Diagonalizability

Observation: Let $T : \mathbb{C}^2 \to \mathbb{C}^2$.

$$T((z_1, z_2)) = (z_1 - z_2,\ z_1 + z_2).$$

basis, $\beta = \{e_1, e_2\} \Rightarrow [T]_\beta = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$

e-values of T are e-values of $[T]_\beta$, hence the e-values are $1+i$, $1-i$

Consider another basis, $\beta' = \{(1,-i), (1,i)\}$

$$[T]_{\beta'} = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$$

clearly e-values of T are $1+i$, $1-i$

Thus, try to chose a basis β of V such that $[T]_\beta$ is as simple as possible.

Diagonalizable Operator: Let $T: V \to V$ linear.

Then T is called diagonalizable if $\exists$ a basis β of V with respect to which $[T]_\beta$ is diagonal.

Diagonalizable Matrix: $A \in M_{n\times n}(\mathbb{F})$ is diagonalizable if $\exists$ an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ such that $P^{-1}AP$ is diagonal, i.e. "A is similar to a diagonal matrix".

Example: (Not every matrix is diagonalizable)

(i) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$

No real e.values.

(ii) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{C})$

Example: (Not every operator is diagonalizable).

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T((x, y)) = (y, -x)$$

Result: (Diagonalizability Criterion)

A operator $T : V \rightarrow V$ is diagonalizable if and only if $\dim(V)$ is equal to sum of the dimensions of the eigen spaces.

That means,

$$\dim(V) = \sum_{i=1}^{m} \dim\left(\operatorname{Ker}(\lambda_i I - T)\right), \text{ where}$$

$\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct e-values of T.

Proof: Let $n = \dim(V)$

($\Rightarrow$) first suppose that T is diagonalizable.

$\Rightarrow \exists$ a basis β of V such that

$$[T]_\beta = \begin{bmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & 0 & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_2 & \\ & 0 & & & & & \ddots \\ & & & & & & \lambda_m \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_m \end{bmatrix}$$

λ_i is repeated r_i-times.

diagonal matrix

$$\Rightarrow \quad n = r_1 + r_2 + \cdots + r_m$$

$$\Rightarrow \dim(V) = \sum_{i=1}^{m} \dim\left(\operatorname{Ker}\left(\lambda_i I - [T]_\beta\right)\right)$$

$$= \sum_{i=1}^{m} \dim\left(\operatorname{Ker}\left(\lambda_i I - T\right)\right).$$

($\Leftarrow$) Conversely, let

$$\dim(V) = \underbrace{\dim\left(\operatorname{Ker}(\lambda_1 I - T)\right)}_{= r_1} + \cdots + \underbrace{\dim\left(\operatorname{Ker}(\lambda_m I - T)\right)}_{= r_m}$$

Let β_i basis of $\operatorname{Ker}(\lambda_i I - T)$ and

let $\beta = \bigcup_{i=1}^{m} \beta_i$

Since e-spaces corresponding to distinct e-values are independent, we have that

β is a basis of $\ker(\lambda_1 I - T) + \dots + \ker(\lambda_m I - T)$

Since $\sum_{i=1}^{m} \ker(\lambda_i I - T) \subseteq V$ subspace.

and $\dim\left(\sum_{i=1}^{m} \ker(\lambda_i I - T)\right) = \sum_{i=1}^{m} \dim\left(\ker(\lambda_i I - T)\right)$

$= \dim(V)$

$$\Rightarrow \quad V = \sum_{i=1}^{m} \ker(\lambda_i I - T).$$

$\Rightarrow$ β is a basis of V.

Also.

$$[T]_\beta = \begin{bmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & 0 & \\ & & \lambda_1 & & & & & \\ & & & \lambda_2 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_2 & & \\ & 0 & & & & & \ddots & \\ & & & & & & \lambda_m & \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_m \end{bmatrix}$$

r_1-times

r_2-times

r_m-times

$\Rightarrow$ T is diagonalizable.

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Example: Determine whether $T: \mathbb{F}^3 \to \mathbb{F}^3$ defined by $T((x,y,z)) = (2x, x+2y, 4x+3z)$ is diagonalizable?

Solution: Step-1 find the distinct e-values of T.

they are $\lambda_1 = 2$, $\lambda_2 = 3$

Step-2 e-spaces corresponding to distinct e-values.

$\lambda = 2$, $\ker(2I - T) = \{(0,s,0), s \in \mathbb{F}\}$

$\lambda = 3$, $\ker(3I - T) = \{(0,0,s), s \in \mathbb{F}\}$

$\Rightarrow \dim(\mathbb{F}^3) = 3 \neq 2 = \dim(\ker(2I - T)) + \dim(\ker(3I - T))$

$\Rightarrow$ T is not diagonalizable.

Example: find a basis β such that $[T]_\beta$ is diagonal in case T is diagonalizable.

Also find a matrix P such that

$$[T]_\beta = P^{-1}[T]_s P$$, where s is the standard basis

(i) $T: \mathbb{C}^2 \to \mathbb{C}^2$

$$T((x,y)) = (y,-x)$$

Solution: e-values of T are $i, -i$

e-space for $\lambda = i$

$$E_1 = \mathrm{Ker}(iI - T)$$

$$= \{(x,y) \in \mathbb{C}^2 : T(x,y) = i(x,y)$$

$$\Downarrow$$

$$(y,-x) = (ix, iy)$$

$$\Downarrow$$

$$y = ix$$

$$= \{(x, ix) : x \in \mathbb{C}\}$$

$$= \mathrm{span}\{(1,i)\}$$

e. space for $\lambda = -i$

$$E_2 = \ker(-iI - T)$$
$$= \{ (x,y) \in \mathbb{C}^2 : T(x,y) = -i(x,y) \}$$
$$\Downarrow$$
$$(y,-x) = (-ix, -iy)$$
$$\Downarrow$$
$$y = -ix$$

$$= \{ (x, -ix) : x \in \mathbb{C} \}$$
$$= \operatorname{span}\{ (1,-i) \}.$$

Clearly, $\underbrace{\dim(\mathbb{C}^2)}_{2} = \underbrace{\dim(E_1)}_{1} + \underbrace{\dim(E_2)}_{1}$.

$\Rightarrow$ T is diagonalizable.

Consider $\beta = \{ (1,i), (1,-i) \}$ basis of $\mathbb{C}^2$.

Then $[T]_\beta = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Again consider $P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$, i.e. P is obtained by writing β in-terms of δ.

Then $P^{-1} [T]_\delta P = P^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} P$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = [T]_\beta .$$ //.

Remark: (On matrix diagonalizability).

$A \in M_{n\times n}(\mathbb{F})$.

A is diagonalizable if and only if

$$n = \sum_{i=1}^{m} \dim\left(\operatorname{Ker}(\lambda_i I - A)\right)$$

, where $\lambda_1, \ldots \lambda_m$ are distinct e.values.

On finding P such that $P^{-1}AP$ is diagonal.

Find basis of $\operatorname{Ker}(\lambda_i I - A)$ for each i.

Construct P whose Columns are the basis vectors.

Then $P^{-1}AP$ is diagonal.

Cayley-Hamilton Theorem:

Suppose $A \in M_{n\times n}(\mathbb{F})$ and $p(\lambda) = \det(\lambda I - A)$ is the characteristic poly. of A. Then the matrix $p(A)$ is identically zero.

That means,

> "Every matrix satisfies its char. poly."

Application of Cayley-Hamilton th^m:

Observe that $p(0) = \det(-A) = (-1)^n \cdot \det(A)$.

Since A is invertible $\Leftrightarrow \det(A) \neq 0$

$\Leftrightarrow p(0) \neq 0$.

Thus if $p(0) \neq 0$, or constant term in $p(\lambda)$ is nonzero, then A is invertible.

finding inverse: Let A be invertible.

If $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$,

then by Cayley-Hamilton th^m

$$p(A) = 0$$

$$\Rightarrow A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 \cdot I = 0$$

$$\Rightarrow a_0 I = -a_1 A - a_2 A^2 - \dots - a_{n-1}A^{n-1} - A^n$$

$$\Rightarrow I = \frac{1}{a_0}\left(-a_1 A - a_2 A^2 - \dots - a_{n-1}A^{n-1} - A^n\right)$$

$$= \underbrace{\frac{1}{a_0}\left(-a_1 I - a_2 A - \dots - a_{n-1}A^{n-2} - A^{n-1}\right)}_{A^{-1}} A$$

$$\Rightarrow A^{-1} = \frac{-1}{a_0}\left(a_1 I + a_2 A + \dots + a_{n-1}A^{n-2} + A^{n-1}\right).$$

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

char. poly. $p(\lambda) = \det(\lambda I - A)$

$= (\lambda-1)(\lambda^2-\lambda-1)$

$= \lambda^3 - 2\lambda^2 + 1$

Since $p(0) = 1 \neq 0$

$\Rightarrow$ A is invertible.

By Cayley-Hamilton thm $p(A) = 0$

$\Rightarrow A^3 - 2A^2 + I = 0$

$\Rightarrow I = A(-A^2 + 2A)$

$\Rightarrow A^{-1} = -A^2 + 2A$

$$= -\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

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