ELL100: INTRODUCTION TO ELECTRICAL ENGG.

Complete Response - I

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Complete Response

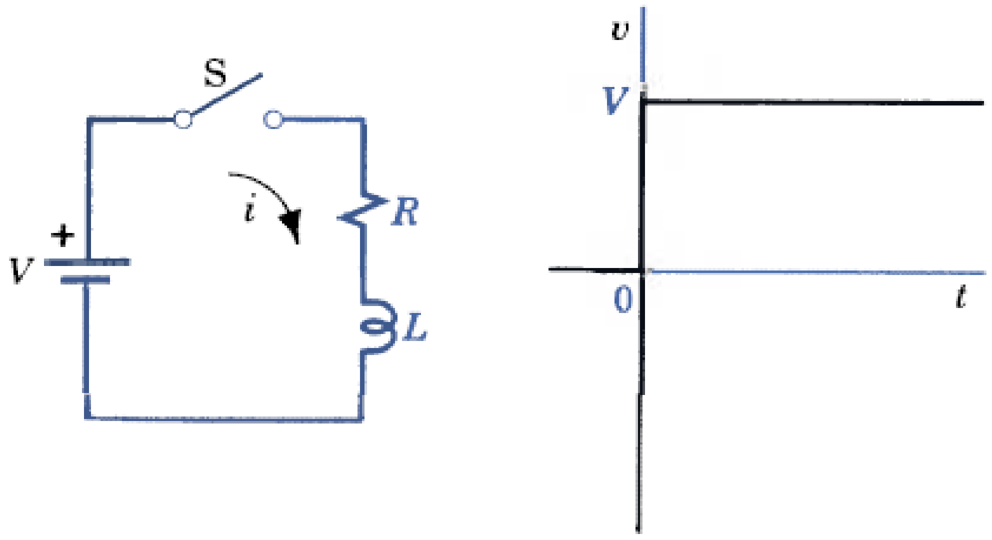
- Natural response
  - Energy storage elements
  - The form of natural response is governed by circuit itself

- Forced response
  - External sources, e.g., batteries or generators
  - The amplitude of forced response is determined by the magnitude of the forcing function and impedance

- Complete response = Natural response + Forced response

Step response of RL circuit

Find i for $t \geq 0$



(a) Circuit (b) Applied voltage

Step response of RL circuit

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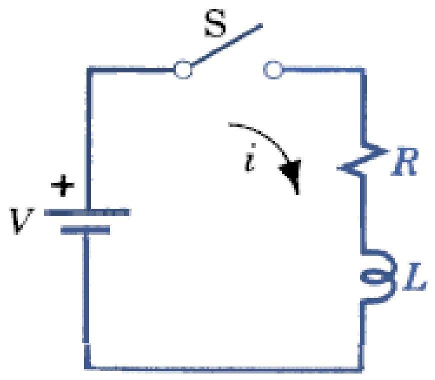
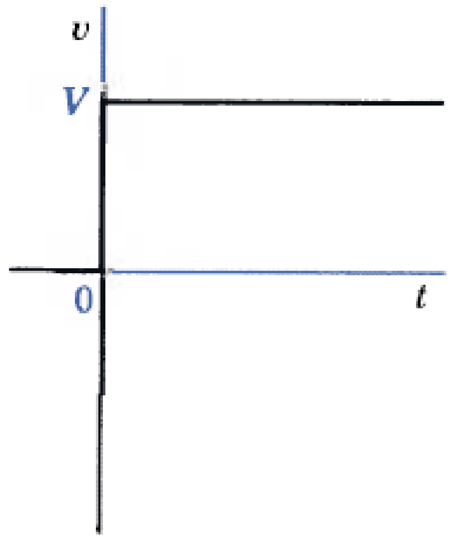


(a) Circuit



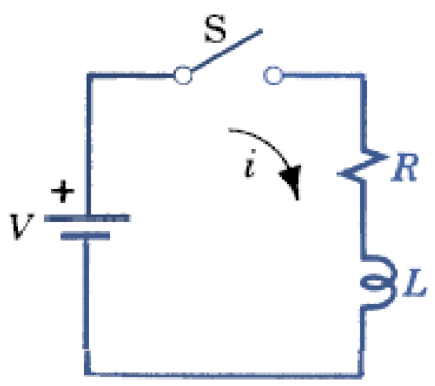
(b) Applied voltage

1. Write the impedance function

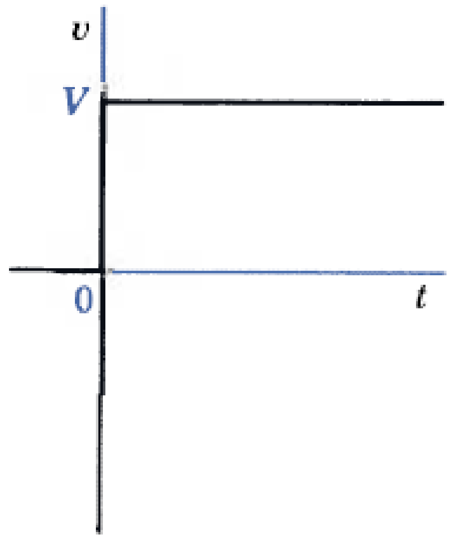
$$Z(s) = R + sL$$

Step response of RL circuit

Find i for $t \geq 0$



(a) Circuit



(b) Applied voltage

1. Write the impedance function

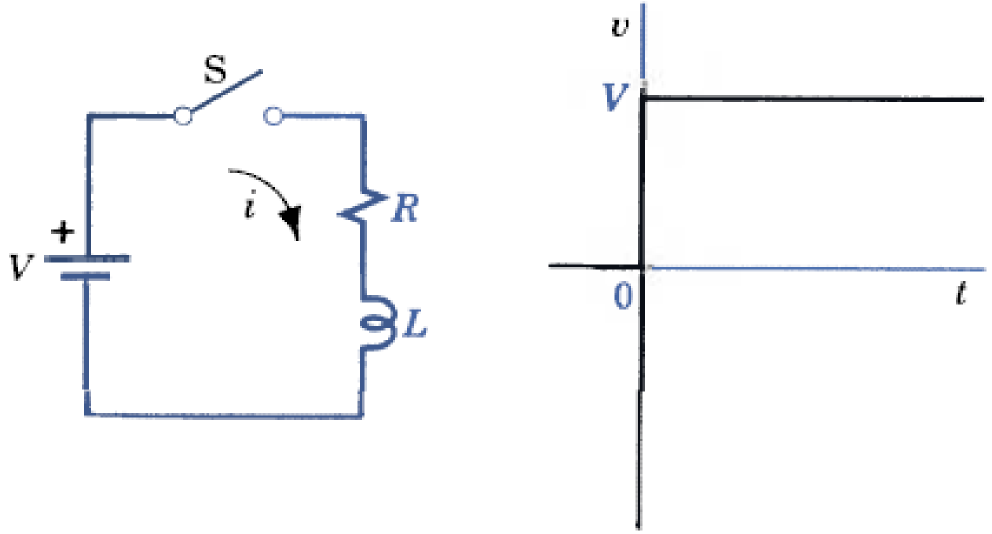
$$Z(s) = R + sL$$

2. Determine forced response: DC $\Rightarrow s = 0$

$$Z(s)\Big|_{s=0} = R,\ i_f = V/R$$

Step response of RL circuit

Find i for $t \geq 0$



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3. Identify natural component: short circiuted, $Z(s) = 0 \Rightarrow s = -\frac{R}{L}$

$$i_n = Ae^{-\frac{R}{L}t}$$

Step response of RL circuit

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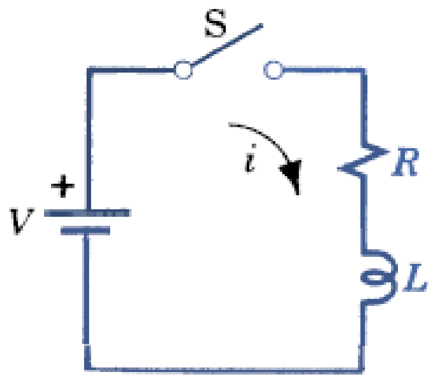
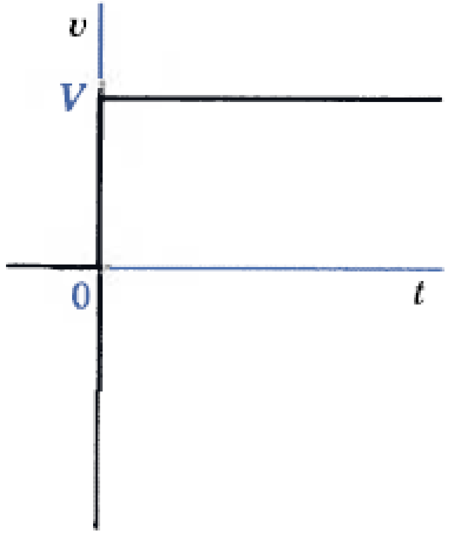


(a) Circuit



(b) Applied voltage

1. Write the impedance function

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2. Determine forced response: DC $\Rightarrow s = 0$

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3. Identify natural component: short circiuted, $Z(s) = 0 \Rightarrow s = -\frac{R}{L}$

$$i_n = Ae^{-\frac{R}{L}t}$$

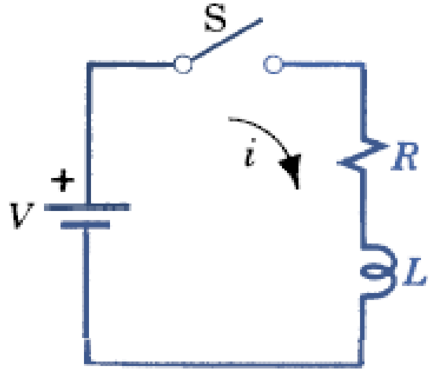
4. Evaluate the undetermined constants

$$i = i_f + i_n = \frac{V}{R} + Ae^{-\frac{R}{L}t} \Rightarrow$$

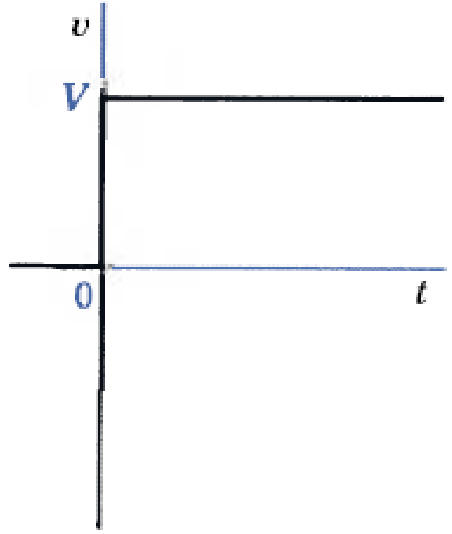
$$i = 0 \ @\ t = 0:\ A = -\frac{V}{R}$$

Step response of RL circuit

Find i for $t \geq 0$



(a) Circuit



(b) Applied voltage

1. Write the impedance function

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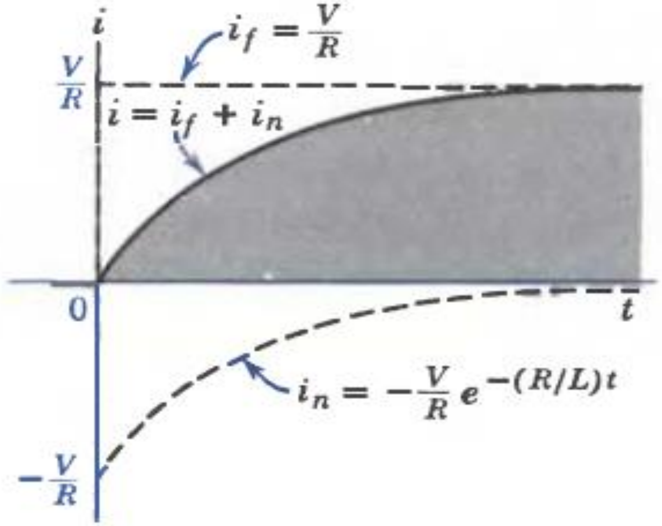
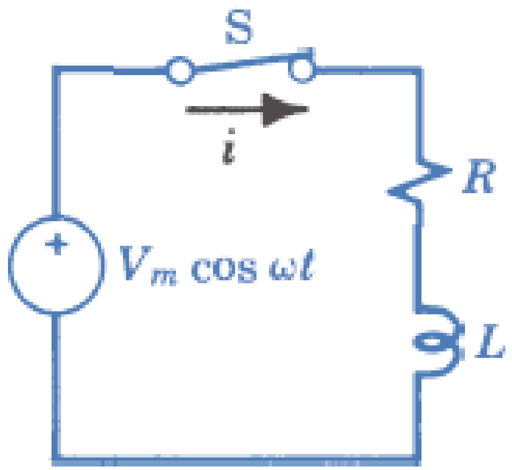


Complete Response

- Write the impedance function (or admittance)
- Determine the forced response from the forcing function
- Identify the natural response from poles and zeros with undetermined constants
- Add the forced and natural responses and evaluate the undetermined constants

Complete Response

Suppose that the switch is closed at $t = 0$



Complete Response

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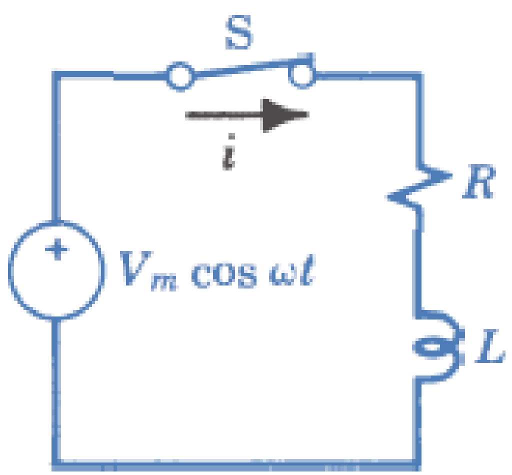


For $t \geq 0$ (forced response)

$$L\frac{di}{dt} + Ri = V_m \cos \omega t$$

$$i_f = I_m \cos(\omega t + \phi)$$

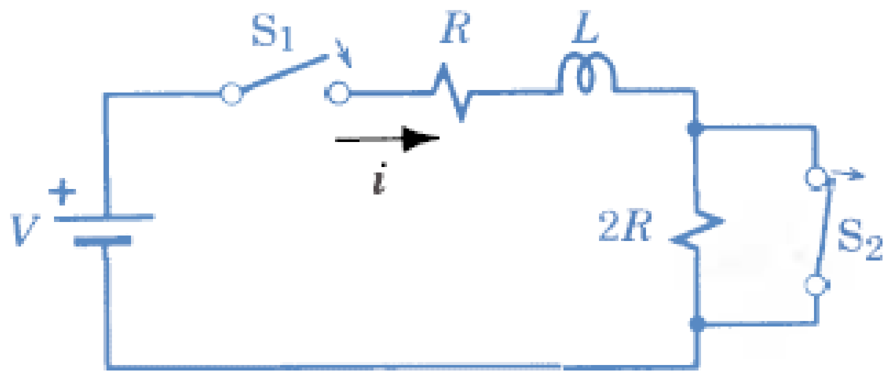
For $t \leq 0$ (natural response)

$$L\frac{di}{dt} + Ri = 0$$

$$i_n = I_n e^{-(R/L)t}$$

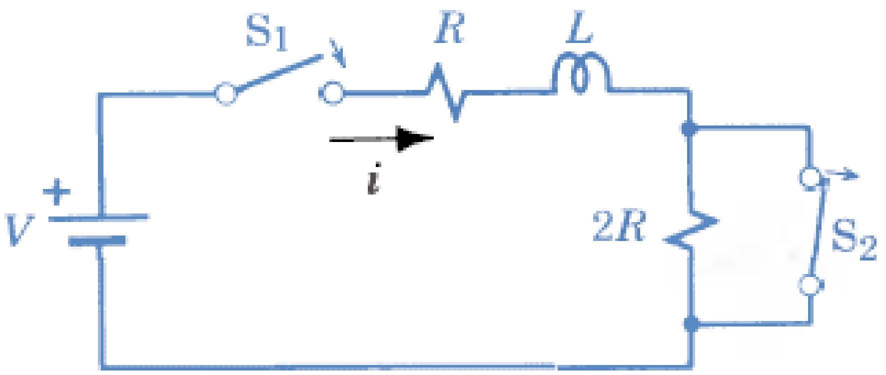
$$i = i_n + i_f = I_m \cos(\omega t + \phi) + I_n e^{-(R/L)t}$$

Example



- For $t < 0$, S_1 open and S_2 closed
- For $0 \leq t < t'$, S_1 closed and S_2 closed
- For $t' \leq t$, S_1 closed and S_2 open

Example



- For $t < 0$, S_1 open and S_2 closed
- For $0 \leq t < t'$, S_1 closed and S_2 closed
- For $t' \leq t$, S_1 closed and S_2 open

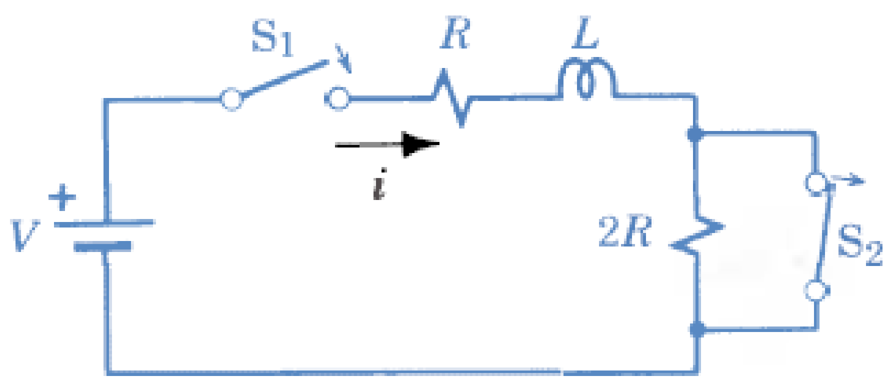
1. Determine the impedance function for $0 \leq t \leq t'$:

$$Z(s) = R + sL$$

2. Forced response: $i_f = \frac{V}{R}$
3. Natural response: $i_n = Ae^{-\frac{R}{L}t}$
4. Complete response: $i = i_n + i_f = \frac{V}{R} + Ae^{-\frac{R}{L}t}$

$$i = 0 \text{ @ } t = 0\text{:}\ \ A = -\frac{V}{R} \qquad\qquad i = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Example



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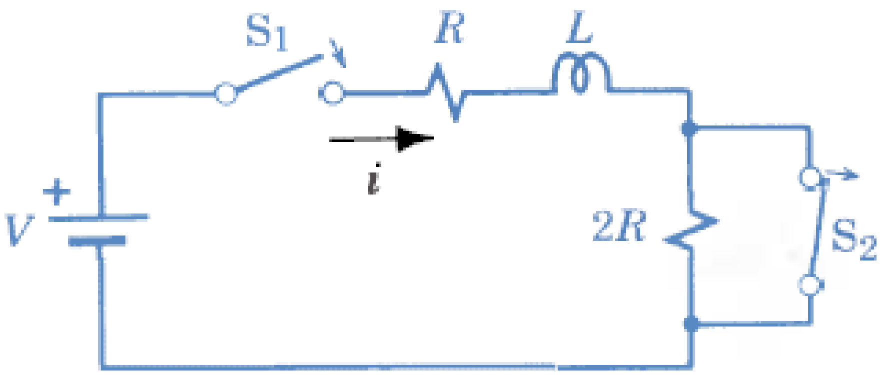
Example



- For $t < 0$, S_1 open and S_2 closed
- For $0 \le t < t'$, S_1 closed and S_2 closed
- For $t' \le t$, S_1 closed and S_2 open

1. Determine the impedance function for $t' \le t$:

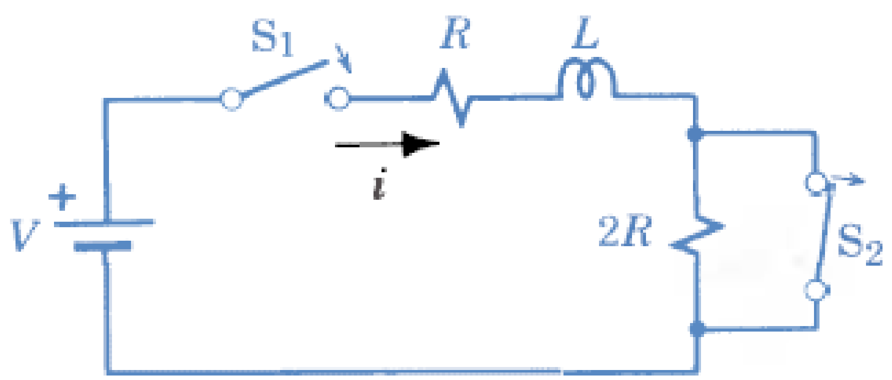
$$Z(s) = R + sL + 2R = 3R + sL$$

2. Forced response: $i_f = \frac{V}{3R}$
3. Natural response: $i_n = A'e^{-\frac{3R}{L}(t-t')}$
4. Complete response: $i = i_n + i_f = \frac{V}{3R} + A'e^{-\frac{3R}{L}(t-t')}\Big|_{t=t'} = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

Previous solution

$\therefore A'$ is determined by $\frac{V}{3R} + A' = \frac{V}{R}(1 - e^{-\frac{R}{L}t'})$

Example



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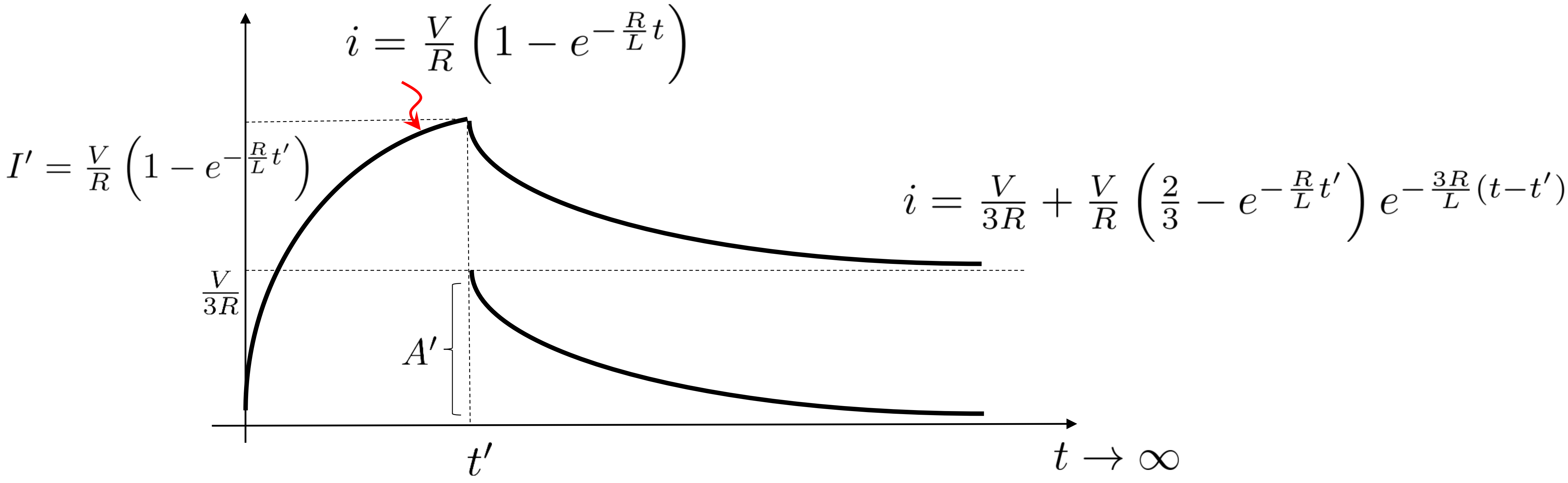
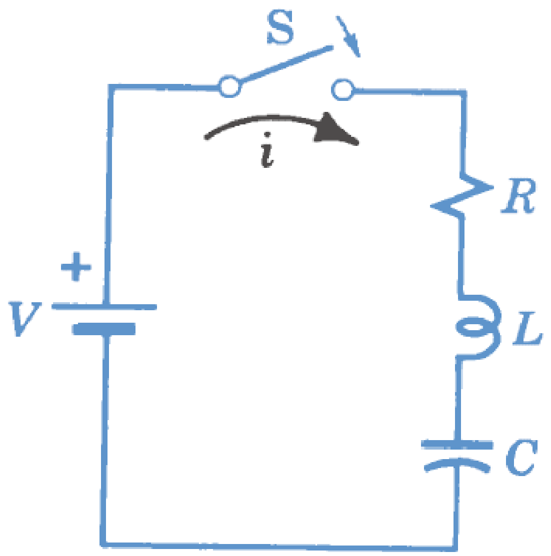


Step response of RLC circuit



- C is initially uncharged
- L carries no current
- Voltage is applied for $t \geq 0$

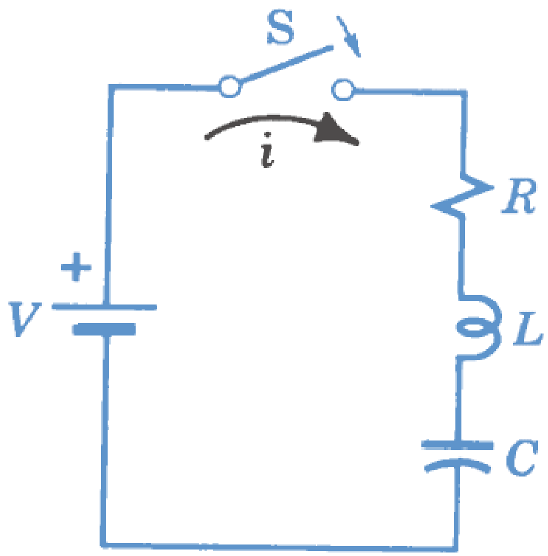
The impedance for this circuit

$$Z(s) = R + sL + \frac{1}{sC}$$

The forced response at DC, i.e., $s = 0$:

$$i_f = \frac{V}{Z(s)}\Big|_{s=0} = \frac{V}{\infty} = 0$$

Step response of RLC circuit



- C is initially uncharged
- L carries no current
- Voltage is applied for $t \geq 0$

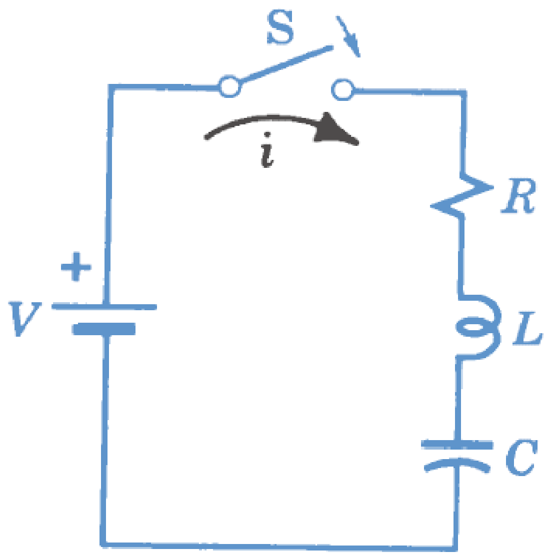
The impedance for this circuit

$$Z(s) = R + sL + \frac{1}{sC} \Rightarrow \frac{s^2 L + sR + \frac{1}{C}}{s}$$

The natural component for i at $Z(s) = 0$:

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -\alpha \pm j\omega$$

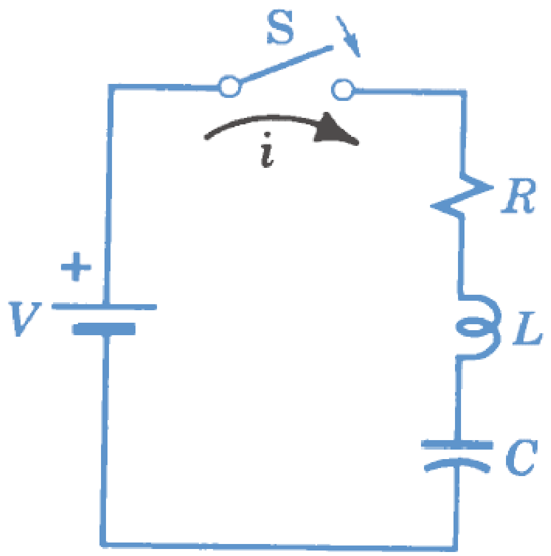
Step response of RLC circuit



- real and distinct roots: $i_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$
- real and equal roots: $i_n = A_1 e^{st} + A_2 t e^{st}$
- complex roots:

$$
\begin{aligned}
i_n &= e^{-\alpha t}(B_1 \cos \omega t + B_2 \sin \omega t) \\
&= A e^{-\alpha t} \sin(\omega t + \theta)
\end{aligned}
$$

Step response of RLC circuit



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- real and equal roots: $i_n = A_1 e^{st} + A_2 t e^{st}$
- complex roots: $i_n = e^{-\alpha t}(B_1 \cos \omega t + B_2 \sin \omega t)$
  $= A e^{-\alpha t} \sin(\omega t + \theta)$

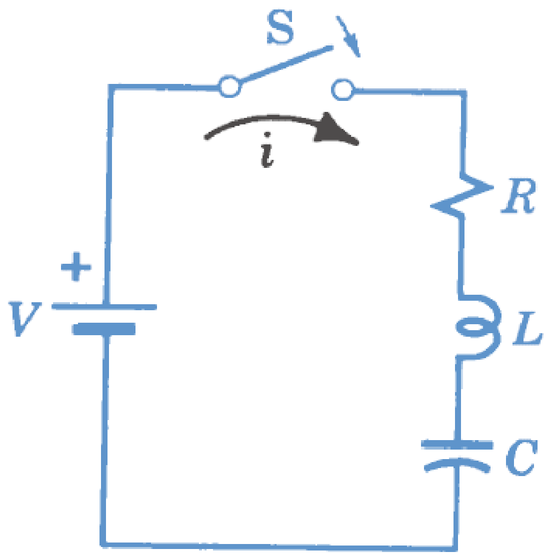
Assuming $i_L = 0$ at $t = 0$:

$$i(0^+) = i_f + i_n = 0$$

Assuming $v_C = 0$ and $v_R = Ri = 0$ at $t = 0$ due to $i = 0$ at $t = 0$

$$v_L(0^+) = L\frac{di}{dt} = V$$

Step response of RLC circuit



Let us assume 'real and distinct roots':

$$i_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The condition $v_L(0^+) = L\frac{di}{dt} = V$ gives

$$v_L(0^+) = L\frac{di}{dt} = L(s_1 A e^{s_1 t} + s_2 A e^{s_2 t})\Big|_{t=0}$$

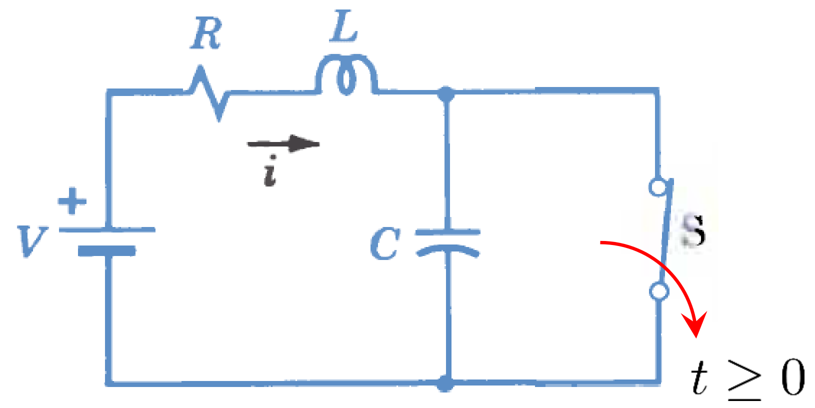
$$\Rightarrow s_1 A_1 + s_2 A_2 = \frac{V}{L}$$

Assuming $i_L = 0$ at $t = 0$:

$$i_n = A_1 + A_2 = 0$$

Example – 1

$V = 20(V),\ R = 4\,(\Omega),\ L = 2\,(\mathrm{H}),\ C = 0.01\,(\mathrm{F}),$



The impedance for this circuit

$$Z(s) = R + sL + \frac{1}{sC} = 4 + 2s + \frac{100}{s}$$

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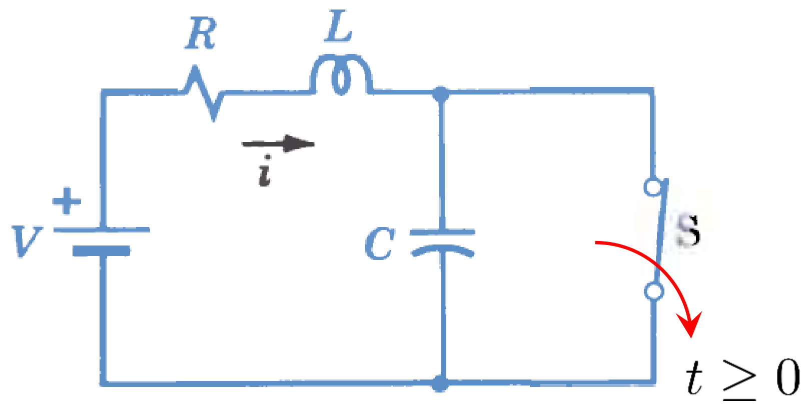

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$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -1 \pm 7j$$

$$i_n = Ae^{-t}\sin(7t + \theta) \Rightarrow i = i_n + i_f = i_n$$

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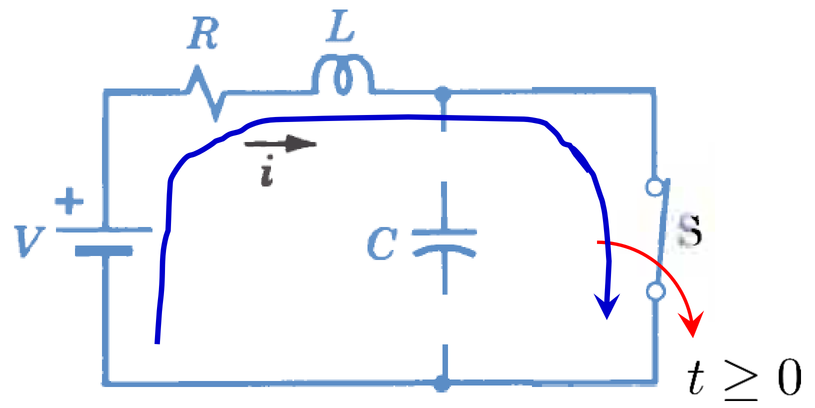

At $t = 0^+,\ i = i_L(0^-) = V/R = 20/4 = 5$

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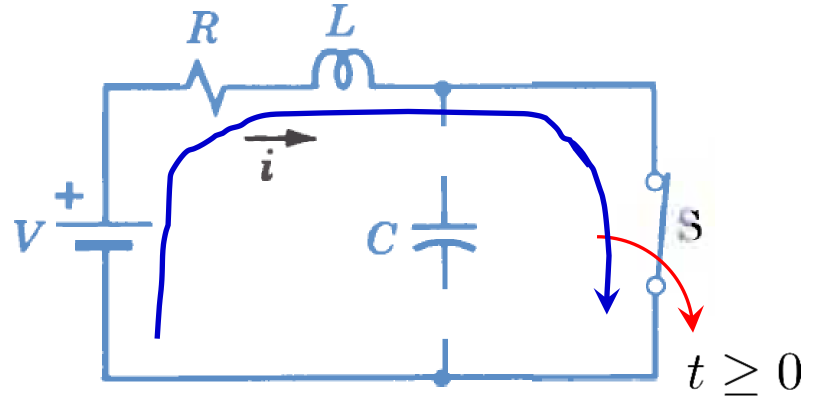

$$\text{At } t = 0^{+},\ i = i_L(0^{-}) = V/R = 20/4 = 5$$

$$i_n = Ae^{-t}\sin(7t+\theta)\Big|_{t=0} = 5$$

$$\Rightarrow A\sin\theta = 5$$

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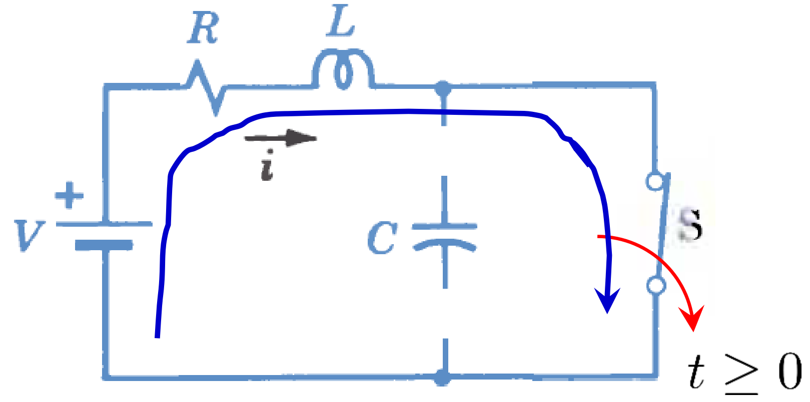

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Another initial condition: $v_C = 0,\ v_R = V\ ,v_L = L\frac{di}{dt}\Big|_{t=0} = 0$

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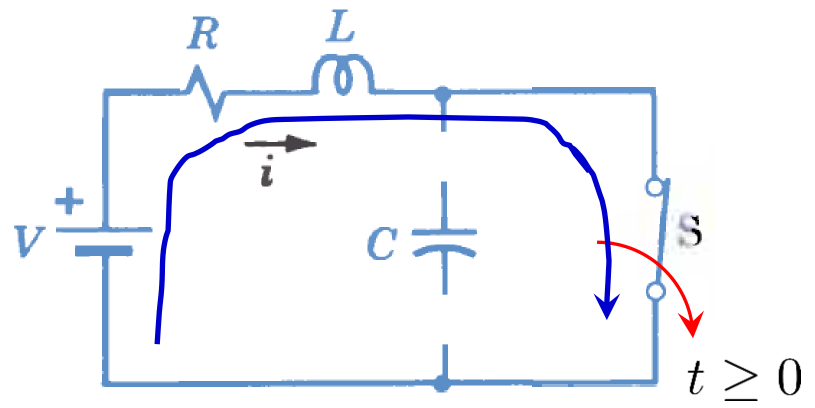

At $t = 0^+,\ i = i_L(0^-) = V/R = 20/4 = 5$

$$i_n = Ae^{-t}\sin(7t+\theta)\big|_{t=0} = 5$$

$$\Rightarrow A\sin\theta = 5$$

Another initial condition: $v_C = 0,\ v_R = V\ ,v_L = L\frac{di}{dt}\big|_{t=0} = 0$

$$\Rightarrow \frac{di_n}{dt} = -Ae^{-t}\sin(7t+\theta) + 7Ae^{-t}\cos(7t+\theta)$$

$$\Rightarrow -A\sin(\theta) + 7A\cos(\theta) = 0$$

$$\Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = 7 = \tan\theta \quad \Rightarrow \arctan(7) = 81.9^\circ$$

$$\Rightarrow A\sin(81.9^\circ) = 5$$

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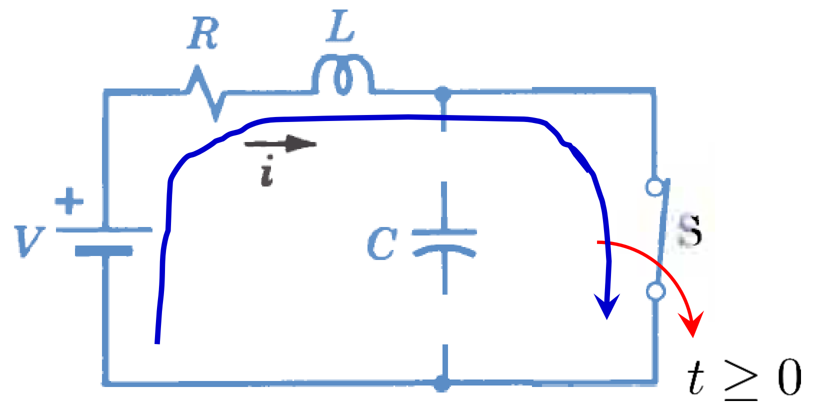

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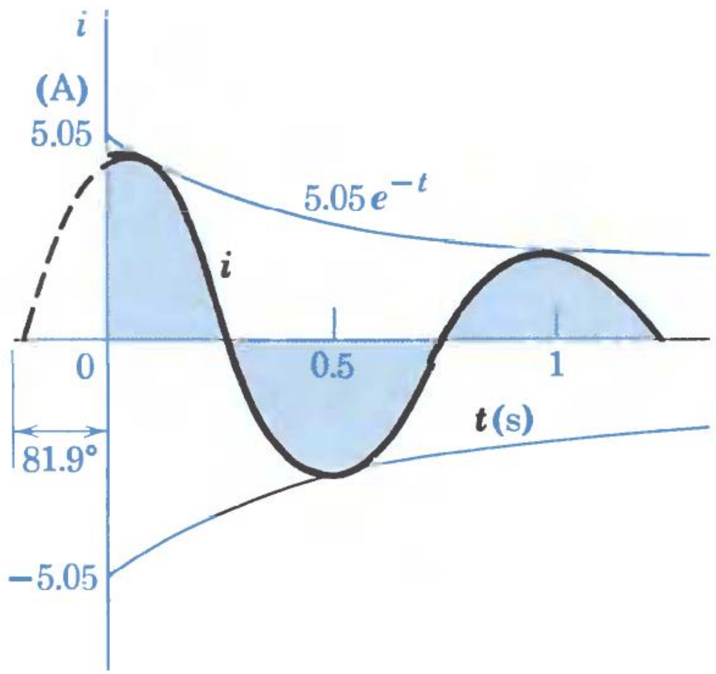


$$\Rightarrow \frac{di_n}{dt} = -Ae^{-t}\sin(7t+\theta) + 7Ae^{-t}\cos(7t+\theta)$$

$$\Rightarrow -A\sin(\theta) + 7A\cos(\theta) = 0$$

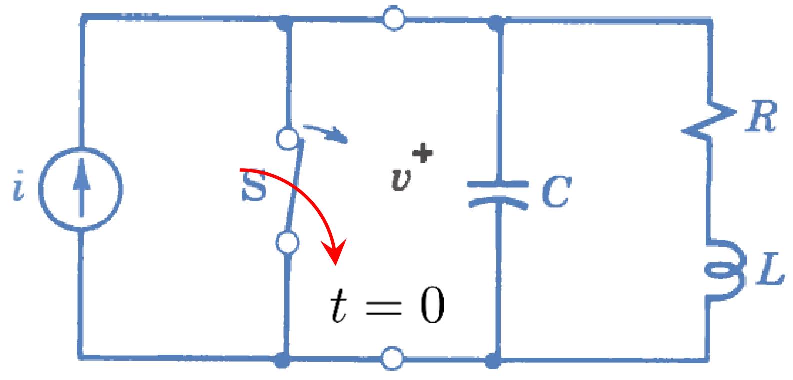
$$\Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = 7 = \tan\theta \quad \Rightarrow \arctan(7) = 81.9^\circ$$

$$\Rightarrow A\sin(81.9^\circ) = 5$$

$$\therefore i_n = 5.05e^{-t}\sin(7t + 81.9^\circ)$$

Example – 2

$i = 8.5\cos(4t)\,(\mathrm{A}),\ R = 2\,(\Omega),\ L = 1\,(\mathrm{H}),\ C = \frac{1}{17}\,(\mathrm{F}),$



The impedance for this circuit

$$Z(s) = \frac{1}{sC} || (R + sL)$$

$$Y(s) = sC + \frac{1}{R+sL} = \frac{s^2LC+sRC+1}{R+sL}$$

Example – 2

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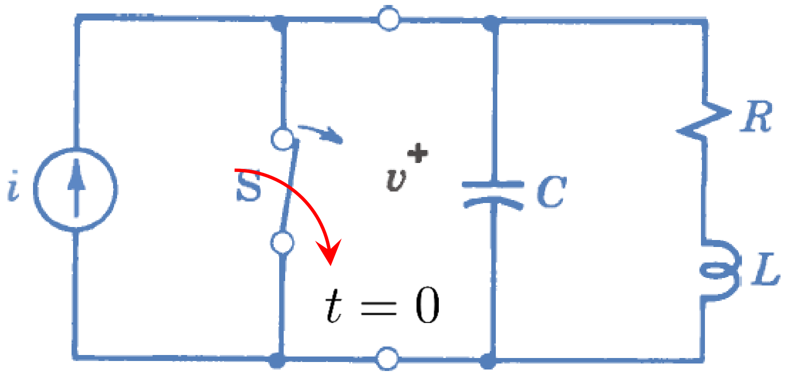

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$$Z(s) = \frac{1}{sC} || (R + sL)$$

$$Y(s) = sC + \frac{1}{R+sL} = \frac{s^2LC+sRC+1}{R+sL}$$

$$\Rightarrow \frac{1-\omega^2LC+j\omega RC}{R+j\omega L} = \frac{1-\frac{16}{17}+\frac{8}{17}j}{2+4j}$$

$$= 0.106\,\underline{/19.5^\circ}$$

$$\mathbf{V}_f = \frac{\mathbf{I}}{\mathbf{Y}} = \frac{(8.5/\sqrt{2})\,\underline{/0^\circ}}{0.106\,\underline{/19.5^\circ}} = \frac{80}{\sqrt{2}}\,\underline{/-19.5^\circ}$$

$$\therefore v_f = 80\cos(4t - 19.5^\circ)$$

Example – 2

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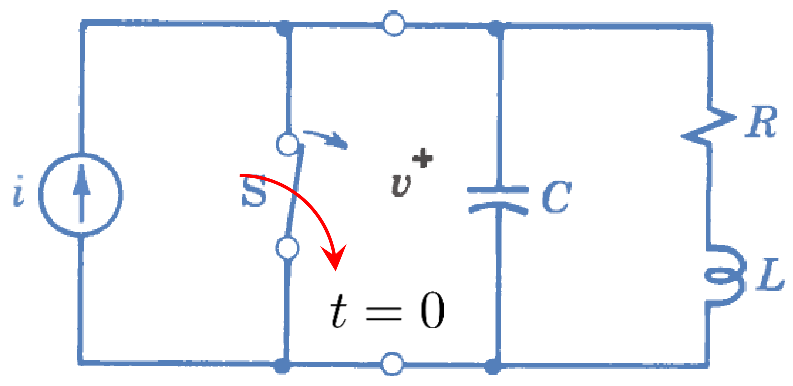

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$$Z(s) = \frac{1}{sC} || (R + sL)$$

$$Y(s) = sC + \frac{1}{R+sL} = \frac{s^2LC + sRC + 1}{R+sL}$$

To find v_n, use the poles of the impedance or the zeros of the admittance

$$Y(s) = 0 \Rightarrow s^2LC + sRC + 1 = 0 \Rightarrow s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

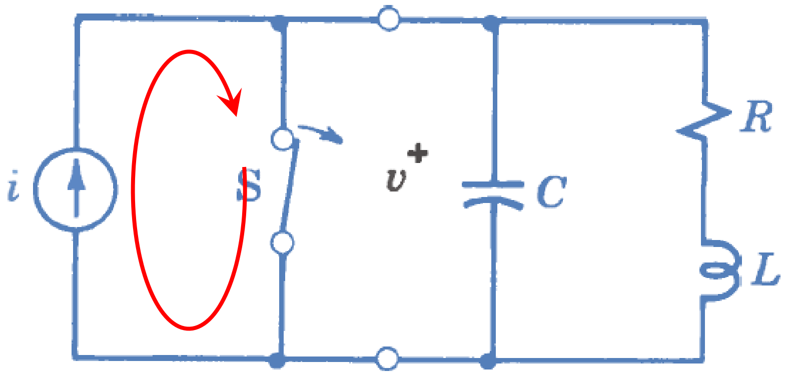
$$= -1 \pm 4j$$

$$\therefore v_n = Ae^{-t}\cos(4t + \theta)$$

$$v = v_f + v_n = 80\cos(4t - 19.5^\circ) + Ae^{-t}\cos(4t + \theta)$$

Example – 2

$i = 8.5\cos(4t)\,(\mathrm{A}),\ R = 2\,(\Omega),\ L = 1\,(\mathrm{H}),\ C = \frac{1}{17}\,(\mathrm{F}),$



$t \le 0$

Determine A and θ:

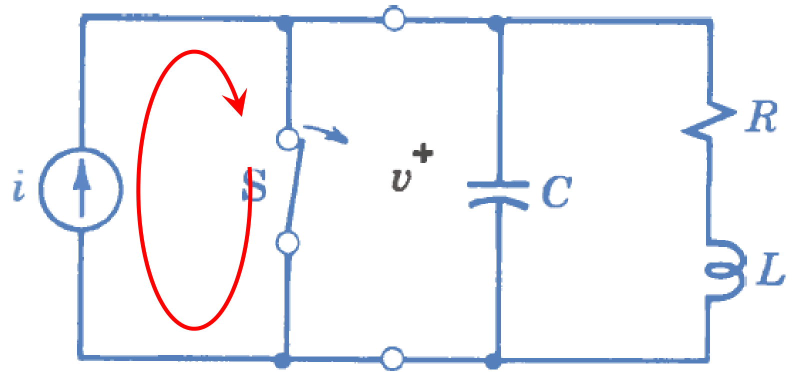
$$v = v_f + v_n = 80\cos(4t - 19.5^\circ) + Ae^{-t}\cos(4t + \theta)$$

At $t = 0^+$, $v_C = v = 0$:

$$80\cos(-19.5^\circ) + A\cos(\theta) = 0 \quad \therefore A\cos(\theta) = -75.5$$

Example – 2

$$i = 8.5\cos(4t)\,(\mathrm{A}),\ R = 2\,(\Omega),\ L = 1\,(\mathrm{H}),\ C = \tfrac{1}{17}\,(\mathrm{F}),$$



$t \le 0$

Determine A and θ:

$$v = v_f + v_n = 80\cos(4t - 19.5^\circ) + Ae^{-t}\cos(4t+\theta)$$

At $t = 0^+$, $v_C = v = 0$:

$$80\cos(-19.5^\circ) + A\cos(\theta) = 0 \quad \therefore A\cos(\theta) = -75.5$$

At $t = 0^+$, $i_L = 0$:

$$v_L = L\frac{di_L}{dt} = 0 \quad \Rightarrow v_R = i_L R = 0$$

At $t = 0^+$, $C\frac{dv_C}{dt} = i_C\big|_{t=0} = 8.5$:

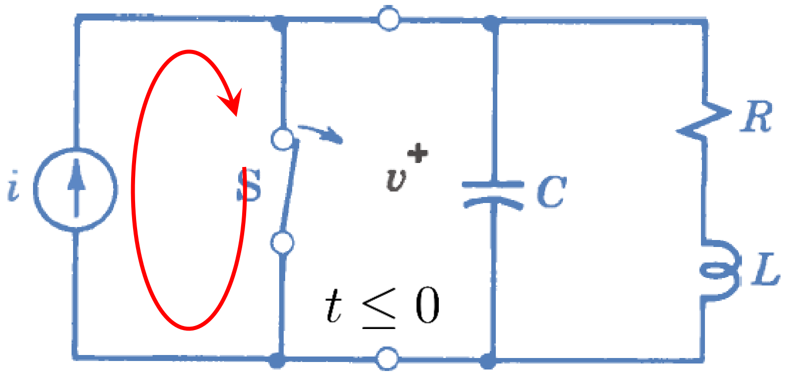
$$\frac{dv_C}{dt} = -320\sin(-19.5^\circ) - A(\cos(\theta) + 4\sin(\theta)) = 8.5 \times 17 = 144.5$$

$$\Rightarrow -4A\sin(\theta) = 144.5 + 320\sin(-19.5^\circ) + A\cos(\theta)$$

$$\therefore A\sin(\theta) = 9.4$$

Example – 2

$i = 8.5\cos(4t)\,(\text{A}),\ R = 2\,(\Omega),\ L = 1\,(\text{H}),\ C = \frac{1}{17}\,(\text{F}),$



$$v = v_f + v_n = 80\cos(4t - 19.5^\circ) + Ae^{-t}\cos(4t + \theta)$$

$$\frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta) = \frac{9.4}{-75.5} = -0.1245$$

$$\therefore \theta = -7.1^\circ$$

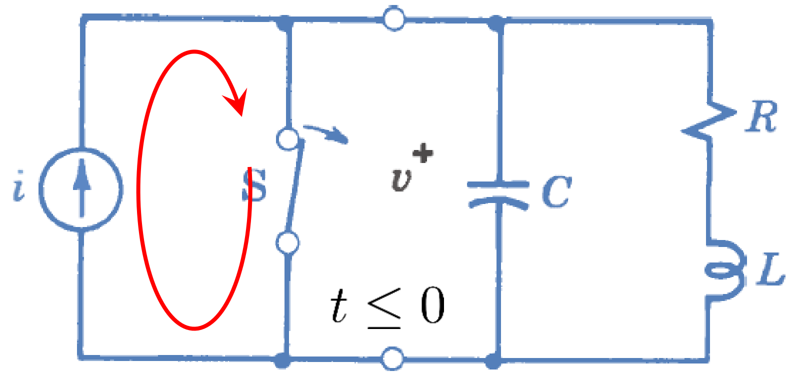
$$\Rightarrow v = 80\cos(4t - 19.5^\circ) + Ae^{-t}\cos(4t - 7.1^\circ)$$

At $t = 0^+$, $v_C = v = 0$:

$$v = 80\cos(-19.5^\circ) + A\cos(7.1^\circ) = 0 \Rightarrow A = \frac{-80\cos(-19.5^\circ)}{\cos(-7.1^\circ)} = -75.994$$

Example – 2

$$i = 8.5\cos(4t)\,(\mathrm{A}),\ R = 2\,(\Omega),\ L = 1\,(\mathrm{H}),\ C = \tfrac{1}{17}\,(\mathrm{F}),$$



$$v = v_f + v_n = 80\cos(4t - 19.5^\circ) + Ae^{-t}\cos(4t + \theta)$$

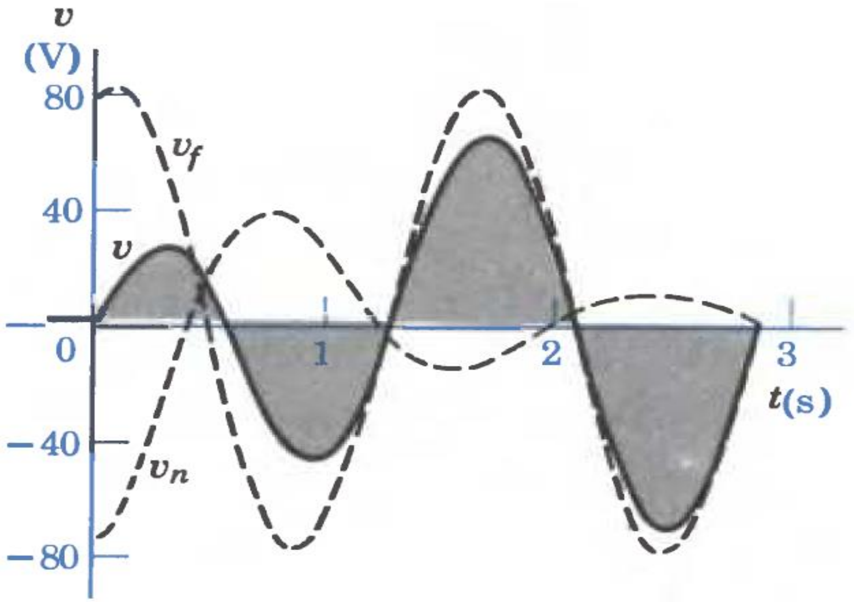
$$\frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta) = \frac{9.4}{-75.5} = -0.1245$$

$$\therefore \theta = -7.1^\circ$$

$$\Rightarrow v = 80\cos(4t - 19.5^\circ) + Ae^{-t}\cos(4t - 7.1^\circ)$$

At $t = 0^+$, $v_C = v = 0$:

$$v = 80\cos(-19.5^\circ) + A\cos(7.1^\circ) = 0 \Rightarrow A = \frac{-80\cos(-19.5^\circ)}{\cos(-7.1^\circ)} = -75.994$$



$$\therefore v = 80\cos(4t - 19.5^\circ) - 75.994e^{-t}\cos(4t + \theta)$$