MTL 103: Practice Sheet 2

1. Consider minimizing $c^T x$ over a polyhedron S. Prove that a point $x \in S$ is a unique optimal solution if and only if $c^T d > 0$ for every nonzero feasible direction d at x to S.

2. Let $S = \{(x_1, x_2, x_3) \ : \ x_1 + x_2 + x_3 = 1, \ x_1, x_2, x_3 \geq 0\}$. Find the feasible directions at $x = (0, 0, 1)$ to S.

3. Consider the linear program: min $c^T x$ subject to $Ax \leq b, \ x \geq 0$, where $c \neq 0$. Suppose that the point x^* is such that $Ax^* < b, \ x^* > 0$. Show that x^* cannot be an optimal solution.

4. Graph the convex hull of points $(0, 5)$, $(3, 5)$, $(6, 3)$, $(5, 0)$, $(3, 3)$, $(2.5, 2.5)$. Which of these points are extreme points of the hull? Express the nonextreme point (among given points), if any, as a convex combination of the extreme points.

5. Express the point $x = (0, 1)$ as a convex combination of the extreme points of the set $\{(x_1, x_2) \ : \ x_1 - x_2 \leq 3, \ 2x_1 + x_2 \leq 4, \ x_1 \geq -3\}$.

6. Find the extreme directions (if any) and extreme points of the set described by $\{(x_1, x_2) \ : \ 5x_1 + 3x_2 \geq 15, \ -x_1 + x_2 \leq 1, \ x_1 \geq 0, \ x_2 \geq (3/2)\}$.

7. Plot the feasible region

   $$S = \{(x_1, x_2) \ : \ -x_1 + x_2 \leq 1, \ x_1 + x_2 \leq 5, \ 4x_1 - 3x_2 \leq 6, \ x_1 - 2x_2 \leq 1, \ x_1, x_2 \geq 0\}.$$

   Find all the basic feasible solutions to the problem. If we move from vertex $(2, 3)$ to vertex $(3, 2)$, then determine the entering and leaving variables.

8. Consider the following linear programming problem, where b and a_i are 3×1 column vectors for $i = 1, 2, 3$.

   min $\sum_{i=1}^{4} c_i x_i$; subject to $\sum_{i=1}^{4} a_i x_i = b$; $x_i \geq 0$.

   Suppose $x_B = (x_1, \ 0, \ x_3, \ x_4)$ is a basic feasible solution, where B is the corresponding basis matrix. Let $d = (d_1, \ 5, \ d_3, \ d_4)$ be such that $x + d$ is a feasible solution of the given LP. Prove that $(d_1, \ d_3, \ d_4) = -5B^{-1}a_2$.

9. Solve the following LPP without using an algorithm max $z = 4x_1 + 5x_2 + 11x_3 + 2x_4$ subject to $21x_1 + 7x_2 - 3x_3 + 10x_4 = 210, \ x_1, \ x_2, \ x_3, \ x_4 \geq 0$.

10. Reduce the solution $(2, 4, 1)$ to a basic feasible solution of $Ax = b, \ x \geq 0$ where $A = [a_1 \ a_2 \ a_3]$, $a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $a_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 18 \end{pmatrix}$

11. The point $(1/2, \ 1/2, \ 1/2)$ is feasible for the system:

    $$\begin{aligned} &x_1 + 2x_2 + x_3 \leq 3 \\ &-2x_1 + 2x_2 + 2x_3 \leq 3 \\ &x_1, x_2, x_3 \geq 0. \end{aligned}$$

    Verify whether it is basic. If not, use then reduce it to a basic feasible solution.