MTL106 Tutorial 9 Solutions

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Problem 1. Consider a CTMC with $Q = \begin{pmatrix} -5 & 3 & 2 \\ 1 & -3 & 2 \\ 2 & 4 & -6 \end{pmatrix}$ and initial distribution $(0, 1, 0)$. Find $P(\tau > t)$ where τ denotes the first transition time of the Markov chain.

Solution. The holding time of the second state is distributed as Exponential$(-q_{22})$. Thus, the probability $P(\tau > t) = e^{-3t}$. □

Problem 2. Suppose the arrival at a counter form a time homogeneous Poisson process with parameter λ and suppose each arrival is of type A or of type B with respective probabilities p and $1 - p$. Let $X(t)$ be the type of the last arrival before time t. Write down the forward Kolmogorov equations for the stochastic process $\{X(t), t \geq 0\}$. Find the time dependent system state probabilities.

Solution. It can be written as a CTMC on space $\{A, B\}$ with generator,

$$Q = \begin{pmatrix} -(1-p)\lambda & (1-p)\lambda \\ p\lambda & -p\lambda \end{pmatrix}.$$

The forward Kolmogorov equation is

$$P'(t) = P(t)Q.$$

Looking at the value of $P_{A,A}(t)$, we see that

$$P'_{A,A}(t) = -(1-p)\lambda P_{A,A}(t) + p\lambda P_{A,B}(t).$$

Since $P_{A,A}(t) + P_{A,B}(t) = 1$,

$$\begin{aligned}
P'_{A,A}(t) &= -(1-p)\lambda P_{A,A}(t) + p\lambda(1 - P_{A,A}(t)) \\
\implies P'_{A,A}(t) &= p\lambda + \lambda P_{A,A}(t) \\
\implies \frac{d(e^{\lambda t} P_{A,A}(t))}{dt} &= p\lambda e^{\lambda t} \\
\implies e^{\lambda t} P_{A,A}(t) - 1 &= p(e^{\lambda t} - 1) \\
\implies P_{A,A}(t) &= p + (1-p)e^{-\lambda t} \\
\implies P(t) &= \begin{pmatrix} p + (1-p)e^{-\lambda t} & (1-p)(1-e^{-\lambda t}) \\ p(1-e^{-\lambda t}) & (1-p) + pe^{-\lambda t} \end{pmatrix}.
\end{aligned}$$

□

Problem 3. (a) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. For any $s, t \geq 0$, find

$$P(N(t+s) - N(t) = k \mid N(u), 0 \leq u \leq t).$$

(b) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate 5. Compute

$$P(N(2.5) = 15, N(3.7) = 21, N(4.3) = 21).$$

Solution. Note that Poisson processes have independent increments so that for disjoint intervals $[a, b)$ and $[c, d)$,

$$P(N(b) - N(a) = x, N(d) - N(c) = y) = P(N(b) - N(a) = x)P(N(d) - N(c) = y).$$

(a) For any $u \in [0, t]$ the intervals $[0, u)$ and $[t, t+s)$ are disjoint

$$\begin{aligned} P(N(t+s) - N(t) = k \mid N(u)) &= P(N(t+s) - N(t) = k \mid N(u) - N(0)) \\ &= P(N(t+s) - N(t) = k) \\ &= e^{-\lambda s} \cdot \frac{(\lambda s)^k}{k!}. \end{aligned}$$

(b)
$$\begin{aligned} &P(N(2.5) = 15, N(3.7) = 21, N(4.3) = 21) \\ =&P(N(2.5) - N(0) = 15, N(3.7) - N(2.5) = 6, N(4.3) - N(3.7) = 0) \\ =&P(N(2.5) - N(0) = 15)P(N(3.7) - N(2.5) = 6)P(N(4.3) - N(3.7) = 0) \\ =&P(N(2.5) - N(0) = 15)P(N(3.7) - N(2.5) = 6)P(N(4.3) - N(3.7) = 0) \\ =&e^{-5\times 2.5}\frac{(5\times 2.5)^{15}}{15!} \cdot e^{-5\times 1.2}\frac{(5\times 1.2)^{6}}{6!} \cdot e^{-5\times 0.6}\frac{(5\times 0.6)^{0}}{0!}. \end{aligned}$$ □

Problem 4. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Let T_1 denote the time of the first event and T_n denote the time between $(n-1)$-th and n-th events. Find $P(T_2 > t \mid T_1 = s)$.

Solution. Since the inter-arrival times in a Poisson process are exponentially distributed i.i.d. variables,

$$P(T_2 > t \mid T_1 = s) = P(T_2 > t) = e^{-\lambda t}.$$ □

Problem 5. Let $\{X(t), t \geq 0\}$ be a Poisson process with parameter λ and $X(0) = j$ where j is a positive integer. Consider the random variable $T_j = \inf\{t : X(t) = j+1\}$, i.e., T_j is the time of occurrence of the first jump after the j-th jump, $j = 1, 2, \ldots$

(a) Find the distribution of T_1.

(b) Find the joint distribution of $(T_{2016}, T_{2017}, T_{2018})$.

Solution. The interarrival time in a Poisson process is exponentially distributed.

(a) So, $T_1 \sim$ Exponential(λ).

(b) The three variables are iid so simply take the product of their pdfs. □

Problem 6. Assume the life times of $N = 400$ soldiers are iid following an exponential distribution with parameter μ, then the process of the number of surviving soldiers by time t, $\{X(t), t \geq 0\}$, is a pure death process with death rates $\mu_i = i\mu$, $i = 1, 2, \ldots, N$. Assume that, $X(0) = N$.

(a) Find $P(X(t) = N-1)$.

(b) Let S_N be the time of the death of the last member of the population, i.e., S_N is the time to extinction. Find $\mathbb{E}[S_N]$.

Solution. (a) From problem 17, we see that $X(t)$ has distribution $B(N, e^{-\mu t})$ hence,

$$P(X(t) = N-1) = \binom{N}{N-1}(e^{-\lambda t})^{N-1}(1-e^{-\lambda t}).$$

(b) Let T_i be the time required to go from state $i \to i-1$. Since it is a pure death process, the transition $i \to i+1$ is not possible and T_i is precisely the holding time of state i. Therefore, $T_i \sim \text{Exponential}(i\mu)$. Now

$$S_N = \sum_{i=1}^{N} T_i$$

as S_N is nothing but the time required to go $N \to N-1 \to \cdots \to 1 \to 0$. Therefore,

$$\mathbb{E}[S_N] = \sum_{i=1}^{N} \mathbb{E}[T_i] = \frac{1}{\mu}\sum_{i=1}^{N}\frac{1}{i}.$$

The official solution given is

$$\frac{1}{\mu}\sum_{i=1}^{N}\frac{(-1)^{i-1}}{i}\binom{N}{i}.$$

This is equivalent to the solution I got as

$$\begin{aligned}
\sum_{i=1}^{N}\frac{(-1)^{i-1}}{i}\binom{N}{i} &= \sum_{i=1}^{N}\binom{N}{i}\int_0^1 (-x)^{i-1}\,dx \\
&= \int_0^1 \frac{1-\sum_{i=0}^{N}\binom{N}{i}(-x)^i}{x}\,dx \\
&= \int_0^1 \frac{1-(1-x)^N}{x}\,dx \\
&= \int_0^1 \frac{1-u^N}{1-u}\,du \\
&= \int_0^1 \sum_{i=1}^{N} u^{i-1}\,du \\
&= \sum_{i=1}^{N}\frac{1}{i}.
\end{aligned}$$

□

Problem 7. Consider a population, denoted by $\{X(t), t \geq 0\}$, in which each individual gives birth after an exponential time of parameter λ, all independently. Suppose $X(0) = 1$. Then, find the mean population size at any $t > 0$.

Solution 1. We model it as a CTMC, the states are $\{1, 2, \dots\}$. The generator matrix is given by

$$q_{i,i} = -i\lambda, \text{ and } q_{i,i+1} = i\lambda.$$

We can do this as if X_j be the time until the the j-th person gives birth then the time required to move from state $i \to i+1$ is

$$\min(X_1, X_2, \dots, X_i).$$

Since $X_j \sim \text{Exponential}(\lambda)$ and they are independent, $\min(X_1, X_2, \ldots, X_i) \sim \text{Exponential}(i\lambda)$ and hence the rate of movement from state $i \to i+1$ is $i\lambda$.

We now write the forward Kolmogorov equation,

$$p'_{i,j}(t) = p_{i,j}(t)q_{j,j} + p_{i,j-1}(t)q_{j-1,j} = \lambda(-jp_{i,j}(t) + (j-1)p_{i,j-1}(t)).$$

We wish to find $p_{1,i}(t)$, simply write it as $p_i(t)$ for convenience. We inductively prove that

$$p_j(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}.$$

For the base case,

$$p'_1(t) = -\lambda p_1(t) \implies p_1(t) = e^{-\lambda t}.$$

For higher values,

$$\begin{aligned}
& p'_j(t) = \lambda(-jp_j(t) + (j-1)p_{j-1}(t)) \\
\iff & e^{j\lambda t}(p'_j(t) + j\lambda p_j(t)) = (j-1)\lambda e^{j\lambda t} p_{j-1}(t) \\
\iff & \frac{d(e^{j\lambda t}p_j(t))}{dt} = (j-1)\lambda e^{j\lambda t} p_{j-1}(t)
\end{aligned}$$

$$\begin{aligned}
\implies p_j(t) &= (j-1)\lambda e^{-j\lambda t} \int_0^t e^{j\lambda s} p_{j-1}(s)\, ds \\
&= (j-1)\lambda e^{-j\lambda t} \int_0^t e^{(j-1)\lambda s}(1 - e^{-\lambda s})^{j-2}\, ds \\
&= (j-1)e^{-j\lambda t} \int_1^{e^{\lambda t}} (u-1)^{j-2}\, du \\
&= e^{-\lambda t}(1 - e^{-\lambda t})^{j-1},
\end{aligned}$$

where we use the substitution $u = e^{\lambda s}$.

Now since $p_j(t) = P(X(t) = j)$, we see that $X(t)$ is distributed geometrically with parameter $e^{-\lambda t}$. Therefore, the mean is given by $e^{\lambda t}$. $\square$

Solution 2. Similar to the previous solution, we model it as a CTMC. Let T_i be the time spent in state i, as described in the previous solution $T_i \sim \text{Exponential}(i\lambda)$.

We see that

$$X(t) \leq N \iff \sum_{i=1}^{N} T_i \leq t.$$

We induct to prove that

$$P\left(\sum_{i=1}^{N} T_i \leq t\right) = (1 - e^{-\lambda t})^N$$

for $t \geq 0$.

The base case $N = 0$ trivially follows, for higher values,

$$
\begin{aligned}
P\left(\sum_{i=1}^{N} T_i \le t\right) &= \int_0^\infty P\left(\sum_{i=1}^{N-1} T_i \le t - x \mid T_N = x\right) f_{X_N}(x)\, dx \\
&= \int_0^\infty P\left(\sum_{i=1}^{N-1} T_i \le t - x\right) f_{X_N}(x)\, dx \\
&= \int_0^t (1 - e^{-\lambda(t-x)})^{N-1} \times N\lambda e^{-N\lambda x}\, dx \\
&= N \int_{e^{-\lambda t}}^1 (u - e^{-\lambda t})^{N-1}\, du \\
&= (1 - e^{-\lambda t})^N
\end{aligned}
$$

where we use the substitution $u = e^{-\lambda x}$.

Now,

$$P(X(t) = j) = P(X(t) \le j) - P(X(t) \le j - 1) = e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}$$

and we get that the mean is $e^{\lambda t}$. □

Solution 3. Model it as a CTMC and define T_i as in the previous solution. We combinatorially find the CDF of $\sum_{i=1}^N T_i$ using the fact that $T_i \sim \text{Exponential}(i\lambda)$.

From the argument in problem 6(b), we can view $\sum_{i=1}^N T_i$ as having the same distribution as the last extinction time for a population of size N in where each component's time to death is independent of each other and is distributed as Exponential(λ), that is we can write

$$\sum_{i=1}^{N} T_i = \max_{i=1}^{N}(S_i)$$

where each S_i represents the i-th component's extinction time, and thus S_i are iid each with distribution Exponential(λ).

Now we see that

$$
\begin{aligned}
P\left(\sum_{i=1}^{N} T_i \le t\right) &= P(\max_{i=1}^{N}(S_i) \le t) \\
&= P(S_1 \le t, S_2 \le t, \ldots, S_N \le t) \\
&= \prod_{i=1}^{N} P(S_i \le t) \\
&= (1 - e^{-\lambda t})^N.
\end{aligned}
$$

From the previous solution, we see that the mean is $e^{\lambda t}$. □

Solution 4. Let $F(t) = \mathbb{E}[X(t)]$ be the mean population at time t. Using the notation and result of problem 1,

$$p_j'(t) = -j\lambda p_j(t) + (j-1)\lambda p_{j-1}(t).$$

Now,

$$
\begin{aligned}
F(t) &= \sum_{j\geq 1} jP(X(t)=j) = \sum_{j\geq 1} jp_j(t) \\
\implies F'(t) &= \sum_{j\geq 1} jp_j'(t) \\
&= \sum_{j\geq 1} -j^2\lambda p_j(t) + j(j-1)\lambda p_{j-1}(t) \\
&= -\lambda\mathbb{E}[X(t)^2] + \lambda\sum_{j\geq 1}((j-1)^2+(j-1))p_{j-1}(t) \\
&= -\lambda\mathbb{E}[X(t)^2] + \lambda\mathbb{E}[X(t)^2] + \lambda\mathbb{E}[X(t)] \\
&= \lambda F(t) \\
\implies \mathbb{E}[X(t)] = F(t) &= e^{\lambda t}
\end{aligned}
$$

as $F(0)=1$. □

Problem 8. A dental surgery has two operation rooms. The service times are assumed to be independent, exponentially distributed with mean 15 minutes. Mr. Ram arrives when both operation rooms are empty. Mr. Rajesh arrives 10 minutes later while Ram is still under medical treatment. Another 20 minutes later Mr. Ajit arrives and both Ram and Rajesh are still under treatment. No other patient arrives during this 30 minute interval.

(a) What is the probability that the medical treatment will be completed for Ajit before Ram?

(b) Derive the distribution function of the waiting time in the system for Ajit?

Solution. Let A, B, C be the time in minutes spent be Ram, Rajesh and Ajit in the operation room, respectively. We know that A, B, C are iid each with distribution Exponential(1/15).

The condition that both Ram and Rajesh are still under treatment is equivalent to conditionining on $(A > 30, B > 20)$.

Let X, Y be the time spent under treatment by Ram and Rajesh after Ajit's arrival under the assumption that they are under treatment when Ajit arrived. That is $X = (A-30) \mid (A > 30)$ and $Y = (B-20) \mid (B > 20)$. By the memoryless property of exponential distribution, we see that $X, Y \sim$ Exponential(1/15).

(a) Ajit will be treated before Ram if $C + Y < X$. Thus, the probability is given by

$$
\begin{aligned}
P(X > C+Y) &= P(X > C+Y \mid X > Y)P(X > Y) + P(X > C+Y \mid X \leq Y)P(X \leq Y) \\
&= P(X > C+Y \mid X > Y)P(X > Y) \text{ as } X > C+Y \text{ is not possible if } X \leq Y \\
&= P(X > C)P(X > Y) \\
&= \tfrac{1}{2}\cdot\tfrac{1}{2} = 1/4.
\end{aligned}
$$

We get that $P(X > C) = P(X > Y) = \frac{1}{2}$ as X, Y, C are iid.

(b) The time spent waiting by Ajit is $\min(X, Y)$ as he occupies the spot as soon as one of Ram and Rajesh leaves. Thus, its distribution is Exponential(2/15) as minimum of independent exponential rvs is an exponential rv. □

Remark. In the first part I assumed that memorylessness of exponential distribution,

$$P(X > a+b \mid X > a) = P(X > b)$$

holds when a, b are random variables.

We have the result that if $X \sim \text{Exponential}(\lambda)$, A and B are random variables such that X, A, B are all independent then

$$P(X > A + B) = P(X > A)P(X > B).$$

If we know that $A \geq 0$, this tells us that

$$P(X > A + B \mid X > A) = P(X > B).$$

For a proof,

$$\begin{aligned}
P(X > A + B) &= \int\int P(X > A + B \mid A = a, B = b)\, dF_A(a)\, dF_B(b) \\
&= \int\int P(X > a + b)\, dF_A(a)\, dF_B(b) \\
&= \int\int e^{-(a+b)\lambda}\, dF_A(a)\, dF_B(b) \\
&= \left(\int e^{-a\lambda}\, dF_A(a)\right)\left(\int e^{-b\lambda}\, dF_B(b)\right) \\
&= \left(\int P(X > A \mid A = a)\, dF_A(a)\right)\left(\int P(X > B \mid B = b)\, dF_B(b)\right) \\
&= P(X > A)P(X > B).
\end{aligned}$$

Problem 9. Four workers share an office that contains four telephones. At any time, each worker is either 'working' or 'on the phone'. Each 'working' period of worker i lasts for an exponentially distributed time with rate λ_i, and each 'on the phone' period lasts for an exponentially distributed time with rate μ_i, $i = 1, 2, 3, 4$. Let $X_i(t)$ equal 1 if worker i is working at time t, and let it be 0 otherwise. Let $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$.

(a) Argue that $\{X(t), t \geq 0\}$ is a continuous-time Markov chain and give its infinitesimal rates.

(b) What proportion of time are all workers 'working'?

Solution. (a) $\{X_i(t), t \geq 0\}$ is a Markov chain as the holding time of each state is exponential. Note that

$$P(X_i(t) \mid X_i(s), X_i(s')) = P(X_i(t) \mid X_i(s)).$$

for $s' < s < t$. By the independence of X_i we see that

$$P(X(t) \mid X(s), X(s')) = \prod_{i=1}^{4} P(X_i(t) \mid X_i(s), X_i(s')) = \prod_{i=1}^{4} P(X_i(t) \mid X_i(s)) = P(X(t) \mid X(s))$$

for $s' < s < t$, and hence $\{X(t), t \geq 0\}$ is a CTMC.

The rate is given by

$$q_{(0,x,y,z),(1,x,y,z)} = \mu_1 \text{ and } q_{(1,x,y,z),(0,x,y,z)} = \lambda_1$$

and 0 for all other values except $q_{v,v}$. Similar rates are true for others.

(b) Looking at the CTMC, $\{X_i(t), t \geq 0\}$. We see that it has the generator,

$$Q = \begin{pmatrix} -\mu_i & \mu_i \\ \lambda_i & -\lambda_i \end{pmatrix}.$$

The limiting distribution satisfies $\pi Q = 0$. Solving this, we see that

$$\pi_0 = \frac{\lambda_i}{\lambda_i + \mu_i} \text{ and } \pi_1 = \frac{\mu_i}{\lambda_i + \mu_i}.$$

Now the proportion of workers working at a large time t is

$$\begin{aligned} P(X(t) = (1,1,1,1)) &\approx P(X(\infty) = (1,1,1,1)) \\ &= \prod_{i=1}^{4} P(X_i(\infty) = 1) \\ &= \prod_{i=1}^{4} \left(\frac{\mu_i}{\lambda_i + \mu_i} \right). \end{aligned}$$

□

Problem 10. Suppose that you arrive at a single-teller bank to find seven other customers in the bank, one being served (First Come First service basis) and the other six waiting in line. You join the end of the line. Assume that, service times are independent and exponential distributed with rate μ. Model this situation as a birth and death process.

(a) What is the distribution of time spend by you in the bank?

(b) What is the expected amount of time you will spend in the bank?

Solution. Let T_i for $1 \leq i \leq 8$, be the time the i-th person in line spends being serviced. It's given that $T_i \sim \text{Exponential}(\mu)$.

(a) The time spent in the bank is $\sum_{i=1}^{8} T_i \sim \text{Gamma}(8, \mu)$.

(b) The expected time is $\mathbb{E}[\sum_{i=1}^{8} T_i] = \sum_{i=1}^{8} \mathbb{E}[T_i] = 8/\mu$. □

Problem 11. Consider a Poisson process with parameter λ. Let T_1 be the time of occurrence of the first event and let $N(T_1)$ denote the number of events occurred in the next T_1 units of time. Find the mean and variance of $N(T_1)T_1$.

Solution. $N(t) \sim \text{Poisson}(\lambda t)$ and $T_1 \sim \text{Exponential}(\lambda)$, therefore the mean

$$\begin{aligned} \mathbb{E}[N(T_1)T_1] &= \mathbb{E}[\mathbb{E}[N(T_1)T_1 \mid T_1]] \\ &= \mathbb{E}[T_1 \mathbb{E}[N(T_1) \mid T_1]] \\ &= \lambda \mathbb{E}[T_1^2] \\ &= \frac{2}{\lambda}. \end{aligned}$$

For the variance, we find the second moment,

$$\begin{aligned} \mathbb{E}[T_1^2 N(T_1)^2] &= \mathbb{E}[T_1^2 \mathbb{E}[N(T_1)^2 \mid T_1]] \\ &= \mathbb{E}[T_1^2((\lambda T_1)^2 + (\lambda T_1))] \\ &= \lambda^2 \mathbb{E}[T_1^4] + \lambda \mathbb{E}[T_1^3] \\ &= \lambda^2 \cdot \frac{4!}{\lambda^4} + \lambda \cdot \frac{3!}{\lambda^3} \\ &= \frac{30}{\lambda^2}. \end{aligned}$$

Thus, the variance $\text{Var}(N(T_1)T_1) = \frac{26}{\lambda^2}$. □

Problem 12. Let $\{X(t), t \geq 0\}$ be a pure birth process with $\lambda_n = n\lambda$, $n = 1, 2, \ldots$, $\lambda_0 = \lambda$; $\mu_n = 0, n = 0, 1, 2, \ldots$. Find the conditional probability that $X(t) = n$ given that $X(0) = i$ $(1 \leq i \leq n)$. Also, find the mean of this conditional distribution.

Solution 1. In problem 7, we proved that the $X(t) \sim \text{Geometric}(e^{-\lambda t})$ if $X(0) = 1$.

Now for $1 \leq j \leq i$, let $X_j(t)$ be the number of descendents of the j-th person at time t. The total number of people at time t, will be

$$X(t) = \sum_{j=1}^{i} X_j(t).$$

Since $X_j(t) \sim \text{Geometric}(e^{-\lambda t})$ and they are iid, $X(t) \sim \text{NB}(i, e^{-\lambda t})$. Its mean $\mathbb{E}[X(t)] = ie^{\lambda t}$. $\square$

Solution 2. From problem 7, we get the Kolmogorov equations,

$$p'_{i,j}(t) = \lambda(-jp_{i,j}(t) + (j-1)p_{i,j-1}(t)).$$

We induct on j to prove that

$$p_{i,j}(t) = \binom{j-1}{i-1}(e^{-\lambda t})^i(1 - e^{-\lambda t})^{j-i}$$

for $j \geq i$ and 0 otherwise.

For $j = i$,

$$p'_{i,i}(t) = -i\lambda p_{i,i}(t) \implies p_{i,i}(t) = e^{-i\lambda t}.$$

For higher values, we do the same manipulations as problem 7 to get,

$$\begin{aligned}
p_{i,j}(t) &= (j-1)\lambda e^{-j\lambda t} \int_0^t e^{j\lambda s} p_{i,j-1}(s)\, ds \\
&= (j-1)\lambda e^{-j\lambda t} \int_0^t e^{j\lambda s} \binom{j-2}{i-1} (e^{-\lambda s})^i (1 - e^{-\lambda s})^{j-1-i}\, ds \\
&= (j-1)\binom{j-2}{i-1} \lambda e^{-j\lambda t} \int_0^t e^{\lambda s} (e^{\lambda s} - 1)^{j-1-i}\, ds \\
&= (j-1)\binom{j-2}{i-1} e^{-j\lambda t} \int_1^{e^{\lambda s}} (u-1)^{j-1-i}\, ds \\
&= (j-1)\binom{j-2}{i-1} e^{-j\lambda t} \int_1^{e^{\lambda s}} (u-1)^{j-1-i}\, ds \\
&= (j-1)\binom{j-2}{i-1} e^{-j\lambda t} \frac{(e^{\lambda s} - 1)^{j-i}}{j-i} \\
&= \binom{j-1}{i-1} e^{-i\lambda t} (1 - e^{-\lambda t})^{j-i}
\end{aligned}$$

where we use the substitution $u = e^{\lambda s}$. $\square$

Problem 13. Suppose the people immigrate into a territory at time homogeneous Poisson process with parameter $\lambda = 1$ per day. Let $X(t)$ be the number of people immigrate on or before time t. What is the probability that the elapsed time between the 100-th and 101-th arrival exceeds two days?

Solution. As the process is Poisson, the interarrival time has distribution Exponential(1) and thus, the probability that it exceeds 2 days is e^{-2}. $\square$

Problem 14. A rural telephone switch has C circuits available to carry C calls. A new call is blocked if all circuits are busy. Suppose calls have duration which has exponential distribution with mean $\frac{1}{\mu}$ and inter-arrival time of calls is exponential distribution with mean $\frac{1}{\lambda}$. Assume calls arrive independently and are served independently. Model this process as a birth and death process and write the forward Kolmogorov equation for this process. Also find the probability that a call is blocked when the system in steady state.

Solution. This can be modelled as a CTMC with $X(t)$ being the number of circuits that are currently busy, the state space is $\{0, 1, \ldots, C\}$ and the generator matrix

$$q_{i,i-1} = i\mu, \text{ and } q_{i,i+1} = \lambda$$

for $0 < i < C$, $q_{0,1} = \lambda$ and $q_{C,C-1} = C\mu$. Everything else other than $q_{i,i}$ is 0.

The limiting distribution satisfies, $\pi Q = 0$ that is

$$\begin{aligned} -\lambda\pi_0 + \mu\pi_1 &= 0, \\ \lambda\pi_{i-1} - (\lambda + i\mu)\pi_i + (i+1)\mu\pi_{i+1} &= 0 \text{ for } 0 < i < C, \\ \lambda\pi_{C-1} - C\mu\pi_C &= 0. \end{aligned}$$

Setting $\rho = \lambda/\mu$, we see that $\pi_i = \rho\pi_{i-1}/i$ which implies

$$\pi_n = \frac{\rho^n}{n!}\pi_0.$$

Since $\sum_{i=0}^{C} \pi_i = 1$,

$$\pi_n = \frac{\rho^n/n!}{\sum_{i=0}^{C} \rho^i/i!}.$$

A call is blocked if all circuits are busy, that is $X(t) = C$, so the probabiliity is

$$\pi_C = \frac{\rho^C/C!}{\sum_{i=0}^{C} \rho^i/i!}.$$

□

Problem 15. Consider the random telegraph signal, denoted by $X(t)$, jumps between two states, -1 and 1, according to the following rules. At time $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $P(X(0) = -1) = P(X(0) = 1) = 1/2$, and let the switching times be decided by a Poisson process $\{N(t), t \geq 0\}$ with parameter λ independent of $X(0)$. At time t, the signal

$$X(t) = X(0)(-1)^{N(t)}, t > 0.$$

Write the Kolmogorov forward equations for the continuous time Markov chain $\{X(t), t \geq 0\}$. Find the time-dependent probability distribution of $X(t)$ for any time t.

Solution 1. As the process is Poisson the interarrival time is Exponential(λ), thus the holding time is Exponential(λ) and the rate is λ. For the state space $\{-1, +1\}$, the rate matrix

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}.$$

The forward Kolmogorov equation tells us that

$$\begin{aligned} P'(t) &= P(t)Q \\ \implies p'_{-1,-1}(t) &= -\lambda p_{-1,-1}(t) + \lambda p_{-1,+1}(t), \\ p'_{-1,+1}(t) &= -\lambda p_{-1,+1}(t) + \lambda p_{-1,-1}(t), \\ p'_{+1,-1}(t) &= -\lambda p_{+1,-1}(t) + \lambda p_{+1,+1}(t), \\ p'_{+1,+1}(t) &= -\lambda p_{+1,+1}(t) + \lambda p_{+1,-1}(t). \end{aligned}$$

Along with this, we have $p_{i,-1}(t) + p_{i,+1}(t) = 1$ and $p_{i,i}(0) = \delta_{i,j}$ for $i, j \in \{-1, +1\}$. So the first equation simply reduces to

$$\begin{aligned}
p'_{-1,-1}(t) &= \lambda(1 - 2p_{-1,-1}(t)) \\
\implies e^{2\lambda t}(p'_{-1,-1}(t) + 2\lambda p_{-1,-1}(t)) &= \lambda e^{2\lambda t} \\
\implies \frac{d(e^{2\lambda t} p_{-1,-1}(t))}{dt} &= \lambda e^{2\lambda t} \\
\implies e^{2\lambda t} p_{-1,-1}(t) - 1 &= \tfrac{1}{2}(e^{2\lambda t} - 1) \\
\implies p_{-1,-1}(t) &= \tfrac{1}{2} + \tfrac{1}{2}e^{-2\lambda t}.
\end{aligned}$$

We get

$$P(t) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} \end{pmatrix}.$$

So,

$$\begin{aligned}
P(X(t) = +1) &= P(X(t) = 1 \mid X(0) = -1)P(X(0) = -1) + P(X(0) = 1 \mid X(0) = 1)P(X(0) = 1) \\
&= \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2\lambda t}\right) + \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{2}e^{-2\lambda t}\right) \\
&= \tfrac{1}{2},
\end{aligned}$$

and similarly $P(X(t) = -1)$. $\square$

Solution 2. Here's an alternate way to get $p_{i,j}(t)$ without solving the Kolmogorov equations,

$$\begin{aligned}
P(X(t) = +1 \mid X(0) = +1) &= P((-1)^{N(t)} = +1) \\
&= P(N(t) \equiv 0 \mod 2) \\
&= \sum_{n \geq 0} P(N(t) = 2n) \\
&= e^{-\lambda t} \sum_{n \geq 0} \frac{(\lambda t)^{2n}}{(2n)!} \\
&= e^{-\lambda t} \left(\frac{e^{\lambda t} + e^{-\lambda t}}{2}\right) \\
&= \frac{1 + e^{-2\lambda t}}{2}.
\end{aligned}$$

$\square$

Problem 16. Same as 3.

Problem 17. The birth and death process $\{X(t), t \geq 0\}$ is said to be a pure death process if $\lambda_i = 0$ for all i. Suppose $\mu_i = i\mu$, $i = 1, 2, 3, \ldots$ and initially $X_0 = n$. Show that $X(t)$ has $B(n, p)$ distribution with $p = e^{-\mu t}$.

Solution 1. Modeling it as a CTMC, $q_{i,i-1} = i\mu$ and $q_{i,i} = -i\mu$. The Kolmogorov equations tell us that

$$p'_{i,j}(t) = i\mu(p_{i-1,j}(t) - p_{i,j}(t)).$$

We induct on i to prove that $p_{i,j}(t) \sim \text{Binomial}(i, e^{-\mu t})$.

For $i = j$,

$$p'_{i,i}(t) = -i\mu p_{i,i}(t) \implies p_{i,i}(t) = e^{-i\mu t}.$$

For the base case $i = 0$, we see that $p_{0,0}(t) = 1$. For higher values of i and $j < i$,

$$
\begin{aligned}
p'_{i,j}(t) &= i\mu(p_{i-1,j}(t) - p_{i,j}(t)) \\
\implies e^{i\mu t}(p'_{i,j}(t) + i\mu p_{i,j}(t)) &= e^{i\mu t} i\mu p_{i-1,j}(t) \\
\implies \frac{d(e^{i\mu t} p_{i,j}(t))}{dt} &= e^{i\mu t} i\mu p_{i-1,j}(t) \\
\implies e^{i\mu t} p_{i,j}(t) &= i\mu \int_0^t e^{i\mu s} p_{i-1,j}(s)\, ds \\
&= i\mu \int_0^t e^{i\mu s} \binom{i-1}{j} (e^{-\mu s})^j (1 - e^{-\mu s})^{i-j-1}\, ds \\
&= i\mu \binom{i-1}{j} \int_0^t e^{\mu s} (e^{\mu s} - 1)^{i-j-1}\, ds \\
&= i\binom{i-1}{j} \int_1^{e^{\mu t}} (u-1)^{i-j-1}\, ds \\
&= i\binom{i-1}{j} \frac{(e^{\mu t} - 1)^{i-j}}{j-i} \\
\implies p_{i,j}(t) &= \binom{i}{j} (e^{-\mu t})^j (1 - e^{-\mu t})^{i-j}
\end{aligned}
$$

where we use the substitution $u = e^{\mu s}$. □

Solution 2. Let $X_i(t)$ be 1 if the i-th agent survives till t, 0 otherwise. Since the holding time of a CTMC is exponential, we see that $X_i(t) \sim \text{Exponential}(\mu)$ or equivalently, $X_i(t) \sim \text{Bernoulli}(e^{-\mu t})$.

Now the total population, $X(t) = \sum_{i=1}^n X_i(t)$. Since $X_i(t)$ are iid Bernoulli variables we see that, $X(t) \sim \text{Binomial}(n, e^{-\mu t})$. □

Problem 18. Consider a taxi station where taxis and customers arrive independently in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are in the system. Moreover, an arriving customer that does not find a taxi also will wait no matter how many other customers are in the system. Note that a taxi can accommodate ONLY ONE customer by first come first service basis. Define

$$
X(t) = \begin{cases} -n & \text{if } n \text{ number of taxis waiting for customers at time } t \\ n & \text{if } n \text{ number of customers waiting for taxis at time } t \end{cases}.
$$

(a) Write the generator matrix Q or draw the state transition diagram for the process $\{X(t), t \geq 0\}$.

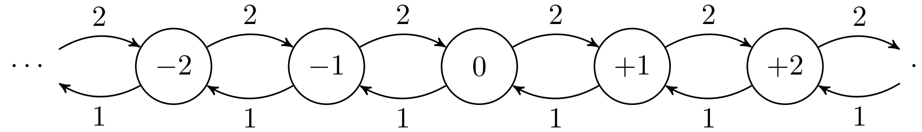
(b) Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$.

(c) Does a unique equilibrium probability distribution of the process exist? Justify your answer.

Solution. (a) The state space is $\mathbb{Z}$ and the generator matrix is

$$
q_{i,i+1} = 2, q_{i,i} = -3, \text{ and } q_{i,i-1} = 1.
$$

The state transition diagram is



(b) The forward Kolmogorov equation is given by $P'(t) = P(t)Q$ which gives,

$$p'_{i,j}(t) = 2p_{i,j-1}(t) - 3p_{i,j}(t) + p_{i,j+1}(t).$$

(c) Since $2/1 > 1$, we see that the embedded markov chain is transient. Therefore, a unique limiting distribution doesn't exist. $\square$

Problem 19. Accidents in Delhi roads involving Blueline buses obey Poisson process with 9 per month of 30 days. In a randomly chosen month of 30 days,

(a) What is the probability that there are exactly 4 accidents in the first 15 days?

(b) Given that exactly 4 accidents occurred in the first 15 days, what is the probability that all the four occurred in the last 7 days out of these 15 days?

Solution. If time is measured in days, it's a Poisson process with mean 9/30.

(a) $N(15) \sim \text{Poisson}(4.5)$, so

$$P(N(15) = 4) = e^{-4.5} \cdot \frac{(4.5)^4}{4!} \approx 0.1898.$$

(b) We wish to find the probability that

$$\begin{aligned}
P(N(15) - N(8) = 4 \mid N(15) = 4) &= P(N(8) = 0 \mid N(15) = 4) \\
&= P(N(15) = 4 \mid N(8) = 0) \cdot \frac{P(N(8) = 0)}{P(N(15) = 4)} \\
&= P(N(15) - N(8) = 4 \mid N(8) = 0) \cdot \frac{P(N(8) = 0)}{P(N(15) = 4)} \\
&= P(N(15) - N(8) = 4) \cdot \frac{P(N(8) = 0)}{P(N(15) = 4)} \\
&= P(N(7) = 4) \cdot \frac{P(N(8) = 0)}{P(N(15) = 4)} \\
&= e^{-7\times 9/30} \frac{(7 \times 9/30)^4}{4!} \cdot \frac{e^{-8\times 9/30}}{e^{-4.5} \frac{(4.5)^4}{4!}} \\
&= (7/15)^4 \approx 0.04742.
\end{aligned}$$

$\square$