

# Electric Fields in Matter

PYL101: Electromagnetics & Quantum Mechanics  
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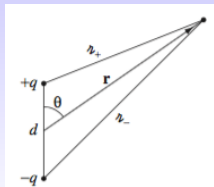
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# References

- ▶ **Introduction to Electrodynamics**, David J. Griffiths [IED]
  - ▶ Chapter IV, Electric Fields in Matter

# Electric Dipole



- ▶ An **electric dipole** consists of two **equal** and **opposite** charges ( $\pm q$ ) separated by a distance  $d$ .
- ▶ Its *potential* at point  $\mathbf{r}$  is<sup>1</sup>,

$$V_{\text{dip.}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{s_+} - \frac{q}{s_-} \right) \approx \frac{1}{4\pi\epsilon_0} \frac{qd \cos \theta}{r^2} \quad (\text{far-field})$$

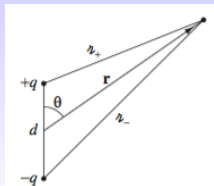
when  $\mathbf{r}$  is **relatively large**, i.e.,  $r \gg d$ .

- ▶ The *far-field* potential of a dipole *falls off* as  $\frac{1}{r^2}$ , as opposed to an isolated point charge, which falls off as  $\frac{1}{r}$ .

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<sup>1</sup>HW: Prove this using the **law of cosines**, i.e.,  $s_+^2 = \left(\frac{d}{2}\right)^2 + r^2 - 2\left(\frac{d}{2}\right)r \cos \theta$ , and  $\cos(\pi - \theta) = -\cos \theta$ .

# The Dipole Moment



- ▶ The **dipole moment** is defined as,

$$\mathbf{p} \equiv q\mathbf{d}$$

where  $\mathbf{d}$  is the displacement vector **pointing from the  $-$  charge to the  $+$  charge**.<sup>2</sup>

- ▶ We may recast the *far-field* potential due to the dipole as,

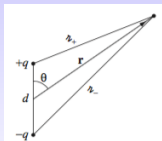
$$V_{\text{dip.}}(\mathbf{r}) \approx \frac{qd \cos \theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}$$

where  $\hat{\mathbf{r}}$  is directed from the **center** of the dipole to the observation point.

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<sup>2</sup>Opposite to how the electric field lines point.

## The Electric Field $\mathbf{E}$ due to $\mathbf{p}$



- Given that,

$$V_{\text{dip.}}(\mathbf{r}) \approx \frac{p \cos \theta}{4\pi\epsilon_0 r^2}$$

and using  $\mathbf{E} = -\nabla V$  expressed in **spherical coordinates** we get,

$$\mathbf{E}_{\text{dip.}}(\mathbf{r}) \approx \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

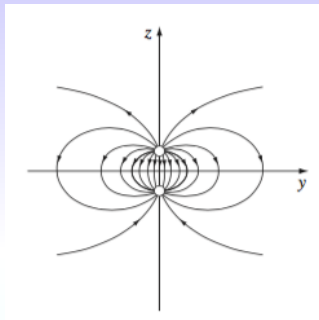
assuming that  $\mathbf{p}$  is oriented along  $\hat{\mathbf{z}}$ .

- Notice that the far-field<sup>3</sup>  $\mathbf{E}_{\text{dip.}}(\mathbf{r})$  falls off as  $\frac{1}{r^3}$ , whereas a single point charge would've fallen off as  $\frac{1}{r^2}$ .

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<sup>3</sup>Again,  $\mathbf{E}_{\text{dip.}}(\mathbf{r})$  is only valid if  $r \gg d$ .

## The **Coordinate-Free** Far Field Dipole Field $\mathbf{E}_{\text{dip.}}$



- **Without** stipulating that the dipole  $\mathbf{p}$  points in  $\hat{\mathbf{z}}$  direction, the dipole field in the *far-field* ( $r \gg d$ ) is<sup>4</sup>,

$$\mathbf{E}_{\text{dip.}}(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0 r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

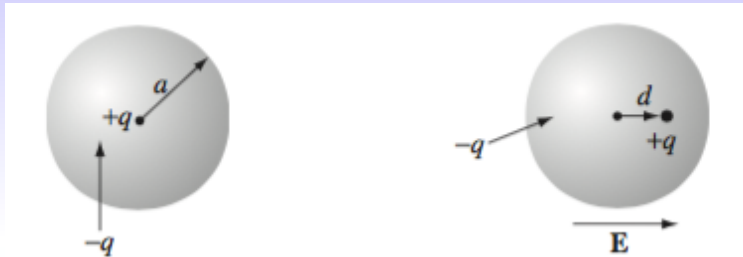
, and is evidently valid for any coordinate system.

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<sup>4</sup>HW: Prove it!

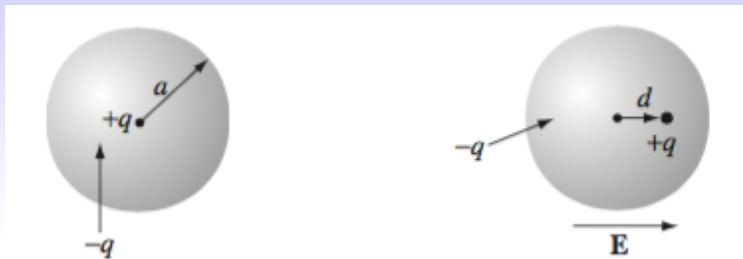


# Induced Dipoles



- ▶ **Problem:** What happens to a **neutral atom** when it's placed in an external electric field  $E_{\text{ext}}$ ?
- ▶ While the atom is electrically neutral as a whole, it consists of a positively nucleus and a negatively charged electron cloud surrounding it.
- ▶ The negatively charged electron cloud is *drawn* to the  $E_{\text{ext}}$ , *relative* to the positively charged nucleus, and when **equilibrium** is reached, the atom is said to be **polarized**.

# Induced Dipoles



- ▶ Each **polarized atom** can be seen as a **tiny dipole** consisting of charges  $+q$  in the nucleus and  $-q$  at the center of the electron cloud.
- ▶ Each such dipole sets up a **small induced electric field**, pointing from the positively charged nucleus to the center of the equally but negatively charged electron cloud.
- ▶ This induced electric field  $E_{\text{ind.}}$ , called a **polarization field**, is **weaker** than and *usually opposite* in direction to  $E_{\text{ext.}}$ .

# Polarizability

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

Figure: Atomic polarizabilities for different isolated atoms. [IEDJ]

- Phenomenologically, each *individual* atom now has a dipole moment  $\mathbf{p}$  proportional to the externally applied electric field,  $\mathbf{E}_{\text{ext.}}$ <sup>5</sup>,

$$\mathbf{p} \propto \mathbf{E}_{\text{ext.}} \mathbf{p} = qd = \alpha \mathbf{E}_{\text{ext.}} \quad (\text{model})$$

pointing in the same direction as  $\mathbf{E}_{\text{ext.}}$ .

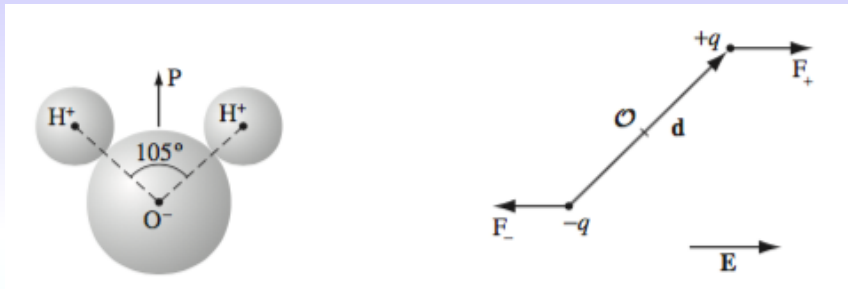
- $\alpha$  is known as the **atomic polarizability**<sup>6</sup>.

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<sup>5</sup>This is an empirical/phenomenological relation *valid* for *relatively small* electric fields.

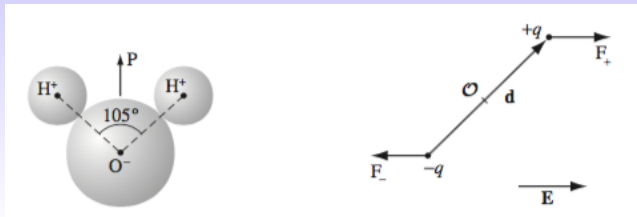
<sup>6</sup>In general,  $\alpha$  is a **tensor** quantity, and thus  $\mathbf{p}$  may not point exactly in the direction of  $\mathbf{E}$ .

# Polar Molecules



- ▶ Thus far we've discussed molecules that had no built-in dipole moment. Their dipole moment was **induced** due to the external electric field.
- ▶ However, there exist molecules like water,  $\text{H}_2\text{O}$  and ethanol that have a **built-in** dipole moment.
- ▶ **Problem:** How does a polar molecule respond to an external field  $\mathbf{E}_{\text{ext}}$ ?

## The Torque on $\mathbf{p}$ due to $\mathbf{E}_{\text{ext}}$ .



- ▶ If  $\mathbf{E}_{\text{ext}}$  is uniform, the figure above suggests a **torque**  $\mathbf{N}$ ,

$$\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) = q\mathbf{d} \times \mathbf{E}_{\text{ext.}} = \mathbf{p} \times \mathbf{E}_{\text{ext.}}$$

- ▶ Notice that the torque  $\mathbf{N}$  is in such a direction as to line up,

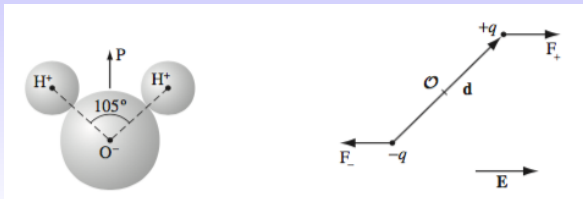
$$\mathbf{p} \parallel \mathbf{E}$$

, i.e., a polar molecule that is free to rotate will swing around until its dipole moment points in the direction of the applied field.<sup>7</sup>

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<sup>7</sup>Why?

## The Force on $\mathbf{p}$ due to $\mathbf{E}_{\text{ext}}$ .



- If the external field is **not** spatially uniform, then  $F_+$  will not exactly balance  $F_-$ , and there will be a **net force**  $\mathbf{F}$  on the dipole,

$$\begin{aligned}
 \mathbf{F} &= (\mathbf{F}^+ + \mathbf{F}^-) = q(\mathbf{E}_{\text{ext.}}^+ - \mathbf{E}_{\text{ext.}}^-) \\
 &= p \underbrace{\lim_{d \rightarrow 0} \frac{\mathbf{E}_{\text{ext.}}(\mathbf{r} + d\hat{\mathbf{n}}) - \mathbf{E}_{\text{ext.}}(\mathbf{r})}{d}}_{\text{directional derivative of a vector function}} \\
 &= p[(\hat{\mathbf{n}} \cdot \nabla)\mathbf{E}_{\text{ext.}}] = (\mathbf{p} \cdot \nabla)\mathbf{E}_{\text{ext.}}
 \end{aligned}$$

assuming the dipole is *relatively short*<sup>8</sup>.

<sup>8</sup>What does a *relatively short* dipole mean in this context?

# The Energy of a Dipole

- **Problem:** What's the energy of a tiny dipole  $\mathbf{p}$  in an external electric field  $\mathbf{E}_{\text{ext.}}$ ?
- For a dipole with charge  $+q$  at  $\mathbf{r} + \mathbf{d}$ , and charge  $-q$  at  $\mathbf{r}$ , its energy in the external field  $\mathbf{E}_{\text{ext.}}$  is<sup>9</sup>,

$$U = q(V_{\text{ext.}}(\mathbf{r} + \mathbf{d})) - q(V_{\text{ext.}}(\mathbf{r})) = q \left[ - \int_{\mathbf{r}}^{\mathbf{r} + \mathbf{d}} \mathbf{E}_{\text{ext.}} \cdot d\mathbf{l} \right]$$

- Since our dipole is relatively **tiny**, *i.e.*,  $d \rightarrow 0$ , we may take  $\mathbf{E}_{\text{ext.}}$  there to be **uniform**, giving us<sup>10</sup>,

$$U = -q\mathbf{E}_{\text{ext.}} \cdot \mathbf{d} = -\mathbf{p} \cdot \mathbf{E}_{\text{ext.}}$$

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<sup>9</sup>How does  $U$  differ from the work required to *assemble* a physical dipole?

<sup>10</sup>Can you justify the negative sign?

# Polarization

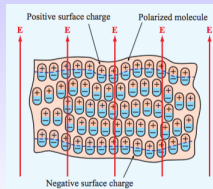
- ▶ **Problem:** What happens to a piece of **dielectric** material when it's placed in an electric field?
- ▶ If the substance consists of **nonpolar molecules/atoms**, the field will induce in **each** of them a tiny dipole moment  $\mathbf{p} \parallel \mathbf{E}$ .
- ▶ If the material is made up of **polar molecules**, each permanent dipole will experience a torque, again tending to line it up  $\mathbf{p} \parallel \mathbf{E}$ .
- ▶ We now define a **macroscopic** measure of polarization for the entire material (**gazillions** of atoms),

$\mathbf{P} \equiv$  dipole moment per unit volume  
called **polarization (density)** such that,

$$\mathbf{p}_{\text{total}} = \sum_{\text{all atoms}} \mathbf{p}_{\text{ind.}} \equiv \int_V \mathbf{P}(\mathbf{r}') d\tau'$$



# Polarization



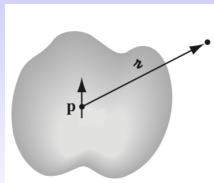
- ▶ Instead of dealing with individual dipoles ( $\mathbf{p}_{\text{ind.}}$ ), we prefer to deal with the dipoles **collectively**,

$$\mathbf{p}_{\text{total}} = \sum \mathbf{p}_{\text{ind.}}$$

and therefore the polarization (density)  $\mathbf{P}$ .

- ▶ This is essentially because the **dipolar field** distribution of an **individual** dipole is quite complex.
- ▶ While the electric field/potential due to the polarization (density)  $\mathbf{P}$  arising from a regular (periodic) arrangement of atoms is quite "well-behaved".
- ▶ Also, once  $\mathbf{P}$  is known, the potential  $V$  (and thus the field  $\mathbf{E}_{\text{dip.}}$ ) due to it becomes **easy to calculate**, as we will see next...

## Bound Charges



- **Problem:** What's the **potential**  $V(\mathbf{r})$  produced due to a collection of dipoles that constitute a polarized medium described by a known polarization (density)  $\mathbf{P}$ ?
- Recall that for a single dipole located at  $\mathbf{r}'$ , the **far-field** potential is,

$$V(\mathbf{r}) \approx \frac{\mathbf{p} \cdot \hat{\mathbf{s}}}{4\pi\epsilon_0 s^2}$$

- Integrating over all the individual dipoles constituting, and using the polarization density  $\mathbf{P}$ , gives us,

$$V(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{s}}}{s^2} d\tau'$$

where  $d\tau'$  is the volume element.

## Bound Charges

- ▶ Using the *mathematical identity*,

$$\nabla' \left( \frac{1}{s} \right) = \frac{\hat{\mathbf{s}}}{s^2}$$

we get,

$$V(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathbf{P} \cdot \nabla' \left( \frac{1}{s} \right) d\tau'$$

- ▶ and then invoking the **divergence theorem**, and **integration by parts** which finally gives,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{1}{s} \mathbf{P} \cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{1}{s} (\nabla' \cdot \mathbf{P}) d\tau'$$

## Bound Charges



$$V(\mathbf{r}) = \underbrace{\frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{1}{s} \mathbf{P} \cdot d\mathbf{a}'}_{\text{first term}} - \underbrace{\frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{1}{s} (\nabla' \cdot \mathbf{P}) d\tau'}_{\text{second term}}$$

- ▶ The *first term* is the potential due to a **surface charge density**

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}' \quad \left[ \frac{C}{m^2} \right]$$

- ▶ The *second term* is the potential due to a **volume charge density**

$$\rho_b \equiv -\nabla' \cdot \mathbf{P} \quad \left[ \frac{C}{m^3} \right]$$

- ▶ In both cases, the **primed notation** ' emphasizes that the relation only applies in the volume in which polarization exists.

## Bound Charges

- ▶ Finally, we may write,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{s} da' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b}{s} d\tau'$$

- ▶ We've concluded that the potential (and, hence also the associated electric field) of a macroscopic, polarized object in the *far-field* is **equivalent** to that produced by,
  - ▶ a surface bound charge density  $\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}'$
  - ▶ a volume bound charge density  $\rho_b \equiv -\nabla' \cdot \mathbf{P}$
- ▶ We're saved from doing the work of **summing** the contributions of each of the **individual** dipoles constituting the material, and instead ...
- ▶ Instead we **first** calculate/find these **bound charges** (both  $\sigma_b$ , and  $\rho_b$ ), and simply calculate the fields they produce using the familiar formula for the potential due to a charge distribution.

## Physical Interpretation of Bound Surface Charges $\sigma_b$

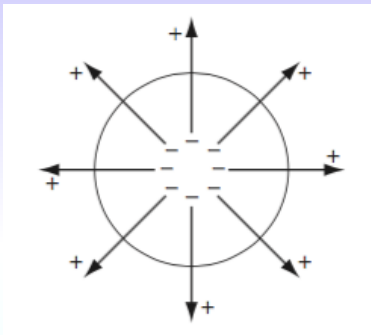


- ▶ Consider the case of **uniform polarization** first, *i.e.*,  $\nabla \cdot \mathbf{P} = 0$ , *e.g.*, a long, linear string of *identical* dipoles.
- ▶ Along the line, the head of one effectively cancels the tail of its neighbor, but at the ends (aka the '*surface*') there are two charges left over: **plus** at the right end and **minus** at the left.
- ▶ These are essentially the **surface bound charges**  $\sigma_b$ .
- ▶ We call the net charges at the ends **bound charges**<sup>11</sup> to emphasize that they **cannot be removed**, as they are always **bound to their respective positively charged ions**.

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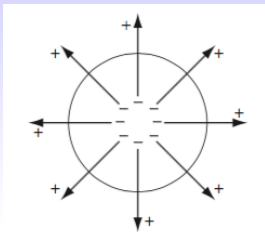
<sup>11</sup>Bound charges may be present even in the **absence** of an external electric field, *e.g.*, an electret.

## Physical Interpretation of Bound Volume Charges $\rho_b$



- ▶ On the other hand, if the polarization is **non-uniform**, i.e.,  $\nabla \cdot \mathbf{P} \neq 0$  we get accumulations of bound volume charge within the material, besides those on the surface due to  $\sigma_b$ .
- ▶ A **diverging**  $\mathbf{P}$  results in a pileup of **negative** charge (at that point) because the higher the  $\nabla \cdot \mathbf{P}$  the more the bound charges would tend to cancel  $\mathbf{E}_{\text{ext}}$ .

## Physical Interpretation of Bound Volume Charges $\rho_b$

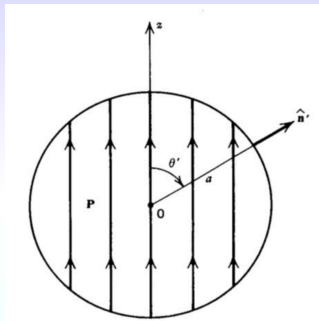


- **Overall charge neutrality** dictates that the overall volume bound charge enclosed in a dielectric  $q_{bv}$  must be **exactly cancelled** by the charge on all its surfaces  $q_{bs}$ , *i.e.*,

$$q_{bs} + q_{bv} = 0$$
$$\oint_{\mathcal{S}} \underbrace{(\mathbf{P} \cdot \hat{\mathbf{n}})}_{\sigma_b} da + \int_{\mathcal{V}} \underbrace{(-\nabla \cdot \mathbf{P})}_{\rho_b} d\tau = 0$$



# The Field Due To A Uniformly Polarized Sphere



- **Problem:** Find the electric field  $\mathbf{E}$ , and potential  $V$  both inside and outside a uniformly polarized  $\mathbf{P} = P\hat{\mathbf{z}}$ , solid sphere of radius  $R$ .

# The Field Due To A Uniformly Polarized Sphere

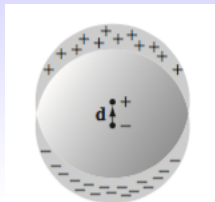
1. Using Gauss' law, and the far-field of a dipole



- ▶ One way to *imagine* bound charges in the **uniformly polarized** case is that we have...
- ▶ *two slightly displaced* uniformly charged spheres: a positive sphere and a negative sphere.
- ▶ Without polarization the two are perfectly superimposed and cancel out completely.
- ▶ This *leftover* charge (at either end) is the bound surface charge  $\sigma_b$ .

# The Field Due To A Uniformly Polarized Sphere

1. Using Gauss' law, and the far-field of a dipole



- The field in the **region of overlap** between two uniformly charged spheres is given by<sup>12</sup>

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0} \frac{q\mathbf{d}}{R^3} \quad (\text{uniform})$$

where  $q$  is the total charge of the positive sphere,  $\mathbf{d}$  is the vector from the negative  $\rightarrow$  positive center.

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<sup>12</sup>Work this out by applying Gauss' law!

# The field due to a uniformly polarized sphere

1. Using Gauss' law, and the far-field of a dipole



- We can express this in terms of the **uniform polarization  $\mathbf{P}$**  of the sphere,

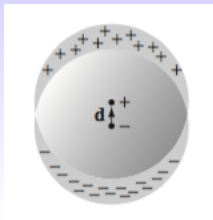
$$\mathbf{p} = q\mathbf{d} = \frac{4}{3}\pi R^3 \mathbf{P}$$

as

$$\mathbf{E} = -\frac{1}{3\epsilon_0} \mathbf{P} \quad (\text{inside})$$

# The field due to a uniformly polarized sphere

1. Using Gauss' law, and the far-field of a dipole



- For points **outside**, it's as though all the charge on each sphere were concentrated at the respective center. We have, at the **far-field**, a simple dipole, with potential

$$V = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

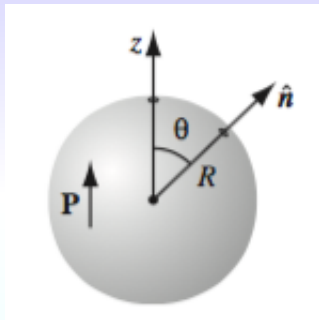
with  $\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P}$ .

- The field outside (**far-field**) the sphere can be determined by

$$\mathbf{E} = -\nabla V$$

# The field due to a uniformly polarized sphere

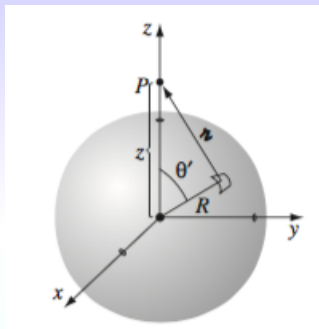
2. Using the method of **bound charges**



- ▶ For convenience, we choose the  $z$  axis to coincide with the direction of polarization  $\mathbf{P}$ .
- ▶ Since  $\mathbf{P}$  is given to be **uniform**, the **volume charge density**  $\rho_b \equiv -\nabla \cdot \mathbf{P} = 0$
- ▶ While the **surface charge density** is  $\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}} = P \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = P \cos \theta$

# The field due to a uniformly polarized sphere

2. Using the method of **bound charges**

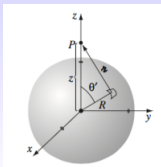


- ▶ We've thus **reduced** the problem to calculating the field due to solely a surface charge density,  $\sigma_b = P \cos \theta$ !
- ▶ From *the law of cosines*,

$$s = \sqrt{R^2 + z^2 - 2Rz \cos \theta}$$

# The field due to a uniformly polarized sphere

2. Using the method of **bound charges**



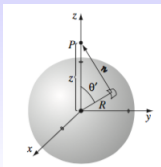
► The potential due to a surface charge density  $\sigma$  is,

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma}{s} da' \\ &= \frac{P}{4\pi\epsilon_0} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{(\cos\theta)R^2 \sin\theta d\theta d\phi}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}} \\ &= \frac{\pi PR^2}{4\pi\epsilon_0 \sqrt{R^2 + z^2}} \int_{\theta=0}^{\pi} \frac{\sin 2\theta}{\sqrt{1 - \frac{2Rz}{R^2 + z^2} \cos\theta}} d\theta \end{aligned}$$



# The field due to a uniformly polarized sphere

2. Using the method of **bound charges**



► We now make use of the result,

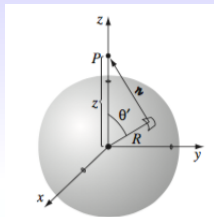
$$\int_{\theta=0}^{\pi} \frac{\sin 2\theta}{\sqrt{1-k\cos\theta}} = -\frac{4\left(2\left(\sqrt{1-k}-\sqrt{k+1}\right)+k\left(\sqrt{1-k}+\sqrt{k+1}\right)\right)}{3k^2}$$

to finally give us,

$$V(r, \theta, \phi) = \begin{cases} \frac{P}{3\epsilon_0} r \cos\theta & r < R \\ \frac{P}{3\epsilon_0} \frac{R^3}{r^2} \cos\theta & r \geq R \end{cases}$$

# The field due to a uniformly polarized sphere

## 2. Using bound charges



- Recognizing that  $r \cos \theta = z$ , the field **inside** the sphere is **uniform**<sup>13</sup>:

$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0} \hat{\mathbf{z}} = -\frac{1}{3\epsilon_0} \mathbf{P} \quad (r < R)$$

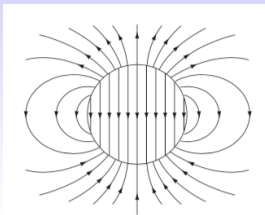
exactly as we found in Method 1.

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<sup>13</sup>Alternatively, use the formula for the gradient in spherical coordinates:  $\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}$

# The field due to a uniformly polarized sphere

2. Using the method of **bound charges**



- **Outside** the sphere, the potential is identical to the *far-field potential* of a dipole at the origin,

$$V = \frac{P}{3\epsilon_0} \frac{R^3}{r^2} \cos\theta = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \quad (r \geq R)$$

whose dipole moment is, *unsurprisingly*, equal to the total dipole moment of the sphere

$$\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P}$$

## Gauss Law in the Presence of Dielectrics

- ▶ Within a dielectric, the **total** charge density could be written as,

$$\rho = \underbrace{\rho_b}_{\text{polarization}} + \underbrace{\rho_f}_{\text{everything else}}$$

where  $\rho_f$  might consist of electrons in a conductor or ions embedded in the dielectric material or whatever; any charge, in other words, **not** a result of polarization!

- ▶ Using **Gauss' Law**, i.e.,  $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ , and the definition of the volume bound charge density  $\rho_b \equiv \nabla \cdot \mathbf{P}$ , we see that,

$$\underbrace{\rho}_{\epsilon_0 \nabla \cdot \mathbf{E}} = \underbrace{\rho_b}_{\nabla \cdot \mathbf{P}} + \rho_f$$

# Gauss Law in the Presence of Dielectrics

- ▶ We **define** the **electric displacement**  $\mathbf{D}$  as,

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

- ▶ and rearrange terms such that,

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \nabla \cdot \mathbf{D} = \rho_f$$

showing that  $\mathbf{D}$  obeys its own **Gauss' law**<sup>14</sup>,

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{or,} \quad \oint \mathbf{D} \cdot d\mathbf{a} = q_f$$

- ▶ But there's a catch...

---

<sup>14</sup>The same **symmetry-related caveats** apply to being able to apply Gauss' law for  $\mathbf{D}$  in practice as was the case with  $\mathbf{E}$ .

## Gauss Law in the Presence of Dielectrics

- ▶ Note that *even in the electrostatic regime, in general,*

$$\nabla \times \mathbf{D} \neq 0$$

since  $\nabla \times \mathbf{P} \neq 0$ , *in general!*

- ▶ Since, *in general*,  $\nabla \times \mathbf{D} \neq 0$ , **Helmholtz's theorem** tells us that there's no **Coulomb's law** associated with  $\mathbf{D}$ , *i.e.*,

$$\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi} \int \frac{\hat{\mathbf{s}}}{s^2} \rho_f(\mathbf{r}') d\tau'$$

, or, *alternatively*, knowledge of  $\rho_f$  alone is **not sufficient** to determine  $\mathbf{D}$ .

## Gauss Law in the Presence of Dielectrics

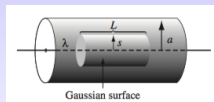
- ▶ While **Gauss' law** **does hold**, *in general*, for  $\mathbf{D}$ , i.e.,

$$\oint \mathbf{D} \cdot d\mathbf{a} = q_f$$

it's **not useful** to actually calculate  $\mathbf{D}$ .

- ▶ Also, *in general*, since  $\nabla \times \mathbf{D} \neq 0$ , we have no scalar potential associated with  $\mathbf{D}$ .
- ▶ However,  $\nabla \times \mathbf{P} = \nabla \times \mathbf{D} = 0$  is guaranteed when **symmetries** such as spherical, cylindrical, or plane are present, and, only in such cases we may use Coulomb's law/Gauss' law to evaluate  $\mathbf{D}$  given  $\rho_f$ .

## Example: $\mathbf{D}$



- **Problem:** A long, straight wire, carrying uniform line charge  $\lambda$ , is surrounded by **rubber** insulation out to a radius  $a$ . Find the electric displacement  $\mathbf{D}$ .
- Since we do have **cylindrical symmetry**, we may use Gauss' Law for *evaluating*  $\mathbf{D}$  given  $\lambda_f = \lambda$ ,

$$D(2\pi sL) = \lambda L$$

and thus,

$$\mathbf{D} = \frac{\lambda}{2\pi s} \hat{\mathbf{s}}$$

- Notice that this formula holds *both* within the insulation **and** outside it.
- Can we determine  $\mathbf{E}$  inside the rubber yet?



## Continuity Conditions for $\mathbf{D}$

- ▶ Using the same procedure of constructing a Gaussian *pillbox* as for the **normal** component  $\mathbf{E}_\perp$  we get<sup>15</sup>,

$$\mathbf{D}_\perp^{\text{above}} - \mathbf{D}_\perp^{\text{below}} = \sigma_f$$

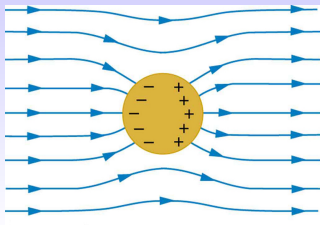
while the continuity of the **tangential** component  $\mathbf{E}_\parallel$  implies,

$$\mathbf{D}_\parallel^{\text{above}} - \mathbf{D}_\parallel^{\text{below}} = \mathbf{P}_\parallel^{\text{above}} - \mathbf{P}_\parallel^{\text{below}}$$

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<sup>15</sup>Think carefully about why we're able to use Gauss' law *without* being guaranteed that  $\nabla \times \mathbf{D} = 0$ .

## Revisiting the Uniformly Polarized Sphere



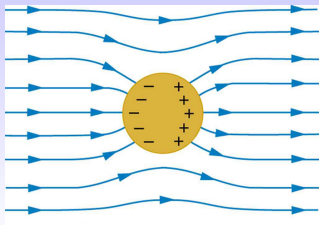
- ▶ **Problem:** For the uniformly polarized sphere with  $\mathbf{P} = P\hat{\mathbf{x}}$ , what's  $\mathbf{D}$  just **outside** the sphere?
- ▶ We'd seen earlier that,

$$\mathbf{E}_{\text{outside}} \neq 0$$

- ▶ Indeed, just outside the sphere we have vacuum, and hence zero polarization, *i.e.*,

$$\mathbf{P}_{\text{outside}} = 0$$

## Revisiting the Uniformly Polarized Sphere



- Using the definition of  $\mathbf{D}$ , we find that

$$\mathbf{D}_{\text{outside}} = \epsilon_0 \mathbf{E}_{\text{outside}} + \mathbf{P}_{\text{outside}} \\ \neq 0$$

despite there being no free charge  $\rho_f$  **anywhere outside!**

- However,

$$\oint \mathbf{D} \cdot d\mathbf{a} = q_f = 0 \quad \text{or,} \quad \nabla \cdot \mathbf{D} = \rho_f = 0$$

still holds for any surface everywhere, yet is **consistent** with  $\mathbf{D} \neq 0$ .

- **Bottom line:** Even if  $\rho_f = 0$ , in general, we **cannot** conclude that  $\mathbf{D} = 0$ .

# Gauss Law in the Presence of Dielectrics

revisited

- **Issue:** In the presence of a dielectric, the **total** charge density  $\rho$  **appears to neglect** the contribution of the **surface charge density**  $\sigma_b$ , and *should* instead be written as,

$$\begin{aligned}\rho &= \rho_b + \rho_{bs} + \rho_f \\ &= \rho_b + \sum_i \sigma_{bi} \delta(\eta - \eta_i) + \rho_f \quad (!!?)\end{aligned}$$

where  $\eta_i$  represents the direction normal to the surface charge  $\sigma_{bi}$ . for say the surface charge lying along the  $x - y$  plane.

- This *issue* arises because in the expression for the potential due to a macroscopic polarization  $\mathbf{P}$ , i.e.,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{s} (-\nabla' \cdot \mathbf{P}) d\tau' + \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{1}{s} \mathbf{P} \cdot d\mathbf{a}'$$

we had not properly accounted for the fact that  $\mathbf{P}$  is a **vector function** and should be defined for **all space**.

- While,  $V(\mathbf{r})$  above is OK, since the *limits of integration* were **restricted to the periphery of the dielectric**, it creates trouble later on...

# Gauss Law in the Presence of Dielectrics

revisited

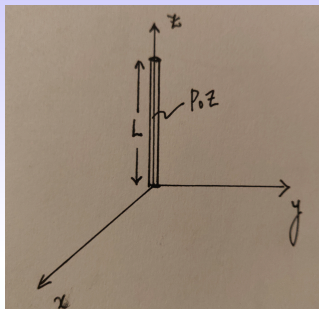
- ▶ A more **mathematically sound** way to write the potential due to a **polarization vector field  $\mathbf{P}$**  is:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{1}{s} (-\nabla \cdot \mathbf{P}) d\tau$$

where the "surface term" **seemingly vanishes**.

- ▶ **Claim:** In this *representation*, the volume charge density  $\rho_b = -\nabla \cdot \mathbf{P}$  actually **contains** the contribution of the bound surface charge density  $\sigma_b$  term, as would be made clear by the next example...

## A 'Uniformly' Polarized Wire



- **Problem:** Consider a **square** wire of length  $L$ , and sides  $a$  with a 'uniform' polarization density  $P_0 = \text{const.}$  along its axis  $\hat{z}$ . **Find the bound charges.**
- We can write the **three-dimensional** polarization **vector field**<sup>16</sup> representing the wire as,

$$\mathbf{P} = P_0 [H(x) - H(x - a)] [H(y) - H(y - a)] [H(z) - H(z - L)] \hat{z}$$

where  $H(x)$  is the Heaviside step function.

---

<sup>16</sup>A vector field must be defined for all space.

## A 'Uniformly' Polarized Wire

- ▶ The negative of the **divergence** is,

$$\begin{aligned}\rho_b &= -\nabla \cdot \mathbf{P} \\ &= 0 - P_0 [H(x) - H(x-a)] [H(y) - H(y-a)] [\delta(z) - \delta(z-L)]\end{aligned}$$

- ▶ The **first term** is **zero** (only for this particular example), and constitutes the bound volume charge density.
- ▶ While the **second term** above is really **two** surface charges  $\sigma_b = P_0 [H(x) - H(x-a)] [H(y) - H(y-a)]$  lying at  $z=0$  and  $z=L$  as,

$$\rho_{bs} = -\sigma_b \delta(z) + \sigma_b \delta(z-L)$$

- ▶ Thus, once you write  $\mathbf{P}$  *properly* as a **vector function**,  $\rho_b = -\nabla \cdot \mathbf{P}$ , actually includes the contribution of **both** the volume and the surface charges!
- ▶ Thus, the relationship,

$$\rho = \rho_f + \rho_b \quad (\checkmark)$$

is **perfectly valid**, and does not neglect the surface charge!

# Susceptibility

- ▶ For many substances, the polarization is proportional to the field, provided  $\mathbf{E}$  is not too strong:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

- ▶  $\chi_e$  is called the **electric susceptibility** of the medium ( $\epsilon_0$  has been extracted to make  $\chi_e$  *dimensionless*).
- ▶ The susceptibility is a **material property**, and substances that obey this *constitutive relation*<sup>17</sup> are called **linear dielectrics**.
- ▶ For such **linear materials**,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi_e) = \epsilon \mathbf{E}$$

where  $\epsilon$  is called the **permittivity** of the material, while

$$\epsilon_r \equiv 1 + \chi_e$$

is known as the **relative permittivity** or **dielectric constant**.

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<sup>17</sup>A **constitutive relation** is a relation between two physical quantities that is specific to a material or substance, and approximates the response of that material to external stimuli.

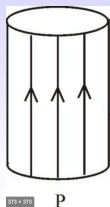


## Common Dielectric Constants

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H <sub>2</sub> )	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N <sub>2</sub> )	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	KTaNbO <sub>3</sub> (0° C)	34,000

- ▶ Notice the variation of nearly 5 orders of magnitude!
- ▶ What explains the wide variation in dielectric constant between materials?

## HW: The Bar Electret



- **Problem:** [Griffiths P.11] A short cylinder, of radius  $a$  and length  $L$ , carries a frozen-in uniform polarization  $\mathbf{P}$ , parallel to its axis<sup>18</sup>. Find the bound charge, and **sketch** the **electric field**  $\mathbf{E}$ , and the **displacement field**  $\mathbf{D}$ .

1.  $L \gg a$
2.  $L \ll a$
3.  $L \approx a$

---

<sup>18</sup>An *electret* is the electrostatic analogue of a permanent magnet.

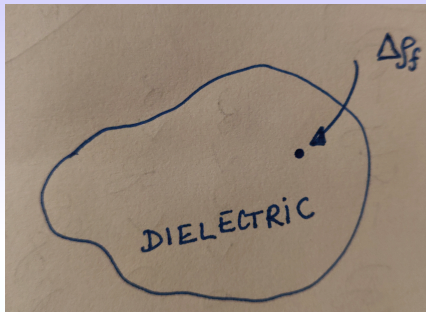
# Energy in Dielectric Systems

- ▶ **Problem:** What's the **work** required to charge up a **dielectric** material?
- ▶ **Claim:** We can simply use the expression derived earlier for the work required to assemble a set of charges, *i.e.*,

$$W = \frac{1}{2} \int \rho V d\tau = \frac{\epsilon_0}{2} \int |\mathbf{E}|^2 d\tau$$

- ▶ **Q:** Is this view correct?
- ▶ **Ans:** It's close but **not** quite, because the work required to charge up a **dielectric** is **not only** of an electrostatic nature, there's also a change in the **internal energy** such as **bond twisting, and stretching** (think of springs) holding the electrons and their ionic cores together, which we must account for.

# Energy in Dielectric Systems



- ▶ Suppose you're given a fixed piece of dielectric having an existing *macroscopic* free charge density  $q_f$ , potential  $V$ , and fields  $\mathbf{E}$  and  $\mathbf{D}$ .
- ▶ We now bring a tiny amount of incremental **free charge**  $\delta q_f$ , to the dielectric.
- ▶ In *response* to  $\delta q_f$ , the polarization  $\mathbf{P}$  (and thus  $\mathbf{E}$ , and  $\mathbf{D}$ ) of the dielectric will change, and therefore the bound charge distribution  $\rho_b$  till **equilibrium** is achieved.

## Energy in Dielectric Systems

- ▶ We will track the **incremental work** done on an **incremental/tiny** amount of **free charge**  $\delta\rho_f$  brought in from  $\infty$  to the dielectric material which provides a potential  $V$  is,

$$\delta W = (\delta q_f)(V - 0)$$

which can be rewritten as,

$$\delta W = \int_{\text{all space}} (\delta\rho_f) V d\tau$$

- ▶ From **Gauss' Law**  $\nabla \cdot \mathbf{D} = \rho_f$ , and for a small change  $\delta$ ,

$$\delta(\nabla \cdot \mathbf{D}) = (\nabla \cdot \delta\mathbf{D}) = \delta\rho_f$$

and thus,

$$\delta W = \int_{\text{all space}} (\nabla \cdot \delta\mathbf{D}) V d\tau$$

- ▶ Using the **vector identity**,

$$\nabla \cdot [(\delta\mathbf{D}) V] = [\nabla \cdot (\delta\mathbf{D})] V + \delta\mathbf{D} \cdot (\nabla V)$$

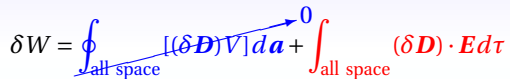
...

## Energy in Dielectric Systems

- ...and then using **integration by parts** we get,

$$\delta W = \int_{\text{all space}} \nabla \cdot [(\delta \mathbf{D}) V] d\tau + \int_{\text{all space}} (\delta \mathbf{D}) \cdot \mathbf{E} d\tau$$

- Applying the **divergence theorem** to the first term gives a **surface integral**,

$$\delta W = \oint_{\text{all space}} [(\delta \mathbf{D}) V] d\mathbf{a} + \int_{\text{all space}} (\delta \mathbf{D}) \cdot \mathbf{E} d\tau$$


- The first term/surface integral then **vanishes** over all space, *i.e.*, even for **point charges**,  $V\delta\mathbf{D} \propto \frac{1}{r} \frac{1}{r^2} \propto \frac{1}{r^3}$ , thus going to zero faster than  $\frac{1}{r^2}$ , and hence the **surface integral** over all space goes to zero.
- Therefore, the **incremental work done** is,

$$\delta W = \int_{\text{all space}} (\delta \mathbf{D}) \cdot \mathbf{E} d\tau \quad (\text{applies to any material})$$

## Energy in Linear Dielectric Systems

- ▶ Now, if the medium is a **linear dielectric**, *i.e.*,  $\mathbf{D} = \epsilon \mathbf{E}$ ,

$$\frac{1}{2} \delta(\mathbf{D} \cdot \mathbf{E}) = \frac{1}{2} \delta(\epsilon E^2) = \epsilon (\delta \mathbf{E}) \cdot \mathbf{E} = (\delta \mathbf{D}) \cdot \mathbf{E}$$

- ▶ We then get,

$$\delta W_{\text{linear}} = \frac{1}{2} \int_{\text{all space}} \delta(\mathbf{D} \cdot \mathbf{E}) \, d\tau$$

- ▶ Finally, we build up the free charge density up from zero to the final configuration:  $\rho_f$ , which corresponds to integrating<sup>19</sup> from zero field up to the final field  $\mathbf{D}(\mathbf{E})$

$$W_{\text{linear}} = \frac{1}{2} \int_{\text{all space}} \int_0^{\mathbf{D}} \delta(\mathbf{D} \cdot \mathbf{E}) \, d\tau$$

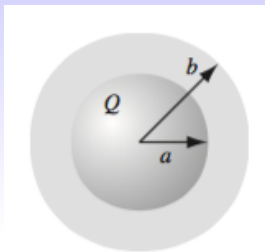
giving us

$$W_{\text{linear}} = \frac{1}{2} \int_{\text{all space}} \mathbf{D} \cdot \mathbf{E} \, d\tau$$

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<sup>19</sup>Such an integration is known as a **functional integration**, in which the domain of an integral is no longer a region of space, but a space of functions..

## Example: metal sphere surrounded by a linear dielectric

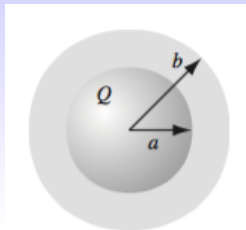


- ▶ **Problem:** A **metal** sphere of radius  $a$  carries a charge  $Q$ . It's surrounded, to radius  $b$ , by a **linear** dielectric of permittivity  $\epsilon$ .
- ▶
  1. Find the potential  $V$  at the center (relative to  $\infty$ ),
  2. the bound charges  $\rho_b$ , and  $\sigma_b$ ,
  3. the total energy stored in the configuration  $W_T$ , and,
  4. the energy stored in the "springs"  $W_S$ , or the internal energy due to polarization inside the dielectric.



## Example: metal sphere surrounded by a linear dielectric

$\mathbf{D}$  and  $\mathbf{E}$  fields



- Using **Gauss' law for  $\mathbf{D}$** , i.e,  $\oint \mathbf{D} \cdot d\mathbf{a} = q_f$ , to **obtain**<sup>20</sup>  $\mathbf{D}$ , we find,

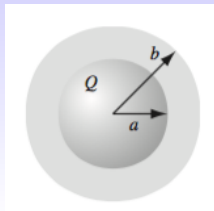
$$\mathbf{D} = \begin{cases} 0 & r \leq a \\ \frac{Q}{4\pi r^2} \hat{\mathbf{r}} & r > a \end{cases}$$

- ...and thus using  $\mathbf{D} = \epsilon \mathbf{E}$ ,

$$\mathbf{E} = \begin{cases} 0 & r \leq a \\ \frac{Q}{4\pi\epsilon r^2} \hat{\mathbf{r}} & a < r < b \\ \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} & r > b \end{cases}$$

## Example: metal sphere surrounded by a linear dielectric

potential  $V$  at center



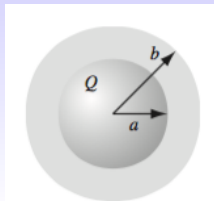
- Calculating the potential at the center  $r = 0$

$$\begin{aligned} V &= - \int_{\infty}^0 \mathbf{E} \cdot d\mathbf{l} = - \int_{\infty}^b \left( \frac{Q}{4\pi\epsilon_0 r^2} \right) dr - \int_b^a \left( \frac{Q}{4\pi\epsilon r^2} \right) dr - \int_a^0 (0) dr \\ &= \frac{Q}{4\pi} \left( \frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right) \end{aligned}$$

- As for  $\mathbf{P}$ , since,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \frac{\epsilon_0 \chi_e Q}{4\pi\epsilon r^2} \hat{\mathbf{r}} \quad (a < r < b)$$

## Example: metal sphere surrounded by a linear dielectric bound charges



- ▶ The bound volume charge density is,

$$\rho_b = -\nabla \cdot \mathbf{P} = 0$$

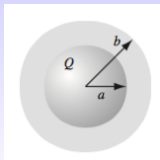
- ▶ While the surface bound charge density is,

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} -\frac{\epsilon_0 \chi_e Q}{4\pi \epsilon a^2} & \text{inner surface} \\ \frac{\epsilon_0 \chi_e Q}{4\pi \epsilon b^2} & \text{outer surface} \end{cases}$$

- ▶ Note that  $\sigma_b(r = a)$  is **negative**, which is *natural*, since the positively charged metal sphere would induce negative charges adjacent to it.

## Example: metal sphere surrounded by a linear dielectric

total stored energy



- In order to calculate the **total stored energy** in the configuration we employ,

$$\begin{aligned} W_T &= \frac{1}{2} \int_{\text{all space}} \mathbf{D} \cdot \mathbf{E} \, d\tau \\ &= \frac{1}{2} \left[ \int_a^b \left( \frac{Q}{4\pi r^2} \hat{\mathbf{r}} \right) \cdot \left( \frac{Q}{4\pi \epsilon r^2} \hat{\mathbf{r}} \right) d\tau + \int_b^\infty \left( \frac{Q}{4\pi r^2} \hat{\mathbf{r}} \right) \cdot \left( \frac{Q}{4\pi \epsilon_0 r^2} \hat{\mathbf{r}} \right) d\tau \right] \\ &= \frac{Q^2}{32\pi^2} (4\pi) \left[ \frac{1}{\epsilon} \int_a^b \frac{1}{r^2} dr + \frac{1}{\epsilon_0} \int_b^\infty \frac{1}{r^2} dr \right] \\ &= \frac{Q^2}{8\pi} \left[ \frac{1}{\epsilon} \left( \frac{1}{a} - \frac{1}{b} \right) + \frac{1}{\epsilon_0} \left( \frac{1}{b} \right) \right] \end{aligned}$$

## Example: metal sphere surrounded by a linear dielectric

'spring' energy

- For the **stored electrostatic energy** in the configuration we invoke,

$$\begin{aligned}W_E &= \frac{\epsilon_0}{2} \int_{\text{all space}} |\mathbf{E}|^2 d\tau \\&= \frac{\epsilon_0}{2} \left[ \int_a^b \left( \frac{Q}{4\pi\epsilon r^2} \right)^2 d\tau + \int_b^\infty \left( \frac{Q}{4\pi\epsilon_0 r^2} \right)^2 d\tau \right] \\&= \frac{Q^2\epsilon_0}{32\pi^2} (4\pi) \left[ \frac{1}{\epsilon^2} \int_a^b \frac{1}{r^2} dr + \frac{1}{\epsilon_0^2} \int_b^\infty \frac{1}{r^2} dr \right] \\&= \frac{Q^2}{8\pi} \left[ \frac{\epsilon_0}{\epsilon^2} \left( \frac{1}{a} - \frac{1}{b} \right) + \frac{1}{\epsilon_0} \left( \frac{1}{b} \right) \right]\end{aligned}$$

- The energy stored in the "springs" is then

$$W_S = W_T - W_E = \frac{Q^2(\epsilon - \epsilon_0)}{8\pi\epsilon^2} \left( \frac{1}{a} - \frac{1}{b} \right)$$