COL 351: Analysis and Design of Algorithms

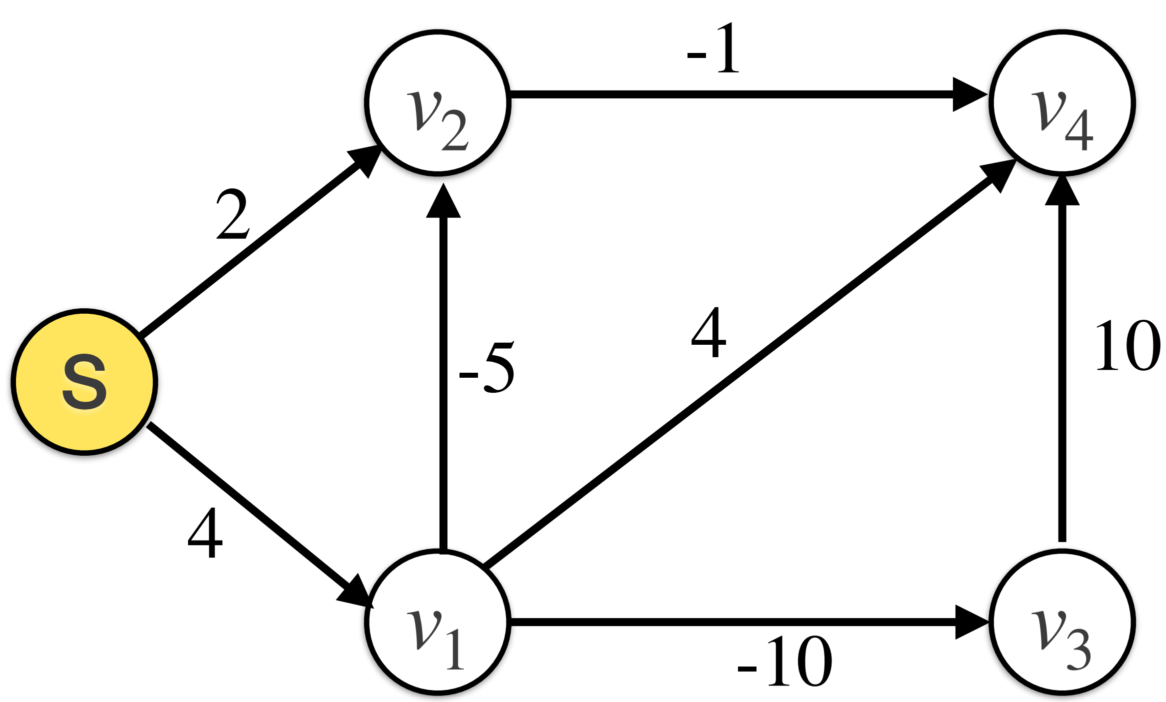
Lecture 15

Single Source Distance Problem

Given: A directed weighted graph $G = (V, E)$ with possibly ***negative*** edge weights, and a source vertex s.

Output: Find either a **shortest-path-tree** rooted at s, or prove that such a tree doesn't exists.

Example:



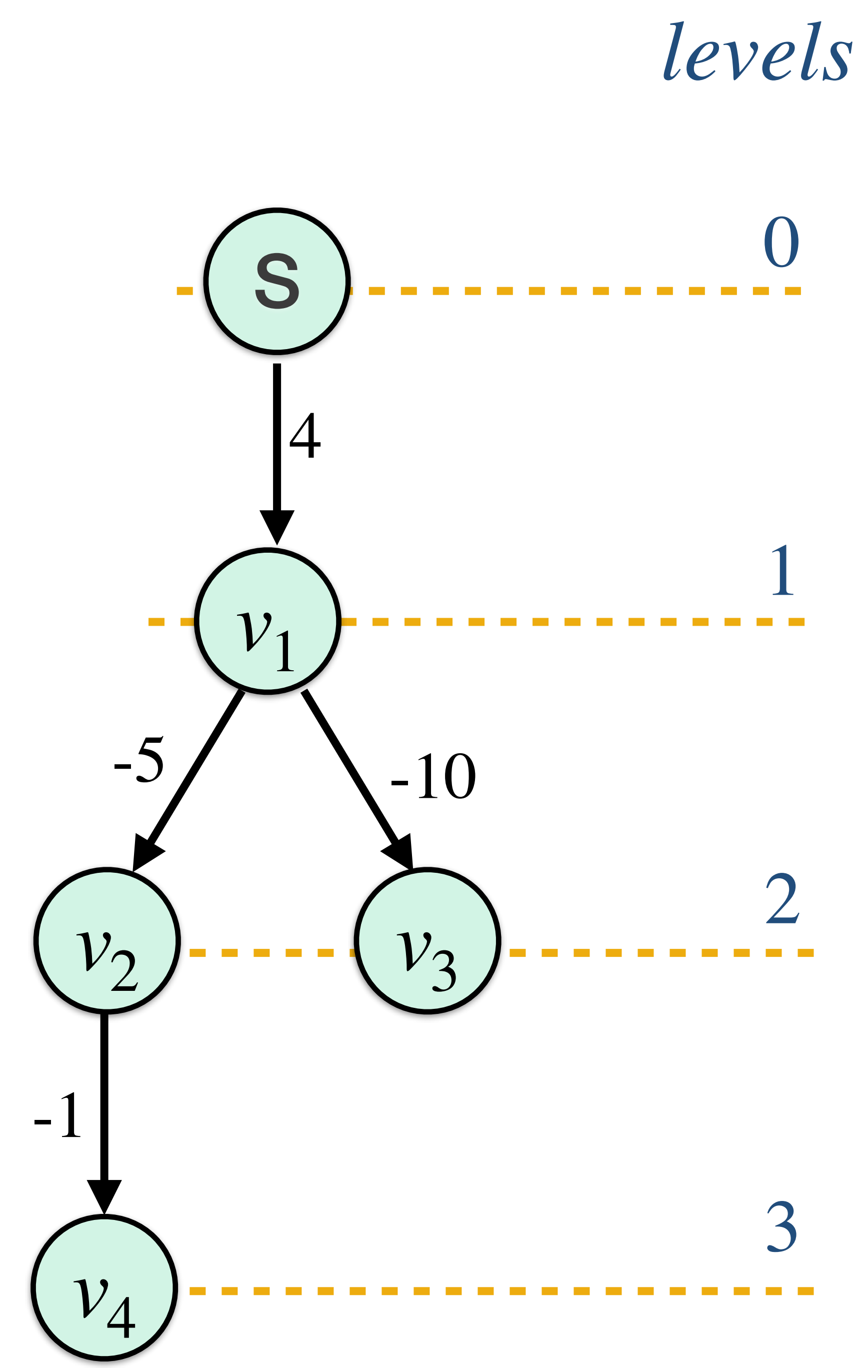
What is shortest path tree from s?

Notation: “Level” of a vertex

Level (v):

Number of edges on $s - v$ shortest path.

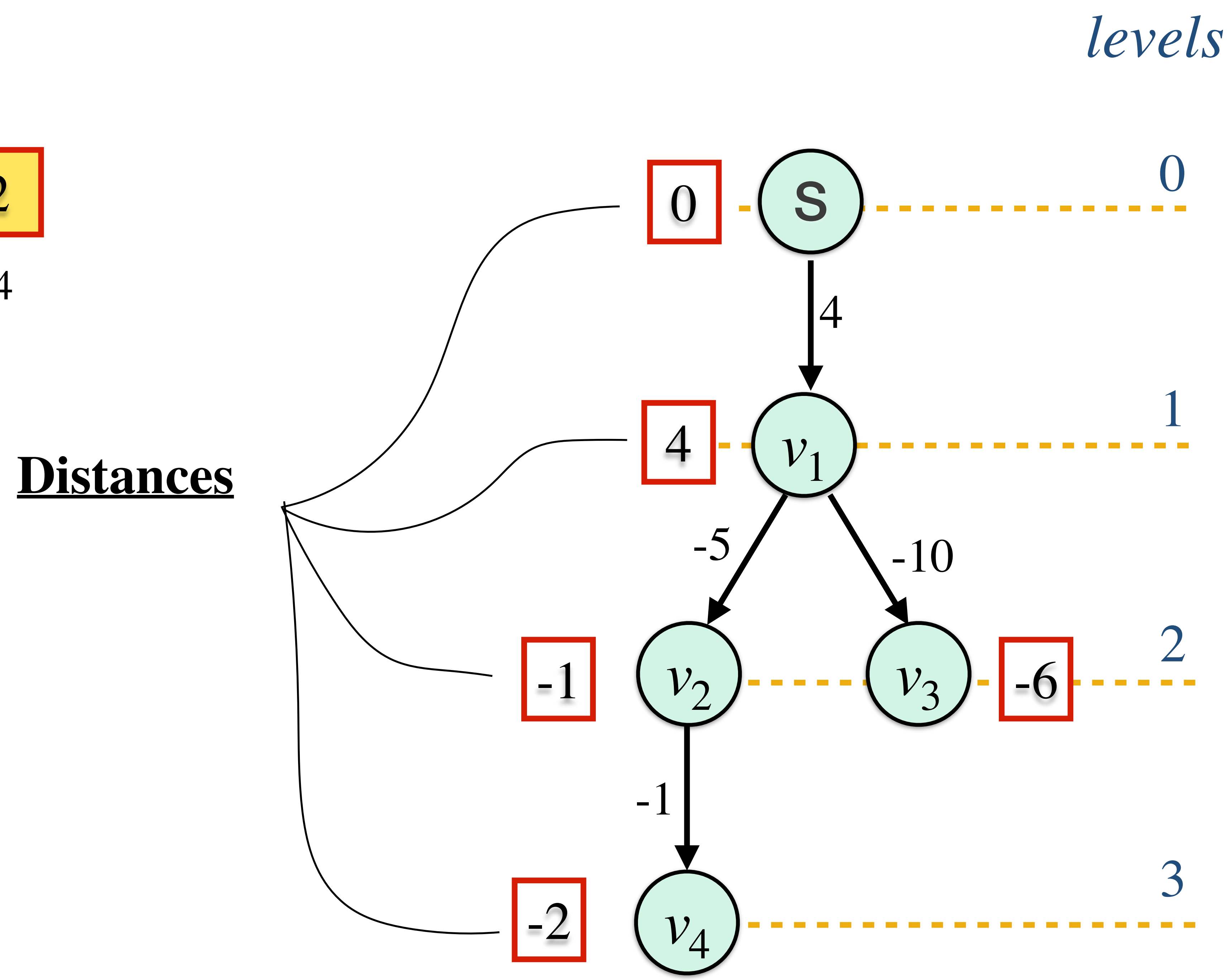
(Break the ties by taking min value).



Shortest path tree, T

Notation: Distance vector

D	0	4	-1	-6	-2
	s	v_1	v_2	v_3	v_4

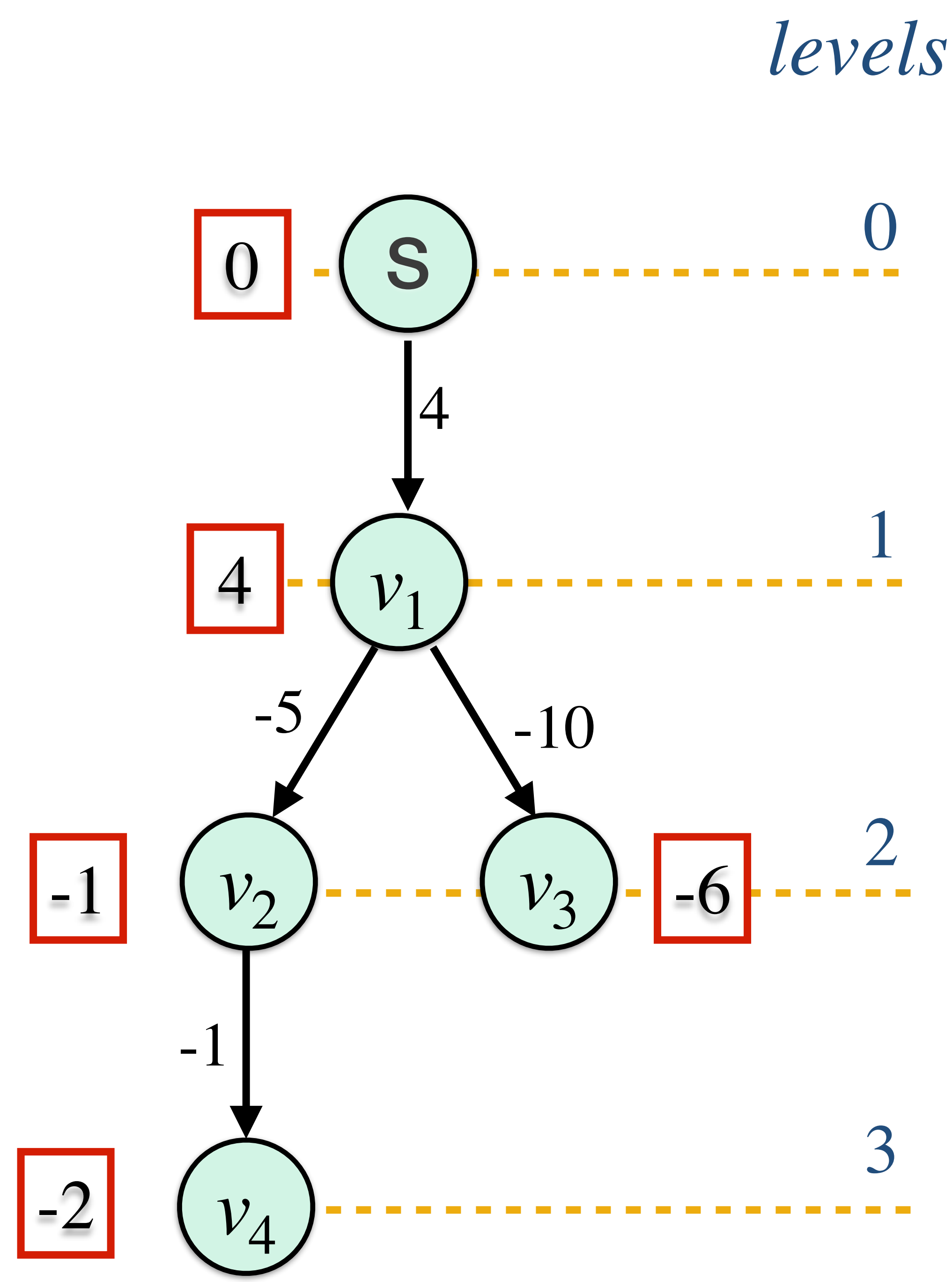


Shortest path tree, T

Simpler Question:

How can one verify if a given 'D' is correct?

D	0	4	-1	-6	-2
	s	v_1	v_2	v_3	v_4



Claim:

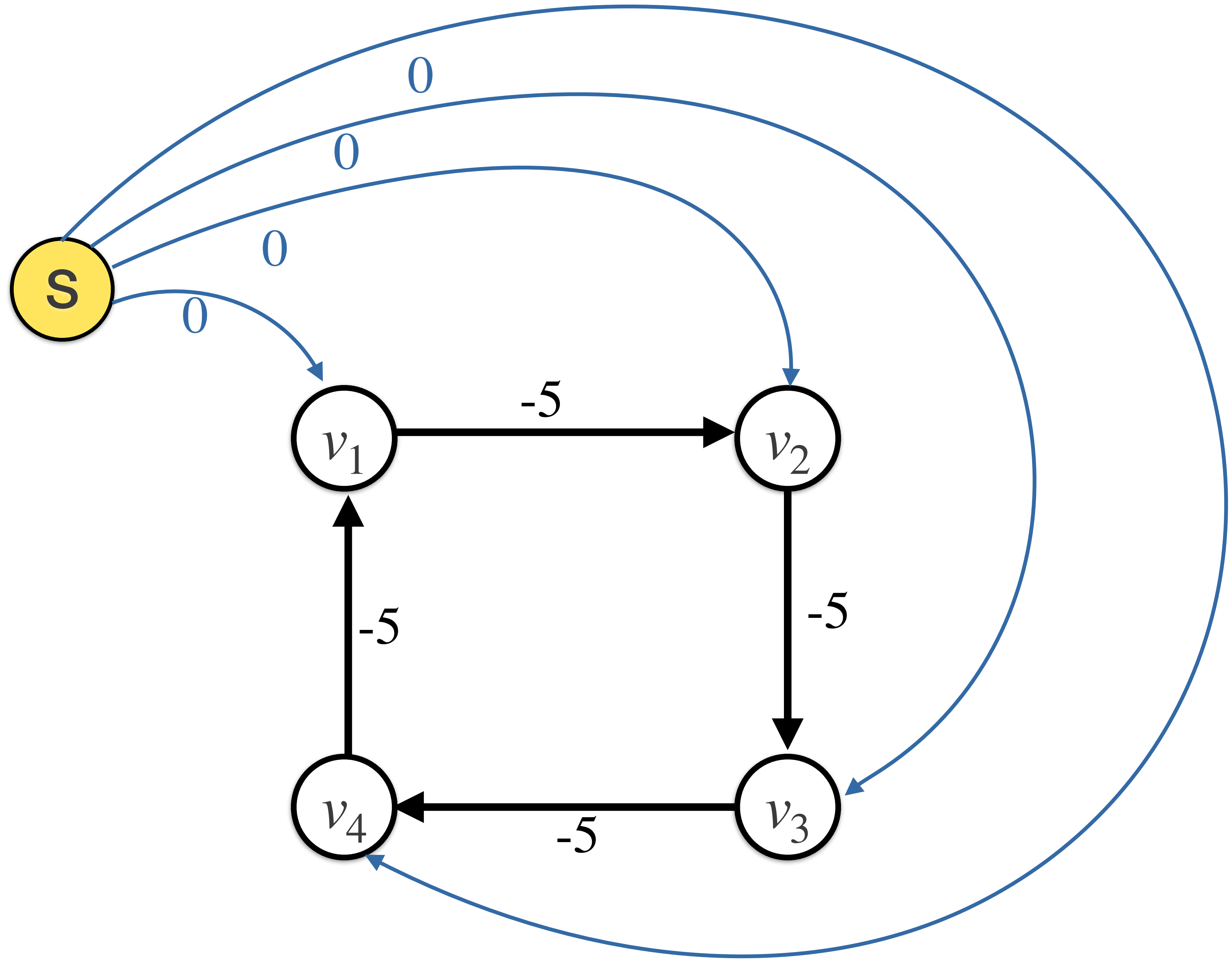
If D is correct then:

For Each $(x, y) \in E$:

$$D[y] \leqslant D[x] + \text{weight}(x, y)$$

Question:

Does a shortest path tree always exist?



Shortest path tree cannot be defined for the graph on right. (Why?)

Homework:
Compute shortest path for all vertices in the cycle.

Definition

An n-sized array D is *valid / correct* upto level i if:

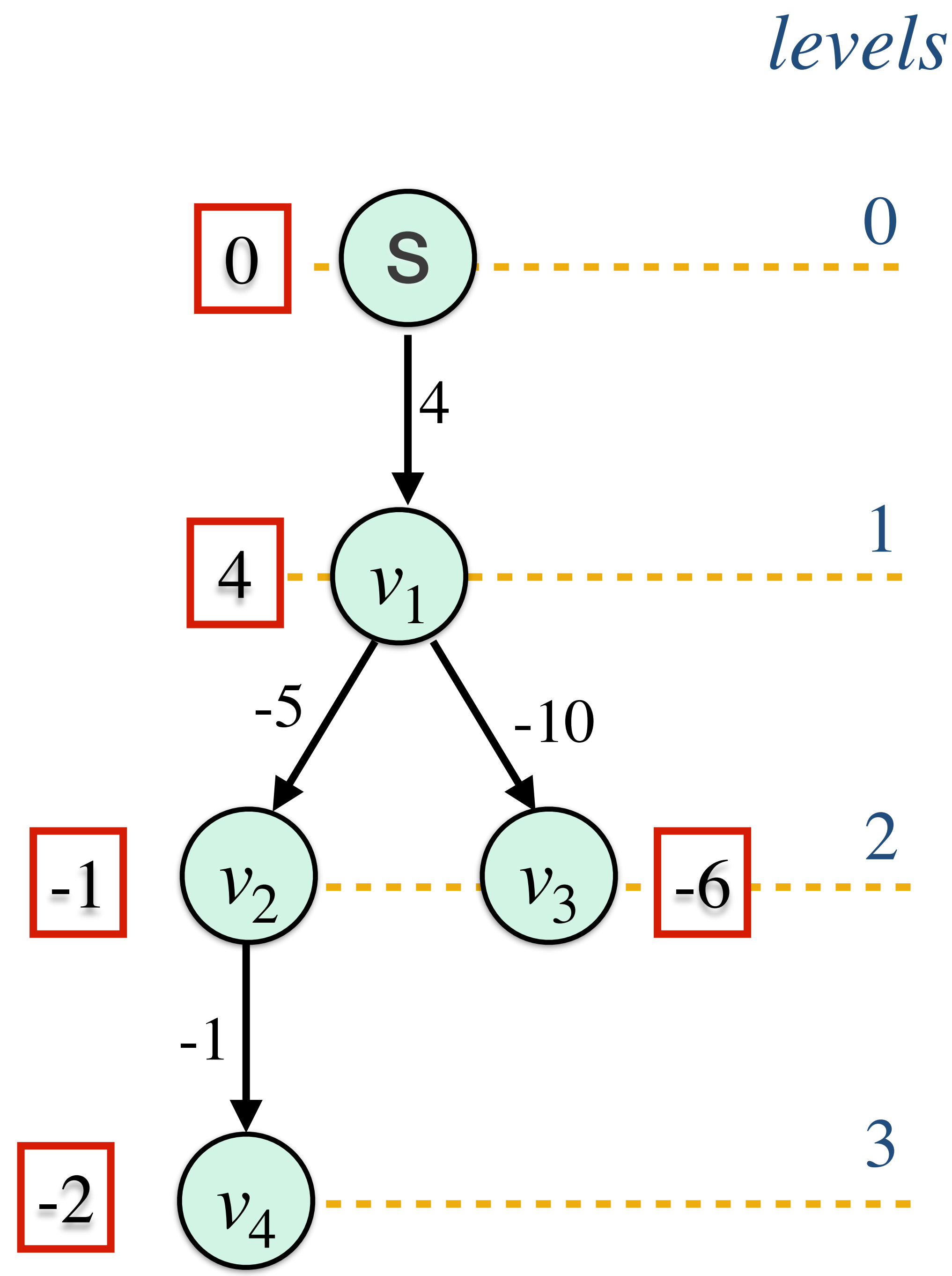
- For each v with level at most i, $D[v] = dist(s, v, G)$.
- For each v with level larger than i, $D[v] \geqslant dist(s, v, G)$.

Valid up to level $i = 2$

D	0	4	-1	-6	99
Levels	0	1	2	2	3
	s	v_1	v_2	v_3	v_4

Valid up to level $i = 1$

D	0	4	-1	66	99
Levels	0	1	2	2	3
	s	v_1	v_2	v_3	v_4



Lemma

Lemma: Let D be an n sized array that is valid upto level i. Then executing the following step ensures that D is valid upto level $i + 1$.

> **For Each** $(x, y) \in E$:
> **If** (D[y] > D[x] + weight(x, y)) **then**
> D[y] = D[x] + weight(x, y)

Proof Sketch:

Consider a vertex y in level $i + 1$. Let x be parent of y in T. Then, level(x) = i.

Now, distance(s, y) = distance(s, x) + weight(x, y), as x is predecessor of y on a $s - y$ shortest path.

The level of x is i which means $D[x]$ is correct, so executing the above code will ensure $D[y] = distance(s, y)$.

Home-work: Argue that for vertices upto level i, there will be no change in D on executing the above code.

Algorithm (assuming no negative weight cycle)

For Each $v \in V$:

  D[v]=∞ and parent[v]=null

D[s]=0

For i = 1 to $n-1$:

  For Each $(x, y) \in E$:

    If (D[y] > D[x] + **weight**(x, y)) **then**

      D[y] = D[x] + **weight**(x, y)

      parent[y] = x

Return D, parent.

Time = $O(mn)$

Question: What if G has a negative weight cycle reachable from s*?*

What if G has negative weight cycle reachable from s?

Lemma: G has 'negative weight cycle' reachable from s if and only if we can make improvement in vector D even in n^{th} round by using the following procedure.

For Each $(x, y) \in E$:
  If (D[y] > D[x] + weight(x, y)) **then**
    D[y] = D[x] + weight(x, y)

Proof:

Home-work

Bellman-Ford Algorithm

For Each $v \in V$:

$\quad$ D[v]=∞ and parent[v]=null

D[s]=0

For i = 1 to $n - 1$:

$\quad$ **For Each** $(x, y) \in E$:

$\quad\quad$ **If** (D[y] > D[x] + **weight**(x, y)) **then**

$\quad\quad\quad$ D[y] = D[x] + **weight**(x, y)

$\quad\quad\quad$ **parent**[y] = x

If ($\exists$ an edge (x, y) satisfying D[y] > D[x] + **weight**(x, y)) **then**

$\quad\quad$ Return "Negative-weight cycle found."

Return D, parent.

O(mn) time algorithm for graphs with negative weights

Challenge Problems

Problem 1: How can you find in $O(mn)$ time a *cycle of negative weight* reachable from s?

Problem 2: Let G be a directed weighted graph with the property that no cycle in it has negative weight. Let s be a source vertex.

Given a parameter L, design an $O(mL)$ total time algorithm to find for each $v \in V$, an $s - v$ path of minimum weight having at most L edges (If it exists).