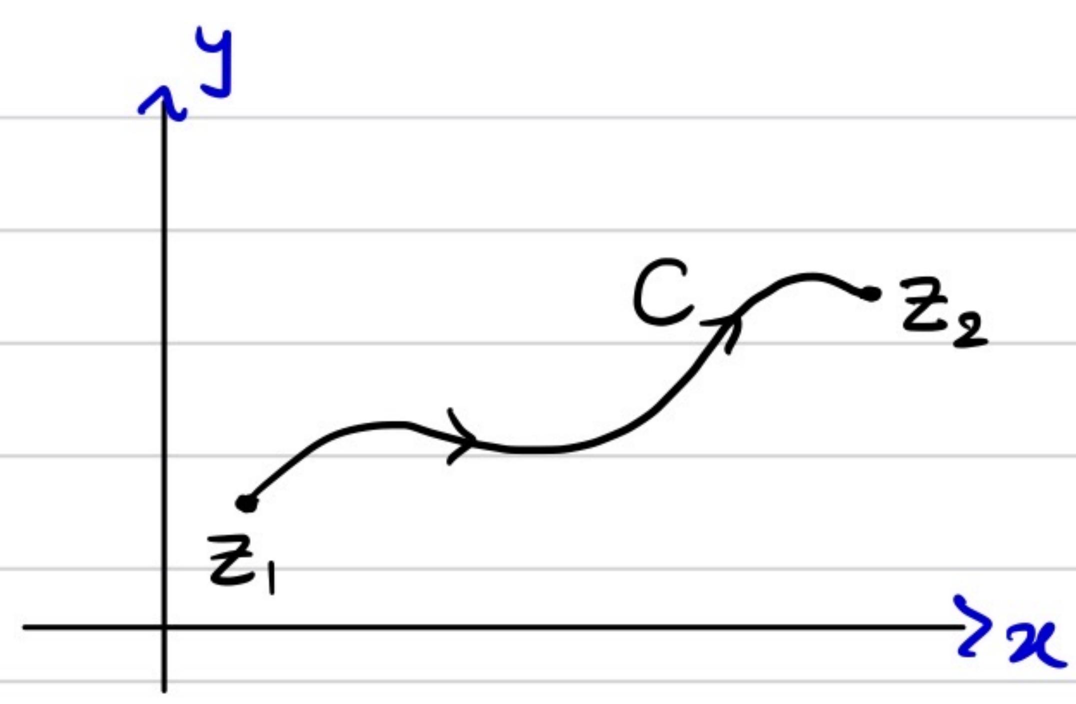
Contour Integrals of complex functions

$$I_C = \int_{z_1}^{z_2} dz\, f(z)$$



C : A smooth non self intersecting curve from z_1 to z_2, referred to as a $\boxed{\text{contour}}$ *

Let t be a real variable $\in (a, b)$ s.t. $\underbrace{z(t)}_{= x(t) + i y(t)}$ goes along the contour C as t goes from a to b with

$$z(t=a) = z_1 \quad \& \quad z(t=b) = z_2$$

$z(t) \longrightarrow$ a <u>parametrization</u> of the contour C.

* {parametrization of a contour is not unique & a given contour can be parametrized in many many different ways}

Let $f(z) = u(x,y) + i\, v(x,y)$ be a complex function with <u>piecewise continuous</u> $u(x,y)$ and $v(x,y)$ along the contour.

i.e. $u(x(t), y(t))$ and $v(x(t), y(t))$ are piecewise continuous functions of t, then

$$\boxed{I_C \equiv \int_C dz\, f(z) \equiv \int_a^b dt\, .\, z'(t)\, .\, f(z(t))} \qquad \boxed{\text{where} \quad z'(t) = \frac{d}{dt} z(t)}$$

is refered to as the <u>contour integral</u> of $f(z)$ along the contour C.

Comments :

1. The value of a complex contour integral in general depends on the contour along which the integral is being evaluated. Contours have direction associated with them.

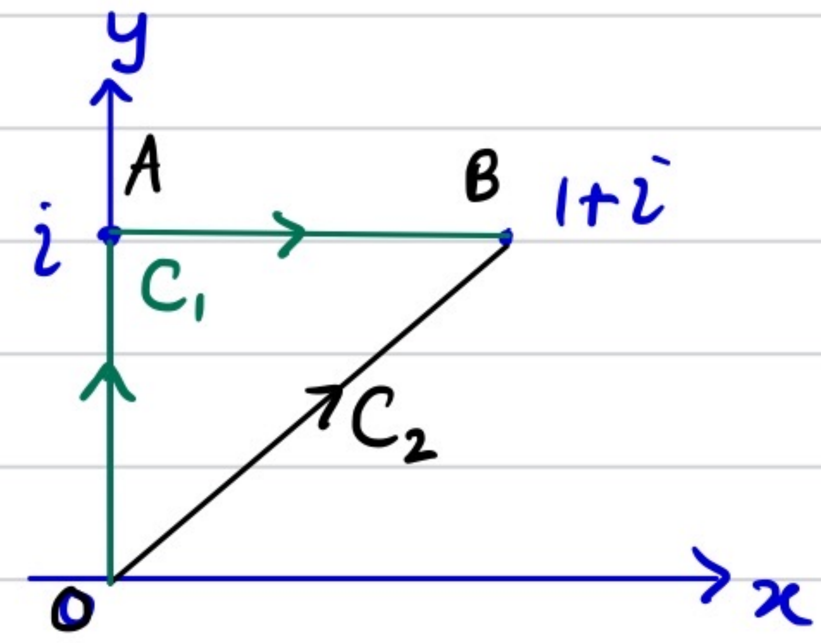
2. It does not depend on the parametrization one uses for the contour. Change in parametrization

$$t \rightarrow \tau(t) \quad w/ \quad \tau'(t) > 0 \;\; \forall \;\; t \in (a, b)$$

are akin to change of variables in real definite integrals.

3. We would be interested in "$\underbrace{\text{Simple}}_{\text{Non self intersecting.}}$" open and closed Contours.

Lets look at some examples!



- $z = x+iy$; $f(z) = y - x - i\,3x^2$

$$\int_{C_1} dz\, f(z) = \int_0^A dz\, f(z) + \int_A^B dz\, f(z)$$

$$= \int_0^1 d(0+it)\, f(0+it) + \int_0^1 d(\omega+i)\, f(\omega+i)$$

$$= i\int_0^1 dt\,(t) + \int_0^1 d\omega\,(1-\omega-i\,3\omega^2)$$

$$= i\,\frac{t^2}{2}\Big|_0^1 + \left(\omega - \frac{\omega^2}{2} - i\,\omega^3\right)\Big|_0^1 = \frac{i}{2} + \left(1-\frac{1}{2}-i\right) = \frac{1-i}{2}$$

$$\int_{C_2} dz\, f(z) = \int_0^1 d(\gamma(1+i))\; f(\gamma(1+i))$$

$$= (1+i)\int_0^1 d\gamma\,(\cancel{\gamma}-\cancel{\gamma}-i\,3\gamma^2) = (1+i)\left[-i\gamma^3\right]_0^1$$

$$= (1+i)(-i) = 1-i$$

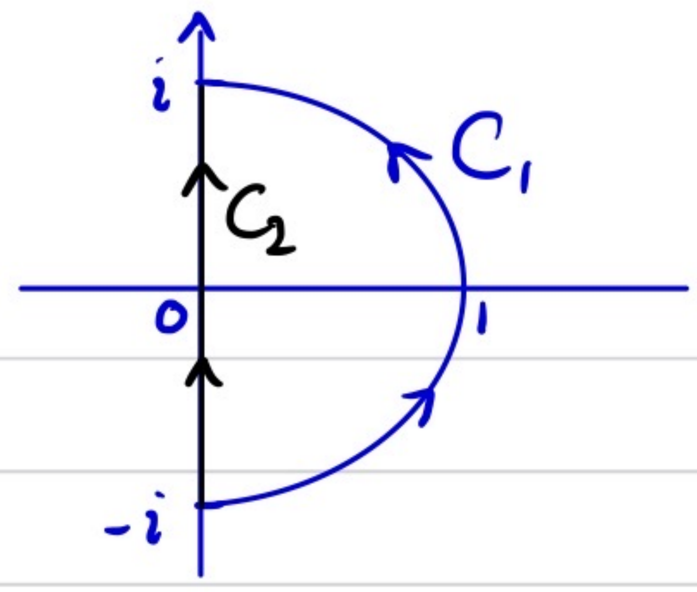
Let us represent by $-C_2$ the contour with direction opposite to C_2 then $C_1+(-C_2)$ defines a closed contour.

Further $I_{(-C_2)} = -I_{C_2} = -(1-i)$

$$\Rightarrow I_{C_1-C_2} = I_{C_1} - I_{C_2} = \frac{1-i}{2} - (1-i) = -\frac{1-i}{2}$$

The result of closed contour $C_1 - C_2$ integral of $f(z)$ above is non vanishing.

- $$f(z) = x^2 - y^2 + i\,2xy = z^2$$



$$I_{C_1} = \int_{C_1} dz\, f(z) = \int_{-\pi/2}^{\pi/2} d(e^{i\theta}) \cdot e^{2i\theta}$$

$$= i\int_{-\pi/2}^{\pi/2} d\theta \cdot e^{i\theta} \cdot e^{2i\theta} = i\int_{-\pi/2}^{\pi/2} d\theta\, e^{3i\theta} = \frac{i}{3i}\, e^{3i\theta}\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{3}\left(e^{i3\pi/2} - e^{-i3\pi/2}\right) = \frac{1}{3}(-i - i) = -\frac{2i}{3}$$

$$I_{C_2} = \int_{C_2} dz\, f(z) = \int_{-1}^{1} d(it) \cdot (it)^2 = -i\int_{-1}^{1} dt \cdot t^2 = -i\,\frac{t^3}{3}\Big|_{-1}^{1}$$

$$= \frac{-i}{3}\left(1 - (-1)\right) = -\frac{2i}{3} \qquad = I_{C_1}$$

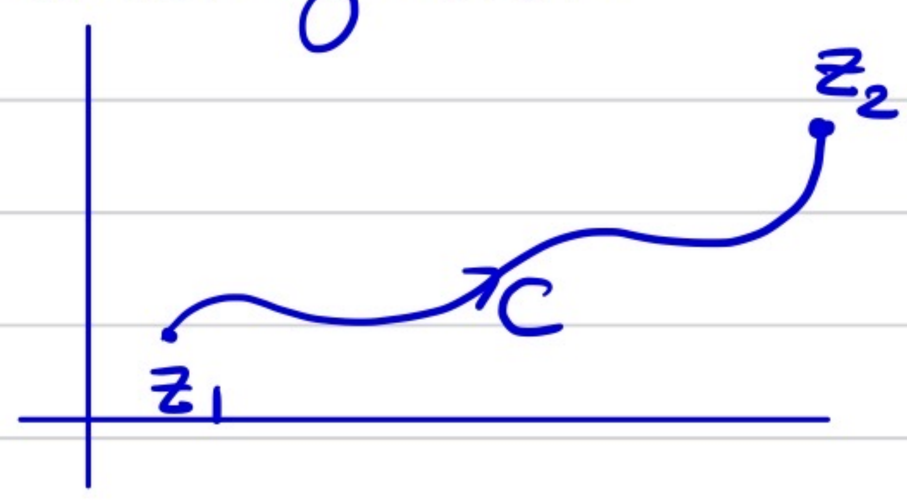
$$I_{C_1 - C_2} = I_{C_1} - I_{C_2} = 0$$

This is a more generally true statement for $f(z) = z^2$, i.e. its integral is independent of the path and only depends on the end points.

We will see later that property of contour integrals being independent of path is true for a class of complex functions which we have encountered earlier, the analytic functions.

- Independence of Contour integrals from parametrization

$$I_C = \int_C dz\, f(z)$$



Let $z(t)$ w/ $t \in (t_1, t_2)$ be parametrization, then

$$I_C = \int_{t_1}^{t_2} dt \left(\frac{d}{dt} z(t)\right) \cdot f\big(z(t)\big)$$

Let us choose another parametrization of C by a new parameter τ which is a smooth monotonic function of t

i.e. $\tau(t)$ s.t. $\frac{d}{dt}\tau(t) > 0 \quad \forall \; t \in (t_1, t_2)$

& $\tau(t_1) \equiv \tau_1$; $\tau(t_2) \equiv \tau_2$

τ is equally good parameter for Contour C

$z(\tau)$ w/ $\tau \in (\tau_1, \tau_2)$ s.t. $z(\tau_1) = z_1$, $z(\tau_2) = z_2$

Thus

$$I_c = \int_{\tau_1}^{\tau_2} d\tau \left(\frac{d}{d\tau} z(\tau)\right) \cdot f(z(\tau))$$

Lets rewrite this back in terms of t using $\tau(t)$

$$= \int_{\tau_1}^{\tau_2} d(\tau(t)) \left[\frac{d}{d\tau} z(\tau(t))\right] \cdot f(z(\tau(t)))$$

$$= \int_{t_1}^{t_2} \left(dt \cdot \cancel{\frac{d\tau(t)}{dt}}\right) \cdot \left[\frac{1}{\cancel{\frac{d\tau(t)}{dt}}} \cdot \frac{dz(\tau(t))}{dt}\right] \cdot f(z(\tau(t)))$$

Since $\frac{d}{dt}\tau(t) > 0 \quad \forall \; t \in (t_1, t_2)$

$$= \int_{t_1}^{t_2} dt \; \frac{dz(t)}{dt} \cdot f(z(t))$$