lecture 9 Fri 11 Aug 2023

$EP_2 \to BFS$ B_2 $m \times m$ (invertible)

$EP_1 \to BFS$ B_1 $m \times m$ (invertible)

$$S = \left\{ \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\}$$

Two extreme points x_1, x_2 of S are adjacent iff every $\hat{x}$ on the line segment joining x_1 and x_2 has a unique convex combination in terms of x_1 & x_2 or a unique convex combination of EP's of S.

In other words, in terms of BFS, x_1 and x_2 are adjacent EP of S. iff the set of column vectors

$$\{a_j : \text{ either } x_j^1 > 0 \text{ or } x_j^2 > 0 \text{ or both}\}$$

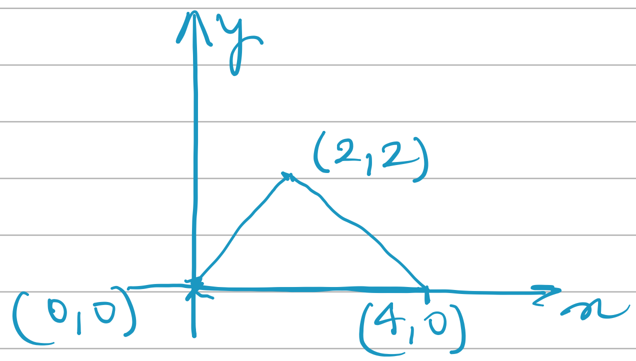
is of cardinality $(n-1)$.

or

In terms of Basis variables x_B^1 or x_B^2. One see only one swap / or exchange of basic variable.

$B^2 = [x_1, x_2]$

$B^1 = [x_1, x_3]$



$x_1 - x_2 \geq 0$

$x_1 + x_2 \leq 4$

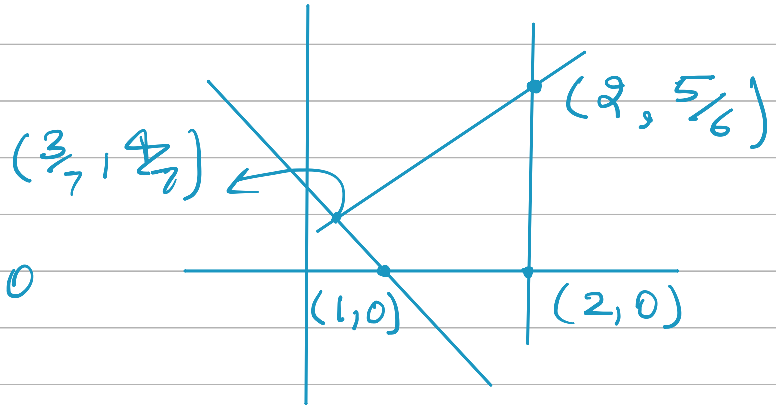
$x_1, x_2 \geq 0$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}_{2\times 4} = A$$

$x_1 - x_2 - x_3 = 0$

$x_1 + x_2 + x_4 = 4$

$x_1, x_2, x_3, x_4 \geq 0$



Result : Let the feasible set S of LPP be $\{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ where A is $m \times n$, $m < n$, rank $A = m$.

Then, for every extreme point of S we have a BFS & for every BFS of system we have an extreme point of S.

Proof: Let $x \in \mathbb{R}^n$ be a BFS of the system $Ax = b$, $x \geq 0$.

To prove: x is an EP of S. We prove it by contradiction. Let x be not an EP of S.

This is like partioning x into 2 parts Basic variables & non basic zero value variables

$\Rightarrow \exists$ two distinct points

x^1 & x^2 in S, and a scalar $\lambda \in (0,1)$ s.t

$x = (1-\lambda)x^1 + \lambda x^2$.

Also, x is a BFS, so.

∃ a basis matrix $B_{m \times m}$ invertible

s.t. $x = \begin{bmatrix} x_B = B^{-1} b \\ x_R = 0 \end{bmatrix} \begin{matrix} m \times 1 \\ (n-m) \times 1 \end{matrix}$

As, $x = (1-\lambda) x^1 + \lambda x^2$.

$$\Rightarrow \begin{bmatrix} x_B \\ x_R = 0 \end{bmatrix}_{n \times 1} = (1-\lambda) \begin{bmatrix} x_B^1 \\ x_R^1 \end{bmatrix}_{n \times 1} + \lambda \begin{bmatrix} x_B^2 \\ x_R^2 \end{bmatrix}_{n \times 1}$$

(indices & order in which we write LHS x & RHS x^1, x^2 are maintained.)

where $x_B^1 = m \times 1$, $x_B^2 = m \times 1$

subvectors x^1 & x^2 corresponding to the indices of x_B in B & in the same order component.

(indices & order in which we write LHS x & RHS x^1, x^2 are maintained.)

$$\Rightarrow x_B = (1-\lambda) x_B^1 + \lambda x_B^2$$

$$0 = (1-\lambda) x_R^1 + \lambda x_R^2$$

≥ 0 ≥ 0 (given in initial condition)

$\because (0 < \lambda < 1)$

So this eq^n $= 0$ only when individual nos are zero.

this gives $x_R^1 = 0$ & $x_R^2 = 0$

multiply both side by B.

$$B x_B = (1-\lambda) B x_B^1 + \lambda B x_B^2.$$

$Ax^1 = b$

$Ax^2 = b$

$$[B \quad R] \begin{bmatrix} x_B^1 \\ x_R^1 \end{bmatrix} = b$$

$$[B \quad R] \begin{bmatrix} x_B^2 \\ x_R^2 \end{bmatrix} = b$$

$$\Rightarrow \left.\begin{array}{l} Bx_B^1 = b \\ Bx_B^2 = b \end{array}\right\} \Rightarrow x_B^1 = B^{-1} b = x^2 B$$

$$\Downarrow$$

$$x_B^1 = x_B^2$$

$$\Rightarrow x^1 = x_B^2$$

(wrong $\because$ both of these vectors are distinct)

$\therefore$ our assump^n is wrong

$\Rightarrow$ x is not an EP of S.