Lecture 16
Signals and Systems (ELL205)

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Towards Fourier Series

Analysis and Synthesis equation

Synthesis

$$x(t) = x(t+T) = \sum_{k} a_k e^{jk\omega_o t}$$

Analysis

$$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\omega_o t}dt$$

Analysis and Synthesis equation

Synthesis $$x(t) = x(t+T) = \sum_{k} a_k e^{jk\omega_o t}$$

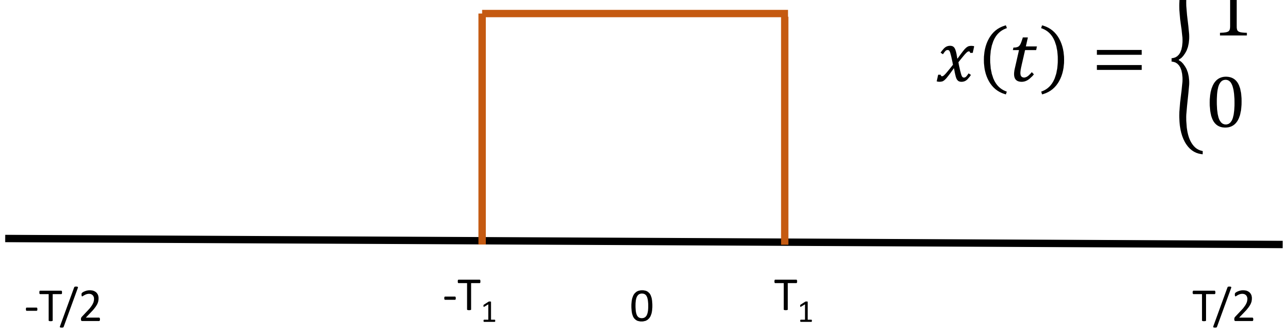
Analysis $$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_o t}dt$$

$x(t)$ real implies $a_k = \overline{a_{-k}}$

(Rectangular) $a_k = B_k + jC_k$ $$x(t) = a_o + \sum_{k=1}^{\infty}(B_k cos k\omega_o t - C_k sin k\omega_o t)$$

(Polar) $a_k = A_k e^{j\theta_k}$ $$x(t) = a_o + \sum_{k=1}^{\infty} A_k cos(k\omega_o t + \theta_k)$$

Question



$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| \le T \end{cases}$$

How many of the following statements are correct?

1. a_k are real and even	2. a_k are imaginary and odd
3. a_k is 0 if k is even but not zero and duty cycle is 50%	4. a_k decreases as k increases

Obs. 1: Even signal has even coefficients

If $x(t) \leftrightarrow a_k$ then $x(-t) \leftrightarrow a_{-k}$

Proof:

$$x(t) = \sum_k a_k e^{jk\omega_o t}$$

$$x(-t) = \sum_k a_k e^{-jk\omega_o t} = \sum_{k'} a_{-k'} e^{jk'\omega_o t} = \sum_k a_{-k} e^{jk\omega_o t}$$

Obs. 2 Real & Even together

If $x(t) \leftrightarrow a_k$ then $x(-t) \leftrightarrow a_{-k}$ & $\overline{x(t)} \leftrightarrow \overline{a_{-k}}$

Even: $a_k = a_{-k}$

Real: $a_k = \overline{a_{-k}}$

Real & Even: $a_k = \overline{a_{-k}} = \overline{a_k}$

Obs. 2 Real & Even together

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Even: $a_k = a_{-k}$

Real: $a_k = \overline{a_{-k}}$

Real & Even: $a_k = \overline{a_{-k}} = \overline{a_k}$

Signal	Coefficients
Real & Even	Real & Even

Obs. 2 Real & Even together

If $x(t) \leftrightarrow a_k$ then $x(-t) \leftrightarrow a_{-k}$ & $\overline{x(t)} \leftrightarrow \overline{a_{-k}}$

Odd: $a_k = -a_{-k}$

Real: $a_k = \overline{a_{-k}}$

Real & Odd: $a_k = \overline{a_{-k}} = -\overline{a_k}$

Signal	Coefficients
Real & Even	Real & Even
Real & Odd	Imaginary & Odd

Obs. 2 Real & Even together

If $x(t) \leftrightarrow a_k$ then $x(-t) \leftrightarrow a_{-k}$ & $\overline{x(t)} \leftrightarrow \overline{a_{-k}}$

Even: $a_k = a_{-k}$

Imag.: $a_k = -\overline{a_{-k}}$

Imag. & Even: $a_k = -\overline{a_{-k}} = -\overline{a_k}$

Signal	Coefficients
Real & Even	Real & Even
Real & Odd	Imaginary & Odd
Imaginary & Even	Imaginary & Even

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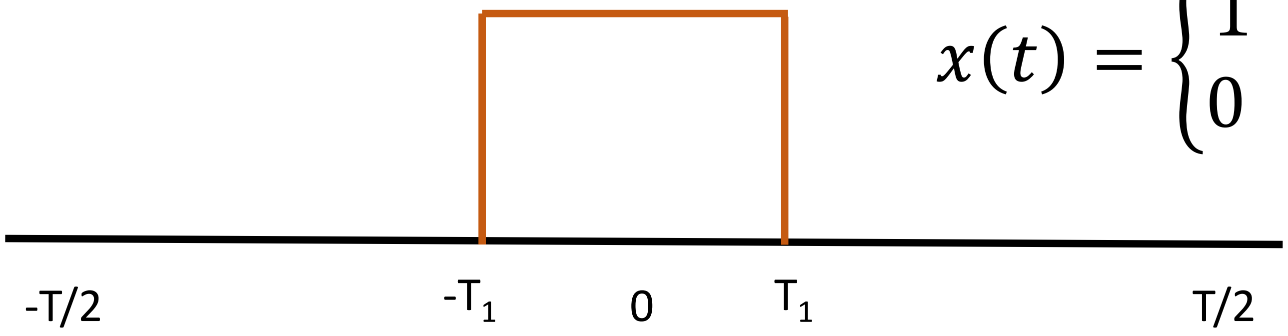
Signal	Coefficients
Real & Even	Real & Even
Real & Odd	Imaginary & Odd
Imaginary & Even	Imaginary & Even
Imaginary & Odd	Real & Odd

Question



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How many of the following statements are correct?

1. a_k are real and even ✓	2. a_k are imaginary and odd $\boxed{\times}$
3. a_k is 0 if k is even but not zero and duty cycle is 50%	4. a_k decreases as k increases

Working out the mechanics

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_o t}\,dt$$

$$a_k = \frac{1}{-jk\omega_o T}\left[e^{-jk\omega_o t}\right]_{-T_1}^{T_1}$$

$$a_k = \frac{1}{-jk\omega_o T}\left[e^{-jk\omega_o T_1} - e^{jk\omega_o T_1}\right]$$

$$a_k = \frac{1}{jk\omega_o T}\left[e^{jk\omega_o T_1} - e^{-jk\omega_o T_1}\right]$$

Working out the mechanics

$$a_k = \frac{1}{jk\omega_o T}\left[e^{jk\omega_o T_1} - e^{-jk\omega_o T_1}\right]$$

$$= \frac{2}{k\omega_o T}\sin(k\omega_o T_1)$$

$$= \frac{2T_1}{T}\frac{\sin(k\omega_o T_1)}{k\omega_o T_1}$$

$$= \frac{2T_1}{T}\text{sinc}(k\omega_o T_1)$$

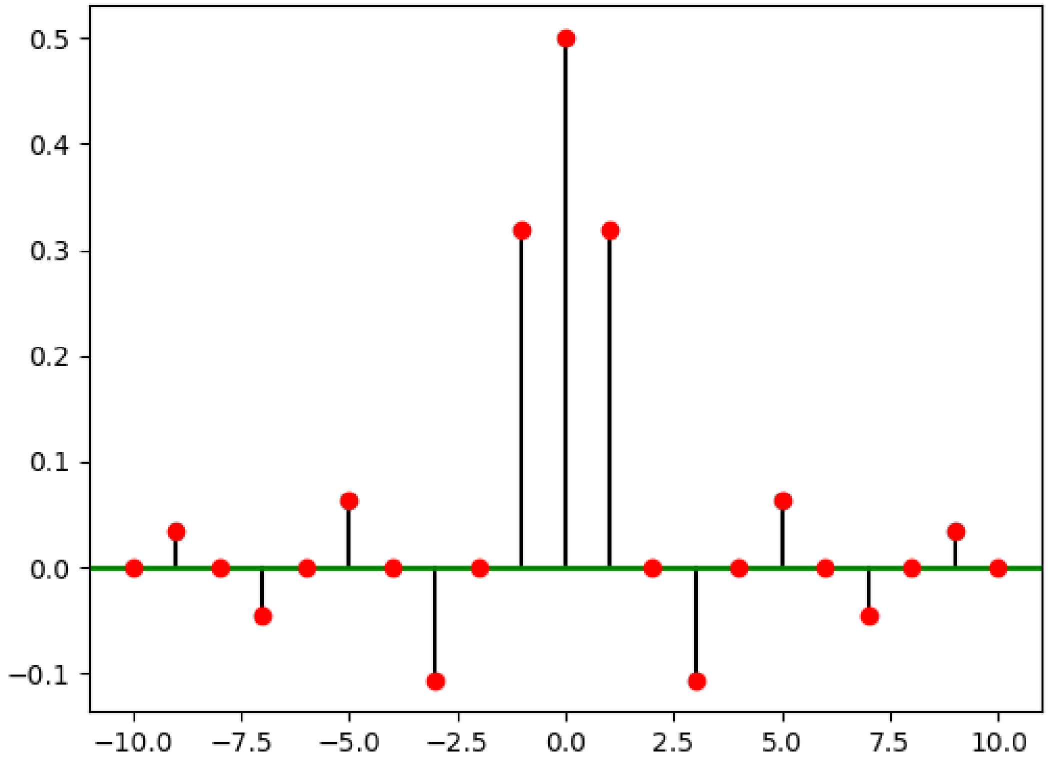
where $\text{sinc}(\theta) = \frac{sin\theta}{\theta}$

$$= D\text{sinc}(k\pi D)$$

where $D = \frac{2T_1}{T}$

Working out the mechanics

$$a_k = \frac{1}{2}\,\mathrm{sinc}(k\pi/2) \qquad \text{where } \mathrm{sinc}(\theta) = \frac{sin\theta}{\theta}$$



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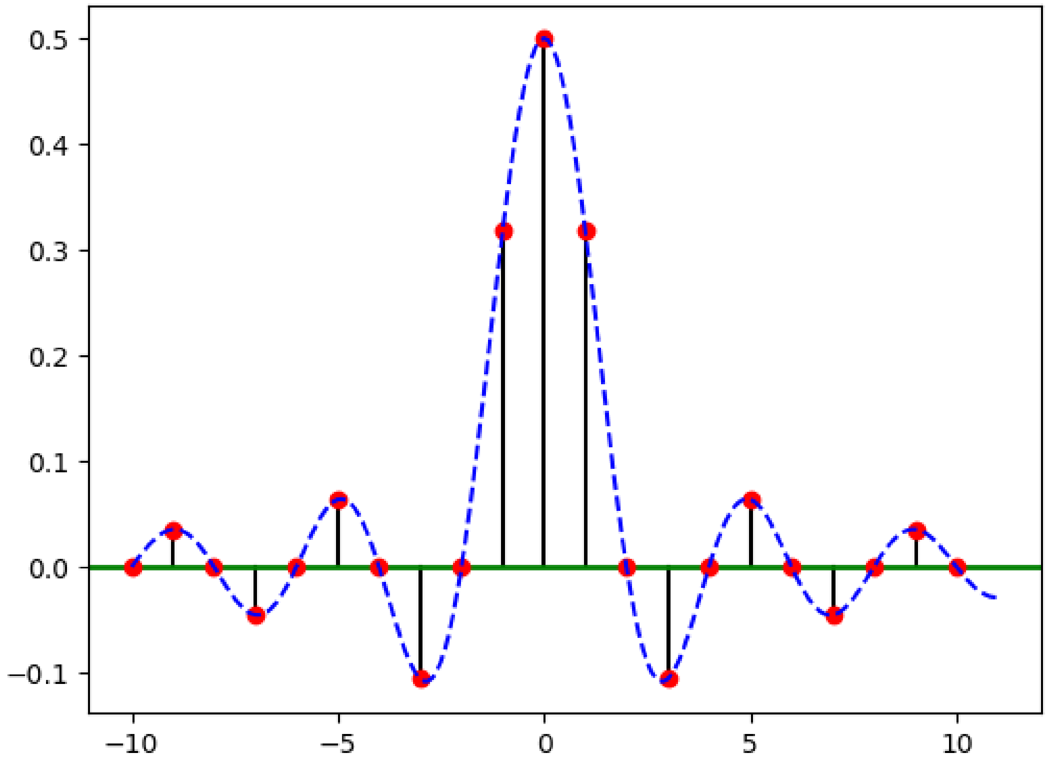
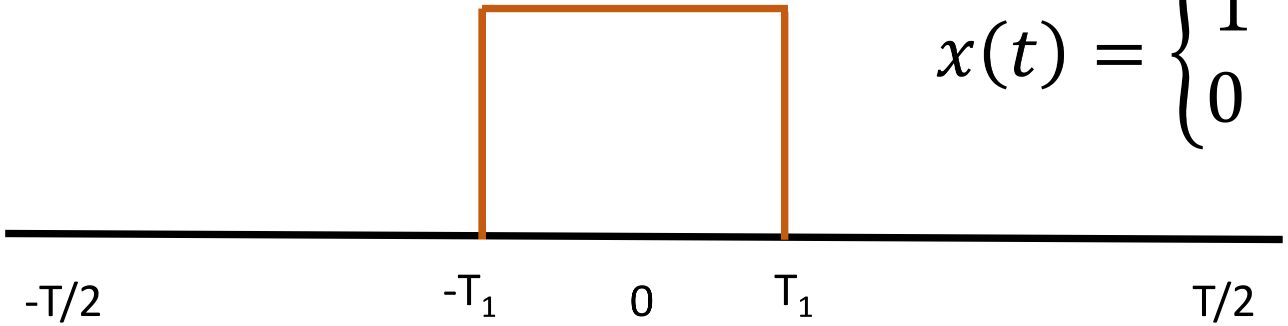


Question



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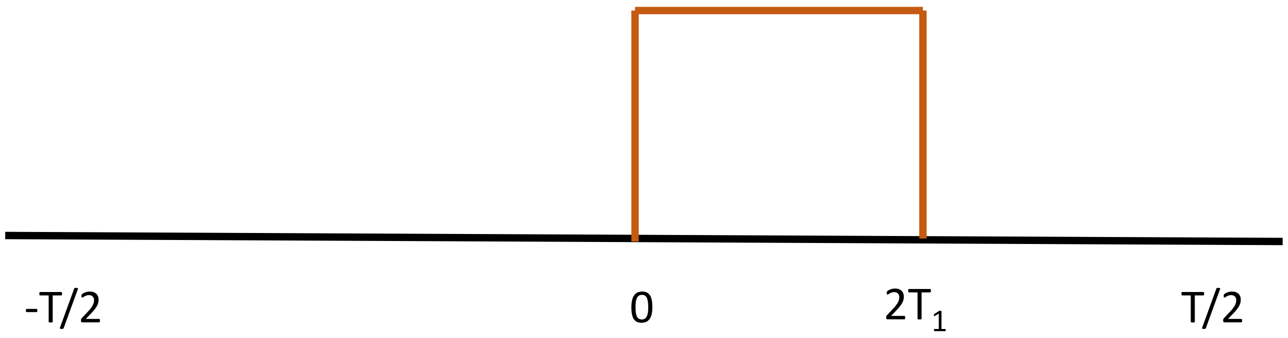


The signal has FS coefficients as b_k. How many of the following statements are correct?

1. b_k have the same magnitude as a_k	2. b_k have the same phase as a_k
3. $b_k = e^{-jk\omega_o T_1} a_k$	4. $b_k = e^{jk\omega_o T_1} a_k$

a_k are the Fourier series coefficients of the previous signal.

Time-shifting

If $x(t) \leftrightarrow a_k$ then $x(t - t_o) \leftrightarrow e^{-jk\omega_o t_o} a_k$

Time-shifting

If $x(t) \leftrightarrow a_k$ then $x(t-t_o) \leftrightarrow e^{-jk\omega_o t_o} a_k$

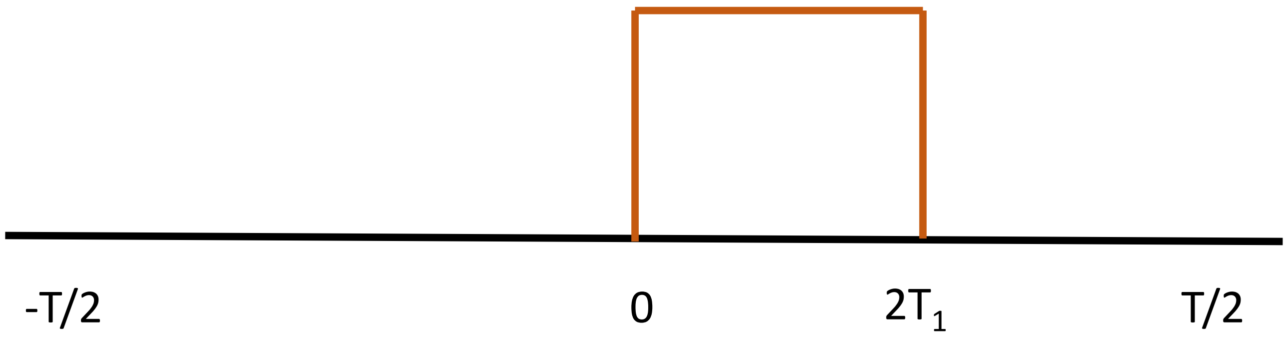
Proof:

Starting from $b_k = \frac{1}{T}\int_T x(t-t_o)e^{-jk\omega_o t}dt$ & changing variable of integration as $t - t_o = \lambda$

we get $$b_k = \frac{1}{T}\int_T x(\lambda)e^{-jk\omega_o(\lambda+t_o)}\,d\lambda$$

On simplification, we get $$b_k = e^{-jk\omega_o t_o} a_k$$

Question



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a_k are the Fourier series coefficients of the previous signal.

List of Properties

1. $x(t) \leftrightarrow a_k$

2. $x(-t) \leftrightarrow a_{-k}$

3. $\overline{x(t)} \leftrightarrow \overline{a_{-k}}$

4. $x(t - t_o) \leftrightarrow e^{-jk\omega_o t_o} a_k$

Signal	Coefficients
Real & Even	Real & Even
Real & Odd	Imaginary & Odd
Imaginary & Even	Imaginary & Even
Imaginary & Odd	Real & Odd

List of Properties

1. $x(t) \leftrightarrow a_k$
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Signal	Coefficients
Real & Even	Real & Even
Real & Odd	Imaginary & Odd
Imaginary & Even	Imaginary & Even
Imaginary & Odd	Real & Odd
Shifted signal	Only phase changes

Convergence of Fourier Series

$$x_N(\mathrm{t}) \triangleq \sum_{k=-N}^{N} a_k e^{jk\omega_o t}$$

$$e_N(\mathrm{t}) \triangleq x(t) - x_N(\mathrm{t})$$

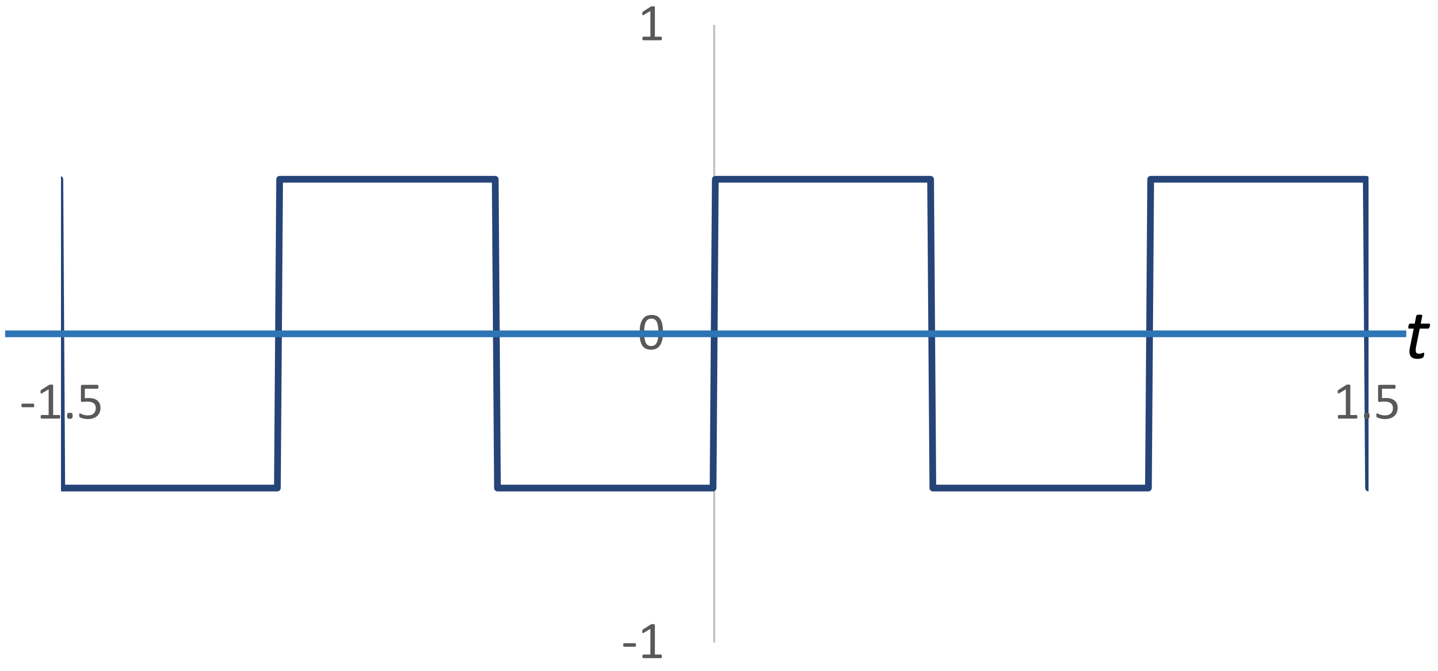
Does $e_N(\mathrm{t})$ decrease as *N* increases?

Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=0}^{0} \frac{1}{j\pi k} e^{jk2\pi t}$$

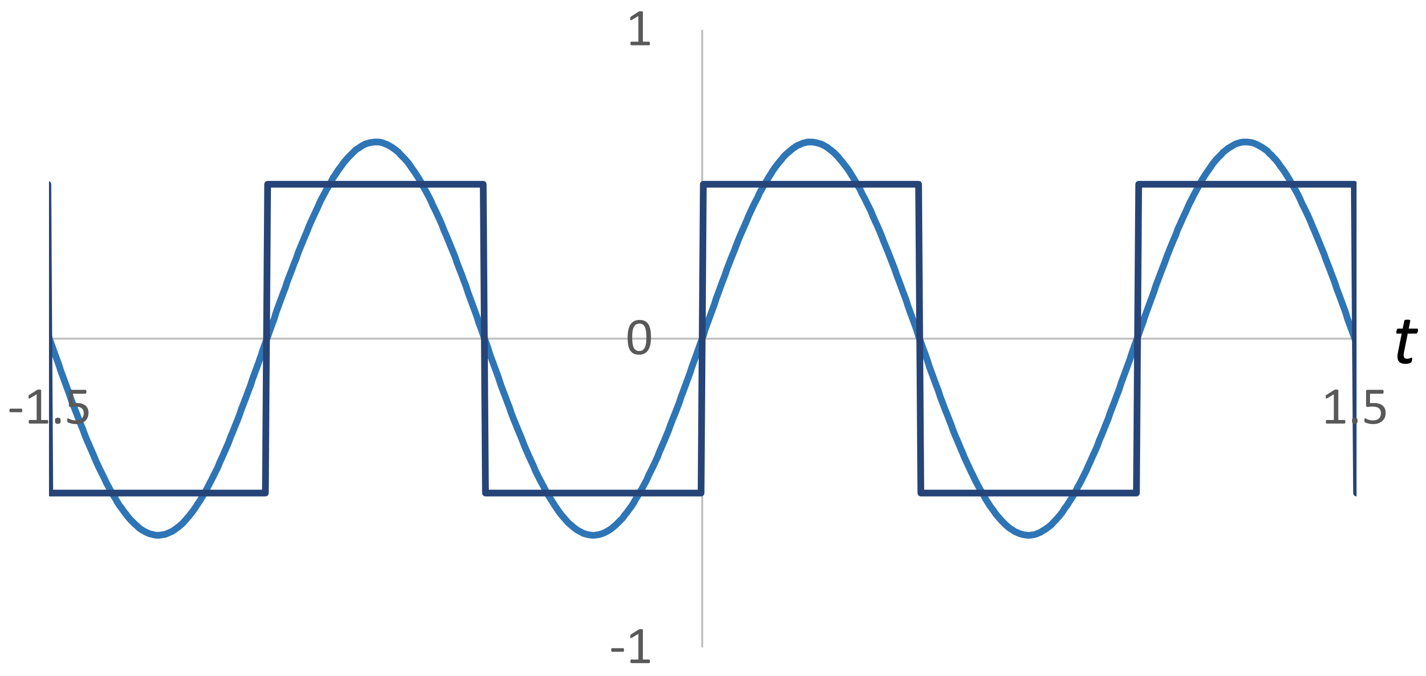


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-1}^{1} \frac{1}{j\pi k} e^{jk2\pi t}$$

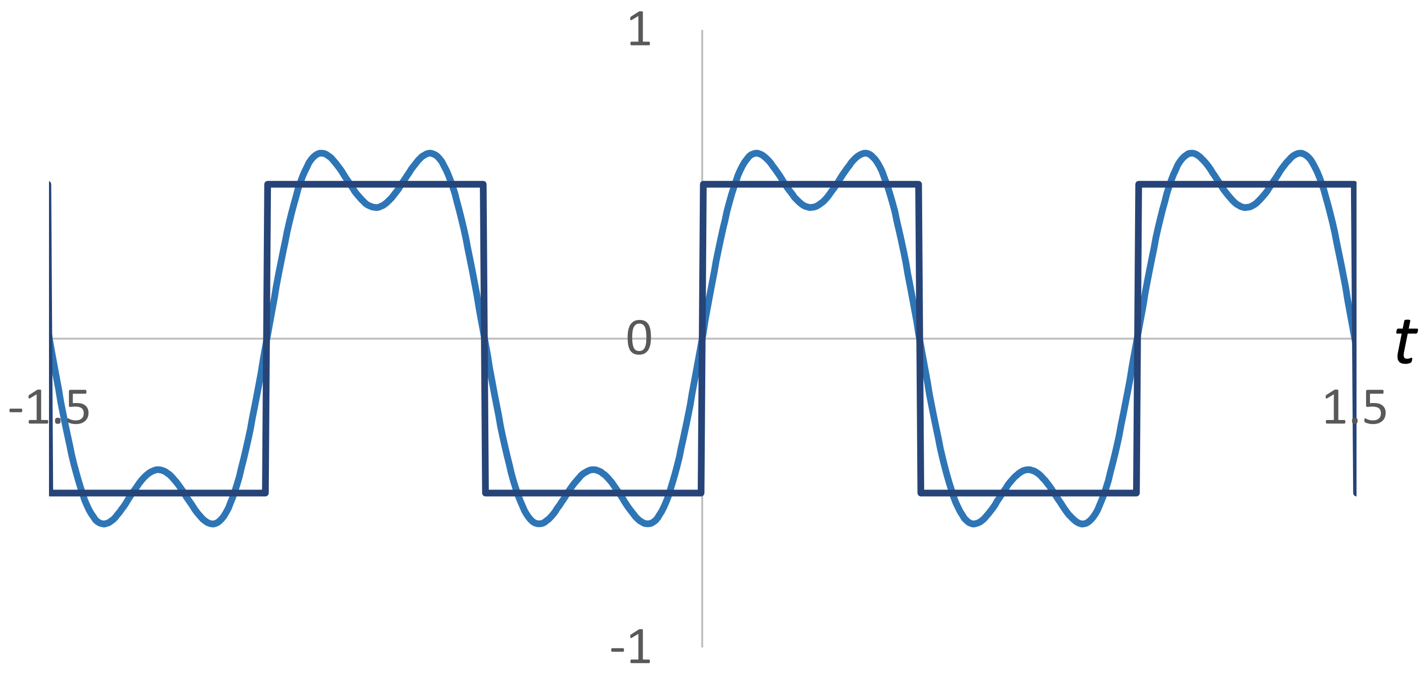


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-3}^{3} \frac{1}{j\pi k} e^{jk2\pi t}$$

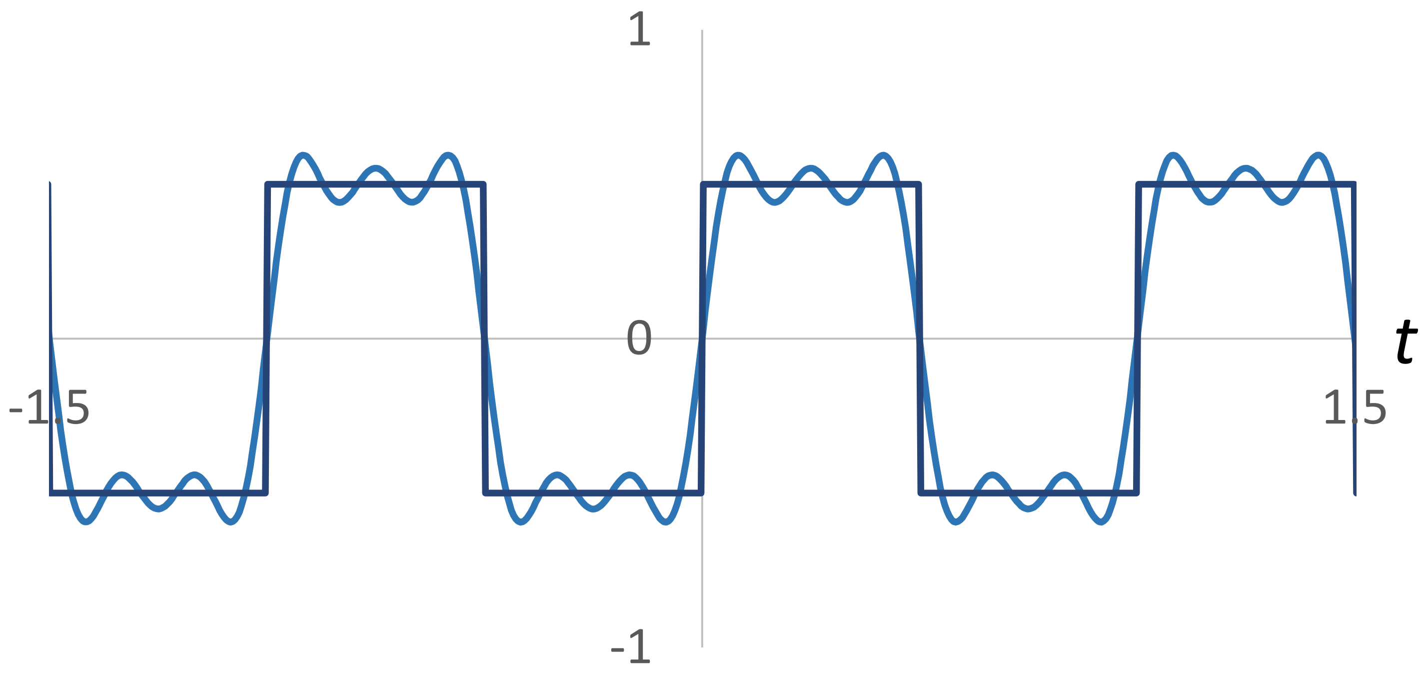


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-5}^{5} \frac{1}{j\pi k} e^{jk2\pi t}$$

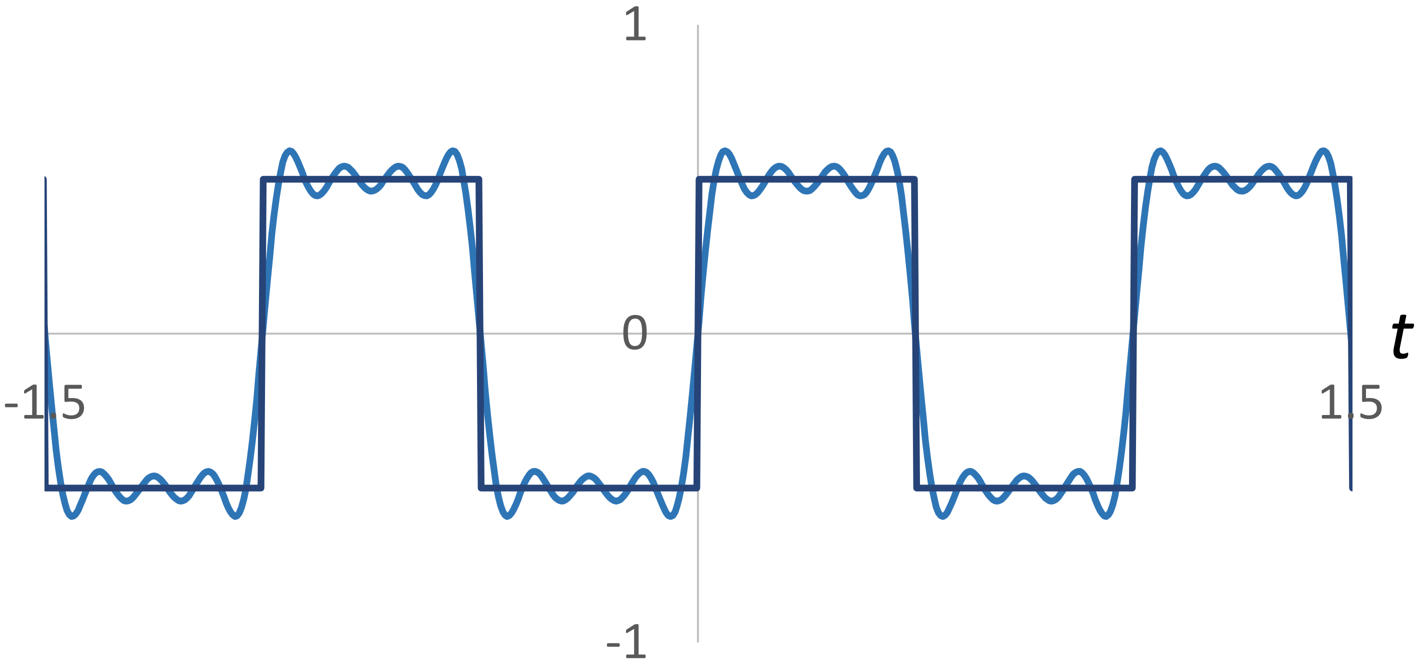


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-7}^{7} \frac{1}{j\pi k} e^{jk2\pi t}$$

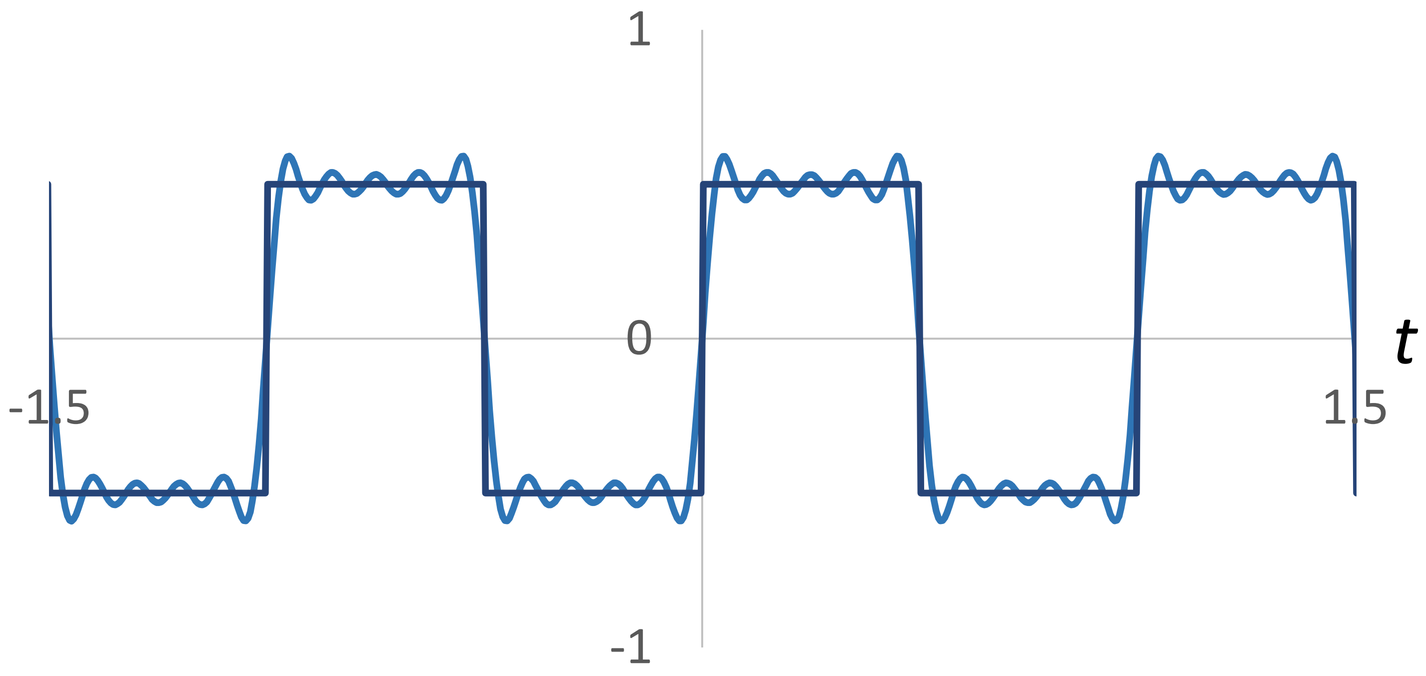


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-9}^{9} \frac{1}{j\pi k} e^{jk2\pi t}$$

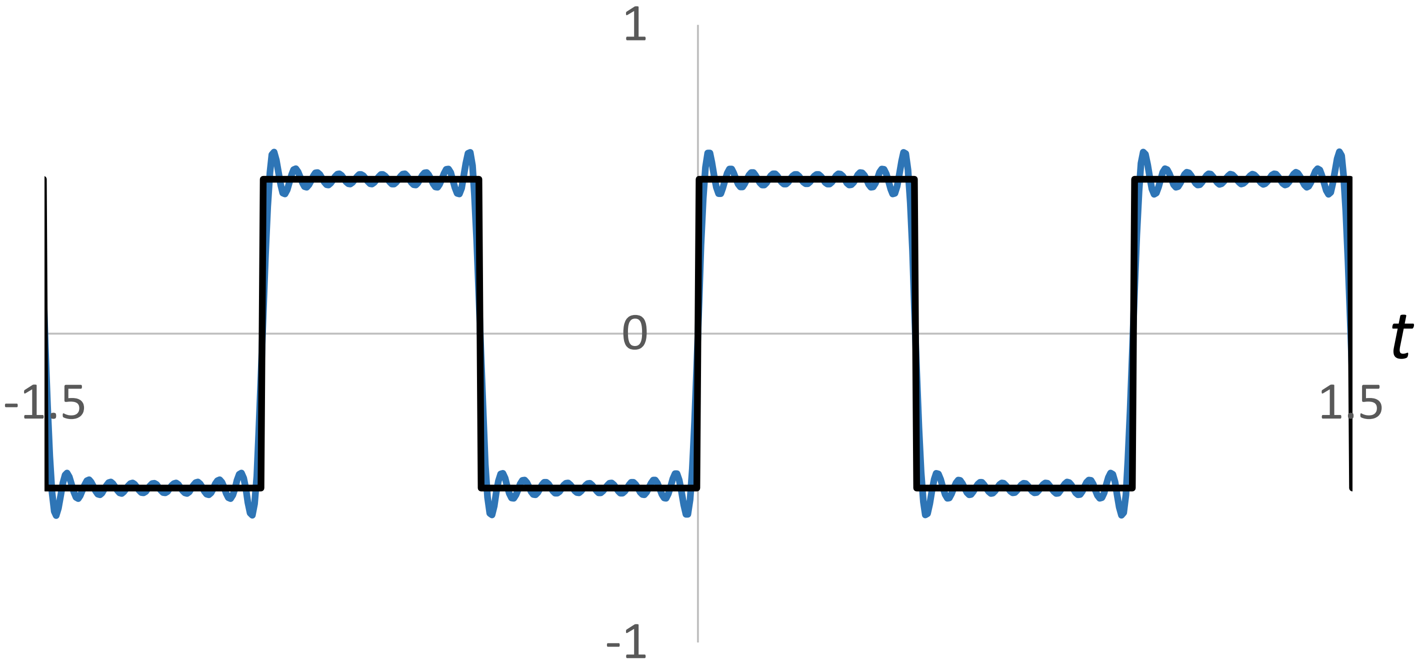


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-19}^{19} \frac{1}{j\pi k} e^{jk2\pi t}$$

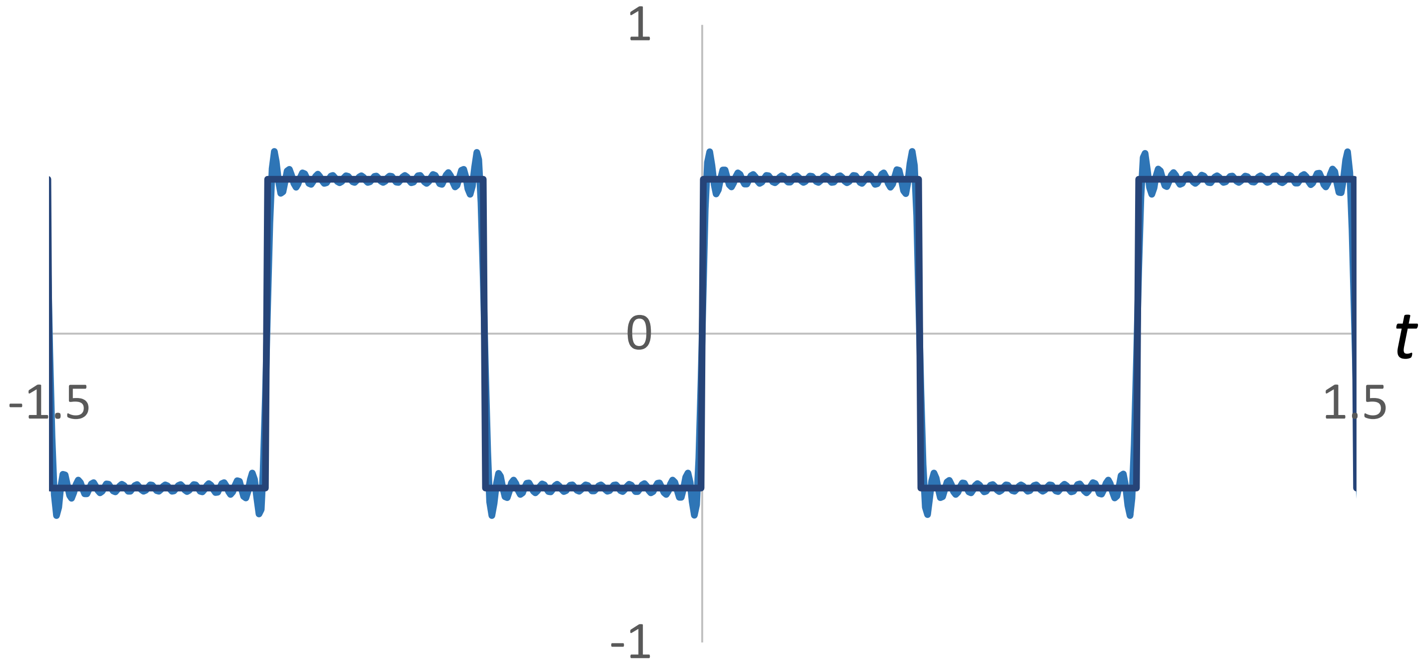


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t)=\sum_{k=-29}^{29}\frac{1}{j\pi k}e^{jk2\pi t}$$

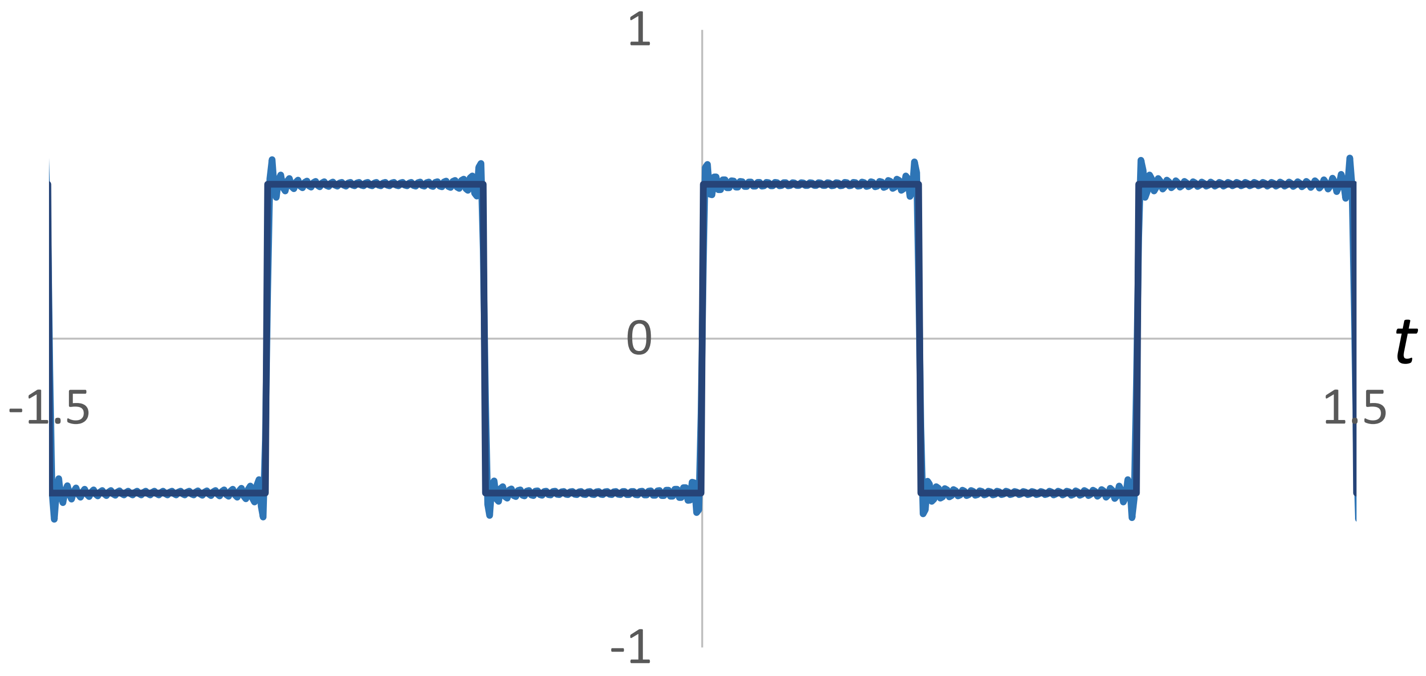


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-49}^{49} \frac{1}{j\pi k} e^{jk2\pi t}$$

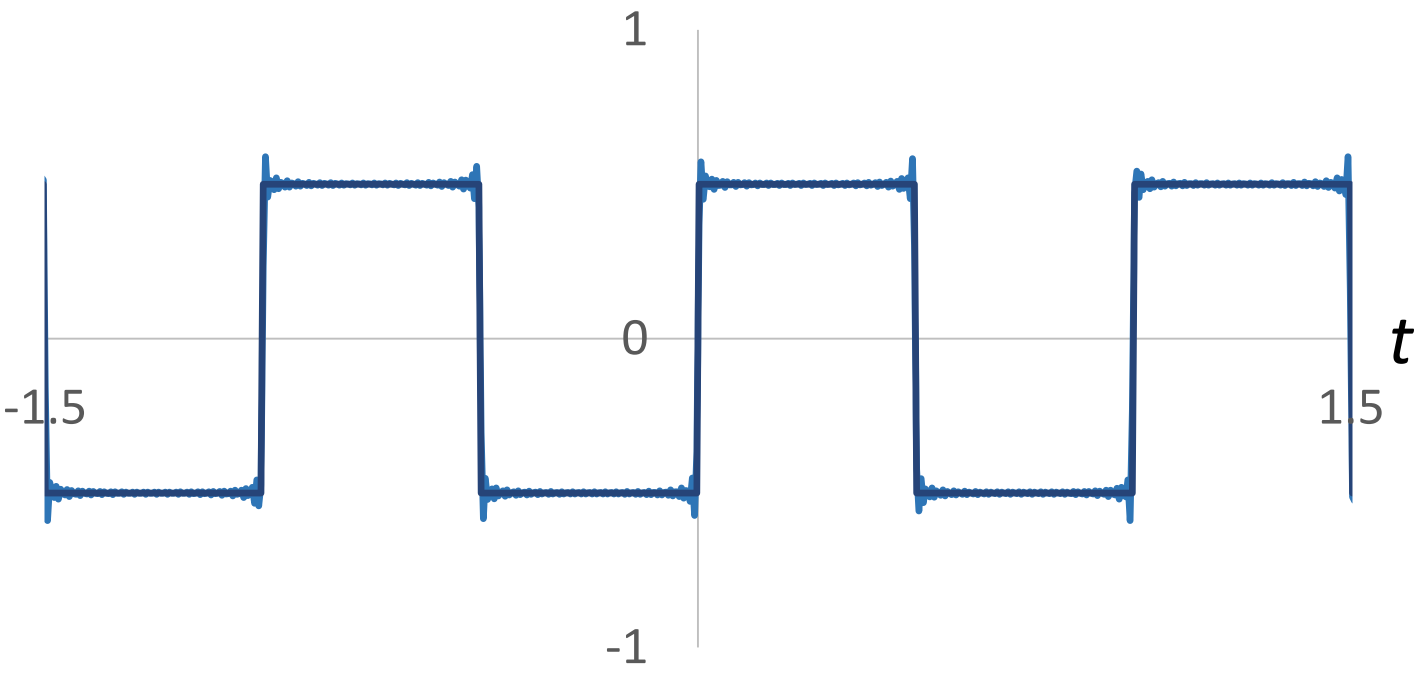


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-79}^{79} \frac{1}{j\pi k} e^{jk2\pi t}$$

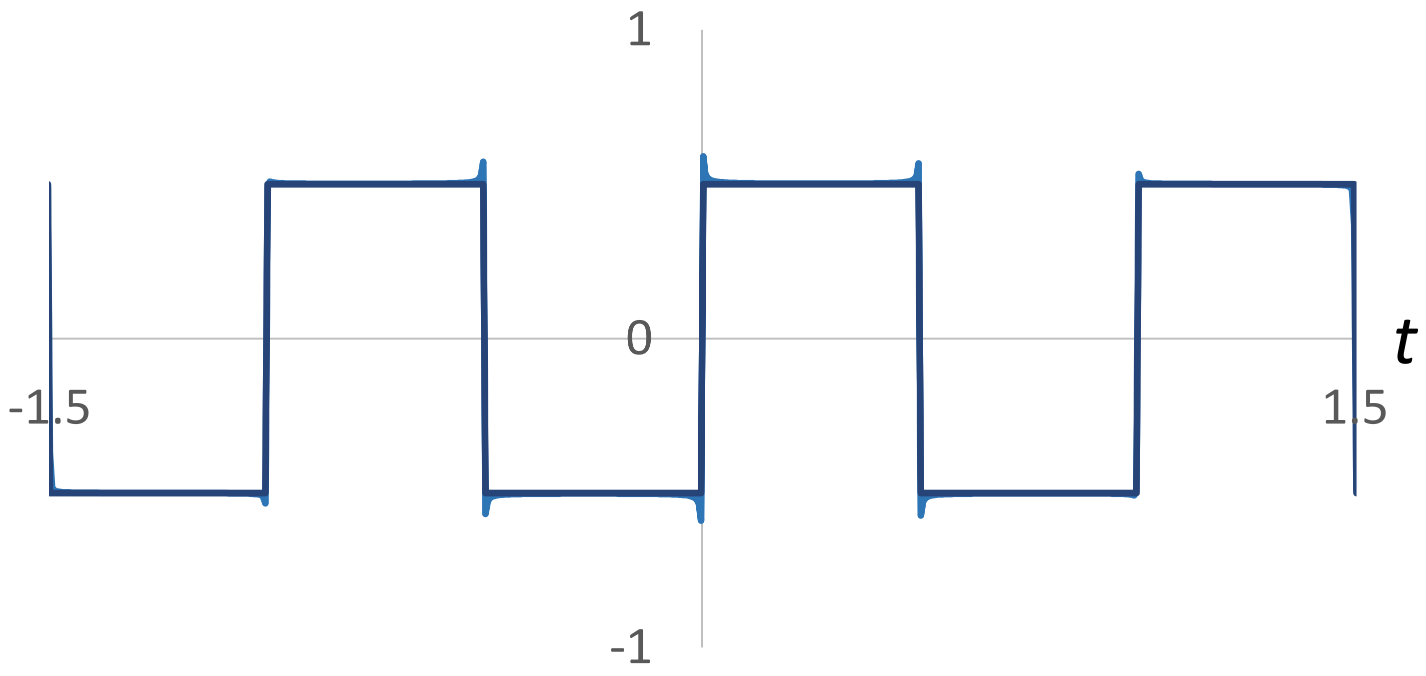


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

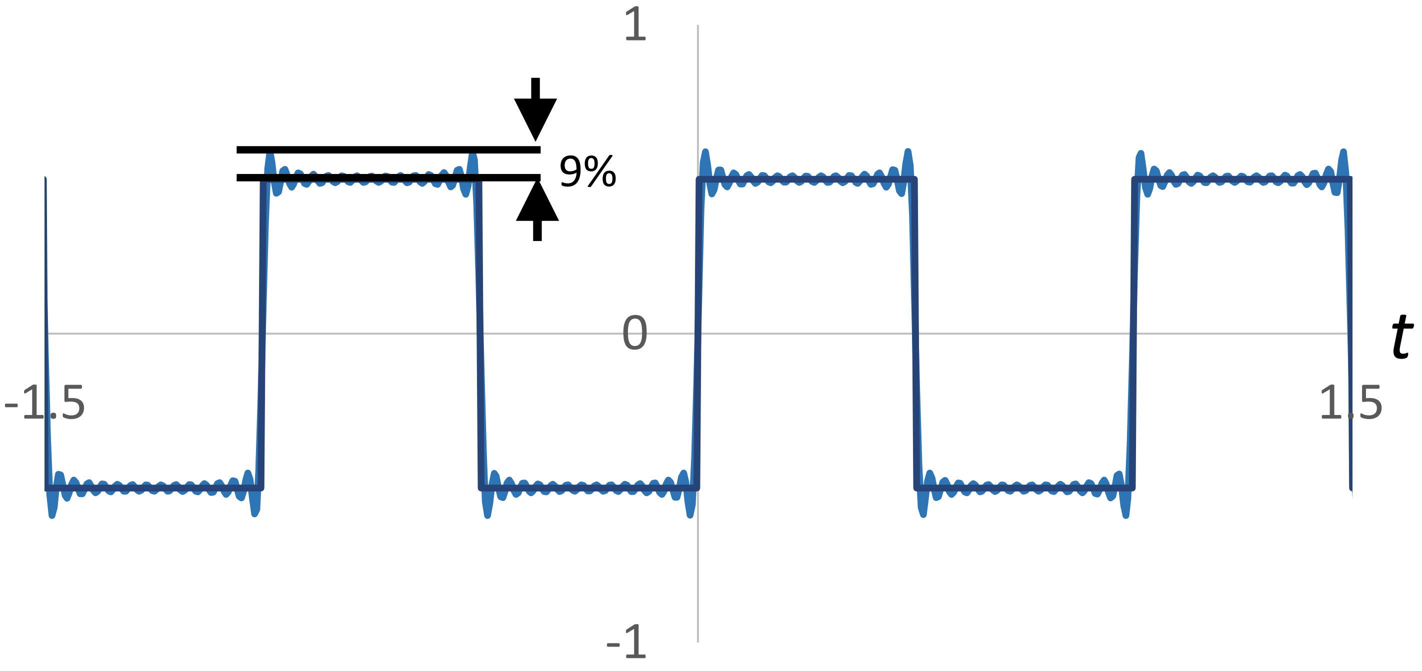
Example: square wave

$$x(t) = \sum_{k=-199}^{199} \frac{1}{j\pi k} e^{jk2\pi t}$$



Convergence of Fourier Series

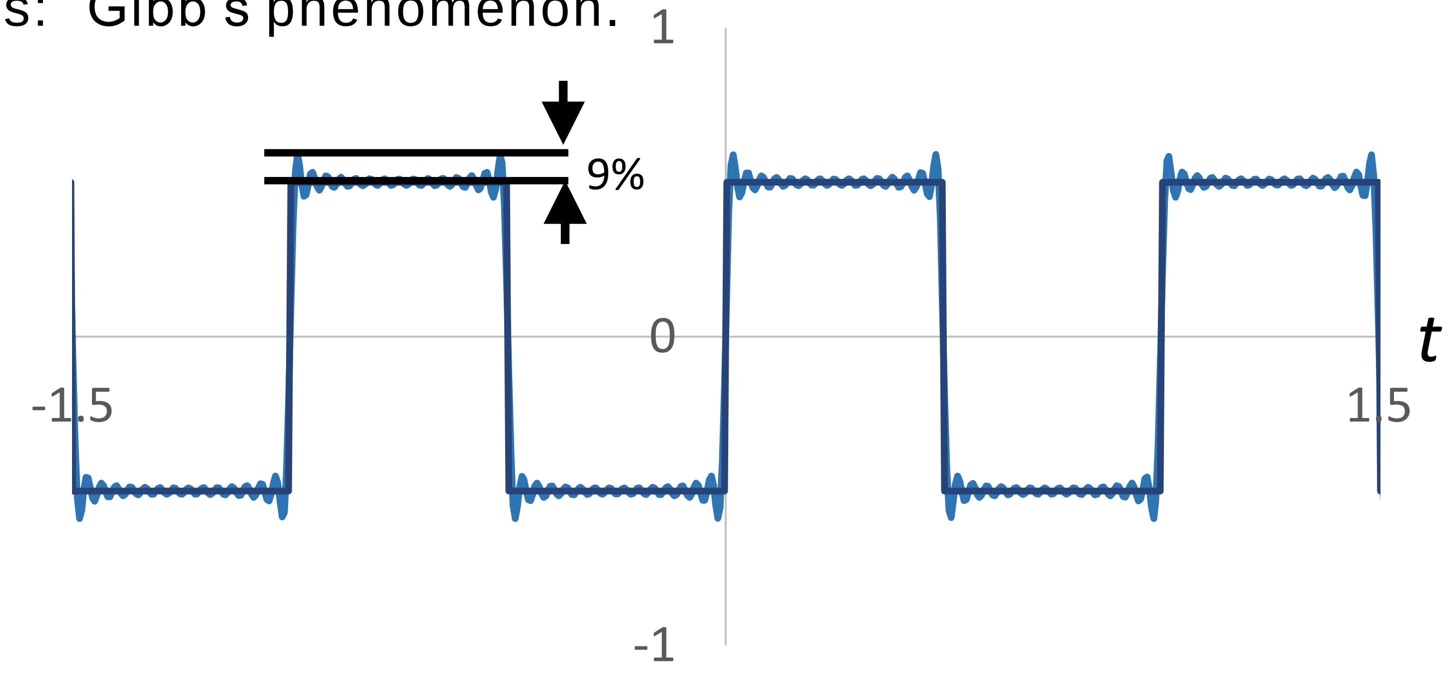
Albert Michelson horror



Convergence of Fourier Series

Albert Michelson horror

Partial sums of Fourier series of discontinuous functions “ring” near discontinuities: Gibb’s phenomenon.



Energy in Error