COL 352 Introduction to Automata and Theory of Computation

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Lecture 12: DFA Minimization

DFA Minimization

- Given a DFA when can we say there are redundant states?

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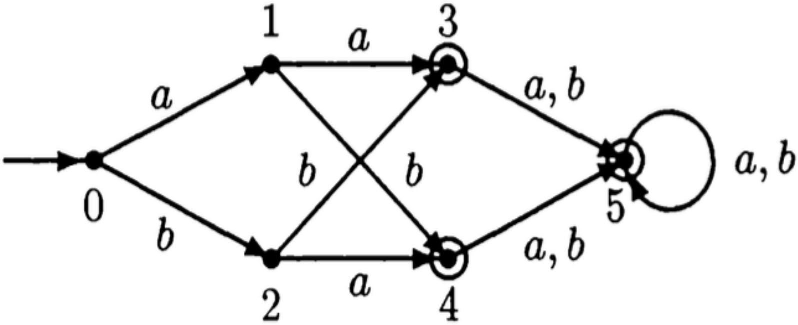
- Rough idea: Given $M = (Q, \Sigma, q_0, \delta, F)$
  - Get rid of inaccessible states.
  - Collapse "equivalent" states.

Minimization : Example 1



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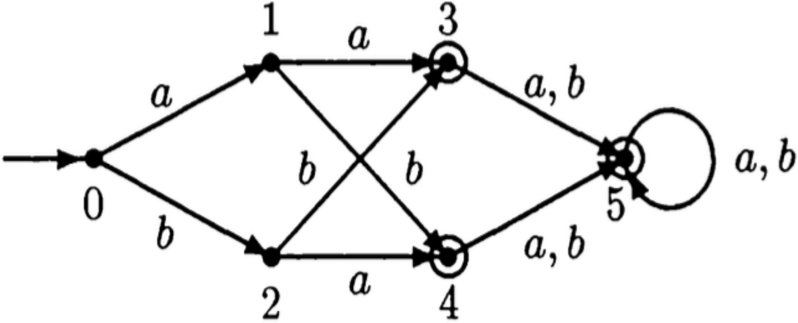

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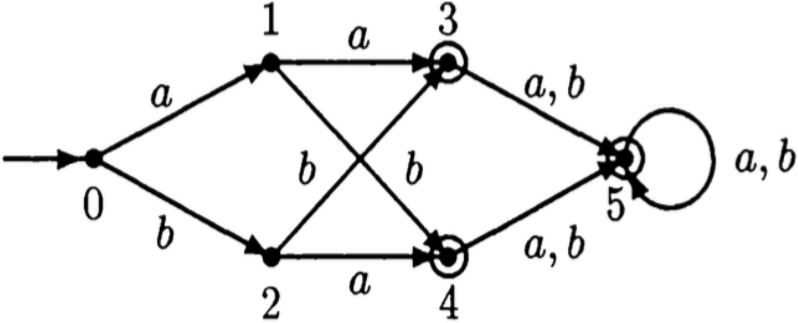

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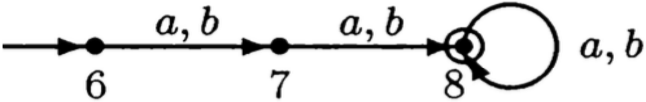

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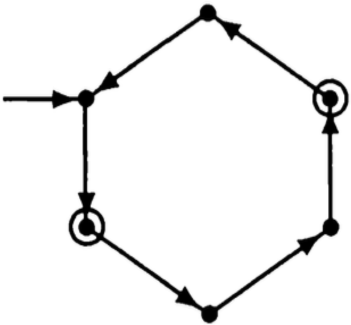

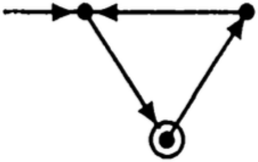

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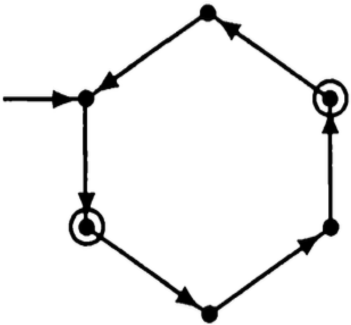

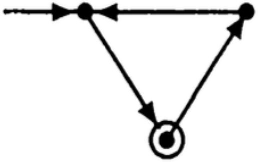

$$L = \{a^m \mid m \equiv 1 \pmod 3\}$$

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- Can we do this formally?
- Is there an efficient algorithm for doing this?
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   If $\hat{\delta}(s, x) = p \in F$ and $\hat{\delta}(s, y) = q \notin F$, so cannot collapse p and q!
2. If we are collapsing p and q, better also collapse $\delta(p, a)$ and $\delta(q, a)$ for all $a \in \Sigma$.

Inductively, these two imply that we cannot collapse p and q if $\hat{\delta}(p, x) \in F$ and $\hat{\delta}(q, x) \notin F$ for some string x.

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Inductively, these two imply that we cannot collapse p and q if $\hat{\delta}(p,x) \in F$ and $\hat{\delta}(q,x) \notin F$ for some string x. Turns out this is necessary and sufficient to decide if a pair of states can be collapsed or not!

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Exericse: Show that $\equiv$ is an equivalence relation.

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Why not define an automaton whose states are just these equivalence classes? (This is exactly the "collapsing states" we wanted!)

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$$\hat{\delta}(\delta(p, a), y) \in F \iff \hat{\delta}(p, ay) \in F$$

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Proof. By induction on $|x|$.

Basis: For $x = \varepsilon$,

$$\begin{aligned} \hat{\delta}'([p], \varepsilon) &= [p] && \text{(by definition of } \hat{\delta}'\text{)} \\ &= [\hat{\delta}(p, \varepsilon)] && \text{(by definition of } \hat{\delta}\text{)} \end{aligned}$$

Induction step: Assume $\hat{\delta}'([p], x) = [\hat{\delta}(p, x)]$ and let $a \in \Sigma$.

$$\hat{\delta}'([p], xa) = \hat{\delta}'(\hat{\delta}'([p], x), a) \quad \text{(by definition of } \hat{\delta}'\text{)}$$

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Claim: $L(M/\equiv) = L(M)$ Exercise!

Cannot Collapse Further

Define

$$[p] \approx [q] \iff \forall x \in \Sigma^* (\hat{\delta}'([p], x) \in F' \iff \hat{\delta}'([q], x) \in F')$$

Apply this relation on $M/\equiv$.

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\end{aligned}$$

The relation $\approx$ is just equality $(=)$!

An algorithm for DFA minimization

Let M be a DFA with no inaccessible states. We will mark (unordered) pairs of states $\{p,q\}$ if we discover a reason why they are not equivalent.

1. Write down a table of pairs $\{p,q\}$, initially unmarked.
2. Mark $\{p,q\}$ if $p \in F$ and $q \notin F$, or vice-versa.
3. Repeat until no change occurs: if there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a),\delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$.
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- Same pair $\{p, q\}$ has to be visited by the algorithm multiple times (status might change because of other ✓ filled in the table)

Finding equivalent states

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Given: DFA A

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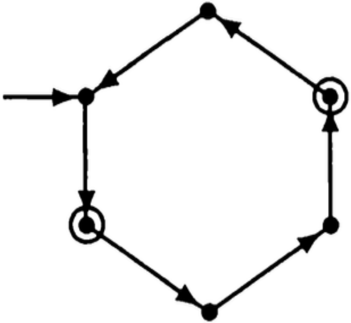
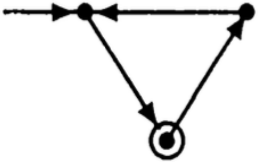
Output: sets of states of A equivalent to each other





Finding equivalent states

Given: DFA A

Output: sets of states of A equivalent to each other

	0	1	2	3	4	5
a	1	2	3	4	5	0

(Red color indicates final states.)

0					
–	1				
–	–	2			
–	–	–	3		
–	–	–	–	4	
–	–	–	–	–	5

Minimization problem

Minimization problem (for fixed Σ)

Given: DFA A

Example

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–	–	–	3		
–	–	–	–	4	
–	–	–	–	–	5

0					
✓	1				
✓	✓	2			
–	✓	–	3		
✓	–	✓	✓	4	
✓	✓	–	–	✓	5

Minimization problem

Minimization problem (for fixed Σ)

Given: DFA A

Output: sets of states of A equivalent to each other

Example

	0	1	2	3	4	5
a	1	2	3	4	5	0

(Red color indicates final states.)

0					
–	1				
–	–	2			
–	–	–	3		
–	–	–	–	4	
–	–	–	–	–	5

0					
✓	1				
✓	✓	2			
–	✓	✓	3		
✓	–	✓	✓	4	
✓	✓	–	–	✓	5

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	0	1	2	3	4	5
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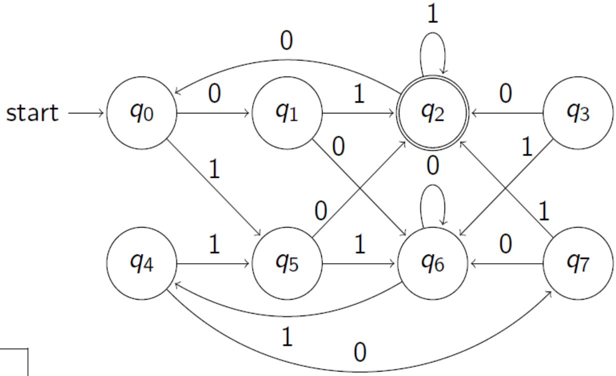
0					
–	1				
–	–	2			
–	–	–	3		
–	–	–	–	4	
–	–	–	–	–	5

0					
✓	1				
✓	✓	2			
–	✓	✓	3		
✓	–	✓	✓	4	
✓	✓	–	✓	✓	5

Example



q_1							
q_2							
q_3							
q_4							
q_5							
q_6							
q_7							
D	q_0	q_1	q_2	q_3	q_4	q_5	q_6

Minimization problem

Minimization problem (for fixed Σ)

Given:	DFA A
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Algorithm

Let $Q = \{q_1, \ldots, q_n\}$.

Minimization problem

Minimization problem (for fixed Σ)

Given: DFA A

Output: sets of states of A equivalent to each other

Algorithm

Let $Q = \{q_1, \ldots, q_n\}$.

1. For each $1 \leq i < j \leq n$, initialize $T(i, j) = --$

Minimization problem

Minimization problem (for fixed Σ)

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2. For each $1 \le i < j \le n$

Minimization problem

Minimization problem (for fixed Σ)

Given:	DFA A
Output:	sets of states of A equivalent to each other

Algorithm

Let $Q = \{q_1, \ldots, q_n\}$.

1. For each $1 \leq i < j \leq n$, initialize $T(i, j) = --$
2. For each $1 \leq i < j \leq n$

   If $(q_i \in F$ AND $q_j \notin F)$ OR $(q_i \in F$ AND $q_j \notin F)$
   $T(i, j) \leftarrow \checkmark$
3. Repeat

Minimization problem

Minimization problem (for fixed Σ)

Given: DFA A

Output: sets of states of A equivalent to each other

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Let $Q = \{q_1, \ldots, q_n\}$.

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   $T(i,j) \leftarrow \checkmark$
3. Repeat

   $\{$ For each $1 \le i < j \le n$

   If $\exists a \in \Sigma, T(\delta(q_i,a), \delta(q_j,a)) = \checkmark$

   then $T(i,j) \leftarrow \checkmark$

   $\}$

   Untill T stays unchanged.

Proof of Correctness

Claim: The pair $\{p, q\}$ is not marked by the algorithm if and only if there exists $x \in \Sigma^*$ such that $\hat{\delta}(p, x) \in F$ and $\hat{\delta}(q, x) \notin F$ or vice-versa, i.e., if and only if $p \not\equiv q$.

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$$\mathcal{Q} = \{\{p, q\} \mid p, q \in Q, p \neq q\}$$

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An automaton view of the algorithm:

$\mathcal{Q} = \{\{p,q\} \mid p,q \in Q, p \neq q\}$

$\Delta : \mathcal{Q} \to 2^{\mathcal{Q}}$ defined as

$$\Delta(\{p,q\},a) := \{\{p',q'\} \mid p = \delta(p',a), q = \delta(q',a)\}$$

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- Step 2 marks elements of $\mathcal{S}$.

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- Step 2 marks elements of $\mathcal{S}$.
- Step 3 marks pairs in $\Delta(\{p, q\}, a)$ when $\{p, q\}$ is marked for some $a \in \Sigma$.

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- Step 2 marks elements of $\mathcal{S}$.
- Step 3 marks pairs in $\Delta(\{p, q\}, a)$ when $\{p, q\}$ is marked for some $a \in \Sigma$.
- Claim above says $p \not\equiv q \iff \{p, q\}$ if and only if $\{p, q\}$ is reachable from $\mathcal{S}$.

Minimization problem

Minimization problem (for fixed Σ)

Given: DFA A

Output: DFA B s.t. $L(A) = L(B)$ and B has the smallest number of states possible for recognizing $L(A)$

Example

	0	1	2	3	4	5
a	1	3	4	5	5	5
b	2	4	3	5	5	5

(Red color indicates final states.)

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	0	1	2	3	4	5
a	1	3	4	5	5	5
b	2	4	3	5	5	5

(Red color indicates final states.)

0					
–	1				
–	–	2			
–	–	–	3		
–	–	–	–	4	
–	–	–	–	–	5

DIY!