MTL 101

Linear Algebra and Differential Equations.

II part of the course: DEs

Lecture I

Prerequisite

- Basic Concepts from Linear Algebra

- Linear Independence
- Basis
- Dimension.
- Eigen values and Eigen vectors

- Basic notions from Calculus
  - Continuity, Differentiability
  - Lipschitz Continuity
  - Convergence

References.

1. "Advanced Engineering Mathematics", Erwin Kreyszig.

2. "Differential Equations with applications and Historical notes", George Simmons.

Why Differential Equations?

Models the physical

Systems.

$y = f(x)$ is a given function

$\frac{dy}{dx}$ — rate of change of y wrto x.

Example:

- Newton's Second law of motion.

$$a = \frac{F}{m}$$

or $F = m\,a$

$$a = \frac{dv}{dt} \quad \text{or} \quad a = \frac{d^2u}{dt^2}.$$

where

v = velocity

u = position function of the object.

$$m \cdot \frac{du}{dt} = F(t, v)$$

$$m \frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right)$$

Modelling

- free fall

$$\frac{d^2u}{dt^2} = g$$

Assume that air exerts a resistance force proportional to velocity

$$\text{Total force} = mg - k.\frac{du}{dt}$$

$$m.\frac{d^2u}{dt^2} = mg - k.\frac{du}{dt}$$

i.e $$\frac{d^2u}{dt^2} + \frac{k}{m}\frac{du}{dt} = g$$

A differential equation

is an equation involving one dependent variable and its derivatives wrto one or more independent variables.

An ordinary differential equation is one in which there is only one independent variable so that all the derivatives occuring in it are ordinary derivatives

Remark

more

1. If we have more than one independent variable, then we end up with partial differential equations.

For instance, suppose

$w = f(x, y, z, t)$ is a function of time and three rectangular coordinates of a point in space, then the following are pde

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

- If we have more than one unknown function, then we have a system of des

x, y — unknown functions

t — independent variable

$$\begin{cases} \frac{dx}{dt} = 4x + 3y + \sin t \end{cases}$$

$$\begin{cases} \dfrac{d}{dt} \\ \dfrac{dy}{dt} = -5x + y + e^t \end{cases}$$

In general form:

$$F\left(t, y(t), y', y'' \ldots y^{(n)}\right) = 0$$

Definition:

The order of a de is the order of

highest derivative appearing in the equation:

For instance

First order de

$$F(t, y, y') = 0 \quad \text{---} \; (*)_1$$

$$y' = f(t, y) \quad \text{---} \; (*)_2$$

There is another notion related to ODE, namely, degree.

loosely, degree is the power of highest order derivative in the de.

eg: (1) $(y')^2 + y'' + y = 0$

order – 2

degree – 1

(2) $(y'')^3 + y' = \sin t$

order – 2

degree – 3

(3) $(y'')^{2/3} = 2 + 3y'$

order – 2

degree – ?

Ratonalizing (3)

$$y''^{2} = (2 + 3y')^{3}$$

order – 2

degree – 2

Defn

When an ODE

involves polynomial in all the derivatives involved, then power to which the highest order derivative is raised is known as degree.

<u>example</u>

$$(y'')^2 = 3y' + \sin t$$

order $-$ 2

degree $-$ 2

$$y'' = 3(y')^{\frac{1}{3}} + t^2$$

order - 2

degree - ?

$$y''' = 3y' + \sin(y'')$$

order - 3

degree - Not defined

Let us see

Initial Value Problem.

Let us start with

$$\frac{dy}{dx} = f(x) \quad \text{——} \ (*)_1$$

integral calculus problem

We solve it by writing

$$y(x) = \int f(x)\, dx + C \quad \text{——} \ (*)_2$$

(Recall that

(Recall

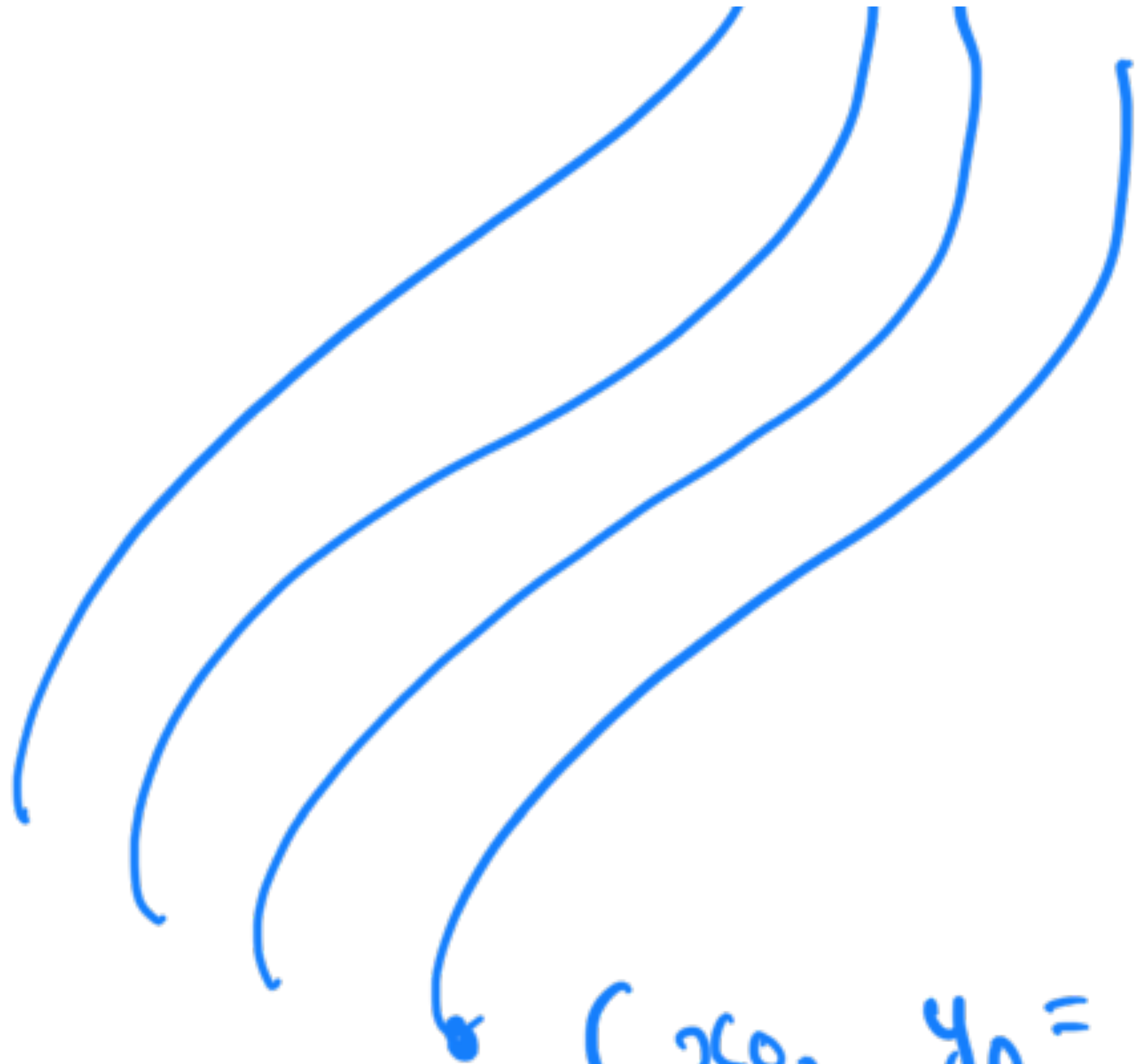
$\int e^{-x^2}\,dx$ and

$\int \frac{\sin x}{x}\,dx$ cannot

be expressed in terms of finite number of elementary functions)

We may write $*_2$

$$y = \int_{x_0}^{x} f(t)\,dt + C$$



$(x_0, \; y_0 = y(x_0))$

$$\text{IVP.} \begin{cases} \dfrac{dy}{dx} = f(x) \\ y(x_0) = y_0 \end{cases} \qquad (*)$$

In general,

$$\frac{dy}{\ } = f(x, y)$$

$$\begin{cases} \dfrac{}{dx} \\ y(x_0) = y_0 \end{cases} \qquad **$$

Consider

$$y'' = g(x, y(x), y'(x))$$

Let us convert into a system

Introduce

$$y_1(x) = y(x)$$

$$y_2(x) = y_1'(x) = y'(x)$$

Then

$$y_1'(x) = y'(x) = y_2(x)$$

$$y_2'(x) = y''(x)$$
$$= g(x, y_1(x), y_2(x))$$

Thus we have

$$\circledast_3 \quad \begin{cases} y_1'(x) = y_2(x) \\ y_2'(x) = g(x, y_1(x), y_2(x)) \end{cases}$$

S... for IVP $\circledast_3$

So needs one condition for y_1 and other condition for y_2

One of the possibility of IVP for second order de is

$$\begin{cases} y'' = f(x, y, y') \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

In general, an IVP for n^{th} order de is

$$\begin{cases} y^{(n)} = f(x, y, y', y'' \dots y^{(n-1)}) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

As far as Second order (and higher order des) de is concerned there exists another interesting class:

Quite often Second order de may be defined on an interval conditions are prescribed not on initial value, but at boundary points

$$
\begin{cases}
y'' = f(x, y, y') \\
\qquad x \in [a, b] \\
\alpha\, y(a) + \beta\, y'(a) = 0 \\
\alpha\, y(b) + \beta\, y'(b) = 0
\end{cases}
$$

— Boundary Value Problems

Solution concepts

By a solution of a de we mean a function that is sufficiently differentiable and satisfies the given equation.

<u>Example</u>

$y(x) = x^{-3/2}$ is a solution of

$$4x^2 y'' + 12x y' + 3y = 0$$ for

0

$x > 0$

We have $y'(x) = -\frac{3}{2} x^{-5/2}$

$y''(x) = \frac{15}{4} x^{-7/2}$

Plug these onto the eqn

$$4x^2 \cdot \frac{15}{4} x^{-7/2} + 12x \cdot -\frac{3}{2} x^{-5/2} + 3 \cdot x^{-3/2} = 0$$

$$15 x^{-3/2} - 18 x^{-3/2} + 3 x^{-3/2} = 0$$

$0 = 0.$

So $y(x) = x^{-3/2}$ satisfy the de and hence it is a solution

why then the condition $x > 0$?

$$y(x) = x^{-3/2} = \frac{1}{\sqrt{x^3}}$$

$x = 0$ to be avoided

$x < 0$ avoid

- A function $y = h(x)$ that is sufficiently differentiable on (a, b) and satisfies the de is called <u>explicit solution</u>

- Sometimes solution may be expressed as a relation connecting

x and y.

<u>eg:</u> $x^2 + y^2 - 1 = 0,$

$y > 0$ is an (implicit)

solution of de

$yy' = -x$ on $(-1, 1)$

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consider

$$\frac{dy}{dx} = f(x).$$

fundamental theorem

From Fundamental ...
of integral calculus

Solution

$$y(x) = \int_{x_0}^{x} f(s)\, ds + C$$

Defn

A solution of a de is called a general solution if it involves as many independent parameters as the order of the de.

A particular Solution of a de is a solution which is obtained from general solution by choosing specific values to the parameters.

Example

1. $y(t) = t^2 + c$ is a general solution of

$y' = 2t$

2. verify that

$y(t) = (c_1 + c_2 t)\, e^{2t}$

is a general solution

of $y'' - 4y' + 4y = 0$

on the other hand,

$y(t) = t\, e^{2t}$ is a

particular solution.

Example

Consided

$$y = x\,y' - (y')^2$$

Clairaut's eqn.

verify that

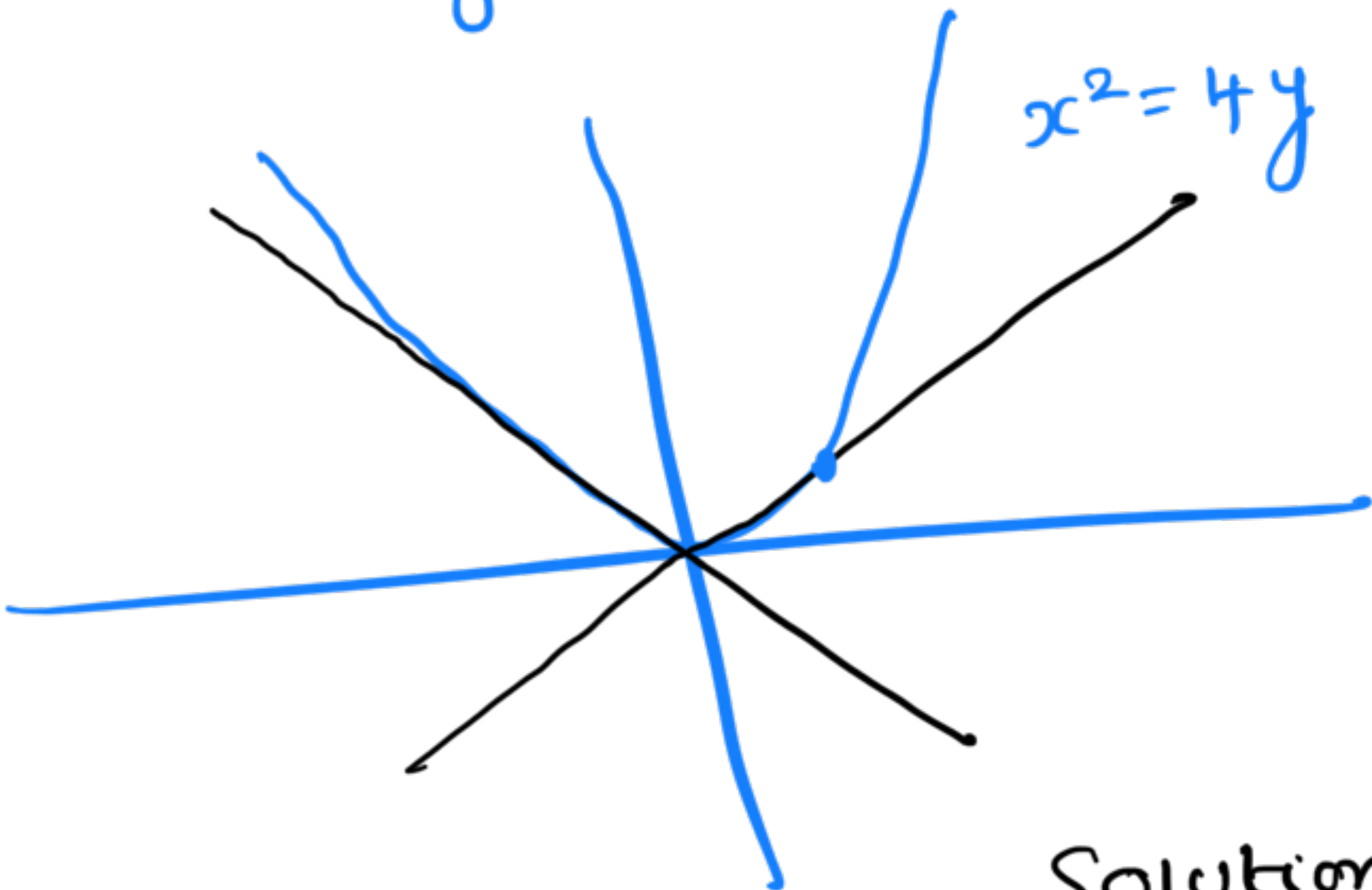
$$y = cx - c^2$$ is

a general solution.

$$(y')^2 - xy' + y = 0$$

$$y' = \frac{x \pm \sqrt{x^2 - 4y}}{2}$$

So if $x^2 - 4y < 0$, No solution



$y = \frac{1}{4}x^2$ is a Solution to given de

Singular Solution

An ODE may Sometimes have an additional solution that cannot be obtained from the general solution by assuming specific values to arbitrary constants involved.

This solution is called Singular Solution