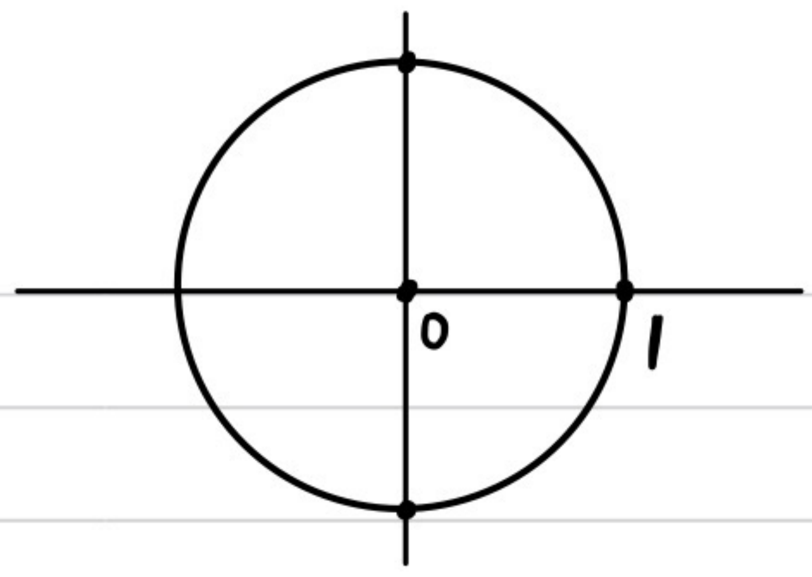
Example of Laurent series :

$$f(z) = \frac{1}{z^2(1-z)}$$



$\rightarrow$ Analytic in the annular region $\boxed{0 < |z| < 1}$

In this region we can write down a Laurent series by simply expanding $\frac{1}{1-z}$ to give

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \frac{1}{z^2}\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots$$

Now lets use Laurant series expression to determine the coefficients to see that the above is indeed correct.

$$f(z) = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i}\oint_c \frac{d\omega\, f(\omega)}{(\omega - z_0)^{n+1}}\right)}_{a_n} (z - z_0)^n + \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{2\pi i}\oint \frac{d\omega\, f(\omega)}{(\omega - z_0)^{-n+1}}\right)}_{b_n\ (\equiv a_{-n})} \frac{1}{(z - z_0)^n}$$

$$a_n = \frac{1}{2\pi i}\oint_c \frac{d\omega}{\omega^{n+1}} \frac{1}{\omega^2(1-\omega)} = \frac{1}{(n+2)!}\left[\frac{d^{n+2}}{d\omega^{n+2}}\left(\frac{1}{1-\omega}\right)\right]_{\omega=0}$$

$$= \frac{1}{(n+2)!} \left.\frac{(+1)(+2)(+3)\cdots(+(n+2))}{(1-\omega)^{n+3}}\right|_{\omega=0} = \frac{(n+2)!}{(n+2)!} = 1$$

$$b_n = \frac{1}{2\pi i}\oint_C \frac{d\omega}{\omega^{-n+1}} \frac{1}{\omega^2(1-\omega)} = \frac{1}{2\pi i}\oint_C \frac{d\omega}{\omega^{-n+3}} \frac{1}{1-\omega} = 0 \ \forall\ n \geq 3$$

by Cauchy's theorem.

$$\frac{n!}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0)^{n+1}} = f^{(n)}(z_0)$$

$$b_1 = \frac{1}{2\pi i}\oint_c \frac{d\omega}{\omega^2(1-\omega)} = \left.\frac{d}{d\omega}\left(\frac{1}{1-\omega}\right)\right|_{\omega=0} = +1$$

$$b_2 = \frac{1}{2\pi i}\oint_c \frac{d\omega}{\omega(1-\omega)} = \left.\frac{1}{1-\omega}\right|_{\omega=0} = +1$$

$$\therefore \quad \frac{1}{z^2(1-z)} = \frac{b_2}{z^2} + \frac{b_1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

$$= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n$$

Isolated singularities & Cauchy's Residue theorem :

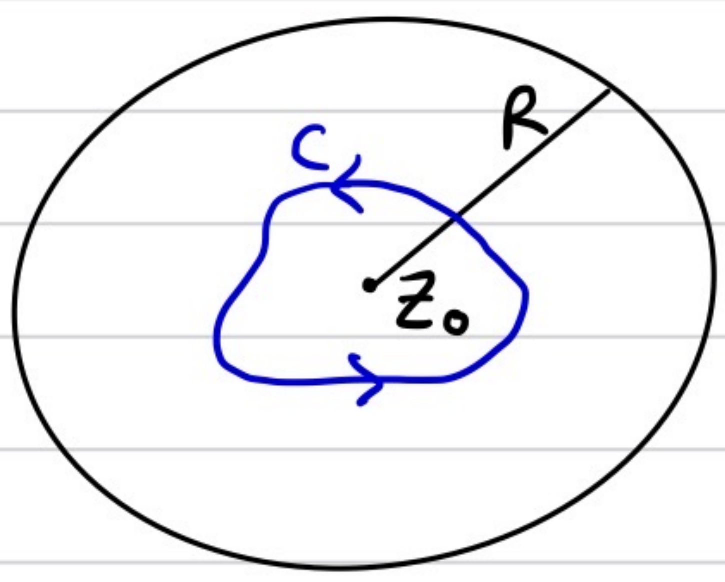
Singularity (Singular point) of $f(z)$: A point z_0 where $f(z)$ is not analytic (differentiable)

Isolated singularity : A singular point z_0 such that $f(z)$ is analytic in the deleted ϵ neighborhood
$$0 < |z - z_0| < \epsilon \qquad \text{for some } \epsilon > 0.$$

- $f(z) = \dfrac{1+z}{z^2(z-1)(z^2+2)}$ has 4 isolated singularities
  $z = 0, 1, \pm i\sqrt{2}$
- $f(z) = \ln(z)$ & $z^{1/2}$ have $z=0$ as singular point but the singularity is not isolated Since in every deleted neighborhood of $z=0$ these functions are still multivalued.

Residue : Let z_0 be an isolated singularity of $f(z)$ such that $f(z)$ is analytic in the "punctured disk" $0 < |z - z_0| < R$, then $f(z)$ has a Laurant series expansion valid in the punctured disk

$$f(z) = \cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

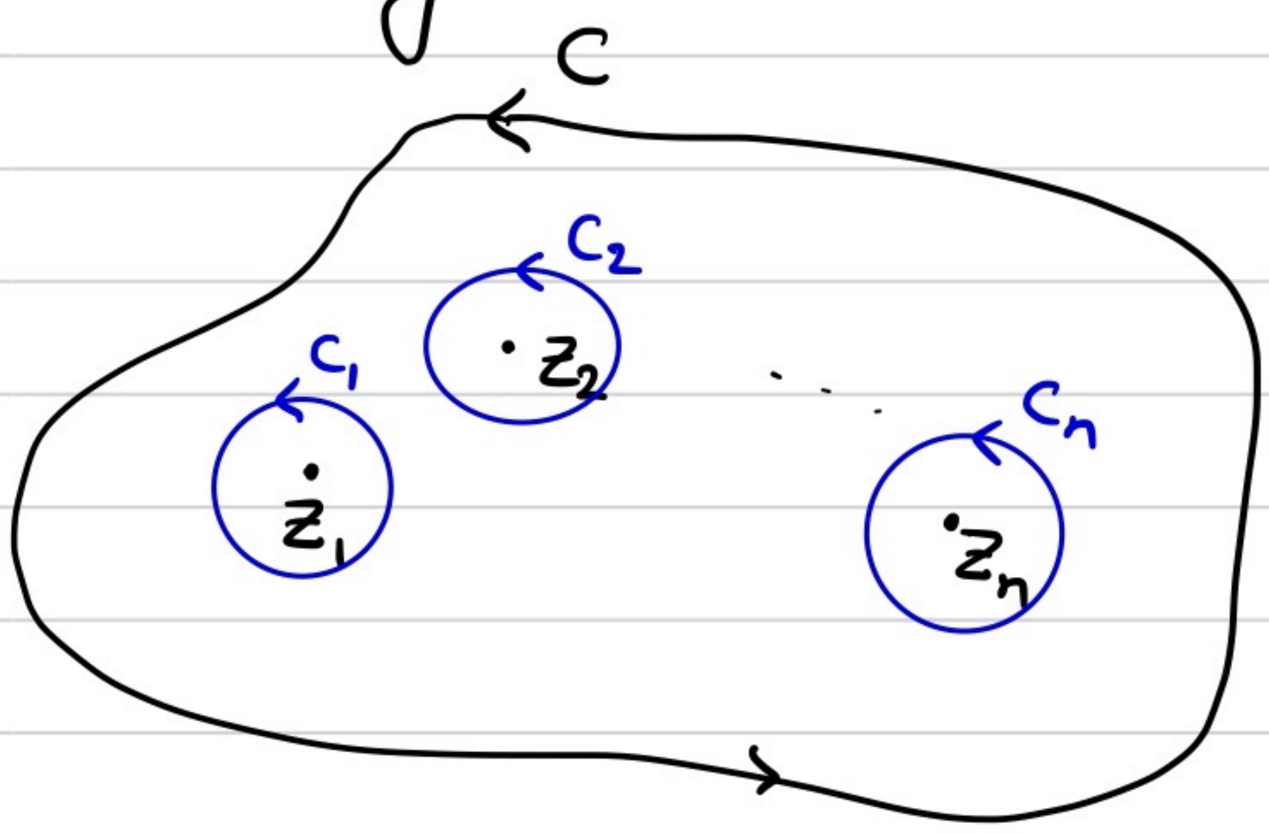


The coefficient $\boxed{b_1 = \dfrac{1}{2\pi i}\oint_C dz\, f(z)}$ is referred to as the Residue of $f(z)$ at $z = z_0$ & denoted as

$$\boxed{\oint_C f(z)\, dz = 2\pi i\, b_1 = 2\pi i \operatorname*{Res}_{z=z_0}\left[f(z)\right]}$$

Cauchy's residue theorem:

Let C be a simple closed counterclockwise contour and $f(z)$ be a complex function which is analytic everywhere inside C except at a finite number of isolated singularities $\{z_1, z_2, \dots z_n\}$, then



$$\boxed{\oint_C dz\, f(z) = 2\pi i \sum_{i=1}^{n} \underset{z=z_i}{\text{Res}} \left[f(z)\right]}$$

→ Follows using straightforward use of Cauchy-Gursat theorem (contour deformation).

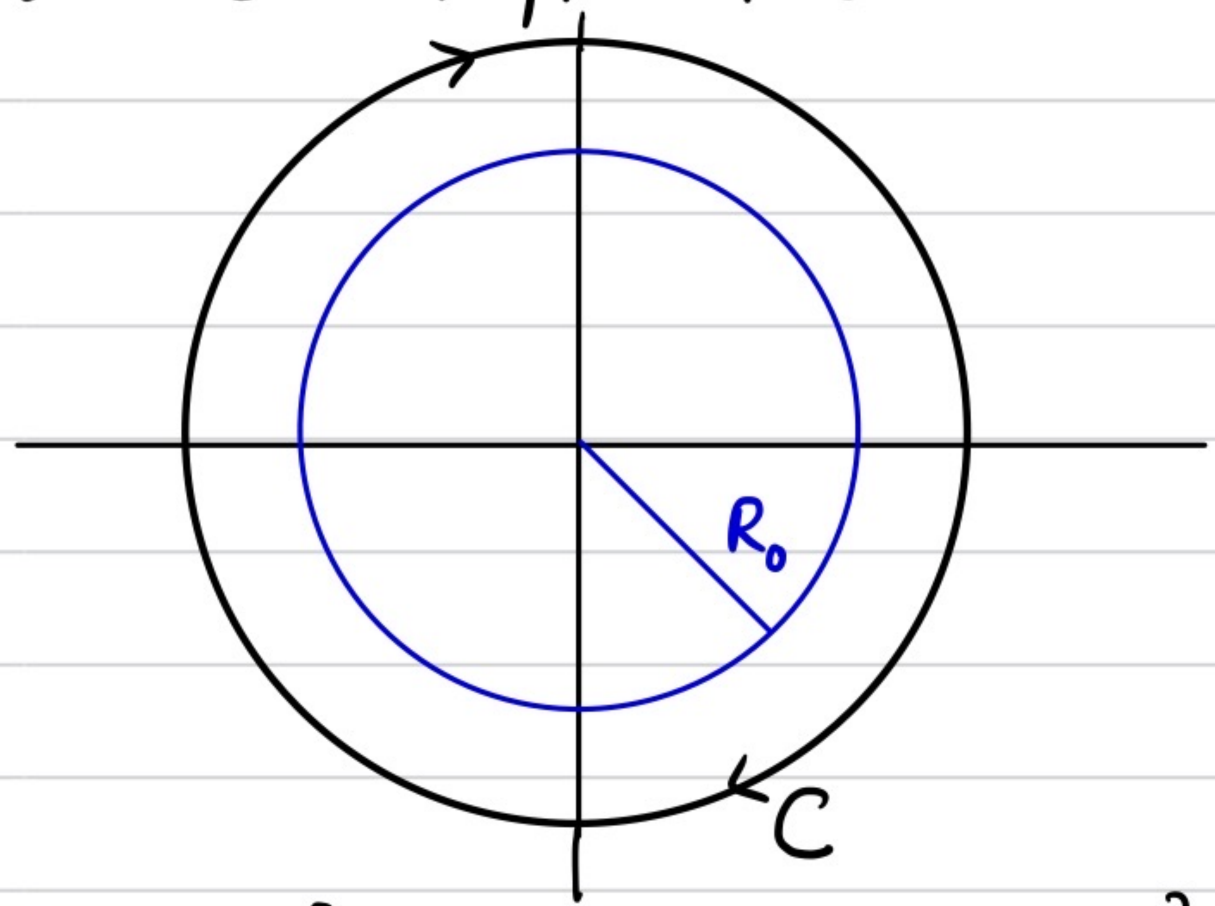
$$\oint_C dz\, f(z) = \sum_{i=1}^{n} \oint_{C_i} dz\, f(z) = 2\pi i \sum_{i=1}^{n} \underset{z=z_i}{\text{Res}} \left[f(z)\right]$$

Residue at ∞:

If $f(z)$ diverges as $z \to \infty$ but is analytic for $R_0 < |z| < \infty$ for some $R_0 > 0$, then $f(z)$ is said to have an isolated singularity at $z = \infty$.

The residue at $z = \infty$ for $f(z)$ is defined as

$$\boxed{\underset{z=\infty}{\text{Res}} \left[f(z)\right] \equiv \frac{1}{2\pi i} \oint_C dz\, f(z)}$$



{ Note that C is clockwise }

a convenient way to rewrite is by making the change of variable

$z = \frac{1}{\omega}$ on R.H.S

$$\Rightarrow \quad dz = -\frac{1}{\omega^2} d\omega$$

$$\underset{z=\infty}{\text{Res}} \left[f(z)\right] = \frac{-1}{2\pi i} \oint_C \frac{d\omega}{\omega^2} f\left(\frac{1}{\omega}\right) \equiv \frac{1}{2\pi i} \oint_{-C} d\omega \underbrace{\left(\frac{1}{\omega^2} f\left(\frac{1}{\omega}\right)\right)}$$

$\rightsquigarrow$ clockwise contour around $\omega = 0$

$$\star \quad \boxed{\underset{z=\infty}{\text{Res}} \left[f(z)\right] = \underset{\omega=0}{\text{Res}} \left[\frac{1}{\omega^2} f\left(\frac{1}{\omega}\right)\right]}$$