Lecture 1 (Infinite Series)

Consider a sequence $(x_n)_{n\geq 1}$

" Add the terms of the sequence $(x_n)_{n\geq 1}$ "

$$\underbrace{x_1 + x_2 + x_3 + \cdots}_{\sum_{n\geq 1} x_n}$$

Series

How to tackle such infinite sums?

$a, b \in \mathbb{R}$

$a+b \in \mathbb{R}$

finitely many real numbers

Consider $x_n = (-1)^n$

$$\sum_{n\geq 1} (-1)^n = -1 + \underbrace{1 - 1} + \underbrace{1 - 1} + \cdots$$

$$= -1 + (1-1) + (1-1) + \cdots$$

Seem to get -1

$$= (-1+1) + (-1+1) + (-1+1) + \cdots$$

Seem to get 0

"convergence of an infinite series"

Defn (sequence of partial sums of a series)

We have $\sum_{n\geq 1} x_n$.

Let
$$S_1 = x_1$$
$$S_2 = x_1 + x_2$$
$$S_3 = x_1 + x_2 + x_3$$
$$\vdots$$
$$S_n = x_1 + x_2 + \cdots + x_n$$

The sequence $(S_n)_{n\geq 1}$ is called the sequence of partial sums of $\sum_{n\geq 1} x_n$.

Defn (convergence of $\sum_{n\geq 1} x_n$)

- (i) We say $\sum_{n\geq 1} x_n$ converges/exists if

  $(S_n)_{n\geq 1}$ converges in $\mathbb{R}$.

  Suppose $S_n \longrightarrow S$ in $\mathbb{R}$,

  then we write

$$\sum_{n\geq 1} x_n = S.$$

- (ii) If $(S_n)_{n\geq 1}$ is not convergent, then we say $\sum_{n\geq 1} x_n$ is not convergent.

(iii) If $(S_n)_{n\geq 1}$ diverges to $\pm\infty$, then we say $\sum_{n\geq 1} x_n$ diverges to $\pm\infty$ accordingly.

<u>**Note**</u> We know,

$$(S_n)_{n\geq 1} \text{ is convergent}$$
$$\Updownarrow$$
$$(S_n)_{n\geq 1} \text{ is Cauchy.}$$

So $\sum_{n\geq 1} x_n$ is convergent iff for given any $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that

$$|s_n - s_m| < \epsilon \quad \forall\ n, m \geq n_0.$$

This is known as "Cauchy criterion" for the convergence of the series $\sum_{n \geq 1} x_n$.

Examples

1) $x_n = (-1)^n$

$$\sum_{n\geq 1} (-1)^n$$

$\downarrow$

does not converge

as $(S_n)_{n\geq 1}$ does not converge.

$S_1 = x_1 = -1$

$S_2 = x_1 + x_2 = -1+1 = 0$

$S_3 = x_1 + x_2 + x_3 = -1+1-1$
$= -1$

$S_4 = 0$

$S_{odd} = -1$

$S_{even} = 0$

2) $x_n = r^n$ where $r \in [-1, 1]$

$\sum_{n \geq 1} r^n$ when $r \in [-1, 1]$

Case-I $r = -1$.

$\sum_{n \geq 1} (-1)^n$ does not converge.

Case-II $r = 1$

$\sum_{n \geq 1} 1 \rightarrow$ diverges to $+\infty$

$S_1 = 1$

$S_2 = 2$

$\vdots$

$S_n = n$

$S_n = n$ diverges to $+\infty$.

<u>Case-III</u> $\sum_{n\geq 1} r^n$ when $|r|<1$.

<u>Show</u> $\sum_{n\geq 1} r^n$ is convergent

and $\sum_{n\geq 1} r^n = \dfrac{r}{1-r}$.

$$\begin{cases} S_n = r + r^2 + \cdots + r^n \\ rS_n = \quad\; r^2 + \cdots + r^n + r^{n+1} \end{cases}$$

$$(1-r)S_n = r - r^{n+1}$$

$$S_n = \frac{r}{1-r} - \frac{r^{n+1}}{1-r}$$

Recall $\quad r^{n+1} \longrightarrow 0$ when $|r| < 1$

$$\therefore \quad S_n \longrightarrow \frac{r}{1-r}.$$

$$\sum_{n \geqslant 1} r^n = \frac{r}{1-r} \quad \text{when } |r| < 1.$$

" Geometric Series "

3) $x_n = \dfrac{1}{n(n+1)}$

$$\sum_{n\geq 1} \frac{1}{n(n+1)}$$

$a_n = \dfrac{1}{n}$ $\quad a_{n+1} = \dfrac{1}{n+1}$

$b_n = a_n - a_{n+1} = \dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{1}{n(n+1)}$

$s_n \to 1$ as $n \to \infty$

$$\therefore \sum_{n\geq 1} \frac{1}{n(n+1)} = 1$$

$S_1 = \dfrac{1}{1\cdot 2}$

$S_2 = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3}$

$S_3 = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4}$

$\vdots$

$$S_n = \underbrace{\frac{1}{1\cdot 2}}_{\shortparallel} + \underbrace{\frac{1}{2\cdot 3}}_{\shortparallel} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Prop Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that $a_n \to L$.

Then $\sum_{n\geq 1} b_n$ where $b_n = a_n - a_{n+1}$ converges to $a_1 - L$.

"Telescoping Series"

Pf $S_n = b_1 + b_2 + \cdots + b_n$

$$= a_1 - a_2 + a_2 - a_3 + \cdots + a_n - a_{n+1}$$

$$= a_1 - a_{n+1}$$

as $n \to \infty$, $S_n \to a_1 - L$. $\therefore \sum_{n\geq 1} b_n = a_1 - L$.

4) $x_n = \frac{1}{n}$

$\sum_{n\geq 1} \frac{1}{n}$ diverges to $+\infty$

Note $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$(S_n)_{n\geq 1}$ is an increasing sequence.

To show that $(S_n)_{n\geq 1}$ diverges to $+\infty$, it is enough to exhibit a subsequence of $(S_n)_{n\geq 1}$ which diverges to $+\infty$.

<u>Show</u> $(S_{2^n})_{n\geq 1}$ diverges to $+\infty$.

$$S_2 = 1 + \frac{1}{2}$$

$$S_{2^2} = S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

as $\frac{1}{3} > \frac{1}{4}$, $S_4 > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$

$$S_8 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{}$$

$$> 1 + \frac{1}{2} + \frac{1}{4} \overset{\text{"}}{+} \frac{1}{4} + \frac{1}{8} + \frac{1}{8} \overset{\text{"}}{+} \frac{1}{8} + \frac{1}{8}$$

$S_8 > 1 + 3/2$

$$S_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \underline{\left(\frac{1}{3} + \frac{1}{4}\right) + \cdots +}$$

There are $2^n - 2^{n-1}$ many terms and each of them is at least $\frac{1}{2^n}$

$$\underbrace{\underline{\left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right)}}$$

general form of the terms that we are clubbing together

$$> \frac{2^n - 2^{n-1}}{2^n} = \frac{1}{2}$$

$$S_{2^n} > 1 + \frac{1}{2} + \frac{n-1}{2} = 1 + \frac{n}{2}$$

S_{2^n} diverges to $+\infty$.

As $(S_n)_{n\geq 1}$ is an increasing sequence and $(S_{2^n})_{n\geq 1}$ diverges to $+\infty$, we get $(S_n)_{n\geq 1}$ diverges to $+\infty$.

$\therefore \sum\limits_{n\geq 1} \frac{1}{n}$ diverges to $+\infty$

$\longrightarrow$ "Harmonic Series"

7) $\sum_{n\geq 1} \frac{1}{n^2}$ is convergent.

$(S_n)_{n\geq 1}$ is monotonically increasing.

$S_1 = 1$

$S_2 = 1 + \frac{1}{2^2}$

$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2}$

$\vdots$

$S_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$

Aim $\exists$ a subsequence of $(S_n)_{n\geq 1}$, which is bounded.

Consider the subsequence $(S_{2^n - 1})_{n\geq 1}$ of $(S_n)_{n\geq 1}$.

$$S_1 = 1$$

$$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2}$$

$$< 1 + \frac{1}{2^2} + \frac{1}{2^2} = 1 + \frac{1}{2}$$

$$S_7 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{7^2}$$

$$= S_3 + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}$$

$$< S_3 + \frac{4}{4^2} < 1 + \frac{1}{2} + \frac{1}{4}$$

$$S_{15} = S_7 + \underbrace{\frac{1}{8^2} + \cdots + \frac{1}{15^2}}_{\leq \frac{8}{8^2}}$$

$$< 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

$$S_{2^n-1} = S_{2^{n-1}-1} + \underbrace{\frac{1}{(2^{n-1})^2} + \cdots + \frac{1}{(2^n-1)^2}}$$

there are total $2^n - 2^{n-1}$ many terms and each of them is $\leq \frac{1}{(2^{n-1})^2}$

$$\leq S_{2^{n-1}-1} + \frac{2^n - 2^{n-1}}{(2^{n-1})^2}$$

$$S_{2^{n-1}-1} < 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-2}}$$

$$S_{2^n-1} < 1+\frac{1}{2}+\cdots+\frac{1}{2^{n-2}}+\underbrace{\frac{1}{2^{n-1}}\{2-1\}}_{\frac{1}{2^{n-1}}}$$

$$= 1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}$$

$$< 2$$

$\therefore$ The sequence $(S_{2^n-1})_{n\geq 1}$ is bounded.

1) Every bounded monotonic sequence is convergent.

2) If a monotonic sequence has a convergent subsequence, then the sequence itself is convergent.

$\underbrace{(S_{2^n-1})_{n\geq 1}}_{}$ is also monotonically increasing bounded

Using fact 1), we can conclude, $(S_{2^n-1})_{n\geq 1}$ is convergent.

Apply Fact 2), to conclude $\sum_{n\geq 1} \frac{1}{n^2}$ is convergent.

Exercise Show that $\sum_{n\geq 1} \frac{1}{n^p}$ is convergent when $p>1$.

Think! What happens if $0<p<1$?

8) $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ is convergent.

Let $(S_n)_{n\geq 1}$ be the sequence of partial sums of this series.

consider, $(S_{2n})_{n\geq 1}$ and $(S_{2n-1})_{n\geq 1}$.

It is enough to show, both these subsequences converge to the same limit.

$$S_{2n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$$

$$= \underbrace{\left(1-\frac{1}{2}\right)}_{>0}+\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{>0}+\cdots+\underbrace{\left(\frac{1}{2n-1}-\frac{1}{2n}\right)}_{>0}$$

$\therefore$ $S_{2n}>0$, $(S_{2n})_{n\geq 1}$ is an increasing sequence.

$$S_{2n-1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots - \frac{1}{2n-2} + \frac{1}{2n-1}$$

$$= 1 - \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{>0} - \underbrace{\left(\frac{1}{4} - \frac{1}{5}\right)}_{>0} - \cdots - \underbrace{\left(\frac{1}{2n-2} - \frac{1}{2n-1}\right)}_{>0}$$

$\therefore$ $S_{2n-1} < 1$ and $(S_{2n-1})_{n \geq 1}$ is a decreasing sequence.

$$\boxed{\begin{array}{l} S_{2n} < S_{2n-1} \\ S_{2n} = S_{2n-1} - \dfrac{1}{2n} \end{array}}$$

$$\therefore \quad 0 < S_{2n} < S_{2n-1} < 1.$$

both are bounded

$\therefore (S_{2n})_{n\geq 1}$ and $(S_{2n-1})_{n\geq 1}$ are convergent.

Suppose, $S_{2n} \to L_1$

and $S_{2n-1} \to L_2$

$$\underbrace{S_{2n-1} - S_{2n}}_{\frac{1}{2n}} \longrightarrow L_2 - L_1$$

$$\frac{1}{2n} \to 0 \quad \text{as } n \to \infty$$

$$\therefore \quad L_2 - L_1 = 0$$

i.e. $L_1 = L_2$.