Higher Order Linear ODEs

ODE — Ordinary differential equation

Plan

(i) Theory of existence & uniqueness of the solutions of second order linear ODEs

(ii) Methods to determine the general solution of homogeneous & non-homogeneous second order linear ODE's

Extension to n^{th} order ODE ($n>2$)

Prerequisite

(i) Theory of first order ODEs

(ii) Linear algebra

Reference Books

(i) E. Kreyszig

(ii) G. Simmons

n^{th} order linear ODE

$x \rightarrow$ dependent variable

$t \rightarrow$ independent variable

The general form

$$x^{(n)} + a_1(t)\, x^{(n-1)} + a_2(t)\, x^{(n-2)} + \cdots + a_{n-1}(t)\, x'(t) + a_n(t)\, x = f(t). \quad \text{--- (1)}$$

$$x' = \frac{dx}{dt}, \quad x'' = \frac{d^2x}{dt^2} \cdots \quad x^{(n)} = \frac{d^nx}{dt^n}$$

In DE (1)

- if the r.h.s. f^n $f(t) \equiv 0$ then DE (1) is called homogeneous o/w is called non-homogeneous

- $$b_0(t)\, x^{(n)} + b_1(t)\, x^{(n-1)} + \cdots + b_n(t)\, x = f(t)$$

where $b_0 \neq 0$, by dividing by $b_0(t)$, we get

$$x^{(n)} + \underbrace{\frac{b_1(t)}{b_0(t)}}_{a_1(t)} x^{(n-1)} + \cdots + \underbrace{\frac{b_n(t)}{b_0(t)}}_{a_n(t)} x = \frac{f(t)}{b_0(t)}$$

$n=2$

Second order linear ODE

$$x'' + a_1(t)x' + a_2(t)x = \underline{f(t)} \quad \text{——— (2)}$$

Exps

(i) $x'' + x' = e^t$ (Linear)

(ii) $x'' + e^t x' + t^3 x = \tan t$ (Linear)

(iii) $x'' + e^t x' + t^2 x = \tan x$ (Non linear)

(iv) $x'' + \underline{x x'} = e^t$ (Non-linear)

(v) $x'' + t x' + \underline{x^2} = t^3$ (Non-linear)

$$x \longmapsto x'' + a_1(t)x' + a_2(t)x$$

$$L(x) := \underline{x''} + a_1(t)\underline{x'} + a_2(t)\underline{x} \quad \text{——— (3)}$$

L satisfies :

(i) $L(cx) = c\,L(x)$

(ii) $L(x_1 + x_2) = L(x_1) + L(x_2)$

We have

- $$L(cx) = (cx)'' + a_1(t)(cx)' + a_2(t)(cx)$$
$$= c\left(x'' + a_1(t)x' + a_2(t)x\right)$$
$$= c\,L(x)$$

- $$L(x_1+x_2) = (x_1+x_2)'' + a_1(t)(x_1+x_2)' + a_2(t)(x_1+x_2)$$
$$= x_1'' + a_1(t)x_1' + a_2(t)x_1 + x_2'' + a_1(t)x_2' + a_2(t)x_2$$
$$= L(x_1) + L(x_2)$$

1 & 2 can be equivalanently stated as

$$L(c_1x_1 + c_2x_2) = c_1\,L(x_1) + c_2\,L(x_2)$$

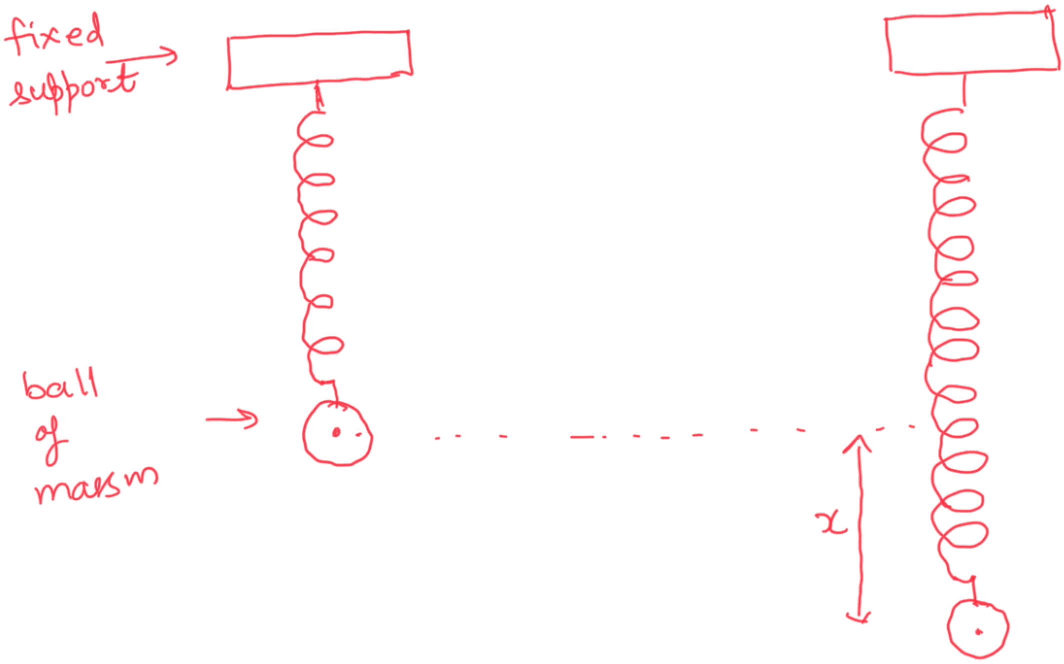
(verify for 3)

Motivation

Expl (Vibration of a spring)

Suppose we have a coiled spring which is suspended vertically from a fixed support and a ball of mass m is attached at its a ball of mass

lower end.



The spring force

$$= -Kx = m\frac{d^2x}{dt^2}$$

(By the Hooke's law spring force $\propto$ stretch in the spring

where $K > 0$ is a constant

$$\boxed{\frac{d^2x}{dt^2} + \frac{K}{m}x}$$

Exp2 (Bending of a beam)

Lets consider a homogeneous elastic beam (B) of length L.

Now we apply a load on beam B in a vertical plane then beam gets bent.

Using the linear elasticity theory, the deflection of the beam is modelled by

$$\frac{d^4y}{dx^4} = y^{iv}(x) = c\,f(x)$$

x – independent variable

Homogeneous Linear Second Order ODE's

IVP → Initial value Problem

Consider

IVP (1)
$$\begin{cases} x''(t) + a(t)\,x'(t) + b(t)\,x(t) = 0 \quad \text{——— (1)} \\ x(t_0) = \alpha \\ x'(t_0) = \beta \end{cases}$$

α, β are given

Define

$$x_1(t) = x(t)$$

$$x_2(t) = x'(t) = x_1'(t)$$

then using ①

$$x_2'(t) = x''(t) = -a(t)x'(t) - b(t)x(t)$$

$$= -a(t)\,x_2(t) - b(t)\,x_1(t)$$

System of first order ODE's

$$\left\{\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -a(t)x_2(t) - b(t)x_1(t) \end{aligned}\right\} \quad — ②$$

② can be rewritten as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b(t) & -a(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Define $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ then last system can be

written as

$$\boxed{X'(t) = A\,X(t)} \quad — ③$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -b(t) & -a(t) \end{bmatrix}$$

Recall

first order ODE $\quad x' = f(t, x)$

Initial condition $\quad x(t_0) = x_0$

Likewise for (3)

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = X(t_0) = X_0 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$x(t_0) = x_1(t_0) = \alpha$$

$$x'(t_0) = x_2(t_0) = \beta$$

$\underline{x'' + a_1(t)x' + a_2(t)x = 0}$

<u>Def^n</u> A function $x = h(t)$ is said to be a sol^n of 2^{nd} order ODE (1) on some open interval I if h is twice differentiable in I & it satisfies the DE (1).

<u>Theorem 1</u> (Existence & Uniqueness theorem)

If $a(t)$ & $b(t)$ are bounded and continuous function on some open interval I then IVP

$$x'' + a(t)x' + b(t)x = 0$$

$$x(t_0) = \alpha, \quad x'(t_0) = \beta, \quad t_0 \in I$$

has a unique solution on I, with given $\alpha, \beta \in \mathbb{R}$.

Exercise Discuss the existence & uniqueness of the soln of the IVP

$$x''(t) + \cos(1/t)\, x'(t) + x(t) = 0 \quad , t \in (0,1)$$

$$x(1/2) = 0 \quad , \quad x'(1/2) = 1 .$$

Defn (General soln)

A soln of a linear ODE is called a general solution if it involves as many arbitrary constants as the order of DE.

Exp Solve $x''(t) - x(t) = 0$ —— (4)

$x_1(t) = e^t$ $\qquad\qquad x_1'' = e^t = x_1$

$x_2(t) = e^{-t}$ $\qquad\qquad x_2'' = e^{-t} = x_2$

x_1 & x_2 are soln's of (4)

And

$x(t) = c_1 x_1(t) + c_2 x_2(t)$, $c_1, c_2 \to$ arbitrary constant

$\boxed{x(t) = c_1 e^t + c_2 e^{-t}}$

is also a soln of (1)

general sol^n of (4) since it involves two arbitrary constants.

Observations

(1) If $x_1(t)$ & $x_2(t)$ are two sol^n of the given 2^{nd} order linear ODE then their linear combination is also a sol^n of the same DE.

(2) We could get the general sol^n of the 2^{nd} order linear ODE if we could get hold on two $\boxed{\text{suitable}}$ sol^ns of same ODE.

Consider $$x''(t) + a(t)x'(t) + b(t)x(t) = 0 \quad \text{—— (5)}$$

Thm 2 If $x_1(t)$ & $x_2(t)$ are two solutions of ODE (5) then $c_1x_1(t) + c_2x_2(t)$ is also a sol^n of ODE (5) for any constants c_1 & c_2.

Proof. $x_1(t)$ & $x_2(t)$ are sol^ns of DE (5), therefore $x_1(t)$ & $x_2(t)$ satisfies

$$x_1'' + a(t)\, x_1' + b(t)\, x_1 = 0$$

$$x_2'' + a(t)\, x_2' + b(t)\, x_2 = 0$$

set

$$\underline{y(t) = c_1 x_1 + c_2 x_2} \qquad \text{——— } (6)$$

We have

$$y''(t) + a(t)\, y'(t) + b(t)\, y(t)$$

$$= (c_1 x_1 + c_2 x_2)'' + a(t)\,(c_1 x_1 + c_2 x_2)' + b(t)\,(c_1 x_1 + c_2 x_2)$$

$$= c_1 \underbrace{(x_1'' + a(t)\, x_1' + b(t)\, x_1)}_{0} + c_2 \underbrace{(x_2'' + a(t)\, x_2' + b(t)\, x_2)}_{0}$$

$$= 0$$

$\Rightarrow$ $y(t)$ is also a solⁿ of ODE (5)

<u>Remark</u>. If we can get hold of any two solⁿs $x_1(t)$ & $x_2(t)$ of DE (5) then (6) provides another solⁿ of DE (5) which involves two arbitrary constants so it <u>may</u> be the general solⁿ of ODE (5).

* Does linear combination of any two sol^{n}s of ODE (5) yields the general sol^n?

NO

if either $x_1(t)$ or $x_2(t)$ is a constant multiple of the other, say

$$\underline{x_1(t) = K x_2(t)}, \quad K \to \text{constant}$$

$$\underbrace{c_1 x_1(t) + c_2 x_2(t)} = c_1 K x_2(t) + c_2 x_2(t)$$

$$= \underbrace{(c_1 + K c_2)} x_2(t)$$

$$= \underline{d\, x_2(t)}$$

hence no more the general sol^n of ODE (5)

RK If neither x_1 nor x_2 is constant multiple of other then $c_1 x_1(t) + c_2 x_2(t)$ will be a general sol^n of ODE (5).

Linear Dependence & linear independence of functions

We call $x_1(t)$ & $x_2(t)$ linearly dependent (L.D.) in some interval I if there exists constants c_1 & c_2 not both zero s.t.

$$\underbrace{c_1 x_1(t) + c_2 x_2(t) = 0}$$ $\forall\ t \in I$

* $x_1(t)$ & $x_2(t)$ are linearly independent (L.I.) if there are no such non-trivial constants c_1 & c_2 which means if we consider

$$c_1 x_1(t) + c_2 x_2(t) = 0 \implies c_1 = 0,\ c_2 = 0.$$

* Suppose $x_1(t)$ & $x_2(t)$ are L.D.

$\Rightarrow \exists$ constants c_1, c_2 not both zero s.t.

$$c_1 x_1 + c_2 x_2 = 0 \quad \text{———} \quad (7)$$

. If $c_1 \neq 0$ then upon dividing (7) by c_1, we get

$$x_1(t) = \underbrace{-\frac{c_2}{c_1}}_{K} x_2(t) = K x_2(t) \text{ on } I$$

- If $c_2 \neq 0$ then (7) yields

$$x_2(t) = \underbrace{-\frac{c_1}{c_2}}_{l} x_1(t) = l x_1(t) \text{ on } I$$

$\Rightarrow$ If $x_1(t)$ & $x_2(t)$ are L.D. then either $x_1(t)$ or $x_2(t)$ can be written as a constant multiple of other.

<u>Exp</u> Show that e^t & $\cos t$ are L.I. on any interval.

<u>Soln</u>

$$c_1 e^t + c_2 \cos t = 0 \quad \forall\, t \in \underline{\underline{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = I}}$$

$$\left.\begin{array}{ll} \text{For } t=0, & c_1 + c_2 = 0 \\ t = \pi/2, & c_1 e^{\pi/2} = 0 \end{array}\right\} c_1 = 0, c_2 = 0$$

$\Rightarrow$ e^t & $\cos t$ are L.I.

<u>Exer</u> Show that e^t, e^{t+2} & $\sin t$ are L.I. on any interval.

Def^n The Wronskian of two f^{n}s $x_1(t)$ & $x_2(t)$ is given by

$$W(x_1, x_2)(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = x_1(t)x_2'(t) - x_1'(t)x_2(t)$$

Thm 3 Suppose $x_1(t)$ & $x_2(t)$ are L.D. & sufficiently differentiable f^{n}s then

$$W(x_1, x_2)(t) = 0 .$$