Electrostatics

PYL101: Electromagnetics & Quantum Mechanics
Semester I, 2020-2021

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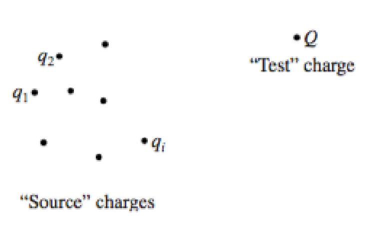
References

- **Introduction to Electrodynamics**, David J. Griffiths [IEDJ]
  1. Chapter II, Electrostatics

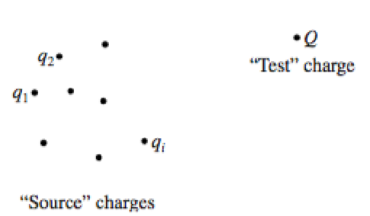
Superposition



- **Problem:** What's the **force** experienced by the test charge Q due to the *source charge distribution* q_i?
- We sum the forces $\boldsymbol{F}_i$ due to the *individual* charges in the source point charge distribution.
- *i.e.*, we use the **principle of superposition**...
- ...is exactly true because Maxwell's equations are **linear** in both the **sources** (ρ and $\boldsymbol{j}$) and the **fields** ($\boldsymbol{E}$ and $\boldsymbol{B}$). *E.g.*, consider *Gauss' law*,

$$\nabla\cdot\boldsymbol{E}_1=\frac{\rho_1}{\epsilon_0} \quad \text{and} \quad \nabla\cdot\boldsymbol{E}_2=\frac{\rho_2}{\epsilon_0} \Longrightarrow \underbrace{\nabla\cdot}_{\text{linear operator}}(\boldsymbol{E}_1+\boldsymbol{E}_2)=\frac{\rho_1+\rho_2}{\epsilon_0} \Longrightarrow \nabla\cdot\boldsymbol{E}_T=\frac{\rho_T}{\epsilon_0}$$

The Meaning of Electrostatics



- However, in general these forces depend not only on the position of the source charges, but also on both their **velocities** and **acceleration**.[1]
- We sweep all these issues under the rug for now, and deal only with source charges that are **stationary**, or *fixed* in space.
- ...which is the domain of **electrostatics**.

[1] With the added complication that EM fields travel at the speed of light c, and so the force experienced by the test charge is due to what happened at the source charge distribution at a time $\sim d/c$ ago.

Coulomb's Law

- Q: What's the force on a test charge Q located at $\boldsymbol{r}$ due to a single point charge q located at $\boldsymbol{r}'$ which is at **rest**?
- *Empirical* observations suggest that this is given by **Coulomb's Law**,

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{qQ}{s^2}\hat{\boldsymbol{s}}$$

  where the separation vector $\boldsymbol{s} = \boldsymbol{r} - \boldsymbol{r}'$
- The constant ϵ_0 is called the **permittivity of free space**, which in SI units is

$$\epsilon_0 = 8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N.m}^2}$$

- The Coulombic force is **repulsive/attractive** when the charges have the same/opposite sign.

The Electric Field

- For **several** source point charges, using the principle of superposition we get,

$$\boldsymbol{F}(\boldsymbol{r}) \equiv \frac{Q}{4\pi\epsilon_0}\sum_{i}^{n}\frac{q_i}{s_i^2}\hat{\boldsymbol{s}}_{\boldsymbol{i}}$$

- The total **electric field** $\boldsymbol{E}$ due to the source charges (at the test charge Q location $\boldsymbol{r}$) is defined as,

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{\boldsymbol{F}(\boldsymbol{r})}{Q}$$

- Giving us,

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i}^{n}\frac{q_i}{s_i^2}\hat{\boldsymbol{s}}_{\boldsymbol{i}}$$

Continuous Charge Distributions

- Q: What if the charges were described via a **continuous** charge distribution?
- In this case, we simply **integrate**,

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{1}{4\pi\epsilon_0} \int \frac{dq}{s^2} \hat{\boldsymbol{s}}$$

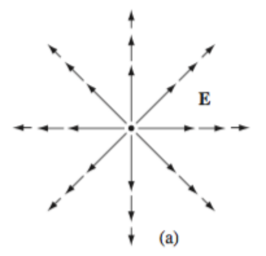
- If the charge distribution is three-dimensional[2], the term dq can instead be written as a product of the **volume charge density** $\rho(\boldsymbol{r}')$ and the volume element[3] $d\tau'$ giving us,

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}') d\tau'}{s^2} \hat{\boldsymbol{s}}$$

[2] How about one- and two-dimensions?

[3] What's the variable that we're integrating over -is it $\boldsymbol{r}$, or $\boldsymbol{r}'$?

Vector Field

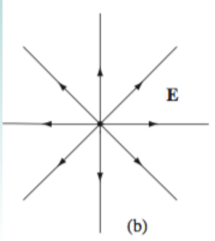


- Consider a *single* point charge q lying at rest at the origin $\mathscr{O}$. Its electric field is given by,

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

- This vector field of $\boldsymbol{E}$ can be represented by vectors that get *shorter* as you go farther away from the origin; and they always point radially *outward*.

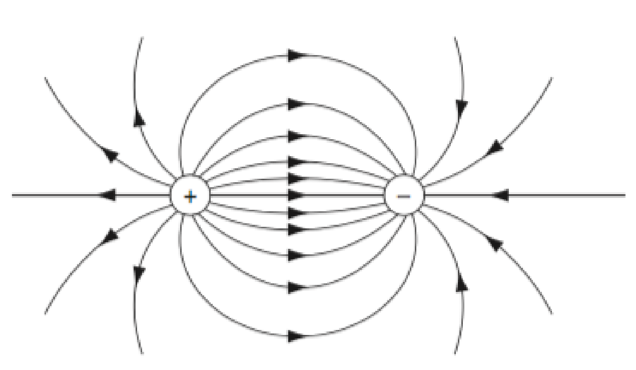
Field Lines



- Instead of drawing the vector field, we can also connect up the arrows, to form **field lines**.
- The *subtle difference* here is that the *magnitude of the field* at a point is indicated by the density of the field lines at that point.
- In this case, the electric field is *stronger near the center* where the field lines are close together, and *weaker farther out*, where they are relatively far apart.
- Field lines can **never** cross each other at a point.[4]

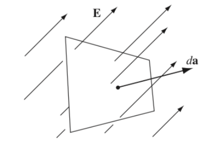
[4] Why?

Field Lines Between Opposite Charges



- Note how the field lines point from positive to negative charges…
- They point *outwards* from positive charges
- They point *into* negative charges
- Q: Can field lines discern between $\frac{\hat{\mathbf{r}}}{r}$ and $\frac{\hat{\mathbf{r}}}{r^2}$?
- Q: Can field lines depict functions that are *increasing* such as $r\ \hat{\mathbf{r}}$?

The Electric Field Flux Φ_E and Gauss' Law



- The **flux** of $\boldsymbol{E}$ through a surface $\mathscr{S}$,

$$\Phi_E = \int_{\mathscr{S}} \boldsymbol{E} \cdot d\boldsymbol{a}$$

is a *measure* of the number of field lines **passing/threading through** $\mathscr{S}$. Because of the dot product with $d\boldsymbol{a}$, electric field lines $\perp$ to the surface plane $\mathscr{S}$ contribute most appreciably to Φ_E.

- This *suggests* **Gauss' Law**,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{\sum q_{\text{encl.}}}{\epsilon_0}$$

- Using the *divergence theorem* we can recast this into,

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

Deriving Gauss' Law from Coulomb's Law

- …if you're still not convinced, let's start from **Coulomb's law** for a 3-d charge distribution,

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r'})d\tau'}{s^2}\hat{\boldsymbol{s}}$$

  and note that the $\boldsymbol{r}$ dependence is contained in $\boldsymbol{s}$ and further that,

$$\nabla_{\boldsymbol{r}} \cdot \left(\frac{\hat{\boldsymbol{s}}}{s^2}\right) = 4\pi\delta^3(\boldsymbol{s})$$

- Evaluating the *divergence* we have,

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\int 4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r'})\rho(\boldsymbol{r'})d\tau' = \frac{\rho(\boldsymbol{r})}{\epsilon_0}$$

  which is the **differential form of Gauss' Law**.

- Applying the **divergence theorem** gives us the integral form of Gauss' law,

$$\oint \boldsymbol{E}\cdot d\boldsymbol{a} = \frac{\sum q_{\text{encl.}}}{\epsilon_0}$$

Canonical Example of Invoking Gauss' Law

- **Problem:** calculate the field $\boldsymbol{E}$ due to a *single* point charge q using **Gauss' law**.
- Gauss' Law states,

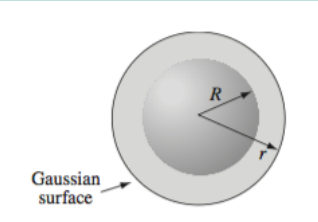
$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{\sum q_{\text{encl.}}}{\epsilon_0}$$

where $\sum q_{\text{encl.}} = q$ and the field due to the point charge is, of course, turns out to be,

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

- Think carefully about the assumptions you're making!
- We've exploited the *symmetry* of the spherical (Gaussian) surface, and the **radial** symmetry of the field due to to the charge.

$\boldsymbol{E}$ due to a solid sphere of total charge q



- **Problem:** What's $\boldsymbol{E}$ **outside** a uniformly charged solid sphere of radius R and total charge q?
- Gauss' Law tells us that the field is simply,

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}} \quad (r \geq R)$$

- A **remarkable** feature of this result is that $\boldsymbol{E}$ outside the sphere is exactly the same as it would have been if all the charge had been concentrated at the center!

Applying Gauss' Law Profitably

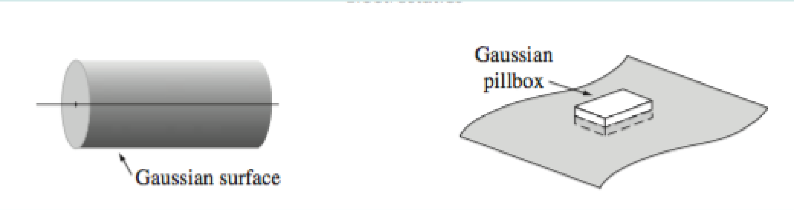
- The previous example should've alerted you to the fact that we required
  - the ***symmetry*** of the *source charge distribution* and,
  - a ***compatible symmetry*** *of the Gaussian surface* $\mathscr{S}$,

  in order to make Gauss' law useful in calculating $\boldsymbol{E}$.
- Symmetry permitting, we usually exploit the fact,

$$\boxed{\int_{\mathscr{S}} \boldsymbol{E}\cdot d\boldsymbol{a} = \int_{\mathscr{S}} |\,\boldsymbol{E}\,|\, da}$$

- When ρ isn't uniform (or doesn't have the requisite symmetry), or if the Gaussian surface is weird, *it's unlikely* that we can apply the integral form Gauss' law profitably to evaluate $\boldsymbol{E}$.

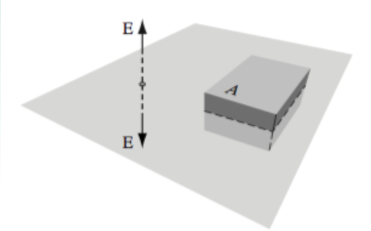
Symmetries That Work with Gauss' Law



- Apart from spherical symmetry, we can exploit Gauss' Law when we have,
  - **Cylindrical symmetry**. Make your Gaussian surface a coaxial cylinder
  - **Plane symmetry**. Use a Gaussian *pillbox* that straddles the surface[5]

[5]Both the cylindrical and plane case only precisely work for *infinitely* long, or large boxes.

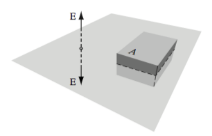
Field due to a large sheet of charge density σ



- **Problem:** Find $\boldsymbol{E}$ due to an infinitely large sheet with uniform charge density σ.
- Applying Gauss' Law,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{\sum q_{\text{encl.}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

Field due to a large sheet of charge density σ



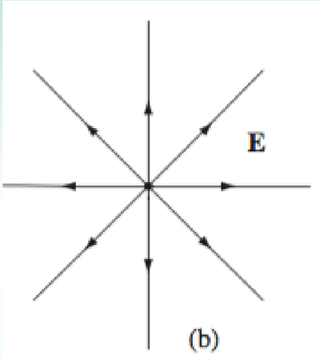
- Symmetry tells us that $\boldsymbol{E}$ points away from the surface plane on each side of the Gaussian surface/*pillbox*.
- Giving us,

$$2A\,|\,\boldsymbol{E}\,| = \frac{1}{\epsilon_0}\sigma A \Rightarrow \boldsymbol{E} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}}$$

, where $\hat{\boldsymbol{n}}$ is the surface normal pointing away from the surface.

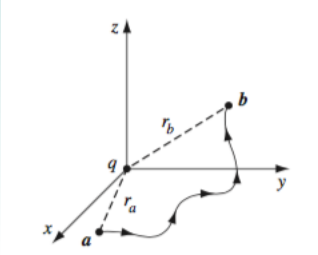
- Why's the field the same regardless of how far away from the sheet we are?
- **Q**: Find the electric field $\boldsymbol{E}$ in, and around a parallel-plate **capacitor** problem (two sheets with opposite charge, a distance d apart.) as HW.

The Curl of $\boldsymbol{E}$ due to a Single Point Charge q



- Recall the *paddlewheel* analogy. Do you expect the $\boldsymbol{E}$ field due to a single stationary point charge to have a **curl**?

$\nabla \times \boldsymbol{E}$ due to a Static Charge Distribution



- Consider the field due to a **single** stationary point charge located at the origin,

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$$

- We now integrate $\boldsymbol{E}$ over an **arbitrary path** $a \to b$ as shown above.
- In *spherical coordinates* $d\boldsymbol{l} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$,

$$\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} = \int_a^b \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

$\nabla \times \boldsymbol{E}$ due to a Static Charge Distribution

- If the **loop is closed**, **i.e.,** $a = b$ we get,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

- Thus using ~~**Stokes' theorem**~~ **the fundamental theorem of gradients**, **and property (2) of second derivatives (*i.e.,* $\nabla \times \nabla V = 0$) we obtain**[6],

$$\boxed{\nabla \times \boldsymbol{E} = 0}$$

- If we had multiple charges q_i, each contributing their own field $\boldsymbol{E}_i$, we'd still get a zero via the static form of Faraday's law (**principle of superposition**),

$$\underbrace{\nabla \times}_{\text{linear operator}} \boldsymbol{E} = \nabla \times (\boldsymbol{E}_1 + \boldsymbol{E}_2 + \ldots) = 0$$

- In fact, $\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$, and $\nabla \times \boldsymbol{E} = 0$ hold for any **static** charge distribution.

[6]See FAQ 4 for details.

FAQ

- Q: Given that
$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0,$$
how can we conclude that $\nabla \times \boldsymbol{E} = 0$?
- Notice that **Stokes' Theorem**, *i.e.*,
$$\int (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{a} = \oint \boldsymbol{E} \cdot d\boldsymbol{l}$$
does not really promise: $\nabla \times \boldsymbol{E} = 0$; that we're only really guaranteed,
$$\int (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{a} = 0$$
while it may be possible that somehow $\nabla \times \boldsymbol{E} \neq 0$.

FAQ

- We know from the **converse** of the **fundamental theorem of gradients**, *i.e.*, "If the integral of $\boldsymbol{F}$ over every closed loop in the domain of $\boldsymbol{F}$ is zero, then $\boldsymbol{F}$ is the gradient of some scalar-valued function." that if

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0 \Longrightarrow \boldsymbol{E} = \nabla T$$

where T is some scalar function.

- From Rule No.(2): The curl of a gradient is always zero we obtain that,

$$\nabla \times \boldsymbol{E} = \nabla \times (\nabla T) = 0 \quad \square$$

The Electric Potential

- We've just showed that for an arbitrary **static** charge distribution,

$$\nabla \times \boldsymbol{E} = 0$$

- Recall that such vector functions, *e.g.*, $\boldsymbol{E}$, can be equivalently expressed as the gradient of a *scalar potential* as,

$$\nabla \times \boldsymbol{E} = 0 \Rightarrow \boldsymbol{E} = -\nabla(V + \text{const.})$$

- *Remarkably*, all the information contained inside $\boldsymbol{E}$ (a vector) is contained inside the scalar potential V.
- This is because the components of $\boldsymbol{E}$ are not indepedent[7],

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

[7] Why?

The Electric Potential

- The scalar potential also obeys the *superposition principle*,

$$V = V_1 + V_2 + V_3 + \ldots$$

- Given $\boldsymbol{E} = -\nabla V$, and the differential form of Gauss' law we obtain **Poisson's equation**,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

- For a **charge free** region[8] we get **Laplace's equation**,

$$\nabla^2 V = 0$$

[8] Shouldn't $\boldsymbol{E}$ be *trivially* zero as dictated by the Helmholtz theorem?

Calculating V from $\boldsymbol{E}$

- We obtain V from $\boldsymbol{E}$ via the line **integral**,

$$\boldsymbol{E} = -\nabla V \Longrightarrow V(\boldsymbol{r}) - \underbrace{\cancel{V(\boldsymbol{r} = \infty)}}_{0} = -\int_{\infty}^{\boldsymbol{r}} \boldsymbol{E}(\boldsymbol{r}') \cdot d\boldsymbol{l}'$$

  where the lower limit is ∞ since we (conveniently) assume the potential to be **zero** there.[9]

- For a collection of stationary source charges described by a charge density ρ, we can get $V(\boldsymbol{r})$ as,

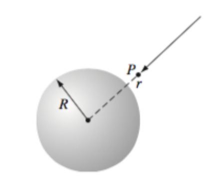
$$\boxed{V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}')}{s} d\tau'}$$

  where s is the **distance** from the infinitesimal charge location $\boldsymbol{r}'$) to the point $\boldsymbol{r}$.[10]

[9] We can get away doing this since the potential V is defined only to within a constant. Adding any constant will not change the value of $\boldsymbol{E}$ derived from the V.

[10] Study Example 8, pg. 86 from [IEDJ].

Potential Due to a Uniformly Charged Spherical Shell



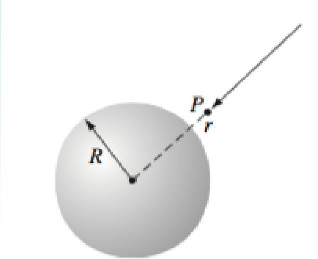
- **Problem**: What's the potential due to a uniformly charged spherical **shell** of radius R?
- Since the entire charge lies solely at the surface of the sphere,

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} 0 & r \leq R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}} & r > R \end{cases}$$

- *Outside* the sphere,

$$V(\boldsymbol{r}) = -\int_{\infty}^{\boldsymbol{r}} \boldsymbol{E}\cdot d\boldsymbol{l}' = -\int_{\infty}^{\boldsymbol{r}} E(r')dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Potential Due to a Uniformly Charged Spherical Shell



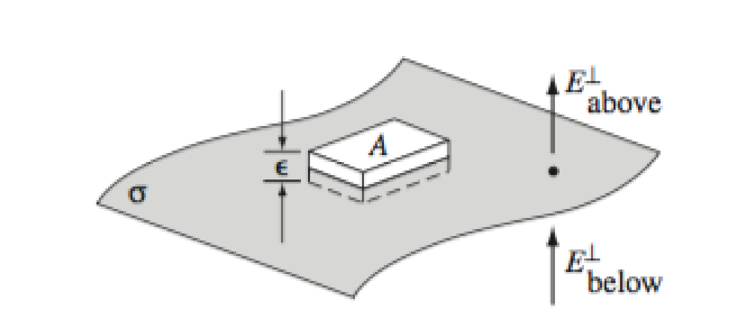
- The potential *inside* $(r < R)$ the sphere is[11],

$$V(\boldsymbol{r}) = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{R}\frac{q}{r'^2}\,dr' - \int_{R}^{r}(0)\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

- Note, that the potential is not zero inside the shell, even though the field is! **Indeed, V is a non-zero constant inside the shell.**

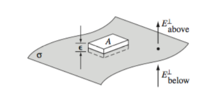
[11] Make sure you understand how to break up the integral $\int_r^{\infty} d\boldsymbol{r'}$.

Discontinuity in the Normal Component of $\boldsymbol{E}_\perp$



- **Problem**: How does the $\perp$ component of $\boldsymbol{E}$ change *across* a surface boundary?

Discontinuity of the Normal Component of $\boldsymbol{E}_\perp$



- Using Gauss' Law,

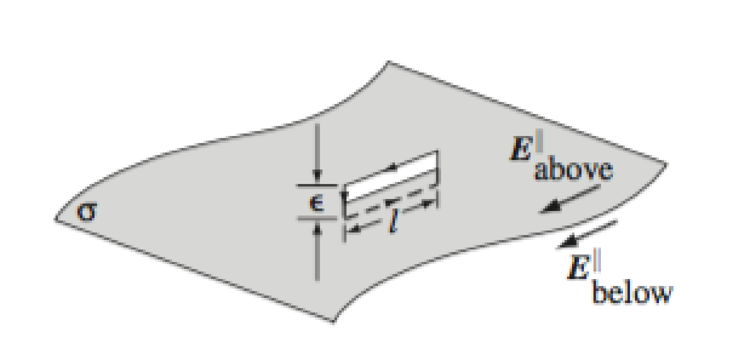
$$\oint_{\mathscr{S}} \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{1}{\epsilon_0} \sum q_{\text{encl.}} = \frac{1}{\epsilon_0} \sigma A$$

  where A is the area of the **Gaussian pillbox**.
- We're free to make the thickness of the pillbox as short as we like, and thus the sides do no contribute anything to the flux giving us,

$$\boxed{\boldsymbol{E}_\perp^{\text{above}} - \boldsymbol{E}_\perp^{\text{below}} = \frac{\sigma}{\epsilon_0}}$$

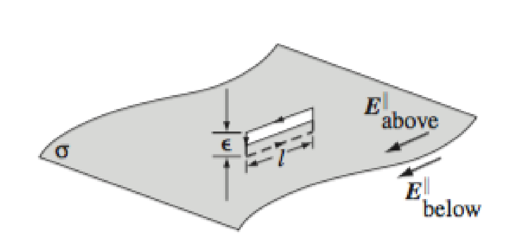
- Of course, when $\sigma = 0$, $\boldsymbol{E}_\perp^{\text{above}} = \boldsymbol{E}_\perp^{\text{below}}$

Continuity of the Tangential Component of $\boldsymbol{E}_{\parallel}$



- **Problem:** How does the $\parallel$ component of $\boldsymbol{E}$ change *across* a surface boundary?
- Which **"law"** do we invoke here?

Continuity of the Tangential Component of $\boldsymbol{E}_\parallel$



- Given that we know that for *purely static* charges,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

we construct an **Amperian loop** as above with shrinking height ϵ giving us,

$$\boxed{\boldsymbol{E}_\parallel^{\text{above}} = \boldsymbol{E}_\parallel^{\text{below}}}$$

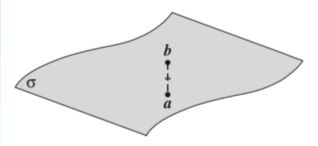
Continuity of $\boldsymbol{E}_{\parallel}$

- We *may* combine the continuity conditions for the $\perp$, and $\parallel$ components of $\boldsymbol{E}$ to write,

$$\boxed{\boldsymbol{E}^{\text{above}} - \boldsymbol{E}^{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}}$$

where is $\hat{\boldsymbol{n}}$ is the unit vector pointing in the direction normal to the bounding surface.

Continuity of V, and Discontinuity of ∇V



- What are the continuity conditions for the scalar potential V?
- Since V is defined as,

$$V_{\text{above}} - V_{\text{below}} = -\int_{P_b}^{P_a} \boldsymbol{E} \cdot d\boldsymbol{l}$$

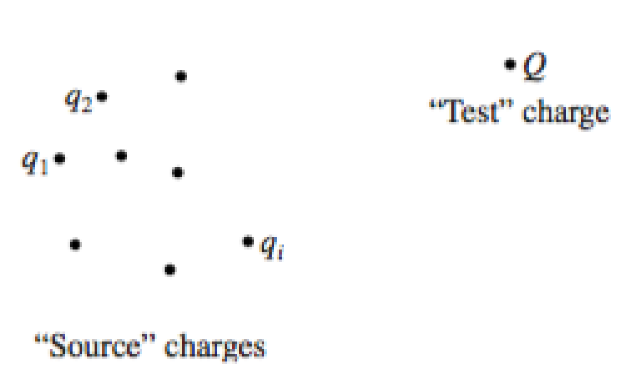
and as we shrink the perpendicular path P_{AB} to zero, so does the RHS and thus,

$$\boxed{V_{\text{above}} = V_{\text{below}}}$$

- However, the **gradient** of V *inherits* the discontinuity from $\boldsymbol{E}$ as,

$$\boxed{\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0}\sigma\hat{\boldsymbol{n}}}$$

The Work It Takes to Move a Test Charge



- **Problem:** *Given a stationary set of source charges*, how much **work** is needed in order to move a test charge Q from points a to b?

The Work It Takes to Move a Test Charge

- Recall the relationship between work W and force $\boldsymbol{F}$,

$$W \equiv \int_a^b \boldsymbol{F} \cdot d\boldsymbol{l}$$

- The force $\boldsymbol{F}$ **you** must exert **against** the electrical force is $\boldsymbol{F} = -Q\boldsymbol{E}$,

$$W = -Q \int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = +Q\ [V(\boldsymbol{b}) - V(\boldsymbol{a})]$$

- The work required W is independent of the path[12] taken from $\boldsymbol{a}$ to $\boldsymbol{b}$. Such forces are known as **conservative** in the language of mechanics.

[12] How do we know?

Assembling a Bunch of Point Charges

- **Problem:** How much work is required (*by us*) to assemble a distribution of point charges brought in from ∞?
- The *first charge* q_1 that we bring from infinity requires **zero** work, because there is no field that it feels.
- The *second charge* q_2 feels q_1 and the *additional/incremental* work needed is simply,[13]

$$W_2 = q_2 \left(\frac{q_1}{4\pi\epsilon_0 s_{12}} - 0 \right)$$

where s_{12} is the final distance between q_1 and q_2.
- Bringing in the *third* q_3, which feels both q_1, and q_2, requires an *additional/incremental*,

$$W_3 = q_3 \left(\frac{q_1}{4\pi\epsilon_0 s_{13}} + \frac{q_2}{4\pi\epsilon_0 s_{23}} - 0 \right)$$

, and so on ...

[13] We somehow keep q_1 (and all other source charges) fixed in place.

Assembling a Bunch of Point Charges

- Adding all these up we get,

$$W = \sum_{i=1}^{n} W_i = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{s_{ij}}$$

- Note to avoid self-interaction between charges we stipulate the condition $i \neq j$ in the second sum $\square$.
- We can recast the above equation as,

$$\boxed{W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\boldsymbol{r}_i)}$$

where $V(\boldsymbol{r}_i)$ is the potential **due to all other charges** at q_i's location.

The Energy of a Continuous Charge Distribution

- The previous expression for total energy, $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\boldsymbol{r}_i)$ can be written in integral form for a *continuous charge distribution* as,

$$W = \frac{1}{2}\int \rho V \, d\tau$$

- By using,
  1. the differential form of Gauss' Law $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$, and,
  2. $\boldsymbol{E} = -\nabla V$, and,
  3. *integration by parts*

  giving us finally an *alternative* expression for the stored electrostatic energy,

$$\boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} |\boldsymbol{E}|^2 \, d\tau}$$

- **Problem**: Find the **total energy** contained in a *uniformly charged* spherical **shell** of total charge q and R, starting from (a) its potential V, and (b) from its field $\boldsymbol{E}$.
- (a) Using the formula containing V we get,

$$W = \frac{1}{2}\int \sigma V da$$

and realizing the potential is a constant $V = \frac{q}{4\pi\epsilon_0 R}$ *precisely* at the shell[14], we get,

$$W = \frac{q^2}{8\pi\epsilon_0 R}$$

- (b) **Alternatively**, we integrate over the field $\boldsymbol{E}$ explicitly, which is simply the field due to a point charge everywhere outside the shell, and so we have,

$$W = \frac{\epsilon_0}{2}\int |\boldsymbol{E}|^2 d\tau = \frac{\epsilon_0}{2}\int_R^\infty \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 r^2 \sin\theta\, d\theta\, d\phi\, dr = \frac{q^2}{8\pi\epsilon_0 R}$$

[14]In fact, it must be continuous there according to the boundary condition derived earlier.

Where's the Electrostatic Energy Stored?

- Note that the integral
$$W = \frac{1}{2}\int \rho V \, d\tau$$
is a sum over the entire charge distribution, and so we had to (*spatially*) only integrate over the shell (*i.e.*, where the charge was located).
- While
$$W = \frac{1}{2}\int_{\text{all space}} |\boldsymbol{E}|^2 \, d\tau$$
is a sum over the electric field which extends from R to ∞.
- Q: **Spatially**, where's the electrostatic energy stored?

Where's the Electrostatic Energy Stored?

- If we stick to *electrostatics*, it's equally **OK** to say that the energy is stored in the charges themselves, <u>or</u> that it's stored in the electric field.
- However, in more advanced theories (*e.g.*, general relativity) only the view that the energy is contained in the **field** is correct.

Can The Total Electrostatic Energy be Negative?

- The expression we just derived,
$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$
implies that the total energy must always be **positive**!
- While,
$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\boldsymbol{r}_i)$$
for two equal but opposite charges the energy
$$-\frac{1}{4\pi\epsilon_0} \frac{q^2}{s}$$
which is clearly **negative**, so what gives?

Can The Total Electrostatic Energy be Negative?

- Consider that we went from a sum $\Longrightarrow$ integral, *i.e.*,

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \Longrightarrow \frac{1}{2}\int \rho(\mathbf{r}')V(\mathbf{r}')d\tau'$$

- The integral form actually introduces a **subtle error** in our estimate for W.
- In the sum above, $V(\mathbf{r}_i)$ is the potential experienced by the point charge q_i located at $\mathbf{r_i}$ (*i.e.*, it **excludes** the contribution of q_i itself)
- *Unfortunately*, in the integral form above $V(\mathbf{r}')$, **also** contains the contribution of the charge lying precisely at the location $\mathbf{r}'$, *i.e.*, $\rho(\mathbf{r}')$.
- For a continuous distribution, the amount of charge right at the point $\mathbf{r}'$, *i.e.*, $\rho(\mathbf{r}')$ is *vanishingly small*, and its contribution to the potential $V(\mathbf{r}')$ is zero, and we can use $W = \frac{\epsilon_0}{2}\int_{\text{all space}} |E|^2 d\tau$ without issue.

Can The Total Electrostatic Energy be Negative?

- However, if we had (*näively*) applied the expression obtained from the integral form,

$$W = \frac{1}{2}\int \rho(\boldsymbol{r'})V(\boldsymbol{r'})d\tau' = \frac{\epsilon_0}{2}\int_{\text{all space}} |E|^2 d\tau$$

for say, a **single point charge** q, it would blow up as,

$$W = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int \left(\frac{q^2}{r^4}\right)(r^2 \sin\theta\, dr\, d\theta\, d\phi) = \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{1}{r^2}dr = +\infty$$

, *i.e.*, the energy of a point charge is actually **positively infinite**!

- **Bottom line**: The energy of a static charge distribution <u>can certainly</u> be **negative**. In the presence of point charges remember that one should use,

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\boldsymbol{r}_i)$$

The Lack of a Superposition Principle

- **Claim**: Because electrostatic energy is **quadratic** in the fields, it does not obey a superposition principle!
- Indeed,

$$
\begin{aligned}
W_{tot} &= \frac{\epsilon_0}{2}\int |(\boldsymbol{E}_1 + \boldsymbol{E}_2)|^2 d\tau \\
&= \frac{\epsilon_0}{2}\int (|E_1|^2 + |E_1|^2 + 2\boldsymbol{E}_1 \cdot \boldsymbol{E}_2) d\tau \\
&= W_1 + W_2 + \epsilon_0 \int \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 d\tau \\
&\neq W_1 + W_2
\end{aligned}
$$

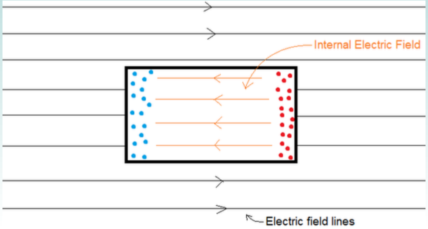
- If we double the charge everywhere, the energy quadruples!
- Electrostatic energy **does not** obey the superposition principle.

Insulators vs. Conductors

- At this *introductory level*, it suffices to think of an **insulator** or **dielectric**, such as glass or rubber, as materials whose valence electrons/clouds/orbitals are stuck (but <u>not</u> frozen) to their atomic cores.
- A **conductor**, such as most metals, on the other hand, have their valence electrons *nearly* free to roam the bulk of the material. Whereas An **idealized conductor** or metal contains an infinite supply of free electrons.

Defining Properties of Idealized Conductors

in *steady state*

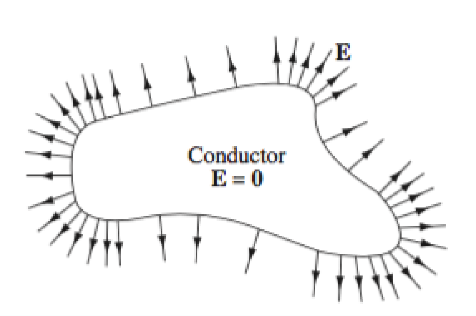


- The induced charges may only reside at the **surface** of the conductor in an *infinitesimally* thin layer.
- $\boldsymbol{E} = 0$ **inside an idealized conductor:** Almost instantaneously, **mobile** charges pile up at each of the edges to completely cancel the external electric field $\boldsymbol{E}_{\text{ext.}}$.
- Since the field in the bulk of the conductor is zero, it necessitates that the (*macroscopic*) charge density is also zero.[15]
- Since there is no field inside the bulk of the conductor, the potential V therein is a constant.

[15] Which Maxwell's law is implied here?

Defining Properties of Idealized Conductors

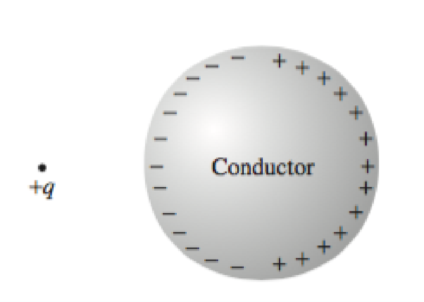
in *steady state*



- ***E*** is **perpendicular to the surface**, *just outside* a conductor. Otherwise, charge will immediately flow around the surface until it kills off the tangential component.
- Both the surface, and the interior (bulk) of the conductor are **equipotential** surfaces.[16]

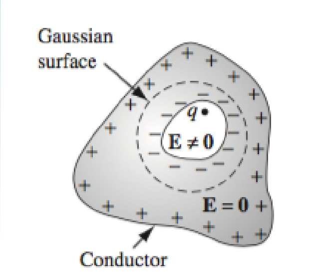
[16] Why's the surface an equipotential?

Induced Charges



- **Observation**: If you hold a charge $+q$ near an uncharged *idealized* conductor, the two will **attract** one another.
- The reason for this is that q will pull minus charges over to the near side and repel plus charges to the far side. Alternatively, we can think of the bulk of the metal reacting in a way to **expel** the field inside it.

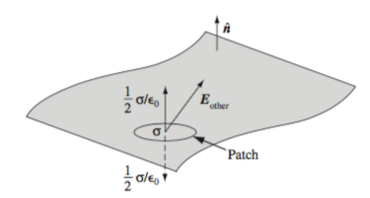
Induced Charges



- Suppose there is some **hollow cavity** inside the idealized conductor, and within that cavity you put some charge *positive* q.
- The field inside the cavity will not be zero[17]
- While the field inside the bulk of the conductor will be **zero**.
- The charge on the outermost surface of the conductor accumulates an overall charge exactly equal to q.

[17] Construct a Gaussian surface around q inside the cavity in order to verify this.

Surface Charge and the Force on a Conductor



- **Problem:** What's the force per unit area $\boldsymbol{f}$ experienced by the surface of an idealized conductor holding an induced surface charge σ?[18]
- Constructing the Gaussian pillbox and applying Gauss' law in integral form,

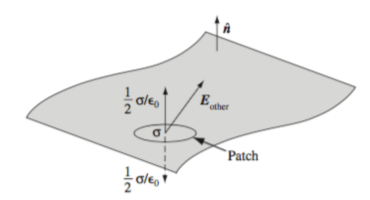
$$\boldsymbol{E}_{\text{outside}} - \boldsymbol{E}_{\text{inside}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}$$

- We already know that $\boldsymbol{E}_{\text{inside}} = 0$ since the field inside the idealized conductor must be **zero** and thus,

$$\boldsymbol{E}_{\text{outside}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}$$

[18] We'll neglect forces due to the charges which created the induced charge.

Surface Charge and the Force on a Conductor



- $\boldsymbol{E}_{\text{outside}}$ consists of **two** parts, attributable to the patch itself, and that due to everything else (other regions of the surface):

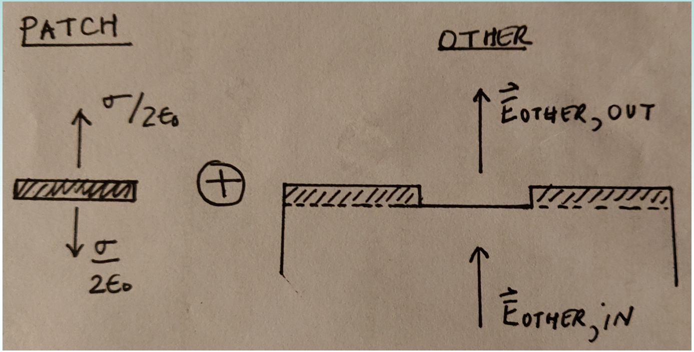
$$\boldsymbol{E}_{\text{outside}} = \boldsymbol{E}_{\text{patch}} + \boldsymbol{E}_{\text{other}}$$

- Now, the patch **cannot** exert a force on itself, and **the force on the patch**, then, is due *exclusively* to $\boldsymbol{E}_{\text{other}}$, *i.e.*,

$$\boldsymbol{f} = \frac{Q_{\text{patch}}\boldsymbol{E}_{\text{other}}}{A_{\text{patch}}}$$

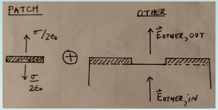
- We still need to somehow deduce $\boldsymbol{E}_{\text{other}}$.

Surface Charge and the Force on a Conductor



- *We look at the system afresh*, as a composite of
  1. (i) the peeled off patch, and,
  2. (ii) the remaining conductor (other),
- We will sum their contributions to get an *alternative* assessment of $\boldsymbol{E}_{\text{inside}}$, and $\boldsymbol{E}_{\text{outside}}$

Surface Charge and the Force on a Conductor



- Thus[19],

$$\boldsymbol{E}_{\text{outside}} = +\frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}} + \boldsymbol{E}_{\text{other,outside}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} \quad \textit{(from earlier)}$$

$$\boldsymbol{E}_{\text{inside}} = -\frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}} + \boldsymbol{E}_{\text{other,inside}}$$

and therefore,

$$\boldsymbol{E}_{\text{other}} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}}$$

- Thus, the **force per unit area** acting on the surface of the conductor is,

$$\boldsymbol{f} = \frac{Q_{\text{patch}}\boldsymbol{E}_{\text{other}}}{A_{\text{patch}}} = \frac{1}{2\epsilon_0}\sigma^2\hat{\boldsymbol{n}}$$

[19] $\boldsymbol{E}_{\text{other,outside}} = \boldsymbol{E}_{\text{other,inside}} = \boldsymbol{E}_{\text{other}}$ because there's no surface charge at the patch location once we have removed the patch.

Surface Charge and the Force on a Conductor

optional

- **Problem:** What's the electric field generated by an **isolated**[20]**, induced**[21] **point charge** lying at the origin of a flat xy surface of an idealized conductor?
- Is it radial outward all around, as usual? Why, or why not?
- Which equation/law do we invoke in order to solve for the field?
- What are the *boundary conditions*?
- Note: This problem is not amenable to treatment via the *method of images*? Why?

[20]While the metal accumulates a whole sheet of induced surface charge, we just isolate one constituent point charge in our mind's eye in order to investigate its field.

[21]We are unconcerned with the external charges that create the induced charge. All that matters is that the induced charges exist.

Mutual Capacitance

- Consider two **adjacent conductors** that are equally, and oppositely charged with $\pm Q$.
- **Mutual Capacitance**[22] is defined as the ratio between the magnitude of charges divided by the voltage difference V between two adjacent conductors,

$$C \equiv \frac{Q}{V}$$

- In SI units, it's measured in **Farads (F)** (Coulomb per Volt).
- The mutual capacitance is a purely[23] **geometrical** quantity, determined by the sizes, shapes, and separation of the two conductors.

[22] Commonly known as the **capacitance**. C is always taken to be positive.
[23] Though it depends on the properties of the material between the charges.

Self Capacitance

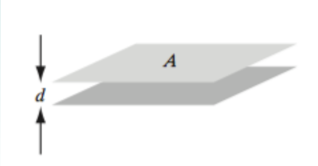
- While mutual capacitance is defined between adjacent conductors. There's another kind of capacitance called the **self-capacitance**, defined for **isolated** conductors.
- **Self capacitance** is defined as the amount of electric charge that must be added to an *isolated conductor* to raise its electric potential by one unit, *i.e.*,

$$C \equiv \frac{dQ}{dV}$$

- *E.g.*, the self capacitance of a conducting sphere of radius R is

$$C = 4\pi\epsilon_0 R$$

The Parallel-Plate Capacitor



- For a **parallel plate capacitor**, let's take the two infinitely large plate conductors to be charged with a uniformly distributed surface charge density $\pm\sigma$
- The mutual capacitance, as you might recall, works out to be,

$$C = \frac{A\epsilon_0}{d}$$

Capacitance of two concentric metal shells

- **Problem:** Find the mutual capacitance of two concentric spherical metal *shells*, with inner radius a and outer radius b? Let the inner shell have a charge $+Q$, the outer $-Q$.
- The field between the spheres is,

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\boldsymbol{r}} \quad (a \le r < b)$$

,

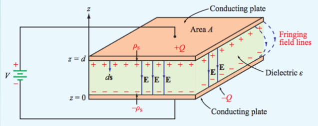
- Thus the potential difference between them is,

$$V = -\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{Q}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

- The **mutual capacitance** is,

$$C = \frac{4\pi\epsilon_0 ab}{b-a}$$

Work Required to Charge a Capacitor



- To charge up a capacitor, your battery has to remove **electrons** from the plate connected to the + terminal of the battery and carry them to the − one.
- You battery fights against the electric field ***E*** **between** the plates, which is pulling them back toward the positive conductor and pushing them away from the negative one.
- Using[24],

$$dW = V\,dq = \left(\frac{q}{C}\right) dq$$

- The work necessary, then, to go from $q = 0$ to $q = Q$, is the familiar expression...

$$W_C = \int_0^Q \left(\frac{q}{C}\right) dq \quad = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2$$

[24] We're using the expression for the work done in a moving a unit charge across a potential difference, $V(\boldsymbol{b}) - V(\boldsymbol{a}) = W/Q$