Multiple Integral - Lecture 4

Change of variables in double integral and triple integral.

\# $\displaystyle\int_a^b f(g(x))\, g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$

$y = g(x)$

$y' = g'(x)$

In case of two variables

$$x = f(u,v) \quad \& \quad y = g(u,v)$$

$$dx\,dx = (?)\,du\,dv$$

In case of polar co-ordinates, we have seen that

$$dx\,dy = (r)\,dr\,d\theta$$

$$x = r\cos\theta = f(r,\theta)$$

$$y = r\sin\theta = g(r,\theta)$$

In case of Cylindrical co-ordinates

$$dv = dx\,dy\,dz = (r)\,dr\,d\theta\,dz$$

In case of Spherical co-ordinates

$$dv = dx\,dy\,dz = (r^2 \sin\phi)\,dr\,d\theta\,d\phi$$

$$x = r \sin\phi \cos\theta = f(r, \phi, \theta)$$
$$y = r \sin\phi \sin\theta = g(r, \phi, \theta)$$
$$z = r \cos\phi = h(r, \phi, \theta)$$

Why do we do change of variables

1. To convert given domain into a nicer domain.

2. Changing variables sometimes makes the integration easy to compute.

Example 1

$$\frac{x^2}{} + \frac{y^2}{36} = 1 \quad \text{— eq}^n \text{ 1}$$

consider $x = \frac{u}{2}$ and $y = 3v$

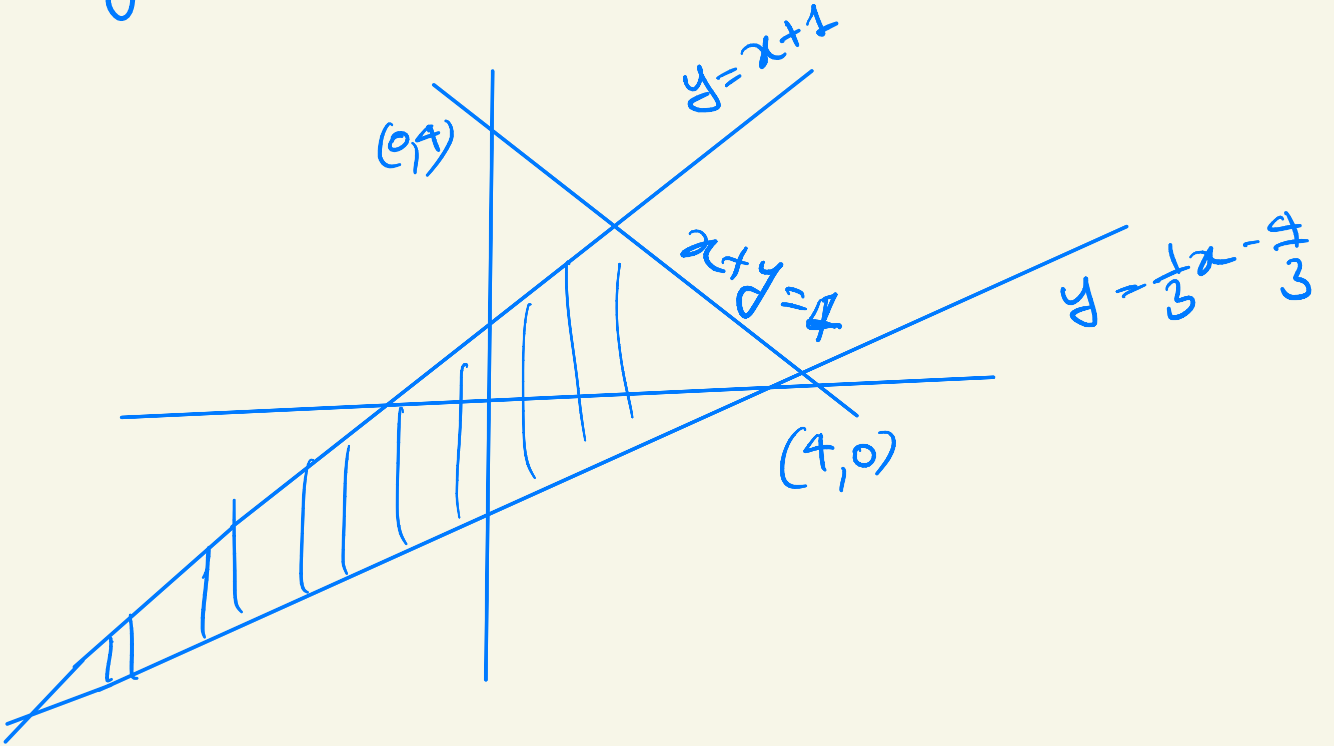
$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1$$

$$\frac{u^2}{4} + \frac{9v^2}{36} = 1$$

$$u^2 + v^2 = 4 \quad \text{— circle}$$

Example ②. Domain bounded by $x+y=4$, $y=x+1$ and $3y = x-4$.

Sol^n: →



Consider the transformation

$$x = \frac{1}{2}(u+v) \quad \text{and} \quad y = \frac{1}{2}(u-v)$$

eqn ① $\quad x + y = 4 \Rightarrow \frac{1}{2}(u+v) + \frac{1}{2}(u-v) = 4$

$$\Rightarrow \quad u = 4.$$

eqn ② $\quad y = x+1 \Rightarrow \frac{1}{2}(u-v) = \frac{1}{2}(u+v) + 1$

$$\Rightarrow \quad v = -1$$

eqn ③ $\quad y = \frac{x}{3} - \frac{4}{3} \Rightarrow \quad v = \frac{u}{2} + 2$ (see)

Suppose we have the transformation given by

$$x = f(u,v) \quad \text{and} \quad y = g(u,v)$$

Then the Jacobian of this transformation is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}_{2\times 2}$$

$$dx\,dy = |J|\,du\,dv.$$

Example ③ In polar co-ordinates we have

$x = r\cos\theta$ and $y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}_{2\times 2}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

$$dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta.$$

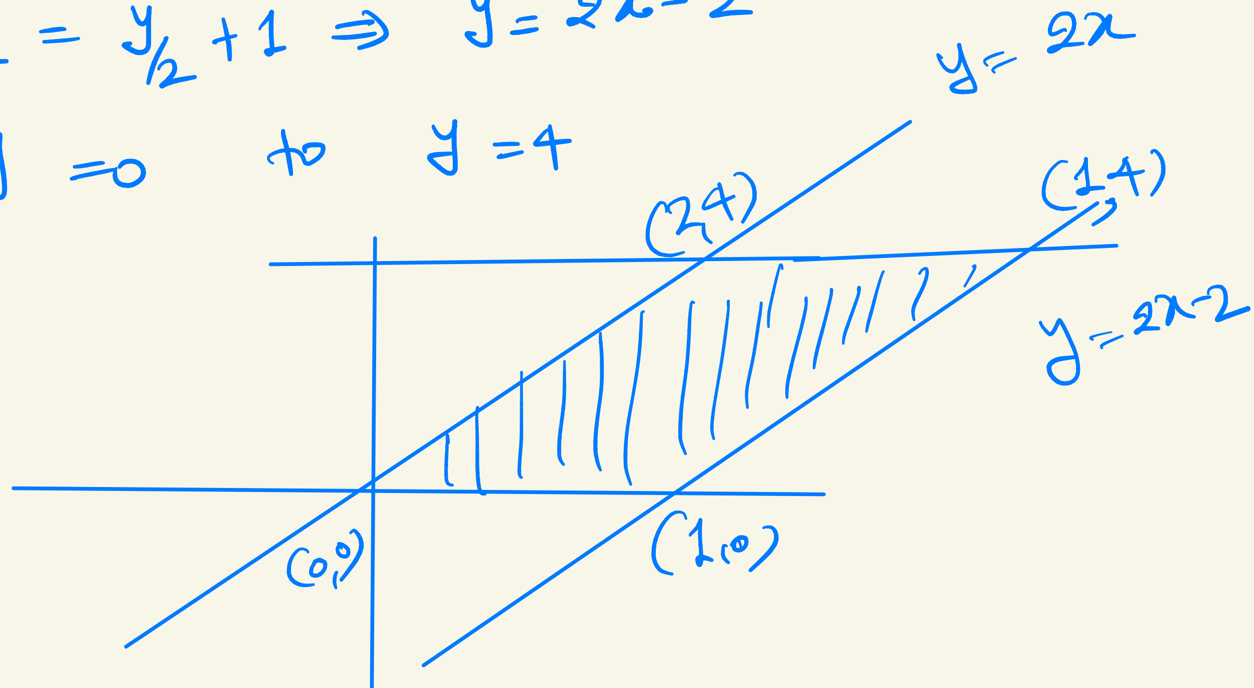
Example (4). Evaluate $\displaystyle\int_0^4 \int_{y/2}^{1+y/2} \frac{2x-y}{2} \, dA$.

Sol^n :→

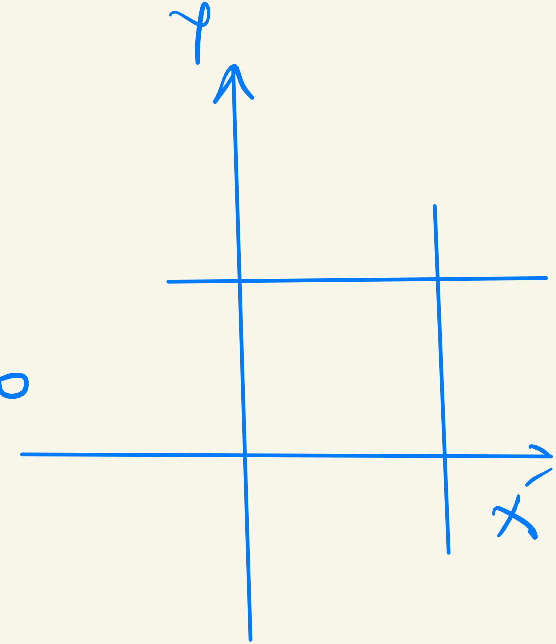
$x = y/2 \Rightarrow y = 2x$

$x = y/2 + 1 \Rightarrow y = 2x - 2$

$y = 0$ to $y = 4$



take the transformation $u = \frac{2x-y}{2}$ & $v = y/2$



eqn-① $y=0 \Rightarrow v=0$

eqn-② $y=4 \Rightarrow v=2$

eqn-③ $x = y/2 \Rightarrow 2x-y=0 \Rightarrow u=0$

eqn-④ $x = 1+y/2 \Rightarrow u=1$

$$\int_0^4 \int_{y/2}^{1+y/2} \frac{2x-y}{2}\, dA = \int_0^2 \int_0^1 u\, |J|\, du\, dv$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Note:

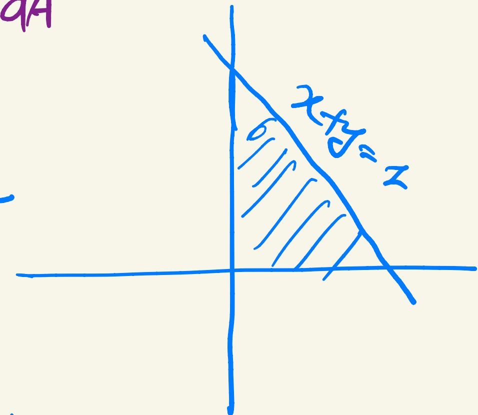
$$\frac{1}{J} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{1}{J} = \begin{vmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \Rightarrow J = 2$$

$$\int_0^2 \int_0^1 2u \, du \, dv = \text{compute.}$$

Example ⑤ $\displaystyle\int_0^1 \int_0^{1-x} \left(\sqrt{x+y}\right)(y-2x)^2 \, dA$

XY-plane



Sol^n :

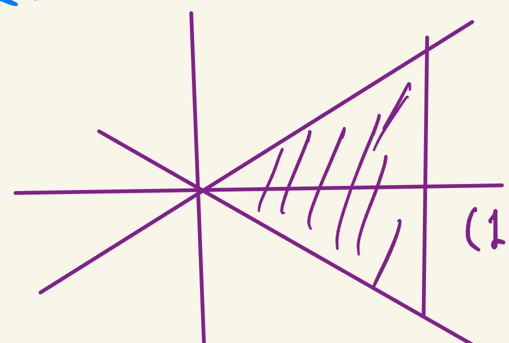
$u = x+y$ and $v = y - 2x$

Domain in XY-plane

$x = 0$, $y = 0$ and $y = 1-x$

$\Downarrow$ $\Downarrow$ $\Downarrow$

$u = v$ $v = -2u$ $u = 1$



$$I = \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dA = \int_0^1 \int_{-2u}^{u} \sqrt{u}\, v^2\, J\, du\, dv$$

$$\frac{1}{J} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 \Rightarrow J = \frac{1}{3}$$

$$I = \int_0^1 \int_{-2u}^{u} \frac{1}{3} \sqrt{u}\, v^2\, du\, dv$$

$$= \text{compute.}$$

In 3-dim, we assume

$x = f(u, v, w)$, $y = g(u, v, w)$ and $z = h(u, v, w)$.

Define

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}_{3\times 3}$$

$$dv = dx\,dy\,dz = |J|\,du\,dv\,dw.$$

Example ⑥ Consider the transformation

$x = r\cos\theta$, $y = r\sin\theta$ and $z = z$.

Sol^n.

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\cdot(r\cos^2\theta + r\sin^2\theta)$$

$$\Rightarrow r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

Example 7. Consider the following transformation

$$x = r\sin\phi\cos\theta,\quad y = r\sin\phi\sin\theta,\quad z = r\cos\phi$$

Sol^n.

$$J = \begin{vmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}_{3\times 3}$$

$$J = \cos\phi\left(-r^2\cos\phi\sin\phi\,\overbrace{(\cos^2\theta+\sin^2\theta)}^{=1}\right) - r\sin\phi\left(r\sin^2\phi\,\overbrace{(\cos^2\theta+\sin^2\theta)}^{=1}\right)$$

$$J = -r^2 \sin\phi \cos^2\phi - r^2 \sin^3\phi$$

$$= -r^2 \sin\phi \left(\cos^2\phi + \sin^2\phi\right)$$

$$= -r^2 \sin\phi$$

$$|J| = r^2 \sin\phi$$

$$dx\,dy\,dz = \left(r^2 \sin\phi\right) dr\,d\theta\,d\theta .$$

Example : 8. Evaluate $\iiint_\Omega (x^2y + 3xyz)\,dv$

where the domain Ω is given by

$$\Omega = \{(x, y, z) \mid 1 \le x \le 2,\ 0 \le xy \le 2,\ 0 \le z \le 1\}$$

Soln :- If we take the transformation

$$u = x, \quad v = xy \quad \text{and} \quad w = z$$

limits

$$1 \le u \le 2 \qquad 0 \le v \le 2 \qquad 0 \le w \le 1$$

$$f(x,y,z) = x^2y + 3xyz$$

$$= x(xy) + 3(xy)z$$

$$= uv + 3vw$$

$$\iiint_{\Omega} (x^2y + 3xyz)\, dv = \iiint_{\Omega} (uv + 3vw)\, J\, du\,dv\,dw = I$$

$$\frac{1}{J} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= x = u$$

$J = \frac{1}{u}$.

$$I = \int_1^2 \int_0^2 \int_0^1 (uv + 3vw) \frac{1}{u} \, du\, dv\, dw.$$

End