ODE – Lecture 4

Some Classes of DEs.

Recall that:

If a function $u(x,y)$ has continuous partial derivatives, then its differential is

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

If we have a family of curves

$$u(x,y) = c,$$

then its differential equation can be written in the form

$$du = 0$$

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$$

Example

The family of curves $x^2 y^3 = c$ has

$$2x\,y^3 \cdot dx + 3x^2 y^2\,dy = 0$$

as its DE.

Now, let as begin with the de

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Suppose there exists a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = M$$

$$\frac{\partial u}{\partial y} = N$$

Then we have

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

or $$du = 0$$

and hence its general solution is

$$u(x, y) = c$$

Definition

A first order de

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called exact if there exists $u(x,y)$ such that

$$\left.\begin{aligned} M &= \frac{\partial u}{\partial x} \\ N &= \frac{\partial u}{\partial y} \end{aligned}\right\}$$

Note

Sometimes it may be possible to check exactness and find u by mere inspection

For instance

$$y\,dx + x\,dy = 0$$

$$d(xy) = 0$$

$\therefore$ General Soln is $xy = c$

What we need is a test for exactness and method for obtaining

u.

Recall

The mixed second partial derivatives of u, namely, $\dfrac{\partial^2 u}{\partial x\, \partial y}$ and $\dfrac{\partial^2 u}{\partial y\, \partial x}$ are equal whenever both exist and are continuous.

Suppose that

$$M\, dx + N\, dy = 0$$ is

exact.

Then there exists u such that

$$\left.\begin{aligned} M &= \frac{\partial u}{\partial x} \\ N &= \frac{\partial u}{\partial y} \end{aligned}\right\}$$

Then

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\, \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\, \partial y}$$

Since $\dfrac{\partial^2 u}{\partial y\, \partial x} = \dfrac{\partial^2 u}{\partial x\, \partial y}$

(under suitable assumptions)

we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus

$M\, dx + N\, dy = 0$ is exact $\implies \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$P \implies Q$$

is equivalent to

not Q $\Longrightarrow$ not P.

Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ is a necessary condition for exactness of

$$M\,dx + N\,dy = 0$$

Next we prove that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ is a sufficient condition as well

ie, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\Longrightarrow M\,dx + N\,dy = 0$$

exact.

is

Assume $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

We wish to construct a function u such that

$$M\,dx + N\,dy$$

$$= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$= du.$$

i.e to construct u such that

$$\begin{cases} \dfrac{\partial u}{\partial x} = M & (1) \\[2ex] \dfrac{\partial u}{\partial y} = N & (2) \end{cases}$$

Integrate (1) w.r.to x treating y as constant

$$\int M\,dx + g(y)$$

③ $u(x,y) = \int$ ...

This reduces our problem to that of finding $g(y)$ such that (3) satisfies (2)

From ③

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int M\,dx + g(y)\right)$$

$$= \frac{\partial}{\partial y}\int M\,dx + g'(y)$$

$$= N \quad (\text{from } ②)$$

$$\therefore\ g'(y) = N - \frac{\partial}{\partial y}\int M\,dx.$$

$$\therefore\ g(y) = \int\left(N - \frac{\partial}{\partial y}\int M\,dx\right)dy,$$

the integrand

provided the integrand is a function of y-alone

This will be true if its derivative w.r.t. x is zero

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M\,dx\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x\,\partial y}\int M\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y\,\partial x}\int M\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= 0$$

Thus

$M\,dx + N\,dy = 0$

is exact

$$\boxed{\iff \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

<u>Example</u>

Test the eqn

$$e^y\,dx + (x e^y + 2y)\,dy = 0$$

for exactness and solve it if it is exact.

Here

$$M(x,y) = e^y$$
$$N(x,y) = x e^y + 2y$$

$$e^y$$

$$\therefore \frac{\partial M}{\partial y} =$$

$$\frac{\partial N}{\partial x} = e^y \cdot 1 = e^y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$\Rightarrow$ Eqn is exact.

Since it is exact

there exists u $\ni$:

$$M\,dx + N\,dy = du$$

$$= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$\begin{cases} \frac{\partial u}{\partial x} = M = e^y & (1) \\ \frac{\partial u}{\partial y} = N = x e^y + 2y & (2) \end{cases}$$

to x

integrating w...

(1) $\Rightarrow$ $u(x,y) = e^y \int dx + g(y)$

$u(x,y) = e^y x + g(y)$ — (3)

From (3)

$\frac{\partial u}{\partial y} = e^y x + g'(y)$

$= x e^y + 2y$

$\Rightarrow g'(y) = 2y$

$\Rightarrow g(y) = y^2$

$\therefore u(x,y) = x e^y + y^2$

$\therefore$ Eqn is

$M\,dx + N\,dy = 0$

$\Rightarrow \quad du = 0$

$\therefore$ Solution is

$$u(x, y) = C$$

$$x e^{y} + y^{2} = c, \text{ Constant}$$

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consider

$$-y \, dx + x \, dy = 0$$

$$M = -y \qquad N = x$$

$$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Eqn is NOT exact.

Let us multiply it by $\frac{1}{x^2}$

$$-\frac{y}{x^2}\,dx + \frac{1}{x^2}\,x\,dy = 0$$

i.e $$-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = 0$$

i.e $$d\left(\frac{y}{x}\right) = 0$$

$\therefore$ Solution is: $\frac{y}{x} = c$

<u>Qn:</u>

If $M\,dx + N\,dy = 0$

is not exact, can

we find a function $\mu(x, y)$ with the property that

$$\mu M \, dx + \mu N \, dy = 0$$

is exact ?

Such a function μ is called an integrating factor.

For instance

$\frac{1}{x^2}$ is an integrating factor (IF) for

$$-y \, dx + x \, dy = 0$$

verify that $\overline{y^2}$, $\overline{xy}$,

$\frac{1}{x^2+y^2}$ are all IF

for $-y\,dx + x\,dy = 0$

Qn: How to find an integrating factor.

consider

$$M\,dx + N\,dy = 0$$

we are searching for a function $\mu(x,y)$ such that

$$\mu M\,dx + \mu N\,dy = 0$$

is exact.

ie $$\frac{\partial}{\;}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\partial y \qquad\qquad \partial x$$

ie

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y}$$

$$= \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

or

$$\frac{1}{\mu}\left(N \frac{\partial \mu}{\partial x} - M \frac{\partial \mu}{\partial y}\right)$$

$$= \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \qquad *$$

Suppose for instance that the given eqn has IF which is a function x-alone

$$\mu = \mu(x).$$

$$\begin{cases} \dfrac{\partial \mu}{\partial x} = \dfrac{d\mu}{dx} \\ \dfrac{\partial \mu}{\partial y} = 0 \end{cases}$$

$\therefore$ (*) reduces to

$$\frac{1}{\mu} N \frac{d\mu}{dx} = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}.$$

$$\frac{1}{\mu} \frac{d\mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \quad \text{---} \ (*)_2$$

$\therefore$ If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a

function of x - alone

then

$$\frac{d}{dx}\left(\log \mu\right) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$\log \mu = \int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$\mu = \exp \int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) dx$$

Thus if

(1) $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a

function of x - alone

then an IF is

given by

$$\mu = \exp \left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx \right)$$

Alternatively if μ is a function y-alone

$$\frac{\partial \mu}{\partial y} = \frac{d\mu}{dy}$$

$$\frac{\partial \mu}{\partial x} = 0$$

$$\mu = \exp \int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \, dy$$

if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y- alone

<u>Example</u> Solve

$$2 \sin(y^2)\, dx + xy \cos(y^2)\, dy = 0$$

$$y(2) = \sqrt{\pi/2}$$

Here $M = 2 \sin(y^2)$

$N = xy \cos(y^2)$

$$\frac{\partial M}{\partial y} = 4y \cos(y^2)$$

$$\frac{\partial N}{\partial x} = y \cos(y^2)$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

$\therefore$ DE is NOT exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$= \frac{4y \cos y^2 - y \cos y^2}{}$$

$$x \, y \quad \cos y^2$$

$= \dfrac{3}{x}$, which is a function of x – alone.

$$\therefore \quad IF = \exp \int \frac{3}{x} \, dx$$

$$= \exp \ln(x^3)$$

$$= x^3.$$

Multiplying th' out by IF we get

$$2x^3 \sin(y^2) \, dx + x^4 y \cos(y^2) \, dy = 0 \quad \text{---} \; (*)$$

$$M_{\text{new}} = 2x^3 \sin(y^2)$$

$$x^4 \, y \quad \cos(y^2)$$

$$N_{new} = x^4 y \cos(y^2)$$

$$\frac{\partial M}{\partial y} = 2x^3 \cos(y^2) \cdot 2y$$

$$= 4x^3 y \cos(y^2)$$

$$\frac{\partial N}{\partial x} = 4x^3 y \cos(y^2)$$

$\therefore$ (*) exact.

$\therefore \exists \; u(x, y)$ such that

$$\begin{cases} \textcircled{1} \quad \dfrac{\partial u}{\partial x} = M = 2x^3 \sin(y^2) \\ \textcircled{2} \quad \dfrac{\partial u}{\partial y} = N = x^4 y \cos(y^2) \end{cases}$$

Intregrating (1) w.r.t. x treating y as constant we obtain

$$u(x,y) = \frac{2x^4}{4} \sin(y^2) + g(y) \quad \text{--- } (3)$$

Diff (3) partially w.r.to y

$$\frac{\partial u}{\partial y} = \frac{x^4}{2} \cos(y^2) \cdot 2y + g'(y)$$

$$= x^4 y \cos y^2$$

$$\Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = c$$

$$\therefore u(x,y) = \frac{x^4}{2} \sin y^2 + c$$

Solution is

$$u(x,y) = K$$

$$\frac{x^4}{2} \sin y^2 = K'$$

$$y(2) = \sqrt{\pi/2}$$

$$\Rightarrow \frac{2^4}{2} \sin\left(\frac{\pi}{2}\right) = K'$$

$$\Rightarrow K' = 2^3.$$

Solution is

$$x^4 \sin(y^2) = 2^4$$

$\square$