_2 + \cdots + 2^n a_{2^n})}_{t_n}$$

$$S_{2^n} \geq \frac{1}{2} a_1 + \frac{1}{2} t_n .$$

If (S_n) is bounded, then (t_n) is also bounded.

$\therefore \sum_{n\geq 1} a_n$ is convergent $\Rightarrow \sum_{n \geq 0} 2^n a_{2^n}$ is convergent.

Next let $(t_n)_{n\geq 1}$ be bounded.

We show, $(S_n)_{n\geq 1}$ is bounded.

$$\begin{aligned} S_{2^n-1} &= a_1 + a_2 + a_3 + \cdots + a_{2^n-1} \\ &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) \\ &\quad + \cdots + (a_{2^{n-1}} + \cdots + a_{2^n-1}) \\ &\leq \underbrace{a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1} a_{2^{n-1}}}_{t_{n-1}} \end{aligned}$$

$$S_{2^n-1} \leq t_{n-1}$$

$\therefore$ (S_{2^n-1}) is bounded as (t_n) is bounded.

Recall, $(S_n)_{n\geq 1}$ is an increasing sequence.

We know, a monotonic sequence is bounded iff one of its subsequences is bounded.

$\therefore$ $(S_n)_{n\geq 1}$ is a bounded sequence.

Applications of Cauchy condensation test

1) $\sum_{n\geq 2} \frac{1}{n\log n}$

$$a_n = \frac{1}{n\log n}$$

Check, $(a_n)_{n\geq 2}$ is a non increasing sequence.

Compare, a_n, a_{n+1}

We know, $\log x > 0$ if $x > 1$.

$\therefore \log(n+1) - \log n$

$= \log\left(1 + \frac{1}{n}\right) > 0$

Also, $(n+1) > n$

We can write that,

$$\underbrace{(n+1)\log(n+1)}_{1/a_{n+1}} > \underbrace{n \log n}_{1/a_n}$$

$\Rightarrow \quad a_{n+1} < a_n$

$\therefore\ (a_n)_{n\geq 2}$ is a decreasing sequence.

$$\sum_{n\geq 1} 2^n a_{2^n} = \sum_{n\geq 1} 2^n \cdot \frac{1}{2^n \log 2^n}$$

$$= \sum_{n\geq 1} \frac{1}{n \log 2}$$

$$= \frac{1}{\log 2} \sum_{n\geq 1} \frac{1}{n}$$

diverges to $+\infty$

diverges to $+\infty$

$\therefore\ \sum_{n\geq 2} \frac{1}{n \log n}$ is not convergent.

Exercise Show that $\sum_{n\geq 2} \frac{1}{n(\log n)^2}$ is convergent.

Hint Use Cauchy condensation test.