# COL 352 Introduction to Automata and Theory of Computation

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Lecture 22:Turing Machines: Variants, CT Thesis

# Recap

- Deterministic single-tape Turing Machines

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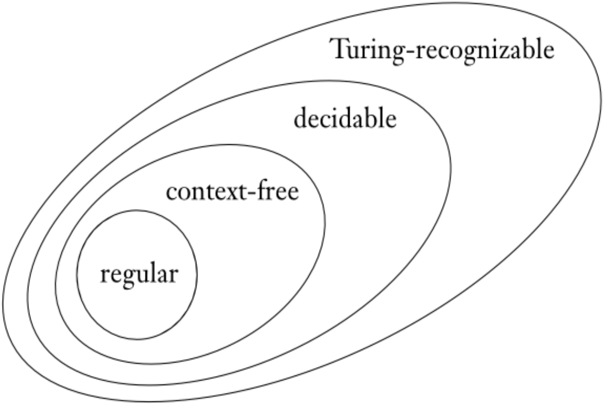

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*Turing decidable languages are closed under the following operations:*

1. *Union*
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3. *Complement*
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- Variants of TM with multiple tapes or with nondeterminism abound.
- Original model of a TM and its variants all have the same computation power, i.e., they recognize the same class of languages.
- Hence, robustness of TM definition is measured by the invariance of its computation power to certain changes in design features of the machine.

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**Exercise:** What about $\delta : Q \times \Gamma \to Q \times \Gamma \times \{R, S\}$

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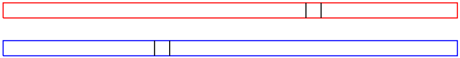

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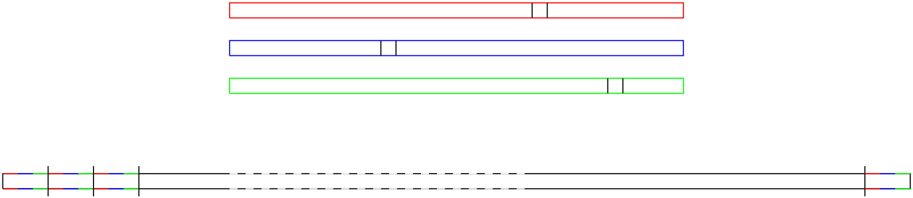

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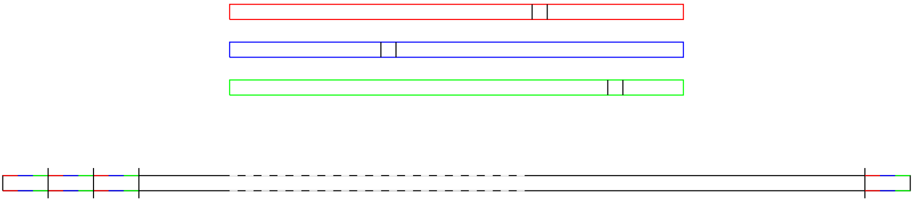

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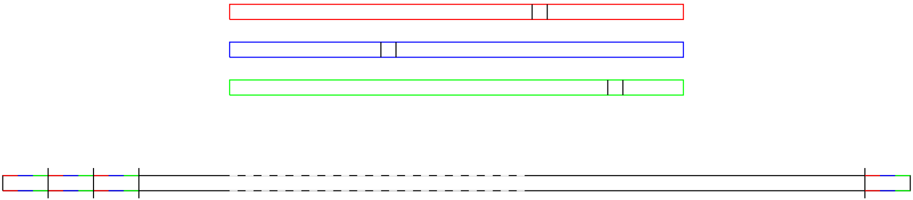

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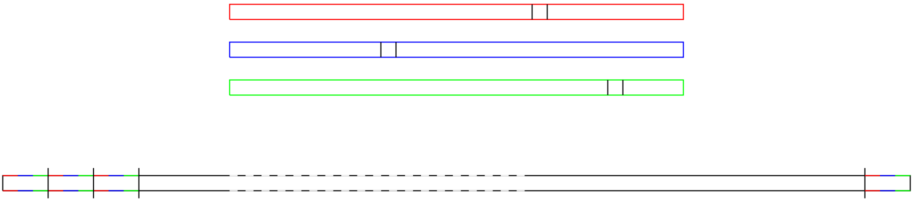

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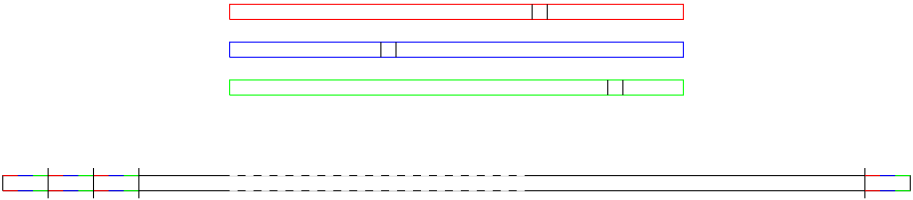

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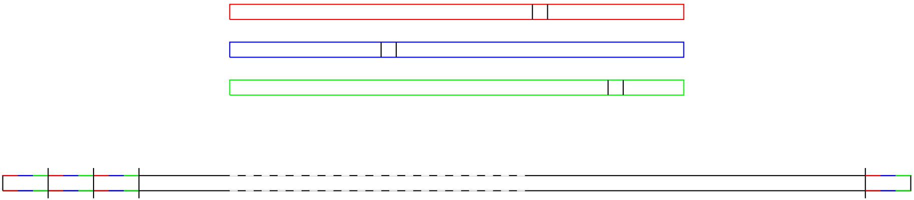

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$\overline{\Gamma}$ symbols used to denote tape head positions.

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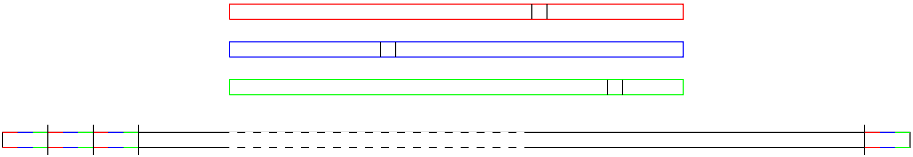

To simulate $1$ step of $M$, $M'$ works follows:

reads the tape left to right once, remembering the marked symbols in its states,

uses $\delta$ to determine the next state,

sweeps the input left to right again to update marked symbols.

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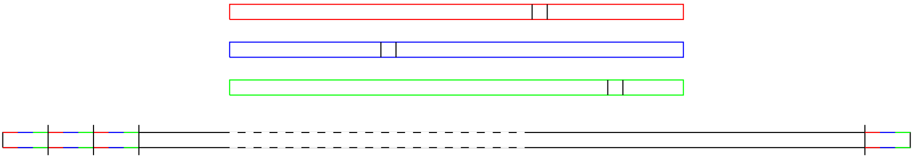

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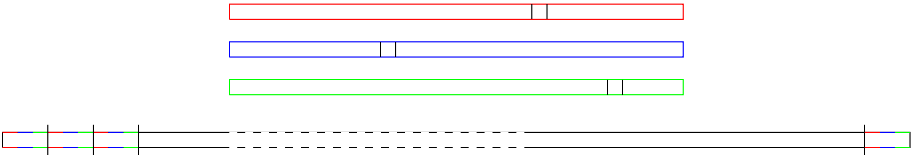

To simulate $1$ step of $M$

# $k$-tape Turing machines

**Theorem**

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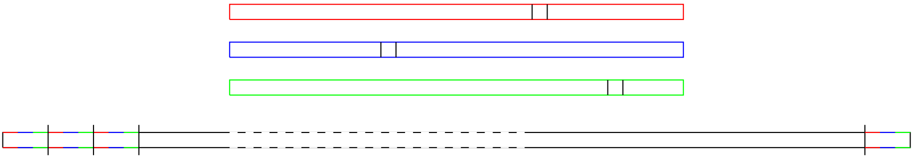

To simulate $1$ step of $M$, $M'$ works follows:

reads the tape left to right once, remembering the marked symbols in its states

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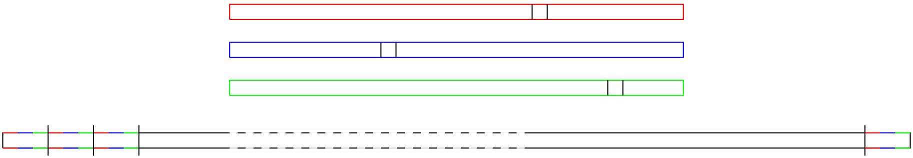

To simulate $1$ step of $M$, $M'$ works follows:

- reads the tape left to right once, remembering the marked symbols in its states,
- uses $\delta$ to determine the next state

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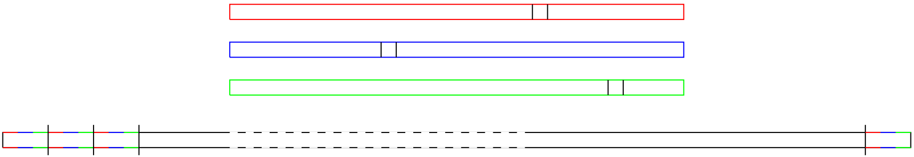
Proof sketch:



To simulate $1$ step of $M$, $M'$ works follows:

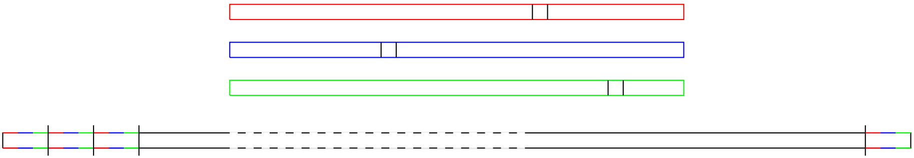
- reads the tape left to right once, remembering the marked symbols in its states,
- uses $\delta$ to determine the next state,
- sweeps the input left to right again

# $k$-tape Turing machines

*Theorem*

*Every $k$-tape Turing machine has an equivalent $1$-tape Turing machine.*

Proof sketch:



To simulate $1$ step of $M$, $M'$ works follows:

reads the tape left to right once, remembering the marked symbols in its states,

uses $\delta$ to determine the next state,

sweeps the input left to right again to update marked symbols.