MTL106 Tutorial 7 Solutions
I Semester 2021-22

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October 2021

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Problem 1. Trace the path of the following stochastic processes:

(a) $\{W_k \mid k \in T\}$ where W_k be the time that the k^{th} has to wait in the system before service and $T = \{1, 2, \dots\}$.

(b) $\{X(t) \mid t \in T\}$ where $X(t)$ be the number of jobs in system at time t, $T = \{t \mid 0 \le t < \infty\}$.

(c) $\{Y(t) \mid t \in T\}$ where $Y(t)$ denote the cumulative service requirements of all jobs in system at time t, $T = \{t \mid 0 \le t < \infty\}$.

Problem 2. Suppose that $X_1, X_2, \dots$ are iid random variables each having $\mathcal{N}(0, \sigma^2)$. Let $\{S_n \mid n = 1, 2, \dots\}$ be a stochastic process where $S_n = \exp(\sum_{i=1}^n X_i - \frac{1}{2}n\sigma^2)$. Find $\mathbb{E}[S_n]$ for all n.

Solution. Note that

$$\mathbb{E}[\exp(X_n - \tfrac{1}{2}\sigma^2)] = \mathbb{E}[\exp(X_n)] \cdot e^{-\sigma^2/2} = M_{X_n}(1)e^{-\sigma^2/2} = 1.$$

Therefore,

$$\mathbb{E}[S_n] = \mathbb{E}\left[\exp\left(\sum_{i=1}^n X_i - \tfrac{1}{2}n\sigma^2\right)\right] = \mathbb{E}\left[\prod_{i=1}^n \exp(X_i^2 - \tfrac{1}{2}\sigma^2)\right] = \prod_{i=1}^n \mathbb{E}[\exp(X_i - \tfrac{1}{2}\sigma^2)] = 1.$$

□

Problem 3. Let $X(t) = A_0 + A_1 t + A_2 t^2$, where A_i's are uncorrelated random variables with mean 0 and variance 1. Find the mean function and covariance function of $X(t)$.

Solution. The mean

$$\mathbb{E}[X(t)] = \mathbb{E}[A_0 + A_1 t + A_2 t^2] = \mathbb{E}[A_0] + \mathbb{E}[A_1]t + \mathbb{E}[A_2]t^2 = 0.$$

The covariance function,

$$\begin{aligned}
\operatorname{Cov}(X(s), X(t)) &= \mathbb{E}[X(s)X(t)] - \mathbb{E}[X(s)]\mathbb{E}[X(t)] \\
&= \mathbb{E}[(A_0 + A_1 s + A_2 s^2)(A_0 + A_1 t + A_2 t^2)] \\
&= \mathbb{E}\left[\sum_{i=0}^{2}\sum_{j=0}^{2} A_i A_j s^i t^j\right] \\
&= \sum_{i=0}^{2}\sum_{j=0}^{2} \mathbb{E}[A_i A_j] s^i t^j \\
&= \sum_{i=0}^{2}\left(\mathbb{E}[A_i^2] s^i t^i + \sum_{\substack{j=0 \\ i\neq j}}^{j=2} \mathbb{E}[A_i A_j] s^i t^j\right) \\
&= \sum_{i=0}^{2} s^i t^i + \sum_{i=0}^{2}\sum_{\substack{j=0 \\ i\neq j}}^{2} \mathbb{E}[A_i]\mathbb{E}[A_j] s^i t^j \\
&= 1 + st + (st)^2.
\end{aligned}$$

□

Problem 4. Consider the process $X_t = A\cos(wt) + B\sin(wt)$ where A and B are uncorrelated random variables with mean 0 and variance 1 and w is a positive constant. Is $\{X_t \mid t \geq 0\}$ co-variance/wide-sense stationary?

Solution. The process is wide-sense stationary as

- The mean of X_t

$$\mathbb{E}[X_t] = \mathbb{E}[A\cos(wt) + B\sin(wt)] = \mathbb{E}[A]\cos(wt) + \mathbb{E}[B]\sin(wt) = 0$$

  is independent of time.

- The covariance function,

$$\begin{aligned}
\operatorname{Cov}(X(s), X(t)) &= \mathbb{E}[X(s)X(t)] - \mathbb{E}[X(s)]\mathbb{E}[X(t)] \\
&= \mathbb{E}[(A\cos(ws) + B\sin(ws))(A\cos(wt) + B\sin(wt)] \\
&= \mathbb{E}[A^2]\cos(ws)\cos(wt) + \mathbb{E}[A]\mathbb{E}[B](\cos(ws)\sin(wt) + \sin(ws)\cos(wt)) \\
&\quad + \mathbb{E}[B^2]\sin(ws)\sin(wt) \\
&= \cos(ws)\cos(wt) + \sin(ws)\sin(wt) \\
&= \cos(w(s-t)) \\
&= \operatorname{Cov}(X(s-t), X(0))
\end{aligned}$$

  only depends on the difference of times.

- And the second moment is finite as

$$\mathbb{E}[(X(t))^2] = \mathbb{E}[X(t)]^2 + \operatorname{Cov}(X(t), X(t)) = \cos(w(t-t)) = 1.$$

□

Problem 5. In a communication system, the carrier signal at the receiver is modeled by $Y(t) = X(t)\cos(2\pi t + \Theta)$ where $\{X(t) \mid t \geq 0\}$ is a zero-mean and wide sense stationary process, Θ is a uniform distributed random variable with interval $(-\pi, \pi)$ and w is a positive constant. Assume that, Θ is independent of the process $\{X(t) \mid t \geq 0\}$. Is $\{Y(t) \mid t \geq 0\}$ wide sense stationary? Justify your answer.

Solution. The process is wide-sense stationary as,

- The mean,
$$\mathbb{E}[Y(t)] = \mathbb{E}[X(t)]\mathbb{E}[\cos(2\pi t + \Theta)] = 0$$
is independent of time.

- The covariance function,
$$\begin{aligned}
\operatorname{Cov}(Y(s), Y(t)) &= \mathbb{E}[Y(s)Y(t)] - \mathbb{E}[Y(s)]\mathbb{E}[Y(t)] \\
&= \mathbb{E}[X(s)X(t)\cos(2\pi t + \Theta)\cos(2\pi s + \Theta)] \\
&= \mathbb{E}[X(s)X(t)]\mathbb{E}[\cos(2\pi t + \Theta)\cos(2\pi s + \Theta)] \\
&= \operatorname{Cov}(X(s), X(t)) \int_{-\pi}^{+\pi} \tfrac{1}{2\pi}\cos(2\pi t + \theta)\cos(2\pi s + \theta)\, d\theta \\
&= \tfrac{1}{2}\cos(2\pi(s-t))\operatorname{Cov}(X(s), X(t)).
\end{aligned}$$
Since X is WSS, $\operatorname{Cov}(Y(s), Y(t)) = \frac{1}{2}\operatorname{Cov}(X(s), X(t)) = \frac{1}{2}\operatorname{Cov}(X(s-t), X(0)) = \operatorname{Cov}(Y(s-t), Y(0)$.

- Finally, the second moment is finite as
$$\mathbb{E}[Y(t)^2] = \operatorname{Cov}(Y(t), Y(t)) + \mathbb{E}[Y(t)]^2 = \tfrac{1}{2}\operatorname{Cov}(X(t), X(t)) = \tfrac{1}{2}\mathbb{E}[X(t)^2]$$
which is finite as X is WSS.

□

Problem 6. Let X and Y be iid random variables each having uniform distribution on interval $[-\pi, +\pi]$. Let $Z(t) = \sin(Xt + Y)$ for $t \geq 0$. Is $\{Z(t), t \geq 0\}$ covariance stationary?

Solution. The process is wide-stationary as,

- The mean,
$$\mathbb{E}[Z(t)] = \mathbb{E}[\sin(Xt + Y)] = \mathbb{E}[\mathbb{E}[\sin(tx + Y) \mid X = x]] = \mathbb{E}[0] = 0$$
is independent of time.

- Note that
$$\int_{-\pi}^{+\pi} \sin(a + x)\sin(b + x)\, dx = \pi\cos(a - b).$$

Therefore, the covariance function

$$\begin{aligned}
\mathrm{Cov}(Z(s), Z(t)) &= \mathbb{E}[Z(s)Z(t)] - \mathbb{E}[Z(s)]\mathbb{E}[Z(t)] \\
&= \mathbb{E}[\sin(Xs+Y)\sin(Xt+Y)] \\
&= \mathbb{E}[\mathbb{E}[\sin(xs+Y)\sin(xt+Y) \mid X = x]] \\
&= \mathbb{E}[\tfrac{1}{2}\cos(X(s-t))] \\
&= \begin{cases} \frac{1}{2} & s = t \\ \frac{1}{2}\cdot\frac{\sin(\pi(s-t))}{\pi(s-t)} & s \neq t \end{cases} \\
&= \begin{cases} \frac{1}{2} & s - t = 0 \\ \frac{1}{2}\cdot\frac{\sin(\pi(s-t))}{\pi(s-t)} & s - t \neq 0 \end{cases} \\
&= \mathrm{Cov}(Z(s-t), Z(0))
\end{aligned}$$

only depends on the difference in times.

- The second moment is finite as

$$\mathbb{E}[Z(t)^2] = \mathrm{Cov}(Z(t), Z(t)) + \mathbb{E}[Z(t)]^2 = \tfrac{1}{2}.$$

□

Problem 7. Consider the random telegraph signal, denoted by $X(t)$, jumps between two states, 0 and 1, according to the following rules. At time $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $P(X(0) = 0) = P(X(0) = 1) = \frac{1}{2}$, and let the switching times be decided by a Poisson process $\{Y(t) \mid t \geq 0\}$ with parameter λ independent of $X(0)$. At time t, the signal

$$X(t) = \frac{1}{2}\left(1 - (-1)^{X(0)+Y(t)}\right), t > 0.$$

If $\{X(t) \mid t \geq 0\}$ covariance/wide-sense stationary?

Solution. The process is wide-sense stationary as,

- The mean,

$$\mathbb{E}[X(t)] = \tfrac{1}{2} - \tfrac{1}{2}\mathbb{E}[(-1)^{X(0)}]\mathbb{E}[(-1)^{Y(t)}] = \tfrac{1}{2} - \tfrac{1}{2}\cdot 0 \cdot \mathbb{E}[(-1)^{Y(t)}] = \tfrac{1}{2}$$

  as independent of time.

- If $X \sim \text{Poisson}(\lambda)$ then,

$$\mathbb{E}[(-1)^X] = \sum_{k\geq 0}(-1)^k \cdot \frac{e^{-\lambda}\lambda^k}{k!} = e^{-2\lambda}.$$

  The covariance function for $s > t$,

$$\begin{aligned}
\mathrm{Cov}(X(s), X(t)) &= \mathbb{E}[X(s)X(t)] - \mathbb{E}[X(s)]\mathbb{E}[X(t)] \\
&= \frac{\mathbb{E}[(1-(-1)^{X(0)+Y(s)})(1-(-1)^{X(0)+Y(t)})] - 1}{4} \\
&= \frac{\mathbb{E}[(-1)^{2X(0)+Y(s)+Y(t)}]}{4} \\
&= \frac{\mathbb{E}[(-1)^{2X(0)}]\mathbb{E}[(-1)^{Y(s)+Y(t)}]}{4} \\
&= \frac{\mathbb{E}[(-1)^{Y(s)-Y(t)}]}{4} \qquad \text{(as } X(0), Y(s), Y(t) \in \mathbb{Z}\text{)} \\
&= e^{-2\lambda(s-t)}. \qquad \text{(as } Y(s) - Y(t) \sim \text{Poisson}(\lambda(s-t))\text{)}
\end{aligned}$$

  depends only on the difference as desired.

- The second moment is finite as

$$\mathbb{E}[X(t)^2] = \operatorname{Cov}(X(t), X(t)) + \mathbb{E}[X(t)]^2 = 1 + \frac{1}{4}.$$

$\square$

Problem 8. Let $\{X(t) \mid 0 \leq t \leq T\}$ be a stochastic process such that $\mathbb{E}[X(t)] = 0$ and $\mathbb{E}[X(t)^2] = 1$ for all $t \in [0, T]$. Find the upper bound of $|\mathbb{E}[X(t)X(t+h)]|$ for any $h > 0$ and $t \in [0, T-h]$.

Solution. We show that it is upper bounded above by 1,

$$\begin{aligned}
\mathbb{E}[X(t)X(t+h)] &= \mathbb{E}\left[\frac{X(t)^2 + X(t+h)^2 - (X(t+h) - X(t))^2}{2}\right] \\
&= 1 - \tfrac{1}{2}\mathbb{E}[(X(t+h) - X(t))^2] \\
&\leq 1. \\
\mathbb{E}[X(t)X(t+h)] &= \mathbb{E}\left[\frac{(X(t+h) + X(t))^2 - X(t)^2 - X(t+h)^2}{2}\right] \\
&= \tfrac{1}{2}\mathbb{E}[(X(t+h) + X(t))^2] - 1 \\
&\geq -1.
\end{aligned}$$

Therefore,

$$|\mathbb{E}[X(t)X(t+h)]| \leq 1.$$

$\square$

Remark. It also trivially follows from Cauchy-Schwarz inequality.

Problem 9. Let A be a positive random variable that is independent of a strictly stationary random process $X(t) \mid t \geq 0$. Show that $Y(t) = AX(t)$ is also strictly stationary random process.

Proof. $\square$

Problem 10. Is the stochastic process $\{X(t) \mid t \in T\}$ stationary, whose probability distribution under a certain condition given by

$$P\{X(t) = n\} = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}} & n = 1, 2, \ldots \\ \frac{at}{1+at} & n = 0 \end{cases}.$$

Solution. The process is not strictly stationary as its second moment is not independent of time.

$$\mathbb{E}[X(t)^2] = \frac{1}{(1+at)^2} \cdot \sum_{n \geq 1} n^2 \left(\frac{at}{1+at}\right)^{n-1} = 2at + 1.$$

If $a = 0$, $P(X(t) = 1)$ is not defined therefore $a \neq 0$. $\square$