Dual vector space : Given a vector space $(V;F)$, consider the set of all <u>linear functionals</u>

$$f : V \to F$$

i.e. $f(|\alpha\rangle) \in F$ and

$$f(a|\alpha\rangle + b|\beta\rangle) = a f(|\alpha\rangle) + b f(|\beta\rangle) \qquad \{\text{Linearity}\}$$

The set of all such Linear functionals forms a vector space over field F.

- If f & g are two linear functions, then $af + bg$ is also a linear functional.
- The additive identity is the zero functional which maps all vector to $0 \in F$
  $$O(|\alpha\rangle) = 0 \quad \forall \; |\alpha\rangle \in V.$$
- The additive inverse of functional f is simply $(-1)f$ where -1 is the additive inverse of multiplicative identity in F.

* The vector space of these <u>linear functionals on (V,F)</u> is called the <u>Dual vector space of (V,F)</u> and represented by V^* (or V^D).

Dimension & Basis for V^* :

1. $\text{Dim}(V^*) = \text{Dim}(V) = n$
2. Let $B = \{|\beta_i\rangle\}_{i=1,\cdots n}$ be a basis for V, then

A choice of Basis for V^* is $\{\bar{\beta}_i\}_{i=1,2,\cdots n}$

such that $\bar{\beta}_i(|\beta_j\rangle) = \delta_{ij}$

<u>Linear independence of $\{\bar{\beta}_i\}$</u> :

Let $0 = a_i \bar{\beta}_i \Rightarrow 0 = a_i \bar{\beta}_i(|\beta_j\rangle) \quad \forall \; j = 1, \cdots n$

$$= a_i \delta_{ij} = a_j$$

$$\Rightarrow a_j = 0 \;\; \forall \; j = 1, 2, \cdots n.$$

Completeness: Let f be an arbitrary linear functional such that $f(|\beta_i\rangle) = f_i$ then

$$\boxed{f = \sum_{i=1}^{n} f_i \bar{\beta}_i}$$

$$f(|\beta_j\rangle) = \sum_{i=1}^{n} f_i \underbrace{\bar{\beta}_i(|\beta_j\rangle)} = \sum_{i=1}^{n} f_i \delta_{ij} = f_j$$

$$\Rightarrow \boxed{\dim(V^*) = n = \dim(V)}$$

Since the dual vector space has same dimension as V, one can define Bijective (one to one & onto) maps from $V \longrightarrow V^*$ {Can be done for any V_F, $\tilde{V}_F$ w/ $\dim(V) = \dim(\tilde{V})$}

We will define here a particular map using the scalar product which defines the dual of a vector in V.

Dual vector: The dual of a vector $|\beta\rangle \in V$ is an element $\bar{\beta} \in V^*$ satisfying

$$\bar{\beta}(|\alpha\rangle) = \langle\beta|\alpha\rangle \quad \forall \ |\alpha\rangle \in V$$

$\longrightarrow$ In terms of the Basis $\{\bar{\beta}_i\}$ of V^* satisfying

$$\bar{\beta}_i(|\beta_j\rangle) = \delta_{ij}$$

, the dual of a vector $\boxed{|\beta\rangle = b_i|\beta_i\rangle}$ is simply $\boxed{\bar{\beta} = b_i^* \bar{\beta}_i}$.

$$\bar{\beta}(|\alpha\rangle) = b_i^* \bar{\beta}_i(a_j|\beta_j\rangle) = b_i^* a_j \, \bar{\beta}_i(|\beta_j\rangle)$$
$$= b_i^* a_j \delta_{ij} = b_i^* a_i$$
$$= \langle\beta|\alpha\rangle.$$

— We will often use the notation $\langle\beta|$ to represent $\bar{\beta}$, the dual of $|\beta\rangle$.

$$|\beta\rangle = b_i|\beta_i\rangle \longleftrightarrow \langle\beta| \equiv b_i^* \underbrace{\langle\beta_i|}_{\equiv \bar{\beta}_i}$$

Linear operators (transformations) on vector spaces:

A Linear operator on a vector space (V, F) to vector space (W, F) is a map

$$A : V \longrightarrow W$$

i.e. $\forall\ |\alpha\rangle \in V$, $(A|\alpha\rangle) \in W$ such that

- $A(a|\alpha\rangle) = a A|\alpha\rangle \qquad \forall\ a \in F$ & $|\alpha\rangle \in V$
- $A(|\alpha\rangle + |\beta\rangle) = A|\alpha\rangle + A|\beta\rangle \qquad \forall\ |\alpha\rangle, |\beta\rangle \in V$

V is reffered to as the Domain of A & the subset of vectors in W which are obtained action of A on some vector in V (i.e. image of V under A) is referred to as the Range of A.

Ex: Show that the Range of a linear operator is a subspace of W.

Composition & Algebra of linear operators:

Though Linear operators can be defined between two arbitrary vector spaces, Let us consider the linear operators from a vector space to itself.

Consider two linear operator $A, B : V_F \rightarrow V_F$. One can naturally define the following operation

Addition : $(A+B)|\alpha\rangle = A|\alpha\rangle + B|\alpha\rangle$

Multiplication: $(A.B)|\alpha\rangle = A(B|\alpha\rangle)$ $\quad \forall\ |\alpha\rangle \in V$

Ex: Note that Both $(A+B)$ & $A.B$ are Linear operators on V.

– Further note that multiplication is distributive over addition

i.e. $\qquad C.(A+B) = C.A + C.B$

— Moreover, $\exists$ Identity (I) & Zero (O) operators

$$I|\alpha\rangle = |\alpha\rangle \quad \forall \; |\alpha\rangle \in V$$
$$O|\alpha\rangle = |0\rangle \quad \forall \; |\alpha\rangle \in V$$

which are multiplicative & additive identities respectively.

★ The space of all linear operators on a vector space V_F itself forms a vector space w.r.t the Addition operator defined above.

Ex. What is the dimension of this vector space of linear transformations on V_F?