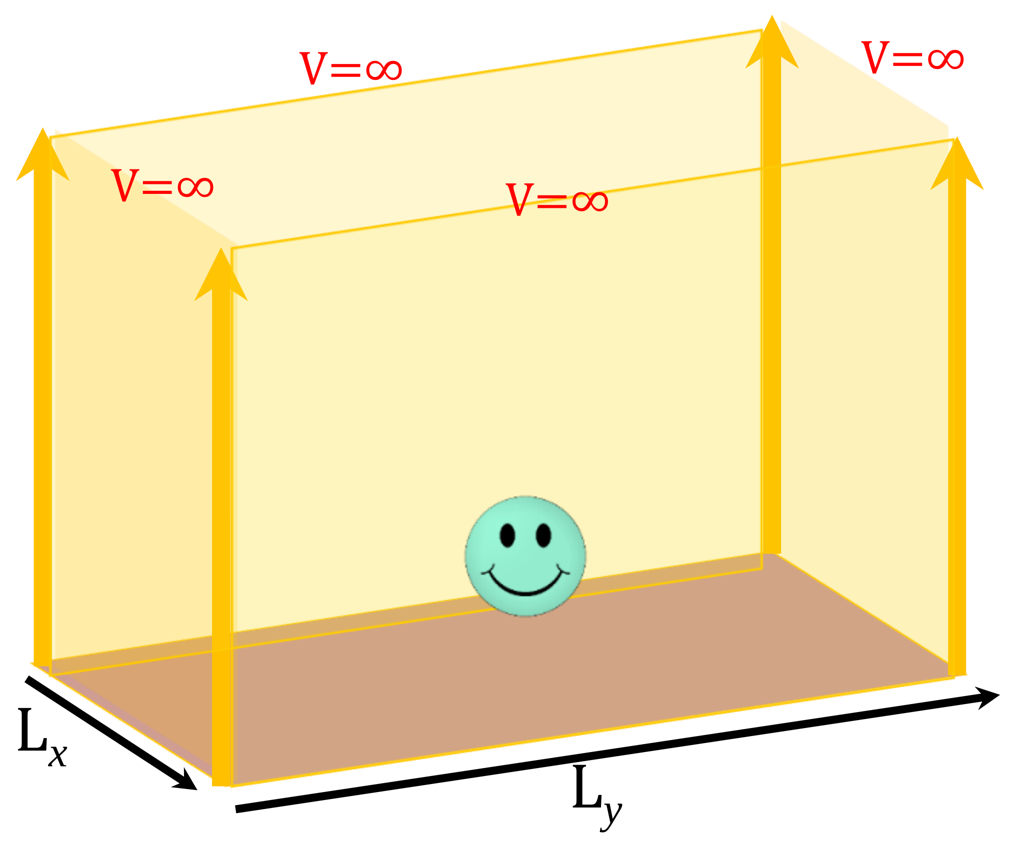
# Particle in a two-dimensional box




$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi(x,y)}{\partial x^2}+\frac{\partial^2\psi(x,y)}{\partial y^2}\right)=E\psi(x,y) \qquad 0\le x\le L_x \;\; and \;\; 0\le y\le L_y \qquad \textbf{(1)}$$

$$\widehat{H} = \widehat{H}_x + \widehat{H}_y$$

**Using Separation of variables:** $\psi(x,y) = X(x)Y(y) = XY$ **(2)**

$$\frac{\partial^2\psi(x,y)}{\partial x^2} = \mathrm{X''Y} \;\; and \;\; \frac{\partial^2\psi(x,y)}{\partial y^2} = \mathrm{XY''}$$

- **From (1)**

$$X''Y + XY'' = -\frac{2mE}{\hbar^2}XY$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = -\frac{2mE}{\hbar^2} \qquad \textbf{(3)}$$

- 1st term is function of $x$ only and second term function of $y$ only

$$\frac{X''}{X} = -\frac{2mE}{\hbar^2} - \frac{Y''}{Y} = -k_x^2 \qquad \textbf{(4)}$$

$$\frac{Y''}{Y} = -\frac{2mE}{\hbar^2} - \frac{X''}{X} = -\frac{2mE}{\hbar^2} + k_x^2 = -k_y^2 \qquad \textbf{(5)}$$

$$\hat{V}(x,y) = 0 \quad for \quad 0\le x\le L_x \text{ and } 0\le y\le L_y$$
$$= \infty \;\; \text{Otherwise}$$

Boundary condition or trivial solution

$$\psi(x,y) = 0 \quad for \quad x,y\le 0 \;\; and \; x\ge L_x \;\&\; y\ge L_y$$

# Particle in a two-dimensional box

$$\frac{X''}{X} = -k_x^2 \quad \text{and} \quad \frac{Y''}{Y} = -k_y^2 \qquad \textbf{(6)}$$

Possible solutions

$$X(x) = A_1 Sin(k_x x) + B_1 Cos(k_x x)$$
$$Y(y) = A_2 Sin(k_y y) + B_2 Cos(k_y y)$$

- **Boundary condition that** $\psi = 0 \; at \; x = 0$

$Sin(x = 0) = 0$ and $Cos(x = 0) = 1 \;\Rightarrow B_1 = 0$

- **Boundary condition that** $\psi = 0 \; at \; y = 0$

$Sin(y = 0) = 0$ and $Cos(y = 0) = 1 \;\Rightarrow B_2 = 0$

$$\Rightarrow \quad X(x) = A_1 Sin(k_x x)$$
$$Y(y) = A_2 Sin(k_y y)$$

- **Boundary condition that** $\psi = 0 \; at \; x = L_x$ **and** $y = L_y$

$$X(x = L_x) = A_1 Sin(k_x L_x) = 0$$
$$Y(y = L_y) = A_2 Sin(k_y L_y) = 0$$

- If we take $A_1 = 0$ or $A_2 = 0$, then $\boldsymbol{\psi = 0}$ for all $x$ *and* $y$.
- This will be in conflict with the Born interpretation that the particle must be somewhere within the box

$$\Rightarrow A_1 \neq 0; \;\; Sin(k_x L_x) = 0$$

$$\Rightarrow k_x L_x = n_x \pi \Rightarrow k_x = \frac{n_x \pi}{L_x} \quad \text{where } n_x = 1,2,\ldots$$

- **Similarly**

$$\Rightarrow A_2 \neq 0; \;\; Sin(k_y L_y) = 0$$

$$\Rightarrow k_y L_y = n_y \pi \Rightarrow k_y = \frac{n_y \pi}{L_y} \quad \text{where } n_y = 1,2,\ldots$$

- **Therefore,**

$$X(x) = A_1 Sin(\frac{n_x \pi}{L_x} x); \qquad Y(y) = A_2 Sin(\frac{n_y \pi}{L_y} y)$$

$$\psi_{n_x, n_y}(x, y) = A_1 A_2 Sin\left(\frac{n_x \pi}{L_x} x\right) Sin\left(\frac{n_y \pi}{L_y} y\right) \qquad \textbf{(10)}$$

# Particle in a two-dimensional box

$$\psi_{n_x,n_y}(x,y) = A_1A_2 Sin\left(\frac{n_x\pi}{L_x}x\right)\ Sin\left(\frac{n_y\pi}{L_y}y\right) = A Sin\left(\frac{n_x\pi}{L_x}x\right)\ Sin\left(\frac{n_y\pi}{L_y}y\right)$$

- **Normalization of the wavefunction**

$$\int_0^{L_x}\int_0^{L_y} \psi^*_{n_x,n_y}(x,y)\,\psi_{n_x,n_y}(x,y)dxdy = 1$$

$$A^2\int_0^{L_x}\int_0^{L_y} Sin^2\left(\frac{n_x\pi}{L_x}x\right)\ Sin^2\left(\frac{n_y\pi}{L_y}y\right)dxdy = 1$$

## Solving separately

$$\Rightarrow \int_0^{L_x} Sin^2\left(\frac{n_x\pi}{L_x}x\right)dx = \frac{1}{2}\int_0^{L_x}\left[1-cos\left(2\frac{n_x\pi}{L_x}x\right)\right]dx = \frac{L_x}{2}$$

$$\Rightarrow \int_0^{L_y} Sin^2\left(\frac{n_y\pi}{L_y}y\right)dy = \frac{1}{2}\int_0^{L_y}\left[1-cos\left(2\frac{n_y\pi}{L_y}y\right)\right]dy = \frac{L_y}{2}$$

**Therefore,**

$$A^2\frac{L_x}{2}\frac{L_y}{2} = 1 \quad \Rightarrow \quad A = \frac{2}{\sqrt{L_xL_y}}$$

$$\psi_{n_x,n_y}(x,y) = \frac{2}{\sqrt{L_xL_y}} Sin\left(\frac{n_x\pi}{L_x}x\right)\ Sin\left(\frac{n_y\pi}{L_y}y\right)$$

**Now,**

$$k_x^2 + k_y^2 = \frac{2mE}{\hbar^2}$$

$$\Rightarrow \left(\frac{n_x\pi}{L_x}\right)^2 + \left(\frac{n_y\pi}{L_y}\right)^2 = \frac{2mE_{n_x,n_y}}{\hbar^2}$$

$$E_{n_x,n_y} = \frac{\hbar^2}{2m}\left(\frac{n_x^2\pi^2}{L_x^2} + \frac{n_y^2\pi^2}{L_y^2}\right)$$

$$E_{n_x,n_y} = \frac{h^2}{8m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right)$$

# Particle in a two-dimensional box

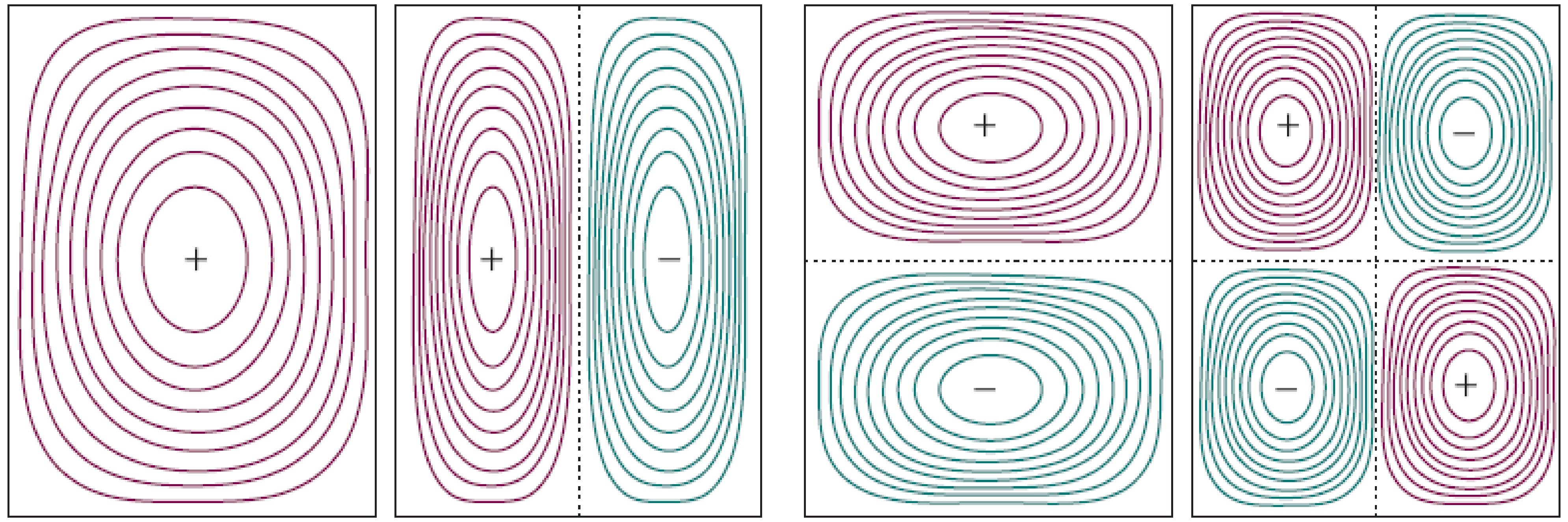
- **If the particle is confined in a square box,** $L_x = L_y = L$

$$E_{n_x,n_y} = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2\right)$$

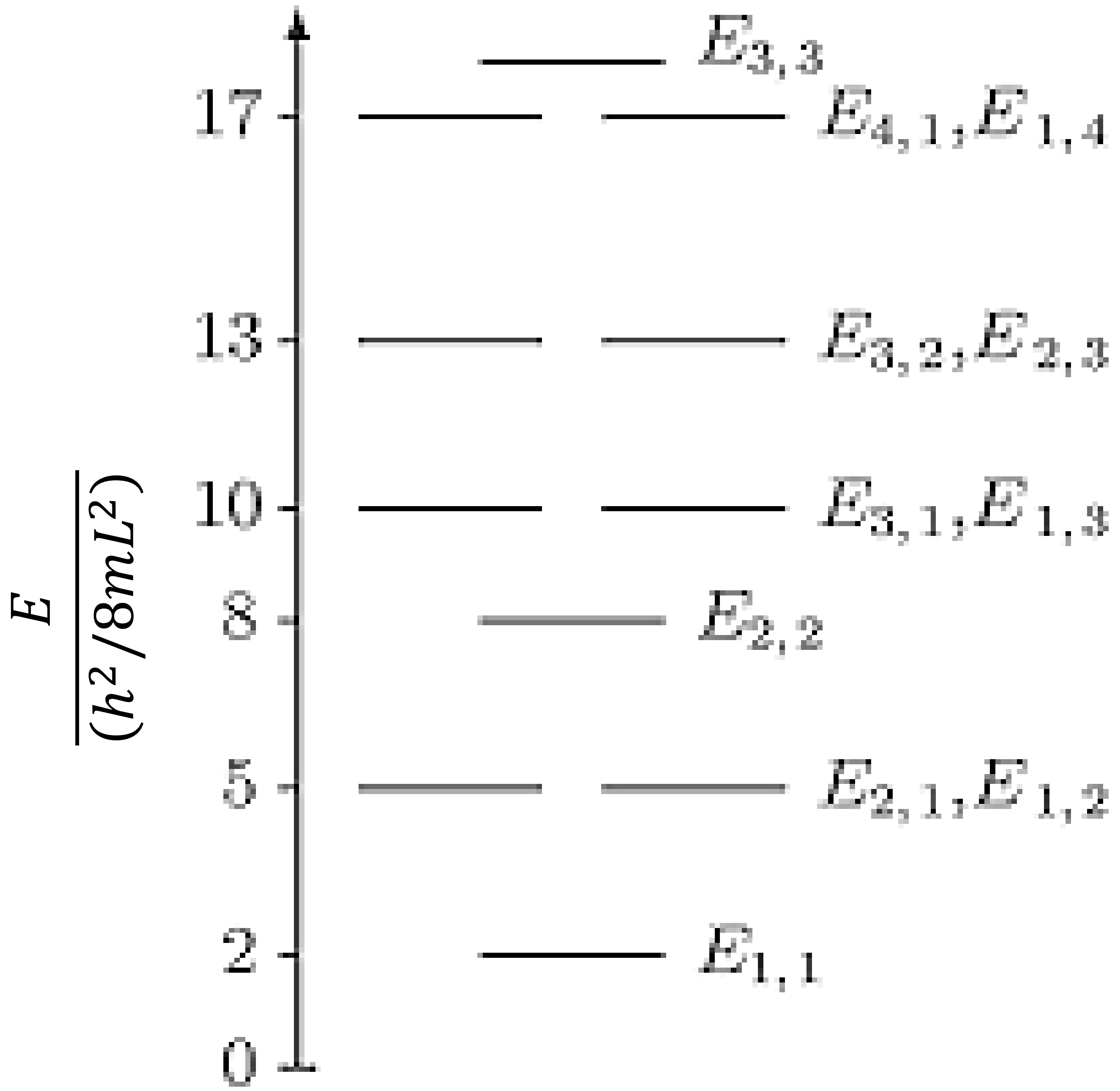
$$\psi_{n_x,n_y}(x,y) = \frac{2}{L} Sin\left(\frac{n_x\pi}{L}x\right)\ Sin\left(\frac{n_y\pi}{L}y\right)$$

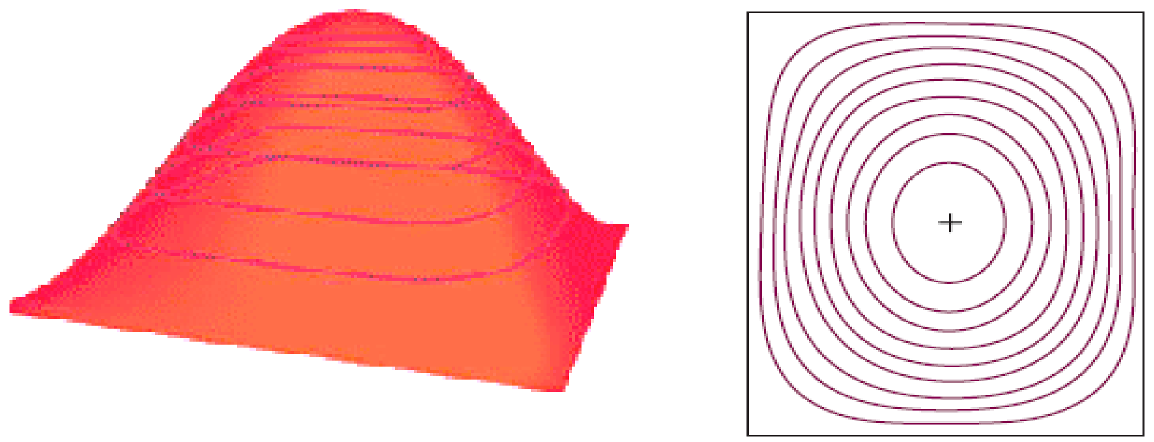
| $n_x$ | $n_y$ | $E_{n_x,n_y}$ | $\psi_{n_x,n_y}$ | Degree of degeneracy |
|---|---|---|---|---|
| 1 | 1 | $E_{1,1} = \frac{2h^2}{8mL^2}$ | $\psi_{1,1}$ | Non-degenerate |
| 1 | 2 | $E_{1,2} = \frac{5h^2}{8mL^2}$ | $\psi_{1,2}$ | 2-fold |
| 2 | 1 | $E_{2,1} = \frac{5h^2}{8mL^2}$ | $\psi_{2,1}$ | |
| 2 | 2 | $E_{2,2} = \frac{h^2}{mL^2}$ | $\psi_{2,2}$ | Non-degenerate |
| 1 | 3 | $E_{1,3} = \frac{10h^2}{8mL^2}$ | $\psi_{1,3}$ | 2-fold |
| 3 | 1 | $E_{3,1} = \frac{10h^2}{8mL^2}$ | $\psi_{3,1}$ | |

The wavefunctions for a particle confined to a rectangular surface depicted as *contours of equal amplitude*.

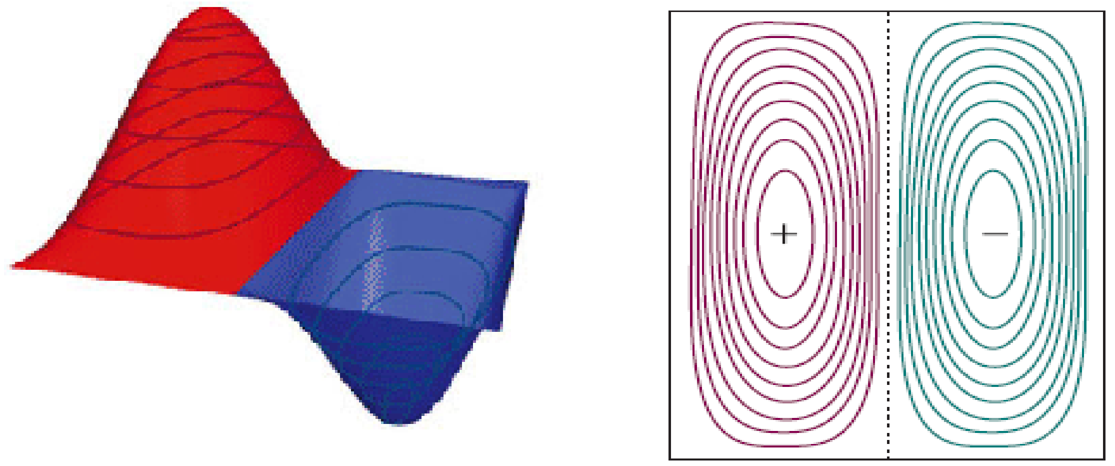



(a) $n_x = 1,\ n_y = 1$ (b) $n_x = 1,\ n_y = 2$ (c) $n_x = 2,\ n_y = 1$ (d) $n_x = 2,\ n_y = 2$

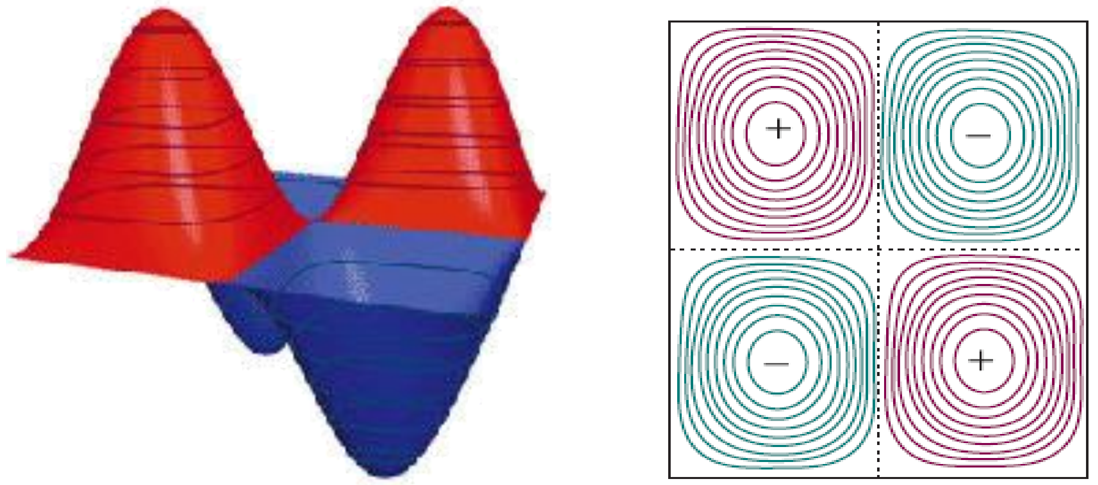




$n_x = 1,\ n_y = 1$

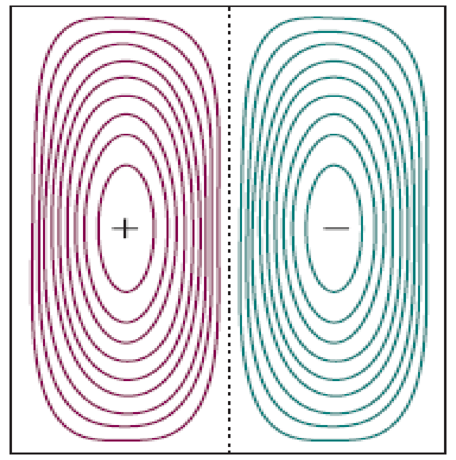


$n_x = 1,\ n_y = 2$



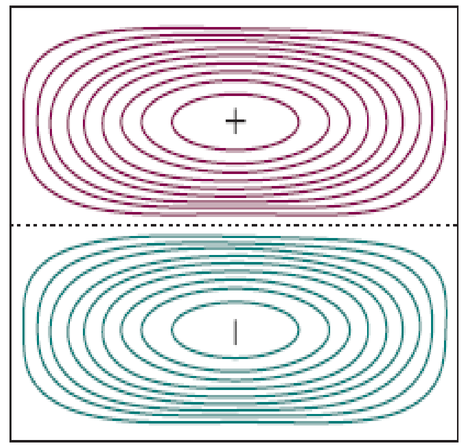
$n_x = 2,\ n_y = 2$

- **Degeneracy**



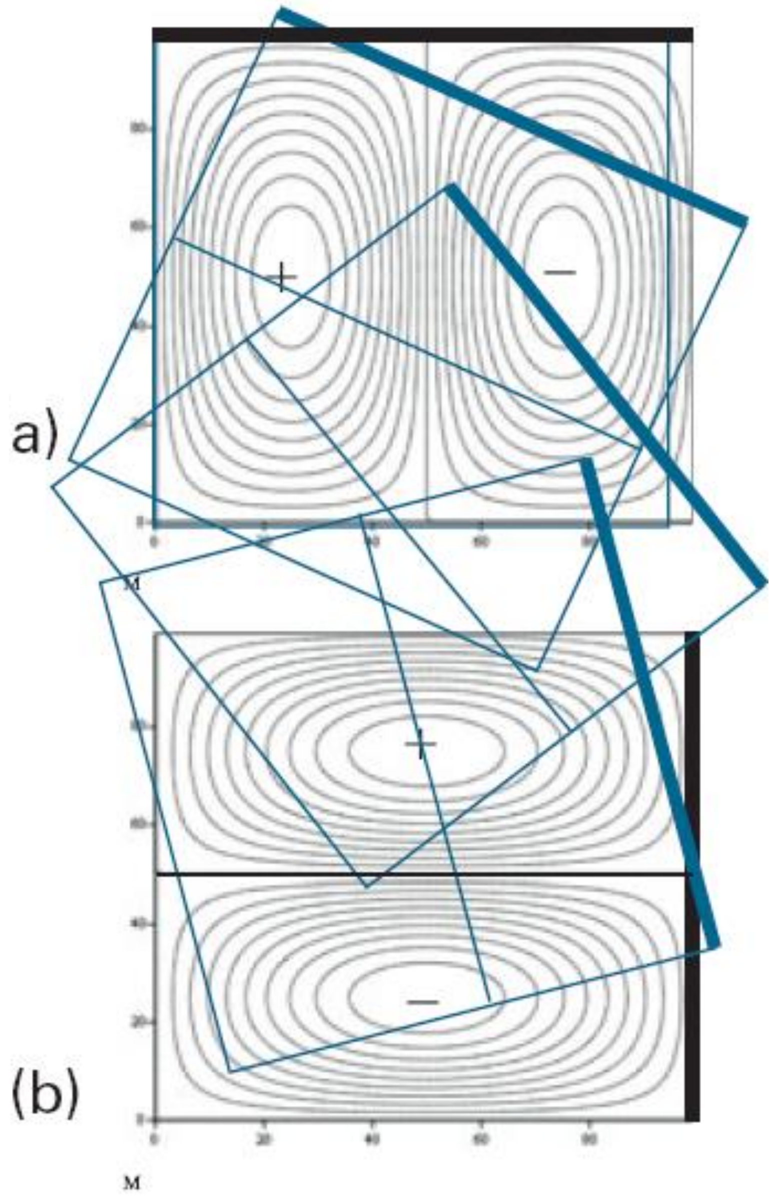
$n_x = 1,\ n_y = 2$

$$\psi_{1,2} = \frac{2}{L} Sin\left(\frac{\pi}{L}x\right)\ Sin\left(\frac{2\pi}{L}y\right); \qquad E_{1,2} = \frac{5h^2}{8mL^2}$$



$n_x = 2,\ n_y = 1$

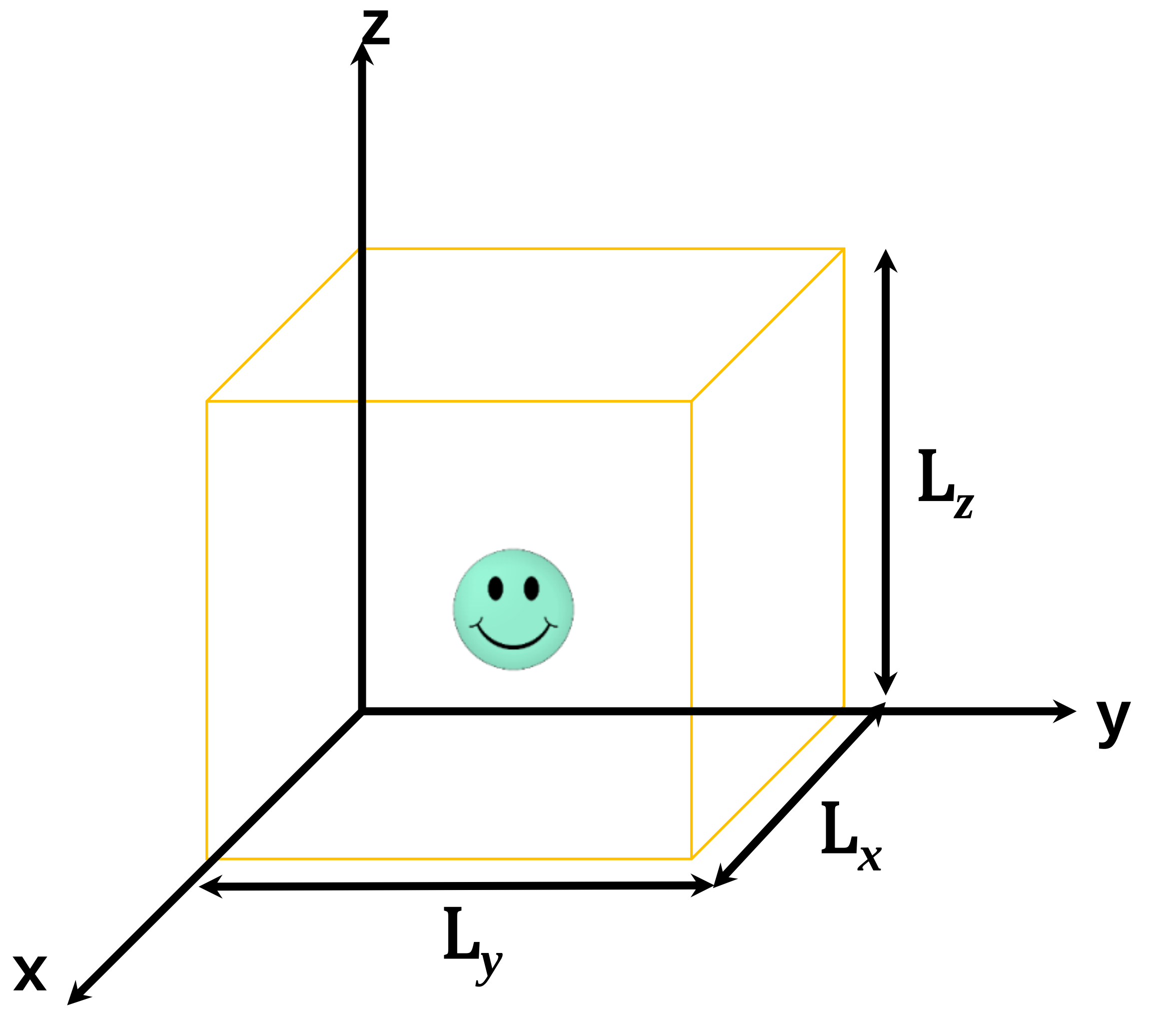
$$\psi_{2,1} = \frac{2}{L} Sin\left(\frac{2\pi}{L}x\right)\ Sin\left(\frac{\pi}{L}y\right); \qquad E_{2,1} = \frac{5h^2}{8mL^2}$$




- **The occurrence of degeneracy is related to the symmetry of the system**

- **Because the box is square, we can convert one wavefunction into the other simply by rotating the plane by 90°.**
- **Interconversion by rotation through 90° is not possible when the plane is not square, and $\psi_{1,2}$ and $\psi_{2,1}$ are then not degenerate.**

# Particle in a three-dimensional box




$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi(x,y,z)}{\partial x^2}+\frac{\partial^2\psi(x,y,z)}{\partial y^2}+\frac{\partial^2\psi(x,y,z)}{\partial z^2}\right)=E\psi(x,y,z)$$

$$\psi_{n_x,n_y,n_z}(x,y,z)=\frac{2\sqrt{2}}{\sqrt{L_xL_yL_z}}Sin\left(\frac{n_x\pi}{L_x}x\right)Sin\left(\frac{n_y\pi}{L_y}y\right)Sin\left(\frac{n_z\pi}{L_z}z\right)$$

$$0\le x\le L_x,\ \ 0\le y\le L_y, 0\le z\le L_z$$

$$\hat{V}(x,y,z)=0\ \ for\ \ 0\le x\le L_x,\ 0\le y\le L_x$$
$$and\ 0\le z\le L_z$$
$$=\infty\ Otherwise$$

$$E_{n_x,n_y,n_z}=\frac{h^2}{8m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}\right)$$

Boundary conditions or trivial solution

$$\psi(x,y,z)=0\ \ for\ \ x,y,z\le 0\ \ and\ x\ge L_x,y\ge L_y\ \&\ z\ge L_z$$

# Particle in a three-dimensional box

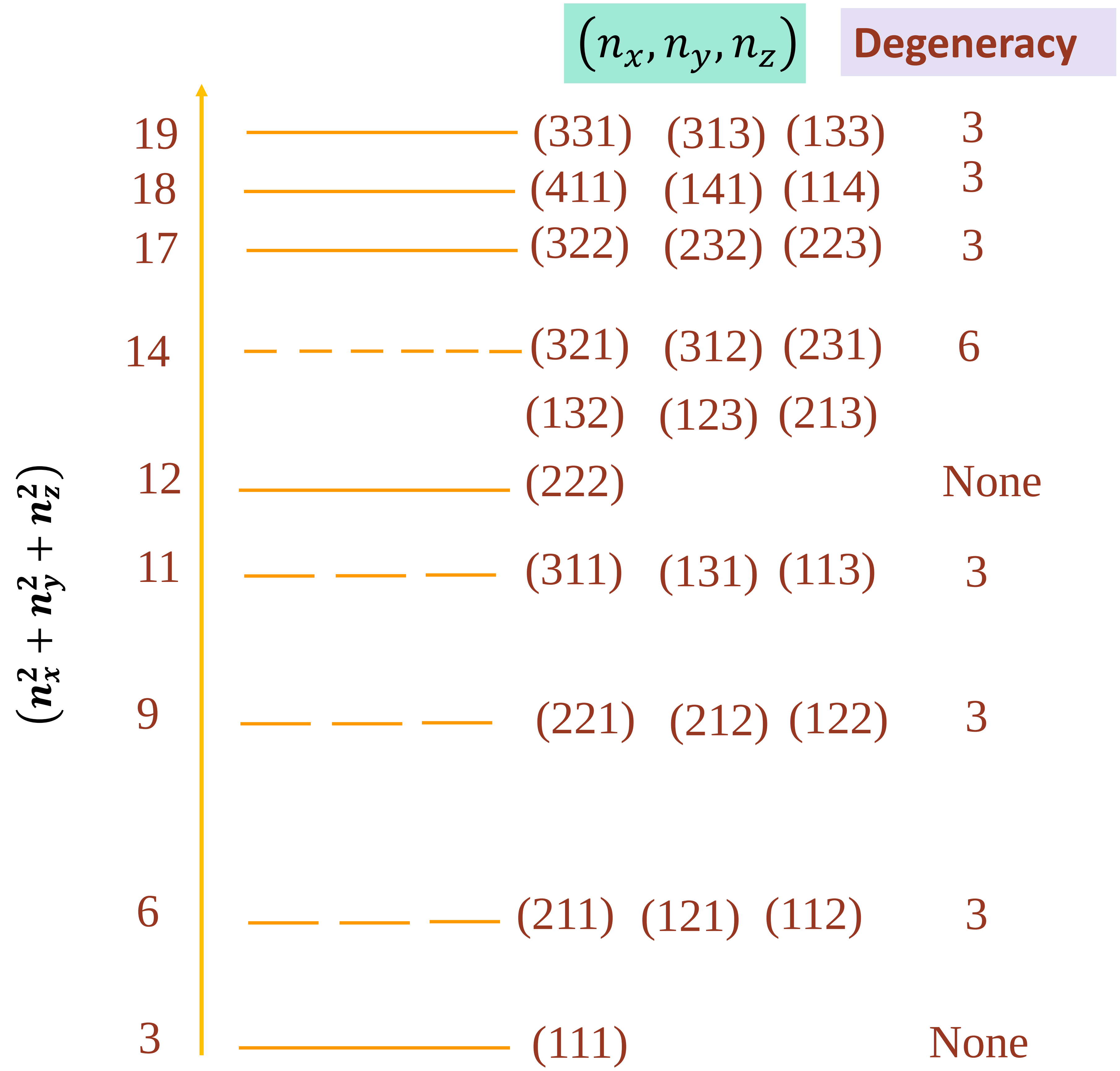
- **If the particle is confined in a cubical box:** $L_x = L_y = L_z = L$

$$\psi_{n_x,n_y,n_z}(x,y,z) = \frac{2\sqrt{2}}{\sqrt{L^3}} Sin\left(\frac{n_x\pi}{L}x\right) Sin\left(\frac{n_y\pi}{L}y\right) Sin\left(\frac{n_z\pi}{L}z\right)$$

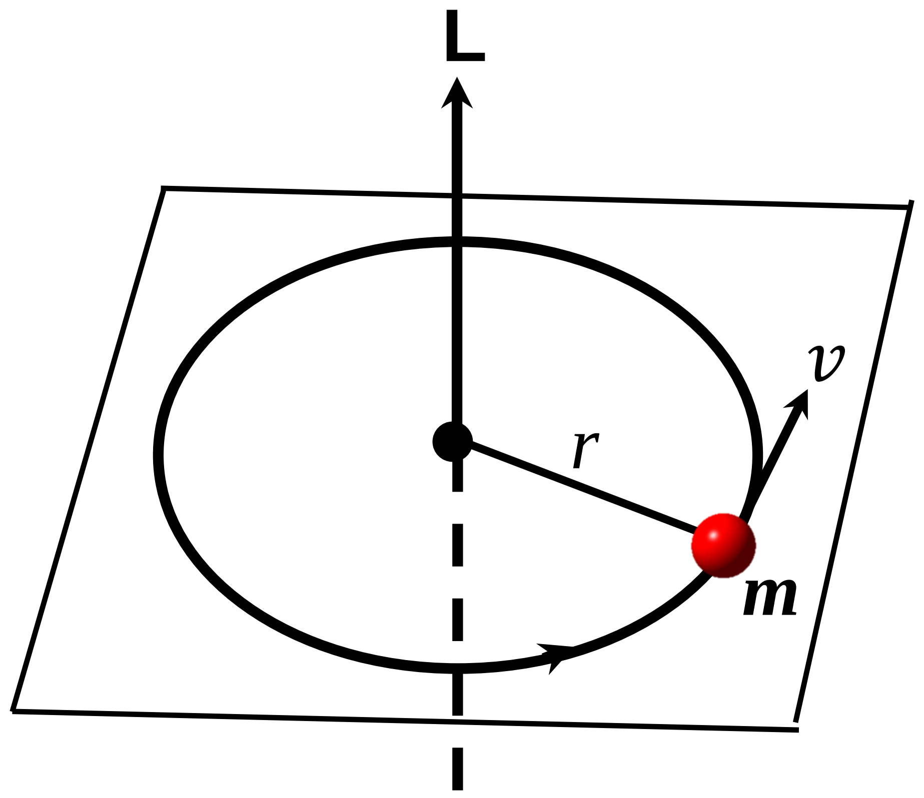
$$0 \le x \le L,\ 0 \le y \le L, 0 \le z \le L$$

$$E_{n_x,n_y,n_z} = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

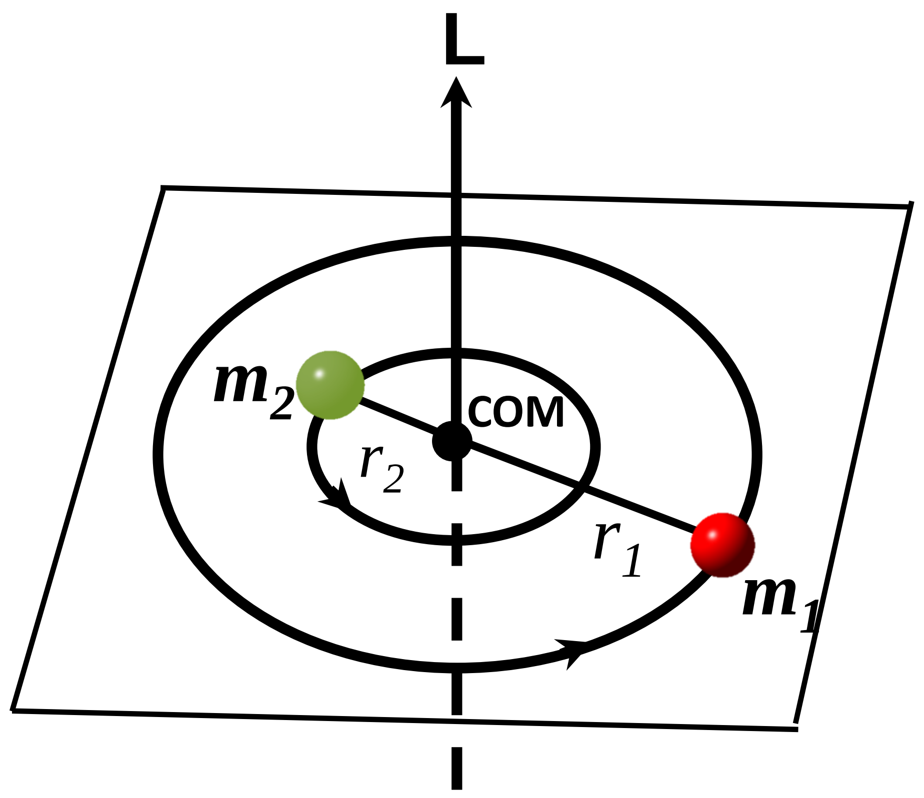
| $n_x, n_y, n_z$ | $E_{n_x,n_y,n_z}$ | $\psi_{n_x,n_y,n_z}$ | Degree of degeneracy |
|---|---|---|---|
| 111 | $E_{1,1,1} = \frac{3h^2}{8mL^2}$ | $\psi_{1,1,1}$ | Non-degenerate |
| 211 | $E_{2,1,1} = \frac{6h^2}{8mL^2}$ | $\psi_{2,1,1}$ | 3-fold |
| 121 | $E_{1,2,1} = \frac{6h^2}{8mL^2}$ | $\psi_{1,2,1}$ | |
| 112 | $E_{1,1,2} = \frac{6h^2}{8mL^2}$ | $\psi_{1,1,2}$ | |



# Particle on a ring/sphere: Quantization of rotations




(a)




(b)

- Particle moving around an axis will possess **angular momentum** $(L = I\omega = mvr = pr)$ and **rotational kinetic energy** $(\mathrm{K} = \frac{mv^2}{2} = \frac{p^2}{2m} = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2 = \frac{L^2}{2I})$

⇒ If no torque is applied, L is conserved

- In a rigid rotor (rigid diatomic molecule), two particles rotate about their COM such that $m_1 r_1 = m_2 r_2$.

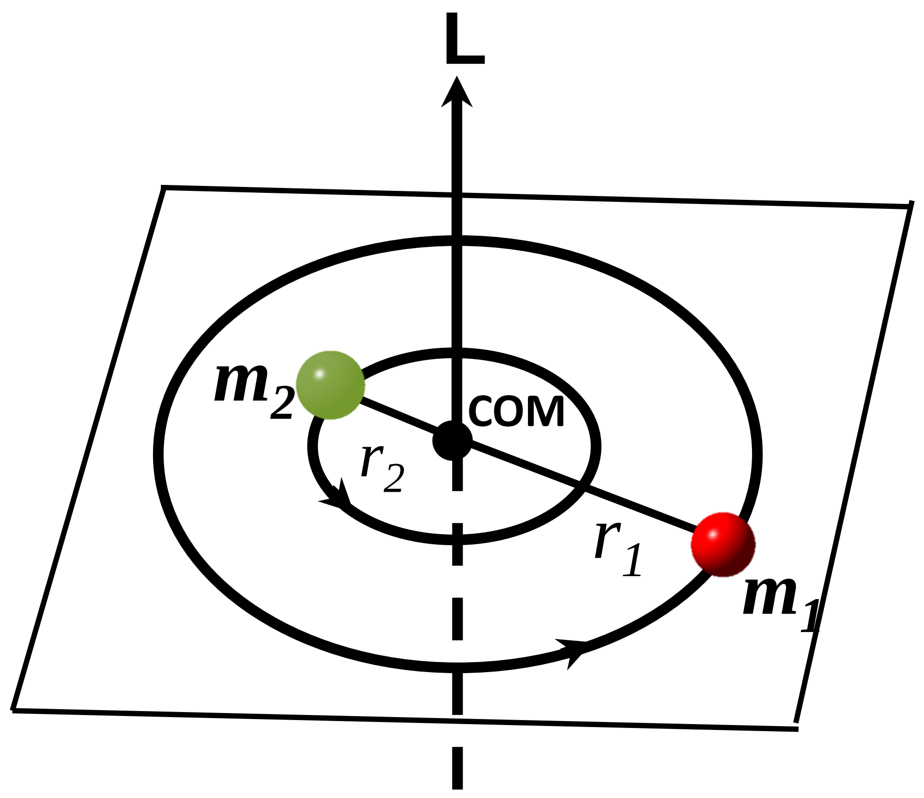
- The equilibrium distance $r = r_1 + r_2$ such that

$$r_1 = \frac{m_2}{m_1+m_2} r \text{ and } r_2 = \frac{m_1}{m_1+m_2} r$$

- Moment of inertia: $I = m_1 {r_1}^2 + m_2 {r_2}^2 = \frac{m_1 m_2}{m_1+m_2} r^2 = \mu r^2$

- Rotational kinetic energy: $\mathrm{K} = \frac{1}{2} m_1 {r_1}^2 \omega^2 + \frac{1}{2} m_2 {r_2}^2 \omega^2$

$$= \frac{1}{2} I\omega^2 = \frac{L^2}{2I} = \frac{L^2}{2\mu r^2}$$

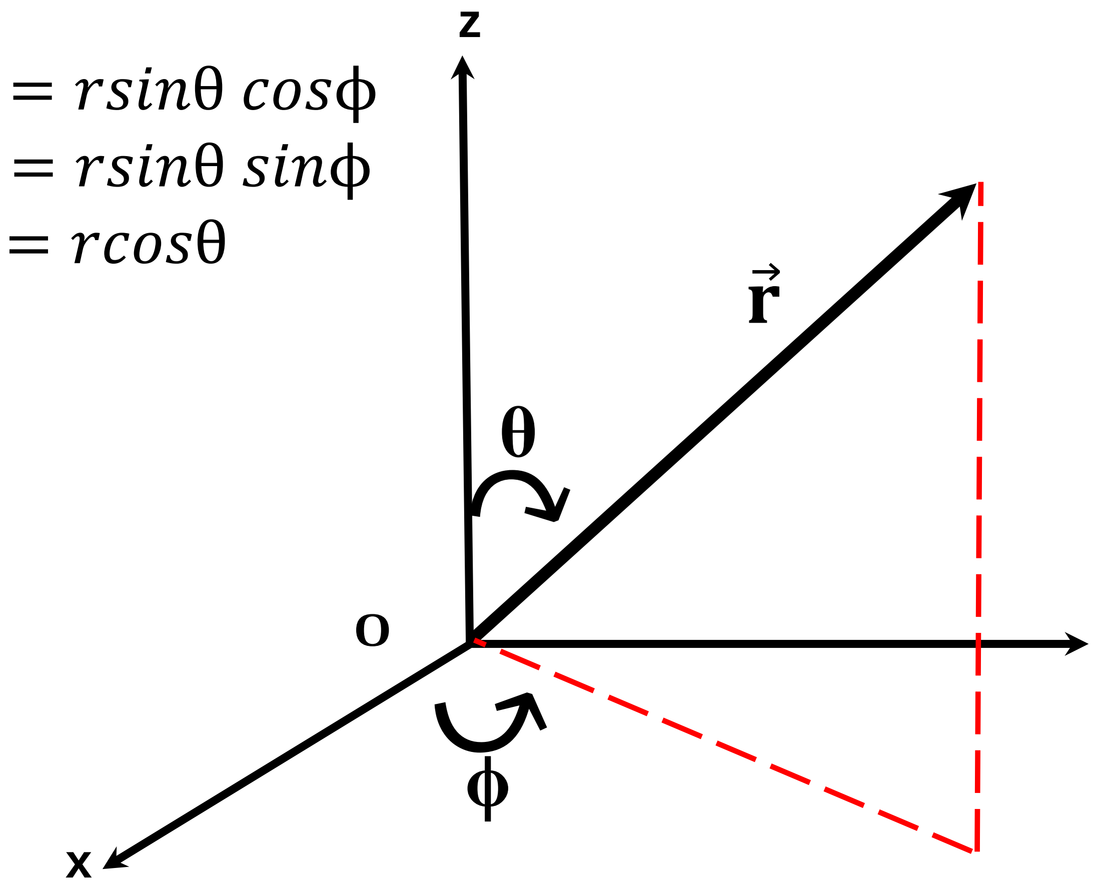



(b)

- Since there is no potential energy, the Hamiltonian operator for a rigid rotor can be written as

$$\hat{H} = \hat{K} = \frac{\hat{L}^2}{2I} = -\frac{\hbar^2}{2\mu r^2}\nabla^2 \qquad (r \text{ constant})$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$




- For rotational motion, it is more convenient to use spherical coordinates and write the Schrödinger equation

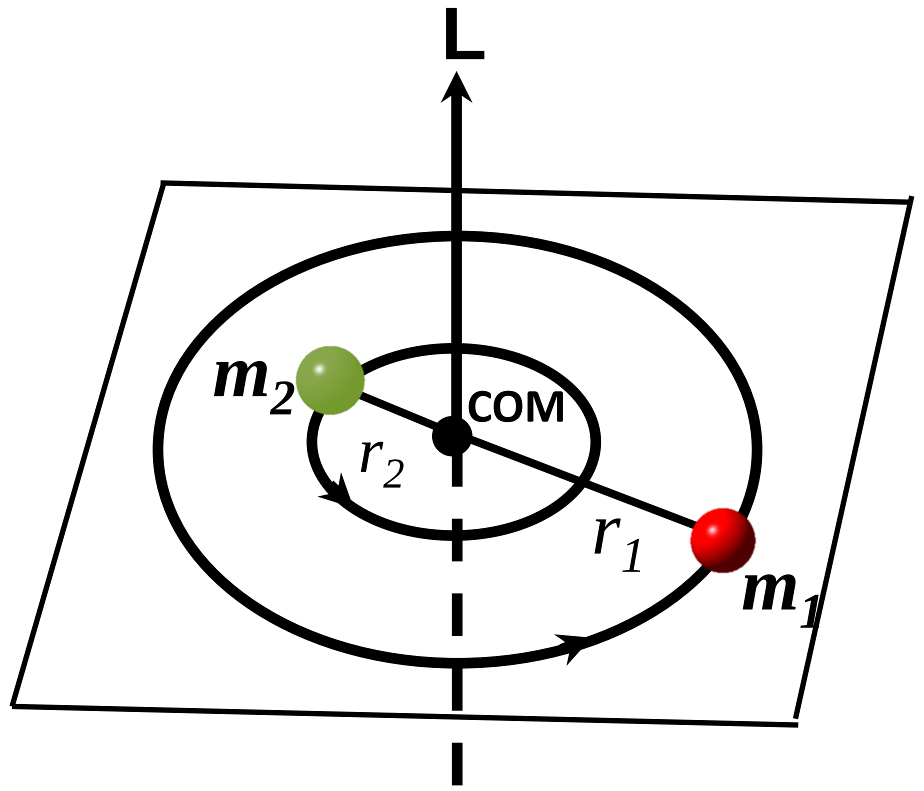
$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$ **[Laplacian operator]**

- Since for rigid rotor, $r = r_1 + r_2 = const.$, we can ignore the derivatives w.r.t. *r* in $\nabla^2$ and also the wavefunction will be function of two variables $\theta$ and $\phi$.

$$-\frac{\hbar^2}{2\mu r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(\theta,\phi) = E\psi(\theta,\phi)$$

$$-\frac{\hbar^2}{2I}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(\theta,\phi) = E\psi(\theta,\phi)$$




(b)

- The Schrödinger equation thus obtained is a standard differential equation whose solutions are spherical harmonics $Y_\ell^{m_\ell}(\theta,\phi)$. Thus,

$$\hat{H}Y_\ell^{m_\ell}(\theta,\phi) = EY_\ell^{m_\ell}(\theta,\phi)$$

$$E = \frac{\ell(\ell+1)\hbar^2}{2I} \quad \text{where } \ell = 0, 1, 2 \ldots$$

$$\hat{H}Y_\ell^{m_\ell}(\theta,\phi) = \frac{\ell(\ell+1)\hbar^2}{2I}Y_\ell^{m_\ell}(\theta,\phi)$$

where $\ell$ is **angular momentum quantum number** and $m_\ell$ is referred to as **magnetic quantum number**

$$E_\ell = \frac{\ell(\ell+1)\hbar^2}{2I}$$

- Note that there is no zero-point energy for rigid rotor

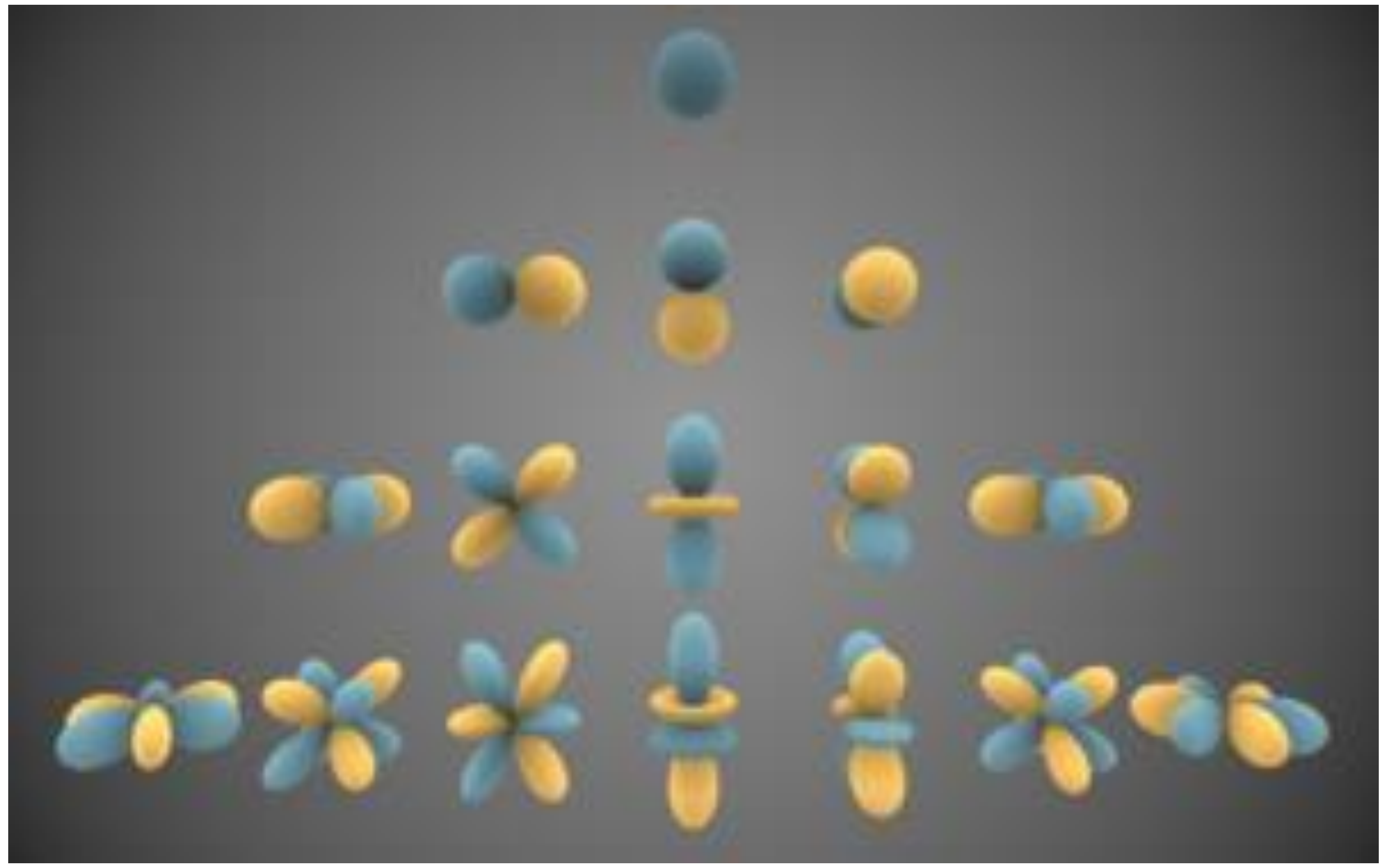
$$Y_0^0 = \frac{1}{(4\pi)^{1/2}} \qquad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$$

$$Y_1^1 = -\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{i\phi} \qquad Y_1^{-1} = \left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{-i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \qquad Y_2^1 = -\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{i\phi}$$

$$Y_2^{-1} = \left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{-i\phi} \qquad Y_2^2 = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{2i\phi}$$

$$Y_2^{-2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{-2i\phi}$$



https://en.wikipedia.org/wiki/Spherical_harmonics

## Construction of operator for square of the angular momentum

- So far, we neglected the fact that the angular momentum is a vector quantity.

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = (yp_z - zp_y)\boldsymbol{i} + (zp_x - xp_z)\boldsymbol{j} + (xp_y - yp_x)\boldsymbol{k} = L_x\boldsymbol{i} + L_y\boldsymbol{j} + L_z\boldsymbol{k}$$

$$\boldsymbol{L} \cdot \boldsymbol{L} = L^2 = L_x^2 + L_y^2 + L_z^2$$

$$\because\ \hat{p}_x = -i\hbar\frac{\partial}{\partial x}\ ;\quad \hat{p}_y = -i\hbar\frac{\partial}{\partial y}\ ;\quad \hat{p}_z = -i\hbar\frac{\partial}{\partial z}\ ,$$

$$\therefore\ \hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right);\ \hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right);\ \hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$\Rightarrow$ The operator for the square of the angular momentum

$$\hat{L}^2 = |\hat{L}|^2 = \hat{L}\cdot\hat{L} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

-

# $\hat{L}^2$ and its eigenfunctions

- It is often more convenient to use the angular momentum operators in spherical coordinates $r, \theta, \phi$

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \qquad \hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\Rightarrow \nabla^2 = -\frac{\hat{L}^2}{r^2\hbar^2}$$

- Since $\hat{H} = \hat{K} = \frac{\hat{L}^2}{2I} = -\frac{\hbar^2}{2\mu r^2}\nabla^2$ ($r$ constant) and $\hat{H}Y_\ell^{m_\ell}(\theta,\phi) = \frac{\ell(\ell+1)\hbar^2}{2I}Y_\ell^{m_\ell}(\theta,\phi)$, we can write

$$\hat{L}^2 Y_\ell^{m_\ell}(\theta,\phi) = \ell(\ell+1)\hbar^2 Y_\ell^{m_\ell}(\theta,\phi)$$

$\Rightarrow$ $Y_\ell^{m_\ell}(\theta,\phi)$ are the eigenfunctions of $\hat{L}^2$ and $\hat{H}$

$$\Rightarrow \left[\hat{L}^2, \hat{H}\right] = 0$$

$\Rightarrow$ Simultaneous measurement of $L^2$ and ‘energy’ with infinite precision is possible for a rigid rotor

# Rigid –Rotor

- It can be shown that

$$[\hat{L}^2,\hat{L}_x] = [\hat{L}^2,\hat{L}_y] = [\hat{L}^2,\hat{L}_z] = 0$$

$$[\hat{L}_x,\hat{L}_y] = i\hbar\hat{L}_z$$
$$[\hat{L}_y,\hat{L}_z] = i\hbar\hat{L}_x$$
$$[\hat{L}_z,\hat{L}_x] = i\hbar\hat{L}_y$$

- Consider the operator $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

$$\hat{L}_z Y_\ell^{m_\ell}(\theta,\phi) = -i\hbar\frac{\partial}{\partial\phi}\mathbf{\Theta}(\theta)e^{im_\ell\phi} = -i\hbar\,(im_\ell)Y_\ell^{m_\ell}(\theta,\phi)$$

$$\hat{L}_z Y_\ell^{m_\ell}(\theta,\phi) = m_\ell\,\hbar Y_\ell^{m_\ell}(\theta,\phi) \qquad |L_z| = m_\ell\hbar$$

⇒ $Y_\ell^{m_\ell}(\theta,\phi)$ are also the eigenfunctions of $\hat{L}_z$ as well as of $\hat{L}^2$ and $\hat{H}$

$$\Rightarrow [\hat{L}_z,\hat{H}] = 0$$

⇒ Simultaneous measurement of $\hat{L}_z$, $L^2$ and 'energy' with infinite precision is possible for a rigid rotor

- Limits on $m_\ell$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \Rightarrow \hat{L}_x^2 + \hat{L}_y^2 = \hat{L}^2 - \hat{L}_z^2$$

$$(\hat{L}_x^2 + \hat{L}_y^2)Y_\ell^{m_\ell}(\theta,\phi) = (\hat{L}^2 - \hat{L}_z^2)Y_\ell^{m_\ell}(\theta,\phi)$$
$$= [\ell(\ell+1) - m_\ell^2]\hbar^2 Y_\ell^{m_\ell}(\theta,\phi)$$

∵ $\hat{L}_x$ and $\hat{L}_y$ are Hermitian, their eigenvalues will be real

∴ The square of the eigenvalues will be real and nonnegative

$$\Rightarrow [\ell(\ell+1) - m_\ell^2]\hbar^2 \geq 0$$

$$\Rightarrow [\ell(\ell+1) - m_\ell^2] \geq 0$$

$$\Rightarrow |m_\ell| \leq \ell$$

∴ The only possible values of $m_\ell$ are

$$m_\ell = 0, \pm1, \pm2, \ldots \pm\ell$$

$E_\ell = \frac{\ell(\ell+1)\hbar^2}{2I}$ ⇒ **Each energy level is $(2\ell+1)$-fold degenerate**

# Construction of operator for square of the angular momentum

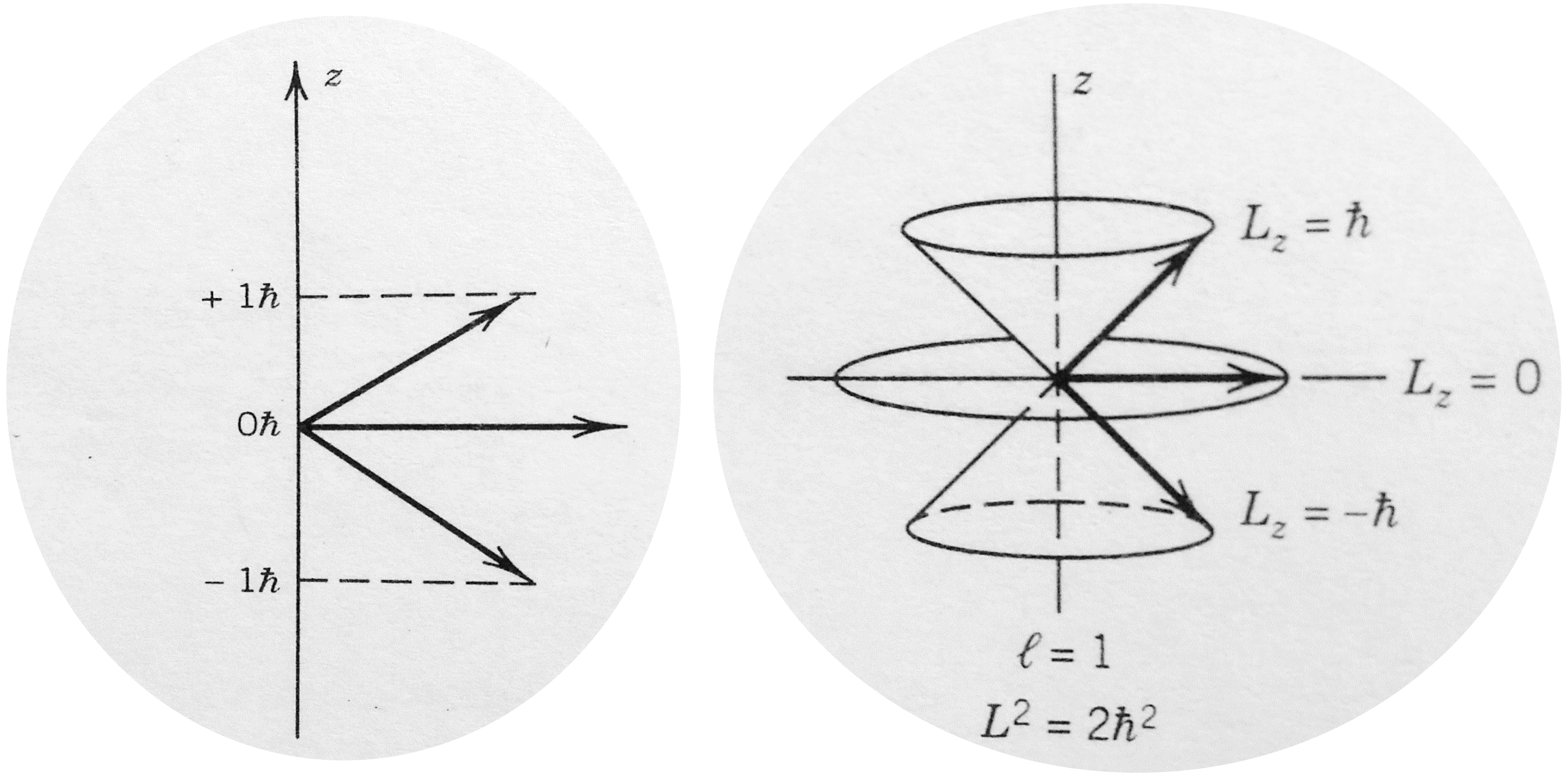
- Example 1: For $\ell$=1, what are the values of $|L|$ and $L_z$?

$$|L| = \sqrt{\ell(\ell+1)}\hbar = \sqrt{2}\hbar$$

$$m_\ell = 0, \pm 1$$

$$|L_z| = m_\ell \hbar = 0, \pm\hbar$$

The angular momentum is same ($\sqrt{2}\hbar$), but its projection on z-axis depends on $m_\ell$




- **Example 2:** For $\ell$=2, what are the values of $|L|$ and $L_z$?

$$|L| = \sqrt{\ell(\ell+1)}\hbar = \sqrt{6}\hbar$$

$$m_\ell = 0, \pm 1, \pm 2$$

$$|L_z| = m_\ell \hbar = 0, \pm\hbar, \pm 2\hbar$$