Vector Analysis - I

PYL101: Electromagnetics & Quantum Mechanics

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Color Codes

- Black: regular text.
- Red: important concept, emphasis,...
- Green: *optional*, but worth thinking about.
- Blue: required HW, test your understanding!
- Orange: in jest; take with a pinch of salt!

References

- **Introduction to Electrodynamics**, David J. Griffiths [IEDJ]
  - Chapter I, 1., Vector Analysis

Applications of Electromagnetism

What is Electromagnetism?

- *Electromagnetism* is a branch of physics which describes the interaction between **charged particles**.
- Charges come in only **two** flavors: positive (+), and negative (−).
- By interaction we mean the (electromagnetic) **forces** which the charges exert on each other.
- The EM force is '*carried/mediated*' by **electromagnetic fields** composed of the electric fields ($\boldsymbol{E}$) and magnetic fields ($\boldsymbol{B}$).

Fields

- Fields?
- **Simplified**: A **field** is a physical quantity, represented by a number, or a vector that has a **value** for each point in space-time, *i.e.*, (x, y, z, t).
    1. *e.g.*, pressure (*scalar field*)
    2. *e.g.*, wind velocity (as seen on a weather report) (*vector field*)
    3. *e.g.*, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields of electromagnetism. (*vector fields*)
- **More precisely**: A **field** is a physical quantity, represented by a tensor[1] (*e.g.*, a number is a rank-0 tensor, a vector is a rank-1 tensor), that has a **value** for each point in space-time, *i.e.*, (x, y, z, t).
    - *e.g.*, stress tensor (*rank-2 tensor field*)

[1] *def*: a **tensor** is an algebraic object (*e.g.*, vector or scalar or other tensors) that describes a **linear mapping** from one set of algebraic objects to another.

How are Electric and Magnetic Fields Produced?

- How are the electric and magnetic fields produced?
- A **stationary** charged particle: produces a *static* **electric field**.
- Similarly, a **steady current** in a wire (also) produces a *static* **magnetic field**.
- An **accelerating** charged particle produces an **electromagnetic (EM) wave/radiation**, *i.e.*, a synchronized oscillation of electric and magnetic fields which have the property that they travel through empty space/vacuum at the **speed of light** c. (*e.g.*, light, X-rays)

The Theoretical Basis of Classical Electromagnetism

- Only these 4 Maxwell's equations in the SI unit convention,

$$
\begin{aligned}
\nabla \cdot \boldsymbol{D} &= \rho_f, && \text{(Gauss' law of electricity)} \\
\nabla \cdot \boldsymbol{B} &= 0, && \text{(Gauss' law of magnetism)} \\
-\nabla \times \boldsymbol{E} &= \frac{\partial \boldsymbol{B}}{\partial t}, && \text{(Faraday's law of induction)} \\
\nabla \times \boldsymbol{H} &= \boldsymbol{J_f} + \frac{\partial \boldsymbol{D}}{\partial t}, && \text{(Ampere's law)}
\end{aligned}
$$

where ρ_f is the free charge density and, $\boldsymbol{J_f}$ is the free current density.
. . . combined with the **Lorentz force law**

$$\boldsymbol{F} = q\boldsymbol{E} + q(\boldsymbol{v} \times \boldsymbol{B})$$

, form the basis of ***classical*** electromagnetism, and optics.

The Theoretical Basis of Classical Electromagnetism

- **Maxwell's equations** in the *SI unit convention*, constitute a set of 4 equations for both [$\boldsymbol{E}$ ($\boldsymbol{D}$)] and [$\boldsymbol{B}$ ($\boldsymbol{H}$)] which are...
  - *coupled*,
  - *first-order*,
  - *partial differential*

Vectors *vs.* Scalars

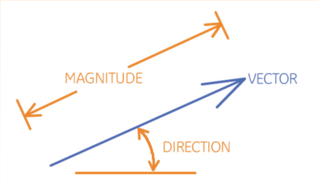


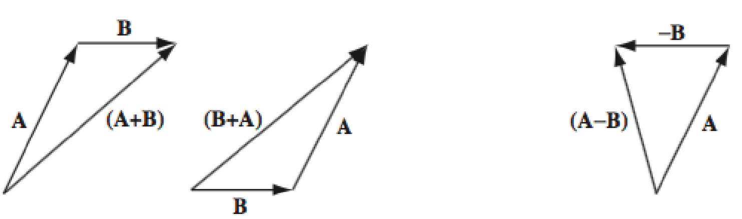
Figure: [Image from `kullabs.com`]

- **Vector** (*noun*): a quantity having a **direction** as well as a **magnitude**.
- *e.g.*, velocity, acceleration, force and momentum.
- **Scalar** (*noun*): a quantity that has a magnitude, but no direction.
- *e.g.*, mass, charge, density, and temperature.
- Vectors have magnitude, and direction, but not *location*.
- *Typographically*, vectors shall be represented by a **bold face**, *e.g.*, $\boldsymbol{A}$.

Vector Operations

- We will encounter **four** kinds of **vector operations**:
  - one *addition* ($\boldsymbol{A} + \boldsymbol{B}$),
  - and three kinds of *'multiplication'*.
    - Multiplication by a scalar ($k\boldsymbol{A}$),
    - Dot product of two vectors ($\boldsymbol{A} \cdot \boldsymbol{B}$),
    - Cross product of two vectors ($\boldsymbol{A} \times \boldsymbol{B}$).

Vector Addition



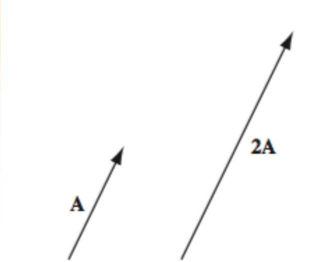
- For *graphically* representing vector addition, recall the *triangle rule* (from high school).
- Vector addition is **commutative**, *i.e.*,

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$$

- Vector addition is also **associative**, *i.e.*,

$$(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$$

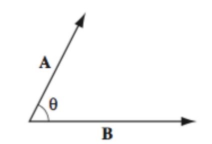
Multiplication by a scalar



- Multiplication of a vector by a positive scalar a multiplies the magnitude but leaves the direction unchanged.
- Scalar multiplication is **distributive**, *i.e.*,

$$a(\boldsymbol{A} + \boldsymbol{B}) = a\boldsymbol{A} + a\boldsymbol{B}$$

Dot Product of Two Vectors



- The **dot product** of two vectors is *defined* ($\equiv$) by,

$$\boldsymbol{A} \cdot \boldsymbol{B} \equiv |A||B|\cos\theta$$

  where θ is the angle they form *when* placed tail-to-tail.
- $\boldsymbol{A} \cdot \boldsymbol{B}$ yields a scalar, hence the *alternative* name **scalar product**.
- The dot product is **commutative**, *i.e.*,

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$$

- The dot product is **distributive**, *i.e.*,

$$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$$

Dot product of two vectors

- Given vectors in **component form**, *i.e.*,

$$\boldsymbol{A} = a_1 \hat{x}_1 + a_2 \hat{x}_2 + a_3 \hat{x}_3 + \ldots = \sum_i a_i \hat{x}_i$$

$$\boldsymbol{B} = b_1 \hat{x}_1 + b_2 \hat{x}_2 + b_3 \hat{x}_3 + \ldots = \sum_i b_i \hat{x}_i$$

- Component-wise we can *define* the dot product as,

$$\boldsymbol{A} \cdot \boldsymbol{B} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + \ldots \equiv \sum_i a_i b_i$$

- Also note that,

$$\boldsymbol{A} \cdot \boldsymbol{A} = |\boldsymbol{A}|^2 = A^2$$

where A^2 or $|A|^2$ are just short hands for $A \cdot A$, or alternatively A^2.

- and,

$$0 \cdot \boldsymbol{A} = 0$$

Orthogonality and Projections

- Two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are **orthogonal**[2] *if and only if*,

$$\boldsymbol{A} \cdot \boldsymbol{B} = 0$$

- The **scalar projection**[3] of $\boldsymbol{B}$ *onto* $\boldsymbol{A}$ is defined as,

$$P_{\boldsymbol{B},\boldsymbol{A}} = \frac{\boldsymbol{B} \cdot \boldsymbol{A}}{A}$$

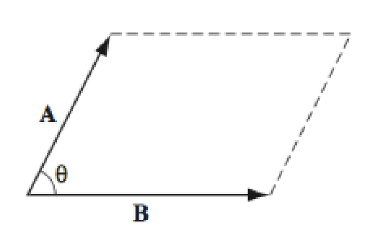
- The **vector projection** of $\boldsymbol{B}$ *onto* $\boldsymbol{A}$ is defined as,

$$\boldsymbol{P}_{\boldsymbol{B},\boldsymbol{A}} = \frac{\boldsymbol{B} \cdot \boldsymbol{A}}{A^2} \boldsymbol{A}$$

[2] The term **perpendicular** describes a property of two vectors, **orthogonal** is a related property of any collection of vectors (*i.e.*, a collection of vectors is orthogonal if and only if all of them are *pairwise* perpendicular), and **normal** is a relation between a vector and an object such as the tangent plane at a point of a smooth surface.

[3] Is $P_{\boldsymbol{A},\boldsymbol{B}} = P_{\boldsymbol{B},\boldsymbol{A}}$?

Cross Product of Two Vectors



- The **cross product** of two vectors is *defined* by,

$$\boldsymbol{A} \times \boldsymbol{B} \equiv |A||B| \sin\theta \ \hat{\boldsymbol{n}}$$

  where θ is the angle they form when placed tail-to-tail, and $\hat{\boldsymbol{n}}$ is a unit vector[4] pointing $\perp$ to the plane of $\boldsymbol{A}$ and $\boldsymbol{B}$.
- The correct *orientation* of $\hat{\boldsymbol{n}}$ is determined by the **right-hand rule**, *e.g.*, $\boldsymbol{A} \times \boldsymbol{B}$ above points <u>into</u> the page.

[4] a hatted $\hat{\boldsymbol{n}}$ denotes a unit vector

Cross Product of Two Vectors

- $\boldsymbol{A} \times \boldsymbol{B}$ is itself a vector (hence the alternative name **vector product**).
- The cross product is **distributive**,

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) + (\boldsymbol{A} \times \boldsymbol{C})$$

- But the cross product is **not** commutative,

$$\boldsymbol{B} \times \boldsymbol{A} \neq \boldsymbol{A} \times \boldsymbol{B}$$

and instead,

$$\boldsymbol{B} \times \boldsymbol{A} = -(\boldsymbol{A} \times \boldsymbol{B})$$

- *Geometrically*, $|\boldsymbol{A} \times \boldsymbol{B}|$ is the **area** of the parallelogram *generated* by $\boldsymbol{A}$ and $\boldsymbol{B}$.

Cross Product of Two Vectors

- Given component-wise **3d** vectors we can conveniently *calculate* the cross product via the **determinant**[5].

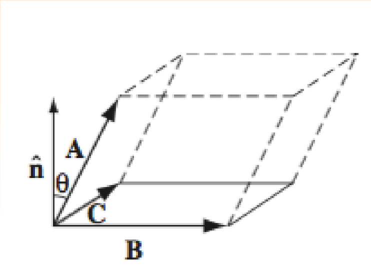
$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

- Expand the above determinant and check that it matches with the expression 13, Chapter I of [IEDJ].
- The vector $\boldsymbol{A} \times \boldsymbol{B}$ is *orthogonal* to both $\boldsymbol{A}$ and $\boldsymbol{B}$.
- Two non-zero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are parallel/anti-parallel *if and only if*

$$\boldsymbol{A} \times \boldsymbol{B} = 0$$

[5] You must be able to calculate determinants for both 3×3, and 2×2 matrices.

Scalar Triple Product



- The **scalar triple product** between vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ is given by

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$$

- *Geometrically*, $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$ is the volume of the parellopiped *generated* by $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$.
- Cyclic order **preserves sign**, *i.e.*,

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$$

Scalar Triple Product

- While non-cyclic permutations reverse sign, *i.e.*,

$$\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{A}\times\boldsymbol{C}) = \boldsymbol{C}\cdot(\boldsymbol{B}\times\boldsymbol{A}) = -\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$$

- In component form, the scalar triple product evaluates to a **determinant**,

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

- The dot and cross can be *interchanged* (keeping the same cyclic order),

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$$

- **Problem:** What's wrong with $(\boldsymbol{A}\cdot\boldsymbol{B})\times\boldsymbol{C}$?

Vector Triple Product

- The **vector triple product** between vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ is given by

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$$

  which geometrically amounts to?
- ..."*a vector in the plane spanned by* $\boldsymbol{B}$ *and* $\boldsymbol{C}$, *which is also perpendicular to* $\boldsymbol{A}$".
- It can be simplified via the BAC-CAB **identity**,

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \equiv \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

- Let's *regroup* the **brackets**. Since cross-products are not associative, *i.e.*,

$$\boxed{(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} \neq \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})}$$

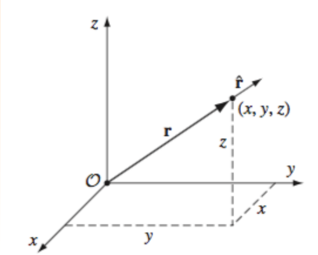
  it turns out that this *regrouping*,

$$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = -\boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C}) + \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C})$$

  is an **entirely different** vector[6]!

[6] Unlike the scalar triple product $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$ where at worst you'll be off by a sign.

The Position Vector



- The *location* of a point in three dimensions can be described by listing its *Cartesian coordinates* (x, y, z)
- The *vector* to that point from the *origin* is called the **position vector**,

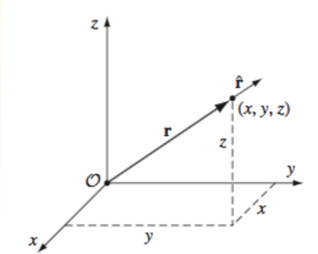
$$\boldsymbol{r} \equiv x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

- Its *magnitude*,

$$r = \sqrt{x^2 + y^2 + z^2}$$

is simply the distance from the origin $\mathcal{O}$.

The Position Vector



- The unit vector

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$$

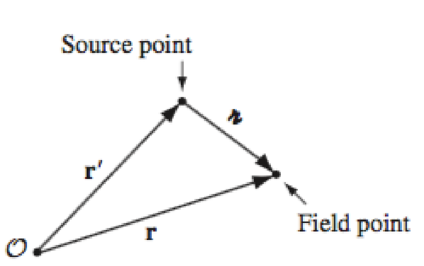
  points *radially outward*.

- The **infinitesimal displacement vector**[7] from (x,y,z) to $(x+dx,y+dy,z+dz)$ is,

$$d\boldsymbol{l} = dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}$$

[7] $d\boldsymbol{l}$ does not point in *any particular* direction since $dx \neq dy \neq dz$, in general.

The Separation Vector $\boldsymbol{s}$



- In *electrodynamics*, one frequently encounters problems involving two points, typically, a **source point**, $\boldsymbol{r}'$, where an electric charge is located, and a **field point**, $\boldsymbol{r}$, at which you are calculating the electric or magnetic field.
- The **separation vector** $\boldsymbol{s}$ from the source point to the field point is then,

$$\boldsymbol{s} \equiv \boldsymbol{r} - \boldsymbol{r}' = (x - x')\hat{\boldsymbol{x}} + (y - y')\hat{\boldsymbol{y}} + (z - z')\hat{\boldsymbol{z}}$$

Ordinary Derivative

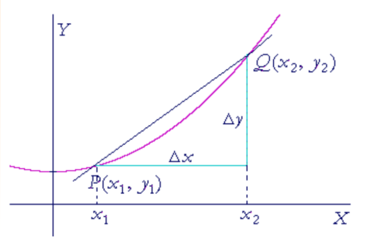


Figure: `www.themathpage.com`

- Given a function $f(x)$, the ordinary derivative df/dx represents the rate of change of f *w.r.t* x.
- *Alternatively*, it tells us how rapidly f varies when we change x by an infinitesimal (tiny) fraction dx.
- *Geometrically*, it gives us the **slope** of the graph of f *vs.* x.

The Partial Derivative

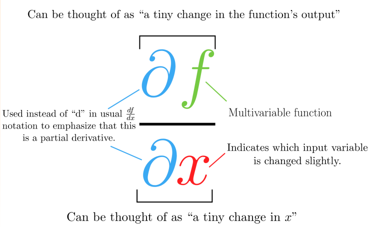


Figure: `www.khanacademy.com`

- The **partial derivative** of a function of *several* variables is its derivative with respect to only one of those variables, with the others held constant (as opposed to the **total derivative**, in which all variables are allowed to vary).
- Q: Given $f(x,y) = x^2y^3$, compute $\frac{\partial f}{\partial y}$, and $\frac{df}{dy}$.
- Ans: $\frac{\partial f}{\partial y} = 3x^2y^2$ and $\frac{df}{dy} = 3x^2y^2 + 2xy^3\frac{dx}{dy}$

The Del ∇ Operator

- The **del** operator is *defined* as,

$$\nabla \equiv \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$$

- Note, ∇ is <u>not</u> a vector! Instead, it's an operator, which in the *grad(ient) case*, maps a differentiable scalar function f <u>to</u> a vector function $\boldsymbol{\nabla} f$.
- <u>Nor</u> does it *multiply* what appears on the right of it. (∇T ??)
- Instead, it's a *vector operator*, or an **instruction** to act on whatever appears to its right.
- It's really just (very) clever notation[8] and acts in 3 ways:
  - On a scalar function: ∇T (*gradient*)
  - On a vector function $\boldsymbol{v}$, via the dot product: $\nabla \cdot \boldsymbol{v}$ (*divergence*)
  - On a vector function $\boldsymbol{v}$, via the cross product: $\nabla \times \boldsymbol{v}$ (*curl*)

[8] So clever that it allows us to use ∇ "like a vector", but being an operator it alone doesn't have a meaning!

The Gradient

- Given a *scalar* function, say $T(x, y, z)$, the change in the quantity T when x, y, z are varied **infinitesimally** *i.e.*, by (dx, dy, dz)[9] is given by the *total differential*,

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz \quad \equiv (\nabla T) \cdot (d\boldsymbol{l})$$

where

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial T}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial T}{\partial z}\hat{\boldsymbol{z}}$$

is, *by definition*, the **gradient** of T, and is evidently a **vector** function.

- Using then the definition of the **dot product** we can of course also write,

$$(\nabla T) \cdot (d\boldsymbol{l}) = |\nabla T||d\boldsymbol{l}|\cos\theta$$

[9] What if the changes in x, y, z were larger (*i.e.*, not infinitesimal)? As a hint consider $f(x)$ and expand it about the point x_0 via a Taylor series as

$$f(x) = f(x_0) + \frac{\partial f}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \ldots$$

The Gradient and the Directional Derivative

- We may *think* of the gradient as "*a multi-variable generalization of the derivative*".
- To make this concrete, consider the definition for the **directional derivative** of a function $f(x, y, z, \ldots)$ along an *arbitrary* unit vector $\hat{\boldsymbol{u}}$ as,

$$D_{\hat{\boldsymbol{u}}} f(\boldsymbol{a}) \equiv \lim_{h \to 0} \frac{f(\boldsymbol{a} + h\hat{\boldsymbol{u}}) - f(\boldsymbol{a})}{h} = \underbrace{\nabla f(\boldsymbol{a})}_{\text{the gradient of } f \text{ at } \boldsymbol{a}} \cdot \hat{\boldsymbol{u}}$$

  which is the rate of change of f at the point $\boldsymbol{a}$ in the direction $\hat{\boldsymbol{u}}$.
- The **directional derivative** can be rewritten as,

$$D_{\hat{\boldsymbol{u}}} f(\boldsymbol{a}) = \nabla f(\boldsymbol{a}) \cdot \hat{\boldsymbol{u}} = |\nabla f(\boldsymbol{a})| \cos\theta$$

  so the largest the $D_{\hat{\boldsymbol{u}}} f(\boldsymbol{a})$ can be is when $\theta = 0$, *i.e.*, when $\hat{\boldsymbol{u}}$ is in the direction of the gradient $\nabla f(\boldsymbol{a})$.
- Thus the gradient $\nabla f(\boldsymbol{a})$ points in the direction of the greatest increase of f, *i.e.*, the direction of **steepest ascent**.

Properties of the Gradient

- The gradient ∇T, or *alternatively*, $\frac{\nabla T}{|\nabla T|}$ at an arbitrary location points in the direction of **maximum** **increase** of the function T around that particular location/point.
- The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.
- When the *gradient vanishes*, *i.e.*, $\nabla T = 0$ at (x_i, y_i, z_i) then the *entire* set, $\{(x_i, y_i, z_i)\}$ represent the **stationary points** or **extrema** of the function $T(x, y, z)$.
- These stationary points/extrema are either:
  - local maxima
  - local minima
  - saddle points
- If you want to *locate* the **extrema(s)** of a scalar function of three (or more) variables, simply set its gradient to zero.

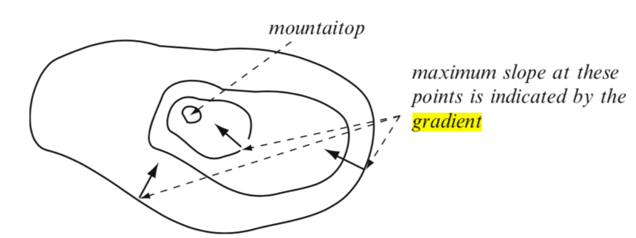
Exercise: The Gradient Of a Hill

- First, assume a very simple **hill** of elevation,

  $$h(x, y) = -x^2 - y^2 + 100$$

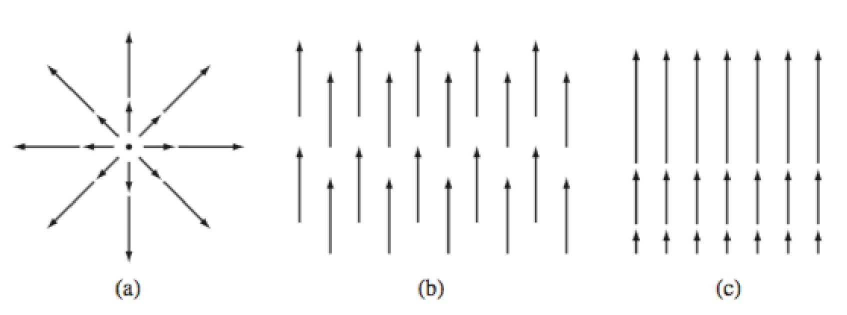
  having a height 100 m. What does the gradient and the contour lines look like?
- Second, draw a more arbitrary, realistic hill using contour lines (*topological map*).
- Q:What does the **gradient** $\nabla h(x, y)$ look like?

The Gradient Of a Hill



- On a *contour/topological map*, the gradient points in a direction which minimizes the distance between adjacent contour lines. Thus the gradient is always $\perp$ to a contour line.
- Do all the gradient vectors point exactly toward the global peak?
- **Ans**: **No**, they do not! The gradient $\nabla h(x, y)$ is a **local quantity** and only gives the magnitude and the direction of maximum increase around the given point (*i.e.*, locally) in question, say (x_i, y_i).
- Dwell on the **mnemonic**: *"water always flows in the direction **opposite** the direction of the gradient."*

The Divergence

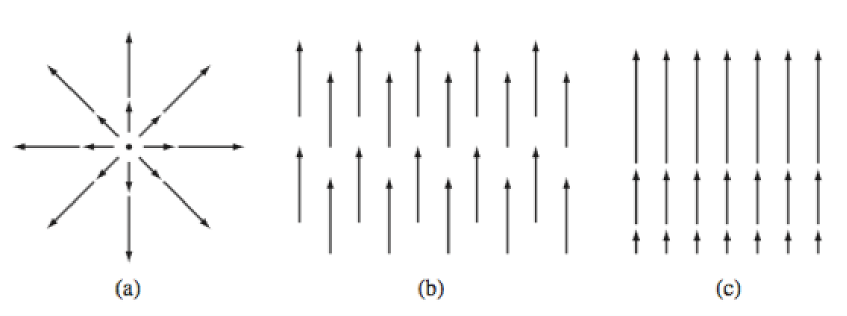


- The **divergence** of a vector function $\boldsymbol{v}$ in 3d-space is defined as:

$$\nabla \cdot \boldsymbol{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

- The divergence $\nabla \cdot \boldsymbol{v}$ is itself a **scalar** function.
- Is $\nabla \cdot \boldsymbol{v}$ defined for each point, or do we have one value per $\boldsymbol{v}$?

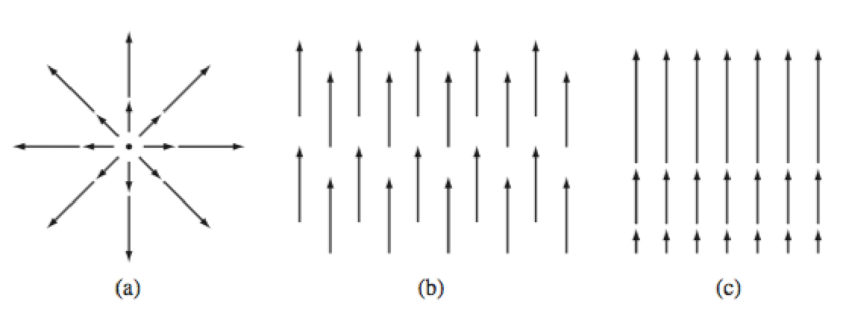
The Divergence



- The diagrams above plot a **vector field**[10] with the (scaled) magnitude and direction of the vector valued function $\boldsymbol{v}$ at selected points/*grid* in Cartesian space.
- *Geometrically*, the divergence measures how much the vector $\boldsymbol{v}$ **spreads out** (diverges) from the point in question.

[10] In *vector calculus*, a **vector field** is an assignment of a vector to each point in a subset (grid point)) of space.

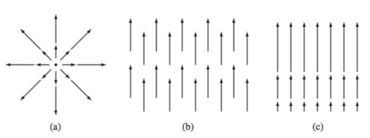
An Intuitive Picture of Divergence



- Consider a **liquid-flow analogy**. If you drop some sawdust[11] at a particular location on the surface and they seem to spread out(in), as opposed to staying stationary, or simply translate, we have location of positive(negative) divergence.
- A point of positive divergence is a **source**, or faucet; a point of negative divergence is a **sink**, or drain.

[11] Why sawdust?

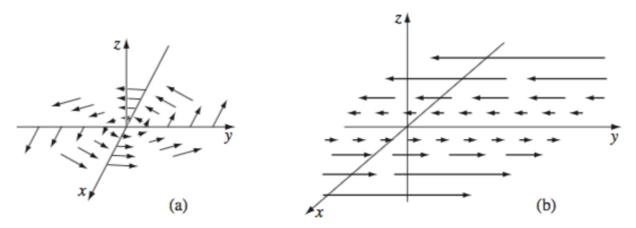
The Divergence: be careful visualizing!



- One might be lulled into thinking that Fig.(a) represents the $\boldsymbol{E}$ due to a single, static charge. However, it cannot possibly represent $\boldsymbol{v} = k\frac{\hat{\boldsymbol{r}}}{r^2}$, since the radial vectors above are getting longer!
- Instead, say if Fig.(a) represents $\boldsymbol{v} = r\ \hat{\boldsymbol{r}}$, then it has a constant divergence[12] everywhere, *i.e.*, $\nabla \cdot \boldsymbol{v} = 3$.
- You might think that the vector function given by Fig.(c) has a zero divergence everywhere. But that would be incorrect, since the arrows/vectors, starting from the bottom are getting longer.

[12]In spherical coordinates $\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \ldots$

The Curl

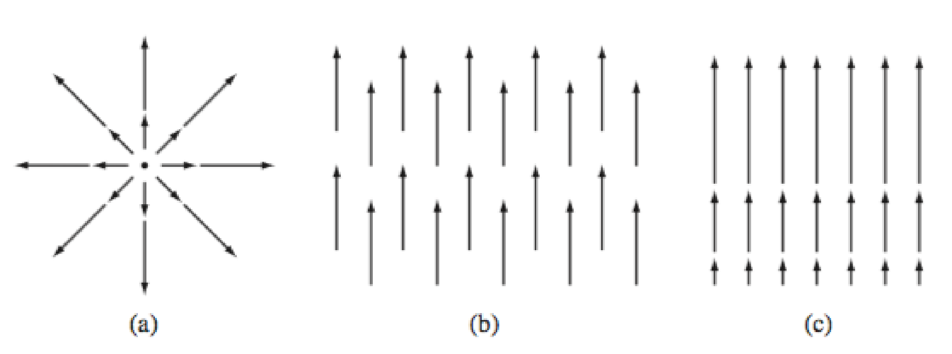


- The **curl** of a vector function $\boldsymbol{v}$ is defined as,

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \quad \text{(vector function)}$$

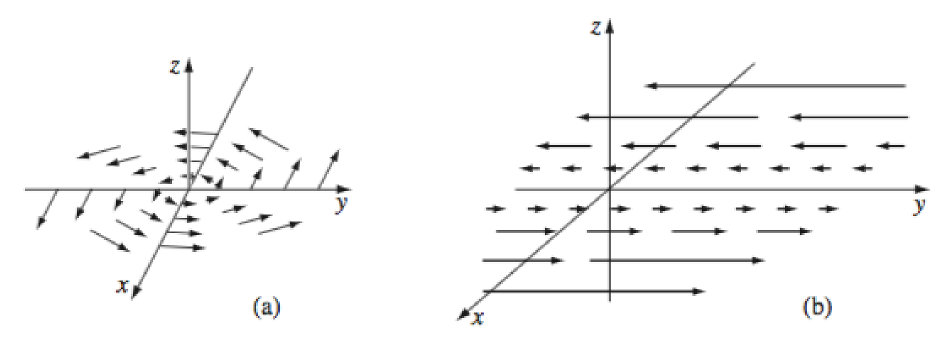
- *Geometrically*, it measures of how much $\boldsymbol{v}$ **swirls** around the point in question.
- *Intuitively*, imagine standing at the edge of a pond. Float a small and light *paddlewheel*; if it starts to **rotate** at that location, then you've placed it at a point of *non-zero* curl.

The Curl



- The three vector fields above <u>all</u> have **zero** curl.

The Curl



- Whereas the functions above have a *substantial* curl, pointing in the $\hat{z}$ direction, as the natural **right-hand rule** would suggest.

Addition and Scalar Multiplication Rules

- Just like with ordinary calculus, certain mathematical **rules** hold for *vector derivatives*,

$$\nabla(f+g) = \nabla f + \nabla g, \qquad \nabla \cdot (\mathbf{A}+\mathbf{B}) = (\nabla \cdot \mathbf{A}) + (\nabla \cdot \mathbf{B}),$$

$$\nabla \times (\mathbf{A}+\mathbf{B}) = (\nabla \times \mathbf{A}) + (\nabla \times \mathbf{B}),$$

and

$$\nabla(kf) = k\nabla f, \qquad \nabla \cdot (k\mathbf{A}) = k(\nabla \cdot \mathbf{A}), \qquad \nabla \times (k\mathbf{A}) = k(\nabla \times \mathbf{A}),$$

- The above rules are fortunately quite *intuitive*!

Product Rules

- While the product rules get a little more *complicated*...[13]

Accordingly, there are *six* product rules, two for gradients:

(i) $$\nabla(fg) = f\nabla g + g\nabla f,$$

(ii) $$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A},$$

two for divergences:

(iii) $$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f),$$

(iv) $$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}),$$

and two for curls:

(v) $$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f),$$

(vi) $$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}).$$

[13] Proving these rules once is advised.

Second Derivatives

- By applying ∇ **twice**, we can construct five species of **second derivatives**

(1) Divergence of gradient: $\nabla \cdot (\nabla T)$.

(2) Curl of gradient: $\nabla \times (\nabla T)$.

The divergence $\nabla \cdot \mathbf{v}$ is a *scalar*—all we can do is take its *gradient:*

(3) Gradient of divergence: $\nabla (\nabla \cdot \mathbf{v})$.

The curl $\nabla \times \mathbf{v}$ is a *vector*, so we can take its *divergence* and *curl:*

(4) Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$.

(5) Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$.

(1) The Divergence of a Gradient *aka* **The Laplacian**

- When given a scalar function T, the **Laplacian** is defined as,

$$\underbrace{\nabla^2 T}_{\textit{just shorthand}} \equiv \nabla \cdot (\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

- Note, the Laplacian of a scalar function T is a **scalar** function!
- On the other hand, given a vector function $\boldsymbol{v}$, the **Laplacian** is then *defined* as,

$$\underbrace{\nabla^2 \boldsymbol{v}}_{\textit{just shorthand}} \equiv (\nabla^2 v_x)\hat{\boldsymbol{x}} + (\nabla^2 v_y)\hat{\boldsymbol{y}} + (\nabla^2 v_z)\hat{\boldsymbol{z}}$$

$$\not\equiv \nabla \cdot (\nabla \boldsymbol{v})$$

and is a **vector** function.

(2) The curl of a gradient

- . . . is always zero. *i.e.,*[14]

$$\nabla \times (\nabla T) = 0$$

[14] Do the proof!

(3) Gradient of a Divergence

- $$\nabla(\nabla \cdot \boldsymbol{v})$$

  ... doesn't show up much in the study of electromagnetism so we won't bother about it.

(4) The divergence of a curl

- ... is also always zero, *i.e.*,[15]

$$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$$

[15] Do the proof!

(5) The curl of a curl

- $$\nabla \times (\nabla \times \boldsymbol{v}) = \underbrace{\nabla(\nabla \cdot \boldsymbol{v})}_{(3)} - \underbrace{\nabla^2 \boldsymbol{v}}_{(1)}$$

  is just the gradient of a divergence minus the Laplacian of a vector, or (3) − (1).