$$L(x(t)) = 0$$

$L \to$ linear differential operator

e.g $\quad L(x) = x^{(n)} + a_1(t)\, x^{(n-1)} + \cdots + a_n(t)\, x$

$\downarrow$ n^{th} order linear differential operator

Non-homogeneous Linear ODE's

$$L(x(t)) = f(t), \quad f(t) \not\equiv 0 \qquad \text{—— ①}$$

$\downarrow$

Linear differential operator

The corresponding homogeneous ODE is

$$\underline{L(x(t)) = 0} \qquad \text{—— ②}$$

$\rightarrow$ learnt methods to find the general sol^n of ②.

Thm $\quad$ The difference of two sol^n of non-homogeneous ODE ① on I is a sol^n of homogeneous ODE ② on I.

Proof: $\quad \ldots\; x^*(t) \ldots \; sol^n \ldots$ ①

Proof

Let $x(t)$ & $x^*(t)$ be two sol's of ①

i.e $\quad L(x) = f$

$\quad\quad L(x^*) = f$

then $\quad L(x - x^*) = L(x) - L(x^*) \quad$ (since L is a linear operator)

$$= f - f$$

$$= 0$$

$\Rightarrow \quad x - x^*$ is a sol^n of DE ②.

Corollary Any sol^n $x(t)$ of non-homogeneous ODE ①

is given by

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t)$ is the general sol^n of ② &

$x_p(t)$ is a particular sol^n of ①.

$x(t) \rightarrow$ sol^n of ①

$x_p(t) \rightarrow$ sol^n of ①

$x(t) - x_p(t) \rightarrow$ sol^n of homogeneous DE ②

$\Rightarrow \quad x(t) - x_p(t) = x_h(t) \longrightarrow$ general sol^n of ②

$$\Rightarrow \boxed{x(t) = \underline{x_h(t)} + \underline{x_p(t)}}$$

the general sol^n of DE ①

Def^n. A general sol^n of non-homogeneous ODE ① in I is a sol^n of the form

$$\boxed{x(t) = \underline{x_h(t)} + \underline{x_p(t)}} \quad \text{——— ③}$$

where $x_h(t)$ is a general sol^n of homogeneous ODE ② on I & $x_p(t)$ is any particular sol^n of DE ①. (containing no constants)

* A particular sol^n of DE ① on I is a sol^n obtained from ③ by assigning specific values to the arbitrary constants occuring in general sol^n $x_h(t)$. (they can fixed by using initial cond^{n}s).

Remark. Since we have already discussed methods of finding $x_h(t)$, it suffices to find the particular $x_p(t)$ to determine general sol^n

$x(t)$ of ①.

① Variation of Parameters method

Lets consider

first order ODE with constant coefficients

$$x'(t) + a\,x(t) = f(t) \quad \text{—— ①}$$

$$\boxed{x'(t) + a(t)\,x(t) = f(t)}$$

, $f(t) \neq 0$

$a \to$ constant

The corresponding homogeneous DE is

$$x'(t) + a\,x(t) = 0 \quad \text{—— ②}$$

General sol^n of ②

$$\boxed{x_h(t) = \underline{c}\,e^{-at}}$$

→ The basic idea of this method is to start with the general sol^n of homogeneous DE ② & replace the constants by functions.

$$x_p(t) = c(t)\,e^{-at}$$

where $c(t)$ is to be determined so that $x_p(t)$ is a sol^n of ①.

$$f(t) = x_p' + a\,x_p$$

$$= c'(t)\,e^{-at} - a\,\underline{c(t)e^{-at}} + \underline{a\,c(t)\,e^{-at}}$$

$$= c'(t)\, e^{-at}$$

$$\Rightarrow \quad c'(t) = e^{at} f(t)$$

$$\Rightarrow \boxed{x_p(t) = e^{-at} \int e^{at} f(t)\, dt}$$

particular solⁿ of DE ①

The general solⁿ of DE ①

$$\boxed{x(t) = x_h(t) + x_p(t)}$$

* $x'(t) + \underline{a}\, x(t) = f(t)$

$\rightarrow x'(t) + \underline{a(t)}\, x(t) = f(t)$

VPM

<u>Exer</u> Solve the following IVP using VPM

$$x' + x \tan t = \sin 2t$$

$$x(0) = 1.$$

2^{nd} order Non-homogeneous linear ODE

$$x'' + a(t)x' + b(t)x = f(t) \quad \text{———— ①}$$

$f(t) \rightarrow$ cts f^n

The corresponding homogeous ODE is

$$x'' + a(t)\,x' + b(t)\,x = 0 \quad \text{--- (2)}$$

The general sol^n of DE (2) is

$$x_h(t) = \underset{\downarrow\, u(t)}{\underline{c_1}}\, x_1(t) + \underset{v(t)}{\underline{c_2}}\, x_2(t) \quad \text{--- (3)}$$

$\{x_1(t), x_2(t)\} \rightarrow$ fundamental set of sol^ns of DE (2)

We determine $u(t)$ & $v(t)$ s.t.

$$\underline{x_p(t)} = u(t)\, x_1(t) + v(t)\, x_2(t)$$

is a particular sol^n of non-homogeous DE (1).

We determine $\underline{u(t)}$ & $\underline{v(t)}$ by substituting x_p in DE (1).

$$x_p' = u' x_1 + u x_1' + v' x_2 + v x_2'$$

$$= u x_1' + v x_2' + \underline{u' x_1 + v' x_2}$$

$$x_p'' = u' x_1' + u x_1'' + v' x_2' + v x_2'' + \underline{(u' x_1 + v' x_2)'}$$

$$x_p'' + a(t)\, x_p' + b(t)\, x_p = f(t) \quad \text{--- (4)}$$

We choose u & v s.t.

$$u'x_1 + v'x_2 = 0 \quad \text{——} \quad (5)$$

This reduces the derivatives x_p' & x_p'' in a <u>simpler form</u>.

$$x_p' = ux_1' + vx_2'$$

$$x_p'' = u'x_1' + ux_1'' + v'x_2' + vx_2''$$

$\therefore$ the DE (4) yields

$$u'x_1' + ux_1'' + v'x_2' + vx_2'' + a(t)[ux_1' + vx_2'] + b(t)[ux_1 + vx_2] = f(t)$$

$$u'x_1' + u\underbrace{(x_1'' + a(t)x_1' + b(t)x_1)}_{=0} + v'x_2' + v\underbrace{(x_2'' + a(t)x_2' + b(t)x_2)}_{=0} = f(t)$$

since x_1 & x_2 are sol^{n}s of homogeneous DDE (2).

$$\begin{cases} u'x_1' + v'x_2' = f(t) & \text{——} \quad (6) \\ u'x_1 + v'x_2 = 0 & \text{——} \quad (5) \end{cases}$$

Eq (5) $\times\, x_2'$ — Eq (6) $\times\, x_2$

$$u'\underbrace{(x_1x_2' - x_1'x_2)}_{W} = -f(t)\,x_2$$

$W = W(x_1, x_2)(t)$

x_1 & x_2 are L.I.

$$u' = \frac{-x_2 f}{\ldots}$$

$u -$ $\frac{}{W}$ $W(t) \neq 0$

$Eq^n ⑥ \times x_1 - Eq^n ⑤ \times x_1'$

$$v'\underbrace{(x_2' x_1 - x_2 x_1')}_{W} = f(t)\,x_1$$

$$\Rightarrow \quad v' = \frac{x_1 f}{W}$$

$$u' = \frac{-x_2 f}{W}, \quad v' = \frac{x_1 f}{W}$$

Upon integrating,

$$u(t) = -\int \frac{x_2(t)\,f(t)}{W(t)}\,dt$$

$$v(t) = \int \frac{x_1(t)\,f(t)}{W(t)}\,dt$$

particular sol^n of ① is

$$\therefore \quad x_p(t) = u(t)\,x_1(t) + v(t)\,x_2(t)$$

$$\boxed{x_p(t) = x_1 \int \frac{-x_2 f}{W} + x_2 \int \frac{x_1 f}{W}} \longrightarrow ⑦$$

The general sol^n of non-homogeneous DE ① is

$$x(t) = x_h(t) + x_p(t)$$

where x_h is given by ③ & x_p is given by ⑦.

$f(t)$

Ex:

Solve $$x'' + x = \sec t \quad \text{--- (8)}$$

The corresponding homogeneous ODE is

$$x'' + x = 0 \quad \text{--- (9)}$$

$\hookrightarrow$ 2nd order linear ODE with constant coefficients

The general soln of DE (9) is

$$x_h(t) = c_1 \cos t + c_2 \sin t$$

$m^2 + 1 = 0$

$m = i, -i$

$x_1(t) = \cos t$

$x_2(t) = \sin t$

To determine $u(t)$ & $v(t)$ s.t.

$$x_p(t) = u(t) \cos t + v(t) \sin t$$

is a soln of DE (8)

$$u(t) = -\int \frac{x_2 f}{W}, \quad v(t) = \int \frac{x_1 f}{W}$$

$$W(x_1, x_2)(t) = x_1 x_2' - x_2 x_1' = \cos t \cos t - \sin t(-\sin t)$$
$$= 1$$

$$u(t) = -\int \frac{\sin t}{\cos t}\, dt = \ln|\cos t| + d_1$$

$$v(t) = \int \cos t \cdot \sec t\, dt = t + d_2$$

$\therefore$ $$\boxed{x_p(t) = \cos t \ln|\cos t| + t \sin t}$$

& the general soln of DE (8) is

$x(t) = x_h(t) + x_p(t)$

$x(t) = c_1 \cos t + c_2 \sin t + x_p(t)$

Exer Solve $x''(t) + 9x(t) = \operatorname{cosec} 3t$

using VPM.