Let X and Y are continuous RVs with joint density $f(x,y)$ and $f_X(x)$ & $f_Y(y)$ are marginals of X and Y respectively. Then, the conditional PDF of X given $Y=y$ is given by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad f_Y(y)>0$$

Similarly, conditional PDF of Y given $X=x$ is given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}, \quad f_X(x)>0$$

Example: Let (X,Y) be jointly distributed with PDF

$$f(x,y) = \begin{cases} 2 & 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

Find $f_{X|Y}(x|y)$ & $f_{Y|X}(y|x)$.

$$f_X(x) = \int_x^1 2\,dy = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & \text{o.w} \end{cases}$$

$$f_Y(y) = \int_0^y 2\,dx = \begin{cases} 2y & 0<y<1 \\ 0, & \text{o.w} \end{cases}$$

For $\underline{0<y<1}$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y}$$

For $\underline{0<y<1}$

$$f_{X|Y}(x|y) = \begin{cases} 1/y & 0<x<y \\ 0 & \text{o.w.} \end{cases}$$

For $0<x<1$

$$f_{Y|X}(y|x) = \frac{2}{2(1-x)} = \frac{1}{1-x}$$

For $0 < x < 1$

$$f_{Y|X}(y|x) = \begin{cases} 1/_{1-x} & x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$P\{Y \geq 1/2 \mid X = 1/2\} = \int_{1/2}^{1} 2\, dy = 1$$

$$P\{X \geq 1/3 \mid Y = 2/3\} = \int_{1/3}^{2/3} 3/2\, dx = 1/2$$

Independent Random Variables :-

Let $(\Omega, \mathcal{F}, P)$ be a probability space. X, Y are random variables defined on $(\Omega, \mathcal{F}, P)$.

Recall the definition of independent events.

$A_1, A_2 \in \mathcal{F}$ are independent we call X, and Y are independent if for every Borel sets B_1, B_2 the events $\underline{\{X \in B_1\}}$ & $\underline{\{Y \in B_2\}}$ are independent, i.e.,

$$P\{X \in B_1,\ Y \in B_2\} = P\{X \in B_1\} \cdot P\{Y \in B_2\}$$

$$\Updownarrow$$

$$P\{X \leq x,\ Y \leq y\} = P\{X \leq x\} \cdot P\{Y \leq y\} \quad \forall\, x, y \in \mathbb{R}$$

__Def^n__: Two RVs X and Y are said to be independent if for any $x, y \in \mathbb{R}$

$$P\{X \leq x, Y \leq y\} = P\{X \leq x\} \cdot P\{Y \leq y\}$$

$$\Updownarrow$$

$$\Downarrow$$

$$\boxed{F(x,y) = F_X(x)\, F_Y(y)}$$

<u>Theorem</u>:

a) A necessary and sufficient condition for RVs, X and Y of discrete type to be independent is that

$$P\{X = x_i, Y = y_j\} = P\{X = x_i\} \cdot P\{Y = y_j\} \quad \forall\, x_i, y_j$$

b) RVs X and Y of continuous type are independent iff

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

<u>Proof</u>: (a) Let X & Y be independent RVs.

$\{x_i\}$ & $\{y_j\}$ are Borel set

$$P\{X = x_i, Y = y_j\} = P\{X = x_i\} \cdot P\{y_j\} \quad \text{--- (+)}$$

Suppose (+) is true

$$P\{X \le x, Y \le y\} = \sum_{x_i \le x} \sum_{y_j \le y} P\{X = x_i, Y = y_j\}$$

$$= \sum_{x_i \le x} P\{X = x_i\} \cdot \sum_{y_j \le y} P\{Y = y_j\}$$

$$P\{X \le x, Y \le y\} = P\{X \le x\} \cdot P\{Y \le y\}$$

(b) Let X & Y be independent.

$$\Rightarrow \quad F(x,y) = F_X(x) \cdot F_Y(y)$$

$$= \int_{-\infty}^{x} f_X(u)\,du \cdot \int_{-\infty}^{y} f_Y(v)\,dv$$

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \underline{f_X(u) \cdot f_Y(v)}\, dv\, du$$

from def^n of CDF.

$$\Rightarrow \quad \underline{f(x,y) = f_X(x) \cdot f_Y(y)}$$

Converse is easy. (exercise).

If X and Y are independent RVs.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x) \cdot \cancel{f_Y(y)}}{\cancel{f_Y(y)}}$$

$$f_{X|Y}(x|y) = f_X(x)$$

$$f_{Y|X}(y|x) = f_Y(y)$$

Result :- Let X and Y be independent RVs and f, g be Borel measurable functions. Then, $\underline{f(X)}$ and $\underline{g(Y)}$ are also independent.

Proof :- $P\{\omega \mid f(X(\omega)) \le x,\ g(Y(\omega)) \le y\}$

$$\|$$

$$= P\{\omega \mid X(\omega) \in f^{-1}((-\infty, x]),\ Y(\omega) \in g^{-1}((-\infty, y])\}$$

$$= \underline{P\{X \in f^{-1}((-\infty, x])\}} \cdot \underline{P\{Y \in g^{-1}((-\infty, y])\}}$$

$$= P\{f(X) \le x\} \cdot P\{g(Y) \le y\}$$

$\underline{f^{-1}(x)}$

$f^{-1}((\underline{-\infty, x}]))$

$= \{y \mid f(y) \le x\}$

$\underline{f^{-1}(B)}$

$= \{y \mid f(y) \in B\}$

$f^{-1}((\underline{-\infty, x}])) = B_1$

$g^{-1}((\underline{-\infty, y}])) = B_2$

are Borel sets

$f : \mathbb{R} \to \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R}$

$$P\{f(X) \le x,\ g(Y) \le y\} = P\{f(X) \le x\} \cdot P\{g(Y) \le y\}$$

$\Rightarrow$ $f(X)$ & $g(Y)$ are independent.

<u>Result</u>: If X and Y are independent

$$E[XY] = E[X] \cdot E[Y]$$

<u>Proof</u>: $$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x y \, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx \cdot \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$$

$$\underline{E[XY] = E[X] \cdot E[Y]}$$

Converse is not true.

<u>Defn</u>: A collection of RVs $X_1, X_2, \ldots, X_n$ is said to be independent if

$$F(x_1, \ldots x_n) = \prod_{i=1}^{n} F_{X_i}(x_i)$$

$$\Updownarrow$$

$$f(x_1, \ldots x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

Def^n : $X_1, \ldots X_n$ are said to be pairwise independent if for every pair (X_i, X_j) $i \neq j$, X_i & X_j are independent.

$$\text{Independence} \overset{?}{\Rightarrow} \text{Pairwise independence}$$

$$\not\Leftarrow ?$$

$$\underline{f(\underline{x_i, x_j})} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots x_n)\, dx_1 \ldots dx_n$$

dx_i, dx_j missing.

$$= f_{X_i}(x_i) \cdot f_{X_j}(x_j) \underbrace{\int_{-\infty}^{\infty} f_{X_1}(x_1)\, dx_1}_{=1} \cdots \underbrace{\int_{-\infty} f_{X_n}(x_n)\, dx_n}_{=1}$$

$$f(x_i, x_j) = f_{X_i}(x_i) \cdot f_{X_j}(x_j)$$