Lecture 4 (Infinite series)

Ratio test

We consider sequences $(a_n)_{n\geq 1}$ of real numbers such that infinitely many a_n's are non-zero, otherwise the series $\sum_{n\geq 1} a_n$ turns out to be a finite sum. For such a sequence $(a_n)_{n\geq 1}$, we denote,

$$a = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| , \quad A = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

If infinitely many a_n's are zero, we take $a = 0$ and $A = +\infty$ (we remove the terms of the form $\frac{0}{0}$).

Theorem

1) If $A < 1$, then $\sum_{n \geq 1} a_n$ converges absolutely.

2) If $a > 1$, then $\sum_{n \geq 1} a_n$ does not converge.

3) The cases where $a \leq 1 \leq A$ are inconclusive.

<u>Proof</u> 1) When $A < 1$.

We can choose b such that $A < b < 1$.

Now $A = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \sup\left\{\left|\frac{a_{k+1}}{a_k}\right| : k \geq n\right\}$$

$\exists\ n_0 \in \mathbb{N}$ such that $\sup\left\{\left|\frac{a_{k+1}}{a_k}\right| : k \geq n\right\} < b \quad \forall\ n \geq n_0.$

In particular, $\left|\frac{a_{k+1}}{a_k}\right| < b \quad \forall\ k \geq n_0$.

i.e. $|a_{k+1}| < b|a_k| \ \forall \ k \geq n_0.$

Now, $|a_{n_0+1}| < b|a_{n_0}|$

and $|a_{n_0+2}| < b|a_{n_0+1}| < b^2|a_{n_0}|.$

Proceeding this way we get,

$|a_{n_0+n}| < b^n|a_{n_0}| \ \forall \ n \geq 1.$

Note that, $\sum_{n\geq 1} b^n$ converges as $b < 1.$

$\therefore$ By comparison test we get,

$\sum_{n\geq 1} |a_{n_0+n}|$ is convergent.

Note that, $\sum_{n\geq 1} |a_{n_0+n}|$ is the (n_0+1) tail of $\sum_{n\geq 1} |a_n|$.

$\therefore$ $\sum_{n\geq 1} |a_n|$ is convergent.

i.e. $\sum_{n\geq 1} a_n$ converges absolutely.

2) When $a > 1$.

$$a = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \inf\left\{ \left|\frac{a_{k+1}}{a_k}\right| : k \geq n \right\}.$$

$\exists\ n_0 \in \mathbb{N}$ such that $\inf\left\{ \left|\frac{a_{k+1}}{a_k}\right| : k \geq n \right\} > 1$

$\forall\ n \geq n_0$.

$\therefore \left|\frac{a_{k+1}}{a_k}\right| > 1 \ \forall\ k \geq n_0$.

Note that since $a > 1$, atmost finitely many a_n's possibly can be zero. Otherwise a would have been zero.

Choose $N \geq n_0$ so that $a_n \neq 0 \; \forall \; n \geq N$.

$\therefore \; \forall \; k \geq N$, we have $\left| \frac{a_{k+1}}{a_k} \right| > 1$.

i.e. $|a_{k+1}| > |a_k| \; \forall \; k \geq N$.

Hence $|a_k| > |a_N| \; \forall \; k > N$.

Since $a_N \neq 0$, we see that $(a_k) \not\to 0$ as $k \to \infty$.

$\therefore \sum_{n\geq 1} a_n$ does not converge.

3) When $a \leq 1 \leq A$.

i) $a = 1 = A$,

ii) $a < 1 = A$,

iii) $a = 1 < A$,

iv) $a < 1 < A$.

Case-i) $a = 1 = A$

Example 1 $a_n = \frac{1}{n}$

Here $a = 1 = A$ and $\sum_{n\geq 1} \frac{1}{n}$ does not converge.

Example 2 $a_n = \frac{1}{n^2}$

Here $a = 1 = A$ and $\sum_{n\geq 1} \frac{1}{n^2}$ converges.

Case - iii) $a < 1 = A$

Example 1 $a_n = \begin{cases} \frac{1}{2^n} & , n \text{ odd} \\ \frac{1}{2^{n-1}} & , n \text{ even} \end{cases}$

$\therefore\ 0 < a_n \leq \frac{2}{2^n}$. Therefore $\sum_{n\geq 1} a_n$ converges. If n is odd, then $\frac{a_{n+1}}{a_n} = 1$ and if n is even then $\frac{a_{n+1}}{a_n} = \frac{1}{4}$. $\therefore\ a = \frac{1}{4}$, $A = 1$.

Example 2 Let $n \geqslant 0$ and $0 \leqslant K < 2^n$.

Consider $a_{2^n+K} = \dfrac{2^n + K}{2^n}$.

Note, $\dfrac{a_{2^n}}{a_{2^n-1}} \to \dfrac{1}{2}$ and

$\dfrac{a_{2^n+K}}{a_{2^n+(k-1)}} \longrightarrow 1$ for $K \neq 0$ as $n \to \infty$.

$\therefore\ a = \frac{1}{2}$, $A = 1$. But $a_{2^n} = 1$.

$\therefore \sum_{n \geqslant 1} a_n$ does not converge.

Case - iii) $a = 1 < A$

Example 1 $a_n = \begin{cases} 2^n, & n \text{ odd} \\ 2^{n-1}, & n \text{ even} \end{cases}$

So $a_n \geq 2$. Hence $\sum_{n\geq 1} a_n$ does not converge. Also note,

if n is odd then $\frac{a_{n+1}}{a_n} = 1$

and if n is even then $\frac{a_{n+1}}{a_n} = 4$.

Here $a = 1$, $A = 4$.

Example 2 Let $n \geq 0$ and $0 \leq k < 2^n$.

Consider $a_{2^n + k} = \dfrac{2^n}{(2^n + k)^3}$.

Note, $\dfrac{a_{2^n}}{a_{2^n - 1}} \to 2$ as $n \to \infty$.

If $k \neq 0$, then $\dfrac{a_{2^n + k}}{a_{2^n + (k-1)}} \to 1$ as $n \to \infty$.

$\therefore\ a = 1,\ A = 2$.

Note, $a_{2^n + k} \leq \dfrac{1}{(2^n + k)^2}$ for any $n \geq 0$ and $0 \leq k < 2^n$.

$\therefore \sum_{n\geq 1} a_n$ converges as

$$0 < a_n \leq \frac{1}{n^2}.$$

Case-iv) $a < 1 < A$.

Example 1 $a_n = \begin{cases} 2/n, & n \text{ odd} \\ 1/n, & n \text{ even} \end{cases}$

Then when n is odd, $\frac{a_{n+1}}{a_n} \to \frac{1}{2}$ as $n \to \infty$.

When n is even, $\frac{a_{n+1}}{a_n} \to 2$ as $n \to \infty$.

So, $a = \frac{1}{2}$, $A = 2$.

Since $a_n \geq \frac{1}{n}$ and $\sum_{n \geq 1} \frac{1}{n}$ does not converge,

$\sum_{n \geq 1} a_n$ does not converge.

Example 2

$$a_n = \begin{cases} 2/n^2, & n \text{ odd} \\ 1/n^2, & n \text{ even} \end{cases}$$

Then when n is odd, $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{2}{n^2}} \to \frac{1}{2}$ as $n \to \infty$

Similarly when n is even,

$$\frac{a_{n+1}}{a_n} \to 2 \text{ as } n \to \infty.$$

$$\therefore \quad a = \frac{1}{2} \text{ and } A = 2.$$

Since $0 < a_n \leq \dfrac{2}{n^2}$,

$\sum_{n\geq 1} a_n$ converges.

Applications of the ratio test

1) $\sum_{n\geq 1} \frac{n}{2^n}$ is convergent.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \left(1+\frac{1}{n}\right)\cdot\frac{1}{2}$$

As $n \to \infty$, $\frac{a_{n+1}}{a_n} \to \frac{1}{2}$.

$\therefore \sum_{n\geq 1} \frac{n}{2^n}$ is convergent.

2) $\sum_{n\geq 1} \frac{n!}{n^n}$ is convergent.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)}{(n+1)^{n+1}} \cdot n^n$$

$$= \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

We know, as $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e$,

so $\frac{a_{n+1}}{a_n} \to 1/e$. $\quad 1/e < 1$

$\therefore \sum_{n\geq 1} \frac{n!}{n^n}$ is convergent.

Root test

Let $\sum_{n\geq 1} a_n$ be an infinite series. Let

$$A = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$ Then

1) if $A < 1$, then $\sum_{n\geq 1} a_n$ is absolutely convergent.

2) if $A > 1$, then $\sum_{n\geq 1} a_n$ does not converge.

3) if $A = 1$, then we cannot conclude anything.

Proof Case 1) $A < 1$.

Choose $A < B < 1$.

Since $\limsup_{n \to \infty} \sqrt[n]{|a_n|} = A$

i.e. $\lim_{n \to \infty} \sup \{ \sqrt[k]{|a_k|} : k \geq n \} = A$,

we get $\exists \; n_0 \in \mathbb{N}$ such that

$\sqrt[k]{|a_k|} < B$ for all $k \geq n_0$.

We have, $|a_k| < B^k$ for all $k \geq n_0$.

Since $B < 1$, $\sum_{k \geq n_0} |a_k|$ is convergent by comparison test.

n_0- tail of $\sum_{k \geq 1} |a_k|$

$\therefore \sum_{k \geq 1} |a_k|$ is convergent.

i.e. $\sum_{k \geq 1} a_k$ is absolutely convergent.

Case 2) $A > 1$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$$

i.e. $\lim_{n\to\infty} \sup\{\sqrt[k]{|a_k|} : k \geq n\} > 1$.

$\therefore \exists\ n_0 \in \mathbb{N}$ such that

$$\sup\{\sqrt[k]{|a_k|} : k \geq n\} > 1 \quad \forall\ n \geq n_0.$$

i.e. $\exists$ infinitely many k such that

$\sqrt[k]{|a_k|} > 1$.

Otherwise $\sup\{\sqrt[k]{|a_k|} : k \geq n\}$ would have been ≤ 1 for n large enough.

Now if $\sqrt[k]{|a_k|} > 1$, then $|a_k| > 1$.

So $a_k \not\to 0$ as $k \to \infty$.

$\therefore \sum_{n\geq 1} a_n$ does not converge.

Case 3)

Example 1

$a_n = \frac{1}{n}$, then $\sqrt[n]{a_n} \to 1$ as $n \to \infty$.

$\therefore A = 1$.

But $\sum_{n \geq 1} \frac{1}{n}$ does not converge.

Example 2

$a_n = \frac{1}{n^2}$, then $\sqrt[n]{a_n} \to 1$ as $n \to \infty$

So $A = 1$, but $\sum_{n \geq 1} \frac{1}{n^2}$ is convergent.

Remark Ratio test is a special case of root test.

Recall,

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \leq \liminf_{n\to\infty}\sqrt[n]{|a_n|}$$

$$\leq \limsup_{n\to\infty}\sqrt[n]{|a_n|}$$

$$\leq \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1.$$

$\therefore \sum_{n\geq 1} a_n$ is absolutely convergent by root test.

if $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1.$$

$\therefore \sum_{n \geq 1} a_n$ is not convergent by root test.

Example $\quad a_n = \begin{cases} 2^{-n}, & n \text{ odd} \\ 2^{-n+1}, & n \text{ even} \end{cases}$

$$\therefore a_n^{1/n} = \begin{cases} \frac{1}{2}, & n \text{ odd} \\ 2^{-1+\frac{1}{n}}, & n \text{ even} \end{cases}$$

$\therefore \quad a_n^{1/n} \to 1/2 \quad$ as $n \to \infty$.

So by root test we can conclude that $\sum_{n \geq 1} a_n$ is convergent.

Now, $\dfrac{a_{n+1}}{a_n} = \begin{cases} 1, & \text{if } n \text{ odd} \\ 1/4, & \text{if } n \text{ even.} \end{cases}$

$\therefore \quad \limsup_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$

$$\liminf_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4}.$$

By ratio test we are not able to conclude anything about the convergence of $\sum_{n\geq 1} a_n$.

<u>Dirichlet test</u> Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ be two sequences of real numbers such that the seq. of partial sums of $\sum_{n\geq 1}^{\infty} a_n$ is bounded and $(b_n)_{n\geq 1}$ is a non-increasing seq. converging to zero. Then $\sum_{n\geq 1} a_n b_n$ is convergent.

Proof the seq. of partial sums of $\sum_{n\geq 1} a_n$ by s_n

and the seq. of partial sums of $\sum_{n\geq 1} a_n b_n$ by t_n

Since $(s_n)_{n\geq 1}$ is bounded, $\exists\ M > 0$ such that $|s_n| \leq M \ \forall\ n \geq 1$.

To show $(t_n)_{n\geq 1}$ is Cauchy.

Let $n > m$.

$$|t_n - t_m| = |a_{m+1}b_{m+1} + a_{m+2}b_{m+2} + \cdots + a_{n-1}b_{n-1} + a_n b_n|$$

$$= |(S_{m+1} - S_m)b_{m+1} + (S_{m+2} - S_{m+1})b_{m+2} + \cdots + (S_{n-1} - S_{n-2})b_{n-1} + (S_n - S_{n-1})b_n|$$

$$= |-S_m b_{m+1} + S_{m+1}(b_{m+1} - b_{m+2}) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_n b_n|$$

Note that $\forall\, i \geq 1$,

$b_i \geq b_{i+1} \geq 0$ as $(b_n)_{n\geq 1}$ is a non-increasing sequence and $b_n \to 0$ as $n \to \infty$.

$\therefore\ |t_n - t_m| \leq Mb_{m+1} + M(b_{m+1} - b_{m+2})$
$+ \cdots + M(b_{n-1} - b_n) + Mb_n$

i.e. $|t_n - t_m| \leq 2Mb_{m+1}$.

Now let $\epsilon > 0$. Since $b_n \to 0$ as $n \to \infty$,
$\exists\ n_0 \in \mathbb{N}$ such that $0 \leq b_n < \frac{\epsilon}{2M}\ \forall\ n \geq n_0$.

$\therefore\ |t_n - t_m| < \epsilon\ \forall\ n > m > n_0$.

$\therefore\ (t_n)_{n \geq 1}$ is Cauchy and thus $\sum_{n \geq 1} a_n b_n$ is convergent.

An application of Dirichlet test

Let $a_n = \sin nx$ and $(b_n)_{n\geq 1}$ be any sequence of real numbers such that $b_n \to 0$ as $n \to \infty$ with $(b_n)_{n\geq 1}$ is nonincreasing. Then $\sum_{n\geq 1} a_n b_n$ is convergent.

Need to show the seq. of partial sums of $\sum_{n\geq 1} \sin nx$ is bounded.

If x is an integer multiple of π.

Then $a_n = \sin nx = 0$.

$$\sum_{n\geq 1} a_n b_n = 0$$

If x is not an integer multiple of π, we consider

$$\sin x + \sin 2x + \cdots + \sin nx.$$

Note that,

$$2\sin\frac{x}{2}\left(\sin x + \sin 2x + \cdots + \sin nx\right)$$

$$= \left(\cos\frac{x}{2} - \cos\frac{3x}{2}\right) + \left(\cos\frac{3x}{2} - \cos\frac{5x}{2}\right) + \cdots + \left(\cos\frac{(n-1)x}{2} - \cos\frac{(n+1)x}{2}\right).$$

$\therefore\ 2\sin\frac{x}{2}\left(\sin x+\cdots+\sin nx\right)$

$= \cos\frac{x}{2} - \cos\frac{(n+1)x}{2}.$

$\therefore\ 2\left|\sin\frac{x}{2}\right|\left|\sin x+\cdots+\sin nx\right| \leq 2$

Hence $\left|\sin x+\cdots+\sin nx\right| \leq \dfrac{1}{\left|\sin\frac{x}{2}\right|}$

$\therefore \sum_{n\geq 1} \sin nx\, b_n$ is convergent.

Corollary (Abel's test)

Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ be two sequences of real numbers such that $\sum_{n\geq 1} a_n$ is convergent and $(b_n)_{n\geq 1}$ is a convergent monotonic sequence. Then $\sum_{n\geq 1} a_n b_n$ is convergent.

Proof Let $b_n \to b$. If (b_n) is non-decreasing then $b - b_n \to 0$ as $n \to \infty$ and $(b-b_n)_{n\geq 1}$ is non-increasing.

$\therefore$ By Dirichlet's test, we get

$\sum_{n\geq 1} a_n(b-b_n)$ is convergent.

Also, $\sum_{n\geq 1} b a_n$ is convergent.

$$\therefore \sum_{n\geq 1} \{b a_n - a_n(b-b_n)\} = \sum_{n\geq 1} a_n b_n$$

is convergent.

Similarly if (b_n) is non increasing, then $(b_n - b) \to 0$ as $n \to \infty$ and $(b_n - b)$ is non increasing.

By Dirichlet's test,

$\sum_{n\geq 1} a_n(b_n - b)$ is convergent.

Also, $\sum_{n\geq 1} b\, a_n$ is convergent.

$\therefore \sum_{n\geq 1} a_n b_n$ is convergent.