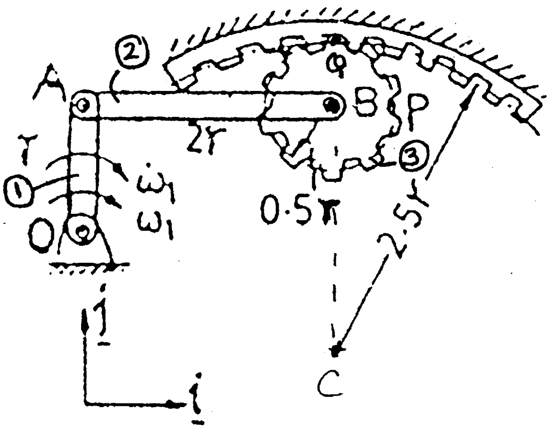
SET 3A

Q1) P 1.24 pg 111

Body ① — Arm OA
Body ② → Arm AB
Body ③ → Pinion BQP.



Although it is not stated explicitly, the centre of the larger gear (rack), C, lies directly below B at this instant $\Rightarrow$ $\vec{CB} = 2r\hat{j} = \vec{r}_{BC}$

$\vec{r}_{AO} = r\hat{j}$; $\vec{r}_{BA} = 2r\hat{i}$; $\vec{r}_{PB} = \frac{r}{2}\hat{i}$; $\vec{r}_{QB} = \frac{r}{2}\hat{j}$.

We have been asked to determine the velocity and acceleration of P. To do this we need the velocity and acceleration of B and $\vec{\omega}_3$ and $\dot{\vec{\omega}}_3$.

Plane motion $\Rightarrow$ $\vec{\omega}_1 = -\omega_1\hat{k}$; $\dot{\vec{\omega}}_1 = -\dot{\omega}_1\hat{k}$
$\vec{\omega}_2 = \omega_2\hat{k}$; $\dot{\vec{\omega}}_2 = \dot{\omega}_2\hat{k}$
$\vec{\omega}_3 = \omega_3\hat{k}$; $\dot{\vec{\omega}}_3 = \dot{\omega}_3\hat{k}$.

O and A lie on body ① : ; $\vec{V}_{O/F} \equiv \vec{V}_O = 0$; $\vec{a}_O = 0$.

$\Rightarrow \vec{V}_A = -\omega_1\hat{k}\times\vec{r}_{AO} = \omega_1 r\hat{i}$; — (I)

$\vec{a}_A = -\dot{\omega}_1\hat{k}\times\vec{r}_{AO} - \omega_1^2\vec{r}_{AO} = \dot{\omega}_1 r\hat{i} - \omega_1^2 r\hat{j}$ — (II)

A and B lie on rigid body ②.

$\therefore \vec{V}_B = \vec{V}_A + \vec{\omega}_2\times\vec{r}_{BA} = \omega_1 r\hat{i} + 2\omega_2 r\hat{j}$. — (III)

To obtain ω_2 we use the constraint specified on the motion of B.

The path taken by B is a circle of radius 2r centred at C, in the ground frame.

Let the speed and rate of change of speed of B along this path be $\dot{s}_B$ and $\ddot{s}_B$.

At this instant $\hat{e}_t = \hat{\imath}$ and $\hat{e}_n = -\hat{\jmath}$.

$\therefore \vec{v}_B = \dot{s}_B \hat{\imath}$ and $\vec{a}_B = \ddot{s}_B \hat{\imath} - \frac{\dot{s}_B^2}{2r}\hat{\jmath}$. — (IV a, b).

Equating III and IV a we have

$\omega_1 r \hat{\imath} + 2\omega_2 r \hat{\jmath} = \dot{s}_B \hat{\imath}$ → Vector equation in 2 scaler unknowns.

$\Rightarrow \dot{s}_B = \omega_1 r$; $\omega_2 = 0$! — (V)

Note: [ω_2 need not be zero if arms OA and AB are not perpendicular to each other].

Again from body 2 we have:

$\vec{a}_B = \vec{a}_A + \dot{\omega}_2 \hat{k} \times \vec{r}_{BA} - \underbrace{\omega_2^2}_{\to 0} \vec{r}_{BA} = \dot{\omega}_1 r \hat{\imath} - \omega_1^2 r \hat{\jmath} + 2\dot{\omega}_2 r \hat{\jmath}$. — (VI)

From IV b. $\vec{a}_B = \ddot{s}_B \hat{\imath} - \frac{\omega_1^2 r}{2}\hat{\jmath}$. — (VII)

From VI and VII we have;

$\ddot{s}_B = \dot{\omega}_1 r$ and $\dot{\omega}_2 = \frac{\omega_1^2}{4}$.

Note that this result is valid even if there is slip at Q. As long as there is continued contact the path of B is unchanged.

From $\dot{s}_B$ and $\ddot{s}_B$ we can now determine ω_3 and $\dot{\omega}_3$.

B and Q belong to 3 and since the gears do not slip ~~so~~ $\vec{v}_Q = 0$.

$\therefore \vec{v}_Q = 0 = \vec{v}_B + \omega_3 \hat{k} \times \vec{r}_{QB} \Rightarrow 0 = \dot{s}_B \hat{\imath} - \frac{\omega_3 r}{2}\hat{\imath}$

$\Rightarrow \omega_3 = \frac{2\dot{s}_B}{r} = 2\omega_1$ at this instant.

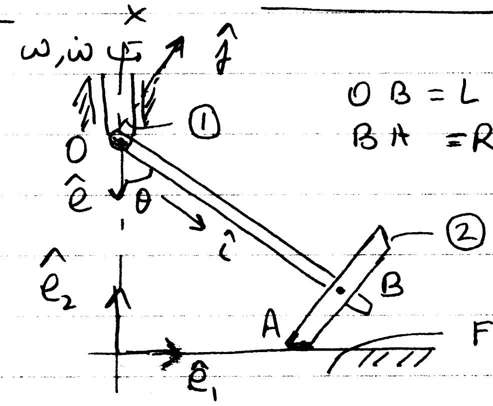
Note $\omega_3 = \frac{2\dot{s}_B}{r}$ is valid $\forall t$ as long as no slip is valid

$\Rightarrow \dot{\omega}_3 = \frac{2\ddot{s}_B}{r} = 2\dot{\omega}_1$ at this instant.

finally we have,

$$\vec{V}_P = \vec{V}_B + \omega_3 \hat{k} \times \vec{r}_{PB} = \omega_1 r \hat{\imath} + 2\omega_1 \hat{k} \times \frac{r}{2}\hat{\imath}$$
$$= \omega_1 r \, (\hat{\imath} + \hat{\jmath})$$
$$\vec{a}_P = \vec{a}_B + \dot{\omega}_3 \hat{k} \times \vec{r}_{PB} - \omega_3^2 \, \vec{r}_{PB}$$
$$= \dot{\omega}_1 r \hat{\imath} - \frac{\omega_1^2 r}{2} \hat{\jmath} + \dot{\omega}_1 r \hat{\jmath} - 2\omega_1^2 r \hat{\imath}$$
$$= (\dot{\omega}_1 r - 2\omega_1^2 r)\hat{\imath} + (\dot{\omega}_1 r - \frac{\omega_1^2 r}{2})\hat{\jmath} .$$

Q2) Q 1.32b pg 108



There are two ways to approach this problem.

a) Following the hint given we can write down the equations relative to frame ①. (This solution is given later.)

b) One can solve it directly in the ground frame (F/G).

Let $\hat{e}_1$ and $\hat{e}_2$ be unit vectors attached to the ground frame. In the figure $\hat{e}_2 = -\hat{e}$

For the motion given (since θ = const.), OB moves with frame ① and can be assumed to belong to frame ①.

$\vec{\omega}_1 = \omega \hat{e}_2$; $\dot{\vec{\omega}}_1 = \dot{\omega} \hat{e}_2$.

The disc spins on axis OB relative to the frame ①.

$\Rightarrow \vec{\omega}_{2/1} = \omega_{2/1} \hat{\imath}$; $\dot{\vec{\omega}}_{2/1} = \dot{\omega}_{2/1} \hat{\imath}$.

$\hat{\imath} = -\cos\theta \, \hat{e}_2 + \sin\theta \, \hat{e}_1$.

To get $\vec{\omega}_{2/1}$ and $\dot{\vec{\omega}}_{2/1}$ we use the no slip condition.

O and B belong to body 1

$$\vec{V}_B = \vec{V}_O + \omega \hat{e}_2 \times \vec{r}_{BO} = 0 + \omega\, \hat{e}_2 \times (L\sin\theta\, \hat{e}_1 - L\cos\theta\, \hat{e}_2)$$
$$= -\omega L \sin\theta\, \hat{e}_3.$$

A and B belong to body 2:

$$\vec{\omega}_2 = \vec{\omega}_1 + \vec{\omega}_{2/1} = \omega\, \hat{e}_2 + \omega_{2/1}\,(\sin\theta\, \hat{e}_1 - \cos\theta\, \hat{e}_2)$$

$$\vec{V}_A = \vec{V}_B + \vec{\omega}_2 \times (\vec{r}_{AB}) = 0 \quad \text{(no slip at A)}.$$
$$\Rightarrow 0 = -\omega L \sin\theta\, \hat{e}_3 + [\omega\, \hat{e}_2 + \omega_{2/1}\,(\sin\theta\, \hat{e}_1 - \cos\theta\, \hat{e}_2)] \times \vec{r}_{AB}$$
$$\vec{r}_{AB} = R\,(-\sin\theta\, \hat{e}_2 - \cos\theta\, \hat{e}_1) = -R\,(\cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_2)$$

$$\therefore 0 = -\omega L \sin\theta\, \hat{e}_3 + [(\omega - \omega_{2/1}\cos\theta)\hat{e}_2 + \omega_{2/1}\sin\theta\, \hat{e}_1] \times (-[R\cos\theta\, \hat{e}_1 + R\sin\theta\, \hat{e}_2])$$
$$= -\omega L \sin\theta\, \hat{e}_3 + (\omega - \omega_{2/1}\cos\theta) R\cos\theta\, \hat{e}_3 - \omega_{2/1} R \sin^2\theta\, \hat{e}_3$$
$$\Rightarrow \omega_{2/1} = -\omega \frac{(L\sin\theta - R\cos\theta)}{R}$$

This expression for $\omega_{2/1}$ is valid as long as no slip is present. $\Rightarrow$

$$\dot{\omega}_{2/1} = -\dot{\omega} \frac{(L\sin\theta - R\cos\theta)}{R}$$

$$\dot{\vec{\omega}}_2 = \dot{\vec{\omega}}_1 + \dot{\vec{\omega}}_{2/1} + \vec{\omega}_1 \times \vec{\omega}_{2/1}$$

$$= \dot{\omega}\, \hat{e}_2 - \frac{\dot{\omega}}{R}(L\sin\theta - R\cos\theta)(\sin\theta\, \hat{e}_1 - \cos\theta\, \hat{e}_2) + \frac{\omega^2}{R}(L\sin\theta - R\cos\theta)\sin\theta\, \hat{e}_3$$

Now;

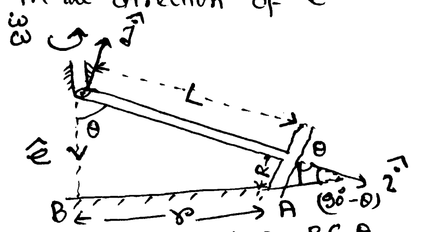
$$\vec{a}_B = \vec{a}_O + \dot{\vec{\omega}}_1 \times \vec{r}_{BO} - \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{BO})$$
$$= 0 - \dot{\omega} L \sin\theta\, \hat{e}_3 - \omega^2 L \sin\theta\, \hat{e}_1$$

$$\vec{a}_A = \vec{a}_B + \dot{\vec{\omega}}_2 \times \vec{r}_{AB} + \vec{\omega}_2 \times (\vec{\omega}_2 \times \vec{r}_{AB})$$

The algebra required to get $\vec{a}_A$ is very tedious by this method. The other method (using frame 1 to do the calculations) is much simpler in algebra as shown below.

Q 1.32b pg 108 Method (a). ⑤

Suppose the ground frame rotates with respect to frame 1 with angular velocity $\vec{\omega}_{3|1}$ in the direction of $\hat{e}$. (Note Frame ③ ≡ Frame F).



$r = L\sin\theta - R\cos\theta$

We know that

$$\vec{\omega}_{3|1} = \vec{\omega}_{3|F} - \vec{\omega}_{1|F} = 0 - (-\omega\hat{e}) = \omega\hat{e}$$

where $\vec{\omega}_{1|F} = -\omega\hat{e}$, is the angular velocity of the frame 1 w.r.t ground [This is given].

Suppose $\vec{\Omega}$ is the angular velocity of the disc 2 in the direction $\hat{i}$ w.r.t frame 1.

Therefore $\vec{\omega}_{2|1} = \Omega\hat{i}$

As the axis $\hat{i}$ of the frame 2 and $\hat{e}$ of frame 3 are rest w.r.t frame 1, Then we can use no slip condition at the point A.

$$\therefore \quad \Omega R = -\omega_{3|1}\, r, \quad \text{where} \quad BA = r = L\sin\theta - R\cos\theta$$

$$\hat{e} = -\sin\theta\,\hat{j} + \cos\theta\,\hat{i}$$

$$\Rightarrow \quad \Omega R = -\omega r \qquad \Rightarrow \quad \dot{\Omega} = -\dot{\omega}\frac{r}{R}$$

Suppose $\vec{\omega}_2$ is the angular velocity of the disc 2 w.r.t ground frame

$$\vec{\omega}_{2|F} = \vec{\omega}_{1|F} + \vec{\omega}_{2|1} = -\omega\hat{e} + \Omega\hat{i} = \Omega\hat{i} - \omega(-\sin\theta\,\hat{j} + \cos\theta\,\hat{i})$$

$$= \Omega\hat{i} + \omega\sin\theta\,\hat{j} - \omega\cos\theta\,\hat{i}$$

$$\dot{\vec{\omega}}_{2|F} = \dot{\vec{\omega}}_{1|F} + \dot{\vec{\omega}}_{2|1} + \vec{\omega}_{1|F} \times \vec{\omega}_{2|1}$$

$$= -\dot{\omega}\hat{e} + \left(-\dot{\omega}\frac{r}{R}\right)\hat{i} + (-\omega\hat{e}) \times \Omega\hat{i}$$

$$= -\dot{\omega}(-\sin\theta\,\hat{j} + \cos\theta\,\hat{i}) + \dot{\Omega}\hat{i} - \omega(-\sin\theta\,\hat{j} + \cos\theta\,\hat{i}) \times \Omega\hat{i}$$

$$= \dot{\omega}\sin\theta\,\hat{j} - \dot{\omega}\cos\theta\,\hat{i} + \dot{\Omega}\hat{i} - \omega\Omega\sin\theta\,\hat{k}$$

The acceleration at the point A w.r. to the ground frame is given by

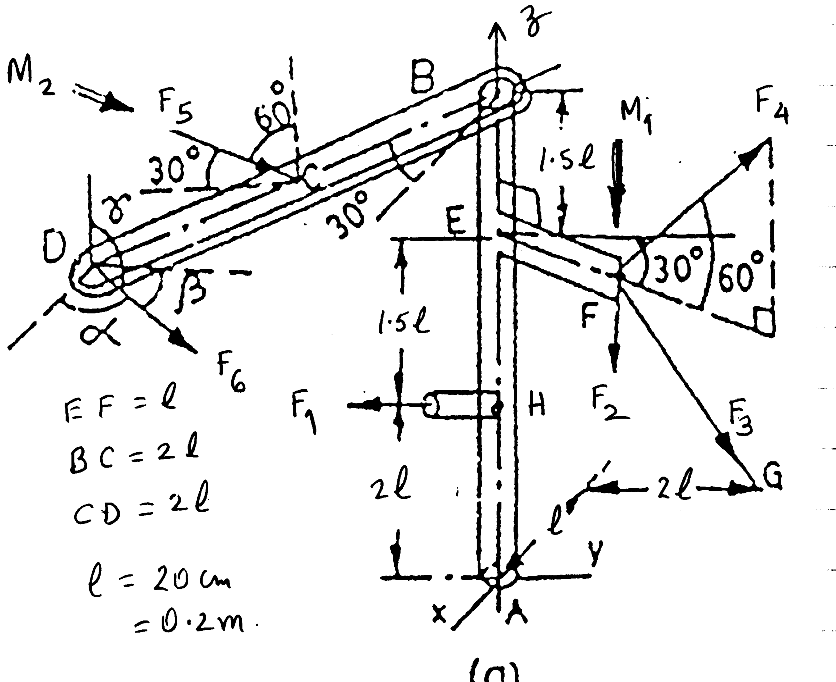
$$a_{A|F} = \vec{a}_{0|F} + \dot{\vec{\omega}}_{2|F} \times \vec{OA} + \vec{\omega}_{2|F} \times (\vec{\omega}_{2|F} \times \vec{OA})$$

$$= \dot{\vec{\omega}}_{2|F} \times \vec{OA} + \vec{\omega}_{2|F} \times (\vec{\omega}_{2|F} \times \vec{OA}), \quad \text{where } \vec{OA} = L\hat{i} - R\hat{j}$$

Set 3 A

P2.9 (a) pg 234

This appears a very complicated problem. The main purpose of this example is to familiarize students with the different ways in which forces and couples can be specified. In all there are six forces (marked F_1 to F_6) and two couples (marked M_1 and M_2).



(a)

Let us consider the resultant system at point A. →
The system consists of a force $\vec{F}_R$ and a moment $\vec{C}_{RA}$.
The magnitudes of the forces and moments are specified on the figure (in the text). We now compute the directions of each of the forces.

$\hat{e}_{F_1} = -\hat{j}$; $\hat{e}_{F_2} = -\hat{k}$; $\hat{e}_{F_4} = [\cos 60° \cos 60° \,\hat{i} + \cos 60° \cos 30° \,\hat{j} + \sin 60° \,\hat{k})$

$$\hat{e}_{F_3} = \frac{\vec{r}_{GA} - \vec{r}_{FA}}{|\vec{r}_{GA} - \vec{r}_{FA}|} = \frac{(-\ell\hat{i} + 2\ell\hat{j}) - [\ell(\cos 60°\hat{i} + \cos 30°\hat{j}) + 3.5\ell\hat{k}]}{|\quad|}$$

$$= \frac{-1.5\hat{i} + (2 - \sqrt{3}/2)\hat{j} - 3.5\hat{k}}{\sqrt{(1.5)^2 + (2-\sqrt{3}/2)^2 + 3.5^2}} = -0.378\hat{i} + 0.285\hat{j} - 0.881\hat{k}$$

$\hat{e}_{F_5} = \cos 30°\,\hat{j} - \cos 60°\,\hat{k}$; $\hat{e}_{F_6} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$.

$\hat{e}_{M_1} = -\hat{k}$; $\hat{e}_{M_2} = \hat{e}_{F_5}$.

We also need the points of application of the various forces.

Since the moments required are about A we choose the origin at A

$$\vec{r}_{FA} \equiv \vec{r}_F \; ; \quad \vec{r}_{HA} \equiv \vec{r}_H \quad \text{etc.}$$

$$\vec{r}_H = 2l\hat{k} \; ; \quad \vec{r}_F = l(+0.5\,\hat{\imath} + 1.13\,\hat{\jmath} + 3.5\,\hat{k})$$

$$\vec{r}_C = 2l\cos 30^\circ\,\hat{\imath} - 2l\sin 30^\circ\,\hat{\jmath} + 5l\hat{k}$$
$$\vec{r}_D = 4l\cos 30^\circ\,\hat{\imath} - 4l\sin 30^\circ\,\hat{\jmath} + 5l\hat{k} \,.$$

$$\vec{F}_R = F_1\hat{e}_{F_1} + F_2\hat{e}_{F_2} \cdots + F_6\hat{e}_{F_6} = \sum_{i=1}^{6} F_i\hat{e}_{F_i}$$

$$\vec{C}_{RA} = -M_1\hat{k} + M_2\hat{e}_{F_5} + \vec{r}_H \times (F_1\hat{e}_{F_1}) + \vec{r}_F \times [-F_2\hat{k} + F_3\hat{e}_{F_3} + F_4\hat{e}_{F_4}]$$
$$+ \vec{r}_C \times F_5\hat{e}_{F_5} + \vec{r}_D \times F_6\hat{e}_{F_6} \,.$$

To get the moments about ~~any~~ AB we choose any point on AB and take the moment of each force about that point and take the dot product with the unit vector along AB.

$$\hat{e}_{BA} = \hat{k} \,.$$

$$M_{AB} = -M_1 + M_2\,\hat{e}_{F_5}\cdot\hat{k} + \overset{\to 0}{\cancel{(\vec{r}_H \times F_1\hat{e}_{F_1})\cdot\hat{k}}} + \{\vec{r}_F \times [F_3\hat{e}_{F_3} + F_4\hat{e}_{F_4}]\}\cdot\hat{k}$$
$$+ (\vec{r}_C \times F_5\hat{e}_{F_5})\cdot\hat{k} + (\vec{r}_D \times F_6\,\hat{e}_{F_6})\cdot\hat{k} \,.$$

(Note: $\vec{F}_2$ is $\parallel$ to AB and hence does not provide a moment about AB. Similarly $\vec{F}_1$ intersects AB and hence its moment should also be zero).