Matrix representation of Linear operators :

Let A be a linear operator on V_F & $B = \{|\beta_i\rangle\}$ be a basis of V_F. Further let

$$A|\beta_j\rangle = A_{ij}|\beta_i\rangle$$

More explicitly

$$A|\beta_1\rangle = A_{11}|\beta_1\rangle + A_{21}|\beta_2\rangle + \cdots A_{n1}|\beta_n\rangle$$
$$A|\beta_2\rangle = A_{12}|\beta_1\rangle + A_{22}|\beta_2\rangle + \cdots A_{n2}|\beta_n\rangle$$
$$\vdots$$
$$A|\beta_n\rangle = A_{1n}|\beta_1\rangle + A_{2n}|\beta_2\rangle + \cdots A_{nn}|\beta_n\rangle$$

The matrix of these coefficients i.e. $\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$ is referred to as the <u>matrix representation of operator A</u> w.r.t. Basis B.

- Once we have the matrix representation of an operator in a given basis B, the action of the operator on an arbitrary vector is simply represented by the <u>matrix multiplication on the coordinates of the vector in same basis</u>.

Let $|\alpha\rangle = a_i|\beta_i\rangle$ where $B \equiv \{|\beta_i\rangle ; i = 1, 2, \ldots n\}$ is a basis of V_F

then $|\gamma\rangle = A|\alpha\rangle = A(a_j|\beta_j\rangle)$

$$= a_j(A|\beta_j\rangle) = a_j A_{ij}|\beta_i\rangle = A_{ij} a_j |\beta_i\rangle$$

Comparing with $|\gamma\rangle = c_i|\beta_i\rangle$

we have $\boxed{c_i = A_{ij}\, a_j}$

Equivalently, we can write the above equation in matrix notation as

$$c = A \cdot a \quad \text{where}$$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

Effect of change of Basis :

Let $B = \{|\beta_i\rangle\}$ & $B' = \{|\beta_i'\rangle\}$ be two basis of V_F related to each other via

$$|\beta_i'\rangle = S_{ij}\,|\beta_j\rangle$$

{ Note that $S = [S_{ij}]$ is an invertible matrix }

then, we have

$$|\beta_i\rangle = (S^{-1})_{ij}\,|\beta_j'\rangle$$

1. On coordinates of vectors

$$|\alpha\rangle = a_i\,|\beta_i\rangle$$
$$= a_i\,\delta_{ij}\,|\beta_j\rangle = a_i\,S^{-1}_{ik}\,S_{kj}\,|\beta_j\rangle$$
$$= a_i\,S^{-1}_{ik}\,|\beta_k'\rangle \equiv a_k'\,|\beta_k'\rangle$$

$$\Rightarrow a_k' = a_i\,S^{-1}_{ik} \xrightarrow[\text{notation}]{\text{In matrix}} a'^T = a^T S^{-1} \xrightarrow{\text{Transpose}} \boxed{a' = S^{-1T} a}$$ ★

2. On matrix representation of Operators

For an arbitrary operator A we have

$$A\,|\beta_i\rangle = A_{ji}\,|\beta_j\rangle$$
$$\Rightarrow A\,(S^{-1})_{in}\,|\beta_n'\rangle = A_{ji}\,(S^{-1})_{jk}\,|\beta_k'\rangle$$
$$A\,|\beta_n'\rangle = S_{ni}\,A_{ji}\,(S^{-1})_{jk}\,|\beta_k'\rangle \equiv A'_{kn}\,|\beta_n'\rangle$$
$$\Rightarrow A'_{kn} = S_{ni}\,A_{ji}\,S^{-1}_{jk}$$
$$A'^T = S \cdot A^T \cdot S^{-1} \Rightarrow \boxed{A' = S^{-1T} \cdot A \cdot S^T}$$ ★

If we redefine the matrix S as $\boxed{S = (M^T)^{-1}}$ then the expressions look a little cleaner as

$$\boxed{\begin{aligned} a' &= M \cdot a \\ A' &= M \cdot A \cdot M^{-1} \end{aligned}}$$

w/ $\boxed{|\beta_i'\rangle = M^{-1}_{ji}\,|\beta_j\rangle}$