$X_1, X_2, \ldots, X_n$ are Random Variables

$g: \mathbb{R}^n \to \mathbb{R}$ is a Borel measurable function

$\Rightarrow$ $g(X_1, X_2, \ldots, X_n)$ is a Random Variable

let $Y = g(X_1, X_2, \ldots, X_n)$

$$P\{Y \le y\} = P\{g(X_1, \ldots X_n) \le y\}$$

$$= P\{\underline{(X_1, \ldots X_n)} \in \underbrace{g^{-1}((-\infty, y])}_{\{x \in \mathbb{R}^n \mid g(x) \le y\}}\}$$

$$= \sum_{\{(x_1, \ldots x_n) \mid g(x_1, \ldots x_n) \le y\}} P\{X_1 = x_1, \ldots X_n = x_n\}$$

$g(x_1, \ldots x_n) = x_1 + x_2 + \ldots + x_n$

$$= \int\int \ldots \int_{\{(x_1, \ldots x_n) \mid g(x_1, \ldots x_n) \le y\} = g^{-1}((-\infty, y])} f(x_1, \ldots x_n)\, dx_1 \ldots dx_n$$

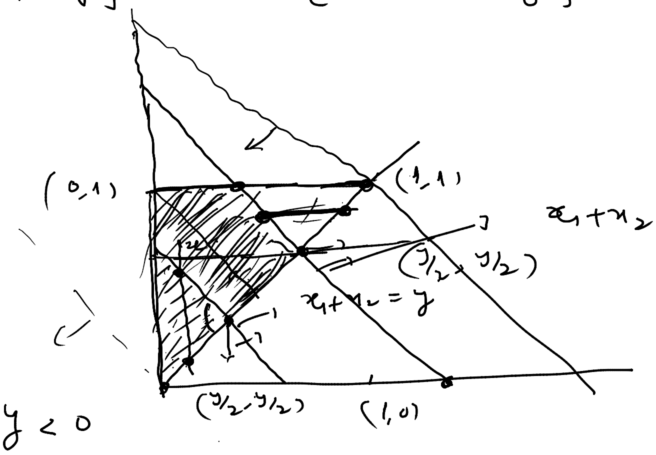
Example: let (X_1, X_2) be a RV whose joint PDF is given by

$$f(x_1, x_2) = \begin{cases} 2 & 0 \le x_1 \le x_2 \le 1 \\ 0 & \text{o.w.} \end{cases}$$

what is PDF of $X_1 + X_2$?

Solⁿ: let $Y = X_1 + X_2$

$$P\{Y \le y\} = P\{X_1 + X_2 \le y\} = \iint\limits_{\{(x_1,x_2) \mid x_1+x_2 \le y\}} f(x_1,x_2)\,dx_1 dx_2$$



$1 \le y \le 2$

$x_2 = y/2$ to 1

$x_1 \to y - x_2$ to x_2

$y < 0$

$P\{Y \le y\} = 0$

for $0 \le y \le 1$

$$P\{Y \le y\} = \int_{x_1=0}^{y/2} \int_{x_2=x_1}^{y-x_1} 2\, dx_2\, dx_1$$

$$= 2\int_0^{y/2} (y - x_1 - x_1)\, dx_1$$

$$= 2\left(y x_1 - x_1^2\right)_0^{y/2}$$

$$= 2\left(\frac{y^2}{2} - \frac{y^2}{4}\right) = \frac{y^2}{2}$$

$$P\{Y \le y\} = y^2/2 \qquad 0 \le y \le 1$$

Let $1 \le y \le 2$

$$P\{Y \le y\} = 1 - \int_{x_2=y/2}^{1} \int_{x_1=y-x_2}^{x_2} 2\, dx_1\, dx_2$$

$$= 1 - 2\int_{y/2}^{1} (x_2 - y + x_2)\, dx_2$$

$$= 1 - 2\left[x_2^2 - x_2 \cdot y\right]_{y/2}^{1}$$

$$= 1 - 2\left[1 - y - y^2/4 + y^2/2\right]$$

$= 1 - 2 + 2y - y^2/2$

$P\{Y \le y\} = -y^2/2 + 2y - 1, \quad 1 \le y \le 2$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^2/2 & 0 \le y \le 1 \\ -y^2/2 + 2y - 1 & 1 \le y \le 2 \\ 1 & y \ge 2 \end{cases}$$

$$f_Y(y) = \begin{cases} y & 0 \le y \le 1 \\ 2 - y & 1 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$X_1, X_2, \ldots X_n$ are RVs.

$g : \mathbb{R}^n \to \mathbb{R}^n$ Borel measurable function.

$g(x) = (g_1(x), \ldots g_n(x))$

what is joint PDF of $(Y_1, Y_2, \ldots, Y_n)$

$Y_1 = g_1(X_1, \ldots X_n)$

$\vdots$

$Y_n = g_n(X_1, X_2, \ldots, X_n)$

<u>Theorem</u>: Let $(X_1, \ldots X_n)$ be an n-dimensional RV of continuous type with joint PDF $f(x_1, \ldots x_n)$. Let

(a) $y_1 = g_1(\underline{x_1, \ldots x_n})$

$\left(g_1(x_1, \ldots x_n) = \Sigma x_i \right)$

(a) $y_1 = g_1(x_1, \dots x_n)$

$y_2 = g_2(x_1, \dots x_n)$

$\vdots$

$y_n = g_n(x_1, x_2, \dots x_n)$

$g_1(x_1, \dots x_n) = \Sigma x_i$

$g_1(X_1, \dots X_n) = \sum_{i=1}^{n} X_i$

be a mapping from $\mathbb{R}^n \to \mathbb{R}^n$ such that there exists inverse transformation

$x_1 = h_1(y_1, \dots y_n)$, $x_2 = h_2(y_1, \dots y_n)$

$\dots$ $x_n = h_n(y_1, \dots y_n)$

(b) Assume that both mapping & its inverse are continuous.

c) Assume that partial derivatives

$\frac{\partial x_i}{\partial y_j}$, $1 \le i \le n$, $1 \le j \le n$

exists and are continuous

d) Assume that the Jacobian determinant of inverse transformation

$$J = \frac{\partial(x_1, \dots x_n)}{\partial(y_1, \dots y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

is non-zero for all $(y_1, \dots y_n)$ in the range of transformation.

Then $(Y_1, \dots Y_n) = (g_1(X_1, \dots X_n), \dots g_n(X_1, \dots X_n))$

has a joint PDF given by

$$w(y_1, \dots y_n) = f(h_1(y_1, \dots y_n), \dots h_n(y_1, \dots y_n)) \, |J|$$

Proof: $P\{ Y_1 \le u_1, \ldots Y_n \le u_n \}$, let $X = (X_1, \ldots X_n)$

$= P\{ \underline{g_1(X)} \le u_1, \ldots, g_n(x) \le u_n \}$ ✓

$= P\{ \underline{(X_1, \ldots X_n)} \in \underbrace{g_k^{-1}(-\infty, u_k]}_{= \{x \in \mathbb{R}^n \mid g_k(x) \le u_k\}} \ \forall\, k = 1, \ldots n \}$

$= \iint \cdots \int_{\{(x_1, \ldots x_n) \mid g_1(x) \le u_1 \ldots g_n(x) \le u_n\}} f(x_1, \ldots x_n)\, dx_1\, dx_2, \ldots dx_n$ ✓

$\overset{!}{=} \int_{-\infty}^{u_n} \int_{-\infty}^{u_{n-1}} \cdots \int_{-\infty}^{u_1} \left(f(h_1(y_1, \ldots y_n), \ldots h_n(y_1, \ldots y_n)) \left| \frac{\partial(x_1, \ldots x_n)}{\partial(y_1, \ldots y_n)} \right| \right) \times dy_1\, dy_2 \ldots dy_n$

$\| $

$F_{Y_1, \ldots Y_n}(u_1, \ldots u_n)$

$\Rightarrow$ Joint PDF of $(Y_1, \ldots Y_n)$ is

$w(y_1, \ldots y_n) = f(h_1(y_1, \ldots y_n), \ldots h_n(y_1, \ldots y_n)) \cdot |J|$

Example: Let X_1, X_2 be independent RVs. with common PDF given by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find joint PDF of $X_1 + X_2,\ 4X_1 - X_2$

Solⁿ. Joint PDF of (X_1, X_2) is

$f(x_1, x_2) = \begin{cases} 1 & 0 < x_1 < 1,\ 0 < x_2 < 1 \end{cases}$

$$f(x_1, x_2) = \begin{cases} 1 & 0 < x_1 < 1 \ , \ 0 < x_2 < 1 \\ 0 & \text{o.w} \end{cases}$$

let $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$

$\Rightarrow x_1 = \frac{y_1 + y_2}{2}$ $\quad x_2 = \frac{y_1 - y_2}{2}$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

$$\omega(y_1, y_2) = \begin{cases} 1/2 \times f\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right) \\ \quad 0 < \frac{y_1+y_2}{2} < 1 \ , \ 0 < \frac{y_1-y_2}{2} < 1 \\ 0 \quad \text{o.w.} \end{cases}$$

$$\omega(y_1, y_2) = 1/2 \quad (y_1, y_2) \in \left\{ (y_1, y_2) \middle| \begin{array}{l} 0 < y_1 + y_2 < 2 \\ 0 < y_1 - y_2 < 2 \end{array} \right\}$$

$$= 0 \quad \text{o.w.}$$

Find PDF of $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$?