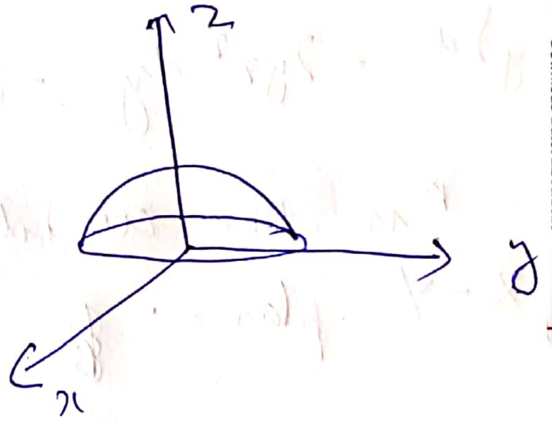
Harshit Mawandia 2020CS10348 GROUP - 4

1) The stokes' theorem states that :

$$\int_S (\bar\nabla \times \bar v) \cdot da = \oint_P \vec v \cdot \vec{dl}$$

Given: $\bar V = \bar A = (2x-y)\hat x - 2yz^2(\hat y) - 2zy^2 \hat z$

We need to verify on the upper hemispherical surface of sphere centered at origin, having radius 2.



$$LHS = \vec\nabla \times \bar v = \begin{vmatrix} \hat x & \hat y & \hat z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 2x-y & -2yz^2 & -2zy^2 \end{vmatrix}$$

$$= \hat x\left(-4zy - (-4zy)\right) - \hat y\,(0-0) + \hat z\,(0-(-1))$$

$$\vec\nabla \times \vec v = \hat z$$

$$\vec{da} = r\,d\theta\,(r\sin\theta\,d\phi)\,\hat{r}$$

where $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$$\therefore\ (\bar{\nabla}\times\bar{v})\cdot d\hat{a} = r_0^2 \sin\theta\cos\theta\,d\theta\,d\phi$$

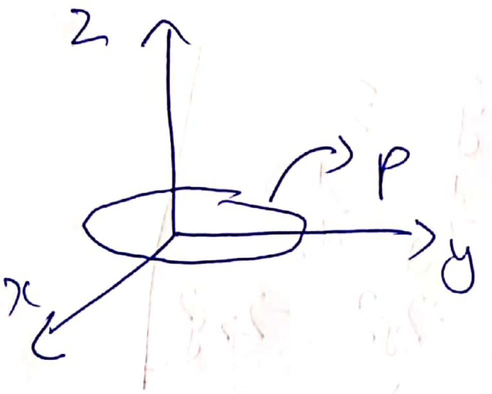
$$\int(\vec{\nabla}\cdot\vec{v})\cdot\vec{da} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r_0^2 \sin\theta\cos\theta\,d\theta\,d\phi$$

$$= \pi r_0^2 = 4\pi \quad (\text{since } r_0 = 2)$$

$$\text{RHS} = \vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$\oint_P \vec{v}\cdot\vec{dl} = \oint_P \left((2x-y)\,dx - 2yz^2\,dy - 2zy^2\,dz\right)$$

Here our periphery P is the circular Projection of hemisphere in X-Y plane.



$\therefore$ For our periphery, $z=0$ implying:

$$\oint_P \vec{v}\cdot\vec{dl} = \oint_P (2x-y)\,dx$$

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$x = r_0 \cos\theta \Rightarrow dx = -r_0 \sin\theta \, d\theta$

$y = r_0 \sin\theta$

$\therefore \oint_P \vec{v} \cdot \vec{dl} = \oint_P (2x - y)\, dx$

$= \oint_P (2 r_0 \cos\theta - r_0 \sin\theta)(-r_0 \sin\theta \, d\theta$

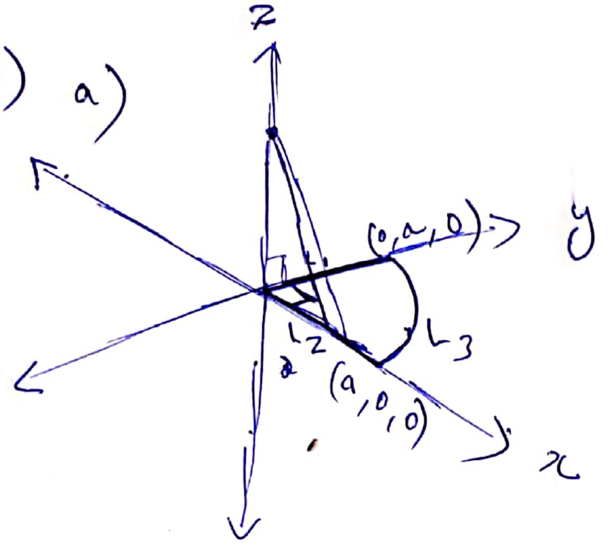
$= r_0^2 \int_0^{2\pi} (\sin^2\theta - 2\sin\theta\cos\theta)\, d\theta$

$= r_0^2 \int_0^{2\pi} \sin^2\theta \, d\theta$

$= \pi r_0^2 = 4\pi$ (since $r_0 = 2$)

$\therefore$ LHS = RHS

Q2) a)



$$V = \int \frac{dq}{4\pi\epsilon_0 r}$$

$= \Sigma$ summation of potentials due to 3 line charges, 2 straight lines + 1 arc.

$$= \int_0^a \frac{\lambda\, dx}{4\pi\epsilon_0 \sqrt{z^2+x^2}} + \int_0^a \frac{\lambda\, dy}{4\pi\epsilon_0 \sqrt{z^2+y^2}}$$

$$+ \int_0^{\pi/2} \frac{r\, d\theta}{4\pi\epsilon_0 \sqrt{r^2+z^2}}$$

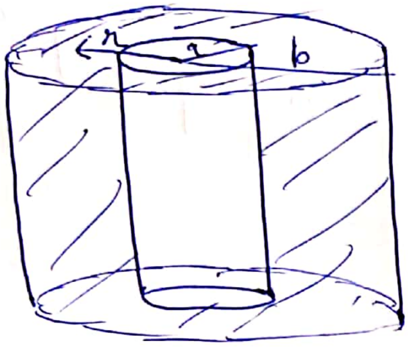
Both are same

$$= \frac{\lambda}{4\pi\epsilon_0} \left(2\left[\ln\left(x + \sqrt{x^2+\cancel{z^2}}\right)\right]_0^a + \left[\frac{\pi r}{2\sqrt{r^2+z^2}}\right]_0 \right.$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left[2 \ln\left(\frac{a + \sqrt{a^2+z^2}}{z} \right) + \frac{\pi r}{2\sqrt{r^2+z^2}} \right]$$

b) No, we cannot determine E from V(0,0,z) because we have already assumed x and y to be 0 and while calculating ~~E = -∇V~~ $V = -\nabla \cdot E$, we ~~also~~ won't get the partial derivatives with respect to x and y.

3) a)



i) $E \cdot 2\pi r \cancel{l} = \frac{\rho_0 \pi (r^2 - a^2)}{\varepsilon} \cancel{l}$

$$E = \frac{\rho (r^2 - a^2)}{2 r \epsilon}$$

$$V = -\int_a^r \frac{\rho_0 (r^2 - a^2)}{2 r \epsilon} dr$$

$$= -\frac{\rho_0}{2 \epsilon} \int_a^r \left(r - \frac{a^2}{r} \right) dr$$

$$= \frac{-\rho_0}{2 \epsilon} \left[\frac{r^2 - a^2}{2} - a^2 \ln r/a \right]$$

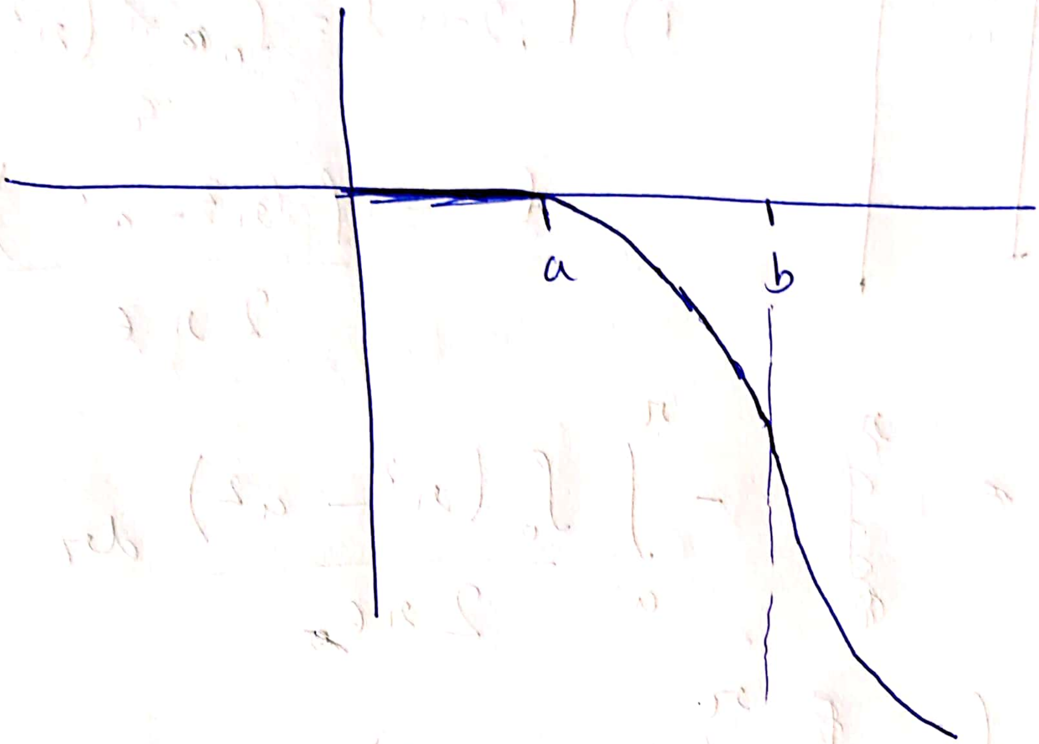
b) $(E \times 2\pi r)\cancel{l} = \frac{\rho_0 \pi (b^2 - a^2) l}{\varepsilon_0} = \frac{\rho_0 (b^2 - a^2)}{2 r \epsilon_0}$

$$V = -\int_a^r E \cdot dr$$

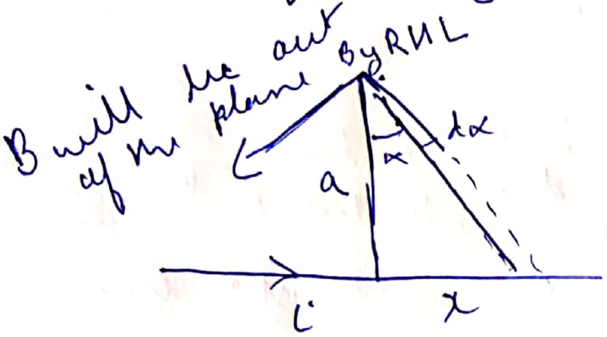
$$= -\int_a^b E \cdot dr - \int_b^r E \cdot dr$$

$$V = -\frac{\rho_0}{2\epsilon_0} \left(\frac{b^2 - a^2}{2} - a^2 \ln(b/a) \right) - \frac{\rho_0 (b^2 - a^2)}{2 \epsilon_0} \ln r/b$$

c)



Q4) We first find magnetic field due to 1 wire of length l ~~at~~ at a distance d



Consider an element of length dx along the wire

Let $d\bar{B}$ be magnetic field induction due to dx

By Biot-Savart Law.

$$d\bar{B} = \frac{\mu_0 i}{4\pi} \cdot \frac{(d\bar{x} \times \bar{r})}{r^3}$$

$$|d\bar{B}| = \frac{\mu_0 i}{4\pi} \frac{dx\,(\cancel{x})}{r^2} \sin\left(\frac{\pi}{2} + \alpha\right)$$

$$= \frac{\mu_0 i}{4\pi} \frac{dx \cos\alpha}{(a/\cos\alpha)^2}$$

$$\frac{x}{a} = \tan\alpha$$

$$dx = a \sec^2\alpha \, d\alpha$$

$$= \frac{\mu_0 i}{4\pi a^{\cancel{2}}} \cancel{a} \cos^{\cancel{3}}\alpha \, \cancel{\sec^2}\alpha \, d\alpha$$

$$|dB| = \frac{\mu_0 i}{4\pi a} \times \cos\alpha \, d\alpha$$

Total Magnetic Field $= \int |d\bar{B}| = \int_{-\theta}^{\theta} \frac{\mu_0 i}{4\pi a} \cos\alpha \, d\alpha$

where $\theta = \tan^{-1}\frac{l}{2a}$

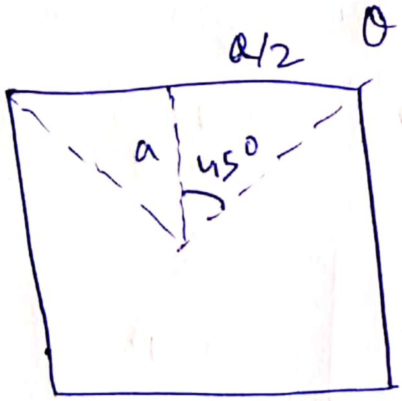
$$\therefore \bar{B}_{net} = \int_{-\tan^{-1}\frac{l}{2a}}^{\tan^{-1}\frac{l}{2a}} \frac{\mu_0 i}{4\pi a} \cos\alpha \, d\alpha$$

$$= \frac{\mu_0 i}{4\pi a}\left[\sin\alpha\right]_{-\theta}^{\theta}$$

$$= \frac{\mu_0 i}{4\pi a} 2\sin\theta = \frac{\mu_0 i}{2\pi a}\sin\theta$$

Now for a square

$\theta = 45^\circ$



and there are 4 such sides

$$\therefore B_{\cancel{tota}\ total} = \frac{\mu_0 i}{2\pi a} \times \frac{1}{\sqrt{2}} \times 4$$

$$= \frac{\sqrt{2}\,\mu_0 i}{\pi a}$$

5) Vector Potential of a single dipole moment m is:

$$A(r) = \frac{\mu_0}{4\pi} \frac{m \times \hat{r}}{r^2}$$

Lets take $\bar{M}$ as mean density of m at any generic point

$\therefore$ $m = M \, d\tau$ so the total potential is:

$$\Rightarrow A(r) = \frac{\mu_0}{4\pi} \int \frac{M(r') \times \hat{r}}{r^2} d\tau$$

$$= \frac{\mu_0}{4\pi} \int \left[M(r') \times \left(\nabla' \frac{1}{r}\right)\right] d\tau$$

Using Product rule: $\left\{\nabla' \frac{1}{r} = \frac{\hat{r}}{r^2}\right\}$

$$A(r) = \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{r}\left(\nabla' \times M(r')\right) d\tau' - \int \nabla' \times \left[\frac{M(r')}{r}\right] d\tau \right\}$$

Using surface integral

$$A(r) = \frac{\mu_0}{4\pi} \int \frac{1}{r} \left[\nabla' \times M(r')\right] d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{r} \left(M(r') \times da'\right)$$

Defining $\nabla \times M = J_B$ (Volume current)

and $M \times \hat{n} = K_B$ (surface current)

$$A(r) = \frac{\mu_0}{4\pi} \int_V \frac{J_B(r')\,d\tau'}{r} + \frac{\mu_0}{4\pi} \oint \frac{K_B(r')\,da'}{r}$$

This means the potential and hence the field of a magnetised object is the same that would be produced by a volume current throughout the material as well as the surface current on the boundary.
This eliminates the need of integrating the contribution of small dipoles.

Physical interpretation :

Surface Current :

Consider a slab of uniform magnetized material. Tiny current loops are like dipoles. It is equivalent to single ribbon of current I along the boundary.

$\bar{m} = \bar{m}\, a r$

$\bar{m} = I a$

$I a = m a t \Rightarrow K_B = \frac{I}{t} = m$

Volume Current.

This is a different kind of current, as not a single charge makes the whole trip, each charge only moves in a tiny little loop within a single atom.

The net effect is a macroscopic current flowing in the volume.

Q6) For a linear medium,
in small fields limits;

$\bar{m} = \chi_m \vec{H}$ (χ_m is the magnetic susceptibility)

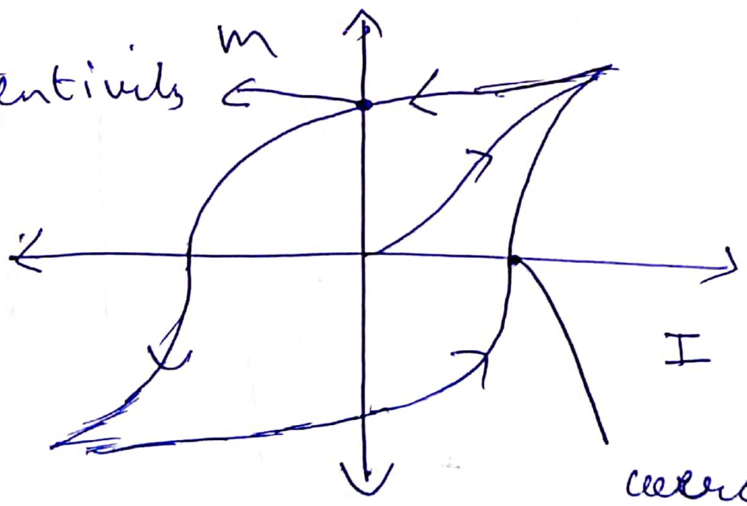
$\Rightarrow \bar{B} = \mu_0(\bar{H} + \bar{m}) = \mu_0(1+\chi_m)\bar{H}$

$\Rightarrow \vec{B} = \mu \vec{H}$, where $\mu = \mu_0(1+\chi_m)$.

μ is permeability of the free space

→ In paramagnetic materials, magnetisation is lost as soon as B_{ext} (or I) vanishes whereas in ferromagnetic material, magnetisation persists even after B_{ext} (or I) is removed.

→ retentivity



Hysteresis Loop

Q7)

a) Physical Interpretation of Maxwell's eq^n

1) ~~B~~ 1st eq^n : Gauss's Law

$$\nabla \cdot \bar{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \bar{D} = \rho_f$$

$$\oint E \cdot da = \frac{q}{\epsilon_0} \qquad \oint D \cdot da = q_f$$

It signifies that electric flux through any closed surface is given by $\frac{1}{\epsilon_0}$ times the charge enclosed by it.

2) 3rd eq^n : Faraday's Law

$$\nabla \times E = \frac{-d\bar{B}}{dt} \quad \text{or} \quad \oint_L \bar{E} \cdot \bar{dl} = \frac{-d\phi_B}{dt}$$

→ This signifies that a time dependant magnetic field generates an electric field.

Also, uniform electric field cannot exist in time dependant magnetic field

3) 2^{nd} eq^n

$$\nabla \cdot \vec{B} = 0$$

$$\oint_S B \cdot d\bar{A} = 0$$

Physically, this means that magnetic flux through a closed surface is always zero.

or, It states that isolated magnetic mono poles cannot exist. They appear only in pairs.

4) 4^{th} equation aka Amper's law

$$\nabla \times B = \mu_0 \epsilon_0 \frac{d\bar{E}}{dt} + \mu_0 \bar{J}$$

$$\oint_L B \cdot dl = \mu_0 i_{enc} + \mu_0 \epsilon_0 \frac{d\phi_E}{dt}$$

$$\nabla \times H = J_F + \frac{d\bar{D}}{dt}$$

$$\oint_P H \cdot dl = i_{F_{enc}} + \frac{d}{dt} \oint_S E \cdot dn$$

It tells us that time varying electric field produces a magnetic field

It also tells that curl of $\bar{H}$ along a closed surface gives the free current passing through it in absence of ~~closed~~ time varying magnetic field.

Physical interpretation of Poynting's theorem:

$$\frac{dw}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B_0^2}{\mu_0}\right) d\tau - \frac{1}{\mu_0}\oint (E \times \bar{B}) \cdot dA)$$

or $$\frac{dw}{dt} = -\frac{d\,U_{em}}{dt} - \oint S \cdot dA$$

The rate of work done on charge by EM waves = rate of decrease of total energy stored in fields + the energy flowing out of the surface per unit time.

Q7) b) Given a charge 'q', which moves an amount dl in time dt, due to magnetic force F acting on it. Then

$$dw = \bar{F}_m \cdot dl$$

$$= q(\vec{v} \times \vec{B}) \cdot (v\,dt) \quad \text{[Lorentz Force]}$$

$$= q\,dt\,[(\bar{v} \times \bar{B}) \cdot \bar{v})] = 0$$

$v \times B$ is always $\perp$ to $\vec{v}$ & $\vec{B}$

$\therefore$ dot product with v will be 0

$\cos 90^\circ = 0$

In general, magnetic force is always perpendicular to velocity of the charge.

Hence it cannot do any work on the charge

$$\boxed{W_{magnetic} = 0}$$

8) a) $\bar{B} = \left|\frac{\bar{E}}{c}\right| (\hat{z} \times \hat{x})$ ← direction of propagation in vaccum

$\bar{E}(z,t) = E_0 \cos(3\omega t) \cos(3kz)\,\hat{x}$

$\bar{B}(z,t) = \frac{E_0}{c} \cos(3\omega t) \cos(3kz)\,\hat{y}$

b) $\bar{S} = \frac{\bar{E} \times \bar{B}}{\mu_0} = \frac{E_0^2}{\mu_0 c} \cos^2(3\omega t) \cos^2(3kz)\,\hat{z}$

$\langle \bar{S} \rangle = \frac{E_0^2}{\mu_0 c} \cos^2(3kz) \times \frac{1}{2} \quad \left(\langle \cos^2 \rangle = \frac{1}{2}\right)$

$= \frac{1}{2} \epsilon c E^2 \cos^2(3kz)$

$\boxed{\frac{1}{\mu_0} = \epsilon_0 c^2}$

the average pointing flux is dependant on the position of the point (z) as this is a standing wave not a travelling wave.

10) $E = \mathrm{Re}\left[E_0 \, e^{i(kz - \omega t)}\right]$

$$= \mathrm{Re}\left[E_0 \, e^{i\left(\frac{2\pi (n+i\beta) z}{\lambda_0} - \omega t\right)}\right]$$

$$= e^{-\frac{2\pi \beta z}{\lambda_0}} \, \mathrm{Re}\left[E_0 \, e^{i(kz - \omega t)}\right]$$

So due to ~~...~~ conducting, the complex refractive index does not induce a phase difference since $n_{rel} = n(1 + i\beta_0)$

this real Part is the part that accounts for phase change and that is 1.

Therefore the phase change will be $\underline{\underline{0}}$ for reflection.

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Ans 11) $k = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2\left(1 + \frac{i\nu}{\omega}\right)}\right)^{1/2}$

where $\omega_p = 1.01 \times 10^{12}\ Hz$

and $\nu = 10^7\ Hz$

a) case I : $\omega \ll \nu$

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2\left(1 + \frac{i\nu}{\omega}\right)}\right) \approx \frac{\omega^2}{c^2}\frac{\omega_p^2}{\omega\nu} i$$

$$\Rightarrow k = \frac{\omega_p}{c}\left(\frac{\omega}{\nu}\right)^{1/2}\left(\frac{1+i}{\sqrt{2}}\right) = k_{real} + i k_{imaginary}$$

Skin depth :

$$\delta = \frac{1}{k_i} = \frac{c}{\omega_p}\left(\frac{2\nu}{\omega}\right)^{1/2}$$

Case II : $\nu \ll \omega \Rightarrow \frac{\nu}{\omega} \longrightarrow 0$

For $\omega < \omega_p \rightarrow$ $k = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} = i\alpha$

where $\alpha = \frac{\omega}{c}\left(\frac{\omega_p^2}{\omega^2} - 1\right)^{1/2}$

$\therefore$ skin depth

$$\delta = \frac{1}{\alpha} = \frac{c}{\sqrt{\omega_p^2 - \omega^2}}$$

b) i) $\omega = 10^3\ Hz \ll \nu$

$\Rightarrow \delta = \frac{c}{\omega_p}\left(\frac{2\nu}{\omega}\right)^{1/2}$

$= \frac{3\times10^8}{1.01\times10^{12}} \times \left(\frac{2\times10^7}{10^3}\right)^{1/2}$

$= \frac{3}{1.01}\times10^{-4}\times\sqrt{2}\times10^2$

$= \frac{3\sqrt{2}}{1.01}\times10^{-2}$

$\approx 4.20\times10^{-2}\,m$ Ans

ii) $\underset{(10^{12})}{\omega} \gg \nu$

$\delta = \frac{c}{\sqrt{\omega_p^2-\omega^2}} = \frac{3\times10^8}{\sqrt{(1.01)^2\times10^{24}-(1)^2\times10^{24}}}$

$= \frac{3\times10^8}{10^{12}\sqrt{2.01\times0.01}}$

$= \frac{3\times10^8}{10^{12}\sqrt{2.01\times0.01}} = \frac{3\times10^{-4}}{0.1\sqrt{2.01}}$

$\approx 2.12\times10^{-3}\,m$ Ans