Tutorial Sheet IV
Duality

1. If x^o is any feasible solution of (P) and w^0 is feasible for (D) such that $w^0 (b - Ax^0) = 0$ and $(w^0A - c)\, x^0 = 0$, then show that x^0 is optimal for (P) and w^0 is optimal for (D).

   (P) $\quad$ Max $z = c\,x$, subject to $Ax \le b$, $x \ge 0$ $\qquad$ (D) $\quad$ Min $w = b\,w$, subject to $Aw \ge c$, $w \ge 0$

2. Show that $wA = c$, $w \ge 0$, is inconsistent iff $Ax \le 0$, $cx > 0$, is consistent.

3. Consider the following problem:

$$\begin{aligned} \text{min} \quad & 3x_1 - 5x_2 - x_3 + 2x_4 - 4x_5 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + 3x_4 + x_5 \le 6 \\ & -x_1 - x_2 + 2x_3 + x_4 - x_5 \ge 3 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

   Write the dual. Find the primal optimal solution from the optimal solution of the dual.

4. Use duality to show that the following LP has an optimal solution

$$\begin{aligned} \text{min} \quad & 2x_1 - x_2 \\ \text{subject to} \quad & 2x_1 - x_2 - x_3 \ge 3 \\ & x_1 - x_2 + x_3 \ge 2 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

5. Use complementary slackness theorem to verify that (n, 0, 0,…,0) is an optimal solution of LPP

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} j\, x_j \\ \text{subject to} \quad & \sum_{j=1}^{i} x_j \ge i, \quad i = 1,\dots,n \\ & x_j \ge 0, \ \forall j \end{aligned}$$

6. Suppose the k^{th} constraint of the primal LPP: Max $z = c^t x$, subject to $Ax = b$, $x \ge 0$, is multiplied by a scalar $\beta \ne 0$. What would be its impact on the dual solution w?

7. Let (D) be the dual of the following primal problem (P) max $c^t x$ subject to $Ax \le b$, $x \ge 0$. Prove that if $b > 0$ and $A > 0$ then both (P) and (D) possess optimal solutions.

8. Are the following statements true? Give reasons for your answer
   1. The primal LP (P) and its dual LP (D), both cannot have unbounded solution.
   2. The primal LP (P) and its dual LP (D), both cannot be infeasible.
   3. The dual(dual(dual)) of a LPP is the primal LPP.
   4. If the primal LP (P) has a unique optimal solution and the dual LP (D) is feasible, then (D) also has a unique optimal solution.

9. Solve the following by dual simplex algorithm

$$\begin{aligned} \text{min} \quad & 80x_1 + 60x_2 + 80x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 \ge 4 \\ & 2x_1 + 3x_3 \ge 3 \\ & 2x_1 + 2x_2 + x_3 \ge 4 \\ & 4x_1 + x_2 + x_3 \ge 6 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$