(1) Constraints

$$-x_1 + x_2 + x_3 = 2$$
$$2x_1 + 3x_2 - x_4 = 6$$
$$x_1 + 2x_2 + x_5 = 8$$
$$2x_1 - x_2 + x_6 = 6$$
$$x_i \geq 0$$

$x^* = (3, 0)$

$\Rightarrow x_1 = 3, \ x_2 = 0, \ x_3 = 5, \ x_4 = 0, \ x_5 = 5, \ x_6 = 0$

BFS : $(3, 0, 5, 0, 5, 0)$

Basic variables (x_1, x_2, x_3, x_5)

$$B = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 0 \end{bmatrix} \qquad B^{-1} = \frac{1}{8}\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & -2 \\ 8 & -1 & 0 & 5 \\ 0 & -5 & 8 & 1 \end{bmatrix}$$

$$y_j = B^{-1} a_j$$
$$y_4 = B^{-1} a_4 = B^{-1}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 \\ -2 \\ 1 \\ 5 \end{bmatrix}$$

$$y_6 = B^{-1} a_6 = \frac{1}{8}\begin{bmatrix} 3 \\ -2 \\ 5 \\ 1 \end{bmatrix}$$

Table

			1	1	0	0	0	0	
c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
1	x_1	3	1	0	0	$-1/8$	0	$3/8$	
1	x_2	0	0	1	0	$-2/8$	0	$-2/8$	
0	x_3	5	0	0	1	$1/8$	0	$5/8$	
0	x_5	5	0	0	0	$5/8$	1	$1/8$	$\rightarrow$
	$z_j - c_j$		0	0	0	$-\frac{3}{8}$	0	$\frac{1}{8}$	

(3) Max $z = x_1 + 2x_2$

s.to $-2x_1 + x_2 + x_3 + x_4 = 2$

$-x_1 + x_2 + x_3 + x_5 = 1$

$x_i \geq 0 \ \forall i$

			1	2	0	0	0
C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5
0	x_4	2	-2	1	1	1	0
0	x_5	1	-1	1	1	0	1 →
			-1	-2 ↑	0	0	0

			1	2	0	0	0
C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5
0	x_4	1	-1	0	0	1	-1
2	x_2	1	-1	1	1	0	1
			-3	0	2	0	2

$z_1 - c_1 < 0$ & $y_{11} < 0,\ y_{12} < 0$

$\therefore$ LPP is unbounded

We want $z = 2000$

current $z = 2$

By theorem (done)

$2000 = 2 - \xi(-3)$

$1998 = 3\xi \Rightarrow \xi = 666$

$\hat{x}_{B_i} = x_{B_i} - \xi y_{ij}$

$\hat{x}_4 = 1 - 666(-1) = 667$

$\hat{x}_2 = 1 - 666(-1) = 667$

$\hat{x}_1 = 666$

$\hat{x} = (666, 667, 0, 667, 0)$ as feasible giving $\hat{z} = 2000.$ | $\hat{z} = 2000$

4. Final table

			6	4	0	0	0	-M
C_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	x_3	14	0	5/3	1	-2/3	0	0
0	x_5	5	0	-1/3	0	1/3	1	-1
6	x_1	8	1	2/3	0	1/3	0	0
			0	0	0	2	0	M

$z_2 - c_2 = 0$ & have +ve y component
x_2 is nonbasic

$\Rightarrow$ LP has alternate solution.
One soln is $(8, 0)$ (from table)
with $z^* = 48$

Let x_2 enters into basis & x_3 leaves

			6	4	0	0	0	-M
C_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
4	x_2	42/5	0	1	3/5	-2/5	0	0
0	x_5	39/5	0	0	1/5	1/5	1	-1
6	x_1	12/5	1	0	-2/5	3/5	0	0
			0	0	0	2	0	M

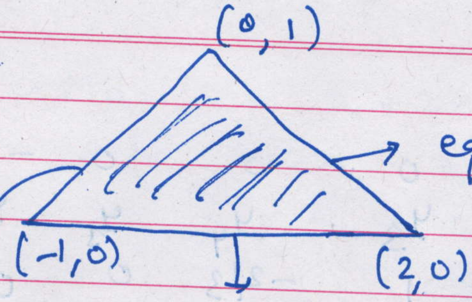
$z_j - c_j \geq 0 \quad \forall j$
Another optimal soln is $\left(\frac{12}{5}, \frac{42}{5}\right)$

with $z = \frac{72}{5} + \frac{168}{5} = 48$

Any convex combination is also an optimal soln

$(1-\lambda)(8, 0) + \lambda\left(\frac{12}{5}, \frac{42}{5}\right)$, $\lambda \in (0, 1)$

(5)



eqn : $y - 0 = \frac{-1}{2}(x-2)$

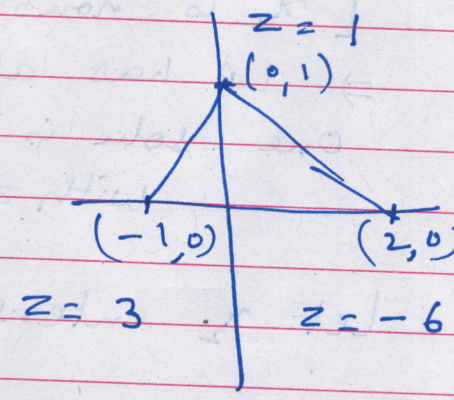
$2y + x = 2$

eqn. $y = 0$

$-1 \le x \le 2$

eqn : $y - 0 = \frac{-1}{-1}(x+1) \Rightarrow y - x = 1$

$$S = \{ (x, y) : x + 2y \le 2,\ -x + y \le 1,\ -1 \le x \le 2,\ y \ge 0 \}.$$



LP: min $-3x + y$
s.to. $(x, y) \in S$

Apply simplex usually & see final optimal answer is $x_1 = 2, x_2 = 0$

minimum is
$z^* = -6.$
$x^* = (2, 0)$ opt. soln

6. (i) Non degerate BFS $\Rightarrow$ $c > 0, f > 0$

Next table has degenerate - ?

$z_6 - c_6 = -4 < 0$ so x_5 enters into basis

The outgoing variable could correspond to row 1 or row 2 when $d > 0$.

If row 1 leaves then 3 is pivot and in the next table

X

$c/3 \quad > 0 \quad$ as $c > 0$

$(3f - cd)/3$

1

$(9+c)/3 \quad > 0 \quad$ as $c > 0$

For degeracy $\quad 3f - cd = 0 \Rightarrow cd = 3f$

as $c > 0, f > 0 \Rightarrow d > 0$.

Also 3 is pivot $d > 0$

$$\Rightarrow \min\left\{\frac{c}{3}, \frac{f}{d}\right\} = \frac{c}{3}$$

$$\Rightarrow \frac{f}{d} \leq \frac{c}{3} \Rightarrow 3f \leq cd.$$

$\therefore$ $c > 0$, $f > 0$, $d > 0$ and $cd = 3f$.

Similarly when row 2 or d is pivot, we get same.

(ii) Non-degerate BFS $\Rightarrow$ $c > 0, f > 0$

Next table has non-degerate means

if row 1 is outgoing or 3 is pivot then

X

$c/3$

$(3f - cd)/3 > 0 \Rightarrow \underline{cd < 3f}$ — 1

1

$(9+c)/3$

If 3 is pivot then $\min\left\{\frac{c}{3}, \frac{f}{d}\right\} = \frac{c}{3}$, and $d > 0$

$\Rightarrow \frac{f}{d} \le \frac{c}{3}$, $d>0$

$\Rightarrow cd \geq 3f$

Not possible with (1)

$\therefore$ $\underline{d<0}$ & hence only first row is candidate to leave.

$\therefore$ $c>0$, $f>0$ $d<0$ (giving obviously $cd<3f$)

Next let second row be outgoing

$\Rightarrow d>0$

$\min\left\{\frac{c}{3}, \frac{f}{d}\right\} = \frac{f}{d}$

$\Rightarrow \frac{c}{3} \geq \frac{f}{d}$ $\Rightarrow$ $\underline{cd \geq 3f}$ (2)

Next table then is

x

$\left(\frac{cd-3f}{d}\right) > 0$ if $\underline{cd>3f}$

$\frac{f}{d} > 0$ as $f>0, d>0$

1

$\frac{3d+f}{d} > 0$ $\because$ $f>0$, $d>0$

$\therefore$ we have $c>0$, $f>0$, $d>0$ & $cd>3f$

<u>Concluding</u> :- Either $c>0$, $f>0$, $d<0$

or $c>0$, $d>0$, $f>0$ and $cd>3f$

Similar way discuss parts (iii) and (iv)

7. Consistency of system means feasibility

$$-6x_1 + x_2 + x_3 + x_4 = 5$$
$$-2x_1 + 2x_2 + 3x_3 - x_5 = 3$$
$$2x_1 - 4x_3 = 1$$
$$x_i \geq 0 \;,\; \forall\; 1 \leq i \leq 5$$

Use artificial variables and take the objective function as follows.

$$\max\; -x_6 - x_7$$
s.to. $-6x_1 + x_2 + x_3 + x_4 = 5$
$-2x_1 + 2x_2 + 3x_3 - x_5 + x_6 = 3$
$2x_1 - 4x_3 + x_7 = 1$
$x_i \geq 0 \;\forall\; 1 \leq i \leq 7.$

Apply usual simplex.
(if all artificial variables takes values $= 0$ then system is consistent else not).

8 On the same lines as in Q7.
Use three artificial variables x_5, x_6, x_7 in three eqns and take objective function as
$$\max\; -x_5 - x_6 - x_7.$$

Apply usual simplex method.

9. Exactly same as Q7 and Q8. Construct the objective fn using the artificial variables.

10 Let ~~us~~ assume that we have to find A^{-1} where A is given.

$$A = \begin{bmatrix} 5 & 1 & 7 \\ 3 & 4 & 8 \\ 2 & 6 & 8 \end{bmatrix}$$

Take $x = (1, 1, 1)^T$ (nothing special, one can take any 3×1 vector x with $x_i > 0 \; \forall i$)

Take $b = Ax = \begin{bmatrix} 13 \\ 15 \\ 16 \end{bmatrix}$ (in our case)

Now consider the problem

$$5x_1 + x_2 + 7x_3 = 13$$
$$3x_1 + 4x_2 + 8x_3 = 15$$
$$2x_1 + 6x_2 + 8x_3 = 16$$
$$x_1, x_2, x_3 \geq 0$$

We know its soln is $(1, 1, 1)^T$. So it is a consistent system. Convert it into LP problem using artificial variables

$$\max \; -x_4 - x_5 - x_6$$

s.to $5x_1 + x_2 + 7x_3 + x_4 = 13$

$3x_1 + 4x_2 + 8x_3 + x_5 = 15$

$2x_1 + 6x_2 + 8x_3 + x_6 = 16$

$x_i \geq 0 \; \forall i = 1, --, 6.$

Write the simplex tables.

In the last optimal table read the matrix corresponding to y_4, y_5, y_6 column &

The idea is start with $[A \mid I]$ and convert it to $[I \mid A^{-1}]$ system by row transformations or pivoting (as done in the simplex method)

12 Usual simplex and matrix corresponding to the identity columns of initial table give B^{-1}.

13 Case of alternate optimal solutions; done in the class.

11 $\left\{ d_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, d_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, d_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a LI set in $\mathbb{R}^3$ (or basis of $\mathbb{R}^3$)

write $a = (1,1,1)^T$ as linear combination of d_1, d_2, d_3

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = y_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y_2 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + y_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} y_1 - 2y_2 &= 1 \\ y_2 &= 1 \end{aligned}\right] \Rightarrow y_1 = 3$$

$$-y_1 + 2y_2 + y_3 = 1 \qquad -3 + 2 + y_3 = 1 \Rightarrow y_3 = 2$$

$$\therefore \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow d_3 = \frac{y_1}{y_3} d_1 + \frac{y_2}{y_3} d_2 - \frac{1}{y_3} a$$

$$= \frac{3}{2} d_1 + \frac{1}{2} d_2 - \frac{1}{2} a.$$

Use this to write $(B^*)^{-1}$

$B^* = (d_1 \; d_2 \; a)$

Sheet 4

1. $\alpha = w^{0T}(b - Ax^0) = 0$, $(w^{0T}A - c^T)\,x^0 = 0 = \beta$

$\Rightarrow \alpha + \beta = 0$

$\Rightarrow w^{0T}b - w^{0T}Ax^0 + w^{0T}Ax^0 - c^Tx^0 = 0$

$\Rightarrow w^{0T}b = c^Tx^0$

$\Rightarrow c^Tx^0 = b^Tw^0$ (zero duality gap)

Let x be any other feasible solu. of (P)
Then by weak duality applied on the feasib pair (x, w^0) we get

$$c^Tx \le b^Tw^0 = c^Tx^0$$

$c^Tx \le c^Tx^0 \Rightarrow x^0$ is optimal for (P)
By same type of argument conclude w^0 is optimal for (D).

2 This is called Farkas lemma
Consider

(P) max c^Tx
s.to $Ax \le 0$ $x \in \mathbb{R}^n$ (unrestricted)

Its dual is

(D) min 0^Tw
s.to $A^Tw = c$
$w \ge 0$.

Note $x = 0$ is a feasible solu of (P).
Also, if $\hat{x}$ is any feasible solu of (P) then $\alpha\hat{x}$ is also a feasible solu of (P) for any $\alpha > 0$.

(i) Let $wA = c$, $w \ge 0$ be consistent
$\Rightarrow$ (D) is feasible and (P) is already feasible

By weak duality

$$C^T x \leq 0^T w$$

$$\Rightarrow C^T x \leq 0 \;\; \& \;\; Ax \leq 0$$

$\therefore$

$Ax \leq 0$ and $C^T x > 0$ is inconsistent.

$\therefore$ if $Ax \leq 0$, $C^T x > 0$ is consistent

$\Rightarrow$ $A^T w = c$, $w \geq 0$ is inconsistent.

(ii) Let $Ax \leq 0$, $C^T x > 0$ be inconsistent

Whenever $C^T x > 0$ then $Ax > 0$

$\Downarrow$

outside the feasible set of (P)

Discarded.

whenever $Ax \leq 0$ then $C^T x \leq 0$

$\Downarrow$

(P) is feasible & its optimal value $\max C^T x = 0$

$\therefore$ By the strong duality theorem, (D) must have optimal soln.

$\Rightarrow$ $A^T w = c$, $w \geq 0$ is consistent

$\therefore$ Whenever $Ax \leq b$ then $C^T x \cancel{\leq} 0$

$\Rightarrow$ (D) is consistent

$\therefore$ if $Ax \leq 0$, $C^T x > 0$ is inconsistent

$\Rightarrow$ $A^T w = c$, $w \geq 0$ is consistent.

(i) + (ii) proves iff result.

3. Dual is

(D) $\max\ -6w_1 + 3w_2$

s.to $w_1 + w_2 \geq -3$

$w_1 + w_2 \geq 5$

$w_1 - 2w_2 \geq 1$

$3w_1 - w_2 \geq -2$

$w_1 + w_2 \geq 4$

$w_1, w_2 \geq 0$

$$\left.\begin{array}{ll} \max & -6w_1 + 3w_2 \\ \text{s.to} & w_1 + w_2 \geq 5 \\ & w_1 - 2w_2 \geq 1 \\ & 3w_1 - w_2 \geq -2 \\ & w_1, w_2 \geq 0 \end{array}\right]$$ Can solve by graphical method or simplex method.

4 Standard question. Solve the dual problem by the simplex or graph and use $z_j - c_j$ values of the dual to get optimal soln of (P)

leave Q 5 and Q 9 (yet to discuss in class)

7 (P) $\max\ c^T x$ s.to $Ax \leq b$, $x \geq 0$ (D) $\min\ b^T w$ s.to $A^T w \geq c$, $w \geq 0$

Let $b > 0$ and $A > 0 \Rightarrow$ each entry of A is strictly positive.

Start argument from (D). Since $c \in \mathbb{R}^n$, if all n-components of c are ≤ 0 then we can always choose any $w \geq 0$ to satisfy $A^T w \geq c$. Then,

$b^T w \geq 0$ (as $b > 0$) and minimum value of (D) is 0 at $w = 0$, with strong duality, (P) then also have an optimal solution.

Let some components of c be strictly positive (or may we assume $c > 0$). Then we can choose $w \geq 0$ vector, large enough components, such that $A^T w \geq c$ (This can be done as $A > 0$, so large enough $w \geq 0$ makes sure that $A^T w \geq 0$ is much larger than c vector, even when some components is c are positive).
Among several such w's possible, choose that one which min $b^T w > 0$ (this is possible as $b^T w$ is a continuous fn. bounded from below by 0 so its minimum would be attained)
$\therefore$ (D) has an optimal soln.
By the strong duality, (P) will also have an optimal solution.

6 Standard and simple. Just an observation from (P) and (D) gives answer.