Multiple Integral - lecture 2

Some properties of double integrals

1. If f and g are two cont^n functions on a bounded domain Ω, then

$$\iint_\Omega (f+g)\,dA = \iint_\Omega f\,dA + \iint_\Omega g\,dA$$

2. $$\iint_\Omega cf\,dA = c\iint_\Omega f\,dA,$$ c is a constant.

(3) If Ω splits into two separate domains Ω_1 and Ω_2. Then

$$\iint_{\Omega} f \, dA = \iint_{\Omega_1} f \, dA + \iint_{\Omega_2} f \, dA$$

(4) If $f(x,y) = h(x)\, g(y)$. Then

$$\iint_{\Omega} f \, dA = \left(\int_X h(x)\, dx \right) \left(\int_Y g(y)\, dy \right)$$

5

Suppose $f(x,y) = 1$. Then

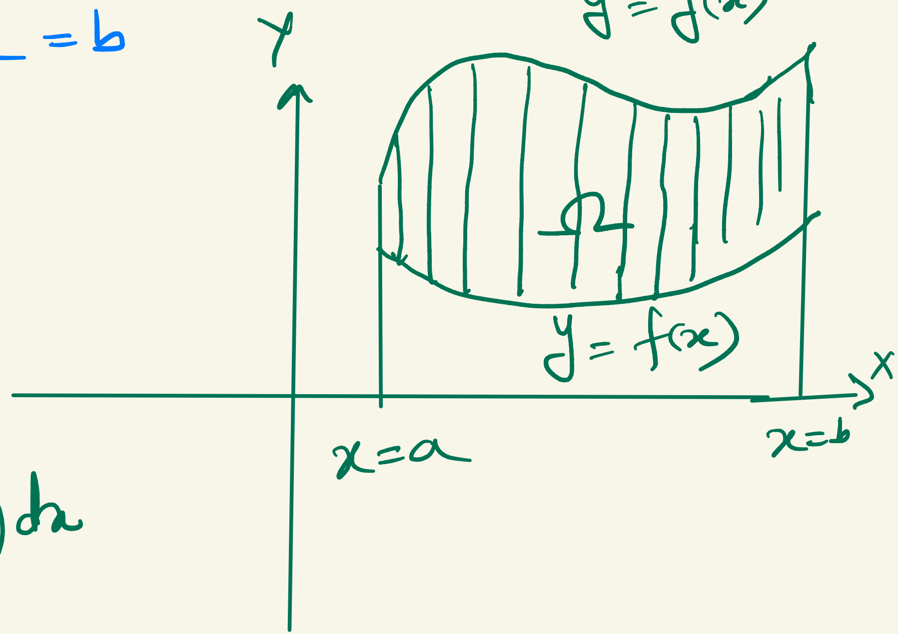
$$\iint_{\Omega} f \, dA = \text{Area}(\Omega).$$

$$\iint_{\Omega} f \, dA \simeq \sum_{k} f(x_k, y_k) \, |\Omega_k|$$

$$= \sum_{k} |\Omega_k| = |\Omega| = \text{Area}.$$

⑥ Area b/w the two curves.

ⓐ Suppose we need to calculate the area b/w the two curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$



$$\text{Area}(\Omega) = \iint_{\Omega} dA$$

$$= \int_a^b \left(\int_{f(x)}^{g(x)} dy \right) dx$$

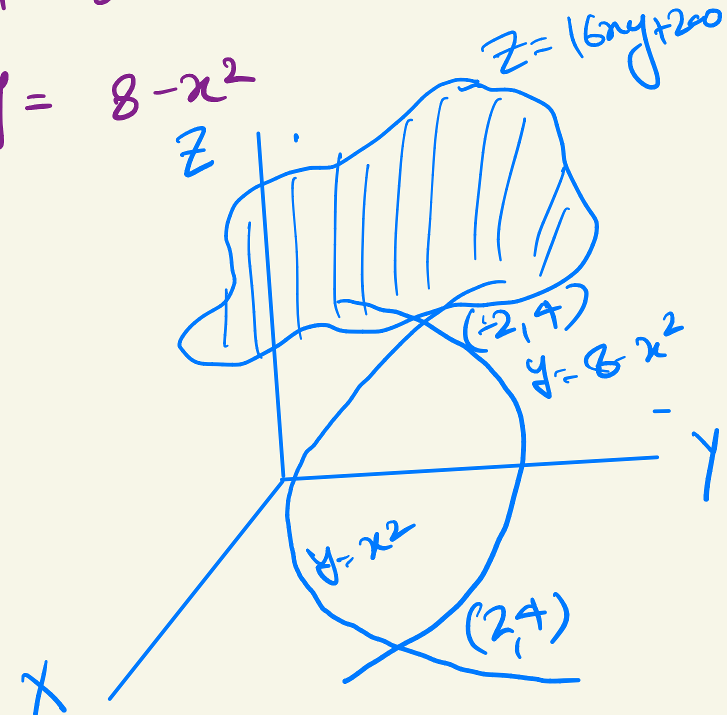
$$\boxed{\text{Area} = \int_a^b \left(g(x) - f(x) \right) dx} \quad \text{———} \circledast$$

If curves are given by eq^n $x = f(y)$ and $x = g(y)$ b/w $y = c$ to $y = d$. Then

$$\boxed{\text{Area} = \int_c^d \left(g(y) - f(y) \right) dy} \quad \text{———} \circledast$$

7 To find the volume of solid.

Example 1 Find volume of the solid below the surface $z = 16xy + 200$ and above the domain Ω bounded by $y = x^2$ and $y = 8 - x^2$



Sol^n.

$$x^2 = 8 - x^2$$
$$x^2 = 4$$
$$x = \pm 2$$
$$y = 4$$

domain Ω is

$$-2 \le x \le 2$$

$$x^2 \le y \le 8 - x^2$$

$$\text{Volume} = \int_{-2}^{2} \left(\int_{x^2}^{8-x^2} (16xy + 200)\, dy \right) dx$$

$$= \text{compute.}$$

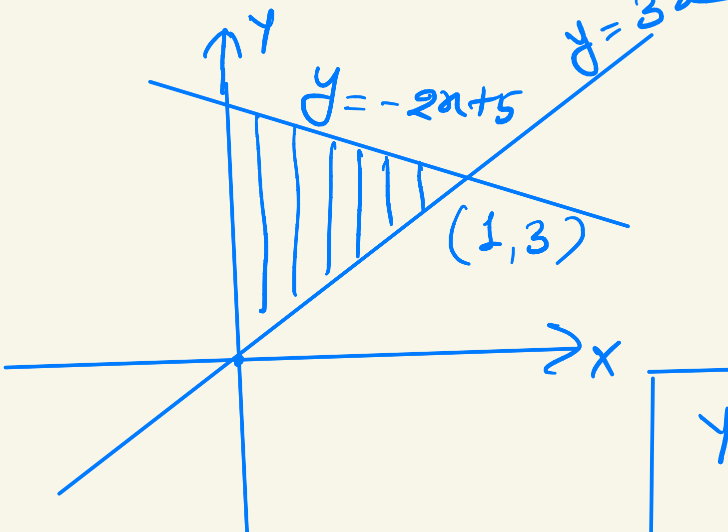
Example (2) Find volume of solid enclosed by the planes $4x + 2y + z = 10$, $y = 3x$, $z = 0$ and $x = 0$.

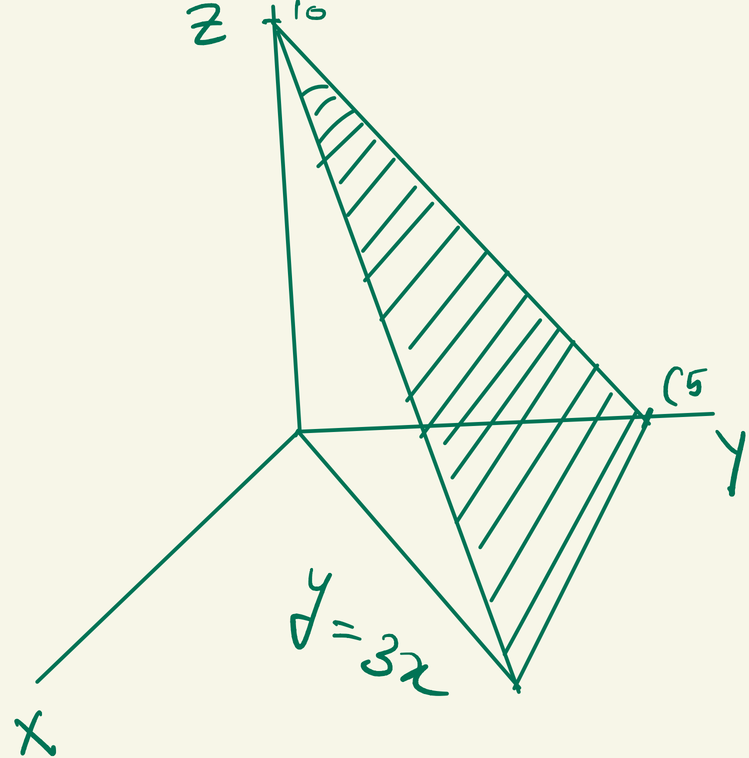
Sol^n. $4x + 2y + z = 10$ — (*)

Ω is given by

$y = 3x$

$4x + 2y = 10$





Y- regular form

$0 \leq x \leq 1$

$3x \leq y \leq -2x+5$

$$\iint_{\Omega} (10-4x-2y)\,dA = \int_0^1 \left(\int_{3x}^{-2x+5} (10-4x-2y)\,dy \right) dx$$

$$= \text{compute.}$$

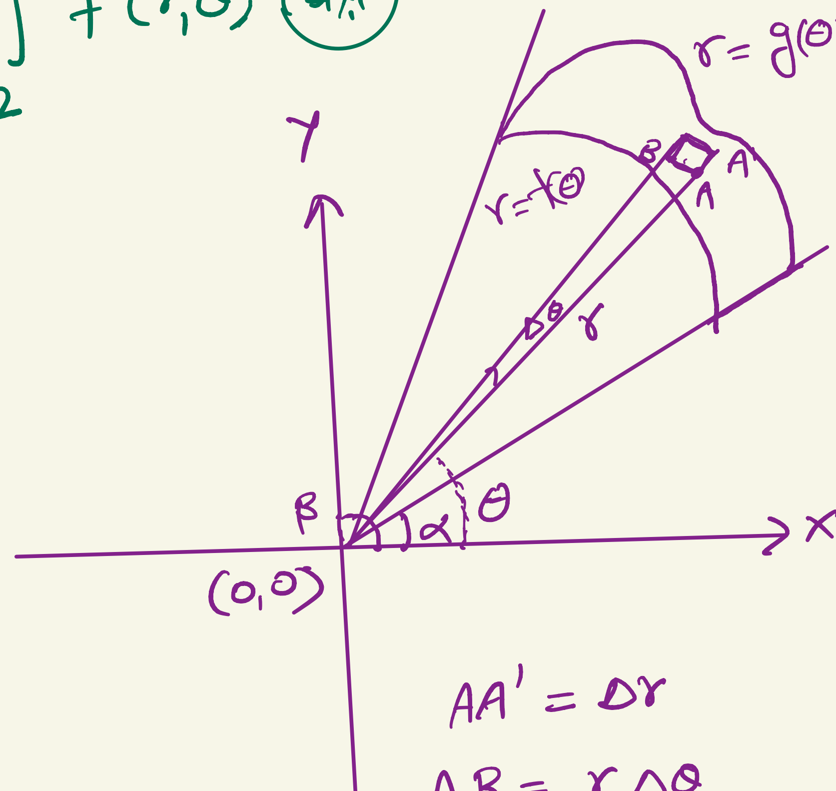
Exercise ① Find the volume of the solid under the paraboloid $z = x^2+y^2$ and over the bounded domain $y=x$, $x=0$ and $x+y=2$.

Exercise ② Find the volume of the solid below the surface $z = x^2$ and above the plane region $y=x$ and $y = 2-x^2$.

Double Integral in Polar Co-ordinates

$$\iint_{\Omega} f(x,y)\, dA = \iint_{\Omega} f(r,\theta)\, dA \to ???$$

$$dA \Downarrow (dx\,dy)$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$AA' = \Delta r$$
$$AB = r\,\Delta\theta$$

$$\Delta A = (\Delta r)(r\,\Delta\theta)$$

$$dA = (dr)(r\,d\theta)$$

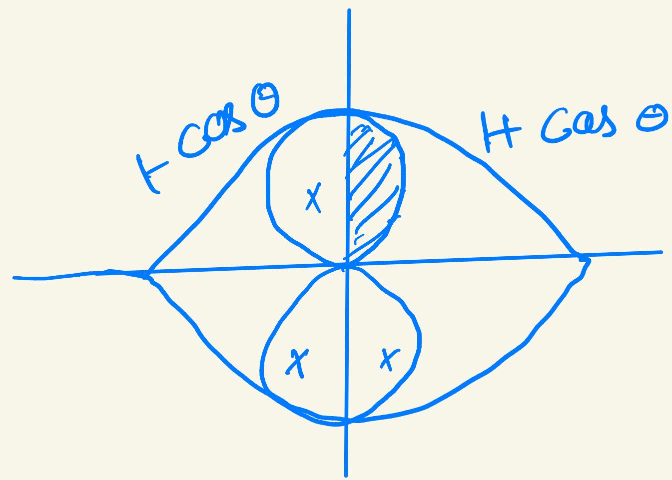
$$dx\,dy = r\,dr\,d\theta$$

$$\boxed{\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Omega} f(r,\theta)\, r\,dr\,d\theta}$$

Example ① Find the area common to the curves $r = 1+\cos\theta$ and $r = 1-\cos\theta$.

Sol^n.

$$\iint_{\Omega} dA = \iint_{\Omega} r\,dr\,d\theta$$



I^{st} quadrant

$0 \le \theta \le \pi/2$

$0 \le r \le 1-\cos\theta$

$$\text{Area} = 4 \int_0^{\pi/2} \left(\int_0^{r\cos\theta} r\,dr \right) d\theta$$

$$= \text{compute.}$$

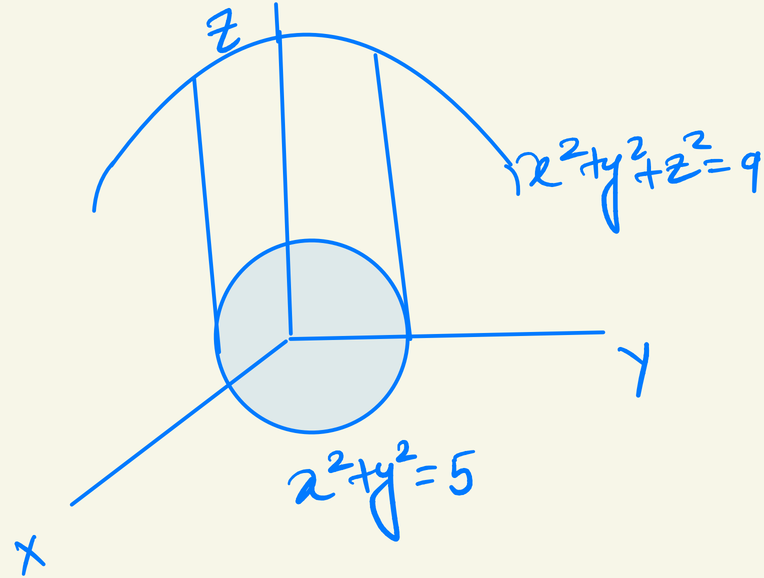
Example (2) Find the volume of the solid lies under the sphere $x^2+y^2+z^2=9$ and above the plane $z=0$ and inside the cylinder $x^2+y^2=5$.

Soln. $z=0$

XY – plane

domain Ω is given by

$$0 \le \theta \le 2\pi$$
$$0 \le r \le \sqrt{5}$$



$$z = \sqrt{9-x^2-y^2}$$
$$= \sqrt{9-r^2}$$

$$\iint f(x,y)\,dx\,dy = \int_0^{2\pi}\left(\int_0^{\sqrt{5}}\left(\sqrt{9-r^2}\right) r\,dr\right) d\theta$$

$=$ compute.

Problem ① Compute $\Gamma(\frac{1}{2})$ by using double integral.

Sol^n

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1}\, dx, \quad p>0 \quad \text{——— eq}^n \ (*)$$

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{-\frac{1}{2}}\, dx$$

Take $x = y^2$, then $dx = 2y\, dy$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-y^2} \frac{1}{y}\, 2y\, dy$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\,dy \Rightarrow \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-y^2}\,dy$$

— eqn ①

$$I = \int_0^\infty e^{-y^2}\,dy.$$

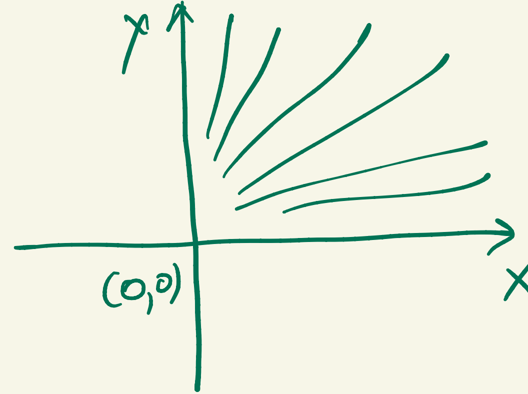
$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

$$= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy \longrightarrow ⊛.$$

In polar co-ordinate

$$0 \le \theta \le \pi/2$$

$$0 \le r < \infty$$



$$I^2 = \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} \, r \, dr \, d\theta$$

$$r^2 = t$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^{\infty} e^{-t} \, dt \, d\theta$$

$$I^2 = \frac{1}{2} \cdot \frac{\pi}{2} \left[-e^{-t} \right]_0^{\infty} = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2} \implies \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \implies \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.}$$

Problem 2 $\qquad \beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

Sol^n.

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1}\, dx$$

$$x = y^2 \Rightarrow dx = 2y\,dy$$

$$\Gamma(m) = 2\int_0^\infty e^{-y^2} y^{2m-1}\, dy$$

$$\boxed{\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx} \quad \text{— eq}^n \text{ 1}$$

$$\Gamma(m)\,\Gamma(n) = \left(2\int_0^\infty e^{-x^2} x^{2m-1}\,dx\right)\left(2\int_0^\infty e^{-y^2} y^{2n-1}\,dy\right)$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, x^{2m-1}\, y^{2n-1}\, dx\,dy$$

$$x = r\cos\theta \quad \& \quad y = r\sin\theta$$

$$\Gamma(m)\Gamma(n) = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r^{2(m+n)-1} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, dr\,d\theta$$

$$= \left(2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\,dr\right)\left(2\int_0^{\pi/2} \cos^{2m-1}\theta \cdot \sin^{2n-1}\theta\, d\theta\right)$$

Note that

$$2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\,dr = \Gamma(m+n)$$

and

$$\beta(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta \,\sin^{2n-1}\theta \,d\theta$$

Hence

$$\Gamma(m)\,\Gamma(n) = \Gamma(m+n)\,\beta(m,n)$$

$$\Rightarrow \quad \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$