# LIMIT & CONTINUITY

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LECTURE 1 : Limit of functions

LECTURE 2 : Limit of functions

LECTURE 3 : Continuity

LECTURE 4 : Continuity

LECTURE 5 : Uniform Continuity.

**Definition :-** Let $f$ be a real valued function defined on $(a,b)$ except possibly at $c \in (a,b)$. We say that limit of $f$ is $L \in \mathbb{R}$ as $x$ approaches to $c$ (or, at $x=c$), written $\lim_{x \to c} f(x) = L$ if for every sequence $(x_n)_{n=1}^{\infty}$ in $(a,b)$ with $x_n \neq c$ $\forall n$ and $x_n \longrightarrow c$, we have $f(x_n) \longrightarrow L$ as $n \longrightarrow \infty$.

**Generalization :-**

(i) Let $c \in \mathbb{R}$ be such that $f$ is defined on $(a,c)$ but $f$ is not defined on $(c,b)$ for some $a,b \in \mathbb{R}$. We say $\lim_{x \to c} f(x) = L$ if for every sequence $(x_n)_{n=1}^{\infty}$ in $(a,c)$ with $x_n \to c$, we have $f(x_n) \to L$ as $n \to \infty$.

(ii) Let $c \in \mathbb{R}$ be such that $f$ is defined on $(c,b)$ but $f$ is not defined on $(a,b)$ for some $a,b \in \mathbb{R}$. We say $\lim_{x \to c} f(x) = L$ if for every sequence $(x_n)_{n=1}^{\infty}$ in $(c,b)$ with $x_n \to c$, we have $f(x_n) \to L$ as $n \to \infty$.

Note :-

(i) The function may not be defined at 'c'.

(ii) Even if $f(c)$ is defined, $f(c)$ may not be equal to $L$.

(iii) If $\lim\limits_{x \to c} f(x) = L \in \mathbb{R}$ then we say the limit of $f$ at $x = c$ exists and if finite.

By 'a real valued function $f$' we mean a function $f : A \to \mathbb{R}$ where $A$ is an interval $(a, b)$, $(a, \infty)$, $(-\infty, b)$, $(a, b]$, $(-\infty, \infty) = \mathbb{R}$, etc. (union of intervals)

In this case a limit point of an interval $(a,b)$ is either a point of the interval or it is a boudary point of the interval.

Since limit of a sequence is unique, by definition it follows that limit of a function (if exists) is unique.

Example :- (i) $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$

$\lim_{x \to 2} f(x) = ?$

Let $(x_n)$ be a sequence in $\mathbb{R}$ s.t. $x_n \neq 2$ & $x_n \to 2$

$\Rightarrow x_n^2 \to 2^2 = 4$

$\Rightarrow f(x_n) \to 4$

$\therefore \lim_{x \to 2} f(x) = 4 \neq f(2)$

(ii) $f : (0,1) \to \mathbb{R}$ $\quad f(x) = x$

$\lim_{x \to 0} f(x) = ?$

$\therefore (x_n)$ in $(0,1)$ s.t. $x_n \to 0 \Rightarrow f(x_n) \to 0$

$\therefore \lim_{x \to 0} f(x) = 0$ but $f(0)$ is not defined.

**Limit at $\infty$** :- Let $f:(a,\infty) \to \mathbb{R}$ be a function. We say the limit of the function $f(x)$ is $L \in \mathbb{R}$ as $x$ approaches to $\infty$, written $\lim_{x\to\infty} f(x) = L$ if for any sequence $(x_n)$ in $(a,\infty)$ with $x_n \to \infty$, we have $\lim_{n\to\infty} f(x_n) = L$.

[Similarly, we can define limit at $-\infty$.]

**Infinite limit** :- Let $A \subseteq \mathbb{R}$ be any set, and '$a$' be a limit point of $A$. Let $f: A \to \mathbb{R}$ be a function. We say the limit of $f(x)$ is $\infty$ as $x$ approaches to '$a$' written $\lim_{x\to a} f(x) = \infty$ if for any sequence $(x_n)$ in $A$ with $x_n \neq a$ & $x_n \to a$, we have $f(x_n) \to \infty$

**Examples** :-

(i) $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$ $\quad f(x) = \frac{1}{x}$.

Find $\lim_{x\to 0} f(x)$.

Take $x_n = \frac{1}{n}$ then $f(x_n) = n$

Here $x_n \to 0$ & $f(x_n) \to \infty$

If we take, $y_n = -\frac{1}{n}$ then $f(x_n) = -n$

Here $y_n \to 0$ & $f(y_n) \to -\infty$

$\therefore \lim_{x \to 0} f(x)$ does not exist.

(ii) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $\quad f(x) = \frac{1}{x^2}$, Find $\lim_{x \to 0} f(x)$

$\therefore$ For any sequence $(x_n)$ in $\mathbb{R} \setminus \{0\}$ with $x_n \to 0$

$\Rightarrow x_n^2 \to 0$

$\Rightarrow \frac{1}{x_n^2} \to \infty$ as $n \to \infty$

$\Rightarrow \lim_{n \to \infty} f(x_n) = \infty$ or $f(x_n) \to \infty$

$\therefore \lim_{x \to 0} f(x) = \infty.$

(iii) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $f(x) = \sin \frac{1}{x^2}$. Find $\lim_{x \to 0} f(x)$.

$\therefore$ Let $x_n = \frac{1}{\sqrt{2\pi n}}$ & $y_n = \frac{1}{\sqrt{2\pi n + \frac{\pi}{2}}}$

$\therefore$ Then $x_n \to 0$ & $y_n \to 0$

$f(x_n) = \sin(2\pi n) = 0 \to 0$

$f(y_n) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1 \to 1$

Thus $\lim_{x \to 0} f(x)$ does not exist.

(iv) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $f(x) = x \sin \frac{1}{x^2}$. Find $\lim_{x \to 0} f(x)$

Note that, $-1 \leq \sin \frac{1}{x^2} \leq 1$ $\forall\, x \in \mathbb{R} \setminus \{0\}$

$\Rightarrow 0 \leq \left|\sin \frac{1}{x^2}\right| \leq 1$

$\Rightarrow 0 \leq \left|x \sin \frac{1}{x^2}\right| \leq |x|$

Let $(x_n)$ be a sequence in $\mathbb{R} \setminus \{0\}$ s.t. $x_n \longrightarrow 0$

$$\Rightarrow \quad 0 \leq \left| x_n \sin\left(\frac{1}{x_n^2}\right) \right| \leq |x_n|$$

$$\Rightarrow \quad 0 \leq | f(x_n) | \leq \underset{\substack{\downarrow \\ 0}}{|x_n|}$$

Since $|x_n| \longrightarrow 0$ as $n \to \infty$, we have $|f(x_n)| \longrightarrow 0$ as $n \to \infty$

$\therefore \quad f(x_n) \longrightarrow 0$ as $n \to \infty$

$\therefore \quad \lim_{x \to 0} f(x) = 0.$

(v) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = [x]$ or $\lfloor x \rfloor$

$[x]$ or $\lfloor x \rfloor$ is the greatest integer $\leq x$.

$\therefore$ $f(x) = n$ if $n \leq x < n+1$, $n \in \mathbb{Z}$.

Find $\lim\limits_{x \to 1} f(x)$.

Solⁿ:- Let $x_n = 1 + \frac{1}{n+1}$. Then $x_n \to 1$ as $n \to \infty$.

$f(x_n) = [x_n] = 1 \to 1$ as $n \to \infty$.

If we chose, $y_n = 1 - \frac{1}{n+1}$. Then $y_n \to 1$ as $n \to \infty$

$f(y_n) = [y_n] = 0 \to 0$ as $n \to \infty$

$\therefore$ $\lim\limits_{x \to 1} f(x)$ does not exist.

**One sided limit :-** For simplicity, assume $f$ is a real valued function defined on an open interval $(c, d)$ for some $c, d \in \mathbb{R}$. Then the right hand limit of $f(x)$ at $x = c$ is $L$ if for any sequence $(x_n)$ in $(c, d)$ with $x_n \to c$ implies $f(x_n) \to L$.

We write $\lim\limits_{x \to c+} f(x) = L$.

Let $f$ be a real valued function defined on $(b, c)$ for some $b, c \in \mathbb{R}$. Then we define the left hand limit of $f(x)$ at $x = c$ is $L$ if for any sequence $(x_n)$ in $(b, c)$ with $x_n \to c$ as $n \to \infty$ implies $f(x_n) \to L$ as $n \to \infty$.

We write $\lim\limits_{x \to c-} f(x) = L$.

Example :- $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = [x]$.

$$\lim_{x \to n+} f(x) \neq \lim_{x \to n-} f(x) \text{ for } n \in \mathbb{Z}.$$

Solⁿ :- $f(x) = n$ if $n \le x < n+1$ $\forall\, n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$. Let $x_m \in (\underset{c}{n}, \underset{d}{n+1})$ be such that $x_m \to n$ as $m \to \infty$.

$$f(x_m) = [x_m] = n \to n \text{ as } m \to \infty$$

$$\therefore \lim_{x \to n+} f(x) = n$$

Let $y_m \in (\underset{b}{n-1}, \underset{c}{n})$ be such that $y_m \to n$ as $m \to \infty$

$$f(y_m) = [y_m] = n-1 \longrightarrow n-1 \text{ as } m \to \infty$$

$$\therefore \lim_{x \to n-} f(x) = n-1$$