MULTIVARIABLE CALCULUS
LECTURE 22

1. Differentiability

1.1. **Differentiability.** It is clear from the previous examples that the concept of differentiability of two (or several variables) should be stronger than the mere existence of partial derivatives of the function. Let us recall the definition of differentiability for one variable function : $f : \mathbb{R} \to \mathbb{R}$, we define

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. In case, $f : \mathbb{R}^2 \to \mathbb{R}$, the above definition of differentiability of a function of one variable can not be generalised as we can not divide by an element of $\mathbb{R}^2$. Let us generalised the above definition in one variable. Let $f : \mathbb{R} \to \mathbb{R}$. Then f is differentiable at x iff there exists $\alpha \in \mathbb{R}$ such that

$$\frac{|f(x+h) - f(x) - \alpha.h|}{|h|} \to 0, \quad \text{as } h \to 0.$$

We shall now define the notion of derivative as in the above way.

Definition 1.1. Let D be an open set of $\mathbb{R}^2$. Then a function $f : D \to \mathbb{R}$ is differentiable at a point (a, b) of D if there exist $\alpha = (\alpha_1, \alpha_2)$ and $\varepsilon_1 = \varepsilon_1(h, k)$, $\varepsilon_2 = \varepsilon_2(h, k)$ such that

$$f(a+h, b+k) - f(a, b) = h\alpha_1 + k\alpha_2 + h\varepsilon_1(h, k) + k\varepsilon_2(h, k) \tag{1}$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(h, k) \to (0, 0)$.

Theorem 1.1. *Suppose f is differentiable at a point (a, b). Then the partial derivatives $\frac{\partial f}{\partial x}$ or f_x and $\frac{\partial f}{\partial y}$ or f_y exist at (a, b) and $\alpha = (\alpha_1, \alpha_2) = (f_x(a, b), f_y(a, b))$ in the above definition (1.1).*

Proof. Since f is differentiable at a point (a, b) of D then there exist $\alpha = (\alpha_1, \alpha_2)$ and $\varepsilon_1 = \varepsilon_1(h, k)$, $\varepsilon_2 = \varepsilon_2(h, k)$ such that

$$f(a+h, b+k) - f(a, b) = h\alpha_1 + k\alpha_2 + h\varepsilon_1(h, k) + k\varepsilon_2(h, k)$$

Substituting $k = 0$ in the above

$$f(a+h, b) - f(a, b) = h\alpha_1 + h\varepsilon_1(h, k)$$

which in turn implies

$$\lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} = \alpha_1$$

And by definition $f_x(a, b) = \frac{\partial f}{\partial x}(a, b) = \alpha_1$.

Similarly, we can prove $f_y(a, b) = \frac{\partial f}{\partial y}(a, b) = \alpha_2$ □

Remark 1.1. Suppose f is differentiable at a point (a,b). Then there exist $\varepsilon_1 = \varepsilon_1(h,k)$, $\varepsilon_2 = \varepsilon_2(h,k)$

$$f(a+h,b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b) + h\varepsilon_1(h,k) + k\varepsilon_2(h,k). \tag{2}$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$.

Theorem 1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$. If f is differentiable at (a,b), then f is continuous at (a,b).*

Proof.

$$\begin{aligned} |f(a+h,b+k) - f(a,b)| &= |f(a+h,b+k) - f(a,b) - (h,k).(f_x(a,b), f_y(a,b)) + (h,k).(f_x(a,b), f_y(a,b))| \\ &\leq |h||\varepsilon_1| + |k||\varepsilon_2| + |h||f_x(a,b)| + |k||f_y(a,b)| \to 0, \end{aligned}$$

as $(h,k) \to (0,0)$. Hence continuous at (a,b). ☐

Let us compute some examples.

Example 1.1. *Consider the function $f(x,y) = x^2 + y^2 + xy$. Then $f_x(0,0) = f_y(0,0) = 0$. Also*

$$f(h,k) - f(0,0) = h^2 + k^2 + hk = 0.h + 0.k + h\varepsilon_1(h,k) + k\varepsilon_2(h,k)$$

where $\varepsilon_1 = h+k$, and $\varepsilon_2 = k$. So $\varepsilon_1, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$. Therefore f is differentiable at $(0,0)$.

Example 1.2. *Show that the following function $f(x,y)$ is is not differentiable at $(0,0)$. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} x\sin\dfrac{1}{x} + y\sin\dfrac{1}{y} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

Clearly, f is continuous at $(0,0)$. Indeed, we get $|f(x,y)| \leq |x| + |y| \leq \sqrt{2}\sqrt{x^2+y^2}$ implies that f is continuous at $(0,0)$. Moreover it is easy to see that $f_x(0,0) = f_y(0,0) = 0$.

Suppose $f(x,y)$ is differentiable at $(0,0)$, then there exist $\varepsilon_1 = \varepsilon_1(h,k)$, $\varepsilon_2 = \varepsilon_2(h,k)$ such that

$$f(h,k) - f(0,0) = h\varepsilon_1(h,k) + k\varepsilon_2(h,k)$$

and $\varepsilon_1, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$. Taking $(h,k) \to (0,0)$ along the line $h = k$, we have

$$\begin{aligned} f(h,h) - f(0,0) &= h\varepsilon_1(h,h) + h\varepsilon_2(h,h) \\ 2h\sin\frac{1}{h} &= h(\varepsilon_1(h,h) + \varepsilon_2(h,h)) \\ 2\sin\frac{1}{h} &= (\varepsilon_1(h,h) + \varepsilon_2(h,h)) \to 0 \quad \text{as } h \to 0 \end{aligned}$$

But $\lim_{h\to 0}\sin\frac{1}{h}$ does not exist. Hence a contradiction. Hence $f(x,y)$ is not differentiable at $(0,0)$.

Example 1.3. *Show that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at the origin $(0,0)$.*

The function f is continuous at $(0,0)$. Moreover, $f_x(0,0) = \lim \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$, and similarly $f_y(0,0) = 0$.

Suppose f is differentiable at $(0,0)$, then there exist ε_1 and ε_2 such that

$$f(h,k) - f(0,0) = h\varepsilon_1(h,k) + k\varepsilon_2(h,k)$$

and $\varepsilon_1, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$. Taking $(h,k) \to (0,0)$ along the line $h = k$, we have

$$\begin{aligned} & f(h,h) - f(0,0) = h\varepsilon_1(h,k) + h\varepsilon_2(h,k) \\ \Rightarrow\ & |h| = h(\varepsilon_1(h,k) + \varepsilon_2(h,k)) \\ \Rightarrow\ & \frac{|h|}{h} = \varepsilon_1(h,k) + \varepsilon_2(h,k) \to 0 \quad \text{as } h \to 0 \end{aligned}$$

But $\lim_{h\to 0} \frac{|h|}{h}$ does not exist. Hence a contradiction. Therefore, $f(x,y)$ is not differentiable at $(0,0)$.

1.2. **Equivalent condition for Differentiability.** Let us fix some notations.

(1) $\Delta f(a,b) = f(a+h, b+k) - f(a,b)$,the total variation of f.
(2) $df(a,b) = hf_x(a,b) + kf_y(a,b)$, the total differential of f.
(3) $\rho = \sqrt{h^2+k^2}$.

Theorem 1.3. Equivalent condition for differentiability: *f is differentiable at (a,b) $\iff \lim_{\rho\to 0} \dfrac{\Delta f(a,b) - df(a,b)}{\rho} = 0$.*

Proof. Suppose f is differentiable at a point (a,b). Then then there exist $\varepsilon_1 = \varepsilon_1(h,k)$, $\varepsilon_2 = \varepsilon_2(h,k)$

$$f(a+h, b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b) + h\varepsilon_1(h,k) + k\varepsilon_2(h,k).$$

and $\varepsilon_1, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$.

$$\Delta f(a,b) - df(a,b) = h\varepsilon_1(h,k) + k\varepsilon_2(h,k).$$

Now $\frac{|h|}{\sqrt{h^2+k^2}} \le 1$ and $\frac{|k|}{\sqrt{h^2+k^2}} \le 1$. So

$$\begin{aligned} \frac{\Delta f(a,b) - df(a,b)}{\rho} &= \frac{h}{\rho}\varepsilon_1(h,k) + \frac{k}{\rho}\varepsilon_2(h,k) \\ &\le \frac{|h|}{\rho}|\varepsilon_1(h,k)| + \frac{|k|}{\rho}|\varepsilon_2(h,k)| \\ &\le |\varepsilon_1(h,k)| + |\varepsilon_2(h,k)| \quad \to 0 \ \text{ as } \ (h,k) \to (0,0) \end{aligned}$$

Since, $\rho \to 0 \Leftrightarrow (h,k) \to (0,0)$, hence

$$\lim_{\rho\to 0} \frac{\Delta f(a,b) - df(a,b)}{\rho} = 0.$$

Conversely, let $\lim_{\rho\to 0} \frac{\Delta f(a,b) - df(a,b)}{\rho} = 0$. We need to prove f is differentiable at (a,b). Then

$$\begin{aligned}\Delta f(a,b) &= df(a,b) + \rho\,\varepsilon(\rho) \qquad \text{where } \varepsilon(\rho) \to 0 \text{ as } \rho \to 0\\ &= df(a,b) + \rho\,\varepsilon(h,k) \to 0 \text{ as } (h,k) \to (0,0).\end{aligned}$$

since $\rho \to 0 \Leftrightarrow (h,k) \to (0,0)$. Now for $(h,k) \neq (0,0)$, we have

$$\begin{aligned}\varepsilon\rho &= \varepsilon\rho\frac{|h|+|k|}{|h|+|k|}\\ &= \varepsilon(h,k)\rho\frac{|h|}{|h|+|k|} + \varepsilon(h,k)\rho\frac{|k|}{|h|+|k|}\\ &= h\varepsilon(h,k)\rho\frac{sgn(h)}{|h|+|k|} + k\varepsilon(h,k)\rho\frac{sgn(k)}{|h|+|k|}\end{aligned}$$

So we take $\varepsilon_1 = \varepsilon(h,k)\rho\dfrac{sgn(h)}{|h|+|k|}$ and $\varepsilon_2 = \varepsilon(h,k)\rho\dfrac{sgn(k)}{|h|+|k|}$. Clearly $\varepsilon_1(h,k) \to 0$ and $\varepsilon_2(h,k) \to 0$, as $(h,k) \to (0,0)$. This proves the theorem.

□

Example 1.4. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0)\\ 0 & (x,y) = 0\end{cases}$$

Partial derivatives exist at $(0,0)$ and $f_x(0,0) = 0$, $f_y(0,0) = 0$, so $df(0,0) = 0$. Hence

$$\Delta f(0,0) = f(h,k) - f(0,0) = \frac{h^2k^2}{h^2+k^2} = \frac{h^2k^2}{\rho^2}$$

By taking $h = \rho\cos\theta$, $k = \rho\sin\theta$, we get

$$\frac{\Delta f(0,0) - df(0,0)}{\rho} = \frac{\rho^4\sin^2\theta\cos^2\theta}{\rho^3} = \rho\sin^2\theta\cos^2\theta \;\to 0 \text{ as } \rho \to 0$$

Hence $\lim_{\rho\to 0}\frac{\Delta f(0,0) - df(0,0)}{\rho} = 0$. Therefore by previous theorem f is differentiable at $(0,0)$.

Example 1.5. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \dfrac{x^2y}{x^2+y^2} & (x,y) \neq (0,0)\\ 0 & (x,y) = 0\end{cases}$$

Partial derivatives exist at $(0,0)$ and $f_x(0,0) = 0$, $f_y(0,0) = 0$, so $df(0,0) = 0$. Hence

$$\Delta f(0,0) = f(h,k) - f(0,0) = \frac{h^2k}{h^2+k^2} = \frac{h^2k}{\rho^2}$$

By taking $h = \rho\cos\theta$, $k = \rho\sin\theta$, we get

$$\frac{\Delta f(0,0) - df(0,0)}{\rho} = \frac{\rho^3 \sin\theta\cos^2\theta}{\rho^3} = \sin\theta\cos^2\theta\,(\neq 0, \theta = \frac{\pi}{4})$$

Therefore by previous theorem f is not differentiable at $(0,0)$.

1.3. **The Sufficient condition for differentiability:** Following theorem is on the Sufficient condition for differentiability:

Theorem 1.4. *Suppose $f_x(x,y)$ and $f_y(x,y)$ exist in an open neighborhood containing (a,b) and both functions are continuous at (a,b). Then f is differentiable at (a,b).*

Proof. Since $\frac{\partial f}{\partial y}$ is continuous at (a,b), there exists a neighborhood N_δ(say) of (a,b) at every point of which f_y exists. We take $(a+h, b+k)$, a point of this neighborhood so that $(a+h,b), (a,b+k)$ also belongs to N_δ. We write

$$f(a+h,b+k) - f(a,b) = f(a+h,b+k) - f(a+h,b) + f(a+h,b) - f(a,b).$$

Consider a function of one variable $\phi(y) = f(a+h,y)$. Since f_y exists in N_δ, $\phi(y)$ is differentiable with respect to y in the closed interval $[b, b+k]$ and as such we can apply Lagrange's Mean Value Theorem, for function of one variable y in this interval and thus obtain

$$\phi(b+k) - \phi(b) = k\phi'(b+k\theta) = kf_y(a+h, b+k\theta)$$

where $0 < \theta < 1$. Hence

$$f(a+h,b+k) - f(a+h,b) = kf_y(a+h,b+k\theta), \quad 0<\theta<1.$$

Now, if we write

$$f_y(a+h,b+k\theta) - f_y(a,b) = \varepsilon_2(h,k)$$

then from the fact that f_y is continuous at (a,b). We obtain $\varepsilon_2(h,k) \to 0$ as $(h,k) \to (0,0)$.

Again because f_x exists at (a,b) implies

$$f(a+h,b) - f(a,b) = hf_x(a,b) + h\varepsilon_1(h,k),$$

where $\varepsilon_1(h,k) \to 0$ as $(h,k) \to (0,0)$. Combining all these we get

$$\begin{aligned} f(a+h,b+k) - f(a,b) &= k[f_y(a,b) + \varepsilon_2(h,k)] + h[f_x(a,b) + \varepsilon_2(h,k)] \\ &= hf_x(a,b) + kf_y(a,b) + h\varepsilon_1(h,k) + k\varepsilon_2(h,k) \end{aligned}$$

where $\varepsilon_1, \varepsilon_2(h,k) \to 0$ as $(h,k) \to (0,0)$. This proves that $f(x,y)$ is differentiable at (a,b). □

Remark 1.2. There are functions which are differentiable but the partial derivatives need not be continuous. For example,

Example 1.6. *$f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} x^3 \sin\frac{1}{x^2} + y^3\sin\frac{1}{y^2} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

Here

$$f(x,y) := \begin{cases} 3x^2 \sin \dfrac{1}{x^2} - 2\cos \dfrac{1}{x^2} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

Although $f_x(0,0) = 0$*, but* $\lim_{(x,y)\to(0,0)} f_x(x,y)$ *does not exist, so the partial derivative is not continuous. Moreover*

$$f(h,k) - f(0,0) = f(h,k) = h.0 + k.0 + h^3 \sin \frac{1}{h^2} + k^3 \sin \frac{1}{k^2} = h\varepsilon_1 + k\varepsilon_2$$

where $\varepsilon_1 = h^3 \sin \frac{1}{h^2}$ *and* $\varepsilon_2 = k^3 \sin \frac{1}{k^2}$ *and both goes to* 0 *as* $(h,k) \to (0,0)$*. Hence* f *is differentiable at* $(0,0)$*.*