Lecture - 5

Riemann Integration

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Absolute Convergence

The improper integral $\int_a^{\infty} f(x)\,dx$ is said to be $\boxed{\text{absolutely convergent}}$ if the integral $\int_a^{\infty} |f|(x)\,dx$ be $\boxed{\text{convergent}}$.

Theorem: If the integral $\int_a^{\infty} |f(x)|\,dx$ converges, then the integral $\int_a^{\infty} f(x)\,dx$ converges.

Proof: Note that,

$$0 \leq f(x) + |f(x)| \leq 2|f(x)|.$$

Now given that,

$$\int_a^{\infty} |f(x)|\,dx \text{ converges.} \quad (1)$$

So by Comparison test,

$$\int_a^\infty \left(f(x) + |f(x)|\right) dx \text{ converges.} \quad \text{——— } 2$$

Then for any $b > a$

$$\int_a^b f(x)\,dx = \int_a^b \left(f(x) + |f(x)|\right) dx - \int_a^b |f(x)|\,dx$$

Using 1, 2 and taking $b \to \infty$, we get

$$\int_a^\infty f(x)\,dx \text{ converges.}$$

$\boxed{\text{Proved}}$

$\boxed{\text{Converse is not true.}}$

Example

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

We have seen in the previous lecture that $\int_0^\infty \frac{\sin x}{x}\,dx$ is $\boxed{\text{convergent.}}$.

Now let's check.

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx.$$

Let

$$f(x) = \left|\frac{\sin x}{x}\right|, \quad x > 0$$

$$= 1, \quad x = 0$$

Then f is $\boxed{\text{continuous}}$ and hence $\boxed{\text{integrable}}$ on $[0, b]$ for all $b > 0$.

Therefore $\left|\frac{\sin x}{x}\right|$ is integrable on $[0, b]$, $\forall\, b > 0$.

We now consider,

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx, \qquad n \in \mathbb{N}.$$

$$= \int_0^{n\pi} \frac{|\sin x|}{x}\, dx$$

$$= \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx$$

We look into

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx$$

$\left[x = (r-1)\pi + u\right]$

$$= \int_0^{\pi} \frac{|\sin u|}{(r-1)\pi + u} du$$

$$= \int_0^{\pi} \frac{\sin u}{(r-1)\pi + u} du$$

For all $u \in [0, \pi]$,

$$(r-1)\pi + u \leq r\pi$$

Therefore,

$$\int_0^{\pi} \frac{\sin u}{(r-1)\pi + u} du \geq \frac{1}{r\pi} \int_0^{\pi} \sin u \, du$$

$$= \frac{2}{r\pi}$$

that is,

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx \geq \frac{2}{r\pi}.$$

Hence

$$\int_{0}^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \sum_{r=1}^{n} \frac{2}{r\pi}$$

$$= \frac{2}{\pi}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

Taking $n \to \infty$ we see that RHS is a divergent series.

Hence

$$\lim_{n\to\infty} \int_{0}^{n\pi} \left|\frac{\sin x}{x}\right| dx$$

is infinite.

Hence,
$$\int_0^\infty \left| \frac{\sin x}{x} \right| dx$$

diverges, that is,

$$\int_0^\infty \frac{\sin x}{x} dx$$ is

$\boxed{\text{not absolutely convergent.}}$

Improper Integral of 2nd kind

Definition

I) Convergence of the improper integral, $\int_a^b f(x)\,dx$ when 'a' is the only point of discontinuity.

Let f be defined on $(a, b]$ & $f \in R[a+\epsilon, b]$ $\forall\ \epsilon > 0$, $(0 < \epsilon < b-a)$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x)\,dx.$$

If the limit exists (finite) then

$$\int_a^b f(x)\,dx$$ converges. Otherwise diverges.

Example:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

'0' is the point of infinite discontinuity.

The integrand, $f(x) = \frac{1}{\sqrt{x}}$ is integrable on $[\epsilon, 1]$, $\forall \epsilon > 0$.

$$\lim_{\epsilon \to 0} \int_\epsilon^1 \frac{1}{\sqrt{x}}\,dx = 2 \ (<\infty)$$

$\therefore$ $$\int_0^1 \frac{dx}{\sqrt{x}} = 2,$$ converges.

II) Convergence of $\int_a^b f(x)\,dx$ when 'b' is the only pt of infinite discontinuity.

Let f be defined on $[a, b)$ & $f \in R[a, b-\epsilon]$, $\forall \epsilon > 0$, $(0 < \epsilon < b-a)$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_a^{b-\epsilon} f(x)\,dx.$$

If the limit exists (finite) then $\int_a^b f(x)\,dx$ converges, otherwise diverges.

Example

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

'1' is the point of infinite discontinuity.

Then

$$\lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \lim_{\epsilon \to 0} \left[\sin^{-1}(1-\epsilon)\right] = \pi/2 \; (<\infty)$$

Hence $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is convergent

and $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/2$.

III) Convergence of the improper integral $\int_a^b f(x)\,dx$ when 'a' & 'b' are only points of infinite discontinuities.

Let f be defined on (a,b) & $f \in \mathcal{R}[a+\epsilon, b-\epsilon']$, $\forall \epsilon, \epsilon' > 0$ satisfying $0 < \epsilon < b-a$, $0 < \epsilon' < b-a$.

Let $c \in (a,b)$

If $\int_a^c f(x)\,dx$ & $\int_c^b f(x)\,dx$ both are convergent then $\int_a^b f(x)\,dx$ converges &,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example: $\displaystyle\int_0^2 \frac{dx}{\sqrt{x(2-x)}}$

'0' & '2' are points of infinite discontinuity.

Now, $0<\epsilon<2$, $0<\epsilon'<2$ &

$$\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{\sqrt{x(2-x)}}\,dx = \pi/2$$

$$\lim_{\epsilon'\to 0}\int_{1}^{2-\epsilon'}\frac{dx}{\sqrt{x(2-x)}} = \pi/2$$

Hence $\displaystyle\int_0^2 \frac{dx}{\sqrt{x(2-x)}}$ converges & equal to π.

$\boxed{\text{Note}}$ Suppose $a_1, a_2, \ldots, a_n$ are finitely many discontinuities of $f(x)$ in $[a, b]$. Then

$$\int_a^b f(x)\,dx = \int_a^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx + \ldots + \int_{a_n}^{b} f(x)\,dx$$

If all the improper integrals on the right converge, then we say the improper integral of f over $[a,b]$ converges. Otherwise we say it diverges.

Tests of Convergence:

Comparison Test :

Suppose $0 \le \phi(x) \le f(x)$ for all $x \in [a, c)$ and are discontinuous at c.

1. If $\int_a^c f(x)dx$ converges then

$$\int_a^c \varphi(x)\, dx \text{ converges.}$$

2. $\int_a^c \varphi(x)\,dx$ diverges then $\int_a^c f(x)\,dx$ diverges.

Example:

1.) $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$,

$e^x < e$, $x \in (0,1)$

$\Rightarrow \frac{e^x}{\sqrt{x}} < \frac{e}{\sqrt{x}}$ & $\int_0^1 \frac{dx}{\sqrt{x}}$ coverges.

By, C T. $\int_0^1 \frac{e^x}{\sqrt{x}}$ converges.

2) $\int_0^1 \frac{1}{x^{1/3}(1+x)}\,dx \longrightarrow$ converges by C.T.

$$\frac{1}{x^{1/3}(1+x)} \leq \frac{1}{x^{1/3}}, \quad x \in (0,1)$$

& $\int_0^1 \frac{1}{x^{1/3}}$ converges (Check)

Limit Comparison Test :

Suppose $0 < f(x), g(x)$ be continuous in $[a, c)$ and $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = L$.

1) $\boxed{0 < L < \infty}$.

Then $\int_a^c f(x)\,dx$ and $\int_a^c g(x)\,dx$ converge or diverge together.

2) $\boxed{L = 0}$

$\int_a^c g(x)\,dx$ converges $\Longrightarrow$ $\int_a^c f(x)\,dx$ converges.

3) $\boxed{L = \infty}$

$\int_a^c g(x)\,dx$ diverges $\Longrightarrow$

$\int_a^c f(x)\,dx$ diverges.

Example

$$\int_0^{\pi/2} \log \sin x \, dx$$

Let, $f(x) = \log \sin x$, $x \in (0, \pi/2]$

- '0' is the pt of infinite discontinuity.

- $-f(x) \geq 0$, $\forall x \in [0, \pi/2]$ (Check)

We know,

$$\lim_{x\to 0^+} \sqrt{x}\log x = 0 \quad -①$$

&

$$\lim_{x\to 0^+} \sqrt{x}\log\left(\frac{\sin x}{x}\right) = 0 \quad -②$$

Ex.

Combining ① & ②

$$\lim_{x\to 0^+} \sqrt{x}\left(\log x + \log\frac{\sin x}{x}\right) = 0$$

$$\Rightarrow \boxed{\lim_{x\to 0^+} \sqrt{x}\log\sin x = 0} \quad -③$$

Let $g(x) = \dfrac{1}{\sqrt{x}}$, $x \in (0, \pi/2]$

then $g(x) > 0$.

and,

$$\lim_{x \to 0^+} \frac{-f(x)}{g(x)}$$

$$= \lim_{x \to 0^+} \frac{-\log \sin x}{\frac{1}{\sqrt{x}}}$$

$$= \lim_{x \to 0^+} -\left(\sqrt{x} \log \sin x\right)$$

$$= 0$$

Also we know,

$$\int_0^{\pi/2} g(x)\,dx \text{ is convergent.}$$

So then by LCT we can conclude,

$$\int_0^{\pi/2} -f(x)\,dx \text{ converges,}$$

ie, $$\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \log \sin x\,dx$$

converges.

Beta & Gamma Functions

Consider the $\boxed{\text{Beta function}}$

for $p > 0$, $q > 0$,

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx.$$

- $\boxed{p \geq 1,\ q \geq 1}$

$$f(x) = x^{p-1}(1-x)^{q-1}$$

continuous in $[0,1]$ &

$$\int_0^1 f(x)\,dx \quad \text{exists.}$$

- $\boxed{p<1 \quad , \quad q \geq 1}$

'0' is the only point of infinite discontinuity.

- $\boxed{q<1, \quad p \geq 1}$

'1' is the only point of infinite discontinuity.

Consider,

$$I_1 = \int_0^{1/2} x^{p-1} (1-x)^{q-1} \, dx.$$

$$\& \quad I_2 = \int_{1/2}^{1} x^{p-1} (1-x)^{q-1} \, dx$$

when $\boxed{p<1 \ \& \ (\text{or}) \ q<1.}$

Check the convergence of I_1

We take, $f(x) = x^{p-1}(1-x)^{q-1}$, $p<1$

& $g(x) = x^{p-1}$

then $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = 1$

& $\int_0^{1/2} x^{p-1}\,dx$ converges,

then, I_1 converges by CT.

Check the convergence for I_2.

Let $f(x) = x^{p-1}(1-x)^{q-1}$, $q<1$.

We take $g(x) = (1-x)^{q-1}$.

By LCT (again) we can conclude I_2 converges.

Some identities

1) $\beta(m, n) = \beta(n, m)$

Substitute $t = 1-x$ in the defn of $\beta(m,n)$.

2) $$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Take $x = \sin^2\theta$ in $\beta(m,n)$

$$\beta(m,n) = \int_0^{\pi} \cos^{2m-2}\theta \sin^{2n-2}\theta \cos\theta \sin\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta$$

Gamma function

For $p>0$,

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} \, dx$$

We consider,

$$I_1 = \int_0^1 x^{p-1} e^{-x} \, dx$$

$$I_2 = \int_1^\infty x^{p-1} e^{-x} \, dx.$$

We take, $f(x) = x^{p-1} e^{-x}$

Convergence at 0 $(p<1)$

$$f(x) > 0 \quad , \forall x \in (0,1]$$

Let

$$g(x) = x^{p-1} \quad , x \in (0,1]$$

Then $g(x) > 0$, $\forall x \in (0,1]$.

& , $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = 1 \quad (<\infty)$

Moreover,

$\displaystyle\int_0^1 g(x)\,dx$ is convergent.

Then by LCT we have

$\displaystyle\int_0^1 f(x)\,dx$ converges.

Convergence at ∞

$f(x) = x^{p-1} e^{-x}$.

Let $g(x) = \frac{1}{x^2}$

Then, $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = \lim\limits_{x \to \infty} x^{2+p-1} e^{-x}$

$= \lim\limits_{x \to \infty} x^{p+1} e^{-x}$

$= 0$

& also, $\int_1^\infty \frac{dx}{x^2}$ converges.

By LCT,

$\int_1^\infty f(x)\, dx$ converges.

Some identities

1) $\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$

2) $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

(Ex)

3) $\Gamma(1/2) = \sqrt{\pi}$ (Done later)

4) $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ [Believe Now prove Later!!]