Ordinary Differential Equations

Lecture 2

Existence and uniqueness of solution of I order IVP

Consider

$$\begin{cases} x' = f(t, x) \end{cases}$$

(1) $\left\{ x(t_0) = x_0, \right.$

where $f : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function.

$F(t, x, x') = 0$

Examples

1. $$(x')^2 + x^2 + 1 = 0$$
$$x(0) = 1$$

This IVP has no Solution.

2. $$|x'| + |x| = 0$$
$$x(0) = 1$$

The only possible solution is $x \equiv 0$

$$(\text{ie } x(t) = 0 \quad \forall \; t)$$

This does not satisfy $x(0) = 1$

Thus, the given IVP has no solution.

3. $$\begin{cases} x'(t) = 2t \\ x(0) = 1 \end{cases}$$

Solution of this de is $x(t) = t^2 + c$

$$\Rightarrow \quad 1 = 0 + c$$

$x(0) = 1 \implies$

$\implies \quad c = 1$

Thus, given IVP has unique solution $x(t) = t^2 + 1$

4. $$\begin{cases} t\, x' = x - 1 \\ x(0) = 1 \end{cases}$$

Here $x(t) = 1 + \alpha t, \ \alpha \in \mathbb{R}$ is a solution of the de $t\, x' = x - 1$

Further it satisfies the IVP

Thus IVP has infinite number of solutions.

Questions

(1) under what conditions the IVP for I order ODE given in (1) has a solution?

(2) under what conditions the IVP given in (1) has a unique solution?

Back ground.

Definition

A function $f = f(\theta)$ $f : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a Lipschitz function if there exists a constant $L > 0$ such that

$$|f(\theta) - f(\theta')| \leq L\,|\theta - \theta'|$$

$$\forall \ \theta, \theta' \in \Omega$$

Remark:

Recall that a function $f : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a point $\theta_0 \in \Omega$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(\theta) - f(\theta_0)| < \varepsilon$$

whenever $|\theta - \theta_0| < \delta$

If $f: \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz, then f is continuous.

Definition

A function $f = f(t, x)$ $f: \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is said to be Lipschitz if there exists $L > 0$ such that

$$|f(t,x) - f(t^*, x^*)|$$

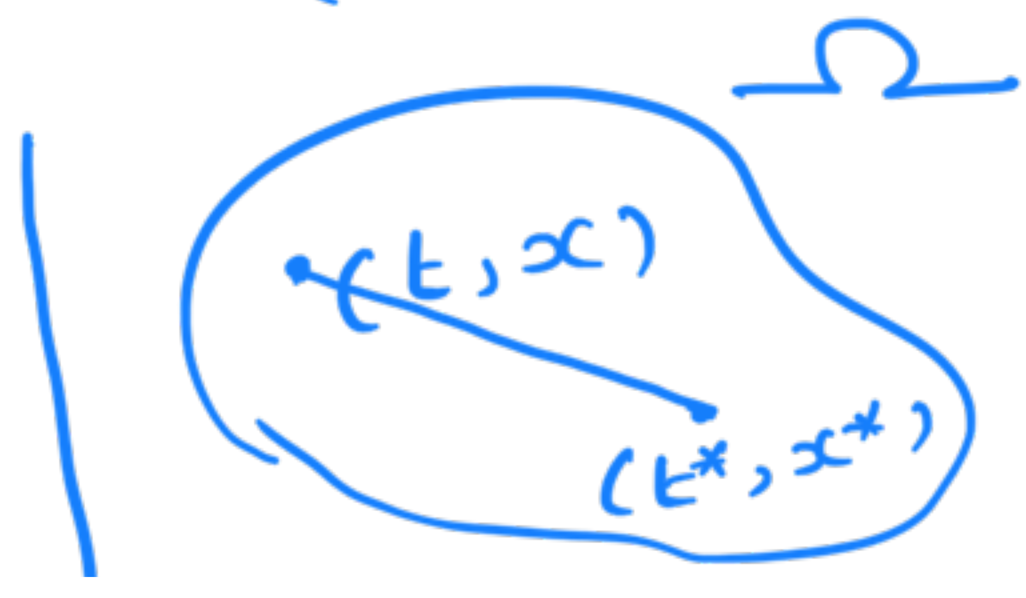
$$\leq L \; d\big((t,x), (t^*, x^*)\big)$$

$$\forall \; (t,x), (t^*, x^*) \in \Omega$$

For instance,

$$d\big((t,x), \; (t^*, x^*)\big)$$

$$= \sqrt{(t - t^*)^2 + (x - x^*)^2}$$



Definition

A function $f = f(t, x)$

$f : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

is said to be

Lipschitz in variable x

if $\exists$ $L > 0$ such that

$$|f(t, x) - f(t, x^*)| \leq L \, |x - x^*|$$

$\forall \ (t, x) \ (t, x^*) \in \Omega$

Examples

1. $f(t, x) = t^2 + \sin x$

is Lipschutz in variable x.

$$|f(t,x) - f(t, x^*)|$$

$$= |(t^2 + \sin x) - (t^2 + \sin x^*)|$$

$$= |\sin x - \sin x^*|$$

$$\overset{?}{\leq} |x - x^*|$$

2. $f(t, x) = t^2 + x^2$

Look at

$$|f(t,x) - f(t, x^*)|$$

$$= |x^2 - x^{*2}|$$

$$= |x - x^*| \, |x + x^*|$$

If Ω is such that

x-variable is bounded,

then we can bound

$|x + x^*|$ by a constant

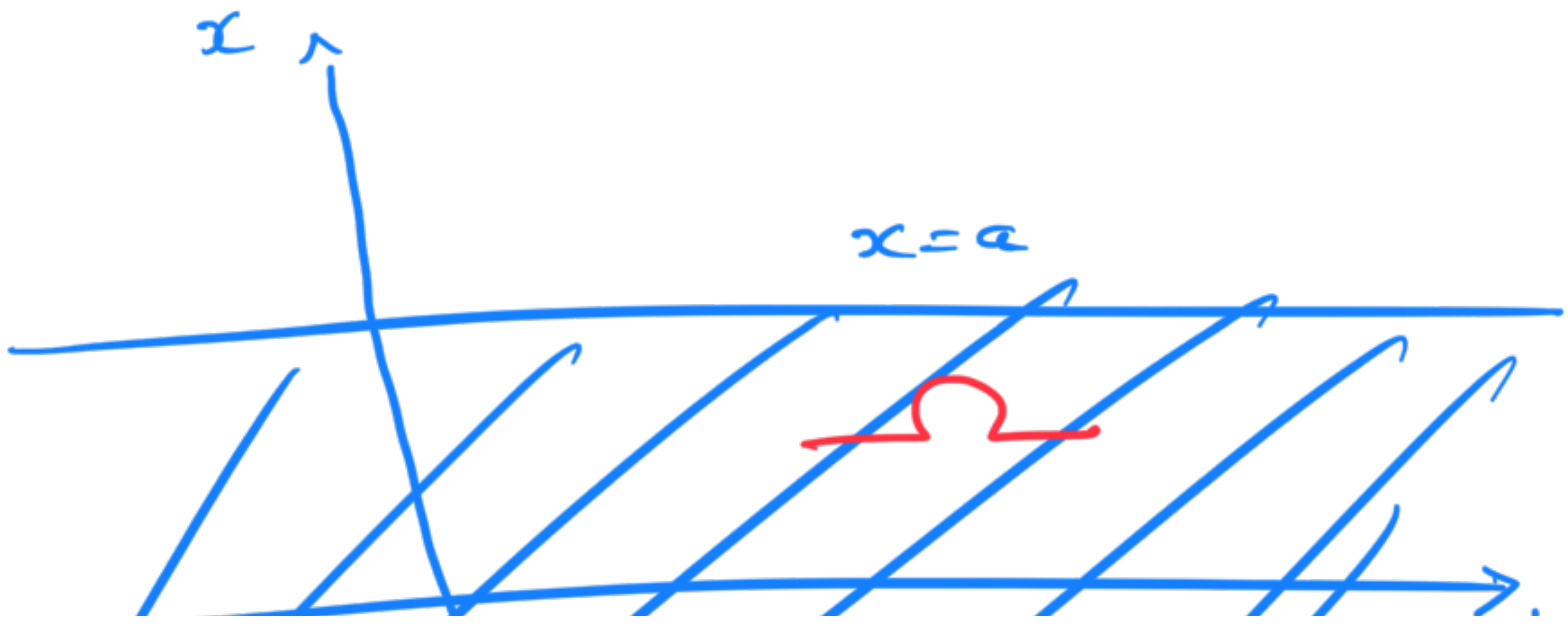
Consequently, f is Lipschitz

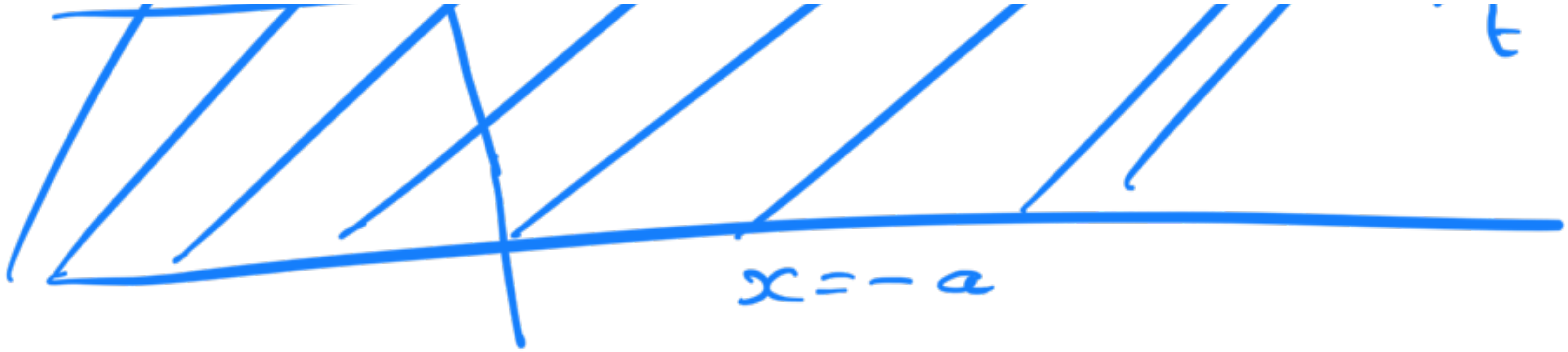
w.r.t. x-variable on Ω.

For instance, let

$$\Omega = \left\{ (t, x) : \begin{array}{l} t \in \mathbb{R} \\ |x| \leq a, \end{array} \right\}$$

for some fixed $a > 0$





Then

$$|x + x^*| \le |x| + |x^*|$$
$$\le 2a \quad \text{on } \Omega$$

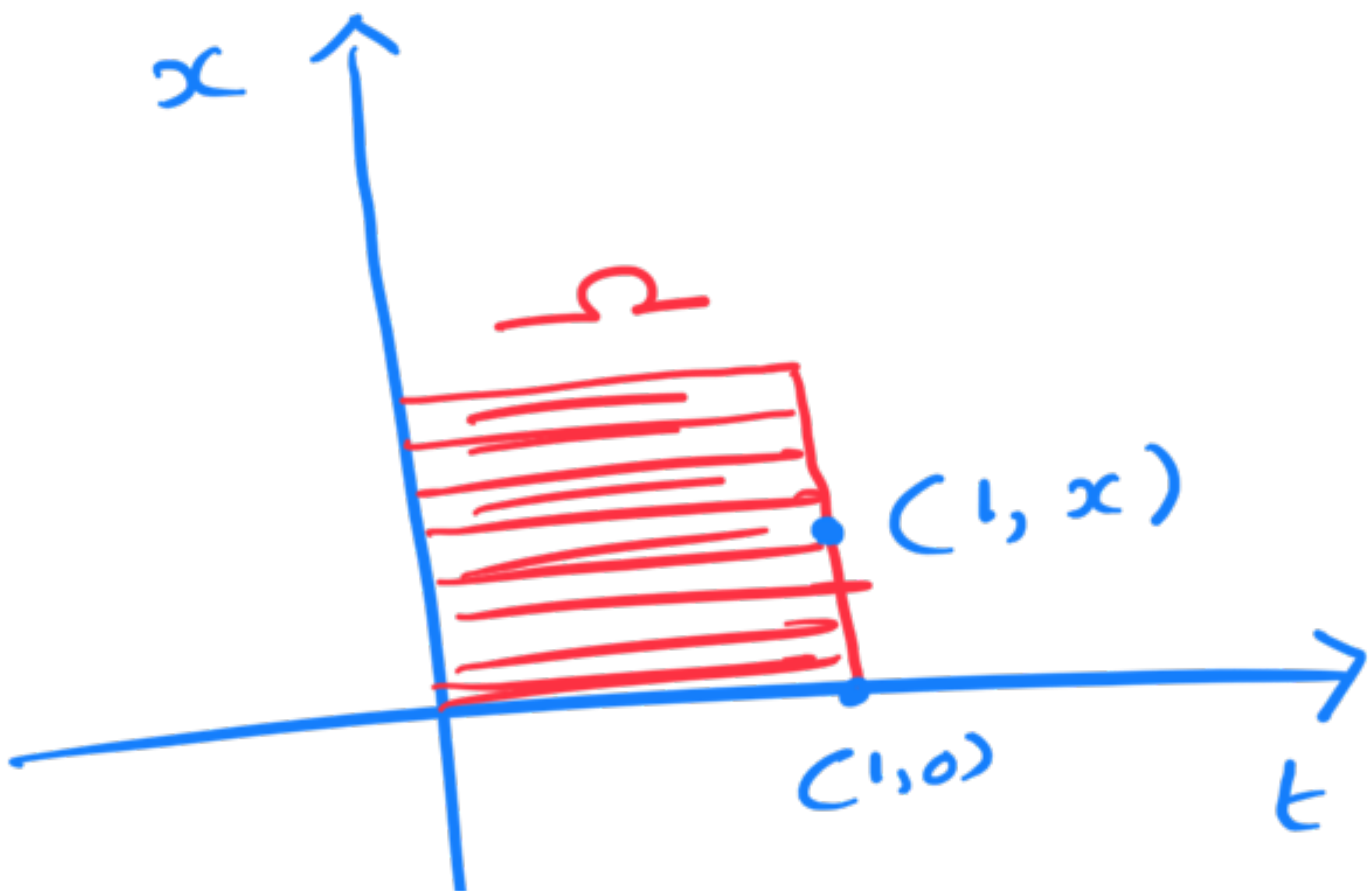
$$\therefore \; |f(t,x) - f(t,x^*)|$$
$$\le 2a \, |x - x^*|$$

$\therefore$ f is Lipschitz in

x - variable on Ω

3. $f(t, x) = t \sqrt{x}$

$$\Omega : \left\{ (t, x) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \leq t \leq 1 \\ 0 \leq x \leq 1 \end{array} \right\}$$



$$|f(t,x) - f(t, x^*)|$$

$$= |t\sqrt{x} - t\sqrt{x^*}|$$

$$= |t| \, |\sqrt{x} - \sqrt{x^*}|$$

$$= |t| \frac{|x - x^*|}{|\sqrt{x} + \sqrt{x^*}|}$$

If f is Lipschutz on Ω

then

$$|f(t,x) - f(t,x^*)|$$

$$|f(t,x) - f(t,x^*)|$$

$$\leq L\ |x - x^*|$$

$$\forall (t,x), (t,x^*) \in \Omega$$

for some $L > 0$.

choose

$$(t,x) = (1,x)$$

$$(t,x^*) = (1,0)$$

$$|f(1,x) - f(1,0)|$$

$$\leq L\ |x|$$

$$|\sqrt{x}| \leq L \ |x|$$

ie $$\frac{1}{\sqrt{x}} \leq L$$

$$\forall \quad 0 < x < 1,$$

which is a contradiction

Hence f is NOT Lipschitz in x-variable.

Sufficient condition for existence of solution.

Theorem (Existence)

Consider the IVP

$$\left.\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0 \end{aligned}\right\} \quad (1)$$

Suppose that $f = f(t,x)$:

$\Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is

continuous in some

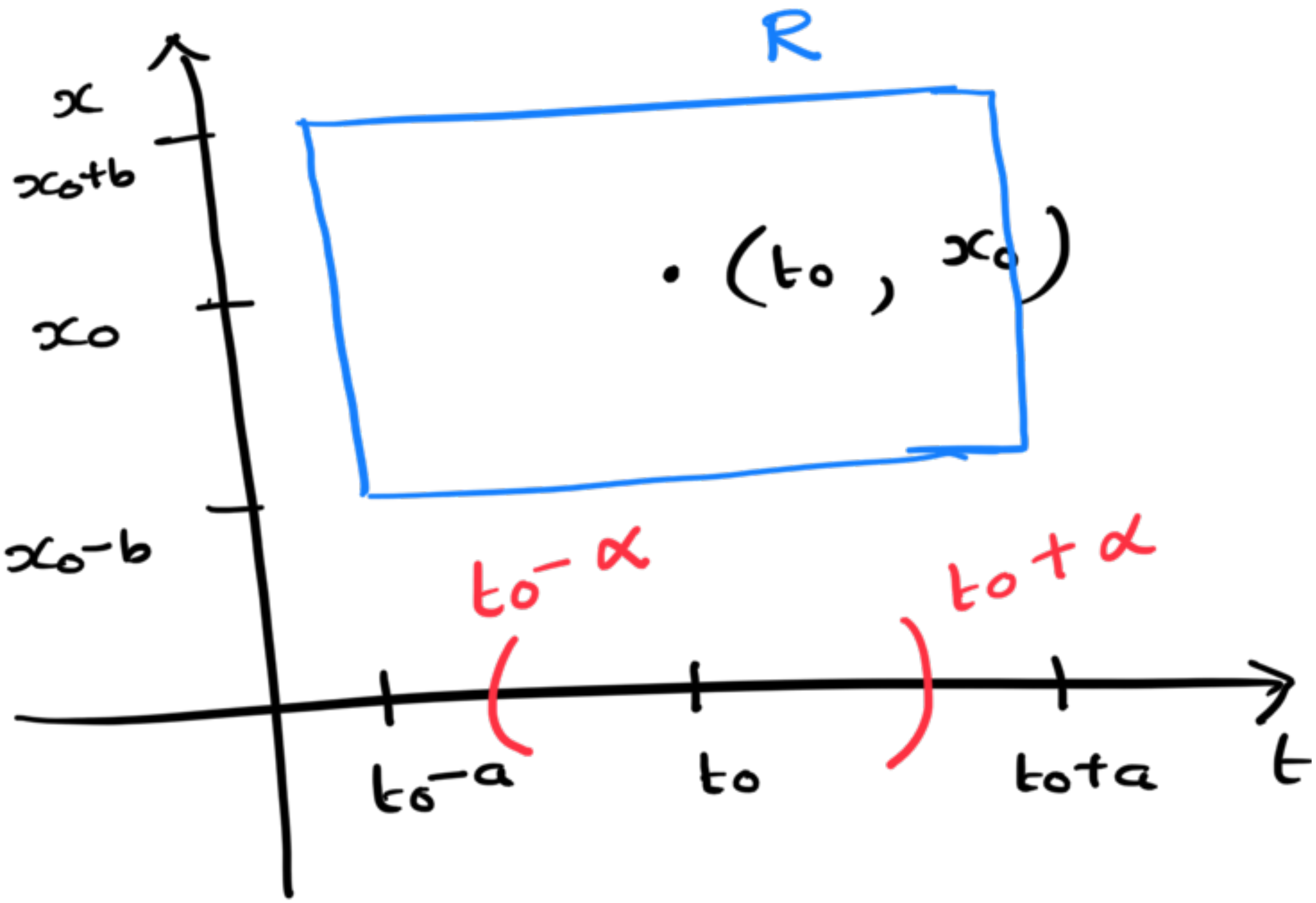
region

$\Omega = \{ (t,x) : \quad |t - t_0| \leq a$

$R = \Big\{ (\ldots$

$|x - x_0| \le b \Big\}$

for some $a, b > 0$.



[Note that as a consequence of continuity of f on

... bounded subset

a closed [cut off]
R of $\mathbb{R}^2$, it follows
that $\exists$ $K>0$ such that

$$|f(t,x)| \leq K$$
$$\forall \ (t,x) \in R].$$

The IVP (1) has a
solution $x = x(t)$
defined in the interval

$$|t - t_0| \leq \alpha, \text{ where}$$

$$\alpha = \min\left\{a, \frac{b}{K}\right\}$$

Theorem (uniqueness)

consider the IVP

$$① \begin{cases} x' = f(t,x) \\ x(t_0) = x_0 . \end{cases}$$

Assume that

(1) $f = f(t,x)$ is continuous on a rectangle R

$$R = \left\{ (t,x) : \; |t - t_0| \leq a \; |x - x_0| \leq b \right\}$$

(2) f is Lipschitz in x-variable on R

Then the IVP ①

has a unique Solution

$x = x(t)$ valid in

the interval

$$|t - t_0| \leq \alpha,$$

where

$$\alpha = \min \left\{ a, \frac{b}{K} \right\}$$

Remark

The theorems above help to make prediction on the interval in which solution exists. However, interval on which solution is ensured by the theorem is smaller than $|t - t_0| \leq a$, where $f(t, x)$ is defined and continuous

Thus the theorems stated above are local in nature

Examples

Let us discuss the IVP

$$x' = 1 + x^2$$

$$x(0) = 0$$

in the light of

existence and uniqueness theorem.

Here $f(t,x) = 1+x^2$

$$(t_0, x_0) = (0,0).$$

Note that f os continuous on a rectangle

Say

$$R = \{(t,x) : |t| \leq a \\ |x| \leq b\}$$

for any fixed $a, b > 0$

Further f is Lipschitz is Lipschitz in x-variable on R.

By the existence and uniqueness theorem, given IVP has a unique solution $x = x(t)$ in the interval $|t| \leq \alpha$, where

$(\quad b \quad)$

$$\alpha = \min\left\{ a, \frac{\quad}{K} \right\}$$

and

$$|f(t, x)| = 1 + x^2$$
$$\leq 1 + b^2$$

we can take $K = 1 + b^2$

$$\therefore \alpha = \min\left\{ a, \frac{b}{1+b^2} \right\}$$

Note that $\dfrac{b}{1+b^2} \leq \dfrac{1}{2}$

$\therefore \alpha \leq \dfrac{1}{2}$ choosing "a" large

Unique Soln is ensured in the interval

$$|t| \leq \frac{1}{2}$$

For instance take

$a = 500$

$$R = \left\{ (t, x) : \begin{array}{l} |t| \leq 500 \\ |x| \leq b \end{array} \right\}$$

Then $f(t, x)$ is continuous and Lipschitz in second variable over R, but

theorem ensures unique

solution in a smaller

interval $|t| \leq \frac{1}{2}$

Note that

$x(t) = \tan t$ is a

solution of IVP; and

it is valid in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Sufficient condition

Lipschitzness:

for Lipschitz ...

Thm

Suppose $f = f(t,x) : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\frac{\partial f}{\partial x}$ exists and bounded in Ω.

Then f is Lipschitz w.r.t. x-variable.

proof

Let (t,x) $(t, x^*) \in \Omega$

$f(t, x^*) |$

$$\left| f(t, x) - \quad \quad \right|$$

$$= \left| \frac{\partial f}{\partial x}\bigg|_{(t,\xi)} (x - x^*) \right|$$

$$= \left| \frac{\partial f}{\partial x}\bigg|_{(t,\xi)} \right| |x - x^*| \quad \text{(By MVT)}$$

$$\leq M |x - x^*|$$

$\therefore f$ is Lipschitz

in x - variable.

Remark:

Consider $f(t,x) = t + |x|$

in $\Omega \begin{cases} -1 \leq t \leq 1 \\ -1 \leq x \leq 1 \end{cases}$

Note that f is Lipschitz in x - variable

However $\dfrac{\partial f}{\partial x}$ is not defined on Ω

$\square$

Theorem (Global existence)

Consider the IVP

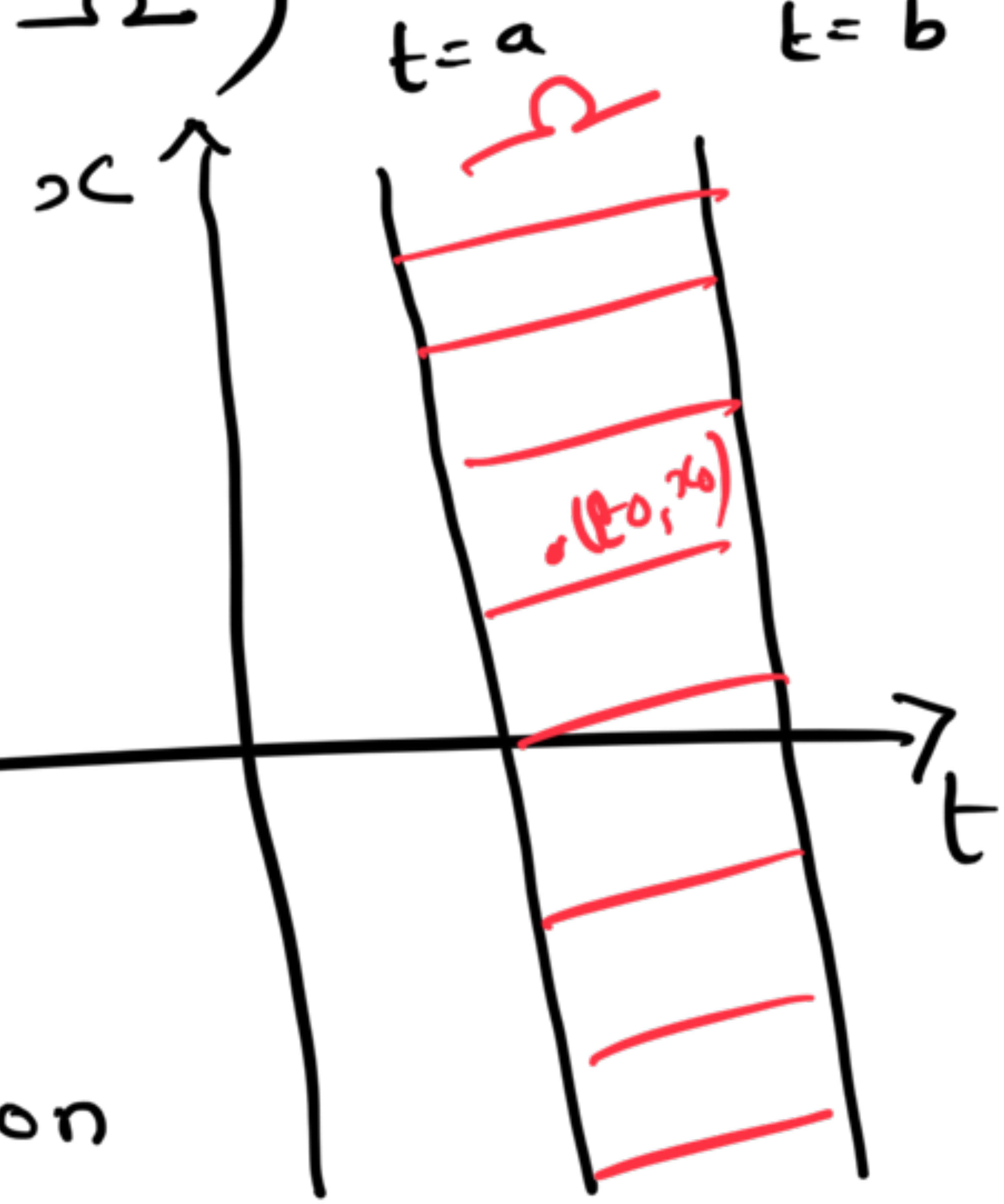
$$\begin{cases} x'(t) = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

Suppose that f is continuous and Lipschitz in x-variable in

$\ldots\,] \times \mathbb{R}$

$\Omega = [a, b] \times \mathbb{R}$

(Here (t_0, x_0) is a point in Ω)



Then the IVP has unique solution in the interval $[a, b]$