PYL101

Electromagnetic Waves and Quantum Mechanics

Tutorial Sheet 3 (L5-L6)

Problem 1:

Consider the two states $|\psi> = i\,|\phi_1> +3i\,|\phi_2> -|\phi_3>$ and $|\chi> = |\phi_1> -i|\phi_2> +5i|\phi_3>$, where $|\phi_1>$, $|\phi_2>$ and $|\phi_3>$ are orthonormal. Then calculate:

$<\psi|\psi>$, $<\chi|\chi>$, $<\psi|\chi>$, $<\chi|\psi>$ and $<\psi+\chi|\psi+\chi>$.

Solution:

Here given, $<\phi_i|\phi_j> = \delta_{ij}$,

$$|\psi> = i\,|\phi_1> +3i\,|\phi_2> -|\phi_3> \text{ and}$$

$$|\chi> = |\phi_1> -i|\phi_2> +5i|\phi_3>$$

Corresponding bras are, $<\psi| = -i<\phi_1| - 3i<\phi_2| - <\phi_3|$ and

$$<\chi| = <\phi_1| + i<\phi_2| - 5i<\phi_3|$$

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Hence,

$$<\psi|\psi> = (-i<\phi_1| - 3i<\phi_2| - <\phi_3|)(i\,|\phi_1> +3i\,|\phi_2> -|\phi_3>)$$

$$= (-i)(i)<\phi_1|\phi_1> +(-3i)(3i)<\phi_2|\phi_2> +(-1)^2<\phi_3|\phi_3>$$

$$= |i|^2 + |3i|^2 + |-1|^2 = 11$$

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$$<\psi|\psi> = (-i<\phi_1| - 3i<\phi_2| - <\phi_3|)(i\,|\phi_1> +3i\,|\phi_2> -|\phi_3>)$$

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$$= |i|^2 + |3i|^2 + |-1|^2 = 11$$

Similarly,

$$<\chi|\chi> = |1|^2 + |-i|^2 + |5i|^2 = 27$$

Solution of problem 1 continued…

$$< \psi|\chi > = (-i < \phi_1| - 3i < \phi_2| - < \phi_3|)(|\phi_1 > -i\ |\phi_2 > +5i|\phi_3 >)$$

$$= (-i) < \phi_1|\phi_1 > +(-3i)(-i) < \phi_2|\phi_2 > +(-1)(5i) < \phi_3|\phi_3 >$$

$$= -i - 3 - 5i$$

$$= -3 - 6i$$

Solution of problem 1 continued…

$$< \psi|\chi > = (-i < \phi_1| - 3i < \phi_2| - < \phi_3|)(\,|\phi_1 > -i\,|\phi_2 > +5i|\phi_3 >)$$

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$$= -3 - 6i$$

$$< \chi|\psi > = (< \phi_1| + i < \phi_2| - 5i < \phi_3|)(\,i|\phi_1 > +3i\,|\phi_2 > -|\phi_3 >)$$

$$= (i) < \phi_1|\phi_1 > +(i)(3i) < \phi_2|\phi_2 > +(-5i)(-1) < \phi_3|\phi_3 >$$

$$= i - 3 + 5i$$

$$= -3 + 6i$$

Solution of problem 1 continued…

$$< \psi|\chi > = (-i < \phi_1| - 3i < \phi_2| - < \phi_3|)(|\phi_1 > -i\, |\phi_2 > +5i|\phi_3 >)$$

$$= (-i) < \phi_1|\phi_1 > +(-3i)(-i) < \phi_2|\phi_2 > +(-1)(5i) < \phi_3|\phi_3 >$$

$$= -i - 3 - 5i$$

$$= -3 - 6i$$

$$< \chi|\psi > = (< \phi_1| + i < \phi_2| - 5i < \phi_3|)(i|\phi_1 > +3i\, |\phi_2 > -|\phi_3 >)$$

$$= (i) < \phi_1|\phi_1 > +(i)(3i) < \phi_2|\phi_2 > +(-5i)(-1) < \phi_3|\phi_3 >$$

$$= i - 3 + 5i$$

$$= -3 + 6i$$

$$< \psi + \chi|\psi + \chi > = < \psi|\psi > + < \psi|\chi > + < \chi|\psi > + < \chi|\chi >$$

$$= 11 - 3 - 6i - 3 + 6i + 27$$

$$= 32$$

Problem 2:

Find the constant α so that the states $|\psi> = \alpha|\phi_1> +5|\phi_2>$ and $|\chi> = 3\alpha|\phi_1> -4|\phi_2>$ are orthogonal. $|\phi_1>$ and $|\phi_2>$ are orthonormal wave functions.

Solution:

Since $|\psi>$ and $|\chi>$ are orthogonal, therefore

$$<\psi|\chi> = 0$$

$$(\alpha <\phi_1| + 5 <\phi_2|)(3\alpha|\phi_1> -4|\phi_2>) = 0 \qquad (\because \alpha \text{ is a real constant})$$

$$3\alpha^2 <\phi_1|\phi_1> -20 <\phi_2|\phi_2> = 0 \qquad (\because <\phi_i|\phi_j> = \delta_{ij})$$

$$3\alpha^2 - 20 = 0$$

$$\alpha^2 = \frac{20}{3}$$

$$\alpha = \pm 2.58$$

Problem 3:

Consider a state which is given in terms of three orthonormal vectors $|\phi_1>$, $|\phi_2>$ and $|\phi_3>$

$|\psi>= \frac{1}{\sqrt{15}}|\phi_1> + \frac{1}{\sqrt{3}}\ |\phi_2> + \frac{1}{\sqrt{5}}\ |\phi_3>$, where $|\phi_n>$ are eigenstates to an operator $\hat{B}$ which satisfies the relation $\hat{B}|\phi_n> = (3n^2 - 1)|\phi_n>$, where n = 1, 2, 3. Then

(a) Find the norm of $|\psi>$.

(b) Find the expectation value of $\hat{B}$ with respect to $|\psi>$

Solution:

(a) Norm of the state $|\psi>$ is given by,

$$< \psi|\psi > = \left(\frac{1}{\sqrt{15}} < \phi_1 | + \frac{1}{\sqrt{3}} < \phi_2 | + \frac{1}{\sqrt{5}} < \phi_3 |\right)\left(\frac{1}{\sqrt{15}}\ |\phi_1 > \frac{1}{\sqrt{2}}\ |\phi_2 > + \frac{1}{\sqrt{5}}|\phi_3 >\right)$$

$$= \frac{1}{15} + \frac{1}{3} + \frac{1}{5}$$

$$= \frac{3}{5}$$

Solution of problem 3 continued…

(b) Expectation value of $\hat{B}$ with respect to $|\psi>$

$$<\hat{B}> = \frac{<\psi|\hat{B}|\psi>}{<\psi|\psi>}$$

$$<\psi\left|\hat{B}\right|\psi> = \left(\frac{1}{\sqrt{15}}<\phi_1| + \frac{1}{\sqrt{3}}<\phi_2| + \frac{1}{\sqrt{5}}<\phi_3|\right)\hat{B}\left(\frac{1}{\sqrt{15}}|\phi_1> \frac{1}{\sqrt{3}}|\phi_2> + \frac{1}{\sqrt{5}}|\phi_3>\right)$$

$$= \left(\frac{1}{\sqrt{15}}<\phi_1| + \frac{1}{\sqrt{3}}<\phi_2| + \frac{1}{\sqrt{5}}<\phi_3|\right)$$

$$\left(\frac{1}{\sqrt{15}}(3(1)^2-1)|\phi_1> + \frac{1}{\sqrt{3}}(3(2)^2-1)|\phi_2> + \frac{1}{\sqrt{5}}(3(3)^2-1)|\phi_3>\right)$$

$$= \frac{2}{15} + \frac{11}{3} + \frac{26}{5} = 9$$

$(\because \hat{B}|\phi_n> = (3n^2-1)|\phi_n)$

Solution of problem 3 continued…

(b) Expectation value of $\hat{B}$ with respect to $|\psi>$

$$< \hat{B} > = \frac{<\psi|\hat{B}|\psi>}{<\psi|\psi>}$$

$$< \psi\big|\hat{B}\big|\psi > = \left(\frac{1}{\sqrt{15}} < \phi_1 \,| + \frac{1}{\sqrt{3}} < \phi_2 \,| + \frac{1}{\sqrt{5}} < \phi_3 \,|\right)\hat{B}\left(\frac{1}{\sqrt{15}}\,|\phi_1 > \frac{1}{\sqrt{3}}\,|\phi_2 > + \frac{1}{\sqrt{5}}|\phi_3 >\right)$$

$$= \left(\frac{1}{\sqrt{15}} < \phi_1 \,| + \frac{1}{\sqrt{3}} < \phi_2 \,| + \frac{1}{\sqrt{5}} < \phi_3 \,|\right)$$

$$\left(\frac{1}{\sqrt{15}}(3(1)^2 - 1)\,|\phi_1 > + \frac{1}{\sqrt{3}}(3(2)^2 - 1)\,|\phi_2 > + \frac{1}{\sqrt{5}}(3(3)^2 - 1)|\phi_3 >\right)$$

$$= \frac{2}{15} + \frac{11}{3} + \frac{26}{5} = 9$$

$$(\because\ \hat{\mathrm{B}}|\phi_{\mathrm{n}} > = (3\mathrm{n}^2 - 1)|\phi_{\mathrm{n}})$$

$$< \hat{B} > = \frac{<\psi|\hat{B}|\psi>}{<\psi|\psi>}$$

$$= \frac{9}{3/5} = 15$$

Problem 4:

Show that the operator $|\psi><\psi|$ is a projection operator only when $|\psi>$ is normalized.

Solution:

A projection operator $\hat{P}$ must satisfy that

$$\hat{P}^2 = \hat{P}$$

Thus,

$$(|\psi><\psi|)^2 = (|\psi><\psi|)(|\psi><\psi|)$$

$$= |\psi><\psi|\psi><\psi|$$ (Not idempotent)

If $<\psi|\psi> = 1$, that is $|\psi>$ is normalized, then

$$(|\psi><\psi|)^2 = |\psi><\psi|$$

"If the state $|\psi>$ is normalized, the product of the ket $|\psi>$ with the bra $<\psi|$ is a projection operator."

Problem 5:

Check whether the operators $\hat{x},\ d/dx$ and $i\,d/dx$ are Hermitian operators.

Solution:

"An operator $\hat{A}$ is said to be Hermitian if it is equal to its adjoint $\hat{A}^{\dagger}$. That is

$$\hat{A} = \hat{A}^{\dagger} \Longrightarrow < \psi|\hat{A}\psi > = < \hat{A}\psi|\psi >$$"

$$< \psi|\hat{x}\psi > = \int_{-\infty}^{+\infty} \psi^*(x)(\hat{x}\psi(x))dx$$

$$= \int_{-\infty}^{+\infty} \psi^*(x)(a\psi(x))dx$$

$$= \int_{-\infty}^{+\infty} a\psi^*(x)\psi(x)dx$$

$$= \int_{-\infty}^{+\infty} (a\psi(x))^*\psi(x)dx$$

$$= \int_{-\infty}^{+\infty} (\hat{x}\psi(x))^*\psi(x)dx$$

$$= < \hat{x}\psi|\psi >$$

$\Longrightarrow \hat{x}$ is a Hermitian.

Solution of problem 5 continued…

$$< \psi | \frac{d}{dx}\psi > = \int_{-\infty}^{+\infty} \psi^*(x)(\frac{d}{dx}\psi(x))dx$$

$$= \psi^*(x)\psi(x)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{d}{dx}\psi^*(x)\psi(x)dx$$

$$= 0 - \int_{-\infty}^{+\infty} \frac{d}{dx}\psi^*(x)\psi(x)dx$$

$$= - \int_{-\infty}^{+\infty} (\frac{d}{dx}\psi(x))^*\psi(x)dx$$

$$= - < \frac{d}{dx}\psi | \psi >$$

$\Longrightarrow \frac{d}{dx}$ is anti-Hermitian.

Solution of problem 5 continued…

$$< \psi | \frac{d}{dx}\psi > = \int_{-\infty}^{+\infty} \psi^*(x)(\frac{d}{dx}\psi(x))dx$$

$$= \psi^*(x)\psi(x)|_{+\infty}^{-\infty} - \int_{-\infty}^{+\infty} \frac{d}{dx}\psi^*(x)\psi(x)dx$$

$$= 0 - \int_{-\infty}^{+\infty} \frac{d}{dx}\psi^*(x)\psi(x)dx$$

$$= - \int_{-\infty}^{+\infty} (\frac{d}{dx}\psi(x))^*\psi(x)dx$$

$$=-< \frac{d}{dx}\psi|\psi >$$

$$\Longrightarrow \frac{d}{dx} \text{ is anti-Hermitian.}$$

Now,

$$(i\frac{d}{dx})^\dagger = \left(-\frac{d}{dx}\right)(-i)$$

$$= i\frac{d}{dx}$$

$$\Longrightarrow i\frac{d}{dx} \text{ is Hermitian}$$

Problem 6:

Consider a system whose Hamiltonian is given by $\hat{H} = \alpha(|\phi_1 >< \phi_2| + |\phi_2 >< \phi_1|)$, α is a real number having the dimensions of energy and $|\phi_1 >$, $|\phi_2 >$ are normalized eigenstates of a Hermitian operator $\hat{A}$ that has no degenerate eigenvalues.

(a) Check whether $|\phi_1 >$ and $|\phi_2 >$ are eigenstates of $\hat{H}$

(b) Calculate the commutators $[\hat{H}, |\phi_1 >< \phi_1|]$ and $[\hat{H}, |\phi_2 >< \phi_2|]$

Solution:

Since $|\phi_1 >$ and $|\phi_2 >$ are eigenstates of the Hermitian operator $\hat{A}$, they must be orthogonal, $< \phi_1|\phi_2 > = 0$

(a)

$$\hat{H}|\phi_1 > = \alpha(|\phi_1 >< \phi_2| + |\phi_2 >< \phi_1|)|\phi_1 >$$

$$= \alpha|\phi_1 >< \phi_2|\phi_1 > + \alpha|\phi_2 >< \phi_1|\phi_1 >$$

$$= \alpha|\phi_2 >$$

Problem 6:

Consider a system whose Hamiltonian is given by $\hat{H} = \alpha(|\phi_1 >< \phi_2| + |\phi_2 >< \phi_1|)$, α is a real number having the dimensions of energy and $|\phi_1 >$, $|\phi_2 >$ are normalized eigenstates of a Hermitian operator $\hat{A}$ that has no degenerate eigenvalues.

(a) Check whether $|\phi_1 >$ and $|\phi_2 >$ are eigenstates of $\hat{H}$

(b) Calculate the commutators $[\hat{H}, |\phi_1 >< \phi_1|]$ and $[\hat{H}, |\phi_2 >< \phi_2|]$

Solution:

Since $|\phi_1 >$ and $|\phi_2 >$ are eigenstates of the Hermitian operator $\hat{A}$, they must be orthogonal, $< \phi_1|\phi_2 > = 0$

(a)

$$\begin{aligned} \hat{H}|\phi_1 > &= \alpha(|\phi_1 >< \phi_2| + |\phi_2 >< \phi_1|)|\phi_1 > \\ &= \alpha|\phi_1 >< \phi_2|\phi_1 > + \alpha|\phi_2 >< \phi_1|\phi_1 > \\ &= \alpha|\phi_2 > \end{aligned}$$

$$\begin{aligned} \hat{H}|\phi_2 > &= \alpha(|\phi_1 >< \phi_2| + |\phi_2 >< \phi_1|)|\phi_2 > \\ &= \alpha|\phi_1 >< \phi_2|\phi_2 > + \alpha|\phi_2 >< \phi_1|\phi_2 > \\ &= \alpha|\phi_1 > \end{aligned}$$

$\Rightarrow |\phi_1 >$ and $|\phi_2 >$ are not the eigenstates of $\hat{H}$.

Solution for problem 6 continued…

(b)

$$\left[\hat{H}, |\phi_1 >< \phi_1|\right] = \hat{H}\ |\phi_1 >< \phi_1| - |\phi_1 >< \phi_1|\hat{H}$$

$$= \alpha|\phi_2 >< \phi_1| - \alpha|\phi_1 >< \phi_2| \quad \text{(Using results from part (a))}$$

$$= \alpha(|\phi_2 >< \phi_1| - |\phi_1 >< \phi_2|)$$

Solution for problem 6 continued…

(b)

$$\left[\hat{H}, |\phi_1 >< \phi_1|\right] = \hat{H}\, |\phi_1 >< \phi_1| - |\phi_1 >< \phi_1|\hat{H}$$

$$= \alpha|\phi_2 >< \phi_1| - \alpha|\phi_1 >< \phi_2|$$ (Using results from part (a))

$$= \alpha(|\phi_2 >< \phi_1| - |\phi_1 >< \phi_2|)$$

$$\left[\hat{H}, |\phi_2 >< \phi_2|\right] = \hat{H}\, |\phi_2 >< \phi_2| - |\phi_2 >< \phi_2|\hat{H}$$

$$= \alpha|\phi_1 >< \phi_2| - \alpha|\phi_2 >< \phi_1|$$ (Using results from part (a))

$$= \alpha(|\phi_1 >< \phi_2| - |\phi_2 >< \phi_1|$$

Problem 7:

Consider an operator $\widehat{D_x}$ to be $\frac{\partial}{\partial x}$ and the wave function of the system to be $\psi(x) = A\sin(\frac{n\pi x}{a})$, then calculate

(a) $\widehat{D_x}\,\psi(x)$ and $\widehat{D_x^2}\psi(x)$

(b) Which one of these forms an eigenvalue problem and what is the corresponding eigenvalue.

Solution:

(a)

$$\hat{D}_x\psi(x) = \frac{\partial}{\partial x}\psi(x)$$
$$= \frac{\partial}{\partial x}\left(A\,sin(\frac{n\pi x}{a})\right)$$
$$= A\left(\frac{n\pi}{a}\right)cos(\frac{n\pi x}{a})$$

$$\hat{D}_x{}^2\psi(x) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\psi(x)\right] = A\left(\frac{n\pi}{a}\right)\frac{\partial}{\partial x}\left(cos(\frac{n\pi x}{a})\right)$$
$$= -\mathrm{A}\left(\frac{n\pi}{a}\right)^2\sin(\frac{n\pi x}{a}) = -\left(\frac{n\pi}{a}\right)^2\psi(x)$$

Problem 7:

Consider an operator $\widehat{D_x}$ to be $\frac{\partial}{\partial x}$ and the wave function of the system to be $\psi(x) = A\sin(\frac{n\pi x}{a})$, then calculate

(a) $\widehat{D_x}\,\psi(x)$ and $\widehat{D_x^2}\psi(x)$

(b) Which one of these forms an eigenvalue problem and what is the corresponding eigenvalue.

Solution:

(a)

$$\widehat{D}_x\psi(x) = \frac{\partial}{\partial x}\psi(x)$$

$$= \frac{\partial}{\partial x}\left(A\,sin(\frac{n\pi x}{a})\right)$$

$$= A\left(\frac{n\pi}{a}\right)cos\left(\frac{n\pi x}{a}\right) = \frac{n\pi}{a}\psi'(x)$$

$$\widehat{D}_x{}^2\psi(x) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\psi(x)\right] = A\left(\frac{n\pi}{a}\right)\frac{\partial}{\partial x}\left(cos(\frac{n\pi x}{a})\right)$$

$$= -\mathrm{A}\left(\frac{n\pi}{a}\right)^2\sin(\frac{n\pi x}{a}) = -\left(\frac{n\pi}{a}\right)^2\psi(x)$$

(b) $\widehat{D}_x{}^2\psi(x)$ is an eigenvalue problem with eigenvalue $= -\left(\frac{n\pi}{a}\right)^2$.

Problem 8:

If the function $e^{-\alpha x^2}$ represent an eigenfunction of the operator $\hat{A} = \left(\frac{d^2}{dx^2} - Bx^2\right)$, then find the value of B.

Solution:

$$\hat{A}\left(e^{-\alpha x^2}\right) = \left(\frac{d^2}{dx^2} - Bx^2\right)\left(e^{-\alpha x^2}\right)$$

$$= \frac{d^2}{dx^2}\left(e^{-\alpha x^2}\right) - Bx^2\left(e^{-\alpha x^2}\right)$$

$$= (4\alpha^2 x^2 - 2\alpha)e^{-\alpha x^2} - Bx^2\left(e^{-\alpha x^2}\right)$$

$$= ((4\alpha^2 - B)x^2 - 2\alpha)e^{-\alpha x^2}$$

For $e^{-\alpha x^2}$ to represent an eigenfunction of the operator $\hat{A}$, $(4\alpha^2 - B)x^2 - 2\alpha$ should be independent of x.

Thus,

$$\mathrm{B} = 4\alpha^2$$

Problem 9:

The state of system at $t = 0$ is given by $|\psi(0)> = \frac{1}{\sqrt{3}}|\phi_1> + A|\phi_2> + \frac{1}{\sqrt{6}}|\phi_3>$, where $|\phi_1>, |\phi_2>$ and $|\phi_3>$ are orthonormal and A is a real constant.

(a) Find A so that $|\psi(0)>$ is normalized.

(b) Write down the state of the system $|\psi(t)>$ at any later time t. Given E_1, E_2 and E_3 are the energies corresponding to $|\phi_1>$, $|\phi_2>$ and $|\phi_3>$ respectively.

Solution:

(a) If $|\psi(0)>$ is normalized, then the sum of the modulus square of the coefficients of $|\phi_1>, |\phi_2>$ and $|\phi_3>$ should be equal to 1.

$$\left|\frac{1}{\sqrt{3}}\right|^2 + |A|^2 + \left|\frac{1}{\sqrt{6}}\right|^2 = 1$$

$$A = \frac{1}{\sqrt{2}}$$

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(a) If $|\psi(0)>$ is normalized, then the sum of the modulus square of the coefficients of $|\phi_1>, |\phi_2>$ and $|\phi_3>$ should be equal to 1.

$$\left|\frac{1}{\sqrt{3}}\right|^2 + |A|^2 + \left|\frac{1}{\sqrt{6}}\right|^2 = 1$$

$$A = \frac{1}{\sqrt{2}}$$

(b) the state of the system $|\psi(t)>$ at any later time t is given as

$$|\psi(t)> = |\psi(0)> e^{-\frac{iE_n t}{\hbar}}, \text{n = 1, 2, 3}$$

$$= \frac{1}{\sqrt{3}}|\phi_1> e^{-\frac{iE_1 t}{\hbar}} + \frac{1}{\sqrt{2}}|\phi_2> e^{-\frac{iE_2 t}{\hbar}} + \frac{1}{\sqrt{6}}|\phi_3> e^{-\frac{iE_3 t}{\hbar}}$$

Problem 10:

If $\psi(x) = Aexp(-x^4)$ is the eigenfunction of one-dimensional Hamiltonian with eigenvalue $E = 0$, then calculate the potential $V(x)$ (in units where $\hbar = 2m = 1$).

Solution:

The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

Using this, we can write,

$$-\frac{d^2}{dx^2}(Aexp(-x^4)) + V(x)(Aexp(-x^4)) = 0$$

$$(12x^2 - 16x^6)(Aexp(-x^4) + V(x)(Aexp(-x^4)) = 0$$

$$(12x^2 - 16x^6)\psi(x) + V(x)\psi(x) = 0$$

$$12x^2 - 16x^6 + V(x) = 0$$

$$\Rightarrow \qquad V(x) = 16x^6 - 12x^2$$