Absorption and Dispersion of EM Waves

PYL101: Electromagnetics and Quantum Mechanics

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References

- Introduction to Electrodynamics, David J. Griffiths (3rd ed.)
  - Chapter 9, 9.4.1 Electromagnetic Waves in Conductors
  - Chapter 9, 9.4.2 Reflection at a Conducting Surface
  - Chapter 9, 9.4.3 Frequency Dependence of Permittivity

Maxwell's Equations in Matter: Recap

- Maxwell's equations in a linear conducting medium read,

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{\epsilon}\rho_f \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu \mathbf{J}_f + \mu\epsilon \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}
$$

Conductors

- According to Ohm's law, in a conducting medium,

$$\mathbf{J}_f = \sigma \mathbf{E}$$

- Also from charge continuity equation

$$\frac{\partial \rho_f}{\partial t} = -\nabla \cdot \mathbf{J}_f = -\nabla \cdot (\sigma \mathbf{E}) = -\sigma(\nabla \cdot \mathbf{E})$$

$$or\ \frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\epsilon}\rho_f$$

- This describes an exponential time decay of the charge density

$$\rho_f(t) = e^{-(\sigma/\epsilon)t}\rho_f(0)$$

Conductors

- Any accumulated free charge on a conductor flows out to the edges in a characteristic time $\tau = \epsilon/\sigma$, which is infinitesimally small for a good conductor.
- This time constant is a measure of how good a conductor is.
- For a perfect conductor $\sigma = \infty$ which leads to $\tau = 0$.
- For a good conductor $\tau \ll 1/\omega$ whereas for a poor conductor $\tau \gg 1/\omega$, where $1/\omega$ corresponds to the characteristic times in the problem.
- Hence after time τ one can assume $\rho_f = 0$.

Conductors

- For a good conductor like copper, $\epsilon \approx \epsilon_0 = 8.86x10^{-12}$ F/m and $\sigma \approx 6x10^8 (\Omega\ m)^{-1}$

$$\begin{aligned}\tau &= \epsilon/\sigma \\ &\approx 8.86x10^{-12}/6x10^8 \\ &\approx 10^{-20}s\end{aligned}$$

- For a poor conductor like water, $\epsilon \approx \epsilon_0 = 80 * 8.86x10^{-12}$ F/m and $\sigma \approx 4x10^{-6} (\Omega\ m)^{-1}$

$$\begin{aligned}\tau &= \epsilon/\sigma \\ &\approx 80x8.86x10^{-12}/4x10^{-6} \\ &\approx 10^{-4}s\end{aligned}$$

Maxwell's Equations in Conducting Media

- So Maxwell's equations in a linear conducting medium read,

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}
$$

Electromagnetic Wave Equation in Conducting Media

- Now if we take curl of the curl equation of E (Faraday's law), and use the Gauss's law

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})$$

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}$$

- Similarly, the wave equation for magnetic field **B** (HW).

$$\nabla^2 \mathbf{B} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t}$$

Plane Wave Solutions in a Conducting Medium

- The plane wave solutions in the complex notation can be written as:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 \ e^{i(\tilde{k}z-\omega t)}$$
$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 \ e^{i(\tilde{k}z-\omega t)}$$

where $\tilde{k}$ is the propagation vector.

- Substituting the above plane wave expressions in the wave equations, we get

$$(i\tilde{k})^2 \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)} = \mu\epsilon(-i\omega)^2 \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)} + \mu\sigma(-i\omega)\tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}$$
$$\rightarrow \ \tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

- The wave vector / propagation vector is complex in the case of a conducting medium.

EM Wave in a Conducting Medium: Complex Wave Vector

- Let us find out the real and imaginary parts, say k_{re} and k_{im} of the complex wave vector $\tilde{k}$.

$$\tilde{k}^2 = (k_{re} + ik_{im})^2 = k_{re}^2 - k_{im}^2 + 2ik_{re}k_{im} = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

- Separating real and imaginary parts on both sides one obtains

$$k_{re}^2 - k_{im}^2 = \mu\epsilon\omega^2; \; 2k_{re}k_{im} = \mu\sigma\omega$$

- Solving the above equations, we get the real and imaginary parts of the wave vector

$$k_{re} = \sqrt{\frac{\mu\epsilon\omega^2}{2}\left[\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} + 1\right]}; \; k_{im} = \sqrt{\frac{\mu\epsilon\omega^2}{2}\left[\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} - 1\right]}$$

- For a non-conducting medium $\sigma = 0$, which leads to

$$\boxed{k_{im} = 0, k_{re} = \sqrt{\mu\epsilon\omega^2} = \sqrt{\omega^2/v^2} = \frac{\omega}{v}}$$

EM Wave in a Conducting Medium: Attenuation

- The plane wave solutions can now be written as:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 \; e^{i(\tilde{k}z-\omega t)} = \tilde{\mathbf{E}}_0 \; e^{i((k_{re}+ik_{im})z-\omega t)}$$

$$\boxed{\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 \; e^{-k_{im}z} \; e^{i(k_{re}z-\omega t)}}$$

- Similary for magnetic field

$$\boxed{\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 \; e^{-k_{im}z} \; e^{i(k_{re}z-\omega t)}}$$

- The imaginary part of the complex wave vector which originates due to finite conductivity of the medium causes an attenuation in the wave amplitude as it propagates.

EM Wave in a Conducting Medium: Skin Depth

- The plane wave solutions at $z = 1/k_{im}$:

$$\tilde{\mathbf{E}}(z,t) = \frac{1}{e}\tilde{\mathbf{E}}_0 \; e^{i(k_{re}z - \omega t)}$$
$$\tilde{\mathbf{B}}(z,t) = \frac{1}{e}\tilde{\mathbf{B}}_0 \; e^{i(k_{re}z - \omega t)}$$

- The distance at which the field amplitudes in a conducting medium are reduced by a factor of 1/e is called the skin depth of that medium and is given by

$$d = \frac{1}{k_{im}}$$

EM Wave in a Conducting Medium: Skin Depth

- The skin depth

$$\boxed{d = \frac{1}{k_{im}} = \sqrt{\frac{2}{\mu\epsilon\omega^2}}\left[\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}-1\right]^{-\frac{1}{2}}}$$

- For good conductors $\frac{\sigma}{\epsilon\omega} \gg 1(say \gtrsim 100)$, d is small (few wavelengths).
- For bad conductors $\frac{\sigma}{\epsilon\omega} \ll 1(say \lesssim 0.01)$, d is large (several wavelengths).
- For quasi-conductors $0.01 \lesssim \frac{\sigma}{\epsilon\omega} \lesssim 100$
- A quasi-conductor is a material which could act as a good conductor or a bad conductor depending on the frequency.

EM Wave in a Conducting Medium: Skin Depth

- Example: Fresh water has very low conductivity $(1x10^{-3}(\Omega m)^{-1})$ and a dielectric constant of 80.

$$\Rightarrow \frac{\sigma}{\epsilon\omega} = \frac{1x10^{-3}}{80x8.854x10^{-12}x\omega} = \frac{1.4x10^{6}}{\omega}$$

- At 100 Hz, $\omega = 2\pi x100/s$

$$\frac{\sigma}{\epsilon\omega} = \frac{1.4x10^{6}}{200\pi} \approx 2x10^{3} \gg 1$$

- At 100 MHz, $\omega = 2\pi x10^{8}/s$

$$\frac{\sigma}{\epsilon\omega} = \frac{1.4x10^{6}}{2\pi x10^{8}} \approx 2x10^{-3} \ll 1$$

- So fresh water acts as a good conductor for $\nu \lesssim 10^{3} Hz$ and as a bad conductor for $\nu \gtrsim 10^{7} Hz$.

EM Wave in a Conducting Medium: Skin Depth

- Example: Copper has very high conductivity $(5.8x10^7 (\Omega m)^{-1})$ and a dielectric constant of ~ 1.

$$\Rightarrow \frac{\sigma}{\epsilon\omega} = \frac{5.8x10^7}{8.854x10^{-12}x\omega} = \frac{6.55x10^{18}}{\omega}$$

- At 100 Hz, $\omega = 2\pi x 100/s$

$$\frac{\sigma}{\epsilon\omega} = \frac{6.55x10^{18}}{200\pi} \approx 10^{16} \gg 1$$

- At 100 MHz, $\omega = 2\pi x 10^8/s$

$$\frac{\sigma}{\epsilon\omega} = \frac{6.55x10^{18}}{2\pi x 10^8} \approx 10^{10} \gg 1$$

- So for both low as well as high frequencies Copper acts as a good conductor.

EM Wave in a Conducting Medium: Skin Depth

- The frequency dependence of skin depth in Copper

$$d = \sqrt{\frac{2}{\sigma\mu\omega}} = \sqrt{\frac{2}{2\pi\nu x 4\pi x 10^{-7} x 5.8 x 10^{7}}}$$
$$\approx \frac{0.066}{\sqrt{\nu}}$$
$$\approx 6.6 \ mm \ at \ \nu = 100 \ Hz$$
$$\approx 6.6 \ \mu m \ at \ \nu = 100 \ MHz$$

- The penetration decreases with increase in frequency.

EM Waves in a Conducting Medium

- Now for an EM wave propagating in z-direction inside a conducting medium

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 \; e^{-k_{im} z} \; e^{i(k_{re} z - \omega t)}; \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 \; e^{-k_{im} z} \; e^{i(k_{re} z - \omega t)}$$

- Using Maxwell's divergence equations

$$\nabla \cdot \tilde{\mathbf{E}} = 0; \nabla \cdot \tilde{\mathbf{B}} = 0 \;\rightarrow\; \tilde{\mathbf{E}_{0z}} = 0; \tilde{\mathbf{B}_{0z}} = 0$$

- EM waves in conducting medium are transverse waves.
- And from Faraday's law

$$\nabla \times \tilde{\mathbf{E}} = -\frac{\partial \tilde{\mathbf{B}}}{\partial t} \rightarrow \tilde{\mathbf{B}}_0 = \frac{\tilde{k}}{\omega}\left(\hat{\tilde{k}} \times \tilde{\mathbf{E}}_0\right)$$

- Fields **E** and **B** are mutually perpendicular to each other.

EM Waves in a Conducting Medium

- An EM wave polarized in x-direction and propagating in z-direction inside a conducting medium can therefore be written as

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 \; e^{-k_{im} z} \; e^{i(k_{re} z - \omega t)} \hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega} \tilde{E}_0 \; e^{-k_{im} z} \; e^{i(k_{re} z - \omega t)} \hat{\mathbf{y}}$$

- Writing wave vector in complex notation as $\tilde{k} = K e^{i\phi}$ where

$$K = \sqrt{k_{re}^2 + k_{im}^2} = \omega \sqrt{\mu\epsilon \sqrt{1 + \frac{\sigma^2}{\epsilon^2 \omega^2}}} \; and \; \phi \quad = tan^{-1} \left(\frac{k_{im}}{k_{re}} \right)$$

EM Waves in a Conducting Medium

- Complex amplitudes of electric and magnetic fields, $\tilde{E}_0$ and $\tilde{B}_0$ are related by

$$B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E}$$

- The electric and magnetic fields are not in phase.

$$\boxed{\delta_B - \delta_E = \phi}$$

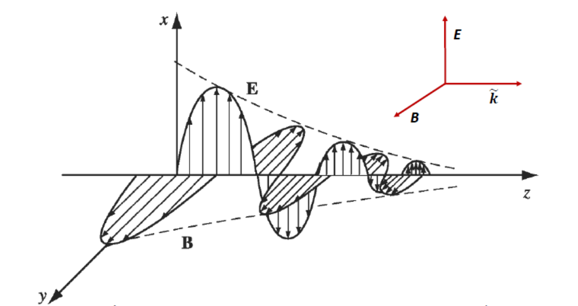
- The expressions for real electric and magnetic fields are then

$$\mathbf{E}(z,t) = E_0 \; e^{-k_{im} z} \; cos(k_{re} z - \omega t + \delta_E)\hat{\mathbf{x}}$$

$$\mathbf{B}(z,t) = \frac{K}{\omega} E_0 \; e^{-k_{im} z} \; cos(k_{re} z - \omega t + \delta_E + \phi)\hat{\mathbf{y}}$$

EM Waves in a Conducting Medium

- The fields associated with an EM wave polarized in x-direction and propagating in z-direction in a conducting medium look like



(x-polarized EM wave traveling in z-direction)

Reflection at a Conducting Surface : Boundary Conditions

- The boundary conditions at the interface between a non-conducting and a conducting media having permittivities ϵ_1 and ϵ_2, and permeabilities μ_1 and μ_2, respectively.

$$(i)\ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$$

$$(ii)\ B_1^{\perp} = B_2^{\perp}$$

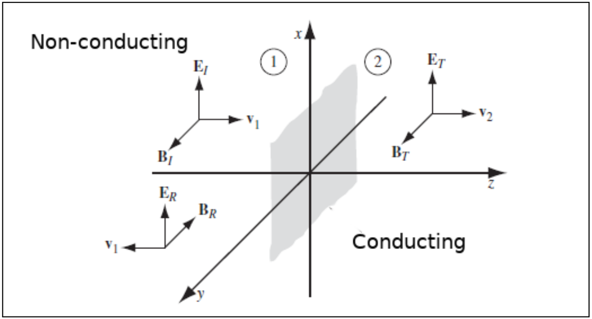
$$(iii)\ \mathbf{E}_1^{||} = \mathbf{E}_2^{||}$$

$$(iv)\ \frac{1}{\mu_1}\mathbf{B}_1^{||} - \frac{1}{\mu_2}\mathbf{B}_2^{||} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

where σ_f is the free surface charge density, $\mathbf{K}_f$ is the free surface current, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface, pointing from medium (2) into medium (1).

Reflection at a Conducting Surface: Normal Incidence

- Consider a plane wave of frequency ω, having polarization in x-direction, is traveling in z-direction in a linear non-conducting medium '1'. It encounters an interface (in xy-plane) between medium '1' and another medium '2' which is a conducting one.



Reflection at a Conducting Surface: Normal Incidence

- The electric and magnetic fields for incident, reflected and transmitted waves:

Incident Wave

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x}; \qquad \tilde{\mathbf{B}}_I(z,t) = \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{y}$$

Reflected Wave

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x}; \qquad \tilde{\mathbf{B}}_R(z,t) = -\frac{\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)} \hat{y}$$

Transmitted Wave

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0T} e^{i(\tilde{k}_2 z - \omega t)} \hat{x}; \qquad \tilde{\mathbf{B}}_T(z,t) = \frac{\tilde{k}_2}{\omega} \tilde{E}_{0T} e^{i(\tilde{k}_2 z - \omega t)} \hat{y}$$

which is attenuated as it penetrates into the conducting medium.

Reflection at a Conducting Surface: Normal Incidence

- At the interface $z = 0$ electric and magnetic fields will satisfy following boundary conditions:

$$E_1^{||} = E_2^{||}$$
$$\frac{B_1^{||}}{\mu_1} = \frac{B_2^{||}}{\mu_2}$$

as $\mathbf{K}_f = 0$ for Ohmic conductors.

- The field components normal to the interface plane are zero in case of normal incidence.
- In terms of incident, reflected and transverse components, the above conditions read

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$
$$\frac{1}{\mu_1}(\tilde{B}_{0I} + \tilde{B}_{0R}) = \frac{1}{\mu_2}\tilde{B}_{0T}$$

Reflection at a Conducting Surface: Normal Incidence

- Writing B in terms of E

$$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{0I} - \tilde{E}_{0R}\right) = \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0T}$$
$$\Rightarrow \tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta}\tilde{E}_{0T}$$

where

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

- Solving for electric field components one obtains

$$\boxed{\tilde{E}_{0R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0I} \;\; and \;\; \tilde{E}_{0T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0I}}$$

Reflection from a Perfect Conductor: Normal Incidence

- For a perfect conductor, $\sigma = \infty$, $k_2 = \infty$, so $\tilde{\beta} = \infty$ and

$$\tilde{E}_{0R} = -\tilde{E}_{0I}, \; \tilde{E}_{0T} = 0$$

- The wave is totally reflected, with a 180° phase shift.
- Therefore excellent conductors make good mirrors. Usually a thin coating of silver is painted at the rear of a glass pane.[1]
- Incident and reflected electric fields for this case

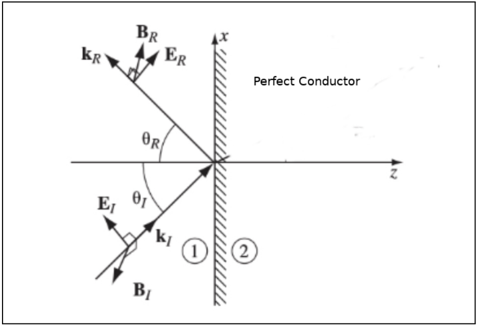
$$\begin{aligned} \mathbf{E}_I &= \mathbf{E}_{0I} cos(k_1 z - \omega t)\hat{x} \\ \mathbf{E}_T &= -\mathbf{E}_{0I} cos(k_1 z + \omega t)\hat{x} \end{aligned}$$

- The incident and reflected waves superimpose to give a standing wave.

[1] You don't need a very thick layer of silver paint as skin depth in silver is on the order of $100A^\circ$ for optical frequencies (HW: Verify).

Reflection from a Perfect Conductor: Oblique Incidence

- Consider a plane wave of frequency ω, having polarization parallel to the plane of incidence (x-z plane), is incident at an angle θ_I with the normal of the interface (in xy-plane) between a non-conducting medium '1' and a conducting medium '2'.



Reflection from a Perfect Conductor: Oblique Incidence

- As the EM fields inside a perfect conductor are zero, the interface completely reflects the incident plane wave.
- The incident and reflected fields are given by

Incident Wave

$$\mathbf{E}_I(z,x,t) = E_{0I}\left(cos\theta_I \hat{x} - sin\theta_I \hat{z}\right) e^{i(k_1(xsin\theta_I + zcos\theta_I) - \omega t)}$$

$$\tilde{\mathbf{B}}_I(z,x,t) = \hat{y}\frac{\tilde{E}_{0I}}{v_1} e^{i(k_1(xsin\theta_I + zcos\theta_I) - \omega t)}$$

Reflected Wave

$$\mathbf{E}_R(z,x,t) = E_{0R}\left(cos\theta_R \hat{x} + sin\theta_R \hat{z}\right) e^{i(k_1(xsin\theta_R - zcos\theta_R) - \omega t)}$$

$$\tilde{\mathbf{B}}_I(z,x,t) = -\hat{y}\frac{\tilde{E}_{0R}}{v_1} e^{i(k_1(xsin\theta_R - zcos\theta_R) - \omega t)}$$

Reflection from a Perfect Conductor: Oblique Incidence

- Due to infinite conductivity, the tangential electric field component at the interface is zero.

$$E_I(0,x,t) + E_R(0,x,t) = 0$$

- This leads to

$$E_{0I} = -E_{0R} \textit{ and } \theta_I = \theta_R$$

- The total electric field can now be written as

$$\begin{aligned}
\mathbf{E}(z,x,t) &= \mathbf{E}_I(z,x,t) + \mathbf{E}_R(z,x,t) \\
&= E_{0I}\left[(cos\theta_I\hat{x} - sin\theta_I\hat{z})e^{ik_1zcos\theta_I}\right. \\
&\left. -(cos\theta_I\hat{x} + sin\theta_I\hat{z})e^{-ik_1zcos\theta_I}\right] e^{i(k_1xsin\theta_I - \omega t)} \\
&= 2E_{0I}\left[\hat{x}icos\theta_I sin(k_1zcos\theta_I) - \hat{z}sin\theta_I cos(k_1zcos\theta_I)\right] * \\
&\quad e^{i(k_1xsin\theta_I - \omega t)}
\end{aligned}$$

Reflection from a Perfect Conductor: Oblique Incidence

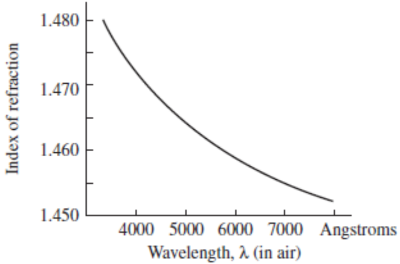
- And the total magnetic field

$$\begin{aligned}\mathbf{B}(z,x,t) &= \mathbf{B}_I(z,x,t) + \mathbf{B}_R(z,x,t)\\ &= \hat{y}\frac{2E_{0I}}{v_1}cos(k_1 zcos\theta_I)e^{i(k_1 xsin\theta_I - \omega t)}\end{aligned}$$

- It represents a standing wave pattern along z-direction while a propagating wave in x-direction.
- As magnetic field always remain perpendicular to the propagation direction whereas electric field has a component along the propagation, it is called a "transverse magnetic (TM)" wave.
- Similarly, one can obtain the transverse electric (TE) wave for the perpendicular polarization of the incident EM wave.

Frequency Dependent Permittivity

- In a dielectric material, permittivity and therefore the index of refraction depends on the frequency of the wave under consideration.
- A prism bends blue light more than red and thus spreads white light out into VIBGYOR spectrum.



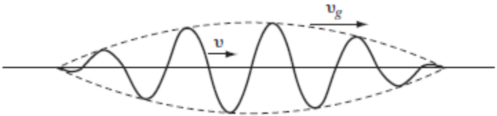
Frequency Dependent Permittivity

- In a dispersive medium waves of different wavelengths propagate at different (phase) velocities.

$$v = \frac{\omega}{k}$$

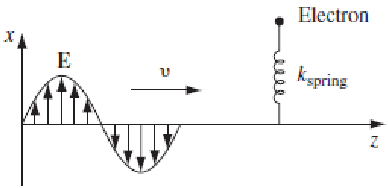
- The wavepacket travels at the so called group velocity.

$$v_g = \frac{d\omega}{dk}$$



Electrons in Dielectrics as Damped Harmonic Oscillators

- Consider an electron attached to an imaginary spring, with force constant k_{spring}.



- If the electron is displaced by a small amount 'x' from its equilibrium, along the length of the spring, the restoring force will be

$$F_{binding} = -k_{spring}x = -m\omega_0^2 x$$

where m is the mass of the electron and $\omega_0 = \sqrt{k_{spring}/m}$ is the natural oscillation frequency.

Electrons in Dielectrics as Damped Harmonic Oscillators

- The oscillating electron will emit radiation which will cause the oscillations to damp (radiation damping [2]).
- The damping force can be modeled by the following simple expression

$$F_{damping} = -m\gamma \frac{dx}{dt}$$

- In the presence of an EM wave of frequency ω, polarized in the x-direction, the electron also experiences a driving force

$$F_{driving} = qE = qE_0 cos(\omega t)$$

where 'q' is the charge of the electron and E_0 is the amplitude of the wave at the location of electron.

[2] chapter 11, Introduction to Electrodynamics, David Griffiths

Electrons in Dielectrics as Damped Harmonic Oscillators

- The equation of motion of electron

$$m\frac{d^2x}{dt^2} = F_{total} = F_{binding} + F_{damping} + F_{driving}$$

OR

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0 cos\omega t$$

- Above equation of motion describes the electron as a damped harmonic oscillator, driven at frequency ω assuming the massive nuclei to remain at rest.
- In complex notation

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t}$$

- In the steady state, the system oscillates at the driving frequency

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$$

Electrons in Dielectrics as Damped Harmonic Oscillators

- Substituting the above expression in equation of motion

$$(-i\omega)^2\tilde{x}_0 + \gamma(-i\omega)\tilde{x}_0 + \omega_0^2\tilde{x}_0 = \frac{q}{m}E_0$$

- This gives the amplitude of damped oscillations as

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0$$

- The dipole moment in complex notation

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 e^{-i\omega t}$$

Electrons in Dielectrics as Damped Harmonic Oscillators

- Differently situated electrons experience different natural frequencies and damping coefficients.
- Say there are f_j electrons with frequency ω_j and damping γ_j in each molecule.
- If there are N molecules per unit volume, the polarization is given by

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m} \left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right) \tilde{\mathbf{E}}$$

- Comparing with $\tilde{\mathbf{P}} = \epsilon_0 \tilde{\chi}_e \tilde{\mathbf{E}}$, the complex susceptbility

$$\tilde{\chi}_e = \frac{Nq^2}{m\epsilon_0} \left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right)$$

Electrons in Dielectrics as Damped Harmonic Oscillators

- Hence the complex dielectric constant according to this damped harmonic oscillator model

$$\tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$$

- Ordinarily the imaginary term in the denominator is negligible, however, when ω is very close to one of the resonant frequencies ω_j, it plays an important role.
- Now in a dispersive medium the wave equation for a given frequency is

$$\nabla^2\tilde{\mathbf{E}} = \tilde{\epsilon}\mu_0\frac{\partial^2\tilde{\mathbf{E}}}{\partial t^2}$$

- and corresponding plane wave solutions

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}$$

with the complex wave number $\tilde{k} = \sqrt{\tilde{\epsilon}\mu_0}\omega$.

Electrons in Dielectrics as Damped Harmonic Oscillators

- Now writing $\tilde{k}$ in terms of real and imaginary parts as $\tilde{k} = k_{re} + ik_{im}$, the plane wave solution becomes

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-k_{im}z} e^{i(k_{re}z - \omega t)}$$

- The intensity will be

$$\begin{aligned} I &\propto E^2 \\ &\propto e^{-2k_{im}z} \\ &\propto e^{-\alpha z} \end{aligned}$$

  Here $\alpha = 2k_{im}$ is called the absorption coefficient.
- Also the refractive index is given by $n = ck_{re}/\omega$ where k_{re} is the real part of the complex wave vector given by

$$\tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \cong \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

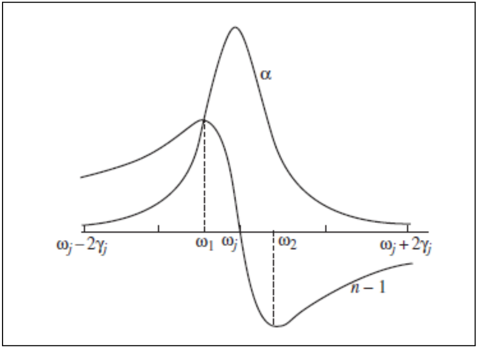
- So the refractive index and the absorption coefficient can be written as

$$n = \frac{ck_{re}}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

$$\alpha = 2k_{im} \cong \frac{Nq^2\omega^2}{m\epsilon_0 c} \sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

Electrons in Dielectrics as Damped Harmonic Oscillators

- The refractive index and the absorption coefficient, in the vicinity of one of the resonances, look like



- In the neighbourhood of a resonance the index of refraction drops sharply. This is called anomalous dispersion.

Electrons in Dielectrics as Damped Harmonic Oscillators

- Region of anomalous dispersion coincides with that of maximum absorption, due to large oscillations and subsequent large damping.
- Away from the resonances damping can be ignored and

$$n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

- For transparent materials, resonant frequencies lie in the ultraviolet region so that $\omega < \omega_j$.

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \cong \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right)$$

Electrons in Dielectrics as Damped Harmonic Oscillators

- So that the refractive index can be written as

$$\boxed{n = 1 + \left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2\left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^4}\right)}$$

or in terms of wavelength in vacuum

$$\boxed{n = 1 + A\left(1 + \frac{B}{\lambda^2}\right)}$$

- This is known as Cauchy's formula. A is called coefficient of refraction and B is called coefficient of dispersion.
- Cauchy's formula applies to most gases in the optical region.