Harshit Mawandia 2020CS10348

Q1) The sequences $(a_n)_{n\geq1}$ and $(b_n)_{n\geq1}$ are cauchy sequences

The sequence $(Z_n)_{n\geq1} = (a_1, b_1, a_2, b_2, \ldots\ldots, a_n, b_n, \ldots)$

To show that sequence $(Z_n)_{n\geq1}$ is cauchy:

We know that a_n and b_n are subsequences of Z_n.

We know that:

A sequence is cauchy if and only if it is convergent.

We also know that:

If a sequence is converging to a limit, say L, all its subsequences should converge to the same limit l.

$\therefore$ If Z_n is cauchy and $\lim_{n\to\infty} Z_n = L$

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$

Hence, Prooved.

Scanned with CamScanner

Q2) Harshit Mawandia 2020CS10348

$(x_n)_{n \geq 1}$ is a sequence defined by

$x_1 = 1$ and $x_{n+1} = x_n \left(1 + \frac{\sin n}{2^n}\right)$, $n \geq 1$

Let us consider sequence $\frac{x_{n+1}}{x_n}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \left(1 + \frac{\sin n}{2^n}\right) \leq$$

also

as $\sin n \leq 1$

$$\lim_{n\to\infty} \left(1 + \frac{1}{2^n}\right) = 1$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n$$

($x_n \neq 0$ because $\sin \neq 0$ for $n \in \mathbb{N}$)

Hence the sequence converges.

Scanned with CamScanner

Harshit Mawandia 2020CS10348

Q3) a)

$$S_n = \frac{1^2}{2} + \frac{2^2}{1} + \frac{3^2}{2^2} + \frac{4^2}{2} + \frac{5^2}{2^3} + \frac{6^2}{2^2} + \frac{7^2}{2^4} + \frac{8^2}{2^3}$$

Let the odd terms be a_n and even terms be b_n

$$\sum a_n = \sum \frac{(2n-1)^2}{2^n}$$

$$\sum b_n = \sum \frac{(2n)^2}{2^{n-1}}$$

~~We know that both~~

$\sum a_n$ will be convergent only if

$\lim_{n\to\infty} (a_n)^{1/n} < 1$ (by root test)

$$\lim_{n\to\infty} \frac{(2n-1)^{2/n}}{2} = \frac{1}{2} < 1 \quad \left(\lim_{n\to\infty} (2n-1)^{2/n} = 1\right)$$

~~which is~~

There fore $\sum a_n$ is convergent

as it can be written as $\lim (2n)^{2/n} = \lim 4^{1/n} \times \lim n^{2/n} = 1 \times 1 = 1$

Similarly for $\sum b_n$:

$$\lim_{n\to\infty} \frac{(4n^2)^{1/n}}{2^{1-1/n}} = \frac{1}{2}$$

There for $\sum b_n$ is also convergent

Scanned with CamScanner

Harshit Mawandia 2020CS10348

$\therefore \sum_{n=1}^{\infty}(a_n + b_n)$ will also be convergent

Now,

$$S_{2n} = \sum_{n=1}^{n}(a_n + b_n)$$

$$S_{2n+1} = \sum_{n=1}^{n}(a_n + b_n) + a_{n+1}$$

$$S_{2n+1} = S_{2n} + \frac{(2n-1)^2}{2^n}$$

Now applying Limit $n \to \infty$

$$\frac{(2n-1)^2}{2^n} \longrightarrow 0$$ (since Σa_n was convergent, $\therefore$ at $n \to \infty \lim a_n = 0$)

$$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n}$$

Since, both the limit of ~~subsequence~~ sub series (which exhaust the ~~sequence~~ series) exist and are equal therefore the series is convergent.

Harshit Mawandia 2020CS10348

Q3) b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\, 3^{2n}}$

Applying ratio test

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2(n+1))!}{(n+1)! \times 3^{2(n+1)}} \times \frac{n! \times 3^{2n}}{(2n)!}$

$= \lim_{n\to\infty} \left(\frac{(2n+1) \times \cancel{(2n+2)}\, 2}{\cancel{(n+1)} \times 9} \right)$

$= \lim_{n\to\infty} \frac{2}{9}(2n+1) = \infty$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$

$\therefore$ The series is not convergent

Harshit Mawandia 2020CS10348 ~~Part~~

Q4)

$$f(x) = \begin{cases} \sin x & \text{if } x \in [0,\pi] \cap Q \\ 0 & \text{if } x \in [0,\pi] \setminus Q \end{cases}$$

Case 1:

Let a be a rational number in $[0,\pi]$ and $a \neq 0$

Then $f(a) = \sin a \neq 0$ $(a \neq \pi$, π is irrational) ~~and $a \neq \pi$~~

Now lets consider a sequence $x_n = a + \frac{\sqrt{2}}{n}$

as $n \to \infty$, $x_n \to a$, but x_n is irrational

$\therefore \lim_{n\to\infty} (f(x_n)) = 0 \neq f(a)$ (proved above)

Therefore $f(x)$ is not continuous in $x \in (0,\pi) \cap Q$

Case 2:

Lets consider $a = a_0.a_1a_2a_3\ldots\ldots$ (an irrational number but $a \neq \pi$)

where $a_0 = N \cup \{0\}$ and $a_i = \{0,1,\ldots,9\}$ for $i \geq 1$

Lets consider another number $y_n = a_0.a_1a_2\ldots a_n$ (rational)

$f(a) = 0$ (a is irrational)

& $\lim_{n\to\infty} y_n = a$

& but $\lim_{n\to\infty} y_n = \sin a \neq 0$ since $(a \neq \pi)$

$\therefore f(x)$ is not continuous in $x \in (0,\pi) \setminus Q$

Harshit Mawandia   2020CS10348   [signature]

Case 3: $a = 0$

$\therefore f(a) = \sin 0 = 0$

~~Lets consider any seque~~

Let x_n be any sequence $\in [0, \pi]$, such that as $n \to \infty$ $x_n \to 0$.

$$\lim_{n\to\infty} f(x_n) = \begin{cases} \sin 0 = 0 & \text{if } x \in [0,\pi] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,\pi] \setminus \mathbb{Q} \end{cases}$$

Since ~~limit~~ $\lim_{n\to\infty} f(x_n) = 0 = f(0)$

$\therefore f(x)$ is continuous at 0.

Case 4: $a = \pi$ (irrational)

$f(a) = 0$

Let x_n be any sequence $\in [0, \pi]$ such that as $n \to \infty$ $x_n \to \pi$

$$\lim_{n\to\infty} f(x_n) = \begin{cases} \sin \pi = 0 & \text{if } x \in [0,\pi] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,\pi] \setminus \mathbb{Q} \end{cases}$$

Since $\lim_{n\to\infty} f(x_n) = 0 = f(\pi)$

$\therefore f(x)$ is continuous at π

Scanned with CamScanner

Q 5) Navsheet Marwardia 2020CS10348

8) $\sqrt{x}\log x$ is continuous at $x \in (0, \infty)$
since $\sqrt{x}$ and $\log x$ both are continuous in $(0, \infty)$

$$\lim_{x \to 0} \sqrt{x}\log x = \lim_{x \to 0} \frac{\log x}{\frac{1}{\sqrt{x}}}$$

by L'Hopital rule

$$\lim_{x \to 0} \frac{\log x}{\frac{1}{\sqrt{x}}} = \frac{\frac{1}{x}}{-\frac{1}{2}\frac{1}{x^{3/2}}} = \lim_{x \to 0} -2\sqrt{x} = 0$$

Therefore $\sqrt{x}\log(x)$ has removable discontinuity at 0 so $\sqrt{x}\log(x)$ can be made continuous in $[0, b]$ where $b \in (0, \infty)$

now $\sqrt{x}$ and $\log x$ both are uniformly continuous at $[1, \infty)$

therefore $\sqrt{x}\log x$ will be uniformly continuous in $[1, \infty)$

now for $b \geq 1$ $[0, b] \cap [1, \infty) \neq \phi$

therefore $\sqrt{x}\log x$ will be uniformly continuous in $(0, \infty)$ as $(0, \infty) \subset [0, \infty)$

Scanned with CamScanner

Harshit Marwandia 2020CS10348

Q5) b) $\sin x \sin\left(\frac{1}{x}\right)$ is continuous in $(0, 1]$

$\lim_{x \to 0} \sin x \sin \frac{1}{x} = 0$

since $\lim_{x \to 0} \sin x = 0$

and $-1 \leq \sin \frac{1}{x} \leq 1$

$\therefore \lim_{x \to 0} \sin x \sin \frac{1}{x} = 0$

Since $\sin x \sin\left(\frac{1}{x}\right)$ has removable discontinuity at 0, we can define

$$f(x) = \begin{cases} 0 & \text{at } x = 0 \\ \sin x \sin\left(\frac{1}{x}\right) & x \in (0, 1] \end{cases}$$

since, $f(x)$ will be uniformly continuous in $[0, 1]$

$\therefore f(x)$ will be uniformly continuous in $(0, 1)$

$\therefore \sin x \sin \frac{1}{x}$ will be uniformly continuous in $(0, 1)$

Hence Proved.

Harshit Mawandia 2020CS10348

Q6) If $f(x)$ is differentiable in R ~~...~~ then $f'(x)$ is continuous at R (otherwise (RHL $\neq$ LHL where $f'(x)$ is not continuous so $f(x)$ will not be differentiable)

Now if $f'(x) > 1$ for all $x \neq 0$

then lets consider a ~~...~~ sequence $x_n = \frac{1}{n}$

where $n \in N$

$\lim_{n \to \infty} x_n = 0$

$\therefore f'(0) = 1$

$\lim_{n \to \infty} f'(x_n) > 1 \neq \lim_{x_n \to 0} f'(x_n) = 1$

but $f'(x)$ is continuous at 0

$\therefore \lim_{x \to 0} f(x)$ should be 0

Therefore $f'(x)$ cannot be > 1 for all $x \neq 0$

(Proved by contradiction)

Scanned with CamScanner

Harshit Mawandia 2020CS10348

Q7)

To show that $(1+x)^p \leq 1+x^p$

for all $x > 0$

Lets assume $g(x) = (1+x)^p - (1+x^p)$

$$g'(x) = p(1+x)^{p-1} - p x^{p-1}$$

$$= p\left((1+x)^{p-1} - x^{p-1}\right)$$

$p \in (0,1)$

$$\therefore g'(x) = p\left(\frac{1}{(1+x)^{1-p}} - \frac{1}{x^{1-p}}\right)$$

$1+x > x$

$$\therefore \frac{1}{(1+x)^{1-p}} < \frac{1}{x^{1-p}}$$

$\therefore g'(x) < 0$ for all $x \in (0, \infty)$

since $g'(x) \neq 0$ $g(x)$ is monotonic in decreasing nature.

now

$$g(0) = 1^p - 1 = 0$$

$g(x) \leq g(0)$ $\forall x > 0$

$$\therefore (1+x)^p - (1+x^p) \leq 0$$

$$\therefore (1+x)^p \leq (1+x^p)$$ Hence prooved