MTL-100 Lec-6

limsup and liminf (continued)

For a bounded sequence $(a_n)_{n=1}^{\infty}$,

$\alpha = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \alpha_n$, where $\alpha_n = \sup_{k\geq n} a_k$

$\beta = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} \beta_n$, where $\beta_n = \inf_{k\geq n} a_k$.

Lemma 1: If $\alpha = \limsup_{n\to\infty} a_n$, then for any $\varepsilon>0$, $\exists\, N\in\mathbb{N}$ s.t. $a_n < \alpha+\varepsilon \quad \forall\, n\geq N$.

Lemma 2: If $\beta = \liminf_{n\to\infty} a_n$, then for any $\varepsilon>0$, $\exists\, N\in\mathbb{N}$ s.t. $a_n > \beta-\varepsilon \quad \forall\, n\geq N$

Created with Doceri

Proof of Lemma 1:

Since $\lim_{n\to\infty} \alpha_n = \alpha$, given $\varepsilon > 0$,

$\exists\, N \in \mathbb{N}$ s.t. $\alpha - \varepsilon < \alpha_n < \alpha + \varepsilon \quad \forall n \geq N$.

Now $\alpha_n = \sup_{k \geq n} a_k$.

So, if $n \geq N$, then $a_n \leq \alpha_n < \alpha + \varepsilon$

$\therefore \quad a_n < \alpha + \varepsilon \quad \forall n \geq N$.

Similarly, we can prove Lemma 2.

Next, we want to show that limsup is the supremum of all limits of convergent subsequences and liminf is the infimum of all limits of convergent subsequences.

Created with Doceri

Theorem 1: Let (a_n) be a bounded sequence and $(a_{n_k})_{k=1}^{\infty}$ be a convergent subsequence. Then

$$\liminf_{n\to\infty} a_n \le \lim_{k\to\infty} a_{n_k} \le \limsup_{n\to\infty} a_n.$$

Proof: Suppose $L = \lim_{k\to\infty} a_{n_k}$ and $\alpha = \limsup_{n\to\infty} a_n$

We'll show that $L \le \alpha$.

Suppose $L > \alpha$. Then we can find $\varepsilon > 0$ such that $L - \varepsilon > \alpha$.

By Lemma 1, $\exists\, N \in \mathbb{N}$ s.t. $a_n < L - \varepsilon \;\; \forall n \geq N$

$\forall n \geq N$.

$$\Rightarrow \quad |a_n - L| > \varepsilon$$

This contradicts $\lim_{k\to\infty} a_{n_k} = L$.

$\therefore \quad L \le \alpha$.

Similarly, using Lemma 2, we can show that $L \geq \beta = \liminf_{n\to\infty} a_n$.

Created with Doceri

Theorem 2: If $(a_n)_{n=1}^{\infty}$ is a bounded sequence, then there exist subsequences converging to $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

Proof: Let $\alpha = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \alpha_n$,

where $\alpha_n = \sup_{k \geq n} a_k$.

Now $\alpha_1 = \sup_{k\geq 1} a_k \Rightarrow \exists\, n_1 \in \mathbb{N}$ s.t.

$$\alpha_1 - 1 < a_{n_1} \leq \alpha_1$$

Now $\alpha_{n_1+1} = \sup_{k \geq n_1+1} a_k$

$$\Rightarrow \exists\, n_2 \geq n_1+1 > n_1 \text{ s.t. } \alpha_{n_1+1} - \frac{1}{2} < a_{n_2} \leq \alpha_{n_1+1}$$

This way we get $n_1 < n_2 < n_3 < \cdots$ s.t.

$$\alpha_{n_{k-1}+1} - \frac{1}{k} < a_{n_k} \leq \alpha_{n_{k-1}+1}$$

Created with Doceri

Now since $\alpha_n \to \alpha$, $\lim\limits_{k\to\infty} \alpha_{n_k+1} = \alpha$.

$\therefore$ Using the sandwich theorem, we get

$$\lim_{k\to\infty} a_{n_k} = \alpha.$$

Similarly, we can get a subseq. converging to $\liminf\limits_{n\to\infty} a_n$.

Combining Theorem 1 & Theorem 2, we see that limsup is the supremum of all limits of subsequences and liminf is the infimum of all limits of subsequences.

Created with Doceri

Example: Let $a_n = \left(1+(-1)^n+\frac{1}{2^n}\right)^{1/n}$, $n \in \mathbb{N}$.

Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

Solution: $a_n = \frac{1}{2}$ if n is odd

and $a_n = \left(2+\frac{1}{2^n}\right)^{1/n}$ if n is even

Now $2 < 2+\frac{1}{2^n} < 3$

So, $2^{1/n} < \left(2+\frac{1}{2^n}\right)^{1/n} < 3^{1/n}$

$\therefore \lim_{n\to\infty} \left(2+\frac{1}{2^n}\right)^{1/n} = 1$ (by Sandwich thm).

Thus $\frac{1}{2}$ and 1 are the limits of convergent subsequences.

Hence, $\liminf_{n\to\infty} a_n = \frac{1}{2}$ & $\limsup_{n\to\infty} a_n = 1$.

Created with Doceri

Supremum & Infimum of unbounded sets

Definition: If A is a nonempty subset of $\mathbb{R}$ that is not bounded above, then we say

$$\sup(A) = +\infty.$$

If A is a nonempty subset of $\mathbb{R}$ that is not bounded below, then we say $\inf(A) = -\infty$.

e.g. $\sup(0,\infty) = +\infty$, $\inf(0,\infty) = 0$.

$\sup(-\infty, 1) = 1$, $\inf(-\infty, 1) = -\infty$.

$\sup(\mathbb{Z}) = +\infty$, $\inf(\mathbb{Z}) = -\infty$.

Created with Doceri

lim sup and lim inf of unbounded sequences:

For any sequence $(a_n)_{n=1}^{\infty}$, we can define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \alpha_n, \text{ where } \alpha_n = \sup_{k\geq n} a_k$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \beta_n, \text{ where } \beta_n = \inf_{k\geq n} a_k.$$

We say $\limsup_{n\to\infty} a_n = \infty$ if $\alpha_1 = \infty$

(Note that if $\alpha_1 = \infty$, then $\alpha_n = \infty$ $\forall n \in \mathbb{N}$.

and if $\alpha_1 < \infty$, then since $\alpha_1 \geq \alpha_2 \geq \cdots$, $\alpha_n < \infty$ $\forall n \in \mathbb{N}$)

So, if $(a_n)_{n=1}^{\infty}$ is not bounded above, then $\limsup_{n\to\infty} a_n = \infty$.

Also, $\limsup_{n\to\infty} a_n = -\infty$ if $\lim_{n\to\infty} \alpha_n = -\infty$.

Similarly, we can have $\liminf a_n = +\infty$ or $-\infty$.

Created with Doceri

Examples:

① If $\lim\limits_{n\to\infty} a_n = \infty$, then $\liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n = \infty$.

$\lim\limits_{n\to\infty} a_n = \infty \Rightarrow$ given any $M > 0$, $\exists N \in \mathbb{N}$ s.t. $a_n > M \quad \forall n \geq N$.

Then $\beta_n = \inf\limits_{k \geq n} a_k > M \quad \forall n \geq N$

& $\alpha_n = \sup\limits_{k \geq n} a_n > M \quad \forall n$

$\therefore \lim\limits_{n\to\infty} \beta_n = \infty$ and $\lim\limits_{n\to\infty} \alpha_n = \infty$.

So, $\liminf a_n = \infty$, $\limsup a_n = \infty$.

② If $\lim\limits_{n\to\infty} a_n = -\infty$, then $\liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n = -\infty$.

Created with Doceri

③ $(a_n) = (0, 1, 0, 2, 0, 3, \dots)$

$\alpha_n = \sup_{k \geq n} a_k = \infty \quad \forall n \in \mathbb{N}.$

$\Rightarrow \limsup_{n \to \infty} a_n = \infty.$

But $\beta_n = \inf_{k \geq n} a_k = 0 \quad \forall n \in \mathbb{N}.$

$\therefore \liminf_{n \to \infty} a_n = \lim_{n \to \infty} \beta_n = 0.$

④ $(a_n) = (0, -1, 0, -2, 0, -3, \dots)$

$\alpha_n = \sup_{k \geq n} a_k = 0 \quad \forall n$

$\beta_n = \inf_{k \geq n} a_k = -\infty \quad \forall n.$

$\therefore \limsup_{n \to \infty} a_n = 0 \ ; \ \liminf_{n \to \infty} a_n = -\infty.$

Created with Doceri

Remark: If $\limsup_{n\to\infty} a_n$ or $\liminf_{n\to\infty} a_n = +\infty$ or $-\infty$, then also we have subsequence diverging to $+\infty$ or $-\infty$.

- Assume $\limsup_{n\to\infty} a_n = +\infty$.

We'll show $\exists$ a subseq whose limit is $+\infty$.

$\limsup_{n\to\infty} a_n = +\infty \Rightarrow \alpha_n = \sup_{k\geq n} a_k = +\infty \ \forall n \in \mathbb{N}$

Since $\alpha_1 = +\infty$, $\exists\, n_1 \in \mathbb{N}$ s.t. $a_{n_1} > 1$.

Now since $\alpha_{n_1+1} = +\infty$, $\exists\, n_2 \geq n_1+1 > n_1$ s.t. $a_{n_2} > 2$.

Continuing this way, we get $(n_k)_{k=1}^{\infty}$ s.t. $n_1 < n_2 < \cdots$ and $a_{n_k} > k \ \forall k \in \mathbb{N}$.

Thus we get a subseq. $(a_{n_k})_{k=1}^{\infty}$ s.t. $\lim_{k\to\infty} a_{n_k} = +\infty$.

Created with Doceri

Theorem: Let $(a_n)_{n=1}^{\infty}$ be a sequence of nonzero real numbers. Then we have

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \le \liminf_{n\to\infty}|a_n|^{1/n} \le \limsup_{n\to\infty}|a_n|^{1/n} \le \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Proof: Note that $\liminf_{n\to\infty}|a_n|^{1/n} \le \limsup_{n\to\infty}|a_n|^{1/n}$ is obviously true.

Now we show the right inequality.

Let $L = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

We want to show that $\limsup_{n\to\infty}|a_n|^{1/n} \le L$.

If $L = +\infty$, then it is obviously true.

Suppose $L \in \mathbb{R}$. Let $\varepsilon > 0$.

By Lemma 1, $\exists\, N \in \mathbb{N}$ s.t.

$$\left|\frac{a_{n+1}}{a_n}\right| < L+\varepsilon \quad \forall n \ge N.$$

Created with Doceri

Then for any $n \geq N+1$, we have

$$|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right| \cdot |a_N|$$

$$< (L+\varepsilon)^{n-N} |a_N| = (L+\varepsilon)^n \left((L+\varepsilon)^{-N} |a_N|\right).$$

Taking $a = (L+\varepsilon)^{-N}|a_N|$, we get

$$|a_n|^{1/n} < (L+\varepsilon)\, a^{1/n} \quad \forall\, n \geq N+1.$$

$$\Rightarrow \quad \limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} (L+\varepsilon)\, a^{1/n}$$

$$= L+\varepsilon \quad \left(\because \lim_{n\to\infty} a^{1/n} = 1\right).$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq L.$$

Similarly, we can prove the first inequality.

Created with Doceri

Example: Consider $a_n = \begin{cases} 2^n & \text{if } n \text{ is even} \\ 2^{n-1} & \text{if } n \text{ is odd} \end{cases}$

Then $a_n^{1/n} = \begin{cases} 2 & \text{if } n \text{ is even} \\ 2^{1-\frac{1}{n}} & \text{if } n \text{ is odd.} \end{cases}$

$\Rightarrow \lim_{n\to\infty} a_n^{1/n} = 2$

Now, for $n = 2k-1$ odd,

$$\frac{a_{n+1}}{a_n} = \frac{a_{2k}}{a_{2k-1}} = \frac{2^{2k}}{2^{2k-2}} = 4$$

for $n = 2k$ even, $\frac{a_{n+1}}{a_n} = \frac{a_{2k+1}}{a_{2k}} = \frac{2^{2k}}{2^{2k}} = 1.$

$\Rightarrow \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ & $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 4.$

$\therefore \liminf \left|\frac{a_{n+1}}{a_n}\right| < \liminf a_n^{1/n} = \limsup_{n\to\infty} a_n^{1/n}$

$< \limsup \left|\frac{a_{n+1}}{a_n}\right|.$

Created with Doceri