06/01/2016 $\boxed{\text{LECTURE 2}}$

Definition : Probability Space

Let E be a random experiment
Let Ω be the collection of all possible outcomes.
Let F be a σ-field (σ-algebra) on Ω

σ-field :
(i) $\phi \in F$
(ii) If $A \in F$ $\Rightarrow A^c \in F$
(iii) If $A_i \in F$ $\cup A_i \in F$

The set function P defined on F satisfying :
i) $P(A) \geq 0 \quad \forall A \in F$
ii) $P(\Omega) = 1$
iii) If $A_i, i=1,2,\ldots$ disjoint events , then $P(\cup A_i) = \Sigma P(A_i)$

Then the function P is called probability. (Kolmogrov axiomitic def^n of probability)

(Ω, F, P) is called a probability space.

Ex:-
① $\Omega = \{a,b,c\}$
$F_1 = \{\phi, \{a\}, \{b,c\}, \Omega\}$
$F_2 = \{\phi, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \Omega\}$

Both are σ field on Ω

② $\Omega = [0,1]$
$F_1 = \{\phi, \{0\}, (0,1], \Omega\}$

Largest σ-field on R $(-\infty, \infty)$ - "Borel" σ field [all subsets of R]

ex: E : Drawing a card from a deck of 52 cards
③ $\Omega = \{A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2\}$ $\quad P(a) = 1/13 \quad \forall a \in \Omega$
$F_1 = \{\phi, \Omega, \{A,K,Q,J\}, \{10,9,8,7,6,5,4,3,2\}\}$
$P(\phi) = 0 \quad P(\Omega) = 13 \times \frac{1}{13} = 1 \quad P(\{A,K,Q,J\}) = \frac{4}{13}$ & $P(\{10,9,8,7,6,5,4,3,2\}) = \frac{9}{13}$
$\Rightarrow$ Probability Space

08/01/2016 $\boxed{\text{LECTURE 3}}$

Conditional Probability

Definition: (Ω, F, P) , $P(B) > 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{(P of A given B occurs)}$$

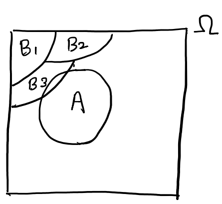
It satisfies: (i) $P(A|B) \geq 0$ (ii) $P(B|B) = 1$ (iii) $P(\bigcup_i P(A_i/B)) = \sum_i P(A_i/B)$

A_i are disjoint

Probability Space: (Ω_B, F_B, P_B)

Remark: (i) When A & B are independent events, $P(A|B) = P(A)$

Definition: Total Probability Rule



For any event $A \in F$

$$P(A) = \sum P(A|B_i).P(B_i)$$

Condition:

$B_i \cap B_j = \phi$

$\bigcup_i B_i = \Omega$

Definition: Baye's Rule

$$P(B_i|A) = \frac{P(A|B_i)\,P(B_i)}{\sum_j P(A|B_j)\,P(B_j)}$$

Example:

(2) $\Omega = (-\infty, \infty)$

$\mathcal{F}$: Borel σ-field

$$P([a,b]) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} . dx$$

$P([a,b]) = 0$ if $a = b$

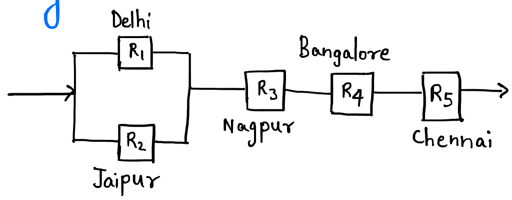
$P([a,b]) = 1$ if $a = -\infty$ & $b = \infty$

i.e. $[a,b] = \Omega$

12/01/2016 LECTURE 4

Reliability

eg:-



$R_1 = P(A_1)$

Reliability of system

$= P((A_1 \cup A_2) \cap A_3 \cap A_4 \cap A_5)$

$= [R_1 + R_2 - R_1R_2] \times R_3 \times R_4 \times R_5$

A_1, A_2, A_3, A_4 & A_5 are mutually independent.

THEOREM

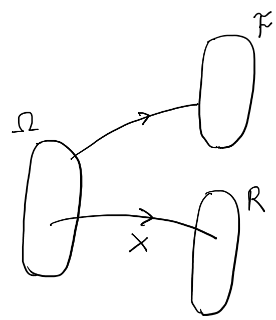
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{A_n, n=1,2 \ldots\}$ be an increasing sequence of elements in $\mathcal{F}$ i.e. $A_n \in \mathcal{F}$ & $A_n \subseteq A_{n+1}$ for $n = 1,2 \ldots$.

Then $P(\lim_{n\to\infty} A_n) = P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$

Similarly, if $\{A_n, n=1,2,\ldots\}$ is a decreasing seq. of elements in $\mathcal{F}$ i.e. $A_n \in \mathcal{F}$ & $A_n \supseteq A_{n+1}$ for $n=1,2\ldots$, then

$P(\lim_{n\to\infty} A_n) = P(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$

Random Variable



X is a real valued function such that $X^{-1}\{(-\infty, x]\} \in \mathcal{F}$ $\forall x \in \mathbb{R}$. Then x is a random variable w.r.t. $\mathcal{F}$.

eg:- $\Omega = \{a,b,c\}$, $\mathcal{F} = \{\phi, \{a\}, \{b,c\}, \Omega\}$

Define $X(\omega) = \begin{cases} -10 & \omega = \{a\} \\ 5 & \omega = \{b\} \\ 5 & \omega = \{c\} \end{cases}$ $\Rightarrow$ $X^{-1}\{(-\infty, x]\} = \begin{cases} \phi & -\infty < x < -10 \\ \{a\} & -10 \le x < 5 \\ \Omega & 5 \le x < \infty \end{cases}$

$\Rightarrow$ Random variable.

Similary for not a random variable, (for some $\mathcal{F}$)

Define $X(w) = \begin{cases} 5 & w = \{a\} \\ -10 & w = \{b\} \\ 5 & w = \{c\} \end{cases} \Rightarrow X^{-1}\{(-\infty, x]\} = \begin{cases} \phi & -\infty < x < -10 \\ \{b\} & -10 \le x < 5 \\ \Omega & 5 \le x < \infty \end{cases}$

$\not\Rightarrow$ Random variable.

Remark: Any real function will become a random variable if we take Power set as the σ-field

13/01/2016 $\boxed{\text{LECTURE 5}}$

Distribution Function

Any real valued function satisfying:

i) $0 \le F(x) \le 1 \quad \forall \; -\infty < x < \infty$

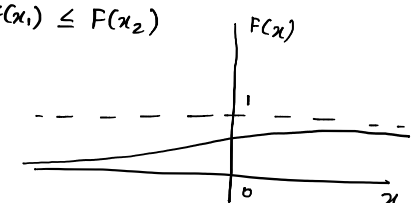
ii) Monotonically increasing in x, i.e., if $x_1 < x_2$ then $F(x_1) \le F(x_2)$

iii) $\lim_{x \to -\infty} F(x) = 0$ & $\lim_{x \to \infty} F(x) = 1$

iv) F is (right) continuous function,

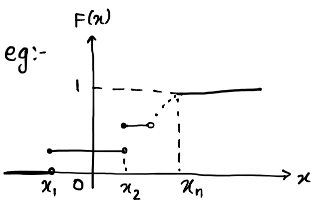
then F is called a distribution function.

eg:-

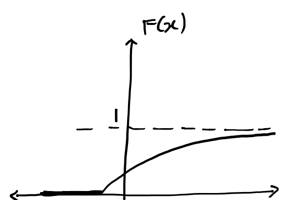


$\frac{\tan^{-1}x}{\pi} + \frac{1}{2}$

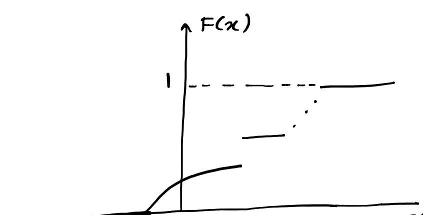
eg:-



1(a)



1 (b)



1 (c)

Cummulative Distribution Function (C.D.F.)

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

$$F(x) = P(X \le x) \quad -\infty < x < \infty$$

is said to be C.D.F of the random variable X.

$$P\{X \le x\} = P\{\underset{w \in \Omega}{w \mid X(w) \le x}\}$$

short hand notation

Classification of Random Variable

1.) Discrete Type Random Variable — Fig. 1(a) above

C.D.F. has countable no. of discontinuities

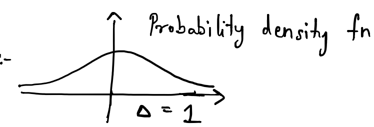
2.) Continuous Type Random Variable — Fig 1(b)

If its C.D.F. is a continuous fn. in x.

$$F(x) = \int_{-\infty}^{x} \underbrace{f(t)}_{\text{probability density fn. (pdf) of r.v. } X} .dt$$

$$\left.\begin{array}{l} f(x) \geq 0 \quad \forall\, x \\ \int_{-\infty}^{\infty} f(t).dt = 1 \end{array}\right] \text{Properties of probability density function}$$

eg:-



3) Mixed Type Random Variable. Fig 1(c)

If it's C.D.F. has countable discontinuities as well as continuous function in some subintervals.

i.e. $$\sum_i P\,|\,x = x_i) + \int_{-\infty}^{\infty} f(x).dx = 1$$

15/01/2016 $\boxed{\text{LECTURE 6}}$

Example :

① Discrete Type Random Variable

a)
$$P(x) = \begin{cases} 0 & x < 0 \\ \dfrac{[x]}{10} & 0 \leq x < 11 \\ 1 & 11 \leq x \end{cases}$$

b) $$F(x) = \sum_{n=1}^{x} \frac{1}{2^n} = 1$$

$$P(x) = \begin{cases} \dfrac{1}{2^x} & x = 1, 2, 3 \ldots \\ 0 & \text{others} \end{cases}$$

② Continuous Type Random Variable

a) $$F(x) = \frac{\tan^{-1} x}{\pi} + \frac{1}{2}$$

$$f(x) = \frac{d\,F(x)}{dx} = \frac{1}{\pi(1+x^2)}$$

b) $$\int_0^{\infty} e^{-x}.dx = 1$$

$$f(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & , \text{others} \end{cases}$$

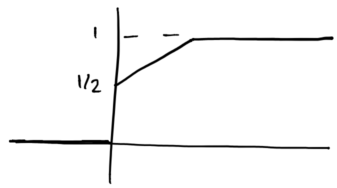
c) $\sum_{k=0}^{n} {}^nC_k \left(1/2\right)^n = 1$

$$P(x) = \begin{cases} {}^nC_k \left(1/2\right)^n & k=0,1,\dots n \\ 0 & \text{others.} \end{cases}$$

c) $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

(Mixed Type Random Variable)

a) $$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{2} + \frac{x}{2} & 0 \le x < 1 \\ 1 & 1 \le x < \infty \end{cases}$$



$$P(x) = \begin{cases} \frac{1}{2} & x=0 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_k P(X=k) + \int_0^1 1/2 \, dx$$

$$1/2 + 1/2 = 1$$

19/01/2016 $\boxed{\text{LECTURE 7}}$

Example : ϵ - Random Experiment

Ω - Collection of all possible outcomes

$\mathcal{F}$ - largest σ- field on Ω

$X: \Omega \to R$ s.t. $X^{-1}\{(-\infty, x]\} \in \mathcal{F} \quad \forall x \in R$

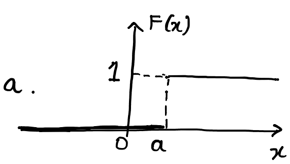
Standard Distributions

- **Discrete Type Random Variable**

① Constant Random Variable

X takes only one value. suppose the point is a.

$\Rightarrow P\{X=a\} = 1$



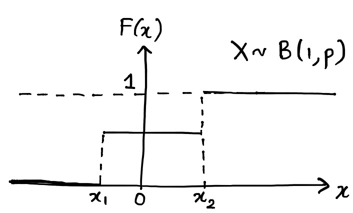
② Bernaulli Distributed Random Variable

$X \sim B(1,p)$

X takes values x_1 & x_2

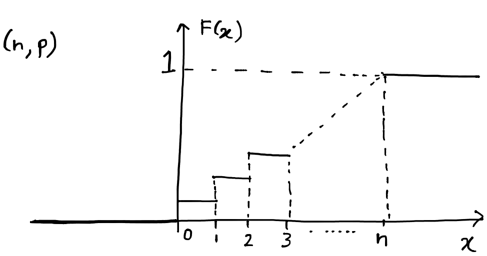
$P\{x=x_1\} = 1 - P\{x=x_2\} = P$

$0 < P < 1$



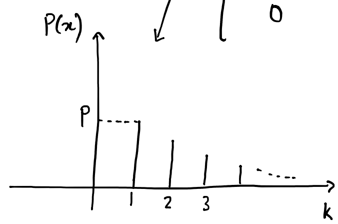
③ Binomial Distributed Random Variable $X \sim B(n,p)$

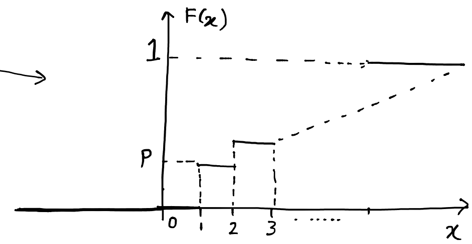
$$P\{X=k\} = \begin{cases} {}^nC_k \, p^k (1-p)^{n-k}, & k=0,1,2,\dots n \\ 0, & \text{otherwise} \end{cases}$$



④ Geometric Distribution

$$P\{x=k\} = \begin{cases} (1-p)^{k-1}p & k=1,2,3,\dots \\ 0 & \text{otherwise} \end{cases}$$





⑤ Negative Binomial or Pascal $X \sim NB(r,p)$

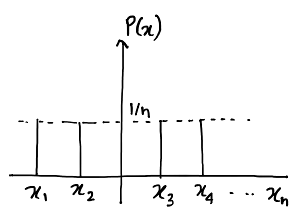
$$P\{x=n\} = \begin{cases} {}^{n-1}C_{r-1} \cdot p^r (1-p)^{n-r} & n=r, r+1, \dots \\ 0 & \text{otherwise} \end{cases}$$

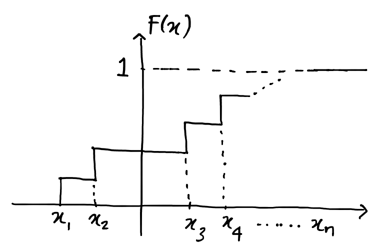
if $r=1 \longrightarrow$ Geometric Distribution

20/01/2016 $\boxed{\text{LECTURE 8}}$

⑥ Discrete Uniform

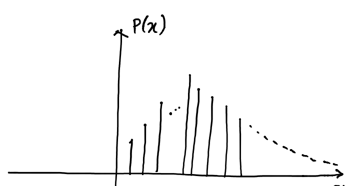
$$P\{x=k\} = \begin{cases} \frac{1}{n} & k=x_1, x_2 \dots x_n \\ 0 & \text{otherwise} \end{cases}$$





⑦ Poisson Distribution

$$P\{x=k\} = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & k=0,1,2\dots \\ 0 & \text{otherwise} \end{cases}$$



For Some λ

⑧ Poisson Process

$\{X(t), t \in T\}$ – <u>Stochastic Process</u>

For fixed t, $x(t) \sim$ Poisson distributed $x(t) \sim P(\lambda t)$

Assume
1.) $X(0) = 0$
2.) $P\{X(t+\Delta t) = 1 / X(t) = 0\} = \lambda \Delta t + o(\Delta t)$
3.) $P\{X(t+\Delta t) = 0 / X(t) = 0\} = 1 - \lambda \Delta t + o(\Delta t)$
4.) $P\{X(t+\Delta t) > 1 / X(t) = 0\} = o(\Delta t)$
5.) Non-overlapping intervals are independant.

Divide the interval $[0,t]$ into n equal parts.

$$P\{X(t)=k\} = \begin{cases} {}^nC_k \left(\lambda.\frac{t}{n}\right)^k \left(1-\frac{\lambda t}{n}\right)^{n-k} & k=0,1\ldots n \\ 0 & \text{otherwise} \end{cases} \quad \text{Binomial}$$

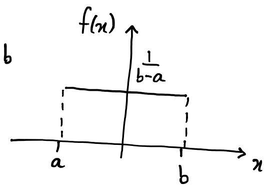
$$\Rightarrow \lim_{n\to\infty} P\{X(t)=k\} = \begin{cases} e^{-\lambda t}\dfrac{(\lambda t)^k}{k!} & k=0,1,2,\ldots\ldots \\ 0 & \text{otherwise} \end{cases} \quad \text{Poisson Distribution}$$

22/01/2016 $\boxed{\text{LECTURE 9}}$

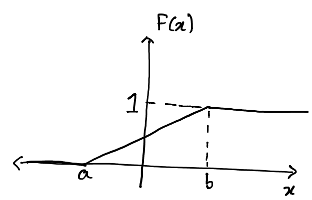
- **Continuous Type Random Variable**

① Uniform Distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{, others} \end{cases}$$

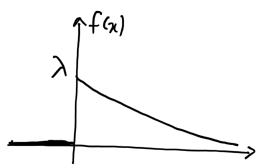


$$F(x) = \begin{cases} 0 & -\infty < x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & b \le x < \infty \end{cases}$$

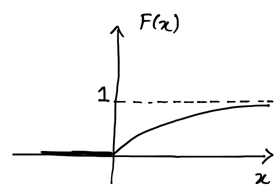


② Exponential Distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$



$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1-e^{-\lambda x} & 0 \le x < \infty \end{cases}$$

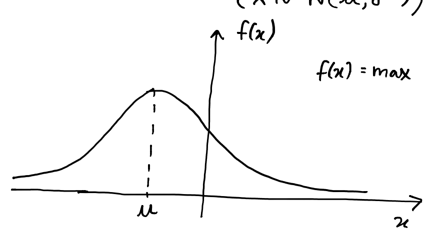


$$P(X > x+t \,/\, X > t) = \frac{P(X > x+t)}{P(X > t)} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x}$$ (independent of $t \Rightarrow$ Memoryless /Marcov Property)

as $P(x>t) = 1-F(x=t) = e^{-\lambda t}$

③ Normal Distribution or Gaussian Distribution $(X \sim N(\mu,\sigma^2))$

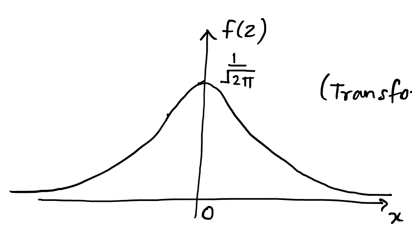
$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$



$f(x) = \max$ at μ

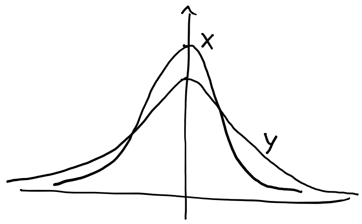
If we take random variable to be $z = \frac{x-\mu}{\sigma}$

$$\Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad -\infty < z < \infty$$



(Transformed $f(x)$)

$X \sim N(0,1)$
$Y \sim N(0,2)$



$f(y)\big|_{y=0} = \frac{1}{2\sqrt{2\pi}}$ $\quad f(x)\big|_{x=0} = \frac{1}{\sqrt{2\pi}}$

X will have higher peak than Y.

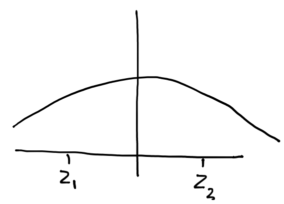
X will interset Y since area = 1 for both.

27/01/2016 $\boxed{\text{LECTURE 10}}$

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-t^2/2}\,dt$$

$$P(a < x \leq b) = F_x(b) - F_x(a)$$

$$= P\left\{\frac{a-u}{\sigma} < \frac{x-u}{\sigma} \leq \frac{b-u}{\sigma}\right\}$$



$$= P\{z_1 < z \leq z_2\}$$

$$= P(z \leq z_2) - P(z \leq z_1)$$

$$= F_z(z_2) - F_z(z_1) = 0.5 + \phi(z_2) - [1 - \phi(z_1) - 0.5]$$

④ $X \sim$ Gamma (r, λ)

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$$

29/01/2016 $\boxed{\text{LECTURE 11}}$

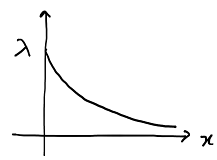
Definition: Let X be a continuous type r.v. with p.d.f $f(x)$

$$E(x) = \int_{-\infty}^{\infty} x f(x).dx \quad \text{provided} \quad \int_{-\infty}^{\infty} |x| f(x)\, dx < \infty$$

i.e., $E(|x|) < \infty$

eg:- $X \sim Exp(\lambda)$

① $E(x) = \int_{-\infty}^{\infty} x f(x).dx = \frac{1}{\lambda}$ $\quad (f(x) = \lambda e^{-\lambda x})$ $\quad 0 < x < \infty$



② $X \sim U(x_1, \ldots x_n)$ [Uniformly Distributed]

$$E(x) = \sum_i x_i P(x = x_i) = \sum x_i \frac{1}{n} = \frac{x_1 + x_2 + \ldots + x_n}{n} \quad \text{(Average)}$$

③ $X \sim P(\lambda)$

$$E(x) = \sum x P(x = x_i) = \lambda$$

④ $X \sim$ Geometric (p)

$$P(x = k) = (1-p)^{k-1} p \quad k = 1, 2, \ldots$$

$$E(x) = \sum k P(x = k)$$

$$= 1. P(x=1) + 2. P(x=2) + 3 P(x=3) + \ldots$$

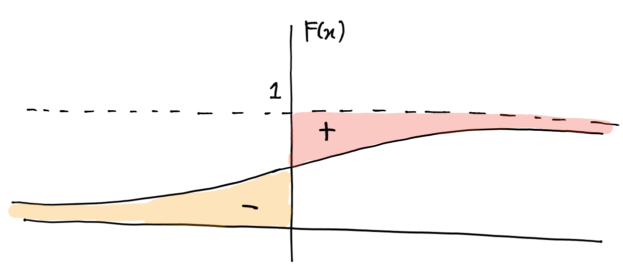
$$\begin{aligned} = P\{x=1\} + P\{x=2\} &+ P(x=3) + \ldots \quad = \sum_{k=1}^{\infty} P\{x \geq k\} = \sum_{0}^{\infty} P\{x > k\} \\ + P\{x=2\} &+ P(x=3) + \ldots \\ &+ P(x=3) + \ldots \quad = \sum_{k=0}^{\infty} (1 - F_x(k)) \end{aligned}$$

In general,

if X is a continuous type r.v.

$$E(x) = \int_0^{\infty} (1 - F_x(x))\, dx - \int_{-\infty}^{0} F_x(x).dx \quad \text{(provided R.H.S. exist)}$$

(● − ● Area wise)



Definition : 2^{nd} order moment

$$\sigma^2 = E[(x-\mu)^2] \quad \text{about the mean (i.e. } \mu = E(x))$$

(Variance of r.v. x)

Remark ① $\sigma^2 \geq 0$

② $\sigma^2 = 0 \Rightarrow Var(x) = 0 \Rightarrow P(x=\mu) = 1$

③ $Var(AX+B) = A^2 \, Var(X)$

④ $\sigma^2 = E[(x-\mu)^2] = E(x^2) - (E(x))^2$

⑤ $E(g(x)) = \begin{cases} \sum g(x) \, P\{x = x_i\} \\ \int_{-\infty}^{\infty} g(x) \, f_x(x) \, dx \end{cases}$

If $g(x)$ is Borel measurable function then $g(x)$ will become a random variable from R to R

For $g(x)$ to be Borel measurable, every piece wise continuous fn. should be Borel Measurable.

⑥ $E\left(\frac{1}{X}\right) \neq \frac{1}{E(X)}$

⑦ n^{th} order moment about the mean

$$\mu'_n = E((x-\mu)^n)$$

provided R.H.S. exist

⑧ n^{th} order moment exist $\not\Rightarrow$ $(n+1)^{th}$ order moment exist

02/02/2016 $\boxed{\text{LECTURE 12}}$

Given a random variable X, $x^{-1}\{(-\infty, x]\} \in F \;\; \forall x \in R$

g is a function $R \to R$ such that $y = g(x)$ is a random variable.

i.e. $Y^{-1}\{(-\infty, y]\} \in F \;\; \forall y \in R$

What is the distribution of Y ?

g is a Borel Measurable Function in $R \longrightarrow R$

(Special case : Every piecewise continuous function is Borel Measurable)

eg:- $y = x^2$, $\sin x$, $2x$, $\begin{cases} x & x<0 \\ 3 & x>0 \end{cases}$

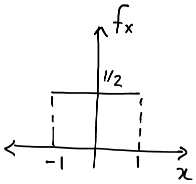
2.) $X \sim B(n,p)$, $Y = n - X$

$$P\{Y=k\} = P\{n-X=k\} = P\{X=n-k\} = {}^nC_{n-k} (p)^{n-k} (1-p)^k \Rightarrow Y \sim B(n, 1-p)$$
$$= {}^nC_k (1-p)^k p^{n-k}$$

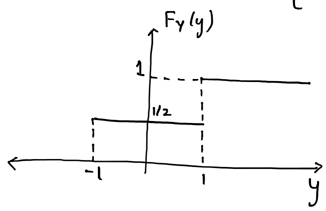
3.)

X	Y = g(X) (Also r.v.) ☆☆
discrete	discrete
continuous	continuous
mixed	mixed

4.) $X \sim U(-1,1)$



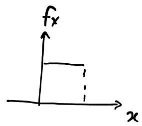
$$Y = \begin{cases} 1 & X < 0 \\ -1 & X \geq 0 \end{cases} \Rightarrow F_Y(y) = \begin{cases} 0 & y < -1 \\ P(X \geq 0) & -1 \leq y < 1 \\ 1 & 1 \leq y \end{cases} = \begin{cases} 0 & y < -1 \\ \int_0^1 \frac{1}{2}\,dx & -1 \leq y < 1 \\ 1 & 1 \leq y \end{cases}$$



$$= \begin{cases} 0 & y < -1 \\ \frac{1}{2} & -1 \leq y < 1 \\ 1 & 1 \leq y \end{cases}$$

5.) $X \sim U(0,1)$

$Y = -\frac{1}{\lambda} \ln(1-X)$



$$F_Y(y) = P(Y \leq y) = P(X \leq 1 - e^{-\lambda y}) = \begin{cases} 0 & y < 0 \\ \int_0^{1-e^{-\lambda y}} 1 \cdot dx & y \geq 0 \end{cases} = \begin{cases} 0 & y < 0 \\ 1 - e^{-\lambda y} & y \geq 0 \end{cases}$$

03/02/2016 $\boxed{\text{LECTURE 13}}$

Theorem : ☆

Let X be a continuous type random variable with p.d.f. $f_X(x)$. If h is strictly monotonic function & differentiable
Then the p.d.f. of $Y = h(x)$ is given by

$$\boxed{f_Y(y) = \begin{cases} f_X(h^{-1}(y)) \left| \frac{d\, h^{-1}(y)}{dy} \right| & , \text{ if } y = h(x) \\ 0 & y \neq h(x) \end{cases}}$$

eg :- $X \sim U(0,1)$, $Y = -\frac{1}{\lambda} \ln(1-X)$
①

$$f_Y(y) = f_X(1 - e^{-\lambda y}) \left| \frac{d}{dy} (1 - e^{-\lambda y}) \right| = \begin{cases} 1 \cdot \lambda \cdot e^{-\lambda y} & , 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Proof :

$$F_Y(y) = P(Y \leq y) = P(h(X) \leq y) = P(X \leq h^{-1}(y)) \quad (\text{if } h(x) \text{ is monotonically increasing})$$
$$= F_X(h^{-1}(y))$$

On differentiating, $f_Y(y) = f_X(h^{-1}(y)) \frac{d}{dy} h^{-1}(y)$

eg :- $X \sim N(\mu, \sigma^2)$ $\quad Z = \frac{X - \mu}{\sigma}$ $\quad f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$
②

$$h(x) = \frac{x-\mu}{\sigma} \Rightarrow h^{-1}(z) = \sigma z + \mu \Rightarrow f_Y(y) = f_X(h^{-1}(y))\left|\frac{d}{dz} h^{-1}(y)\right| = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{z^2}{2}} \times \sigma \quad \forall z \in R$$
$$= \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

③ $X \sim U(-1,1) \qquad Y = |X|$

$$F_Y(y) = P(|X| \le y) \;(y<0) = P(-y \le X \le y) = F_X(y) - F_X(-y) \Rightarrow f_Y(y) = f_X(y) + f_X(-y)$$
$$\Rightarrow f_Y(y) = \begin{cases} 1 & y>0 \\ 0 & y<0 \end{cases}$$

Alternate :

Corollary : Let h be a piecewise strictly montonic function & differentiable. Further there exists intervals $I_1, I_2, \dots I_n$

Then the pdf of $Y = h(x)$ is given by

$$f_Y(y) = \begin{cases} \sum_{k=1}^{n} f_X(h_k^{-1}(y)) \left|\frac{d}{dy}(h_k^{-1}(y))\right| \\ 0 & \text{otherwise} \end{cases}$$

$I_1 \to (-\infty, 0) \qquad I_2 \to [0, \infty)$

$y = -x \quad h^{-1}(y) = -y \qquad y = x \quad h^{-1}(y) = y$

$$f_Y(y) = \begin{cases} f_X(-y)\,|-1| + f_X(y)\,|1| & y>0 \\ 0 & y<0 \end{cases} = \begin{cases} 1 & y \ge 0 \\ 0 & y<0 \end{cases}$$

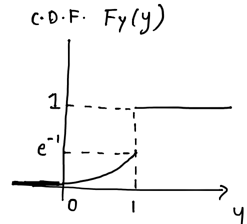
④ $X \sim N(\mu, \sigma^2)$, $Y = e^X$ Log normal

⑤ $X \sim Exp(1)$, $Y = \begin{cases} 1 & X \le 1 \\ \frac{1}{X} & X > 1 \end{cases}$

$f_X(x) = e^{-x}$

$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{X} \le y\right) \text{ If } (y<1) \;/\; 1 \text{ If } (y>1)$$
$$= P\left(X \ge \frac{1}{y}\right) \text{ If } (0<y<1)$$
$$= 0 \text{ If } y<0$$
$$= \begin{cases} 1 & y>1 \\ \int_{1/y}^{\infty} e^{-x}\,dx & 0<y<1 \\ 0 & y<0 \end{cases} = \begin{cases} 1 & y>1 \\ e^{-1/y} & 0<y<1 \\ 0 & y<0 \end{cases}$$

C.D.F. $F_Y(y)$



$$\Rightarrow f_Y(y) = \begin{cases} \frac{e^{-1/y}}{y^2} & 0<y<1 \\ 0 & \text{otherwise} \end{cases} \qquad P_Y(y) = \begin{cases} 1-e^{-1} & y=1 \\ 0 & \text{otherwise} \end{cases}$$

05/02/2016 LECTURE 14

GENERATING FUNCTIONS

Uniqueness Theorem :
1. P.G.F. $G_X(t)$ Probability Generating Function
2. M.G.F. $M_X(t)$ Moment Generating Function
3. C.F. $\phi_X(t)$ Characteristic Function

If $G/M/\phi_X(t) = G/M/\phi_Y(t) \;\; \forall \; t \;\; \Rightarrow \;\; X \overset{d}{=} Y$

Probability Generating Function (P.G.F.)

Let X be a non-negative integer values r.v. with p.m.f. $P_k = P\{X=k\}, k=0,1,2,\ldots$

Then the prb. gen. func. (pgf) be defined as

$$G_X(t) = \sum_{k=0}^{\infty} P\{X=k\}\, t^k \qquad |t| \le 1$$

e.g:- $X \sim B(n,p) \;\; \Rightarrow \;\; G_X(t) = (pt + 1-p)^n$

Remarks : (1) $G_X(1) = 1$

(2) $G_X(t) = E(t^X)$

(3) $P_k = P\{X=k\} = \dfrac{1}{k!} \left.\dfrac{d^k G_X(t)}{(dt)^k}\right|_{t=0}$

(4) $\underbrace{E(X(X-1))}_{\text{Factorial moments}} = \left.\dfrac{d^2 G_X(t)}{dt^2}\right|_{t=1}$

Moment Generating Function (M.G.F.)

Let X be a r.v. s.t. $E(e^{tX})$ is finite for some interval in $(-\infty,\infty)$ including the point 0. Then the M.G.F. of X is defined as

$$M_X(t) = E(e^{tX}) = E\left(1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \cdots + \frac{t^nX^n}{n!} + \cdots\right)$$

eg:- $X \sim P(\lambda)$

① $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} \dfrac{e^{-\lambda}\lambda^k}{k!} e^{tk} = e^{-\lambda} \sum_{k=0}^{\infty} \dfrac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{-\lambda(1-e^t)} \qquad t \in (-\infty,\infty)$

② $X \sim Exp(\lambda)$

$$M_X(t) = \int_0^\infty e^{xt} \lambda e^{-\lambda x} \, dx = \frac{\lambda}{t-\lambda}(-1) = \frac{\lambda}{\lambda - t} \quad \forall \; t \in (-\infty, \lambda)$$

Remarks : (1.) $E(X^k) = \left. \frac{d^k M_X(t)}{(dt)^k} \right|_{t=0}$

③ Find distribution of X if $M_X(t) = \frac{e^t}{2} + \frac{e^{-t}}{3} + \frac{1}{6}$

X	-1	0	1
$P\{X=k\}$	$1/3$	$1/6$	$1/2$

$M_X(t) = E(e^{tX})$

Characteristic Function (C.F.)

$$\psi_X(t) = E(e^{itX}) \qquad i = \sqrt{-1}$$

Remarks : (1) $\psi_X(0) = 1$

(2.) $|\psi_X(t)| \leq 1$

(3.) If $E(X^k)$ exist,

$$E(X^k) = \frac{1}{i^k} \left. \frac{d^k \psi_X(t)}{dt^k} \right|_{t=0}$$

Ex :- ① $X \sim \mathcal{N}(\mu, \sigma^2)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left\{\frac{x-\mu}{\sigma}\right\}^2} . dx \quad \cdots\cdots = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

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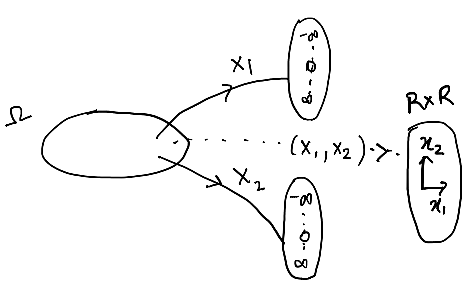
MINOR 1 SYLLABUS

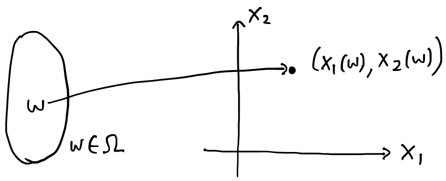
09/02/2016 $\boxed{\text{LECTURE 15}}$

TWO & HIGHER DIMENSIONAL RANDOM VARIABLES



If $X^{-1}\{(-\infty, x]\} \in \mathcal{F} \quad \forall \; x \in \mathbb{R}$

$X = (X_1, X_2)$ random vector of 2 dimension



In general,

$(X_1, X_2, \ldots\ldots X_n)$ n-dim random variable

C.D.F. (x_1, x_2)

$$F_{X_1,X_2}(x_1, x_2) = \text{Prob}\{X_1 \le x_1, X_2 \le x_2\}$$
$$= \text{Prob}\{\omega \mid X_1(\omega) \le x_1, X_2(\omega) \le x_2\}$$
$\omega \in \Omega$ $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$

It satisfies

1.) $0 \le F(x_1, x_2) \le 1 \quad \forall\ x_1, x_2$

2.) $\lim_{x_1 \to -\infty,\ x_2 \to -\infty} F(x_1, x_2) = 0$

3.) $\lim_{x_1 \to \infty,\ x_2 \to \infty} F(x_1, x_2) = 1$

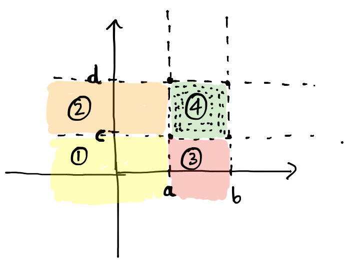
4) $F(x_1, x_2)$ is montonically increasing in both x_1 & x_2.

5.) $\forall\ a < b,\ c < d$

$F_{X_1X_2}(b,d) - F(a,d) - F(b,c) + F(a,c) \ge 0$ (Similar for multi-dimension)

i.e. $(1+2+3+4) - (1+2) - (1+3) + 1$
$= 4$
$F(4) \ge 0$

eg:- ① $F(x_1, x_2) = \begin{cases} 1 - e^{-(x+y)} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$



② $F(x_1, x_2) = \dfrac{\tan^{-1}(x_1 + x_2)}{\pi} + \dfrac{1}{2}$ $x \in \mathbb{R}$, $y \in \mathbb{R}$

- **Two Dimensional Discrete Type Random Variable**

$$P_{X_1,X_2}(x_1, x_2) = \text{Prob}\{X_1 = x_1, X_2 = x_2\} \quad \text{(Mass function)}$$
$$= \text{Prob}\{\omega \mid X_1(\omega) = x_1, X_2(\omega) = x_2\} \quad \omega \in \Omega$$

It satisfies

1.) $0 \le P(x_1, x_2) \le 1 \quad \forall\ x_1, x_2$

2.) $\sum_{x_1} \sum_{x_2} P(x_1, x_2) = 1$

$$P_{X_1}(x_1) = \sum_{x_2} P(x_1, x_2) \ ; \quad P_{X_2}(x_2) = \sum_{x_1} P(x_1, x_2)$$

10/02/2016 $\boxed{\text{LECTURE 16}}$

- Two Dimensional continuous type Random vector (X_1, X_2)

$$F_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \underbrace{f_{X_1,X_2}(t,s)}_{\text{joint pdf}}\, ds\, dt$$

Joint pdf satisfies

1.) $f(x_1, x_2) \geq \forall\ x_1, x_2$

2.) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,s)\, ds\, dt = 1$

eg :- 1.) $f(x,y) = \begin{cases} e^{-(x+y)} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$

2.) $f(x,y) = \begin{cases} \frac{1}{2\pi} e^{-\frac{1}{2}[x^2+y^2]} & -\infty < x < \infty,\ -\infty < y < \infty \end{cases}$

eg:-

E : Tossing a coin Thrice

X : # of heads

Y : Difference (in absolute) of # of heads & # of tails

$\Omega = \{(HHH), \ldots\ldots, (T,T,T)\}$ $\quad N_q(\Omega) = 8$

joint p.m.f. of (X,Y)

$X \downarrow$ / $Y \rightarrow$	1	3	$P(X=x)$
0	0	1/8	1/8
1	3/8	0	3/8
2	3/8	0	3/8
3	0	1/8	1/8
$P(Y=y)$	6/8	2/8	

Individually : (Marginal Distrubution of X,Y)

Y	P
1	2/8
3	6/8

X	P
0	1/8
1	3/8
2	3/8
3	1/8

- Independent Random Variable

We say that two random variables are independent iff

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y) \quad \forall\ x,y$$

i.e. if (X,Y) - discrete type $\Rightarrow P_{X,Y}(x,y) = P_X(x)\, P_Y(y) \quad \forall\ x,y$

if (X,Y) - continuous type $\Rightarrow f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad \forall (x,y)$

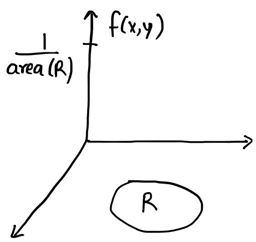
eg:- Multivariable Normal Distribution (x, y, z)

$$f_{X,Y,Z}(x,y,z) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}(x^2+y^2+z^2)} \quad \begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{matrix}$$

(Independent)

eg:- (x, y) - joint pdf 'R' in x-y plane
Uniform Distribution.

(Independent)



3.) $$f_{X,Y}(x,y) = \begin{cases} 2 & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Not independent.

16/02/2016 LECTURE 17

Password - Markov

CONDITIONAL DISTRIBUTION

$P(A|B)$; $P(B) > 0$

Defn: Let (X,Y) be a 2 dimension discrete type R.V. with joint pmf $P_{X,Y}(x,y)$.
Then the conditional dist. of X given $y = y_j$ is defined as

$$\boxed{P\{X = x_i / Y = y_j\} = \frac{P\{X = x_i, Y = y_j\}}{P\{Y = y_j\}}}$$

$\forall i$
provided $P\{Y = y_j\} > 0$

Note:- $\sum_i P(X = x_i / Y = y_j) = \frac{\sum_i P\{X = x_i, Y = y_j\}}{P\{Y = y_j\}} = 1$

Defn: Let (X,Y) be a 2 dimension continuous type R.V. with joint pdf $f_{X,Y}(x,y)$.
Then the conditional dist. of X given $y = y_j$ is defined as

$$\boxed{f_{X/Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}} \qquad \forall\, x \text{ provided } f_Y(y) > 0$$

C.D.F. of $X/Y=y_j = \int_{-\infty}^{x} f_{X/Y}(t/y_j)\,.dt$

eg:- $f_{X,Y}(x,y) = \begin{cases} 2 & 0<y<x<1 \\ 0 & \text{otherwise} \end{cases}$

p.d.f. of

(i) $X/Y \Rightarrow f_{X/Y}(x,y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{2}{2(1-y)} & y<x<1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow X/y \sim U(y,1)$

$f_Y(y) = \int_y^1 2.dx = 2(1-y)$

ii) $f_X(x) = \int_0^x 2.dy = 2x$

$Y/x = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{2}{2x} & 0<y<x<1 \\ 0 & \text{otherwise} \end{cases} \qquad Y \sim U(0,x)$

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17/02/2016 LECTURE 18

FUNCTIONS OF RANDOM VARIABLES

eg:- Let $X \sim B(n,p)$, $Y \sim B(n,p)$ Assume that X & Y are independent r.v.'s.
Define $Z = X+Y$, dist of $Z = ?$

$P(Z=k) = \sum_i P(X=i, Y=k-i) = \sum_i P(x=i).P(y=k-i)$

$= \sum_i {}^nC_i\, p^i (1-p)^{n-i} . {}^nC_{k-i}\, p^{k-i}. (1-p)^{n-k+i}$

$= \sum_i {}^nC_i\, {}^nC_{k-i}\, p^k (1-p)^{2n-k}$

$= \left(\sum_i {}^nC_i\, {}^nC_{k-i}\right) p^k (1-p)^{2n-k}$

$= {}^{2n}C_k\, p^k\, p^{2n-k}$

In general, $X_i \sim B(n_i, p)$, $i = 1,2,3,\dots,k$ & X_i's are independent r.v.

$$Z = \sum_{i=1}^{k} X_i \sim B\left(\sum_{i=1}^{k} n_i, p\right)$$

iid - independent + identically distributed

② $X_i \sim B(1,p)$, $i=1,\dots,n$ X_i - iid's

$$Z = \sum_{i=1}^{n} X_i \sim B(n,p)$$

ΣX_i's - same distribution

$\Rightarrow$ Reproductive Property

③ $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

X & Y are independent

$$Z = X+Y \sim N(\mu_1+\mu_2, \dots)$$

④ $X \sim P(\lambda_1)$, $Y \sim P(\lambda_2)$

$Z = X+Y$

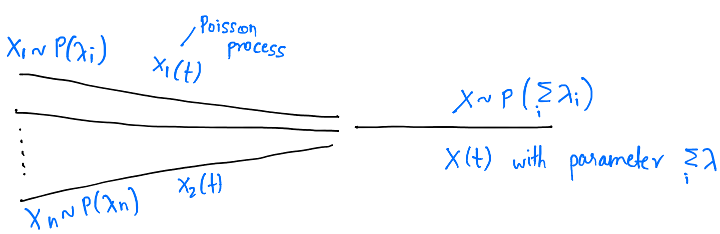
$$P(Z=k) = \sum_i P(X=i, Y=k-i)$$

$$= \sum_i P(X=i)\,P(Y=k-i) = \sum_i \frac{e^{-\lambda_1}(\lambda_1)^i}{i!} \times \frac{e^{-\lambda_2}(\lambda_2)^{k-i}}{(k-i)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^k}{k!} \Rightarrow Z \sim P(\lambda_1+\lambda_2) \quad \text{(Reproductive)}$$

In general, $X_i \sim (\lambda_i)$ $i=1,2,\dots n$ X_i's are idd.

$$Z = \sum_i X_i \sim P\left(\sum_i \lambda_i\right)$$



MOMENTS OF RANDOM VARIABLE

Defn: Let $x, x_2, \dots x_n$ be R.V.'s

$$E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} E(x_i)$$

$$Var\left(\sum x_i\right) = \sum_{i=1}^{n} Var(x_i) + 2 \sum_{i}\sum_{j}^{i<j} Cov(x_i, x_j)$$

Defn: $Cov(x,y)$

$$Cov(x,y) = E\left((x-E(x))(y-E(y))\right)$$

1^{st} order cross moment about the mean.

$$\mu_{i,j} = E\left((x-E(x))^i (y-E(y))^j\right)$$ $(i,j)^{th}$ order cross moment about the mean.

$E(x_i, x_j) \longrightarrow$ about the zero.

$E\left((x-E(x))^2\right) \longrightarrow Var(x)$ $\quad [\mu_{(2,0)}]$

Variance Covariance Matrix of n-variables

$$\begin{bmatrix} Var(x_1) & Cov(x_1,x_2) & \cdots & Cov(x_1, x_n) \\ Cov(x_2,x_1) & \cdots & \cdots & \vdots \\ \vdots & & & \vdots \\ \vdots & \cdots & \cdots & Cov(x_{n-1}, x_n) \\ Cov(x_n,x_1) & \cdots & \cdots & Var(x_n) \end{bmatrix}$$

$Cov(x, y) = Cov(y, x)$

Remark: If x & y are independent r.v.'s

$$Cov(x,y) = E\left((x - \underset{\mu_1}{\underset{\downarrow}{E(x)}})(y - \underset{\mu_2}{\underset{\downarrow}{E(y)}})\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_1)(y-\mu_2) f_{x,y}(t,s)\, dt.ds$$

$$= E(x-E(x))\ E(y-E(y))$$

$$= [E(x)-E(x)][E(y)-E(y)]$$

$$= 0$$

$\Rightarrow$ If x & y are independant $\Longrightarrow$ $Cov(x,y) = 0$

$\not\Longleftarrow$

Vice - versa is not true.

Defn :- Correlation Coefficient

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$ provided R.H.S. exist.

X & Y are independent, then $\rho_{X,Y} = 0$
$\not\Leftarrow$

Remark :-
(i) $|\rho_{X,Y}| \leq 1$
(ii) If $\rho_{X,Y} = \pm 1$
$\Rightarrow Y = aX + b$
$a > 0 \Rightarrow \rho_{X,Y} = 1$
$a < 0 \Rightarrow \rho_{X,Y} = -1$

23/02/2016 $\boxed{\text{LECTURE 19}}$

eg:- ① $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

Game overs whenever X or Y reaches 1.
Assume X & Y are independent.

Define $Z = \min\{x, y\}$

$$P(Z > z) = P(\min\{x,y\} > z) = P(x, y > z)$$
$$= P(x > z) \cdot P(y > z) = \int_z^\infty \lambda e^{-\lambda x} \cdot dx \int_z^\infty \mu e^{-\mu y} dy = e^{-(\lambda+\mu)z}$$

$$P(Z \leq z) = 1 - e^{-(\lambda+\mu)z}$$

$$F_Z(z) = \begin{cases} 1 - e^{-(\lambda+\mu)z} & z \geq 0 \\ 0 & z < 0 \end{cases} \Rightarrow Z \sim Exp(\lambda + \mu)$$

In general, $X_i \sim Exp(\lambda_i)$, $i = 1, 2, \dots, n$, X_i independent r.v.'s

$$Z = \min\{X_i, i = 1, 2, \dots n\} \sim Exp\left(\sum_{i=1}^n \lambda_i\right)$$

$$W = \max\{X_i \dots\} \not\sim Exp$$
$$(1 - e^{-\lambda w})(1 - e^{-\mu w})$$

② A camera runs on 2 batteries. We have 6 batteries (Battery life $X_i \sim Exp(\lambda)$)

a) What is average usage time

Suppose battery 1 & 2 are running. For one to fail, $Y = \min\{X_1, X_2\} \sim Exp(2\lambda)$
$\Rightarrow$ Avg Time for one battery to fade $= \frac{1}{2\lambda}$. Suppose battery 2 fails. Then put battery 3, then using memoryless property on battery 1, again $1/2\lambda$.

$$\Rightarrow \text{Total avg. time} = \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{5}{2\lambda}$$ (till 5 batteries fail)

(b) C.D.F. of $Z = \underset{\sim Exp(2\lambda)}{\underset{\downarrow}{Y_1}} + Y_2 + Y_3 + Y_4 + Y_5 = Gamma(5, 2\lambda)$

□ **Theorem**: Let (X,Y) be a two dimensional cont. type random variable with a joint p.d.f. $f_{X,Y}(x,y)$. Define $Z = H_1(x,y)$ & $W = H_2(x,y)$

Assume (i) $z = h_1(x,y)$ & $w = h_2(x,y)$ can be solved uniquely for x,y in terms of z & w, i.e. $x = g_1(z,w)$, $y = g_2(z,w)$

(ii) $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}, \frac{\partial y}{\partial w}$ exist and are continuous function

Then (Z,W) is 2-dim. cont. type r.v. with joint p.d.f.

$$f_{Z,W}(z,w) = \begin{cases} f_{X,Y}(g_1(z,w), g_2(z,w)) \, |J(z,w)| & x = g_1(z,w) \\ & y = g_2(z,w) \\ 0 & \text{otherwise} \end{cases}$$

where $$J(z,w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} \neq 0$$

eg:-

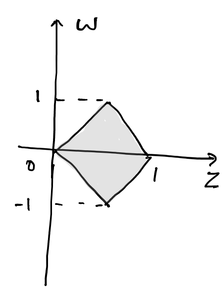
1.) Let (X,Y) - 2 dim cont. type. X & Y are idd r.v.'s

$$f_X = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$Z = X + Y$
$W = X - Y$ $\Rightarrow$ $X = \frac{Z+W}{2}$ & $Y = \frac{Z-W}{2}$

$X \in (0,1)$ & $Y \in (0,1)$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$



$$f_{Z,W}(z,w) = \begin{cases} 1 \cdot 1 \cdot \frac{1}{2} & 0 < \frac{z+w}{2} < 1 \ \& \ 0 < \frac{z-w}{2} < 1 \\ 0 & \text{otherwise} \end{cases}$$

Note: $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

But $f_{Z,W}(z,w) \neq f_Z(z) \cdot f_W(w) \Rightarrow z$ & w are not independent

24/02/2016 $\boxed{\text{LECTURE 20}}$

ex: 2) $X_i \sim N(0,1)$, $i = 1,2,\dots n$, iid r.v.'s

$Z = \sum_{i=1}^{n} X_i$

Assume that M.G.F. of Z exists $M_X(t)$

$$M_Z(t) = M_{\sum_{i=1}^{n} X_i}(t) = E\left(e^{\sum_{i=1}^{n} x_i t}\right) = \left(e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right)^n$$

$$= e^{\frac{1}{2} n t^2}$$

$\Rightarrow \quad Z \sim N(0, n)$ (Other alternatives lengthy !!)

CONDITIONAL EXPECTATION

Defn. Let (X,Y) be a 2 dim. r.v.

$$E(X/y) = \begin{cases} \sum_i x_i \, P\{X = x_i / Y = y\} & (X,Y) \text{ - disc.} \\ \int_{-\infty}^{\infty} x \, f_{X/y}(x/y) \, dx & (X,Y) \text{ - Cont.} \end{cases}$$

provided in absolute sense, R.H.S. exist.

Remark : ① If X & Y are independent random variables, $E(X/y) = E(x)$; $E(Y/x) = E(y)$

② $E(E(X/y)) = \int_{-\infty}^{\infty} E(X/y) \cdot f_Y(y) \cdot dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, f_{X/y}(x/y) \cdot dx \cdot f_Y(y) \cdot dy$

$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_{X/y}(x/y) \cdot f_Y(y) \cdot dy} \, dx = \int_{-\infty}^{\infty} x \, f_X(x) \cdot dx = E(X)$

$\Rightarrow \quad E(E(X/y)) = E(X)$

③ $E(X/y)$ is a function of y ; $E(X/y)$ is a r.v.

④ $E(XY/y) = yE(X/y)$

Martingale Property

⑤ $\{X_1, X_2, \ldots, X_n, \ldots\}$

$$E(X_{n+1} / X_1 = x_1, X_2 = x_2, \ldots X_n = x_n) = \begin{cases} > x_n \\ < x_n \\ = x_n & \text{Fair game} \end{cases}$$

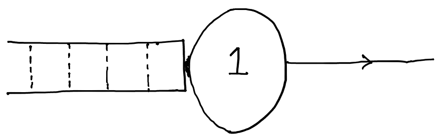
where

$X_i = X_{i-1} + Y_i$

$Y_i \downarrow$ i^{th} game

Y_i's are iid r.v.'s

26/02/2016 $\boxed{\text{LECTURE 21}}$



queing system

Random Sum

X : total time spent in the system

$= \underbrace{X_1}_{\text{Residual service time}} + X_2 + \dots + \underbrace{X_n}_{\text{his own service time}}$

$X_i \sim Exp(\lambda)$ iid r.v.'s

where even n is a random variable.

$$E(X) = E\left(E(X/N=n)\right)$$

$$= \sum_n E(X/n)\, P(N=n)$$

(But X_i are independent with N)

$\Rightarrow E(X/n) = n\,E(X_i)$

$$= \sum_n n\,E(X_i)\, P(N=n)$$

$$E(X) = E(X_i) \cdot E(N)$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = E\left(E(X^2/N)\right)$$

dist of X, given dist of X_i, N

$P(N=n) = (1-p)p^n$

$n = 0, 1, 2, \dots$

$$f_X(x) = \sum_n \underbrace{f_{X/N=n}(x/n)}_{\text{cont.}} \cdot \underbrace{P(N=n)}_{\text{discrete}}$$

$\downarrow$
Gamma

$$f_{(X,N)}(x,n) = f_{X/N}(x/n) \cdot P(N=n)$$

$$\boxed{f_{X,Y}(x,y) = f_{X/Y}(x/y) \cdot f_Y(y)} \quad \text{Always } \forall x, y$$

08/03/2016 $\boxed{\text{LECTURE 22}}$

1.) Inequalities

2.) Law of Large Numbers

3.) Limiting Distribution

4.) Central Limit Theorem

Defn.: Markov Inequality

Let X be non-negative random variable with E(X) exists and is known.

For fixed $t > 0$

$$\boxed{\underbrace{P(X > t)}_{\text{event}} \leq \frac{E(X)}{t}}$$

$\{w \mid X(w) > t \quad w \in \Omega\}$

Define $Y = \begin{cases} 0 & , x \leq t \\ t & x > t \end{cases}$ $\quad P\{Y = 0\} = P\{x \leq t\}$ $\quad$ Y is a discontinuous type r.v.

$P\{Y = t\} = P\{x > t\}$

$E(Y) = t \cdot P\{x > t\}$ —— ①

Now $X \geq Y$

$\therefore \; E(X) \geq E(Y)$ —— ②

$$\Rightarrow \quad P\{x > t\} \leq \frac{E(X)}{t}$$

Defn.: Chebyshev's Inequality

Let X be a r.v. with $E(X) = \mu$, $Var(X) = \sigma^2$ exist & are known.

Then for any positive number t,

$$P\{|X - \mu| > t\} \leq \frac{\sigma^2}{t^2}$$

or

$$P\{|X - \mu| \leq t\} > 1 - \frac{\sigma^2}{t^2}$$

Proof :- X is a r.v. $\Rightarrow (X - \mu)^2$ is a non neg. r.v.

where $E((X - \mu)^2) = \sigma^2$

Apply Markov Inequality. Hence Proved.

eg:- Let X be a r.v. with $E(X) = \frac{1}{2}$, $Var(X) = \frac{1}{12}$. Find the lower bound for

$$P\left\{|X - \tfrac{1}{2}| < 2\sqrt{\tfrac{1}{12}}\right\} \Rightarrow \frac{3}{4}$$

Let X be a uniform distribution in interval $[0,1]$.

$$\frac{1}{2} + 2\sqrt{\frac{1}{12}} > 1 \quad \& \quad \frac{1}{2} - 2\sqrt{\frac{1}{12}} < 0 \quad \Rightarrow \quad P\left\{|X - 1/2| < 2\sqrt{\frac{1}{12}}\right\} = 1$$

Remark: In Chebyshev's Inequality.

μ can be replaced by any number c.

In that case, $\sigma^2 \rightarrow E\left((X-c)^2\right)$

09/03/2016 $\boxed{\text{LECTURE 23}}$

LIMITING DISTRIBUTIONS / PROBABILITIES / THEOREMS

$X_1, X_2, X_3, \ldots\ldots \rightarrow X \qquad (\Omega, \mathcal{F}, P)$

① $X_i \sim U(-1,1)$, $i = 1, 2, \ldots$ iid's

Define $Y_1 = X_1$, $Y_2 = X_1 + X_2$, ... $Y_n = \sum_{i=1}^{n} X_i$

What is the convergence of Y series.

Y's will be polygons. $\Rightarrow$ Increasing no. of sides $\Rightarrow$ Normal distribution.

② $X_i \sim B(1,p)$ $i = 1, 2, \ldots$ iid r.v. s

$Y_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$

For fixed n, $Y_n \sim B(n,p)$

For large n, $n \rightarrow \infty$

normal distribution

Theorem: Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent random variables defined on $(\Omega, \mathcal{F}, P)$ with $E(X_i) = \mu_i$, $i = 1, 2, \ldots$ and $Var(X_i) = \sigma_i^2 > 0$, $i = 1, 2, \ldots$

Define
$$\boxed{Z_n = \frac{\sum_{i=1}^{n} X_i - E\left(\sum_{i=1}^{n} X_i\right)}{\sqrt{Var\left(\sum_{i=1}^{n} X_i\right)}}} \qquad n = 1, 2, \ldots$$

Then for larger n, Z_n approaches standard normal distribution.
(approximately)

i.e. $$P(Z_n \le x) \underset{n \to \infty}{\simeq} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt \qquad x \in \mathbb{R}$$

CENTRAL LIMIT THEOREM

11/03/2016 $\boxed{\text{LECTURE 24}}$

MODES OF CONVERGENCE

1) IN PROBABILITY

for given $\epsilon > 0$, $X_n \xrightarrow{P} X$

if $\quad P\{|X_n - X| > \epsilon\} \longrightarrow 0 \quad$ as $n \to \infty$

2.) IN DISTRIBUTION

$$X_n \xrightarrow{d} X$$

if $\quad F_{X_n}(x) \longrightarrow F(x) \qquad \forall\ x \in R$

3.) IN RTH MOMENT

$$X_n \xrightarrow{r^{th}\text{ moment}} X \qquad r = 1, 2, 3, \ldots$$

$$E\left(|X_n - X|^r\right) \longrightarrow 0 \quad \text{as } n \to \infty$$

4.) IN ALMOST SURELY

$$X_n \xrightarrow{a.s.} X$$

if $\quad P\{\lim_{n\to\infty} X_n = X\} = 1$

eg:- Let $\{X_n, n = 1, 2, 3, \ldots\}$ be a sequence of r.v.'s

s.t. $\quad P\{X_n = 0\} = 1 - \frac{1}{n}\ ;\ P\{X_n = n\} = \frac{1}{n} \qquad n = 1, 2, 3, \ldots$

check $\quad X_n \xrightarrow{P} 0$

$$\lim_{n\to\infty} P\{|X_n - 0| > \epsilon\} = \lim_{n\to\infty} \frac{1}{n} = 0$$

check $\quad X_n \xrightarrow{d} 0$

H.W.

FLOWCHART

$$\begin{tikzcd}
 & & X_n \xrightarrow{2} X \arrow[d, Rightarrow] \\
X_n \xrightarrow{a.s.} X \arrow[dr, Rightarrow] & & X_n \xrightarrow{1} X \arrow[dl, Rightarrow] \\
 & X_n \xrightarrow{P} X \arrow[d, Rightarrow] & \text{1st moment} \\
 & X_n \xrightarrow{d} X &
\end{tikzcd}$$

eg :- $\{X_n, n=1,2,3,\dots\}$ be a seq. of iid r.v.'s with $E(X_i)=\mu$; $Var(X_i)=\sigma^2$, $i=1,2,\dots$

Define $S_n = X_1 + X_2 + \dots + X_n$

Check (1) $\frac{S_n}{n} \xrightarrow{\text{2nd order}} \mu$ (2) $\frac{S_n}{n} \xrightarrow{P} \mu$

$$\lim_{n\to\infty} E\left(\left|\frac{S_n}{n}-\mu\right|^2\right) = \lim_{n\to\infty}\left[\frac{1}{n^2}E(S_n^2) - \frac{2\mu}{n}E(S_n) + \mu^2\right]$$

$$= \sigma^2$$

15/03/2016 $\boxed{\text{LECTURE 25}}$

• LAW OF LARGE NUMBERS (Bernoulli Law)

Let ϵ be a random experiment and A be an event. Consider n independent trials.

Define n_A = # of times event A occurs in n-trials

Let $f_A = \frac{n_A}{n}$; $p(A) = p$, constant.

Then $f_A \xrightarrow{P} p$

Proof :

$n_A = B(n,p)$

$$E(n_A) = \sum n_A \cdot {}^nC_{n_A}\, p^{n_A} \cdot (1-p)^{n-n_A} = np \sum {}^{n-1}C_{n_A-1}\, p_A^{n_A-1}(1-p)^{n-n_A}$$
$$= np$$

$$E(f_A) = E\left(\frac{n_A}{n}\right) = \frac{1}{n}\cdot np = p$$

$$Var(f_A) = Var\left(\frac{n_A}{n}\right) = \frac{1}{n^2}Var(n_A) = \frac{p(1-p)}{n}$$

Apply Chebyshev's Inequality,

$$P\{|f_A - p| > \epsilon\} \le \frac{p(1-p)}{n\epsilon^2}$$

$$\lim_{n\to\infty} P\{|f_A-p| > \epsilon\} = 0 \Rightarrow f_A \xrightarrow{P} p$$

eg :- ϵ : die what is min. n for given $\epsilon = 0.01$ s.t.

A : getting #6 $P\{|f_A - p| < \epsilon\} \ge 0.95$

Sol^n : $p = \frac{1}{6}$ $\quad P\{|f_A - p| > \epsilon\} < 0.05$

$$\Rightarrow \quad n = \frac{p(1-p)}{(0.05)\,\epsilon^2} \approx 27{,}778$$

Proof of CENTRAL LIMIT THEOREM

Assume : (i) X_i's are iid r.v.'s

(ii) M.G.F. of X_i's exist

$$M_{Z_n}(t) = M_{\frac{\Sigma X_i - n\mu}{\sigma\sqrt{n}}}(t) = E\left(e^{\left(\frac{\Sigma X_i - n\mu}{\sigma\sqrt{n}}\right)t}\right) = e^{-\frac{\sqrt{n}\mu t}{\sigma}} E\left(e^{\frac{\Sigma X_i}{\sigma\sqrt{n}}t}\right)$$

$$= e^{-\frac{\sqrt{n}\mu t}{\sigma}} \left[M_{X_1}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

$$M_X(t) = 1 + \underbrace{\frac{\mu t}{1!} + \frac{E(X^2)t^2}{2!} + \dots}_{x}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \qquad |x|<1$$

$$\ln M_{Z_n}(t) = -\frac{\sqrt{n}\mu t}{\sigma} + n \ln\Big[1 + \underbrace{\frac{\mu t}{\sigma\sqrt{n}} + \frac{(\sigma^2+\mu^2)t^2}{2!\, n\sigma^2} \dots}_{x}\Big]$$

$$= -\frac{\sqrt{n}\mu t}{\sigma} + n\Big[\underbrace{\frac{\mu t}{\sigma\sqrt{n}} + \frac{(\sigma^2+\mu^2)t^2}{2n\sigma^2}}_{x} \underbrace{- \frac{1}{2}\left(\frac{\mu^2 t^2}{n\sigma^2} + \dots\right)}_{-x^2/2}\Big]$$

$$= \frac{(\sigma^2+\mu^2)t^2}{2\sigma^2} - \frac{1}{2}\frac{\mu^2 t^2}{\sigma^2} = \frac{t^2}{2}$$

$$\Rightarrow \lim_{n\to\infty} \ln M_{Z_n}(t) = \frac{t^2}{2} \quad \text{or} \quad \lim_{n\to\infty} M_{Z_n}(t) = e^{t^2/2}$$

$$Z_n \sim N(0,1)$$

16/03/2016 $\boxed{\text{LECTURE 26}}$

eg:- Let $X_1, X_2 \dots X_n$ be a sequence of iid r.v. with common p.m.f.

$$P(x_1) = \begin{cases} p & x_1 = 1 \\ 1-p & x_1 = 0 \end{cases} \qquad p = 1/2$$

Let $X = X_1 + X_2 + \dots + X_{10}$

(a) Find $P(X \le 8)$ exactly (distribution)

(b) $P\{X \le 8\}$ lower bound (Inequalities)

(c) $P\{X \le 8\}$ approximately (CLT)

Sol^n (a) $P(X \le 4) = \sum_{0}^{8} {}^{10}C_i \, p^i (1-p)^{10-i}$

(b) Markov's inequality

$$P\{X \le 8\} \ge 1 - \frac{E(X)}{8} = 1 - \frac{5}{8} = \frac{3}{8}$$

(c) $$P\{X \le 8\} = P\left\{ \frac{X - \overbrace{5}^{\mu = np}}{\underbrace{\sqrt{2.5}}_{\text{Variance } np(1-p)}} \le \frac{8-5}{\sqrt{2.5}} \right\} = P\Big\{ \underbrace{Z}_{\text{normal}} \le \underbrace{\frac{3}{\sqrt{2.5}}}_{1.89} \Big\} = F_Z(1.89) = \underbrace{0.9698}_{\text{from table}}$$

eg:- Find $\lim_{n \to \infty} \sum_{k=0}^{n} e^{-n} \frac{n^k}{k!}$ approximately using C.L.T.

Sol^n $X_i \sim P(1)$

$X = \sum_{i=1}^{n} X_i \sim P(n)$ $\quad P(X \le n) = P\left(\frac{X-n}{\sqrt{n}\sigma} \le \frac{n-n}{\sqrt{n}\sigma} \right) = P(Z \le 0) = \frac{1}{2}$ //

$Z = \frac{X - n}{\sqrt{n}\,\sigma}$

18/03/2016 **LECTURE 27**

TUTORIAL 5

28/03/2016 **LECTURE 28**

Defn:

Let $(\Omega, \mathcal{F}, P)$ be a prob space. The collection of random variables $\{x(t), t \in T\}$ be fixed on the probability space $(\Omega, \mathcal{F}, P)$ is called a stochastic process.

Parameter Space, T

state space, S

S \ T	disc	cont
disc		
cont		

eg:- Temp at time t in a city
① $\{x(t), t \geq 0\}$ $T = \{t \mid 0 \leq t \leq \infty\}$ cont. state, cont. time stoch. process
$S = \{x \mid a \leq x \leq b\}$

② # of vehicles parked in the main gate upto t
$\{Y(t), t \geq 0\}$ $T = \{t \mid 0 \leq t \leq \infty\}$ $S = \{0, 1, 2, \dots\}$ disc. state cont. time stoch. process

Time Series :
- Observed information / data over the time.

Sample Path is right continuous.

29/03/2016 **LECTURE 29**

PROPERTIES OF STOCHASTIC PROCESS

1) Independent Increments $\{x(t), t \geq 0\}$

for arbitrarily $0 < t_0 < t_1 < \dots < t_n < \dots$
of the rvs $\forall n$

$X(t_1) - X(t_0),\ X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$ are mutually independent.

2) Wide sense stationary / Covariance stationary

If (i) $E(X(t))$ is not a function of time t
(ii) $E(X^2(t)) < \infty$
(iii) $Cov(X(t), X(s))$ depends only on $|t-s|$.

then process is wide sense stationary

3) Strict sense stationary

$t_1 < t_2 < t_3 \dots < t_n$

$$(X(t_1), X(t_2), \dots, X(t_n)) \stackrel{d}{=} (X(t_1+h), X(t_2+h), \dots, X(t_n+h)) \quad \forall n \ \forall h > 0$$

Then it is time variant.

4.) Markov Property

$0 \leq t_0 < t_1 < \dots < t_n < t$ $\forall n$

If $P\{X(t) \leq x \mid X(t_0) = x_0, X(t_1) = x_1, \dots, X(t_n) = x_n\} = P\{X(t) \leq x \mid X(t_n) = x_n\}$
Then it is Markov Property.

Ex :- Let $\{X_1, X_2, \dots\}$ be a sequence of iid r.v.s with common p.m.f's.

$P\{X_i = 0\} = 1 - p = 1 - P\{X_i = 1\} \quad 0 < p < 1$

Define $S_0 = 0$

$$S_n = \sum_{i=1}^{n} X_i$$

$\{S_n, n = 0, 1, 2, \dots\}$ is a stochastic process

For fixed n, $S_n \sim B(n, p)$

S_n will satisfy only Independent Increment & Markov Property.

$P\{S_{n+m} = k / S_0 = 0, S_1 = i_1, S_2 = i_2, \dots S_n = i_n\}$

$$= \frac{P\{S_{n+m} = k, S_0 = 0, \dots S_n = i_n\}}{P\{S_0 = 0, S_1 = i_1, \dots S_n = i_n\}} = \frac{P\{S_{n+m} - S_n = k - i_n, S_n - S_{n-1} = i_n - i_{n-1}, \dots S_1 - S_0 = i_1 - 0\}}{P\{S_n - S_{n-1} = i_n - i_{n-1}, \dots S_1 - S_0 = i_1 - 0\}}$$

But $S_n - S_{n-1}, \dots S_1 - S_0$ are independent, terms will cancel out to

$$P\{S_{n+m} = k / S_n = i_n\}$$

$$\boxed{S_n = \underbrace{S_{n-1} + X_n}_{\text{independent increment}}}$$

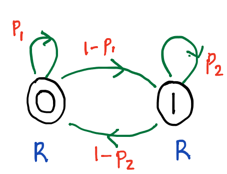
First order dependent or Markov process Auto regressive AR(1)

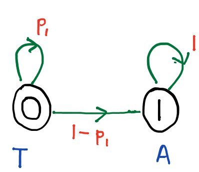
30/03/2016 LECTURE 30

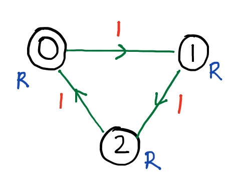
01/04/2016 $\boxed{\text{LECTURE 31}}$

05/04/2016 $\boxed{\text{LECTURE 32}}$

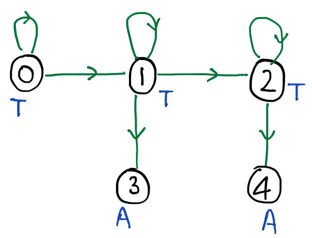
CLASSIFICATION OF STATE

1.) 

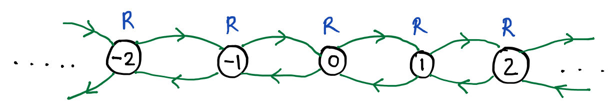
2.) 

3.) 

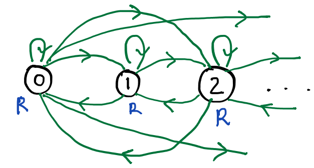
4.)



5.)



6.)



Definitions :

1.) Visit $i \to j$, if $P_{ij}^{(n)} > 0$ for some n.

2.) Communicate $i \to j$ if $P_{i,j}^{(m)} > 0$ for some m

& $P_{j,i}^{(m)} > 0$ for some m

3.) Periodicity of state i

$$d_i = g.c.d\,\{x \geq 1 : P_{ii}^{(n)} > 0\}$$

4.) First visit

$f_{i,j}^{(n)}$ - conditional probability that the system first visit to state j in exactly n steps (with initial state i)

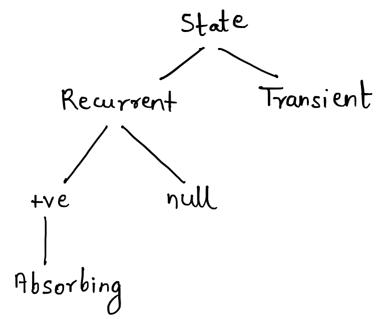
$$P\{X_n = j / X_0 = i,\ X_k \neq j \quad k = 1, 2, \ldots, n-1\}$$

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$ ever visiting state j starting from state i

$$P_{ij}^{(n)} = \sum_{k} f_{ij}^{(k)} P_{jj}^{(n-k)}$$

5.) Recurrent State

iff $f_{ii} = 1$



6.) Transient State iff $f_{ii} < 1$

7.) Absorbing state iff $P_{ii} = 1$

8.) Mean recurrence time $\mu_i = \sum n f_{ii}$

9.) +ve recurrent state if $\mu_i < \infty$

10.) null recurrent state if $\mu_i = \infty$

06/04/2016 $\boxed{\text{LECTURE 33}}$

Definition : 1.) Closed Communicating Class (C.C.C.)

A set of communicating states, say $C \subseteq S$

If no state outside C can be reached from any state in C, we say the set C is C.C.C.

2.) Aperiodic state

$d_i = 1$

3.) Irreducable

if $S = C_1$ where C_1 is only one C.C.C.

Defn : Stationary Distribution

Let P be the one-step transition probability matrix of a (time-homogeneous) DTMC.

A probability distribution $\{\pi_i\}$ $i \in S$ is said to be stationary distribution (or time invariant) for the given DTMC or steady state distribution or equillibrium distribution

if $\pi_j = \sum_i \pi_i p_{ij}$, $j \in S$ $\quad P = [P_{ij}]_{i,j \in S}$

s.t. $\pi_i \geq 0$ & $\sum_i \pi_i = 1$

$$\underbrace{\pi_j}_{\substack{\text{matrix} \\ 1 \times n}} = \underbrace{\pi_i}_{\substack{\text{matrix} \\ 1 \times n}} \times \underbrace{P_{ij}}_{\substack{\text{matrix} \\ n \times n}} = 1 \times n$$

Defn : Limiting Distribution

Let $\pi_j = \lim_{n \to \infty} \text{Prob}\{X_n = j\}$, $j \in S$ if it exist

$\{\pi_j\}$, $j \in S$

Theorem :

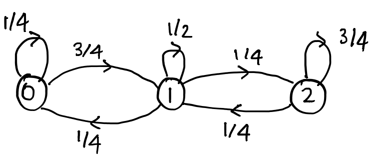
For an irreducable, aperiodic, +ve recurrent DTMC, the limiting distribution exist and is independent of initial distribution.

This is same as the stationary distribution and is given by

$$\pi P = \pi \quad \text{with} \quad \sum_{j \in S} \pi_j = 1$$

eg:-

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 3/4 \end{bmatrix} \end{array}$$



$S = \{0, 1, 2\} = C_1 \Rightarrow$ Irreducable

$$f_{00} = f_{00}^{(1)} + f_{00}^{(2)} + \dots$$

$$= 1/4 + \frac{3}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{1}{2} \times \frac{1}{4} + \dots\dots$$

$$= 1$$

$f_{11} = 1$ & $f_{22} = 1$

Check for aperiodic, +ve recurrent, irreducable to apply the theorem.

$$\Rightarrow \quad (\pi_0 \;\; \pi_1 \;\; \pi_2) = (\pi_0 \;\; \pi_1 \;\; \pi_2) \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \quad \& \quad \sum \pi_j = 1$$

Limiting dist. same as stationary distribution

$$\pi_0 = \frac{1}{4}(\pi_0 + \pi_1) \quad \text{—①} \qquad \Rightarrow \quad \pi_0 = \frac{\pi_1}{3} \;, \; \pi_1 = \pi_2$$

$$\pi_1 = \frac{3}{4}\pi_0 + \frac{\pi_1}{2} + \frac{\pi_2}{4} \quad \text{—②} \qquad \frac{\pi_1}{3} + \pi_1 + \pi_1 = 1 \Rightarrow \pi_1 = \frac{3}{7} = \pi_2 \;, \; \pi_0 = \frac{1}{7}$$

$$\pi_2 = \frac{\pi_1}{4} + \frac{3\pi_2}{4} \quad \text{—③} \qquad (1/7, 3/7, 3/7)$$

08/04/2016 LECTURE 34

FEW IMPORTANT RESULTS

1.) For a finite state space DTMC, if the state is recurrent, then it has to be +ve recurrent.

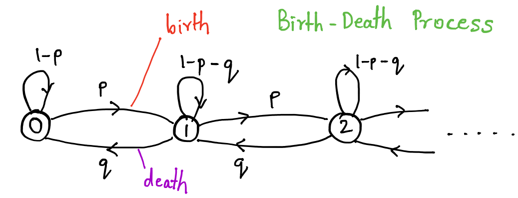
2.) In an irreducible MC, all the states are either +ve recurrent or null recurrent. All states are having the same period.

3.) For an irreducible finite state space DTMC, all the states are +ve recurrent.

4.) For an irreducible aperiodic DTMC, if the states are null recurrent, then $\pi_j = 0 \quad , j \in S$

Birth - Death Process

eg:-



$\{X_n : n = 0, 1, 2, \ldots\}$ with $S = \{0, 1, 2, \ldots\}$

Case (i) $q = 0, \; p > 0$

$T = \{0, 1, 2, \ldots\}$

$\pi_j = \lim_{n \to \infty} \text{Prob}\{X_n = j\} \quad , j = 0, 1, 2, \ldots$
$= 0$

Case (ii) $p = 0, \; q > 0$

$C_1 = \{0\} \quad , T = \{1, 2, \ldots\}$

$\pi_0 = 1 \quad , \pi_j = 0 \quad , j \in S \setminus \{0\}$

Case (iii) $0 < p, q < 1$

$C = \{0, 1, 2, \ldots\} = S$

irreducible, recurrent, aperiodic

It can be +ve/null recurrent

Assume that all states are +ve recurrent.

$\Rightarrow$ Stationary dist. exist

$\pi = \pi P \quad \& \quad \sum_{i \in S} \pi_i = 1$

$$(\pi_0 \; \pi_1 \; \ldots) = (\pi_0 \; \pi_1 \; \ldots) \begin{pmatrix} 1-p & p & 0 & \cdots \\ q & 1-p-q & p & \cdots \\ 0 & q & 1-p-q & p \; \cdots \\ & \vdots & & \end{pmatrix}$$

$\pi_0 = (1-p)\pi_0 + q\pi_1 \quad \Rightarrow \quad \pi_1 = \frac{p}{q}\pi_0$

$\pi_1 = p\pi_0 + (1-p-q)\pi_1 + q\pi_2 \quad \Rightarrow \quad \pi_2 = \frac{p}{q}\pi_1 = \frac{p^2}{q^2}\pi_0$

$\vdots$

III'ly $\pi_n = \frac{p}{q}\pi_{n-1} = \ldots = \left[\frac{p}{q}\right]^n \pi_0 \quad \Rightarrow$

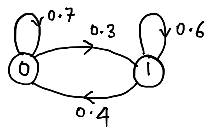
Now, $\sum \pi_i = 1$

For this to occur, $\frac{p}{q} < 1$

or $\boxed{p < q}$ Condition for +ve recurrent

$\& \quad \pi_0 \times \frac{1}{1 - \frac{p}{q}} = 1 \quad$ or $\pi_0 = \frac{q-p}{q}$

eg:- ①



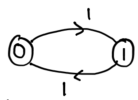
Limiting Dist: $(4/7, 3/7)$
Stationary Dist: $(4/7, 3/7)$

$C = \{0, 1\} = S$ Irreducible
Aperiodic
$\Rightarrow$ Stationary = Limiting

Finite DTMC recurrent $\Rightarrow$ +ve recurrent

$$(\pi_0 \quad \pi_1) = (\pi_0 \quad \pi_1)\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \Rightarrow \pi_0 = \frac{4}{7}, \ \pi_1 = \frac{3}{7}$$

②



Not aperiodic
$\Rightarrow$ Theorem won't hold

Limiting Dist: D.N.E. (Check $P, P^2, \ldots$)
Stationary Dist: $(1/2, 1/2)$

③

Remarks : When the stationary distribution exists,

1.) $X_n \xrightarrow{d} \pi$

2.) $\pi P^n = \pi P \cdot P^{n-1} = \pi \cdot P^{n-1} = \ldots = \pi \quad (n \geq 2)$

3.) When $\sum_i P_{ij} = 1$, $P = [P_{ij}]$ is doubly stochastic matrix.

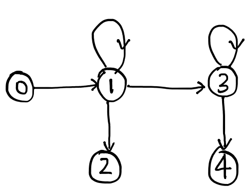
When the DTMC is irreducible finite state space & +ve recurrent & P is doubly stochastic, then

$\pi_j = \frac{1}{N}$, $j \in S$ π is uniformly distributed.

12/04/2016 $\boxed{\text{LECTURE 35}}$

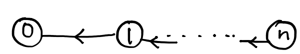
REDUCIBLE MARKOV CHAIN

①



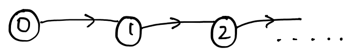
$S = T \cup C_1 \cup C_2$
$T \to \{0, 1, 3\}$, $C_1 \to \{2\}$, $C_2 \to \{4\}$

②



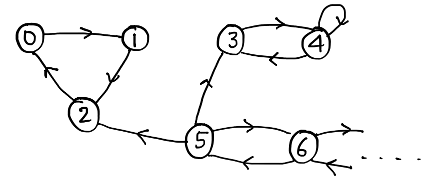
$S = T \cup C_1$
$T \to \{1, 2, \ldots n\}$, $C_1 \to \{0\}$

3



$S = \underbrace{T}_{\{0,1,2,\dots\}}$

4



$S = T \cup C_1 \cup C_2$, where $T = \{5,6,7,\dots\}$, $C_1 = \{0,1,2\}$, $C_2 = \{3,4\}$

FINITE STATE SPACE (say n) REDUCIBLE MARKOV CHAIN

$|S| = n$

$$P = \begin{array}{c} \\ \text{Absorbing or Recurrent} \\ \text{Transient} \end{array} \begin{array}{c} \begin{array}{cc} \text{Ab/Rec} & \text{Transient} \end{array} \\ \begin{pmatrix} R_{r\times r} & O_{r\times n-r} \\ A_{n-r\times r} & B_{n-r\times n-r} \end{pmatrix} \end{array}$$

If all recurrent states are absorbing state

$R_r = I_r$

$$S = \underbrace{C_1 \cup C_2 \cup \dots\dots C_k}_{\text{Recurrent state}} \cup T$$

Define $M = (I-B)^{-1}$ (fundamental matrix)

$= I + B + B^2 + B^3 + \dots$

Theorem : Define

1.) μ_{ij} - Mean # of visits of the system to the state j before reaching an absorbing state given that $X_0 = i$, $i \in T$, $j \in T$

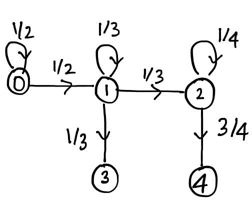
2.) g_{ij} - Conditional property that the system in absorbing state j given that $X_0 = i$, $i \in T$, $j \in S \setminus T$

Then

$$G = (g_{ij}) = (I-B)^{-1} A$$

$$M = (\mu_{ij}) = (I-B)^{-1}$$

Verification :



$$P = \begin{array}{c} \\ 3 \\ 4 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 3/4 & 0 & 0 & 1/4 \end{bmatrix}_{5\times 5} \end{array}$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \\ 0 & 3/4 \end{bmatrix} \quad B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1/4 \end{bmatrix}$$

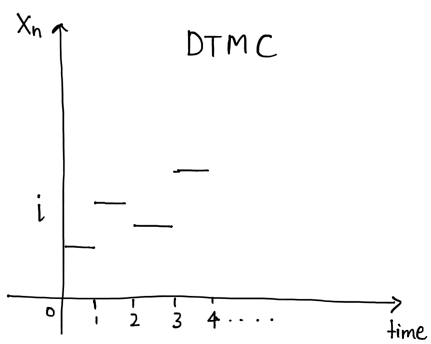
$$I - B = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & 2/3 & -1/3 \\ 0 & 0 & 3/4 \end{bmatrix}$$

$$(I-B)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 3/8 & 3/8 & 0 \\ 1/6 & 1/6 & 1/3 \end{bmatrix}^T * 4 = \begin{bmatrix} 2 & 3/2 & 2/3 \\ 0 & 3/2 & 2/3 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$
G = \begin{bmatrix} 2 & 3/2 & 2/3 \\ 0 & 3/2 & 2/3 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \\ 0 & 3/4 \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} \end{array}
$$

Similarly solve for M.

→ CTMC

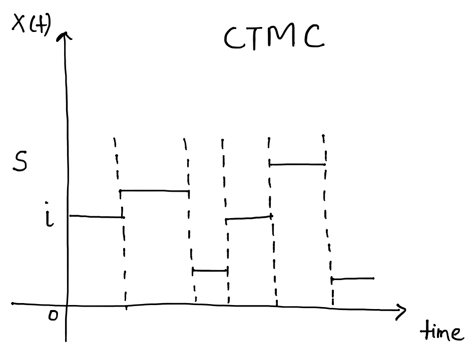


$\{X_n, n = 0, 1, \ldots\}$

Given $\pi(0)$ – initial dist.

$\pi(0) = (\pi_0(0) \;\; \pi_1(0) \;\ldots)$

$P = [P_{ij}]$



$\{X(t), t \geq 0\}$

$\pi(0) = (\pi_0(0) \;\; \pi_1(0) \;\ldots)$

$Q = [q_{ij}] \quad i \in S$

↑ rate

13/04/2016 $\boxed{\text{LECTURE 36}}$

Definitions :

1.) Initial Distribution

$\pi(0) = [\pi_0(0) \;\; \pi_1(0) \;\ldots]$ where $\pi_i(t) = \text{Prob}\{X(t) = i\}, t \geq 0$

2.) Transmission Probability

$P_{ij}(s,t) = \text{Prob}\{X(t+s) = j \,/\, X(s) = i\} \quad i,j \in S$

Since system is time homogeneous,

$P_{ij}(t) = \text{Prob}\{X(t) = j \,/\, X(0) = i\} \quad i,j \in S$

$= \text{Prob}\{X(t+s) = j \,/\, X(s) = i\} \quad \forall s > 0$

3.) Transmission Probability Matrix

$P(t) = [P_{ij}(t)] \;, t \geq 0$

$$\pi_j(t) = \text{Prob}\{X(t)=j\} = \sum_{i\in S} \pi_i(0)\, P_{ij}(t)$$

$\pi_j(t)$ ↓ unconditional $P_{ij}(t)$ ↓ conditional

$$\boxed{\pi(t) = \pi(0)\, P(t)}$$

4.) Generator Matrix

$$Q = [q_{ij}]$$

for $i \neq j$ $\quad q_{ij} = \dfrac{d}{dt} P_{ij}(t) \Big|_{t=0}$

$i = j$ $\quad q_{ii} = -\sum_{i \neq j} q_{ij}$

$P_{ij}(\Delta t) = q_{ij}\,\Delta t + o(\Delta t)$, $j \neq i$

$P_{ii}(\Delta t) = 1 + q_{ii}\,\Delta t + o(\Delta t)$, $j = i$

q_{ij} satisfies

(1) $q_{ij} \geq 0$

(2) $q_{ii} \leq 0$

(3) $\sum_j q_{ij} = 0$

We know that

$$\boxed{P_{ij}(t+T) = \sum_{k\in S} P_{ik}(t) \cdot P_{kj}(T)}$$

CHAPMAN KOLMOGOROV EQUATION

Differenciate w.r.t. T & put T=0

$$P_{ij}'(t) = \sum_{k\in S} P_{ik}(t)\, q_{kj}$$

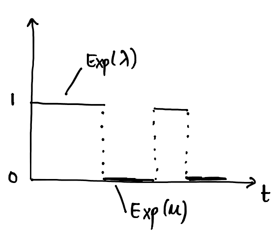
$$\Rightarrow \boxed{P'(t) = P(t) \cdot Q}$$

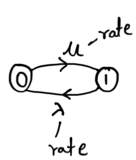
KOLMOGOROV FORWARD Eqn

Since it is a differential equation, initial condn $P(0)$ is required.

$$\pi'(t) = \pi(t)\, Q$$

eg:-





$$Q = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix} \end{array} \qquad \pi(0) = [0 \quad 1]$$

$\pi'(t) = \pi(t) \cdot Q$ $\qquad \pi_0(t) + \pi_1(t) = 1$

$\Rightarrow \pi_0'(t) = -\mu\pi_0(t) + \lambda\pi_1(t)$

$\Rightarrow \pi_0'(t) = (-\mu - \lambda)\pi_0(t) + \lambda$

$$\Rightarrow \frac{\pi_0'(t)}{\pi_0(t) - \frac{\lambda}{\mu+\lambda}} = -(\mu+\lambda)$$

$$\Rightarrow \ln\left(\pi_0(t) - \frac{\lambda}{\mu+\lambda}\right) + c = -(\mu+\lambda)t$$

$$\Rightarrow \pi_0(t) = e^{-(\mu+\lambda)t} \cdot c' + \frac{\lambda}{\mu+\lambda} \Rightarrow \pi_0(t) = \frac{\lambda}{\mu+\lambda}\left[1 - e^{-(\mu+\lambda)t}\right]$$

At $t=0$, $\pi_0(0) = 0 \Rightarrow c' = -\dfrac{\lambda}{\mu+\lambda}$

19/04/2016 $\boxed{\text{LECTURE 37}}$

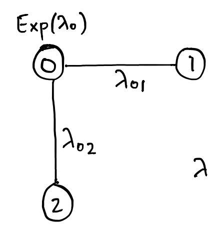
Theorem: Time spent in any state i before moving to another state for (time homogeneous) CTMC is always independent exponential distribution with parameter λ_i, $i \in S$

Proof :

τ - r.v.

$$P(\tau > s+t \,/\, X(0)=i)$$
$$= P(\tau > s+t \,/\, X(s)=i) \,.\, P(\tau > s \,/\, X(0)=i)$$
$$= P(\tau > t \,/\, X(0)=i) \,.\, P(\tau > s \,/\, X(0)=i)$$

eg:-



$$\lambda_0 = \lambda_{01} + \lambda_{02}$$

$$F_\tau^c(s+t) = F_\tau^c(t) \,.\, F_\tau^c(s) \quad , \; s,t > 0$$

$$\Rightarrow \quad e^{-\lambda_i(s+t)} = e^{-\lambda_i s} \,.\, e^{-\lambda_i t}$$

$$\Rightarrow \quad F_\tau(s) = 1 - e^{-\lambda_i s} \;\; \underset{s>0}{} \quad \Rightarrow \quad \tau \sim Exp(\lambda_i)$$

eg:- $\{X(t), t \geq 0\}$, $S = \{0, 1, 2, \dots\}$ $\qquad \Pi_0(0) = 1$

$$0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} \cdots \qquad \Pi'(t) = \Pi(t) \,.\, Q$$

$$\Pi_j(t) = Prob\,\{X(t) = j\}$$

$$\Rightarrow \quad (\Pi_0'(t) \;\; \Pi_1'(t) \;\; \Pi_2'(t) \;\dots) = (\Pi_0(t) \;\; \Pi_1(t) \;\; \Pi_2(t) \;\dots) \begin{array}{c|cccc} & 0 & 1 & 2 & \dots \\ \hline 0 & -\lambda & \lambda & 0 & \dots \\ 1 & 0 & -\lambda & \lambda & \dots \\ 2 & 0 & 0 & -\lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{array}$$

$$\Pi_0'(t) = -\lambda \Pi_0(t)$$

$$\ln(c\,\Pi_0(t)) = -\lambda t$$

At $t=0$, $\Pi_0(0) = 1 \Rightarrow c = 1 \Rightarrow \Pi_0(t) = e^{-\lambda t}$

$$\Pi_1'(t) = \lambda \Pi_0(t) - \lambda \Pi_1(t) \quad \Rightarrow \quad \Pi_1(t) = \lambda t e^{-\lambda t}$$

$$\vdots$$

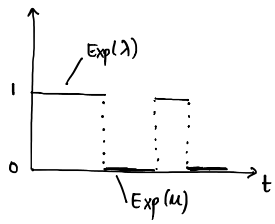
$$\Pi_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad , \; n = 1, 2, \dots$$

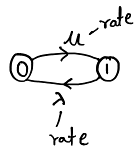
For fixed t, the dist of $X(t)$ is Poisson with parameter λt

$\{X(t), t \geq 0\} \longrightarrow$ Poisson Process (C.T.M.C.)

Consider the previous example

eg:-





$$Q = \begin{array}{c|cc} & 0 & 1 \\ 0 & -\mu & \mu \\ 1 & \lambda & -\lambda \end{array} \qquad \pi(0) = [0 \quad 1]$$

$$\pi'(t) = \pi(t) \cdot Q \qquad \pi_0(t) + \pi_1(t) = 1$$

$$\Rightarrow \pi_0'(t) = -\mu \pi_0(t) + \lambda \pi_1(t)$$

$$\Rightarrow \pi_0'(t) = (-\mu - \lambda)\pi_0(t) + \lambda$$

$$\Rightarrow \frac{\pi_0'(t)}{\pi_0(t) - \frac{\lambda}{\mu+\lambda}} = -(\mu+\lambda)$$

$$\Rightarrow \pi_0(t) = e^{-(\mu+\lambda)t} \cdot c' + \frac{\lambda}{\mu+\lambda} \Rightarrow \pi_0(t) = \frac{\lambda}{\mu+\lambda}\left[1 - e^{-(\mu+\lambda)t}\right]$$

$$\Rightarrow \ln\left(\pi_0(t) - \frac{\lambda}{\mu+\lambda}\right) + c = -(\mu+\lambda)t$$

At $t=0$, $\pi_0(0) = 0 \Rightarrow c' = -\frac{\lambda}{\mu+\lambda}$

Availability $= \pi_1(t)$

Steady state availability $\underset{t\to\infty}{=} \pi_1 = \dfrac{\mu}{\lambda+\mu} = \dfrac{1/\lambda}{1/\mu + 1/\lambda} = \dfrac{\text{Mean Failure Time}}{\text{Mean F Time} + \text{Mean Repair Time}}$

22/04/2016 $\boxed{\text{LECTURE 38}}$

BIRTH DEATH PROCESS

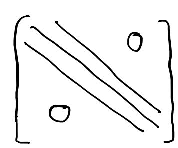
A B.D.P. is a CTMC $\{X(t), t \geq 0\}$ with a state space $S\ (= \{0,1,2,\dots\})$ s.t.

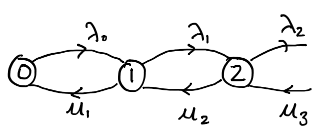
$$q_{i,i+1} = \lambda_i, \quad i = 0,1,2,\dots$$

$$q_{i,i-1} = \mu_i, \quad i = 1,2,\dots$$

$$q_{i,j} = 0 \qquad |i-j| > 1$$

$$q_{i,i} = \begin{cases} -(\lambda_i + \mu_i) & i = 1,2,\dots \\ -\lambda_i & i = 0 \end{cases}$$

$Q =$ 



$\lambda_i \longrightarrow$ birth rate

$\mu_i \longrightarrow$ death rate

eg:- 1. $\{X(t), t \geq 0\}$ — P.P.(λ) is a BDP $\left(\begin{array}{ll}\lambda_i = \lambda & i = 0,1,2,\dots \\ \mu_i = 0 & i = 1,2,\dots\end{array}\right)$

Poison Process

Pure birth process

② Pure death process

$\lambda_i = 0 \quad , \ i = 0,1,2,\dots n$

$\mu_i = \alpha \quad , \ i = 1,2,\dots n$

It should be finite (Has to start from somewhere)

→ Steady state Probability

$$\pi'(t) = \pi(t).Q \qquad \text{F.K.E.}$$

For steady state, $\pi'(t) = 0$ & $\pi(t) = \pi$

$$\Rightarrow \quad 0 = \pi . Q \quad ; \quad \sum_{i \in S} \pi_i = 1$$

$\pi = (\pi_i)_{i \in S}$

$$\Rightarrow \quad 0 = -\lambda_0 \pi_0 + \mu_1 \pi_1$$

$$0 = \lambda_0 \pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2 \pi_2$$

$$\vdots$$

$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$$

$$\pi_2 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1}\pi_0$$

$$\vdots$$

$$\pi_n = \frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_0}{\mu_n . \mu_{n-1} \dots \mu_1}\pi_0$$

$\pi_1, \pi_2, \dots$ in terms of π_0

$\sum \pi_i = 1$ Solve for π_0

$$\pi_0\left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_1\lambda_0}{\mu_2\mu_1} + \dots\right) = 1 \quad \Rightarrow \quad \pi_0 = \frac{1}{1 + \dots} > 0$$

Special case:

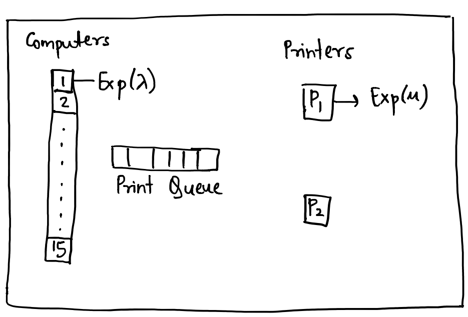
$\lambda_i \to \lambda$

$\mu_i \to \mu$

Condition for steady state to exist:

$\frac{\lambda}{\mu} < 1$ or $\lambda < \mu$ (common ratio of G.P.)

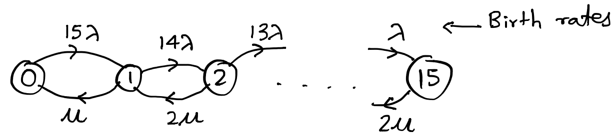
eg:-



$\{X(t), t \geq 0\}$

\# of print tasks in system at time t

$S = \{0, 1, \dots, 15\}$



Since there are finite terms, steady state will always exist.

Probability that system is always busy $= \sum_{i=2}^{15} \pi_i$

26/04/2016 **LECTURE 39**

QUEUEING MODELS

Parameters

1.) Inter arrival — deterministic / probabilistic

2.) service

3.) # of servers

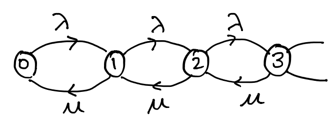
4.) Capacity

$X(t)$ = no. of people in system (queue + service)

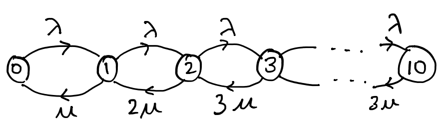
eg:- 1.) M/M/1/∞ — infinite capacity

M → Exp(λ), M → Exp(μ), 1 → one server

For this to be B.D.P., at a small interval of time, maximum of 1 person can enter / leave the system

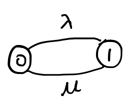


2.) M/M/3/10



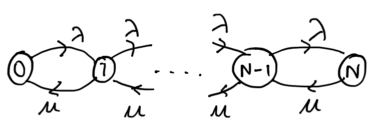
27/04/2016 **LECTURE 40**

3.)



⇒ M/M/1/1

4.)



⇒ M/M/1/N

Note: For a B.D.P.,

$$\boxed{\lambda \cdot E(R) = E(N)}$$ Little's Formula

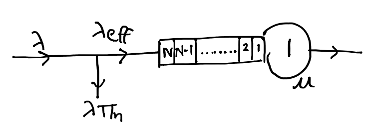
λ: mean arrival rate; E(R): mean spending time; E(N): mean # of customers

⇒ Mean spending time in queue $= E(Q) = E(R) - \frac{1}{\mu}$

$\frac{1}{\mu}$: mean service time

☆ For a N capacity system, at capacity N people are not allowed to enter the system.

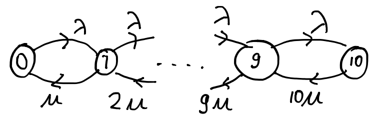
⇒ Effective Arrival Rate $\lambda_{eff} = \lambda(1-\pi_n)$



eg:- What is average queue & spending time for M/M/10/10

Sol^n : $E(Q) = 0$

$E(R) = \frac{1}{\mu}$

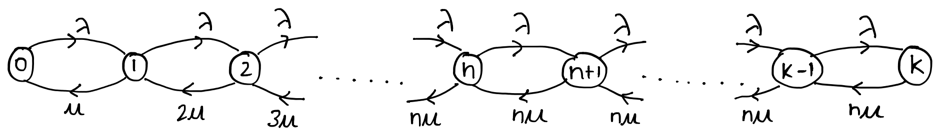


$\pi' = \pi Q = 0$

29/04/2016 $\boxed{\text{LECTURE 41}}$

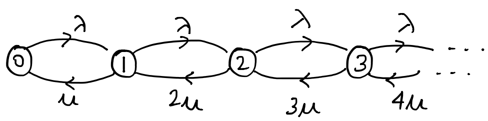
M/M/n/k

$k > n$



$$\pi_n = \frac{1}{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \cdots + \frac{\Pi \lambda_{k-1}}{\Pi \mu_k}} \times \frac{\Pi \lambda_{n-1}}{\Pi \mu_n}$$

M/M/∞



$$\pi_0 = \frac{1}{1 + \rho + \frac{\rho^2}{2!} + \cdots} = e^{-\rho} \Rightarrow \pi_n = \frac{e^{-\rho} \times \rho^n}{n!}$$

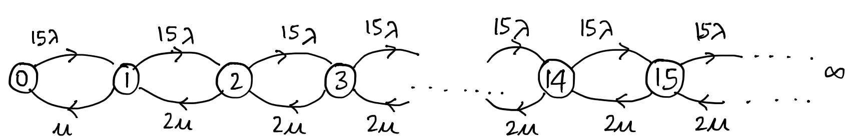
$\Rightarrow$ Poisson distribution with parameter ρ

$\pi_n \sim P(\rho)$

M/M/2/∞ with population size 15

$\sim$ M/M/2/∞ (M $\rightarrow$ Exp(15λ))

M $\rightarrow$ Exp(λ) of individual P.C.

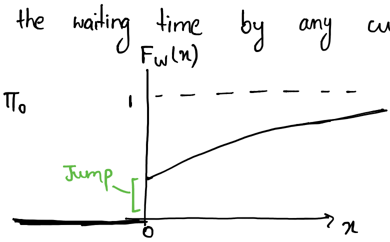


15 Printers each giving print command with rate 15 with any P.C. can give subsequently multiple print command.

M/M/1/∞

Let W be the waiting time by any customer. $\Rightarrow$ Mixed Type R.V. (Jump at 0 when no one in service)

$P\{W=0\} = \pi_0$



$$P\{0 < W \le t\}$$

$$= \sum_{n=1}^{\infty} P\{0 < W \le t / N=n\} . P\{N=n\}$$

$\rightarrow$ Erlang distribution

$$= \rho \left(1 - e^{-(\mu-\lambda)t}\right)$$