# COL 352 Introduction to Automata and Theory of Computation

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Lecture 28: Reductions 3

#### Lemma

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The following two details need to be addressed.

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▶ If M does not accept w, then no matter what x is,  $N_{M,w}$  will accept x, i.e.  $L(N_{M,w}) = \sum_{k=0}^{*} .$ 

### String List Matching Problem

Given two lists  $A = \langle s_1, s_2, \dots, s_n \rangle$  and  $B = \langle t_1, t_2, \dots, t_n \rangle$  of strings of equal length, decide whether there is a sequence of combining elements that produces same string for both lists.

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$$s_{i_1}s_{i_2}\ldots s_{i_n}=t_{i_1}t_{i_2}\ldots t_{i_n}$$

**Example:** Consider the lists  $A = \langle 110, 0011, 0110 \rangle$  and  $B = \langle 110110, 00, 110 \rangle$ .

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**Example:** Consider the lists  $A = \langle 110,0011,0110 \rangle$  and  $B = \langle 110110,00,110 \rangle$ . **Solution:** There is a sequence i=2,3,1 such that  $s_2s_3s_1=t_2t_3t_1$ ,

**Witness:**  $s_2s_3s_1 = 00110110110$  and  $t_2t_3t_1 = 00110110110$ 

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- What about  $A = \{100, 0, 1\}$  and  $B = \{1, 100, 0\}$ ?

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- ▶ What about  $A = \{100, 0, 1\}$  and  $B = \{1, 100, 0\}$ ?(Shortest solution lenth = 75)

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Can you design an algorithm to solve this problem? A semi-algorithm?

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#### Theorem

There is no algorithm for the string-list matching problem. In other words, this problem is undecidable.

### A Domino game

Given a collection of dominos  $\begin{bmatrix} b \\ ca \end{bmatrix} \begin{bmatrix} a \\ ab \end{bmatrix} \begin{bmatrix} ca \\ a \end{bmatrix} \begin{bmatrix} abc \\ c \end{bmatrix}$ 

A match is a list of these dominos (with possible repetitions) such that the string formed in top is same as string formed by bottom row.

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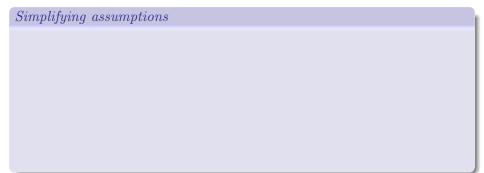
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- Does this collection of dominos have a match?
- Same as the string matching problem
- Called Post's Correspondance Problem or PCP.

#### Theorem

#### PCP is undecidable.

- Encode TM computation histories!
- ▶ Each transition as a domino!
- Simulate the run using the dominos.



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We build instance P' of MPCP in several steps.



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▶ Step 1: fix first domino in *P*′

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  - add all dominos (i.e, for all  $a \in \Gamma \cup \{\#\}$  to P'.  $\begin{bmatrix} a \\ a \end{bmatrix} \begin{bmatrix} \# \\ \downarrow \downarrow \# \end{bmatrix}$

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- ▶ MPCP to PCP?