HomeWork2 Solutions

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Problem 1

Let $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ be any two acyclic graphs (a.k.a. forests) on the same vertex set V such that $|E_1| < |E_2|$. Prove that there exists an edge $e \in E_2 \setminus E_1$ such that $F = (V, E_1 \cup \{e\})$ is also an acyclic graph.

Solution: We proove by contradiction.

We assume there exists no edge $e \in E_2 \setminus E_1$ such that $F = (V, E_1 \cup \{e\})$ is also an acyclic graph.

We know that, in an acyclic graph $G = (V, E)$,

$$|S| = 2 \times |E| \tag{1}$$

where S is the set of vertices lying on the edges of E as there would be no edge connecting the vertices which have already been connected by some other edge (It can also be said that no vertex will be counted twice while counting the number of edges).

From our assumption, we can say that there exists no edge $e \in E_2$ which joins two vertices in V which were not connected by E_1.
If there was such an edge then $F = (V, E_1 \cup \{e\})$ would be acyclic.

We can also see that there exist no edge $e \in E_2$ which connectc a vertice connected by E_1 to a vertice not connected by E_1.
If there were such an edge then $F = (V, E_1 \cup \{e\})$ would be acyclic.

From the above two claims we can see that the set of vertices lying in E_1 , say V_{E_1}, is a superset of the set of vertices lying in E_2, say V_{E_2}.
So from (1), we get,

$$2 \times |E_1| = |V_{E_1}| \geq |V_{E_2}| = 2 \times |E_2|$$

From this we get $|E_1| \geq |E_2|$, which is a contradiction to $|E_1| < |E_2|$.

Hence Proved.

Problem 2

Prove that every connected graph $G = (V, E)$ has a closed walk which traverses every edge in E exactly twice. Hence, prove that every connected graph $G = (V, E)$ has a closed walk of length $2|V| - 2$ which visits every vertex in V at least once.

Solution: Proof by construction.

We define inverse of a path $p = v_1 v_2 \ldots v_n$ as $p^{-1} = v_n \ldots v_2 v_1$.

Now, consider a path, say p_o, from any vertex, say v_1 to another arbitrary vertex say v_n.
Let that path be $v_1 v_2 \ldots v_n$.
Let's say a vertex, say $v_k \in V$, is not in our path.
We now consider a path, say p, from some vertex, say v_i in our path to v_k such that no other vertex of our path lies on it.
Claim: Such a path exists.

As G is connected, there must exist a path from v_1 to v_k.
If some other vertex of our path, say v_i, lies on the path from v_1 to v_k then we will consider the path from v_i to v_k.
And since length of path is a natural number, by well ordering principle, there must exist a shortest path. So this can't keep going infinitely.
So, our claim is true.

We now consider the walk $v_1 v_2 \ldots v_{i-1} p p^{-1} \ldots v_n$.

We keep repeating this process until all elements in V are covered.
Since V is finite, our process will take finite time.

Now we add the inverse of p_o to our resulting walk, which yeilds a walk, $w = v_1 v_2 \ldots v_{n-1} p_o^{-1}$.

We can see that w covers all the vertices in G.
Now, if there is some edge $e = (v_i, v_j) \in E$, which is not traversed by w then in w, we replace v_i by $v_i v_j v_i$.
And we keep doing it till all edges in E are traversed.

We can see that w is a closed walk, and every edge is traversed.

Hence the first part is proved.

Now for the second part, we take the walk, w, obtained when we added p_o^{-1} in the solution to part 1.

We can see that this walk traverses all the vertices in V.
Consider, a graph $G' = (V, E^*)$,where E^* is the set of all edges present in w.
We can see that G' is acyclic.
So, $|E^*| = |V| - 1$.
Since every edge is traversed exactly twice, the length of the walk becomes $2|V| - 2$.

Hence Proved.

Problem 3

A graph is said to be *regular* if the degrees of all its vertices are the same, A matching M in a graph is said to be a *perfect matching* if for every vertex v, M contains an edge incident on v (that is, all vertices are matched by M). Prove that the edge set of every bipartite regular graph can be partitioned into perfect matchings. (Convince yourself that the claim is not true if the graph is (i) regular but not bipartite, even if it has an even number of vertices (ii) bipartite but not regular, even if the graph is connected, both sides of its bipartition contain an equal number of vertices, and the degree of every vertex is at least d, where d is as large a constant as you want.)

Lemma: Every bipartite regular graph, $G = (V, E)$ has a perfect matching.

Proof: We have two sets V_1 and V_2 which make G bipartite.

We take any arbitrary subset of V_1, say S, and the set of its neighbours as $N(S)$.

We can see that $|S| < |N(S)|$ as every edge containing a vertex in S has an endpoint in $N(S)$ but the converse is not true.
Therefore the cardinality of edge set of S, $d|S|$ will be less than the cardinality of edge set of $N(S)$, $d|N(S)|$, where d is the degree of every vertex in G.
So, by Hall's Theorem, there exists a mathing M which matches every vertex in V_1.
Since G is regular, $|V_1| = |V_2|$. So, The mathing M will be perfect matching.

We now prove the original problem by induction on degree of vertices of the graph.

Base Case: When the degree of all vertices is zero then the edge set is empty so it can be partitioned to 0 perfect matchgings.
When the degree of all vertices is one then the edge set is perfect matching as if (v_1, v_2) is an edge then there will be no other edge whose endpoint is either v_1 or v_2 and all vertices will be the endpoint of exactly one edge.

Induction Hypothesis: We assume our claim holds true for a graph whose each vertex has a degree of $k - 1$.

Induction Step: We now consider a bipartite regular graph, say $G = (V, E)$, whose each vertex has a degree k.
Since G is a bipartite regular graph there must exist a perfect matching in G, say M.

Now, consider a graph $G' = (V, E \setminus M)$.
We can see that G' is also a bipartite regular graph with degree of its vertices as $k - 1$ (As every vertex will have exactly one edge in M).

Since G' is a bipartite regular graph with degree $k - 1$, by our induction hypothesis, the edgeset ,i.e., $E \setminus M$ can be partitioned into perfect matchgings.

So the edgeset of G can also be partitioned into perfect matchings with the partition set being $P(E \setminus M) \cup \{M\}$, where $P(S)$ be the partition set of S such that every element in $P(S)$ is a perfect matching.
By the **principle of mathematical induction**, our claim is true for all degrees ≥ 0.

Hence Proved.

Problem 4

Find an expression for the number of perfect matchings in a complete graph on n vertices, and prove your answer. Your expression must involve only a constant number of applications of only the following mathematical operators: addition, subtraction, multiplication, division, exponentiation, and factorial.

Solution: We claim,

$$f(2k) = \begin{cases} \frac{(2k)!}{k! \times 2^k} & \text{if } k \in \mathbb{N} - \{0\} \\ 0 & \text{otherwise} \end{cases}$$

Here $2k$ is the number of vertices and $f(n)$ is the number of perfect matchings in a complete graph of n vertices.

Proof
We can see that $f(2k) = (2k - 1) \times f(2(k - 1))$ when $k > 1$.

If $k \notin \mathbb{N} - \{0\}$ then that means the number of vertices will either be fractional which is not possible or it will odd, in which perfect matching does not exists.
So, we are only left to prove when $k \in \mathbb{N} - \{0\}$.

Now we will use induction on k to proove our claim.

Base Case: When $k = 1$, we only have one perfect matching which is the edge connecting both vertices.

Induction Hypothesis: We assume our claim holds true for some $k = n - 1$.

Induction Step: We consider a graph of $2n$ vertices.
We take any vertex, say v_1, from the graph.
It will only be counted as 1 as every vertex of the graph should be matched in a perfect matching.
Then we have $2n - 1$ edges which match v_1 to distinct vertices.
We arbitrarily choose an edge.
Now, we are left with $2n - 2$ unmatched vertices.
By our induction hypothesis, we know total possible perfect matching for a graph of $2n-2$ vertices will be $f(2n-2)$.
Also since we arbitraly chose an edge from the set of $2n-1$ edges and for each edge we have $f(2n-2)$ possible perfect matchings, we can say the total number of perfect matchings of our original graph will be $f(2n) = (2n-1) \times f(2n-2)$ (By the multiplicative rule of combinatorics).
So, by the **principle of mathematical induction**, we have our claim true, $\forall k \in \mathbb{N} - \{0\}$.

Hence Proved.

Problem 5

Consider a Delhi Metro train consisting of n seats numbered $1, \dots, n$ carrying m (distinct) passengers. The government rules for physical distancing prevent passengers from standing in the compartment during their journey. Moreover, they must leave a gap of at least two seats between themselves: if a seat k is occupied, seats $k-2$, $k-1$, $k+1$, $k+2$ must remain vacant. Find an expression for the number of ways in which the passengers can occupy seats while following the physical distancing norms, and prove your answer. Again, your expression must involve only a constant number of applications of only the following mathematical operators: addition, subtraction, multiplication, division, exponentiation, and factorial.

Solution: We claim,

$$f(n, m) = \frac{(n-2m+2)!}{(n-3m+2)!}$$

Here $f(n, m)$ is the number of ways to arrange m passengers in n seats such that the conditions in question are satisfied.

Proof:
We disregard the number on seats such that the seats become identical.

We arrange the passengers in m seats disregarding any conditions.
That can be done by $m!$ ways.

Now we put 2 seats between every pair of adjecent passengers.
Now we are left with $n - 3m + 2$ seats.

We try to put them in the arrangement we have obtained so that we get the total of n seats.
We consider the leftover seats as balls and the positions where they have to be put, i.e., before first passenger, after last passenger or between 2 adjecent passengers, as bins.
And here we have $m + 1$ "bins".
Due to presence of passengers these "bins" have become distinct and since we have erased the number on seats, the "balls" here are identical.
From class, we have the number of ways to arrange n identical balls in m distinct bins as $\binom{n-m+1}{m-1}$.

So, to put the seats in our arrangement, this can be done in $\binom{n-2m+2}{m}$ ways.

Now we renumber the seats.

So, our total answer becomes $m! \times \binom{n-2m+2}{m}$, which is the same as $f(n, m)$.

Hence Proved.