COL 351: Analysis and Design of Algorithms

Lecture 29

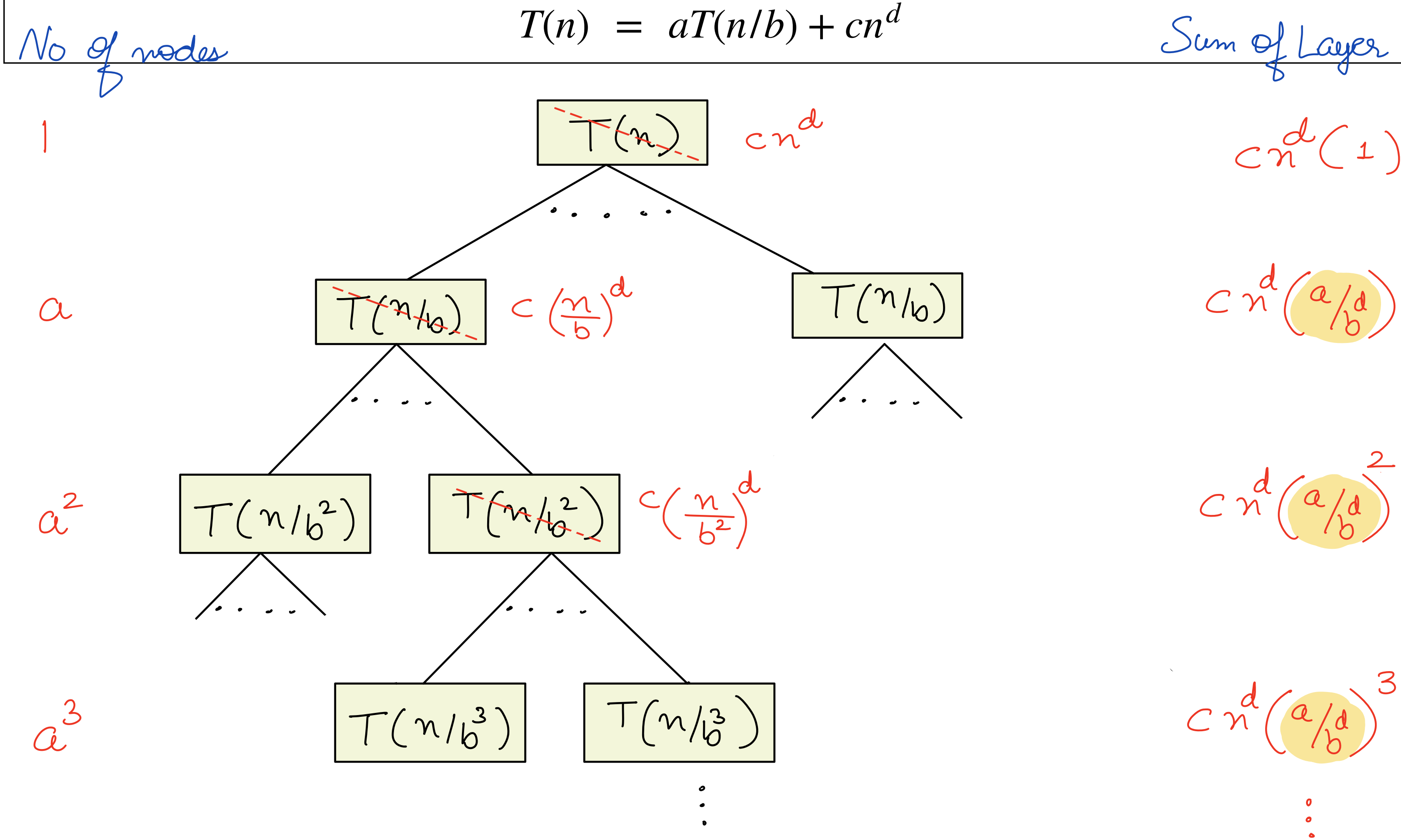
Master's Theorem

$$T(n) \;=\; aT(n/b) + cn^d$$

$$T(n) \;=\; \begin{cases} \Theta(n^d \log_b n) & \textbf{if } \frac{a}{b^d} = 1 \\ \Theta(n^d) & \textbf{if } \frac{a}{b^d} < 1 \\ \Theta(n^{\log_b a}) & \textbf{if } \frac{a}{b^d} > 1 \end{cases}$$

How to prove this?

$$T(n) = aT(n/b) + cn^d$$



$$\text{Total sum} = cn^d\left(1 + x + x^2 + \cdots + x^h\right) \quad \text{where}$$

$$h = \text{height of tree} \quad \text{and} \quad x = \left(a/b^d\right).$$

$$T(n) = aT(n/b) + cn^d$$

$$T(n) = cn^d(1 + x + x^2 + \ldots + x^h) \quad \text{where}$$

$h = \log_b n$ = height of tree and

$x = (a/b^d)$.

Lemma: The sum $1 + x + x^2 + \ldots + x^{h-1}$ is

1. h if $x = 1$ (All layers have same sum)
2. $O(1)$ if $x < 1$ (Root-term is dominant)
3. $O(x^h)$ if $x > 1$ (leaves sum is dominant)

$$T(n) = aT(n/b) + cn^d$$

$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } a/b^d = 1 \\ \Theta(n^d) & \text{if } a/b^d < 1 \\ \Theta\left(n^d \left(a/b^d\right)^{\log_b n}\right) & \text{if } a/b^d > 1 \\ = \Theta\left(n^d \cdot n^{\log_b(a) - d\log_b(b)}\right) & \\ = \underbrace{\Theta\left(n^{\log_b a}\right)} & \end{cases}$$

This term is same as the number of leaves in recursion tree (Why?)

Examples

$$T(n) = aT(n/b) + cn^d \qquad T(n) = \begin{cases} O(n^d \log_b n) & \textbf{if } \frac{a}{b^d} = 1 \\ O(n^d) & \textbf{if } \frac{a}{b^d} < 1 \\ O(n^{\log_b a}) & \textbf{if } \frac{a}{b^d} > 1 \end{cases}$$

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1. Merge Sort

$$T(n) = 2T(n/2) + O(n)$$

$$\Rightarrow T(n) = O(n \log n)$$

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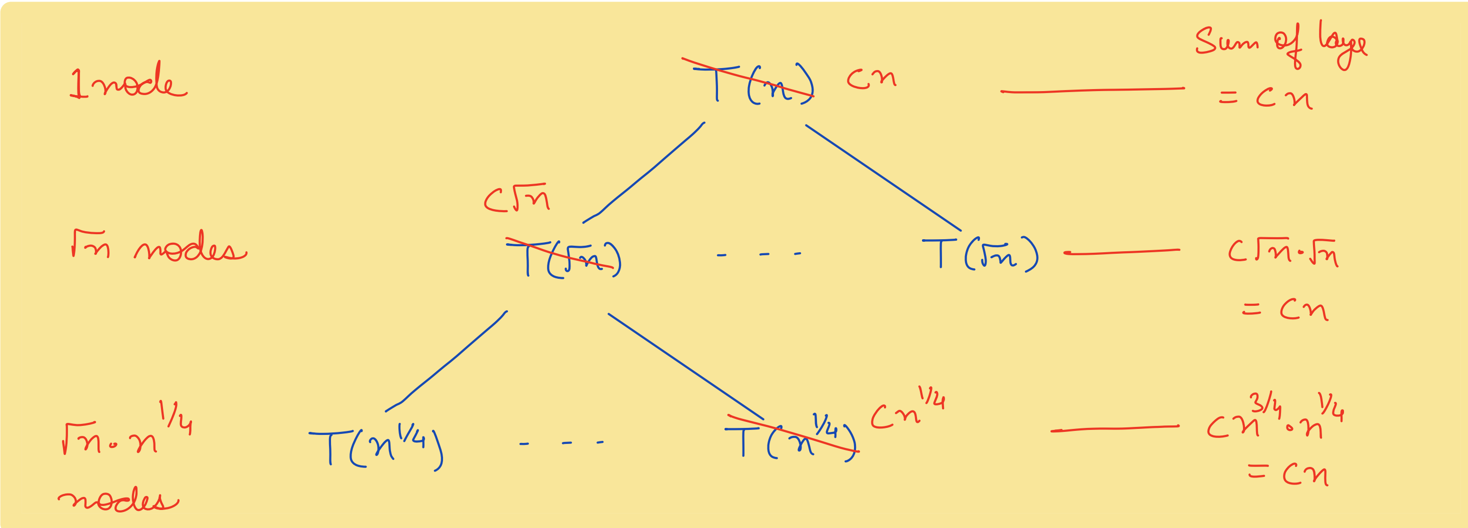
2. Strassen's Algorithm

$$T(n) = 7T(n/2) + O(n^2)$$

$$\Rightarrow T(n) = O(n^{\log_2 7})$$

Example 3:

$$T(n) = \sqrt{n} \cdot T(\sqrt{n}) + cn$$



So, $T(n) = \Theta(n \times \text{height})$, where height h satisfies $n^{1/2^h} = \text{constant}$

$\Rightarrow \log_2 n = 2^h \cdot \text{constant}$

$\Rightarrow h = \log \log n$

$\Rightarrow T(n) = \Theta(n \log \log n)$

Example 4 (Medians of Median):

$$T(n) \;=\; T(n/5) + T(7n/10) + cn$$

H. W.

Prove using recursion tree that $T(n) = \Theta(n)$.