Vector Analysis - II

PYL101: Electromagnetics & Quantum Mechanics
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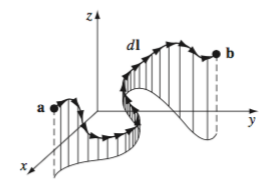
Line Integrals



- A **line integral** is an expression of the form

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{v} \underbrace{\cdot}_{\textit{dot product}} d\boldsymbol{l}$$

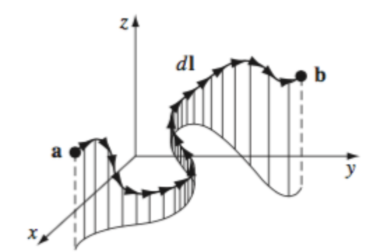
where $\boldsymbol{v}$ is a vector function, and $d\boldsymbol{l}$ is the infinitesimal displacement vector.

- The integral **must be** carried out **along a prescribed path** $\mathscr{P}$ from point $\boldsymbol{a}$ to point $\boldsymbol{b}$, and we should instead write

$$\int_{\boldsymbol{a}\ \mathscr{P}}^{\boldsymbol{b}} \boldsymbol{v} \cdot d\boldsymbol{l}$$

explicitly mentioning the path $\mathscr{P}$.

Line Integrals

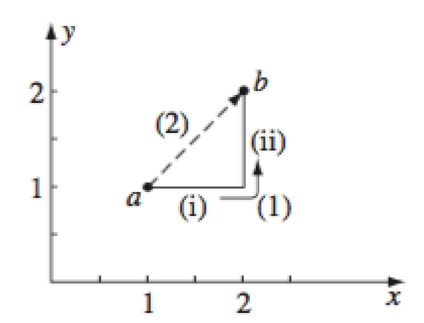


- If the path $\mathscr{P}$ is **closed loop** (*i.e.*, $\boldsymbol{a} = \boldsymbol{b}$) we put a circle around the integral sign as,

$$\oint_{\mathscr{P}} \boldsymbol{v} \cdot d\boldsymbol{l} \neq 0$$

where it's important to note that, in general, closed line integrals don't have to be zero, *in general*!

A Line Integral Example

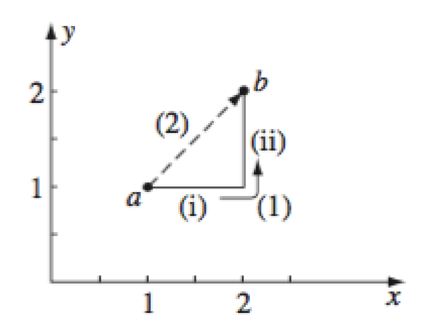


- **Problem**: Calculate the line integral for the vector function

$$\boldsymbol{v} = y^2\hat{\boldsymbol{x}} + 2x(y+1)\hat{\boldsymbol{y}}$$

from the point $a = (1,1,0)$ to $b = (2,2,0)$, along the paths (1) and (2) in the above figure.

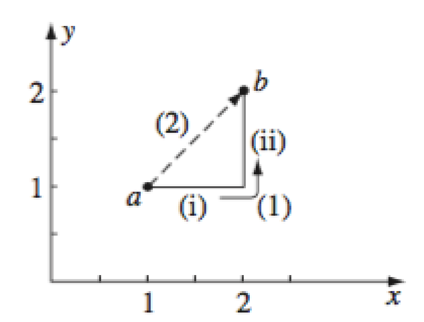
A Line Integral Example: Along path (1)



- Along the ***horizontal* segment of (1)** (*only x varies*) *dy = dz = 0*, so

$$d\boldsymbol{l} = dx\hat{\boldsymbol{x}}, y = 1, \boldsymbol{v} \cdot d\boldsymbol{l} = y^2 dx, \int \boldsymbol{v} \cdot d\boldsymbol{l} = (1)^2 \int_1^2 dx = 1$$

A Line Integral Example: Along path (1)



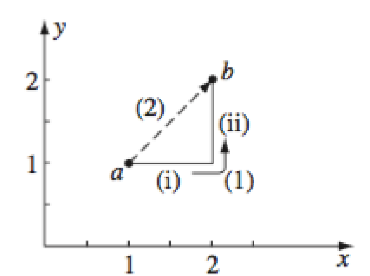
- For the ***vertical* segment of (1)** (*only y varies*) $dx = dz = 0$, so

$$d\boldsymbol{l} = dy\,\hat{\boldsymbol{y}},\, x = 2,\, \boldsymbol{v}\cdot d\boldsymbol{l} = 2x(y+1)\,dy,\, \int \boldsymbol{v}\cdot d\boldsymbol{l} = 4\int_1^2 (y+1)\,dy = 10$$

- Thus, summing the horizontal and vertical parts of (1) we get,

$$\int_{(1)} \boldsymbol{v}\cdot d\boldsymbol{l} = 1 + 10 = 11$$

A Line Integral Example: Along path (2)



- For path (2), $x = y \Rightarrow dx = dy$[1] and, $dz = 0$

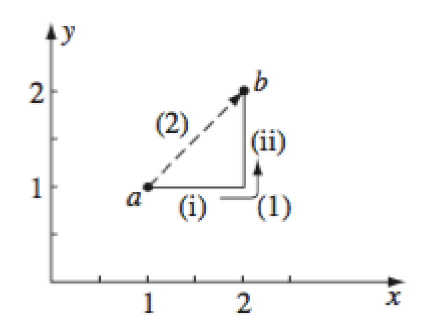
$$d\boldsymbol{l} = dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}}, \boldsymbol{v} \cdot d\boldsymbol{l} = x^2 dx + 2x(x+1)dx = (3x^2 + 2x)dx,$$

, and finally,

$$\int_{(2)} \boldsymbol{v} \cdot d\boldsymbol{l} = 10$$

[1] This step is crucial for a *general* path where variables are interdependent.

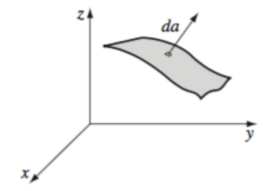
A Line Integral Example: $a \to (1) \to (2) \to a$



- For the *circuitous* path $a \to (1) \to (2) \to a$, we simply **sum** the contributions of paths (1) and (2) and get,

$$\oint \boldsymbol{v} \cdot d\boldsymbol{l} = 11 - 10 = 1$$

Surface/Double and Flux Integrals



- A **surface/double integral** is an expression of the form

$$\int_{\mathscr{S}} f \, da$$

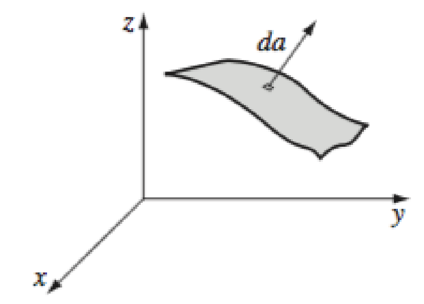
where f is a scalar function, da is an infinitesimal patch of area over a prescribed surface $\mathscr{S}$.

- However, the **flux**[2] of a vector function $\boldsymbol{v}$ through $\mathscr{S}$ is defined as,

$$\int_{\mathscr{S}} \boldsymbol{v} \underbrace{\cdot}_{\text{dot product}} d\boldsymbol{a}$$

[2] in analogy with liquid flow, *i.e.*, if $\boldsymbol{v}$ describes the flow a mass of liquid per unit area per unit time.

Flux Integrals

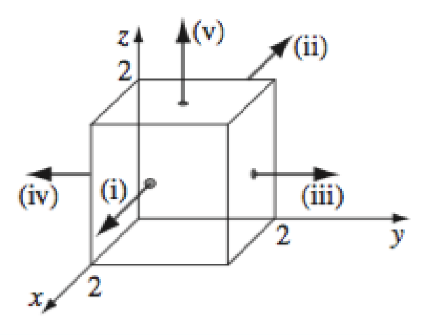


- Since there are two opposite directions for the surface normal, we *choose* the direction that points **radially outward** with the origin as a reference point.
- If the surface is **closed**[3] we write,

$$\oint_{\mathscr{S}} \boldsymbol{v} \cdot d\boldsymbol{a} \neq 0$$

[3] A closed surface in $3-d$ is exemplified by a *balloon*.

A Flux Example

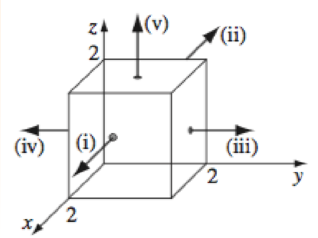


- **Problem**: Calculate the flux

$$\boldsymbol{v} = 2xz\,\hat{\boldsymbol{x}} + (x+2)\,\hat{\boldsymbol{y}} + y(z^2-3)\,\hat{\boldsymbol{z}}$$

over five sides (*excluding the bottom*).

A Flux Example: Surface 1



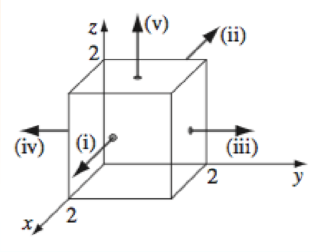
(i) $x = 2$, $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, $\mathbf{v} \cdot d\mathbf{a} = 2xz\,dy\,dz = 4z\,dy\,dz$, so

$$\int \mathbf{v} \cdot d\mathbf{a} = 4 \int_0^2 dy \int_0^2 z\,dz = 16.$$

- Note, that for surface (i) variables x and y are independent of each other[4], and the integrals for each can be carried out independently.

[4] Why?

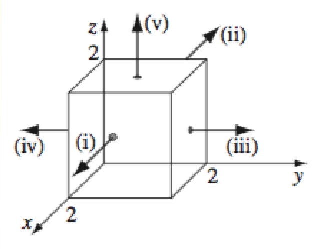
A Flux Example: Surface 2



(ii) $x = 0$, $d\mathbf{a} = -dy\,dz\,\hat{\mathbf{x}}$, $\mathbf{v} \cdot d\mathbf{a} = -2xz\,dy\,dz = 0$, so

$$\int \mathbf{v} \cdot d\mathbf{a} = 0.$$

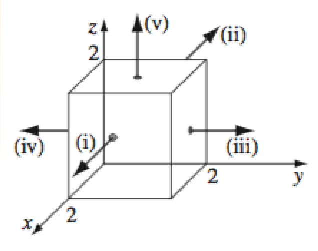
A Flux Example: Surface 3



(iii) $y = 2,\ d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}},\ \mathbf{v}\cdot d\mathbf{a} = (x+2)\,dx\,dz$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^2 (x+2)\,dx \int_0^2 dz = 12.$$

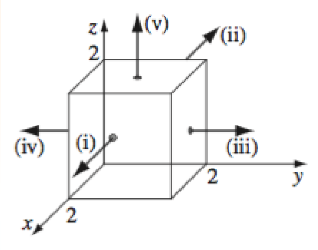
A Flux Example: Surface 4



(iv) $y = 0$, $d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}}$, $\mathbf{v}\cdot d\mathbf{a} = -(x+2)\,dx\,dz$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^2 (x+2)\,dx \int_0^2 dz = -12.$$

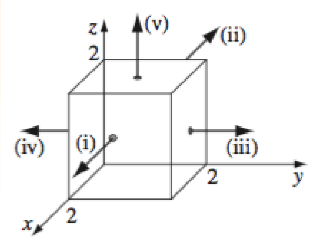
A Flux Example: Surface 5



(v) $z = 2$, $d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}}$, $\mathbf{v} \cdot d\mathbf{a} = y(z^2 - 3)\,dx\,dy = y\,dx\,dy$, so

$$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^2 dx \int_0^2 y\,dy = 4.$$

A Flux Example: Total

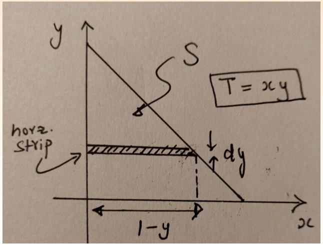


The *total* flux is

$$\int_{\text{surface}} \mathbf{v} \cdot d\mathbf{a} = 16 + 0 + 12 - 12 + 4 = 20.$$

- Note that the surface $\mathscr{S}$ over which the flux is calculated <u>need not</u> be closed.

Evaluating a Surface Integral over a triangular $\mathscr{S}$



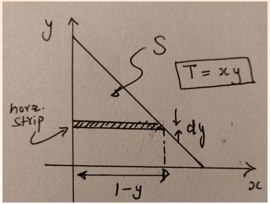
- **Problem**: Evaluate

$$\iint_{\mathscr{S}} xy \; dxdy$$

over the triangular region with $\perp$ sides of unity, as above.

- Our first **strategy** is to:
  1. Calculate $T(x, y = const.)$ × the area of each infinitesimally short **horizontal strip** of height dy [INNER SUM: $x : 0 \to (1-y)$], and then,
  2. Sum these horizontal strips up one by one [OUTER SUM: $y : 0 \to 1$].

Evaluating the Surface Integral over a triangular $\mathscr{S}$



- Thus, keeping y **constant**, let's focus now on the [INNER SUM: $x : 0 \to (1-y)$], *i.e.*,

$$H(y) = \int_{x=0}^{1-y} \underbrace{xy}_{y=\text{const.}} \; dx = \frac{y(1-y)^2}{2}$$

where the $(1-y)$ limit accounts for the fact that the length of these strips varies.

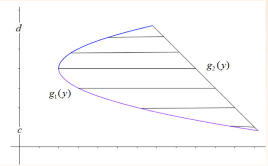
- Finally, we perform the [OUTER SUM: $y : 0 \to 1$] and

$$\int_{y=0}^{1} H(y)\; dy = \int_{y=0}^{1} \frac{y(1-y)^2}{2}\; dy = \frac{1}{24}$$

- HW: Find $\int_{\mathscr{S}} xy\; dxdy$ by instead summing **vertical** strips.

Evaluating the Surface Integral for a Horizontally-Simple Region

optional



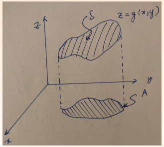
- A **horizontally-simple region** is a region where every horizontal line drawn inside it shares the same left-function $g_1(y)$ and right-function $g_2(y)$.
- It's evaluated as

$$\iint_{\mathscr{S}} f(x,y)\,dA = \int_{y=c}^{y=d} \left[\int_{x=g_1(y)}^{g_2(y)} \underbrace{f(x,y)}_{y=const}\,dx \right] dy$$

- You should be able to *guess* the math for surface integrals over **vertically-simple regions**.

Evaluating the Flux for a surface described by $z = g(x, y)$

optional



- When the surface $\mathscr{S}$ can be described by the equation

$$z = g(x, y)$$

the **flux** of the vector function $\boldsymbol{F}(x, y, z)$ through $\mathscr{S}$ is given by

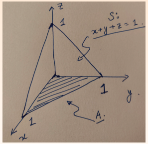
$$\int_{\mathscr{S}} \boldsymbol{F} \cdot d\mathscr{S} = \iint_A \boldsymbol{F} \cdot \left(-\frac{\partial g}{\partial x}\hat{\boldsymbol{x}} - \frac{\partial g}{\partial y}\hat{\boldsymbol{y}} + 1\hat{\boldsymbol{z}}\right)\, dx dy$$

where the region A is the **projection** of $\mathscr{S}$ on the (x, y)-plane[5].

[5]Note the subtelty in the above formula which is that in the integrand of the RHS, after calculating the dot product you must replace any instance of z by $g(x, y)$.

Evaluating the Flux for a surface described by $z = g(x, y)$

optional



- **Problem**: Calculate the flux of $\boldsymbol{F} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ over the surface given by the planar region $\mathscr{S} : x + y + z = 1$ above. The projection of $\mathscr{S}$ along the $(x, y)-$plane is the shaded region A.
- Therefore[6], $z = g(x, y) = 1 - x - y$, and $\frac{\partial g}{\partial x} = -1$, $\frac{\partial g}{\partial y} = -1$

$$\begin{aligned}\int_{\mathscr{S}} \boldsymbol{F} \cdot d\mathscr{S} &= \iint_A (x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}) \cdot (\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}) \; dA \\ &= \iint_A (x + y + z) dx dy \quad (x + y + z = 1 : \mathscr{S}) \\ &= \frac{1}{2}\end{aligned}$$

[6] In the integrand above, $z \neq 0$ despite the projected surface A lying on the (x, y)-plane. Instead replace any mention of the variable $z = g(x, y) = 1 - x - y$.

Calculating Fluxes Numerically

- Let the vector function $\boldsymbol{v}(x,y) = xy^2\hat{\boldsymbol{z}}$, and $\mathscr{S}$ be a square region of sides unity, with its south-west corner located at the origin.
- Divide the square region into:
  - 4 subsquares of length $dx_i = 0.25$ and breath $dy_i = 0.25$
  - 16 subsquares of length $dx_i = 0.0625$ and breath $dy_i = 0.0625$
  - 64 subsquares of length $dx_i = 0.015625$ and breath $dy_i = 0.015625$ ...

  and calculate the value $\boldsymbol{v}(x_i, y_i)$ at the *center* (x_i, y_i) of each of the subsquares i.
- The area element $d\boldsymbol{a} = dx_i dy_i\ \hat{\boldsymbol{z}}$.
- The sum[7] $\sum_i v(x_i, y_i) dx_i dy_i$ is the required answer!
- As you **increase** the number of subsquares, the value obtained from the sum above **converges** to the **exact** value:

$$\int_{\mathscr{S}} \boldsymbol{v} \cdot d\boldsymbol{a} = \frac{1}{6}$$

[7] A practical way to do this is to write a small computer program.

Volume Integrals

- Given a scalar function T, volume integrals are expressed as,

$$\int_{\mathcal{T}} T d\tau$$

and $d\tau$ is an infinitesimal volume element.

- For a vector function $\boldsymbol{v}$, on the other hand, the associated volume integral is,

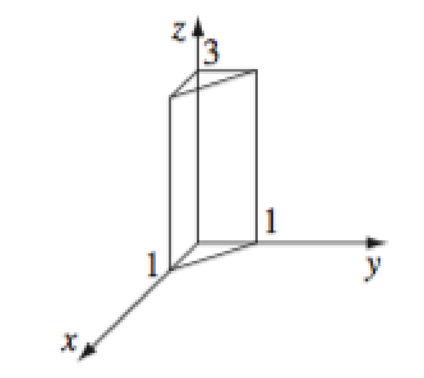
$$\begin{aligned}\int \boldsymbol{v} d\tau &= \int (v_x \hat{\boldsymbol{x}} + v_y \hat{\boldsymbol{y}} + v_z \hat{\boldsymbol{z}}) d\tau \\ &= \hat{\boldsymbol{x}} \int v_x d\tau + \hat{\boldsymbol{y}} \int v_y d\tau + \hat{\boldsymbol{z}} \int v_z d\tau\end{aligned}$$

- In **Cartesian coordinates**,[8]

$$d\tau = dx dy dz$$

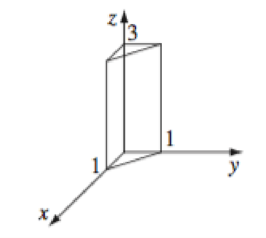
[8] Spherical and cylindrical coordinates are other candidates.

A Volume Integral Example



- **Problem:** Calculate the volume integral of $T = xyz^2$.
- It's easy to see from the prismatic volume that z does not depend on either x or y, and ranges from 0 to 3.
- This means that the integral over z can be **factored out.**

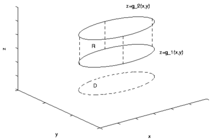
A Volume Integral Example



- $$\int xyz^2 d\tau = \int_0^3 z^2 dz \underbrace{\iint xy\ dxdy}_{\text{done earlier!}}$$
$$= 9\int_{y=0}^{1} y\left[\int_{x=0}^{1-y} xdx\right]dy$$
$$= \frac{9}{2}\int_0^1 y(1-y)^2 dy \qquad = \frac{3}{8}$$

Volume Integrals Over More General Regions

optional



- Ler R be a solid region[9] bounded below and above by the functions $g_1(x,y)$ and $g_2(x,y)$, respectively such that

$$g_1(x,y) \leq z \leq g_2(x,y)$$

- The region D is the **projection** of R onto the xy-plane. The triple integral is then given by

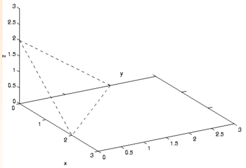
$$\int_R f(x,y,z)\,dV = \iint_D \left[\int_{z=g_1(x,y)}^{g_2(x,y)} \underbrace{f(x,y,z)}_{x,y=\text{const}} dz \right] dA = \iint_D h(x,y)\ dA$$

which is a double-integral over the region D in the (x,y)-plane.

[9] Such a solid region is known as a "z-simple" solid.

Volume Integrals Over General Regions

optional



- **Problem:** Evaluate

$$\int_R (x+2y)\,dV$$

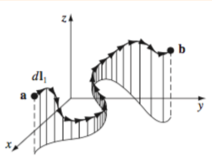
where R **is the tetrahedron** bounded by the planes $x=0, y=0, z=0$ and $x+y+z=2$.

- We can rewrite the equation of the plane $x+y+z=2$ as $z=2-x-y$. Note that $0 \le z \le 2-x-y$. Hence, we have[10]

$$\iint_D \left[\int_0^{2-x-y} \underbrace{(x+2y)}_{x,y=\text{const}}\,dz\right] dA = \iint_D (x+2y)(2-x-y)\ dA = 2$$

[10] The required answer was **incorrectly** given as $\frac{2}{3}$ in the video lecture.

The Fundamental Theorem for Gradients



- Given a **scalar** function $T(x, y, z)$, by changing x, y and z *infinitesimally*, *i.e.*, by dx, dy and dz, the variation in T is,

$$dT = (\nabla T) \cdot d\boldsymbol{l}$$

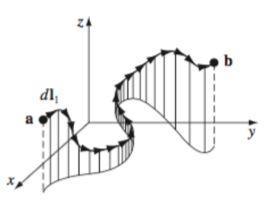
- If we keep advancing from a to b along units of $d\boldsymbol{l}_i$ we can **accumulate** the total change in the scalar function T,

$$\int_a^b (\nabla T) \cdot d\boldsymbol{l} = T(b) - T(a)$$

also known as the **fundamental theorem for gradients**[11].

[11] There is also a direct proof using the integral in the LHS above.

The Fundamental Theorem for Gradients



- *Remarkably*, the RHS, *i.e.*, $T(b) - T(a)$ makes no reference to the actual **path** taken.
- It implies that, $\int_a^b (\nabla T) \cdot d\boldsymbol{l}$ is **independent** of the path taken from a to b.[12]
- It also implies that,

$$\boxed{\oint (\nabla T) \cdot d\boldsymbol{l} = 0}$$

[12] In practice, even though the integral is independent of the path, we **must** pick a specific (if *convenient)* route in order to evaluate it explicitly.

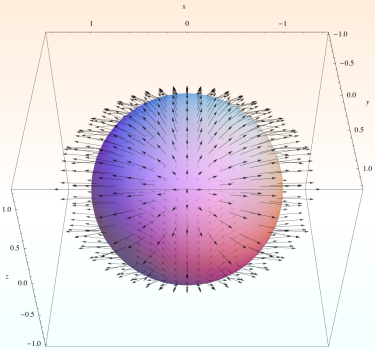
The Fundamental Theorem for Divergences

- The fundamental theorem for divergences states that:

$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{v})\, d\tau = \oint_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{a}}$$

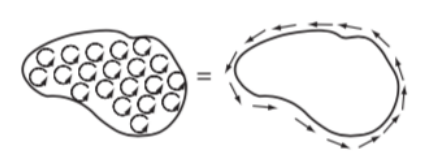
- It is alternatively known as, **Gauss' theorem**, **Green's theorem**, or simply the **divergence theorem**.
- **Physical Analogy:** Imagine we're interested in measuring the flux (RHS) of an *incompressible* fluid through a closed area $\mathcal{S}$.
- The divergence theorem states that instead of measuring the flux directly, we could've *equivalently*, sum up all the (liquid) sources inside the volume $\mathcal{V}$ enclosed by the surface $\mathcal{S}$.

The Fundamental Theorem for Divergences



- Q: Verify the divergence theorem for $\boldsymbol{v} = 2x\hat{\boldsymbol{x}} + y^2\hat{\boldsymbol{y}} + z^2\hat{\boldsymbol{z}}$ for the sphere of radius unity. [Ans: LHS = RHS = $\frac{8\pi}{3}$]

The Fundamental Theorem for Curls



- Also known as **Stoke's theorem** states that,

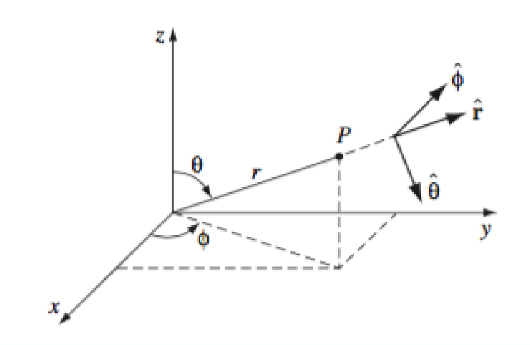
$$\boxed{\int_{\mathscr{S}} (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{a} = \oint_{\mathscr{P}} \boldsymbol{v} \cdot d\boldsymbol{l}}$$

  which says that the integral of a curl (of a vector function) over a surface $\mathscr{S}$ is equal to the value of the function integrated over the boundary $\mathscr{P}$) *enclosing* that surface.
- Sticking with the liquid analogy, since the curl measures the *twist* of $\boldsymbol{v}$, if we are interested in the total *swirl*, we can equivalently just measure how much the flow $\boldsymbol{v}$ follows the closed boundary $\mathscr{P}$ enclosing the surface $\mathscr{S}$.
- As earlier, by convention we select the orientation of $d\boldsymbol{a}$ pointing outward, and the sense of the line $d\boldsymbol{l}$ to be anti-clockwise.

The Fundamental Theorem for Curls

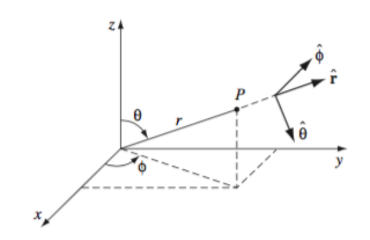
- As a consequence of the theorem, $\int_{\mathscr{S}} (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{a}$ depends only on the boundary line $\mathscr{P}$, but not on the particular surface $\mathscr{S}$ used, as long as it's circumscribed by $\mathscr{P}$. Think of an intact **soap bubble** across a fixed loop. It doesn't matter whether the bubble is convex, concave, or combinations thereof, as long as the loop circumscribing it is fixed.
- As a consequence we may **deform** $\mathscr{S}$ for *mathematical convenience* as long as it satisfies the boundary $\mathscr{P}$.
- $\oint_{\mathscr{S}} (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{a} = 0$ for any *closed* surface, since the boundary line, like the mouth of a balloon, shrinks down to a point.
- HW: Do example 1.11 of [IEDJ].

Spherical Coordinates



- An *alternative* to using the usual Cartesian coordinates described by (x, y, z) is to use **spherical coordinates** described by (r, θ, ϕ) where,
  1. r is the distance from the origin (the magnitude of the position vector $\boldsymbol{r}$)
  2. θ (the angle down from the z axis) is called the **polar angle**
  3. ϕ (the angle around from the x axis) is the **azimuthal angle**.

Spherical Coordinates



- From the above figure we observe that,

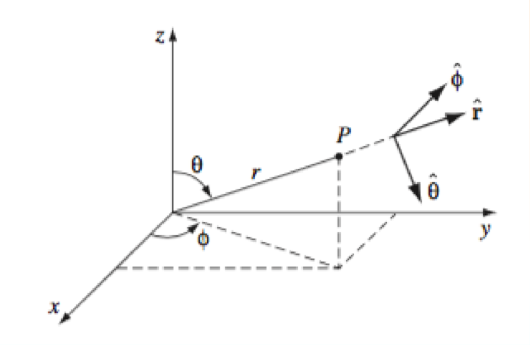
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

- A general vector $\boldsymbol{A}$ can be represented in spherical coordinates as,

$$\boldsymbol{A} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$$

where $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ form an orthogonal basis set.

Spherical Coordinates

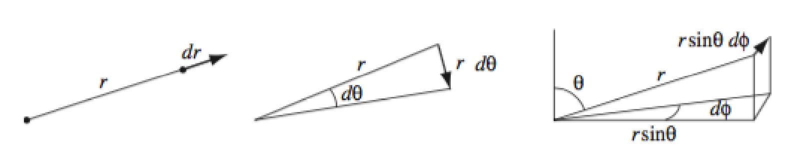


- In terms of Cartesian unit vectors we can write,[13],

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$$

[13] Do the proof.

The inifinitesimal elements in spherical coordinates



- Beware that in the spherical coordinate system,

$$dl \neq dr\hat{r} + d\theta\hat{\theta} + d\phi\hat{\phi}$$
$$d\tau \neq dr\, d\theta\, d\phi$$

- An infinitesimal displacement in the $\hat{\boldsymbol{r}}$ direction is simply dr, and thus,

$$dl_r = dr$$

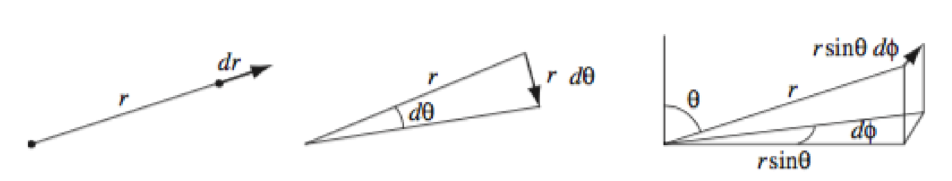
- An infinitesimal element of length in the $\hat{\boldsymbol{\theta}}$ direction is,

$$dl_\theta = r\, d\theta$$

- An infinitesimal element of length in the $\hat{\boldsymbol{\phi}}$ direction is,

$$dl_\phi = r \sin\theta\, d\phi$$

The inifinitesimal elements in spherical coordinates



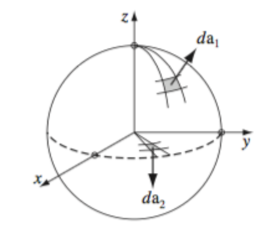
- Thus the general **infinitesimal displacement** $d\boldsymbol{l}$ is,

$$d\boldsymbol{l} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

- The **infinitesimal volume element** $d\tau$, in spherical coordinates, is the *product* of the three infinitesimal displacements:

$$d\tau = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

The surface element in spherical coordinates



- Suppose you're integrating over the *surface* of a sphere of radius r, here,

$$d\boldsymbol{a}_1 = dl_\theta dl_\phi \hat{\boldsymbol{r}} = r^2 \sin\theta d\theta d\phi \hat{\boldsymbol{r}}$$

- On the other hand, if the *surface* lies in the xy plane, as above[14]

$$d\boldsymbol{a}_2 = dl_r dl_\phi \hat{\boldsymbol{\theta}} = r dr \ d\phi \ \hat{\boldsymbol{\theta}}$$

[14] What happened to the $\sin\theta$ term in the RHS of $dl_\phi = r\sin\theta d\phi$?

More Formulae for Spherical Coordinates

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}.$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}.$$

- Things get complicated[15]...

[15] The above formulae will be provided, if required, during the exam, and needn't be memorized.

Cylindrical Coordinates

- Do as HW.

Need for the Dirac Delta Function

- Given a body of mass M located at $\boldsymbol{r}_0$, we may express it in terms of its mass density $\rho(\boldsymbol{r})$ as,

$$M = \int_{\mathcal{V}} \rho(\boldsymbol{r})\, d\tau$$

- But what does the mass density of a ***point mass*** located at $\boldsymbol{r}_0$ look like?
- It can be unequal to zero only at a single point, *i.e.*,

$$\rho(\boldsymbol{r}) = 0 \quad \forall \boldsymbol{r} \neq \boldsymbol{r}_0$$

- The volume integral, however,

$$\int_{\mathcal{V}} \rho(\boldsymbol{r})\, d\tau = M \quad \text{(finite)}$$

is finite provided $\boldsymbol{r}_0$ lies within the volume $\mathcal{V}$.

Need for the Dirac Delta Function

- It was a long time coming.
- Since there was no function that encoded such a property, **Paul A. M. Dirac** invented one, writing $\rho(\boldsymbol{r})$ as,

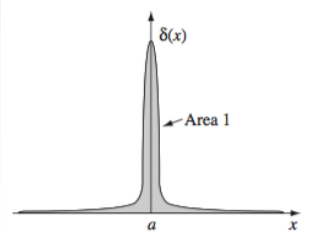
$$\rho(\boldsymbol{r}) = M \times \delta(\boldsymbol{r} - \boldsymbol{r}_0)$$

, and *requiring* that,

$$\int_{\mathcal{V}} \delta(\boldsymbol{r} - \boldsymbol{r}_0) d\tau = \begin{cases} 1 & \text{if } \boldsymbol{r}_0 \in \mathcal{V} \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(\boldsymbol{r} - \boldsymbol{r}_0) = 0 \quad \forall \boldsymbol{r} \neq \boldsymbol{r}_0$$

The One-Dimensional Dirac Delta Function



- The one-dimensional Dirac delta function[16], $\delta(x-a)$, can be pictured as an infinitely high, but infinitesimally narrow spike, with area 1, located at $x = a$, *i.e.*,

$$\delta(x-a) \equiv \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$

and,

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$$

[16] If x has units of length, what's the unit of $\delta(x)$?

Properties of the One-Dimensional Dirac Delta Function

- *Technically*, $\delta(x)$ is not a function at all, since its value is not finite at $x = 0$.
- It's **even**, *i.e.*, $\delta(x) = \delta(-x)$.
- An important characteristic of the Dirac delta function is its **sifting property**[17],

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\,dx = f(x_0)\int_{-\infty}^{\infty} \delta(x - x_0)\,dx = f(x_0)$$

  where, *loosely speaking*, the delta function *picks out* the value of $f(x)$ at $x = x_0$, *i.e.*, $f(x_0)$.
- Another curious property of the Dirac delta function is,

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

  where k is any (non-zero) constant

[17] Prove the sifting property!

The One-Dimensional Dirac Delta Function

- If $f(x)$ is differentiable,

$$\int_{-\infty}^{\infty} f(x)\delta'(x - x_0)\,dx = -f'(x_0)$$

- The $\delta(x)$ may also be seen as the derivative of the **Heaviside step function**[18],

$$\delta(x-a) = \frac{d}{dx}\Theta(x-a)$$

[18] This property might be invoked while discussing square potenial barriers in the QM part of this course.

The Three-Dimensional Dirac Delta Function

- In three dimensions,

$$\underbrace{\delta^3(\boldsymbol{r})}_{\textit{shorthand}} \equiv \delta(x)\delta(y)\delta(z)$$

  and,

$$\int_{\text{all space}} \delta^3(\boldsymbol{r})\, d\tau \equiv 1$$

- Also,

$$\int_{\text{all space}} f(\boldsymbol{r})\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\, d\tau = f(\boldsymbol{r}_0)$$

Divergence of $\hat{\boldsymbol{r}}/r^2$

- Consider the vector function[19],

$$\boldsymbol{v} = \frac{1}{r^2}\hat{\boldsymbol{r}}$$

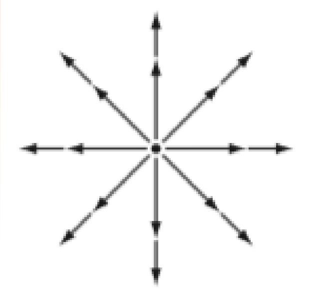
- From the figure, evidently $\boldsymbol{v}$ has a LARGE positive divergence at the center,
- . . . and **yet**. . . (do the math)
-

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0 \quad \text{everywhere!?}$$

which is certainly not what we were expecting!

[19] Which is eerily reminiscent of $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$, the electric field due to a **single**, **static point** charge.

Divergence of $\hat{\boldsymbol{r}}/r^2$



- However, when we consider the **divergence theorem**, *i.e.*,

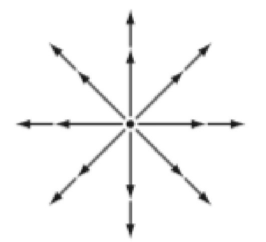
$$\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{v})\, d\tau = \oint_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{a}$$

  the RHS is...

- ...,

$$\oint_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{a} = \int \left(\frac{1}{r^2}\hat{\boldsymbol{r}}\right) \cdot (r^2 \sin\theta\, d\theta\, d\phi\, \hat{\boldsymbol{r}}) = 4\pi$$

Divergence of $\hat{\boldsymbol{r}}/r^2$



- But we'd just *(albeit näively)* found that the LHS, *i.e.*, $\int_{\mathcal{V}}(\nabla\cdot\boldsymbol{v})\,d\tau = 0$, which **contradicts** the **divergence theorem**!
- However, $\int_{\mathcal{V}}(\nabla\cdot\boldsymbol{v})\,d\tau = 0$ is **incorrect** –the **source** of the problem being the point $r = 0$, where $\boldsymbol{v} = \infty$, *i.e.*, it BLOWS UP.
- Indeed, $\nabla\cdot\boldsymbol{v}$ is actually *zero* everywhere **except** the origin.[20]

[20] Beware! The intuition of observing the spreading of sawdust at any of the non-central points seems to suggest a non-zero diverence, at least to my eye.

Divergence of $\hat{\mathbf{r}}/r^2$

- We resolve this *paradox* by realizing that the volume integral of $\nabla \cdot (\hat{\mathbf{r}}/r^2)$ must yield a constant 4π. (Why because the RHS of the **divergence theorem** just said so.)
- Thus, we can write using the definition of the Dirac delta function,

$$\boxed{\nabla \cdot (\hat{\mathbf{r}}/r^2) = 4\pi\delta^3(\mathbf{r})}$$

- *Alternatively*, $\frac{1}{4\pi}\nabla \cdot (\hat{\mathbf{r}}/r^2)$ is a concrete representation of the Dirac delta function $\delta(\mathbf{r})$.

The Scalar Potential V

- *When* $\nabla \times \boldsymbol{E} = 0$ everywhere[21], **Stokes' theorem** tells us that,

$$\oint \boldsymbol{E} \cdot \boldsymbol{dl} = 0$$

- On the other hand, the **fundamental theorem of gradients** allows us to write,

$$\oint -\nabla V \cdot \boldsymbol{dl} = 0$$

- Thus, $\boldsymbol{E}$ can be written as[22] the **gradient** of a **scalar potential** V,

$$\boldsymbol{E} = -\nabla(V + \text{const.})$$

- The potential is **not unique**, *i.e.*, any **constant** can be added to V without affecting its gradient, *i.e.*, the (negative) electric field $-\boldsymbol{E}$.

[21] $\nabla \times \boldsymbol{E} = 0$ is guaranteed only in the electro*static* regime.

[22] The negative sign is purely a matter of convention.

Curl-less (or irrotational) fields $\boldsymbol{v}_{irr}$

- The condition $\nabla \times \boldsymbol{E} = 0$ everywhere is **equivalent** to,
  - $\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l}$ is *independent* of path, for any given end points, a consequence of the fundamental theorem of gradients.
  - $\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$.
  - $\boldsymbol{E}$ is the gradient of some scalar function: $\boldsymbol{E} = -\nabla V$
- Irrotational fields $\boldsymbol{v}_{irr}$, *i.e.*, $\nabla \times \boldsymbol{v}_{irr} = 0$, are also known as **conservative fields**.[23]
- We'll soon **prove** in the chapter on **Electrostatics** that the **work done** in a moving a test charge against a background of static charges is **independent** of the path taken.

[23]In mechanics, a **conservative force** $\boldsymbol{F} = -\nabla U$ is a force with the property that the total work done $W = \int_a^b \boldsymbol{F} \cdot d\boldsymbol{l}$ in moving a particle between point a to b is *independent* of the path taken.

The Vector Potential $\boldsymbol{A}$

- **Maxwell's equations** *guarantee* that

$$\nabla \cdot \boldsymbol{B} = 0$$

- This allows $\boldsymbol{B}$ to be written as the curl of a **vector potential** $\boldsymbol{A}$,

$$\boldsymbol{B} = \nabla \times (\boldsymbol{A} + \nabla T)$$

  since mathematically (4) the divergence of a curl is always zero, *i.e.*, $\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$

- The vector potential $\boldsymbol{A}$ is **not unique**, *i.e., the gradient of any scalar function* $+\nabla T$ can be added to $\boldsymbol{A}$ withough affecting its curl, *i.e.*, the magnetic field $\boldsymbol{B}$, since mathematically (2) the curl of a gradient is *always* zero, *i.e.*, $\nabla \times (\nabla T) = 0$.

Divergence-less (or solenoidal) fields

- The condition $\nabla \cdot \boldsymbol{B} = 0$ everywhere is *equivalent* to,
  - $\oint \boldsymbol{B} \cdot d\boldsymbol{a} = 0$ for any closed surface, a direct consequence of the divergence theorem.
  - $\boldsymbol{B}$ is the curl of some vector function: $\boldsymbol{B} = \nabla \times \boldsymbol{A}$
  - $\int_{\mathscr{S}} \boldsymbol{B} \cdot d\boldsymbol{a}$ is independent of the details of the precise **open** surface $\mathscr{S}(\mathscr{P})$, once its periphery $\mathscr{P}$ is set.[24]

[24] This means we can deform the surface as we please as long as the boundary is kept fixed.

The Helmholtz Theorem

- "A well-behaved (*i.e.*, goes to zero at infinity) vector field is **uniquely** specified by its divergence and curl (and, in the case of a *finite region*, additionally by its normal component over the entire boundary.)"
- *i.e.*, suppose we know over <u>all</u> space.[25]

$$\nabla \cdot \boldsymbol{F}(\boldsymbol{r}) = D(\boldsymbol{r}) \quad \text{and,} \quad \nabla \times \boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{C}(\boldsymbol{r})$$

- Then the **unique** vector field $\boldsymbol{F}$ is given by

$$\boldsymbol{F}(\boldsymbol{r}) = -\nabla U(\boldsymbol{r}) + \nabla \times \boldsymbol{W}(\boldsymbol{r}) \quad \text{Helmholtz Decomposition}$$

where,

$$U(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\text{all space}} d^3r' \, \frac{\overbrace{\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}')}^{D(\boldsymbol{r}')}}{|r - r'|} \quad \text{and,} \quad \boldsymbol{W}(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\text{all space}} d^3r' \, \frac{\overbrace{\nabla' \times \boldsymbol{F}(\boldsymbol{r}')}^{\boldsymbol{C}(\boldsymbol{r}')}}{|r - r'|}$$

where the ∇' denotes that the derivatives are to be taken *w.r.t.* source points $\boldsymbol{r}'$.

[25] Note that since (4): $\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$, we must have $\nabla \cdot \boldsymbol{C} = 0$ for consistency.

Why's Helmholtz Theorem Useful?

- Since a vector field is completely specified once its divergence and curl are known (a purely **mathematical** result), and we know that the study of electromagnetism involves the vector fields $\boldsymbol{E}$ and $\boldsymbol{B}$, we can already *guess*...
- ...the laws of electromagnetism:

$$\begin{aligned} \nabla\cdot\boldsymbol{E} &= \text{something}_1 \\ \nabla\times\boldsymbol{E} &= \text{something}_2 \\ \nabla\cdot\boldsymbol{B} &= \text{something}_3 \\ \nabla\times\boldsymbol{B} &= \text{something}_4 \end{aligned}$$

  which look exactly like **Maxwell's equations**[26], which are thus *mathematically sufficient* to reconstruct $\boldsymbol{E}$, and $\boldsymbol{B}$.
- Even further, we can almost solve for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ via the **Helmholtz decomposition**, even without explicitly knowing what the $RHS = \text{something}_i$ are!

[26] If you're wondering where $\boldsymbol{D}$ and $\boldsymbol{H}$ went, here we're discussing the so-called **microscopic** representation of Maxwell's equations. $\boldsymbol{D}$ and $\boldsymbol{H}$ appear in the **macroscopic** formulation where the material medium is built into the equations. Both formulations are *equally* general.

Why's Helmholtz Theorem Useful?

- Considering the static[27] version of Maxwell's equations

$$\begin{aligned} \nabla\cdot\boldsymbol{E} &= \frac{\rho(\boldsymbol{r})}{\epsilon_0} \\ \nabla\times\boldsymbol{E} &= 0 \\ \nabla\cdot\boldsymbol{B} &= 0 \\ \nabla\times\boldsymbol{B} &= \mu_0\ \boldsymbol{j}(\boldsymbol{r}) \end{aligned}$$

  we're gratified by noting that a given set of **stationary** charges $\rho(\boldsymbol{r})$, and **steady** currents $\boldsymbol{j}(\boldsymbol{r})$, may only generate **one** possible steady $\boldsymbol{E}$, and **one** possible steady $\boldsymbol{B}$ field.

- Similarly, if all the sources ρ, and currents $\boldsymbol{j}$ are **zero everywhere**, then the **only** physical solution is $\boldsymbol{E} = \boldsymbol{B} = 0$.

- This implies that static electric and magnetic fields **cannot** generate themselves, *i.e.*, there must be stationary charges and steady currents generating them!

[27] *i.e.*, all charges are stationary, and currents steady.

Limitations of our Treatment of the Helmholtz Theorem

- Our discussion for the Helmholtz theorem only works for **time-independent** sources, and currents.
- For the **time-dependent** case, while the divergence and curl still uniquely identify the vector field, the **Helmholtz decomposition** looks a bit different, though I will not be writing it down explicitly.
- A second issue is that we haven't grappled with the application of the **boundary conditions** (*i.e., the normal component of the field must be known at all points of the periphery*) that need to be imposed if we're considering a **finite region of space**.