Lecture 7 Mon 7 Aug 2023

$Ax \le b \qquad x \ge 0 \Rightarrow \underline{-x \le 0}$

$$\begin{bmatrix} A \\ -I \end{bmatrix} x \le b$$

$n \times m \qquad G_{n \times n} \longrightarrow Gx = g$

* Basic feasible solution (BFS)

Consider,

$$Ax = b$$
$$x \ge 0$$

Any linear inequality can be converted into equality.

$$3x_1 + x_2 - 4x_3 \le 5$$

$\downarrow$

$\underline{\text{add } \ge 0}$

$$3x_1 + x_2 - 4x_3 + x_4 = 5$$
$$x_4 \ge 0.$$

(additional non-negative variable)

$x_4 \rightarrow$ Slack variable

$$2x_1 + 7x_2 - x_3 \ge 4$$

$$-2x_1 - 7x_2 + x_3 \le -4$$

$$-2x_1 - 7x_2 + x_3 + x_4 = -4$$

$$\Rightarrow 2x_1 + 7x_2 - x_3 - x_4 = 4$$

$$x_4 \ge 0.$$

Slack Variable
(Surplus Variable).

Assume $A : m \times n$, $m < n$.

Rank $A = m$

(1) $\left[\begin{array}{c} Ax = b \\ \downarrow \\ m \times n \end{array} \right]$
(full row rank)

$\exists$ at least one submatrix $B_{m \times m}$ of A s.t. B^{-1} exists.

$\Rightarrow$ There are $n-m$ columns of A which are not part of B.

$$A = [B \cdot R]$$

B: $m \times m$ $\quad$ R: $m \times (n-m)$.

$\downarrow$

for those columns in R, put $x_i = 0$.

$x : n \times 1$

$\downarrow$

out of which $n-m = 0$

Remaining $m \times 1$
call it by $x_B : m \times 1$.

$Ax = b \iff Bx_B = b \iff x_B = B^{-1} b$

If $x_B \geq 0$ then $x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$ is called BFS of (1).

else, $x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$ is a sol^n of $Ax=b$ but not a feasible sol^n of (1).

maximum number of BFS $= {}^nC_m$

$B \rightarrow$ Basis matrix

$x_B \rightarrow$ Basic feasible sol^n

Any Real Matrix,

$M_{m\times n}$ rank $M = m < n$.

Find sol^n of $\underset{\substack{\downarrow \\ x \in \mathbb{R}^n}}{M x} = u$.

$n\times 1$ $m\times 1$

3×7

rank $= 3$

BFS $x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$, $x_B = B^{-1}b \geq 0$

degenerate BFS

$\downarrow$

If atleast one of component of $x_B : m\times 1$, is zero.

non degenerate

$x_B = B^{-1} b$

$m\times 1$ $m\times 1$

$\downarrow$

$m\times m$

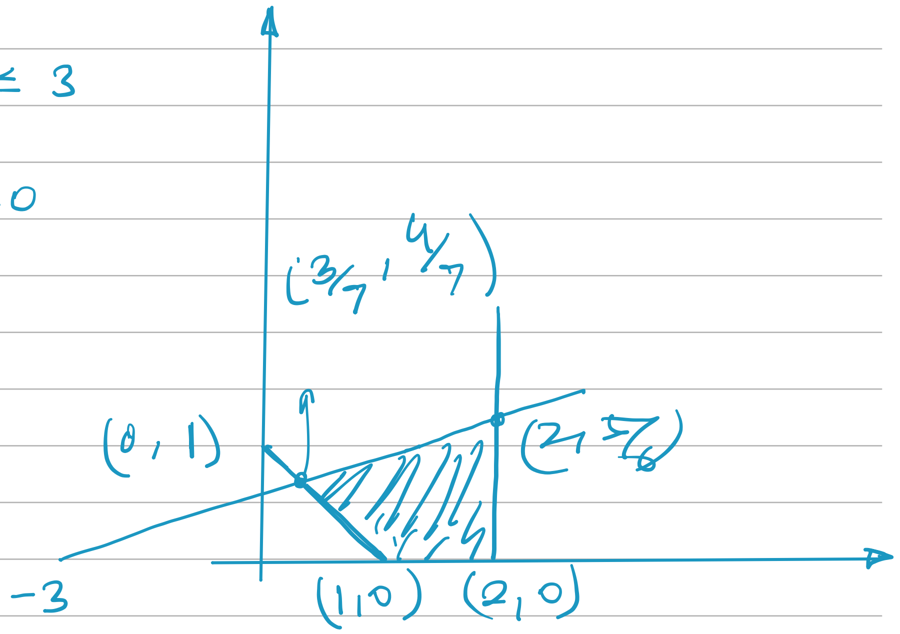
>0

all m components are >0.

If a polyhedron $\{Ax=b, x\geq 0\}$ has at least one degenerate BFS, then polyhedron is called degenerate polyhedron.

Let us consider a system

$$x_1 + x_2 \ge 1$$
$$-x_1 + 6x_2 \le 3$$
$$x_1 \le 2$$
$$x_1, x_2 \ge 0$$



$$x_1 + x_2 - x_3 = 1$$
$$-x_1 + 6x_2 + x_4 = 3$$
$$x_1 + x_5 = 2$$
$$x_1, x_2, \dots, x_5 \ge 0$$

$$\underset{3\times 5}{A} = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 6 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$m = 3$

$n = 5$

# of choices for $B_{3\times 3}$ submatrix of A

$= {}^5C_3 = 10$

(which are invertible).

Let $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

put $x_1 = x_2 = 0$

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$Bx_B = b \Rightarrow x_3 = -1 \longrightarrow$ not a BFS

$x_4 = 3$

$x_5 = 2$

let $B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow$ invertible

$Bx_b = b \Rightarrow x_1 = 1 \qquad x_2 = x_3 = 0$

$x_4 = 4$

$x_5 = 1$

$x_B = \begin{pmatrix} x_1 = 1 \\ x_4 = 4 \\ x_5 = 1 \end{pmatrix}$

non degenerate BFS.

$(1, 0, 0, 4, 1)$