LIMIT & CONTINUITY

LECTURE 1 : Limit of functions

LECTURE 2 : Limit of functions

LECTURE 3 : Continuity

LECTURE 4 : Continuity

LECTURE 5 : Uniform Continuity.

Continuity

Definition :- Let $f : D \to \mathbb{R}$ be a function, where D is an interval. Let $a \in D$. We say f is continuous at 'a' if for every sequence (x_n) in D such that $x_n \to a$ as $n \to \infty$, we have $f(x_n) \longrightarrow f(a)$ as $n \to \infty$.

- In other words, f is continuous at 'a' if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

* If there exists a sequence (x_n) in D such that $x_n \to a$ but $f(x_n) \not\to f(a)$ as $n \to \infty$, then we say f is not continuous at 'a'.

<u>Definition</u> :- Let $f: D \to \mathbb{R}$ be a function. If f is continuous at each point of the set D then we say f is continuous on D.

<u>Example 1</u> :- Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{x}.$$

Let $a \in \mathbb{R} \setminus \{0\}$. Let (x_n) be a sequence in $\mathbb{R} \setminus \{0\}$ such that $x_n \to a$ as $n \to \infty$.

$$\Rightarrow \frac{1}{x_n} \to \frac{1}{a} \quad \left[\begin{array}{l} \text{Since } x_n, a \neq 0 \ \forall n \\ \text{By algebra of sequence} \end{array}\right]$$

$$\Rightarrow f(x_n) \to f(a) \text{ as } n \to \infty$$

Since (x_n) is arbitrary, f is continuous at 'a'.

Since a is arbitrary, f is continuous on $\mathbb{R}\setminus\{0\}$.

Example 2 :- Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

Discuss the continuity of f.

Soln :- Case I, Let $a \in \mathbb{R}$ be such that $a < 0$

$\Rightarrow a \in (-\infty, 0)$.

Let (x_n) be any sequence in $\mathbb{R}$ s.t. $x_n \to a$ as $n \to \infty$.

Thus, $\exists\, N \in \mathbb{N}$ s.t. $x_n \in (-\infty, 0)\ \forall\, n \geq N$.

$\Rightarrow f(x_n) = -1 \quad \forall\, n \geq N$

$\Rightarrow f(x_n) \longrightarrow -1 = f(a)$ as $n \to \infty$

Thus f is continuous at 'a'.

<u>Case 2</u> :- Let $a > 0$. Let (x_n) be a sequence converges to $a \in (0, \infty)$. Therefore, $\exists\ N \in \mathbb{N}$ s.t. $x_n \in (0, \infty)\ \forall\ n \geq N$

$$\Rightarrow f(x_n) = 1 \quad \forall\ n \geq N$$

$$\Rightarrow f(x_n) \longrightarrow 1 = f(a) \text{ as } n \to \infty.$$

Thus f is continuous at 'a'.

<u>Case 3</u>: Let $a = 0$. Then $f(0) = 1$

Let $x_n = -\frac{1}{n}$. Then $f(x_n) = -1$ and $x_n \to 0$

Here $x_n \to 0$ but $f(x_n) \to -1 \neq 1 = f(0)$.

Thus, $\exists$ a sequence (x_n) in $\mathbb{R}$ s.t. $x_n \to 0$ but

$f(x_n) \not\to f(0)$ as $n \to \infty$.

Thus f is not continuous at '0'.

Example 3 :- $f : \mathbb{R} \to \mathbb{R}$ $f(x) = 1 \;, x \in \mathbb{Q}$

$= -1 \;, x \in \mathbb{R} \setminus \mathbb{Q}.$

Discuss the continuity of the function f.

Soln :- Case 1, Let $a \in \mathbb{Q}$. Then $f(a) = 1$

Let $x_n = a + \frac{\sqrt{2}}{n}$. Then $x_n \in \mathbb{R} \setminus \mathbb{Q}$ & $x_n \to a$ as $n \to \infty$.

Therefore $f(x_n) = -1 \quad \forall n$

$\therefore f(x_n) \longrightarrow -1 \neq 1 = f(a)$ as $n \to \infty$.

Thus, $x_n \to a$ but $f(x_n) \not\to f(a)$ as $n \to \infty$

f is not continuous at 'a'.

Case 2 :- Let $a \in \mathbb{R} \setminus \mathbb{Q}$. Then $f(a) = -1$

Let $a = \pm a_0 . a_1 a_2 a_3 a_4 a_5 \cdots\cdots$

where $a_0 \in \mathbb{N} \cup \{0\}$ and $a_i \in \{0, 1, \cdots, 9\}$ $\forall i$

Let $r_n = \pm a_0 . a_1 a_2 \cdots a_n$

Then $r_n \in \mathbb{Q}$ and $r_n \longrightarrow a$ as $n \to \infty$.

Therefore $f(r_n) = 1 \longrightarrow 1 \neq f(a)$ as $n \to \infty$

Thus, f is not continuous at 'a'

Therefore, f is nowhere continuous on the domain $\mathbb{R}$.

Example 4 :- Let $f : \mathbb{R} \to \mathbb{R}$ $f(x) = 1, \ x \in \mathbb{Q}$
$= x, \ x \in \mathbb{R} \setminus \mathbb{Q}$.

Solution :-

Case 1, Let $a \neq 1 \in \mathbb{Q}$. Then $f(a) = 1$

Let $x_n = a + \frac{\sqrt{2}}{n}$. Then $x_n \in \mathbb{R} \setminus \mathbb{Q}$ & $x_n \to a$

But $f(x_n) = a + \frac{\sqrt{2}}{n} \longrightarrow a \neq 1 = f(a)$ as $n \to \infty$

Therefore, f is not continuous at 'a'.

Case 2 : Let $a \in \mathbb{R} \setminus \mathbb{Q}$. Then $f(a) = a$

Let $a = \pm a_0 . a_1 a_2 a_3 a_4 \cdots$, where $a_0 \in \mathbb{N} \cup \{0\}$ and $a_i \in \{0, 1, \cdots, 9\}$

Let $r_n = \pm a_0 . a_1 a_2 \cdots a_n$

Then $r_n \in \mathbb{Q}$ & $r_n \longrightarrow a$. This implies,
$f(r_n) = 1 \longrightarrow 1 \neq a = f(a)$

Therefore f is not continuous at 'a'

<u>Case 3</u>, Let $a = 1$

Let (x_n) be an arbitrary sequence such that

$x_n \to 1$ as $n \to \infty$.

Choose $\varepsilon > 0$. Since $x_n \to 1$ as $n \to \infty$, $\exists$

$N \in \mathbb{N}$ s.t. $|x_n - 1| < \varepsilon$ $\forall$ $n \geq N$.

For any $m \geq N$,

If x_m is rational then $f(x_m) = 1$, Then

$|f(x_m) - 1| = 0 < \varepsilon$

If x_m is irrational then $f(x_m) = x_m$.

Then $|f(x_m) - 1| = |x_m - 1| < \varepsilon$

Therefore, $|f(x_n) - 1| < \varepsilon$ $\forall$ $n \geq N$, i.e., $f(x_n) \to 1 = f(1)$.

f is continuous at 1.

Therefore, f is continuous only at $x = 1$.

Example 5 :- Let $f : (0,1) \to \mathbb{R}$ be a function defined by

$$f(x) = 0 \text{ if } x \in (0,1) \setminus \mathbb{Q}$$
$$= \frac{1}{q} \text{ if } x = \frac{p}{q} \in (0,1) \cap \mathbb{Q}$$

where p & q are relatively prime.

Discuss the continuity of the function f.

Soln :- case 1: Let $r = \frac{p}{q} \in (0,1) \cap \mathbb{Q}$

Then $f(r) = \frac{1}{q} \neq 0$

Let $x_n = r + \frac{\sqrt{2}}{n} \in \mathbb{R} \setminus \mathbb{Q}$

But $x_n \longrightarrow r \in (0,1)$

Thus, $\exists\ N \in \mathbb{N}$ s.t. $x_n \in (0,1)\setminus\mathbb{Q} \quad \forall\ n \geq N$

Then $f(x_n) = 0 \quad \forall\ n \geq N$

Thus $f(x_n) \longrightarrow 0 \neq f(r)$ as $n \to \infty$.

Therefore, f is not continuous at $x = r$, $r \in (0,1)\cap\mathbb{Q}$.

Case 2, Let $a \in (0,1)\setminus\mathbb{Q}$. Then $f(a) = 0$

Let (x_n) be any sequence in $(0,1)$ such that $x_n \longrightarrow a$ as $n \to \infty$.

$\therefore$ Let us choose, $\varepsilon > 0$.

We claim that, $\exists\ N \in \mathbb{N}$ s.t.

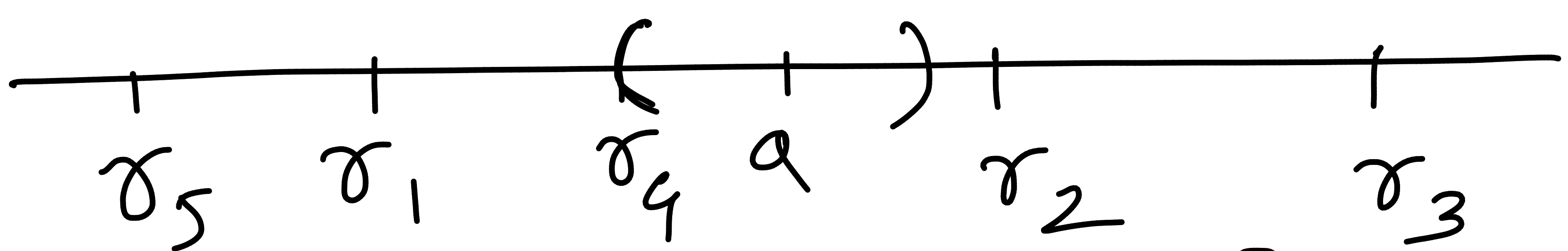
$|f(x_n) - f(a)| = |f(x_n)| < \varepsilon \quad \forall\ n \geq N.$

The number of natural numbers n such that $n \leq \frac{1}{\varepsilon}$ is finite.

Thus, the number of natural numbers n such that $\frac{1}{n} \geq \varepsilon$ is finite.

Thus, the number of rational numbers $\frac{m}{n}$, $1 \leq m < n$ such that $\frac{1}{n} \geq \varepsilon$ is also finite.

Thus, we have only finite number of rational numbers $\frac{m}{n}$ in $(0,1)$ such that $f\left(\frac{m}{n}\right) = \frac{1}{n} \geq \varepsilon$. Let these rational numbers are $r_1, r_2, r_3, \ldots, r_k$.



$r_i \notin (a-\delta, a+\delta) \ \forall i$

Let $\delta = \min\limits_{1 \leq i \leq k} \{ |a - r_i| \}$

Therefore, $r_i \notin (a-\delta, a+\delta)$ $\forall\ i=1,2,\dots,k$.

Thus, if $r = \frac{m}{n} \in (a-\delta, a+\delta)$ then $f(r) = \frac{1}{n} < \varepsilon$.

Therefore, if $c \in (a-\delta, a+\delta)$ then if c is rational say $c = \frac{m}{n}$ then $f(c) = \frac{1}{n} < \varepsilon$ and, if c is irrational then $f(c) = 0 < \varepsilon$.

Since $x_n \to a$ as $n \to \infty$, $\exists\ N \in \mathbb{N}$ s.t.

$$x_n \in (a-\delta, a+\delta)\ \forall\ n \geq N$$

$$\Rightarrow f(x_n) < \varepsilon$$

Thus, $|f(x_n) - f(a)| = |f(x_n)| < \varepsilon$ $\forall\ n \geq N$.

Therefore, $f(x_n) \to f(a)$ as $n \to \infty$

Since, (x_n) is arbitrary, f is continuous at $x = a$, where $a \in (0,1) \setminus \mathbb{Q}$.

Properties :- Let $f:[a,b] \to \mathbb{R}$ be a continuous function.

(i) If $f(a) > 0$ then $\exists$ a $\delta > 0$ s.t. $f(x) > 0$ $\forall$ $x \in [a, a+\delta)$.

(ii) If $f(b) > 0$ then $\exists$ a $\delta > 0$ s.t. $f(x) > 0$ $\forall$ $x \in (b-\delta, b]$

(iii) If $f(c) > 0$ for some $c \in (a,b)$ then $\exists$ a $\delta > 0$ such that $f(x) > 0$ $\forall$ $x \in (c-\delta, c+\delta)$.

Proof of (iii) :- Since $c \in (a,b)$, $\exists$ $N \in \mathbb{N}$ s.t.

$$\left(c - \frac{1}{n}, c + \frac{1}{n}\right) \subset (a,b) \quad \forall \ n \geq N.$$

We claim, $\exists$ $M \geq N$ s.t. $f(x) > 0$ $\forall$ $x \in \left(c - \frac{1}{M}, c + \frac{1}{M}\right)$

If there is no such M then for each $n \geq N$, $\exists$ $x_n \in \left(c - \frac{1}{n}, c + \frac{1}{n}\right)$ s.t. $f(x_n) \leq 0$.

Therefore, $x_n \to c$ as $n \to \infty$

Since f is continuous at c, $f(x_n) \to f(c)$ as $n \to \infty$.

Therefore $f(c) = \lim_{n \to \infty} f(x_n) \leq 0$

This is a contradiction.

$\therefore$ Thus, we must have $M \in \mathbb{N}$ s.t.

$$f(x) > 0 \quad \forall \ x \in \left(c - \frac{1}{M}, c + \frac{1}{M}\right).$$

Theorem :- Let $f : D \to \mathbb{R}$ be a function and $a \in D$. Then f is continuous at 'a' if and only if for any $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $|x - a| < \delta$ and $x \in D$ implies $|f(x) - f(a)| < \varepsilon$.