Laplace Transforms

Lecture 34

Properties of Laplace Transforms

- **Linearity** $ax(t) + by(t) \leftrightarrow aX(s) + bY(s)$ $\supset (R_1 \cap R_2)$

- **Time shifting** $x(t - T) \leftrightarrow e^{-sT} X(s)$ R

- **Time scaling** $x(a\,t) \leftrightarrow \dfrac{1}{|a|} X\left(\dfrac{s}{a}\right)$ aR

Properties of Laplace Transforms

- **Multiply by t** $$tx(t) \leftrightarrow -\frac{dX(s)}{ds}$$ R

- **Differentiation in time** $$\frac{dx(t)}{dt} \leftrightarrow sX(s)$$ $\supset R$

Properties of Laplace Transforms

- **Multiply by $e^{-\alpha t}$** $\quad x(t)\mathrm{e}^{-\alpha t} \leftrightarrow X(s+\alpha)$ $\quad$ $shift\ R\ by\ -\alpha$

- **Convolution** $\quad \int_{-\infty}^{\infty} x(\tau)\,\mathrm{y}(\mathrm{t}-\tau)\,\mathrm{d}\tau \leftrightarrow X(s)Y(s)$ $\quad$ Ↄ $(R_1 \cap R_2)$

- **Integrate in time** $\quad \int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{X(s)}{s}$ $\quad$ Ↄ $(R \cap (Re(s) > 0))$

Properties of Laplace Transforms

If $x(t) = 0\ t < 0+$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0+}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{0+}^{\infty} x(t)e^{-st}dt = [x(t)e^{-st}/(-s)]_{0+}^{\infty} + \int_{0+}^{\infty} \frac{dx(t)}{dt}\frac{e^{-st}}{s}dt$$

$$X(s) = \frac{x(0+)}{s} - \frac{x(\infty)e^{-s\infty}}{s} + \frac{\int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st}dt}{s}$$

Properties of Laplace Transforms

If $x(t) = 0 \; t < 0+$

$$sX(s) = x(0+) - x(\infty)e^{-s\infty} + \int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$\lim_{s \to \infty} sX(s) = x(0+)$ [Initial value theorem]

$\lim_{s \to 0} sX(s) = x(0+) + x(\infty) - x(0+) = x(\infty)$ [Final value theorem]

Properties of Laplace Transforms

- Initial-value theorem

$$x(0+) = \lim_{s \to \infty} sX(s)$$

If $X(s)$ is rational, then for non-trivial output degree of denominator should be 1 plus degree of numerator.

$$X(s) = \frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0} \text{ implies } x(0+) = \frac{a_{n-1}}{b_n}$$

Properties of Laplace Transforms

- Final-value theorem

$$x(\infty) = \lim_{s \to 0} sX(s)$$

If $X(s)$ is rational, then for non-trivial output denominator should contain a factor of s.

$X(s) = \frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s}$ implies $x(\infty) = \frac{a_0}{b_1}$

What is the physical meaning of $x(0)$ and $x(\infty)$?

- They corresponds to $x(t)$, which is continuous at these values.

Properties of Laplace Transforms

- **Output of an LTI system**

$$Y(s) \leftrightarrow X(s)H(s)$$

Causal and Stable LTI system

Choose the right option
I) All poles lie in right-half plane
II) All poles lie in left-half plane
III) Poles can lie anywhere
IV) There are no poles at all
V) I do not care

Causal and Stable LTI system

Choose the right option
I) All poles lie in right-half plane
II) **All poles lie in left-half plane**
III) Poles can lie anywhere
IV) There are no poles at all
V) I do not care

Solving differential equation with Laplace transforms

A causal and stable LTI system with differential equation:

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

has the impulse response:

I $\quad h(t) = e^{-3t}u(t)$	II $\quad h(t) = -e^{-3t}u(-t)$
III $\quad h(t) = e^{3t}u(t)$	IV $\quad h(t) = -e^{3t}u(-t)$

Solving differential equation with Laplace transforms

$$\frac{dy(t)}{dt}+3y(t)=x(t)$$

$$sY(s)+3Y(s)=X(s)$$

$$Y(s)=\frac{1}{s+3}X(s)$$

$$H(s)=\frac{1}{s+3}$$

$$h(t)=e^{-3t}u(t) \quad \surd$$

$$h(t)=-e^{-3t}u(-t)$$

Solving second-order differential equation with L-Transform

$$\frac{d^2y(t)}{dt^2} + 2\varsigma\omega_n \frac{dy(t)}{dt} + \omega_n{}^2 y(t) = \omega_n{}^2 x(t)$$

$$(s^2 + 2\varsigma\omega_n s + \omega_n{}^2)Y(s) = \omega_n{}^2 X(s)$$

$$H(s) = \frac{\omega_n{}^2}{s^2 + 2\varsigma\omega_n s + \omega_n{}^2}$$

Solving second-order differential equation with L-Transform

$$H(s) = \frac{{\omega_n}^2}{s^2 + 2\varsigma\omega_n s + {\omega_n}^2}$$

$$H(s) = \frac{{\omega_n}^2}{(s - c_1)(s - c_2)}$$

$$c_1 = -\varsigma\omega_n + \omega_n\sqrt{\varsigma^2 - 1}$$

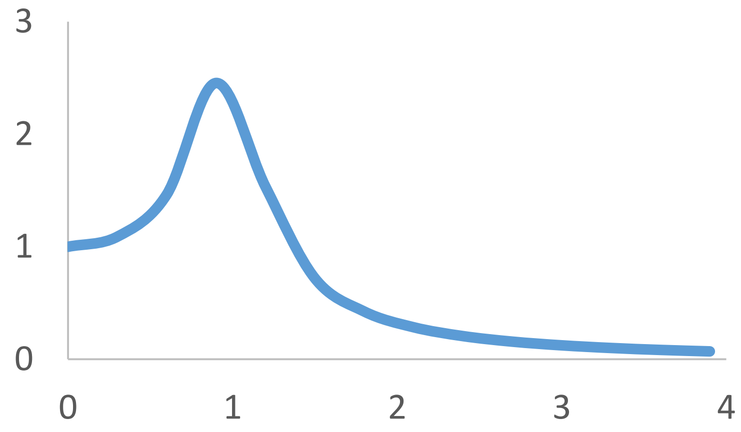
$$c_2 = -\varsigma\omega_n - \omega_n\sqrt{\varsigma^2 - 1}$$

For $\varsigma < 1 \quad c_1 = \overline{c_2}$

$$c_1 = -\varsigma\omega_n + j\omega_n\sqrt{1 - \varsigma^2}$$

$$c_2 = -\varsigma\omega_n - j\omega_n\sqrt{1 - \varsigma^2}$$

Changing real part



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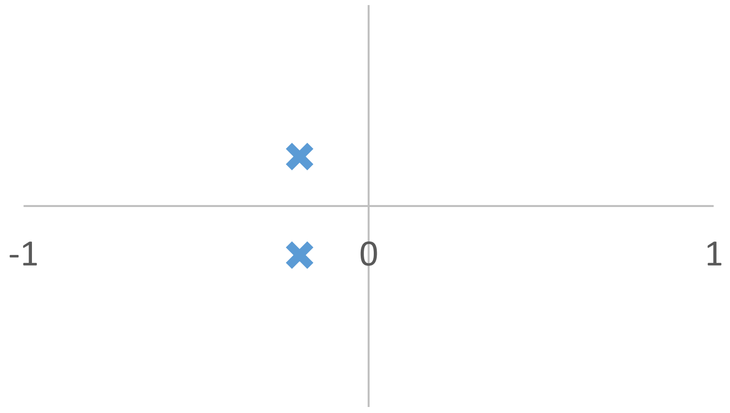

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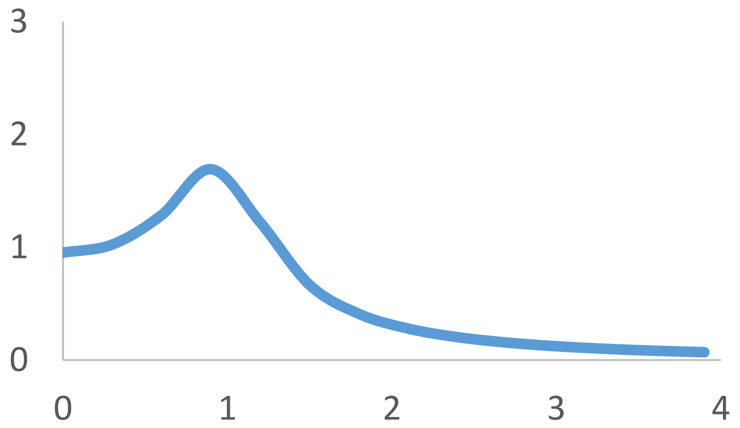
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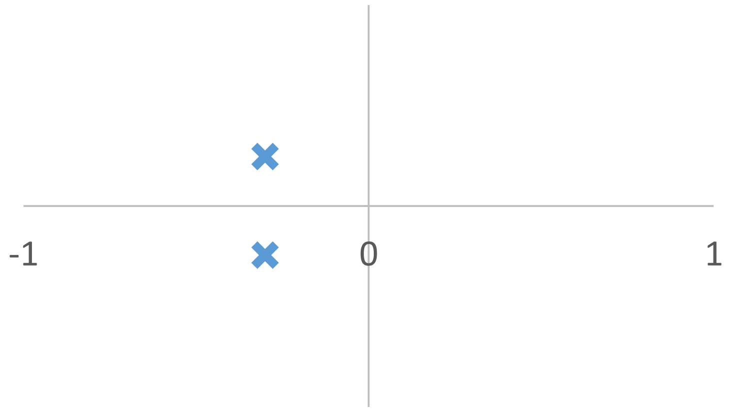
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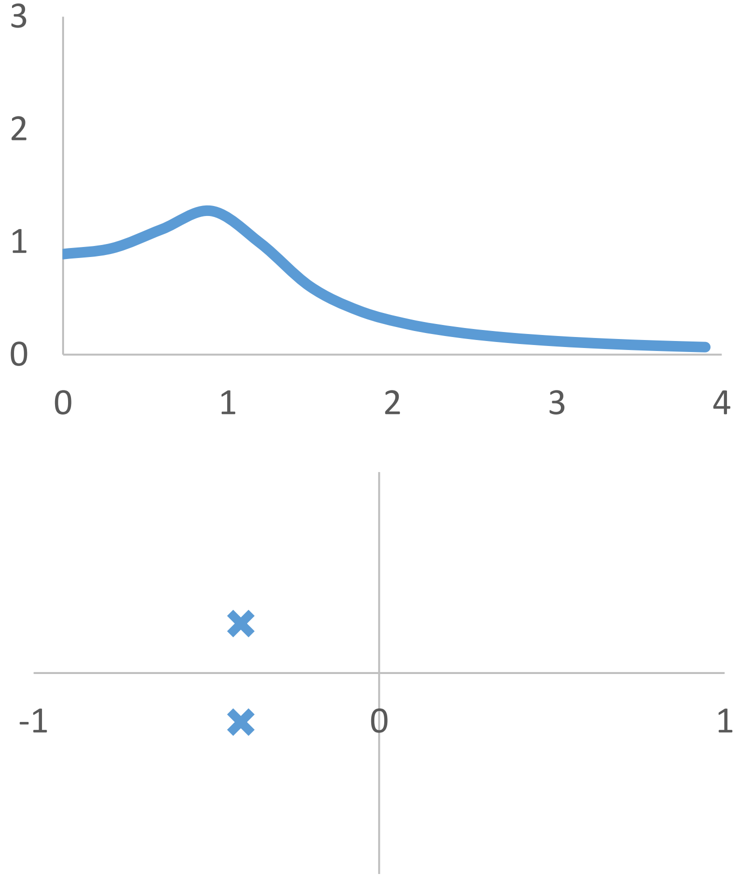

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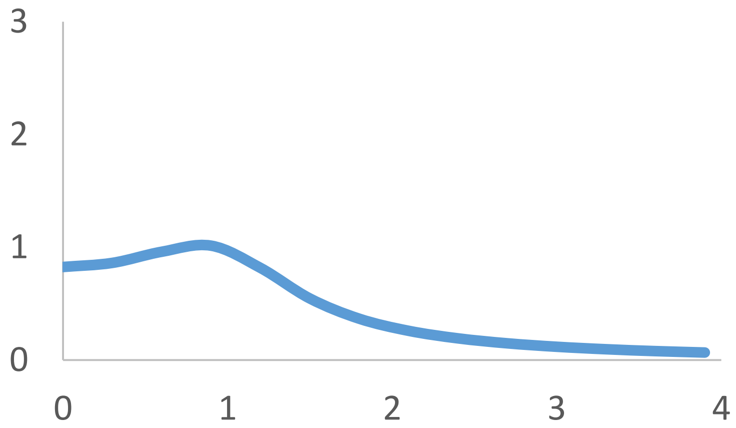
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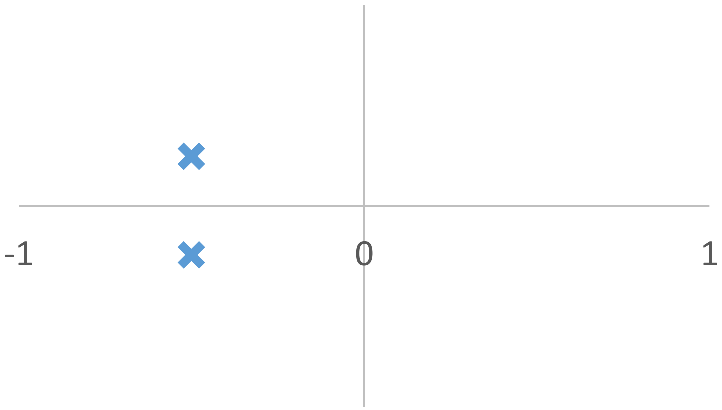
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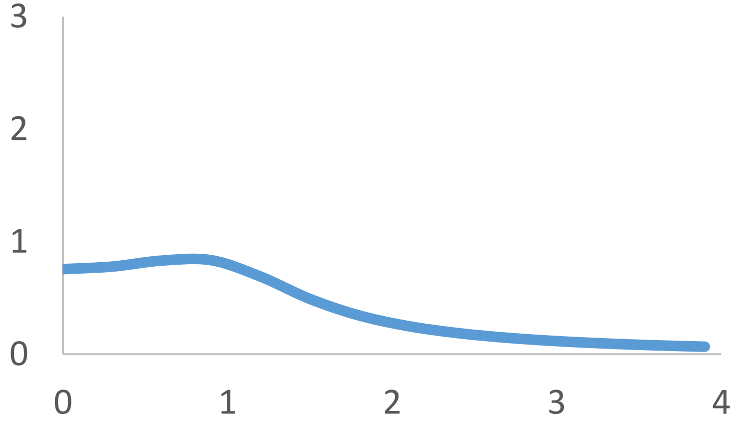
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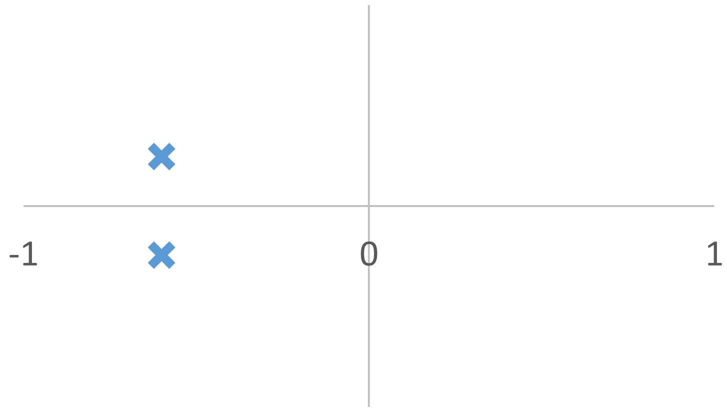
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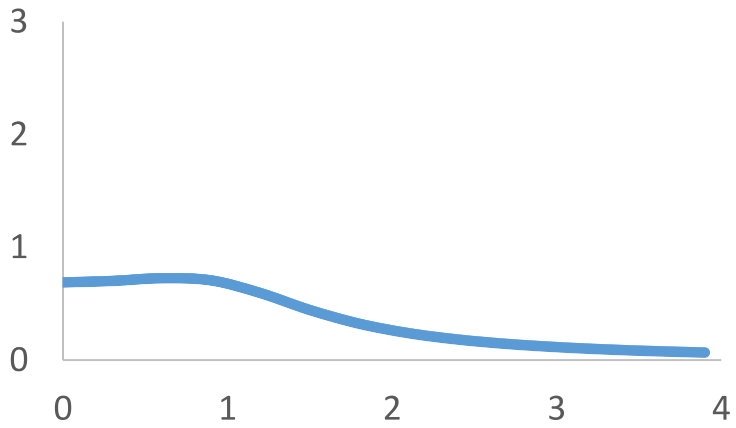
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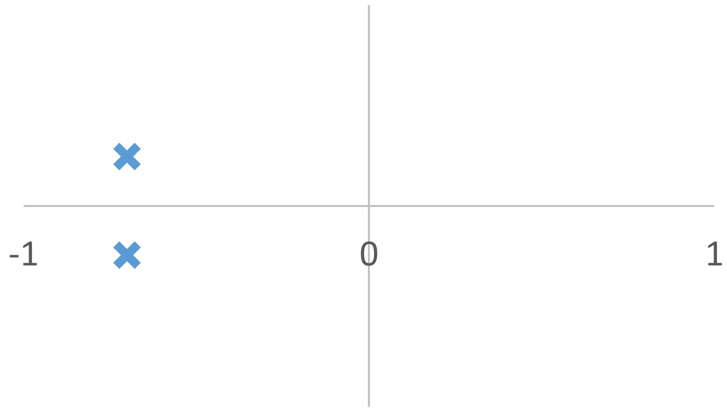
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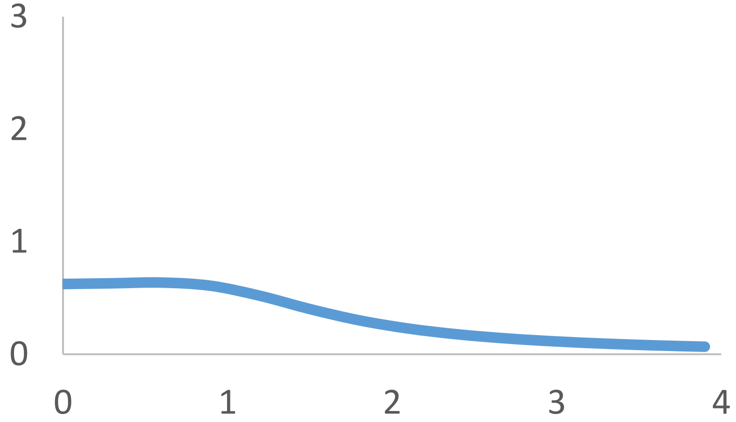
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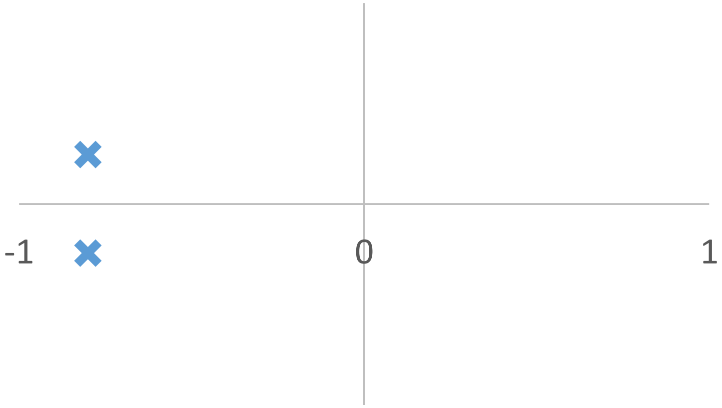
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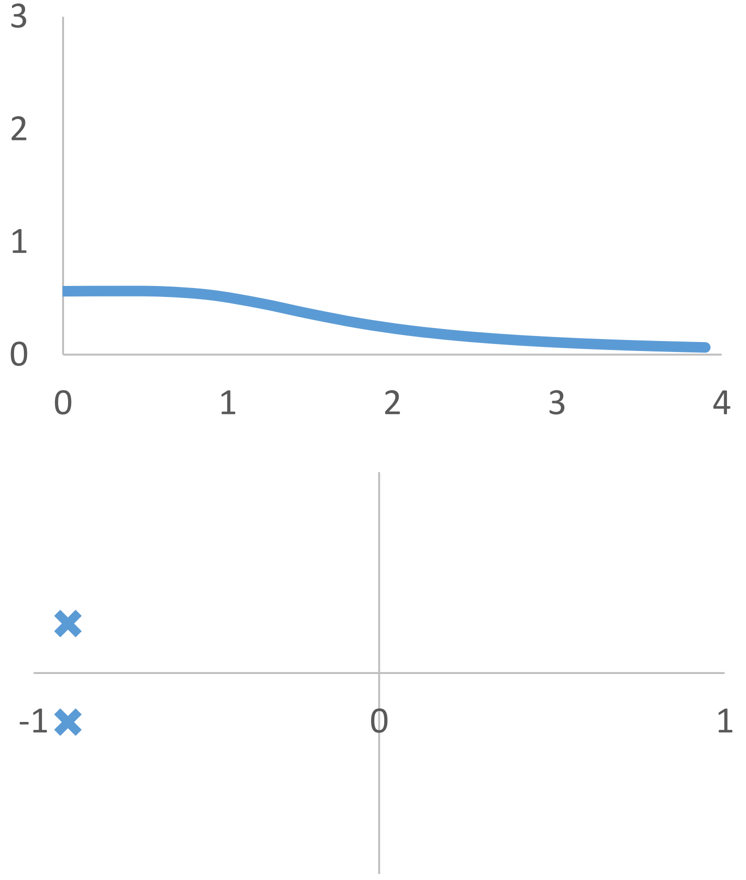

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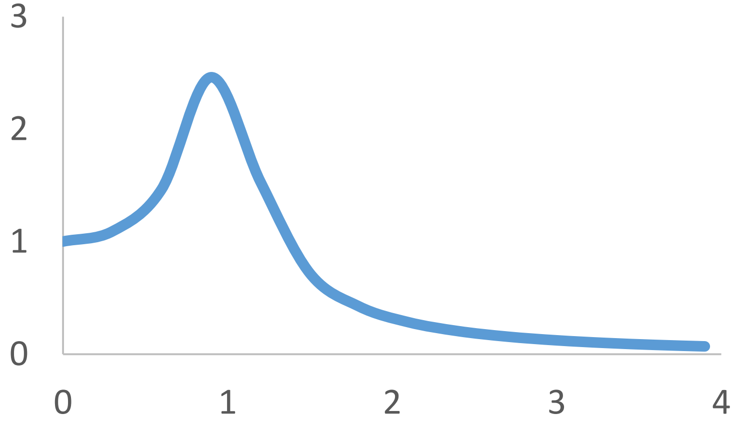
Changing imaginary part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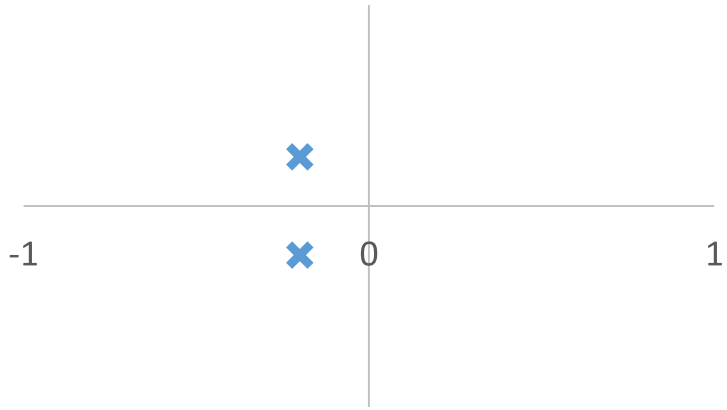
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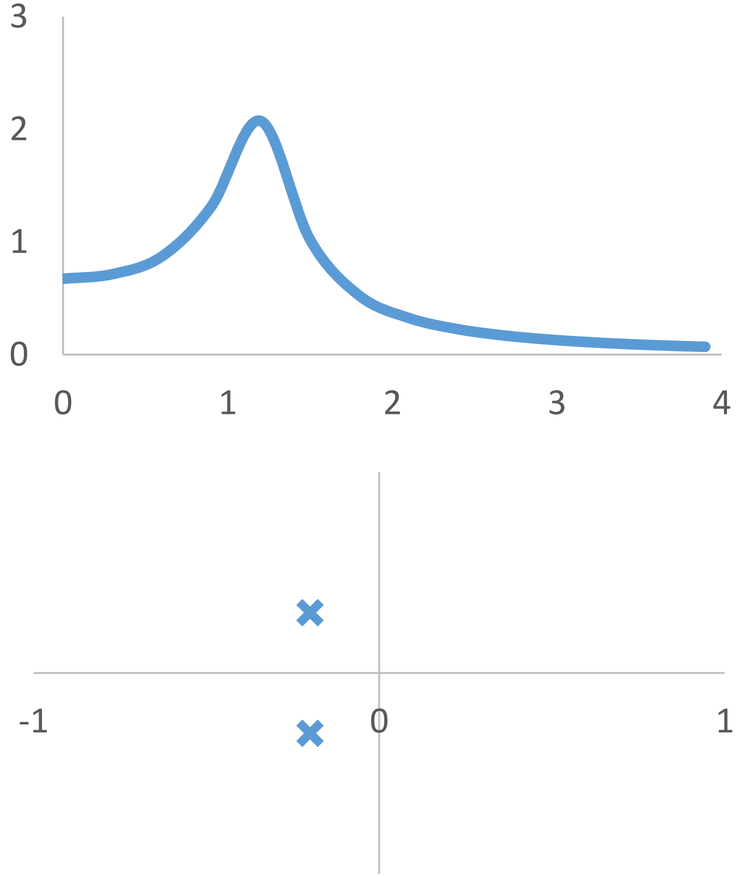
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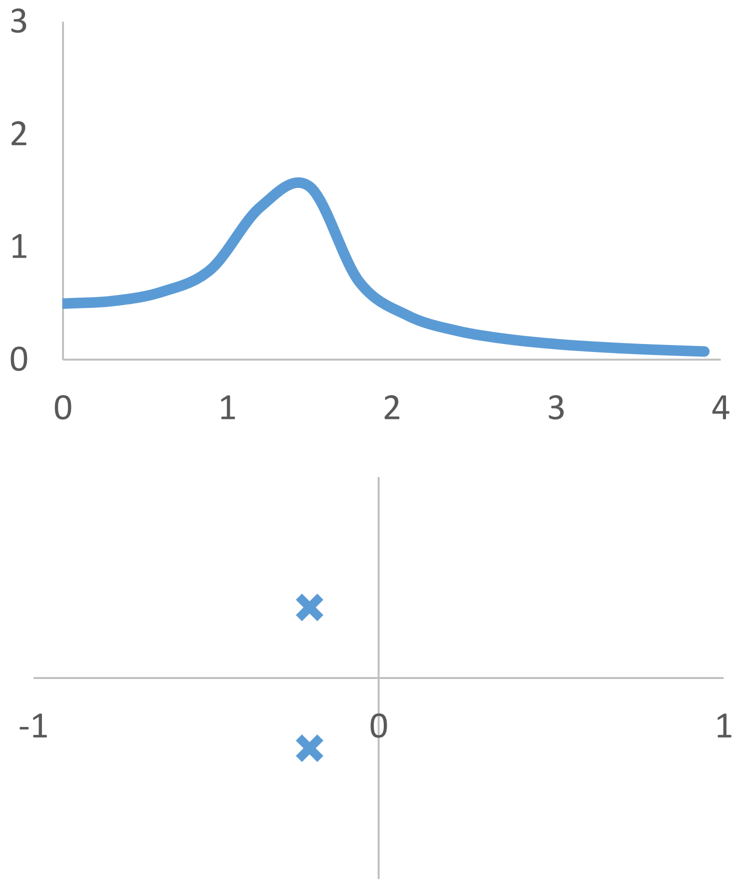
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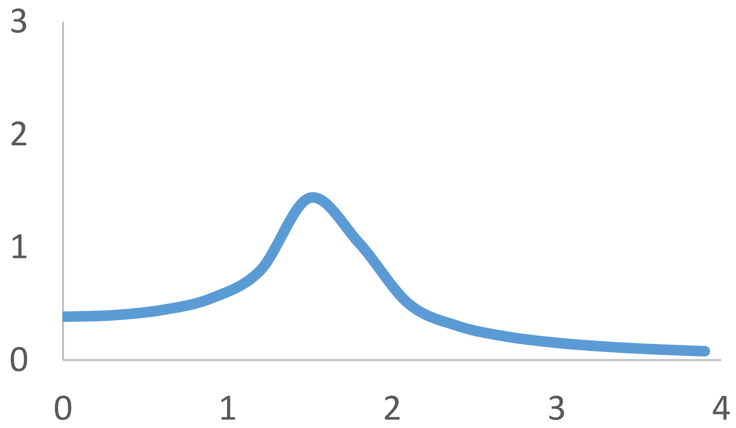
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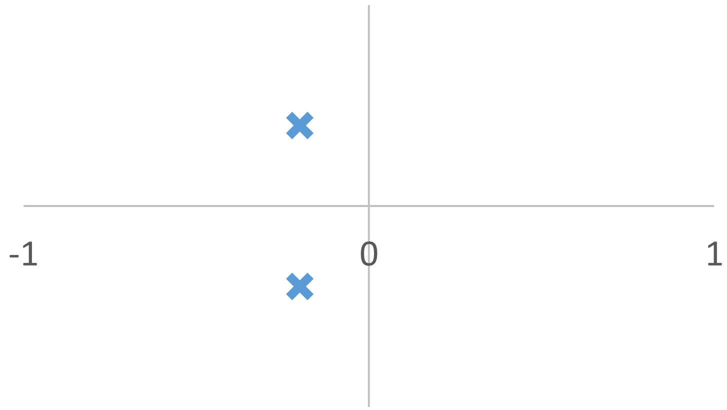
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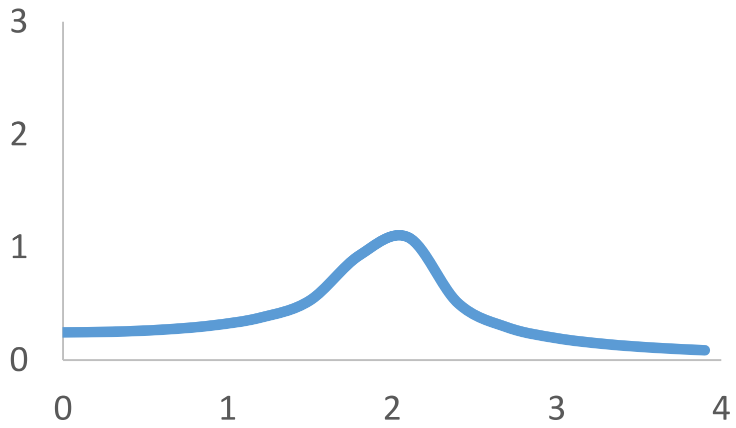
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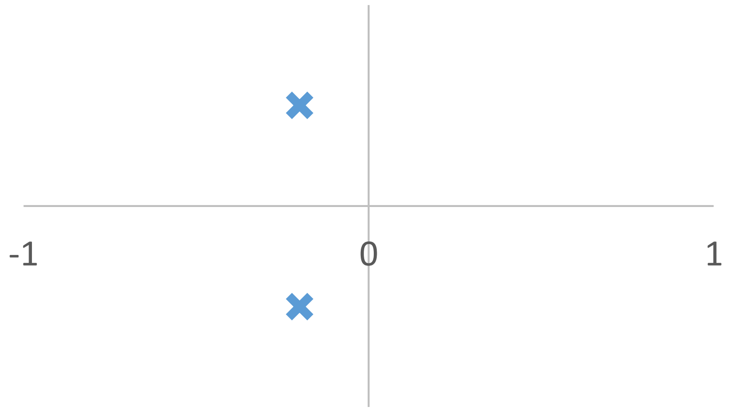
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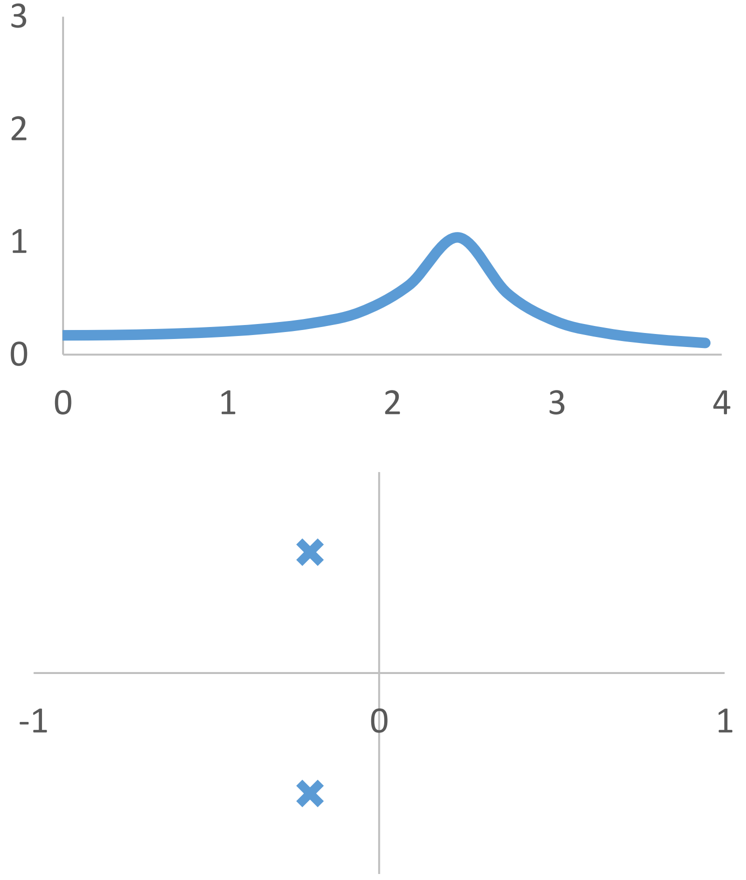
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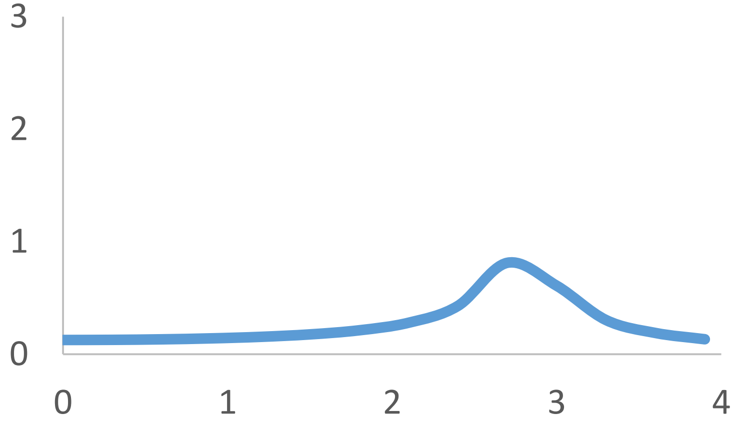
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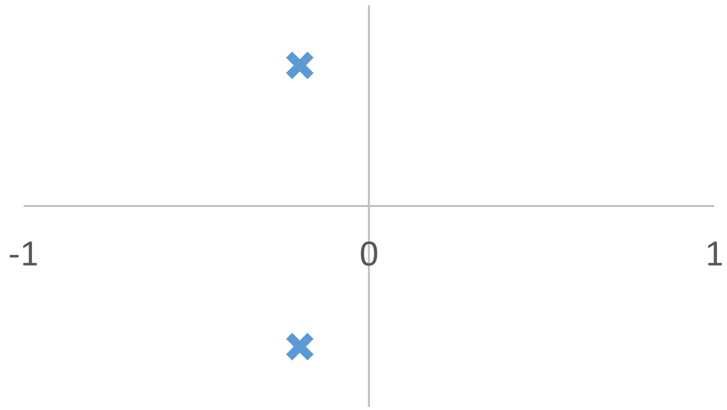
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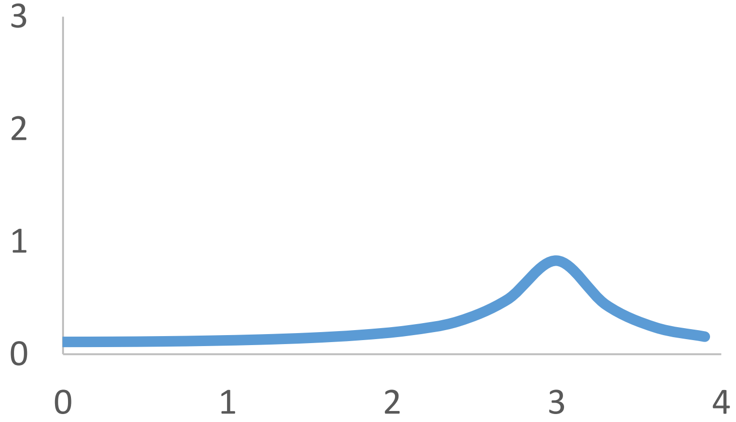
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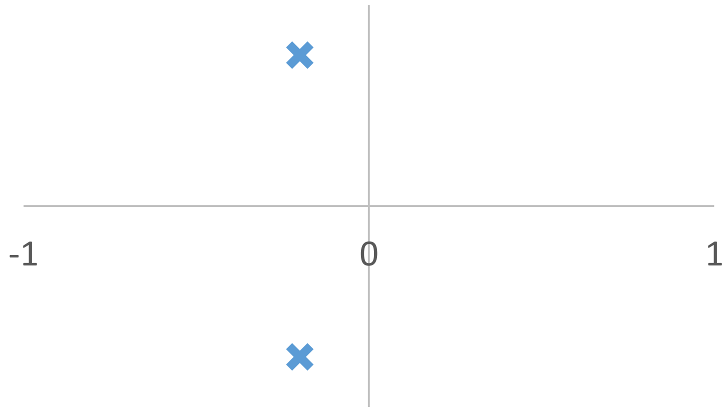
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Unilateral Laplace Transform

$$\mathbb{X}(s) = \int_{0-}^{\infty} x(t)e^{-st}dt$$

$$\mathbb{X}(s) = \int_{0-}^{\infty} x(t)e^{-st}dt = \frac{x(0\,-)}{s} + \frac{1}{s}\int_{0-}^{\infty} \frac{dx(t)}{dt}e^{-st}dt$$

$$\int_{0-}^{\infty} \frac{dx(t)}{dt}e^{-st}dt = s\mathbb{X}(s) - x(0-)$$

Solving DE

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2\mathrm{y}(\mathrm{t}) = \mathrm{x}(\mathrm{t})$$

$$y(0-) = \beta, y'(0-) = \alpha, x(t) = \delta(t)$$

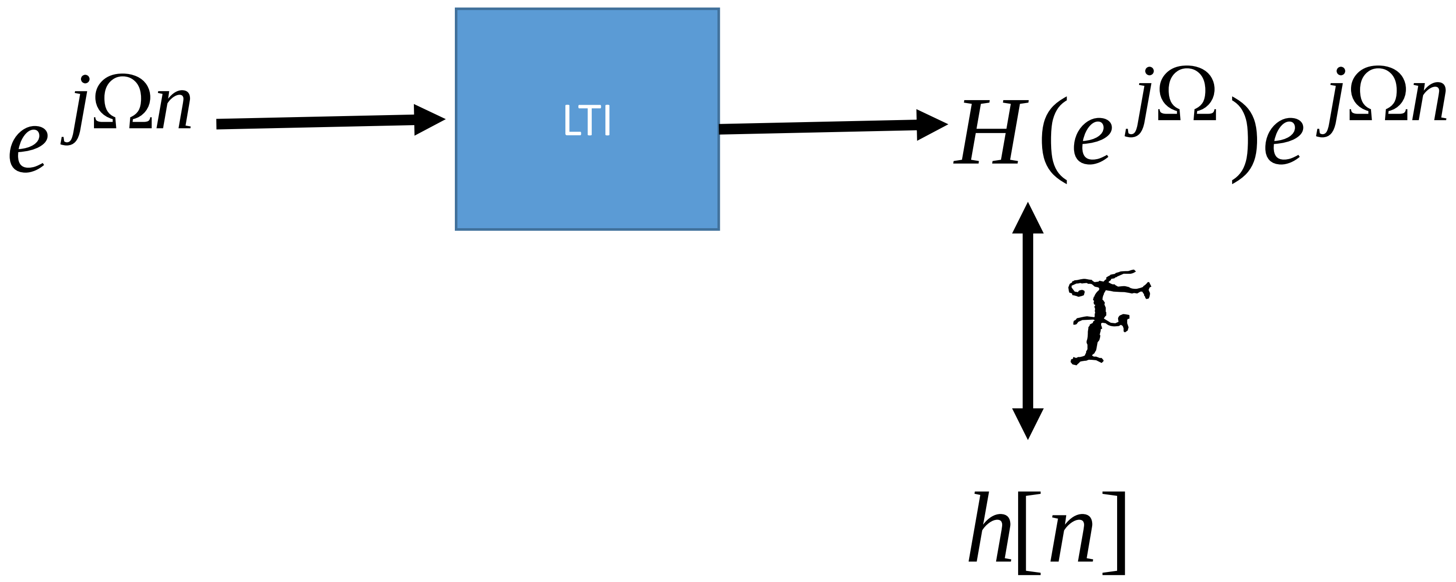
$$s(s\mathbb{Y}(s) - \beta) - \alpha + 3(s\mathbb{Y}(s) - \beta) + 2\mathbb{Y}(s) = \mathbb{X}(s)$$

$$\mathbb{Y}(s) = \frac{1 + \beta(s+3) + \alpha}{s^2 + 3s + 2}$$

Introduction to Z transforms

LTI systems

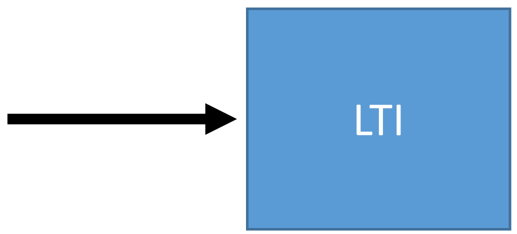
- Impulse response h[n]



LTI systems

- Impulse response h[n]

$$z^n$$



$$z = re^{j\Omega}$$

LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow \sum_{k=-\infty}^{\infty} h[k] z^{n-k}$$

$$z = re^{j\Omega}$$

LTI systems

- Impulse response h[n]

$$z^n$$

$$z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$$z = re^{j\Omega}$$

LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$$z = re^{j\Omega}$$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$$z = re^{j\Omega}$$

$$\boxed{H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}}$$

LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow z^n H(z)$$

$$z = re^{j\Omega}$$

$$\boxed{H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}}$$

Applications of Z transforms

- Scale transformations (images with different resolutions)
- Solving Difference equations with initial conditions

Z-Transform

Bilateral

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

Connection between Z and Fourier Transform

$$X(z)\,|_{r=1} = \mathcal{F}\{x[n]\}$$

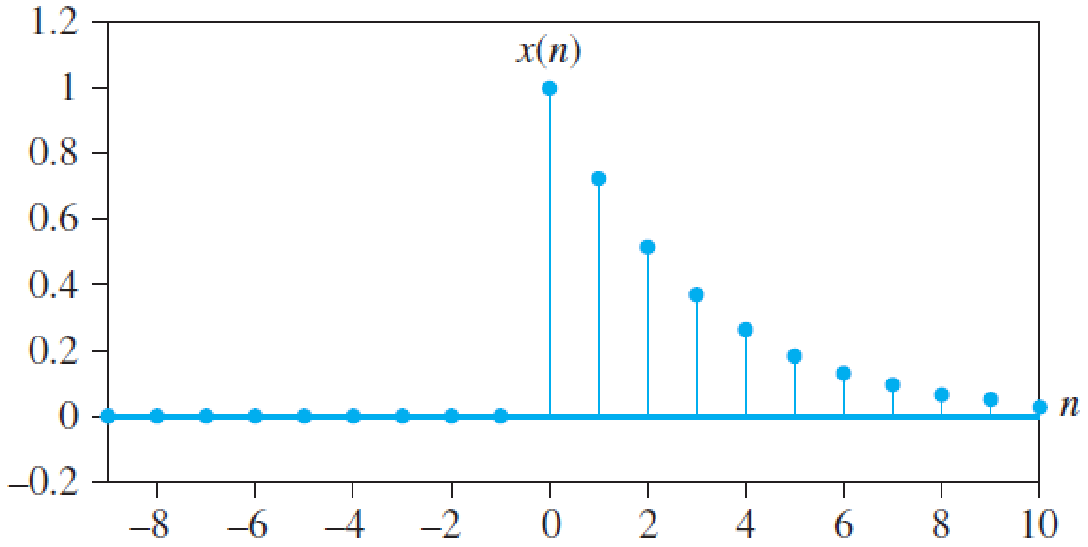
$$X(z) = \mathcal{F}\{x[n]r^{-n}\}$$

$\mathcal{Z}$ may converge when $\mathcal{F}$ does not

Causal Exponential Function

Find the z-transform of $x(n) = a^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$



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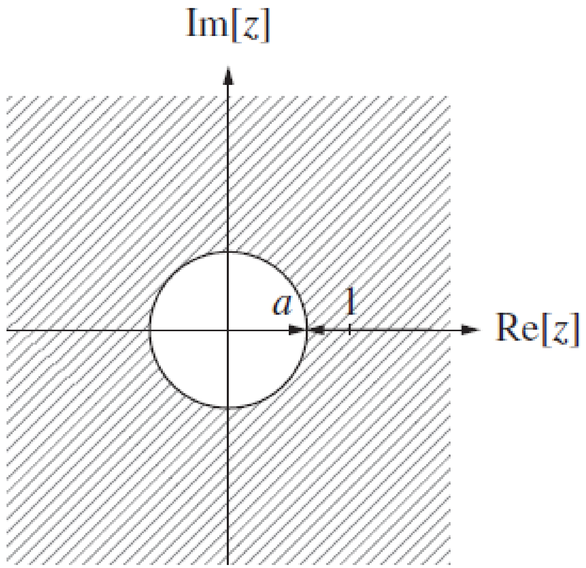

X(z) exists outside a circle with radius |a|.

Anticausal Exponential Function

Find the z-transform of $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-1}^{-\infty} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = -\{\sum_{n=0}^{\infty} a^{-n} z^n - 1\}$$

$$X(z) = \{1 - \sum_{n=0}^{\infty} a^{-n} z^n\} = 1 - \frac{1}{1 - a^{-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{a^{-1}z}{a^{-1}z - 1}$$

$$X(z) = \frac{a^{-1}z}{a^{-1}z - 1} = \frac{z}{z - a} = \frac{1}{1 - az^{-1}} \qquad |z|<|a|$$

Exponential Function

Z-transform of

$x[n] = -a^n u[-n-1]$ $\quad X(z) = \dfrac{1}{1 - az^{-1}}$ $\quad |z| < |a|$

$x[n] = a^n u[n]$ $\quad X(z) = \dfrac{1}{1 - az^{-1}}$ $\quad |z| > |a|$