Lecture 23: Fourier Transforms

Properties of Fourier Transforms

- Conjugate symmetry
- Linearity
- Time shift
- Modulation
- Convolution
- Duality
- Parseval’s theorem
- Integration
- Differentiation

1. Conjugate symmetry

If $x(t)$ is real, $\overline{X(\omega)} = X(-\omega)$

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If $x(t)$ is real, $\overline{X(\omega)} = X(-\omega)$

Proof:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\overline{X(\omega)} = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = X(-\omega)$$

a) If $\overline{X(\omega)} = X(-\omega)$, $|X(\omega)|$ is an even function.

Proof (hint): $|X(\omega)| = \left|\overline{X(\omega)}\right| = |X(-\omega)|$

b) If $\overline{X(\omega)} = X(-\omega)$,$\angle X(\omega)$ is an odd function.

Proof (hint): $\angle X(\omega) = -\angle\overline{X(\omega)} = -\angle X(-\omega)$

2. Linearity

$$x(t) \overset{\mathbb{F}}{\leftrightarrow} X(\omega)$$

$$y(t) \overset{\mathbb{F}}{\leftrightarrow} Y(\omega)$$

$$ax(t) + by(t) \overset{\mathbb{F}}{\leftrightarrow} aX(\omega) + bY(\omega)$$

3. Time-shift

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$$x(t - t_0) \overset{\mathbb{F}}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$$

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Proof:

Assume $X'(\omega) = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$

Put $t - t_0 = \tau$

$$X'(\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} e^{-j\omega t_0} d\tau$$

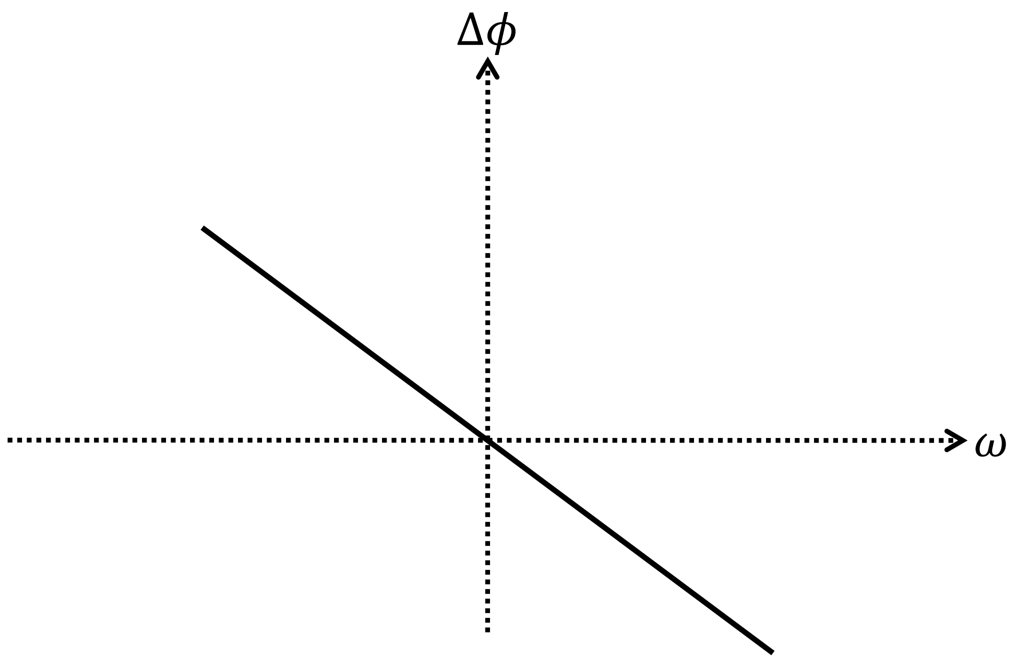
$$X'(\omega) = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$$X'(\omega) = e^{-j\omega t_0} X(\omega)$$

Physical Interpretation of Time-Shift

Assume $x(t - t_0) \overset{\mathbb{F}}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$

The magnitude of $X(\omega)$ does not change when we time shift the signal only phase changes linearly with frequency.



Physical Interpretation of Time-Shift

$$x(t) = \sin(\omega t) + \sin(2\omega t)$$

$$x(t - t_0) = \sin(\omega t - \omega t_0) + \sin(2\omega t - 2\omega t_0)$$

If a signal is delayed by time t_0 then individual components of the signal get delayed by a linear phase shift.

If phase change is non-linear , then different components get delayed differently, leading to pulse broadening, referred to as dispersion.

4. Modulation

$$x(t) \overset{\mathbb{F}}{\leftrightarrow} X(\omega)$$

$$x(t)e^{j\omega_0 t} \overset{\mathbb{F}}{\leftrightarrow} X(\omega - \omega_0)$$

Proof:

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Proof:

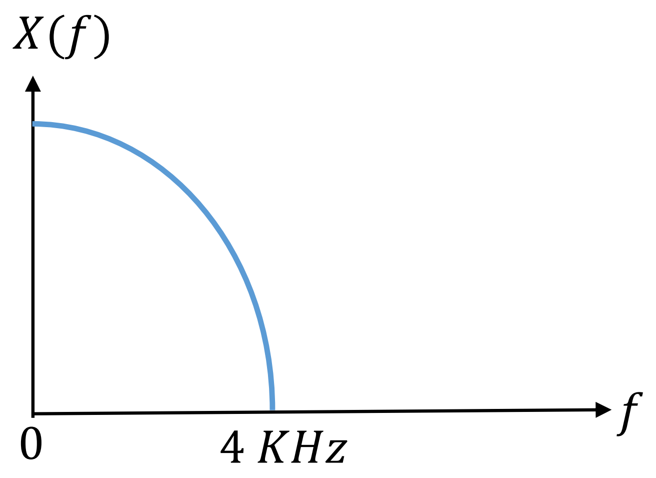
$$X'(\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}\, e^{-j\omega t}\mathrm{dt}$$

$$X'(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}\,\mathrm{dt}$$

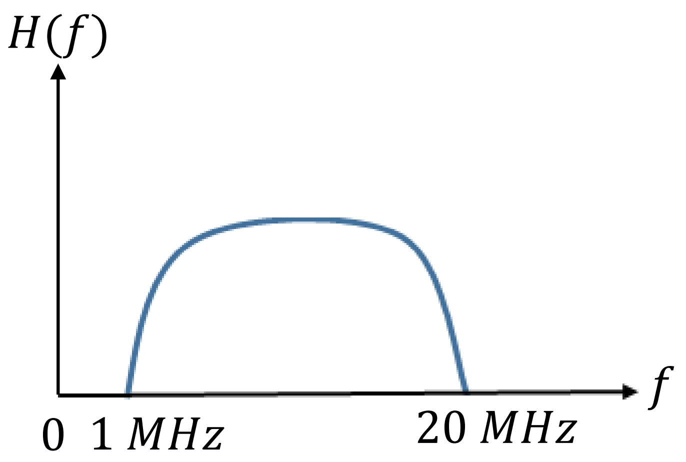
$$X'(\omega) = X(\omega - \omega_0)$$

Modulation

Speech signal (0-4 KHz)



Frequency response $H(f)$ of a twisted pair



Modulation

$x(t)e^{j2\pi\times 10\ MHz\times t}$ shifts the signal spectrum to 10 MHz.

Match the signal characteristics with the channel characteristics.

5. Convolution

$$x(t) \overset{\mathbb{F}}{\leftrightarrow} X(\omega)$$
$$y(t) \overset{\mathbb{F}}{\leftrightarrow} Y(\omega)$$
$$x(t) * y(t) \overset{\mathbb{F}}{\leftrightarrow} X(\omega)Y(\omega)$$

Convolution in time domain is equivalent to multiplication in frequency domain.

Proof:

5. Convolution

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Convolution in time domain is equivalent to multiplication in frequency domain.

Proof:

$$F[x(t) * y(t)] = F\left[\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} y(t-\tau)e^{-j\omega t}dt\right]d\tau$$

Convolution

By Putting, $t - \tau = \lambda$ in equation: $\int_{-\infty}^{\infty} y(t-\tau)e^{-j\omega t}dt$

$$\int_{-\infty}^{\infty} y(t-\tau)e^{-j\omega t}dt \quad becomes \quad \int_{-\infty}^{\infty} y(\lambda)e^{-j\omega\lambda}\, e^{-j\omega\tau}\mathrm{d}\lambda$$

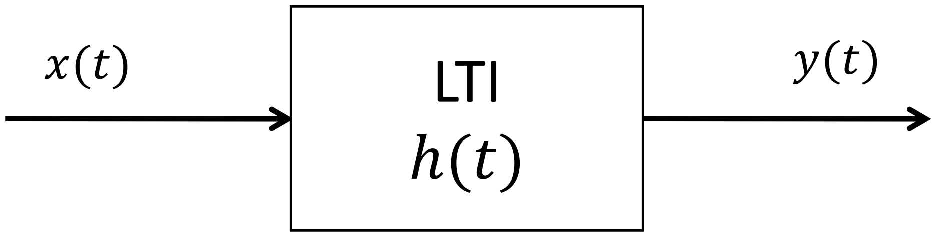
$$F[x(t) * y(t)] = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} y(\lambda)e^{-j\omega\lambda}d\lambda\right]e^{-j\omega\tau}d\tau$$

$$F[x(t) * y(t)] = Y(\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\, d\tau = Y(\omega)\mathrm{X}(\omega)$$

$$F[x(t) * y(t)] = Y(\omega)\mathrm{X}(\omega)$$

Convolution

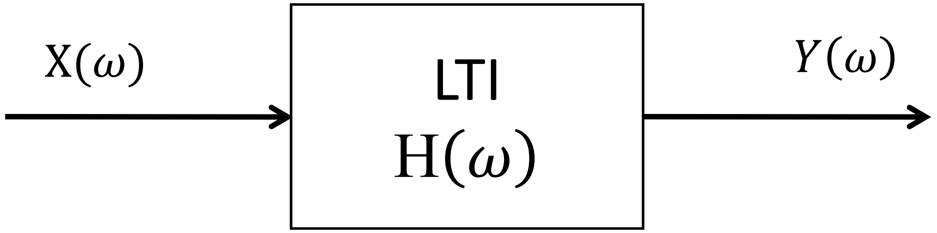
- In time domain,



$$y(t) = x(t) * h(t)$$

where $h(t)$ is the impulse response of the system.

- In frequency domain,



$$Y(\omega) = \mathrm{X}(\omega)\, \mathrm{H}(\omega)$$

Implications

- Computationally efficient
- LTI system cannot generate new frequencies
- Frequency response provides point-wise decoupling

6. Duality

Duality states if

$$x(t) \overset{\mathbb{F}}{\leftrightarrow} X(\omega)$$

$$X(t) \overset{\mathbb{F}}{\leftrightarrow} 2\pi x(-\omega)$$

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Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(u)e^{jut}du$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(u)e^{-ju\omega}du$$

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Prove using duality: $x(t)y(t) \xrightarrow{F.T.} \frac{1}{2\pi}[X(\omega) * Y(\omega)]$ (Multiplication property)

7. Parseval's Theorem

$$x(t) \overset{\mathbb{F}}{\leftrightarrow} X(\omega)$$

$$y(t) \overset{\mathbb{F}}{\leftrightarrow} Y(\omega)$$

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\,\overline{Y(\lambda)}d\lambda$$ Parseval's Theorem

7. Parseval’s Theorem

$$\overline{y(t)} \overset{\mathbb{F}}{\leftrightarrow} \overline{Y(-\omega)}$$

$$x(t)y(t) \overset{\mathbb{F}}{\leftrightarrow} \frac{1}{2\pi}[X(\omega) * Y(\omega)]$$

Parseval's Theorem

$$x(t)\overline{y(t)} \overset{\mathbb{F}}{\leftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\,\overline{Y(\lambda-\omega)}d\lambda$$

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}e^{-j\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\,\overline{Y(\lambda-\omega)}d\lambda$$

Put $\omega = 0$

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\,\overline{Y(\lambda)}d\lambda \quad \text{Parseval's Theorem}$$

Parseval's Theorem

Let $y(t) = x(t)$

$$\int_{-\infty}^{\infty} x(t)\overline{x(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\,\overline{X(\lambda)}d\lambda$$

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\, d\omega$$

$|X(\omega)|^2 \rightarrow$ Energy spectral density

$|x(t)|^2 \rightarrow$ Energy time density.

Parseval's Theorem

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\,\overline{Y(\lambda)}d\lambda \quad \text{Parseval's Theorem}$$

- Inner product (dot product) of two vectors is independent of the choice of unit vectors.

  $x(t).y(t) \rightarrow$ unit vectors are impulse functions.

  $X(\omega).Y(\omega) \rightarrow$ unit vectors are complex exponentials.

- Orthogonality of signals can be checked in time domain or in frequency domain

8. Differentiation

$$x(t) \overset{\mathbb{F}}{\leftrightarrow} X(\omega)$$

$$\frac{dx(t)}{dt} \overset{\mathbb{F}}{\leftrightarrow} j\omega X(\omega)$$

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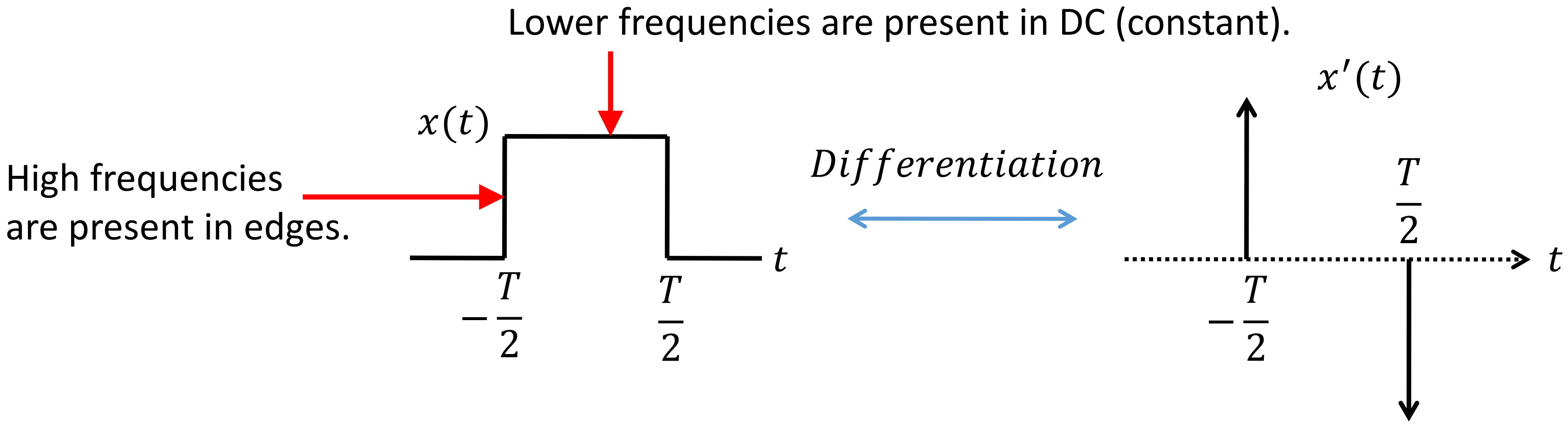
Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)j\omega e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} \overset{\mathbb{F}}{\leftrightarrow} j\omega X(\omega)$$

Differentiation



9. Integration

$$x(t) \overset{\mathbb{F}}{\leftrightarrow} X(\omega)$$

$$\int_{-\infty}^{t} x(t)dt \overset{\mathbb{F}}{\leftrightarrow} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

9. Integration

$$x(t) \overset{\mathbb{F}}{\leftrightarrow} X(\omega)$$

$$\int_{-\infty}^{t} x(t)dt \overset{\mathbb{F}}{\leftrightarrow} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Proof:

$$\int_{-\infty}^{t} x(t)dt = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau$$

$$= x(t) * u(t)$$

$$= X(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

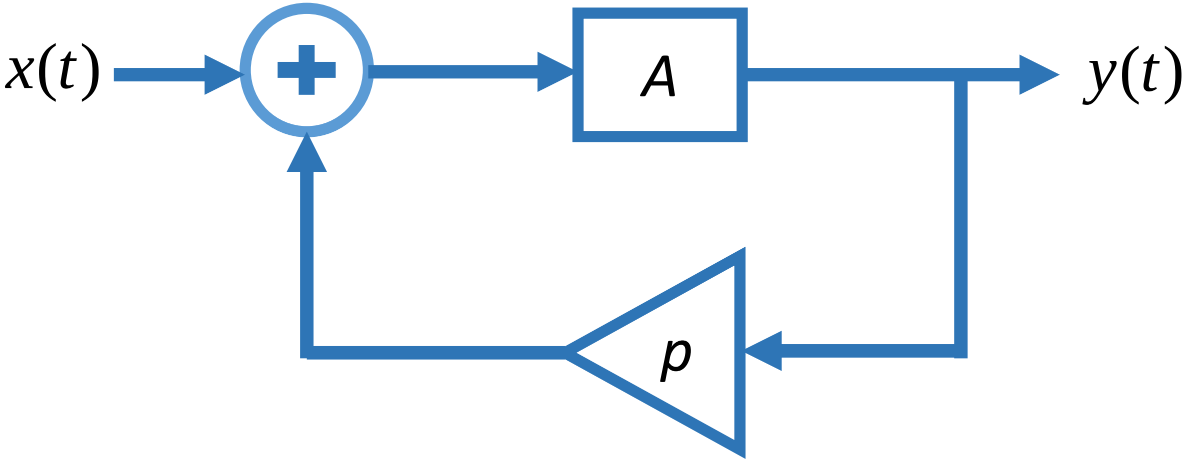
Outline

- Properties
- Differential Equations

Differential equations

LTI and

Causal



$$\sum_{k=0}^{M} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}$$

Differential equations

$$\sum_{k=0}^{M} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}$$

Linear constant coefficient differential equation may not be for LTI systems

$$y(t) = x(t) + 3$$

LTI system may not have a linear constant coefficient differential equation

$$y(t) = x(t-3)$$

Differential equations

$$\sum_{k=0}^{M} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}$$

LTI system, causal and stable $\Rightarrow$

$$\sum_{k=0}^{M} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{N} b_k (j\omega)^k X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{N} b_k (j\omega)^k}{\sum_{k=0}^{M} a_k (j\omega)^k}$$

Differential equations

LTI, causal, and stable

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{N} b_k (j\omega)^k}{\sum_{k=0}^{M} a_k (j\omega)^k}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\quad}{\quad}$$

Differential equations

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Differential equations

LTI, causal, and stable → $$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

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$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1(j\omega)^0}{}$$

Differential equations

LTI, causal, and stable $\Rightarrow$ $$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

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$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1(j\omega)^0 + 1(j\omega)^1}{}$$

Differential equations

LTI, causal, and stable $\Rightarrow$

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Differential equations

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Differential equations

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Differential equations

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Differential equations

LTI, causal, and stable → $$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1(j\omega)^0 + 1(j\omega)^1}{6(j\omega)^0 + 5(j\omega)^1 + 1(j\omega)^2} = \frac{1 + j\omega}{(2 + j\omega)(3 + j\omega)}$$

$$H(\omega) = \frac{-1}{2 + j\omega} + \frac{2}{3 + j\omega}$$

$$\mathrm{h}(t) = -e^{-2t}u(t) + 2e^{-3t}u(t)$$