Lecture 17
Signals and Systems (ELL205)

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Analysis and Synthesis equation

Synthesis

$$x(t) = x(t+T) = \sum_{k} a_k e^{jk\omega_o t}$$

Analysis

$$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\omega_o t}dt$$

List of Properties

1. $x(t) \leftrightarrow a_k$

2. $x(-t) \leftrightarrow a_{-k}$

3. $\overline{x(t)} \leftrightarrow \overline{a_{-k}}$

4. $x(t - t_o) \leftrightarrow e^{-jk\omega_o t_o} a_k$

Signal	Coefficients
Real & Even	Real & Even
Real & Odd	Imaginary & Odd
Imaginary & Even	Imaginary & Even
Imaginary & Odd	Real & Odd
Shifted signal	Only phase changes

Convergence of Fourier Series

$$x_N(\mathrm{t}) \triangleq \sum_{k=-N}^{N} a_k e^{jk\omega_o t}$$

$$e_N(\mathrm{t}) \triangleq x(t) - x_N(\mathrm{t})$$

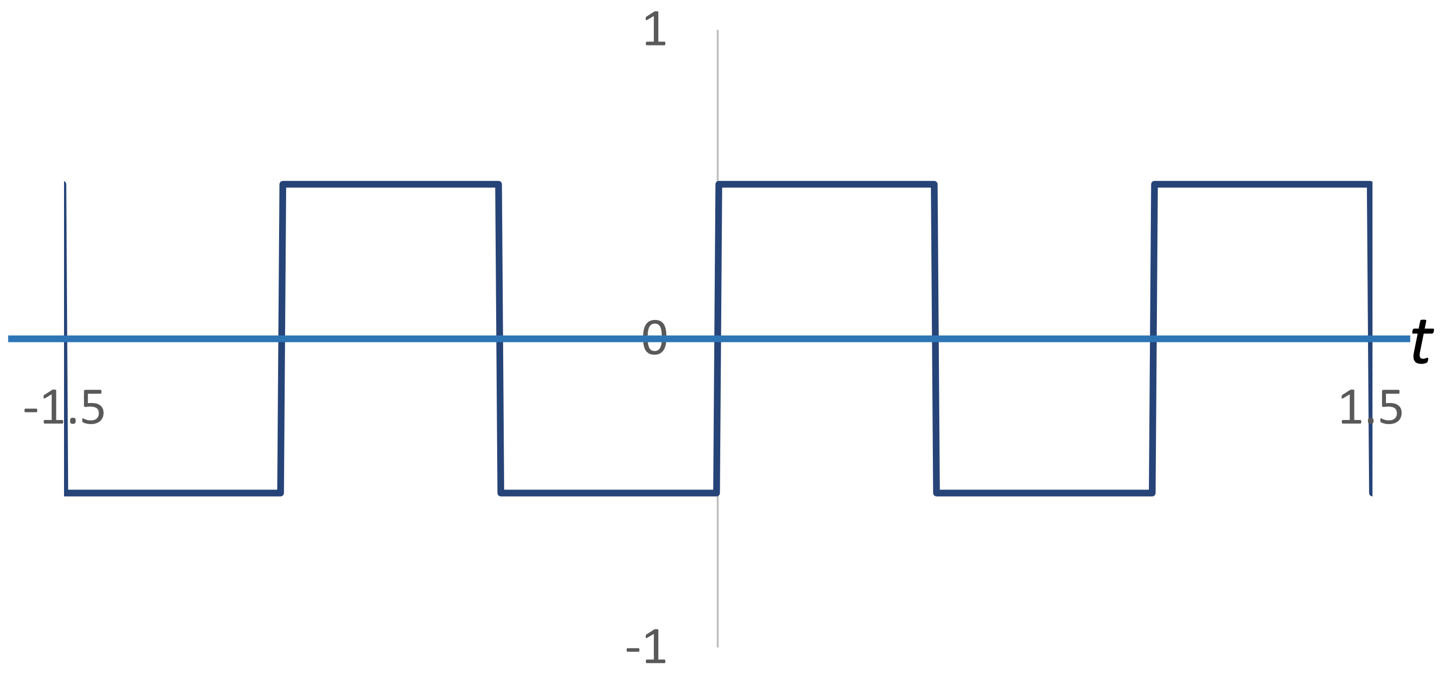
Does $e_N(\mathrm{t})$ decrease as *N* increases?

Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=0}^{0} \frac{1}{j\pi k} e^{jk2\pi t}$$

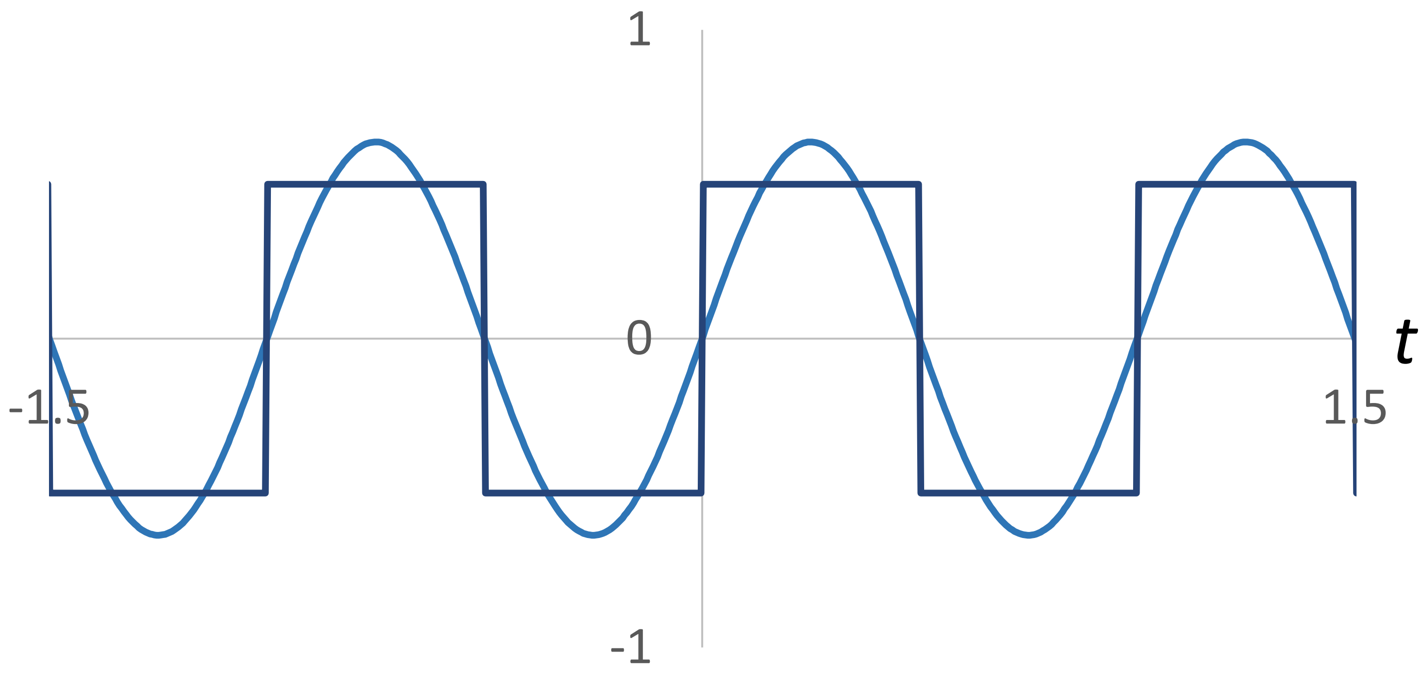


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-1}^{1} \frac{1}{j\pi k} e^{jk2\pi t}$$

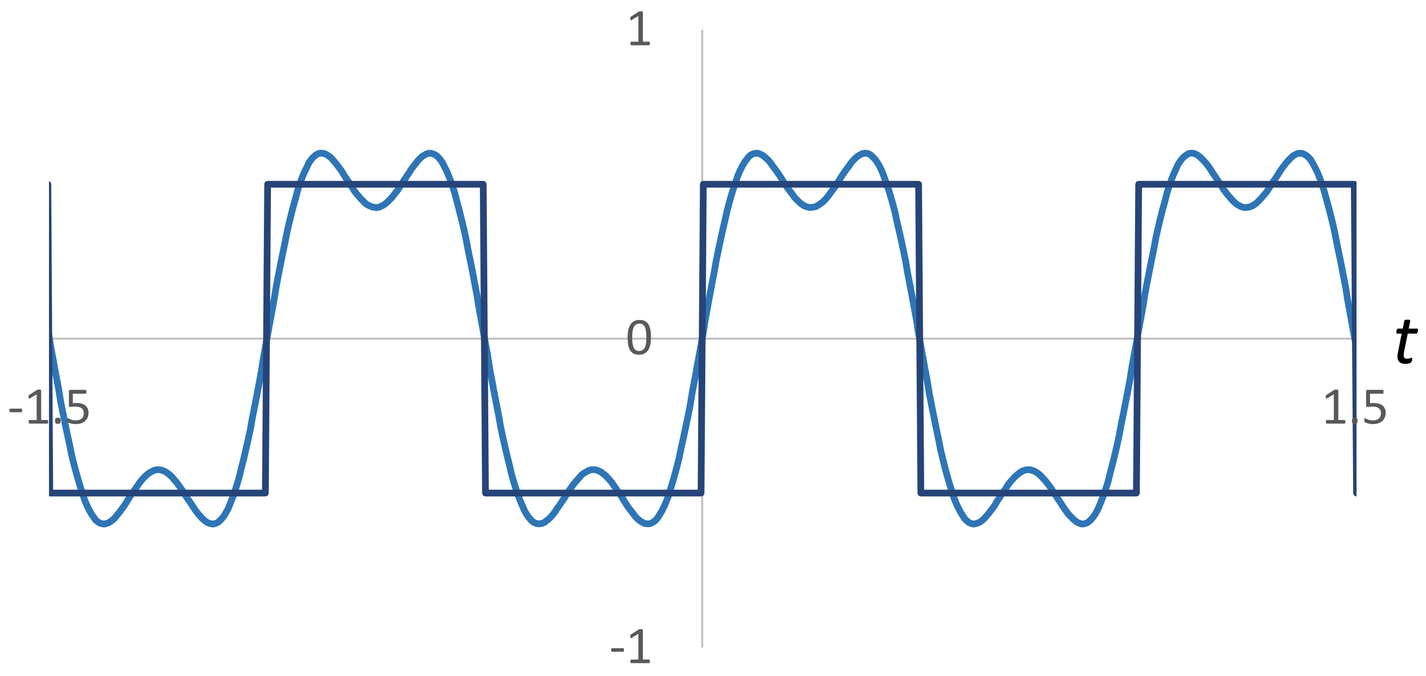


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-3}^{3} \frac{1}{j\pi k} e^{jk2\pi t}$$

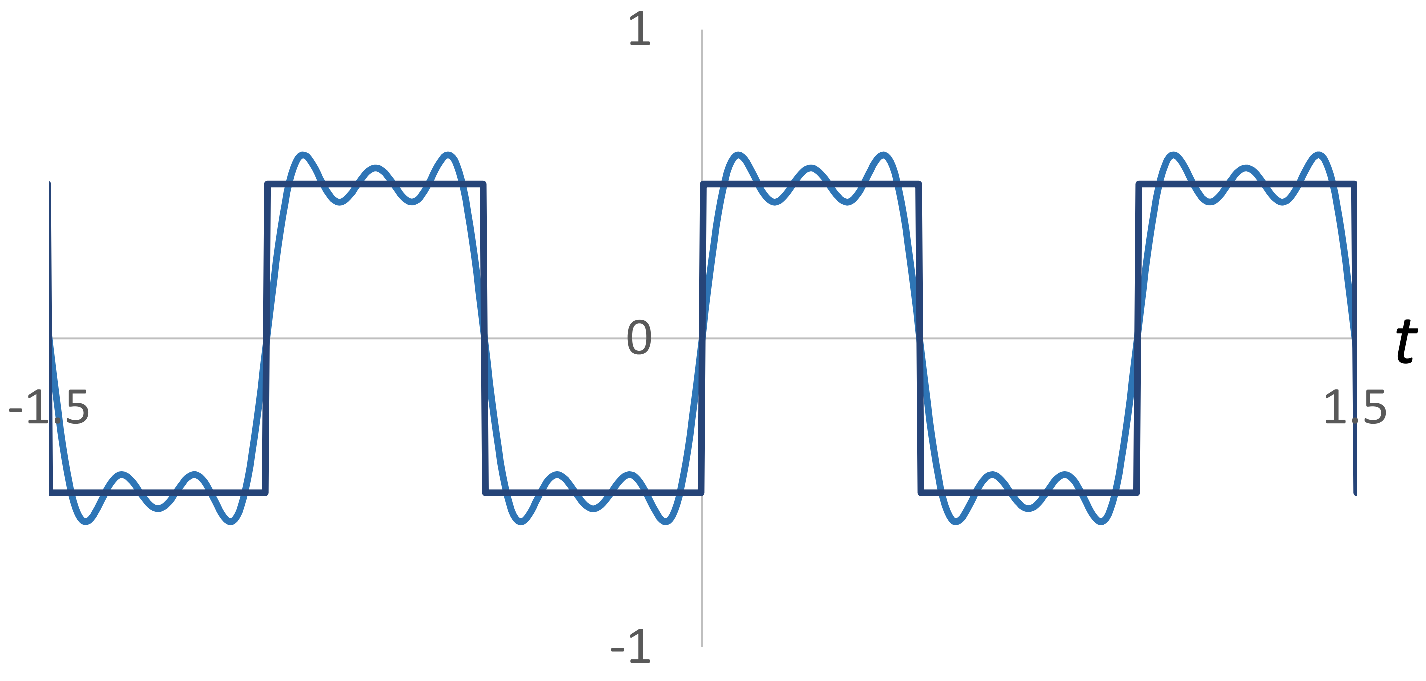


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-5}^{5} \frac{1}{j\pi k} e^{jk2\pi t}$$

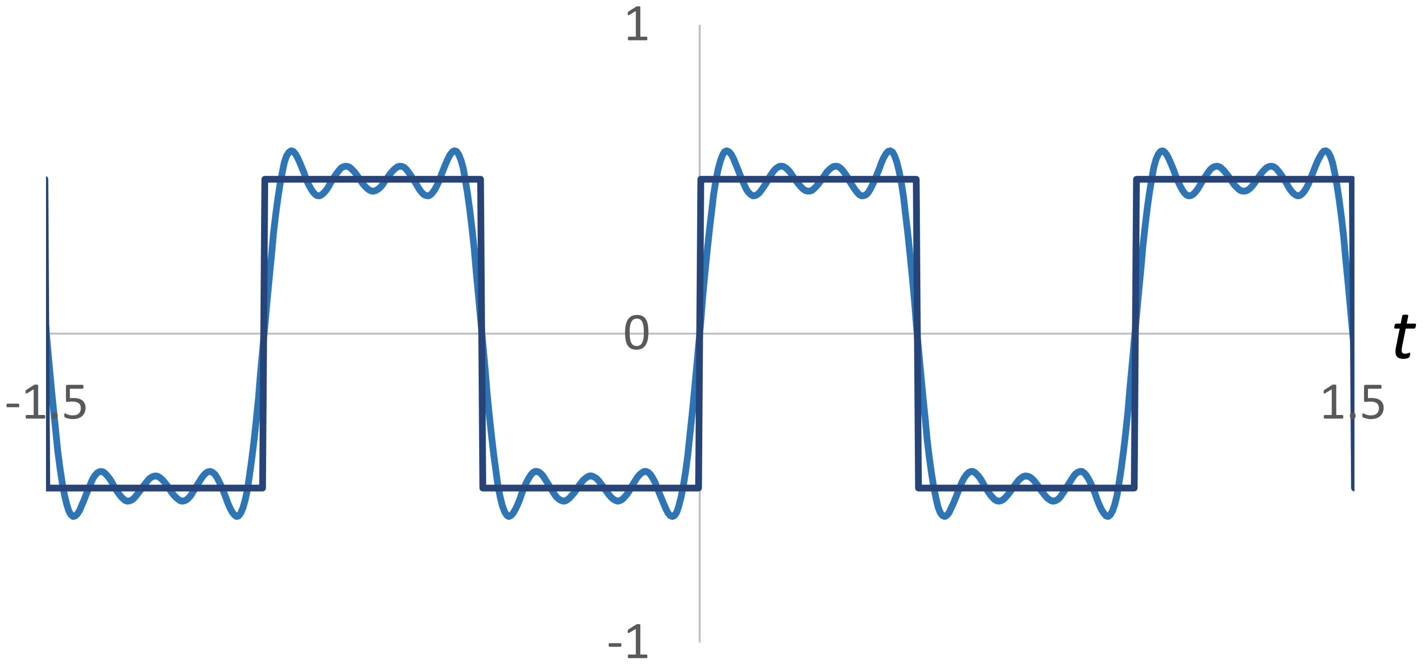


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-7}^{7} \frac{1}{j\pi k} e^{jk2\pi t}$$

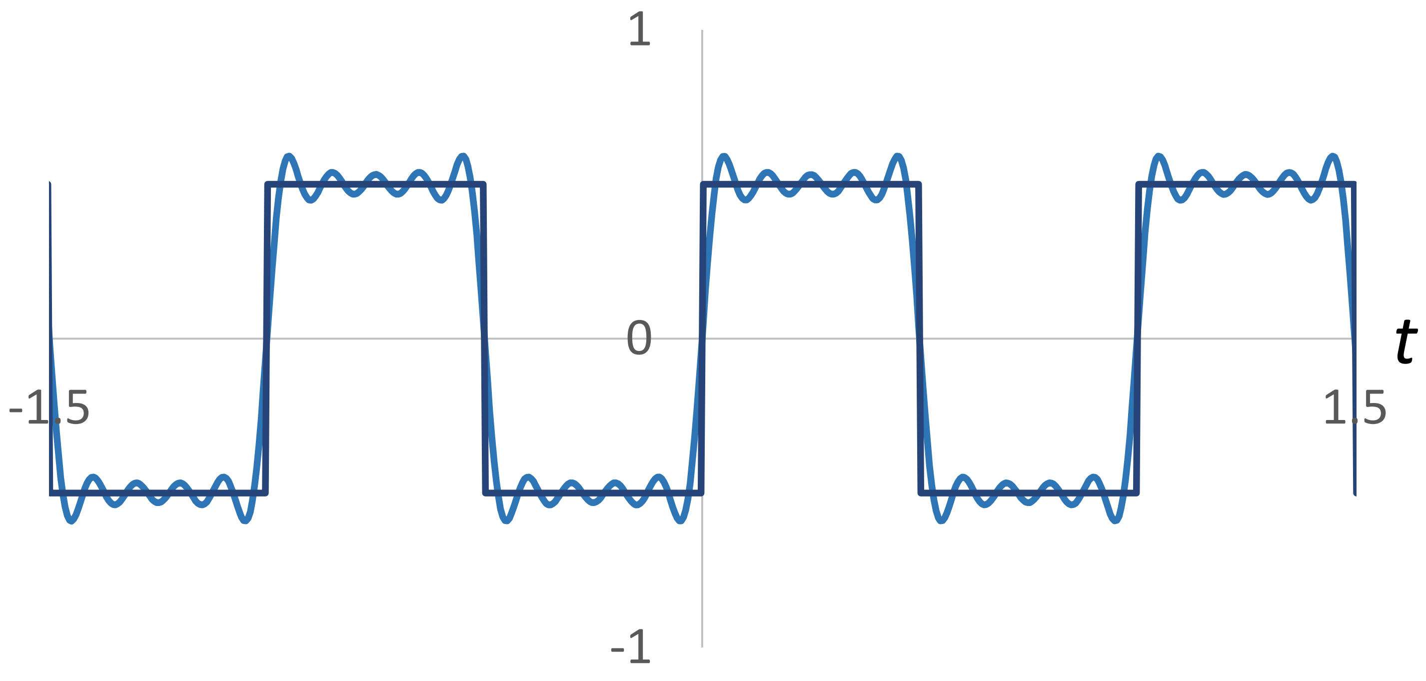


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-9}^{9} \frac{1}{j\pi k} e^{jk2\pi t}$$

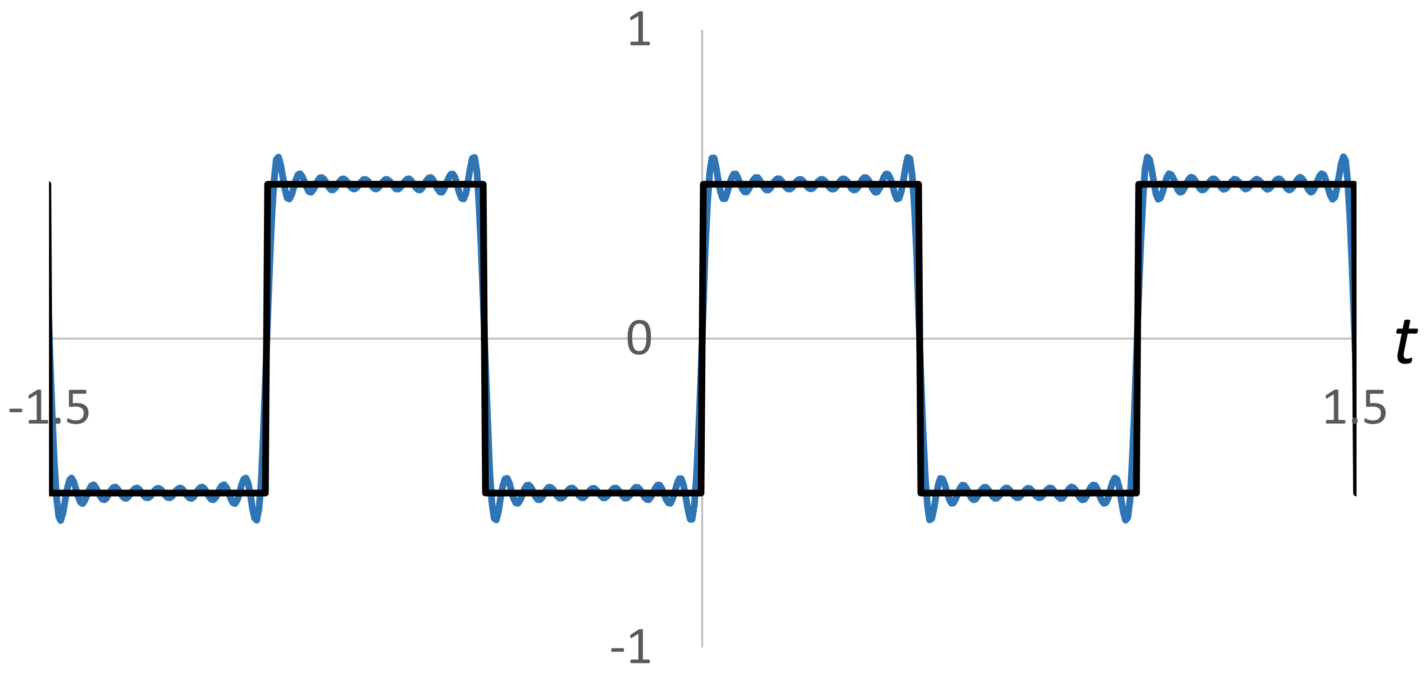


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-19}^{19} \frac{1}{j\pi k} e^{jk2\pi t}$$

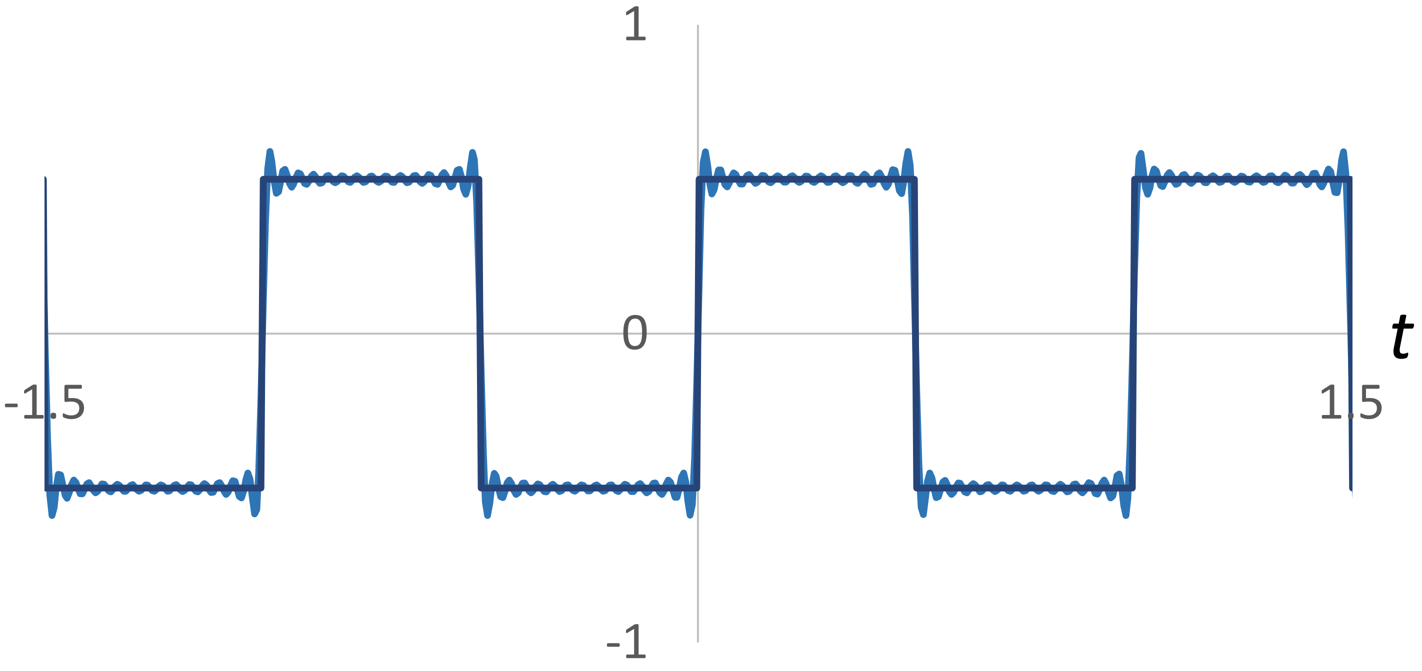


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-29}^{29} \frac{1}{j\pi k} e^{jk2\pi t}$$

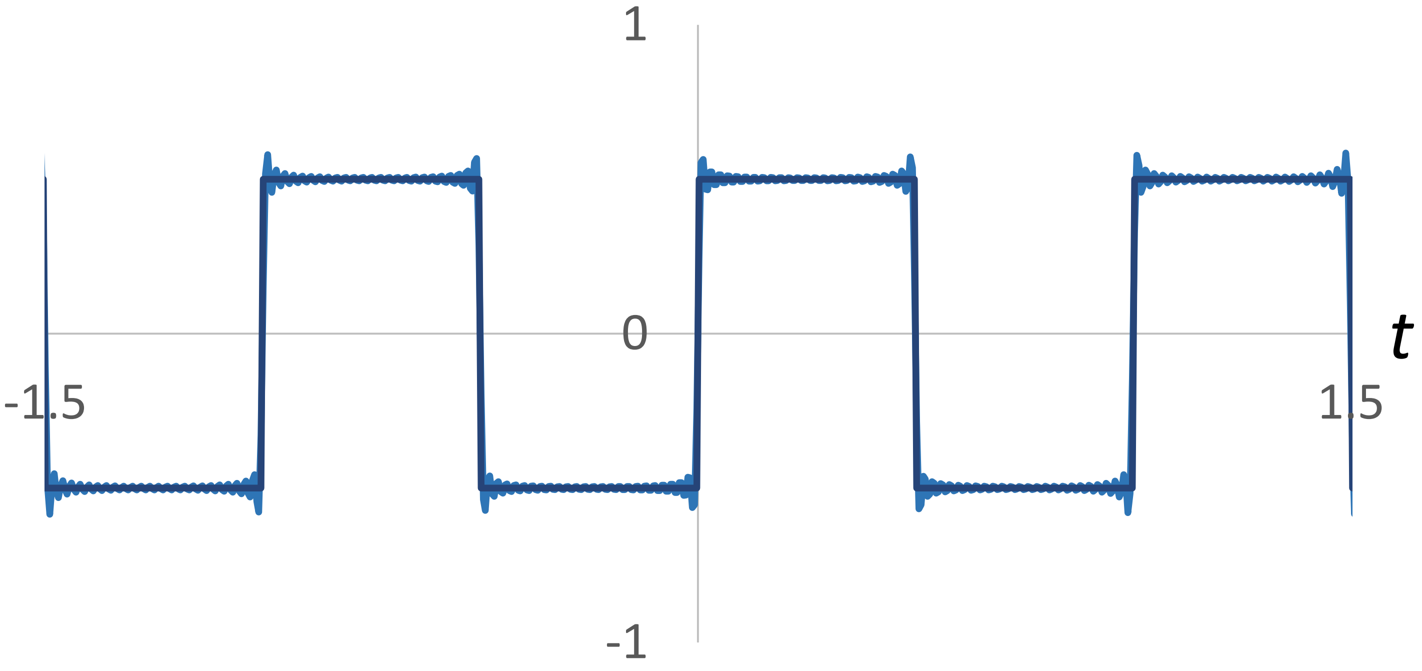


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-49}^{49} \frac{1}{j\pi k} e^{jk2\pi t}$$

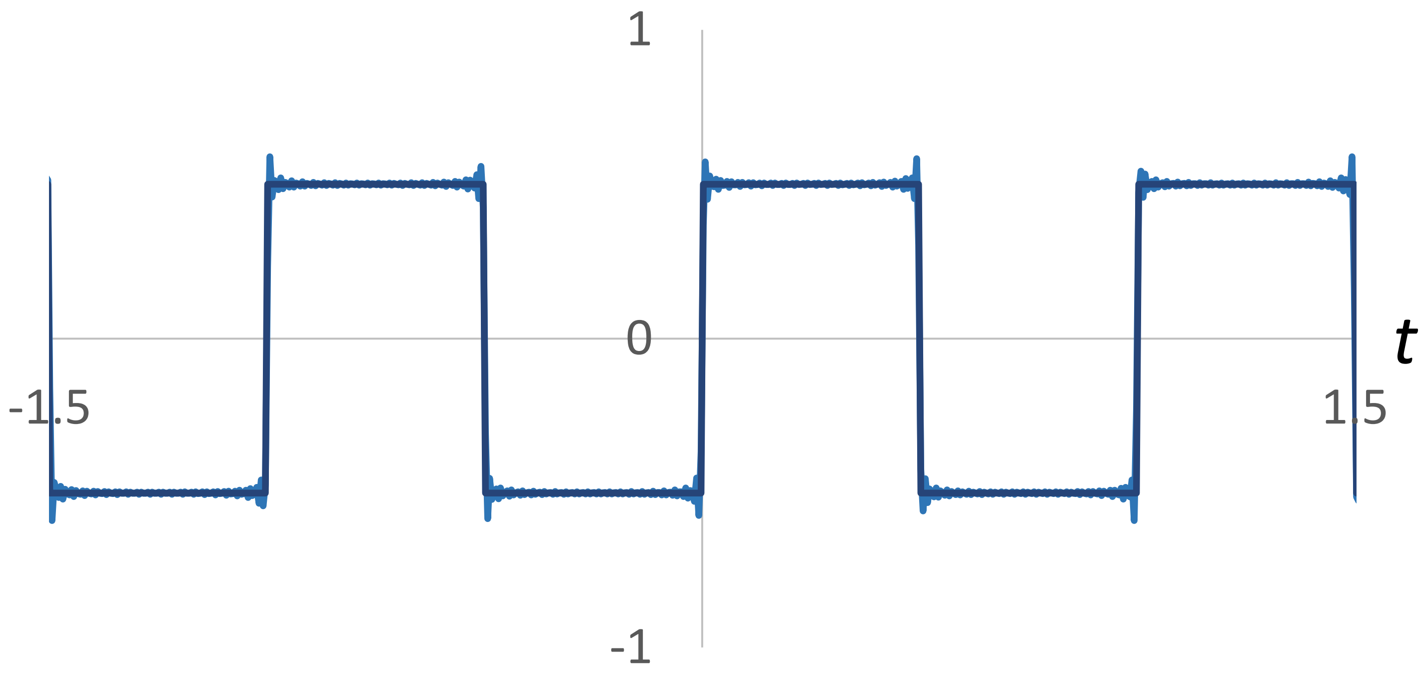


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-79}^{79} \frac{1}{j\pi k} e^{jk2\pi t}$$

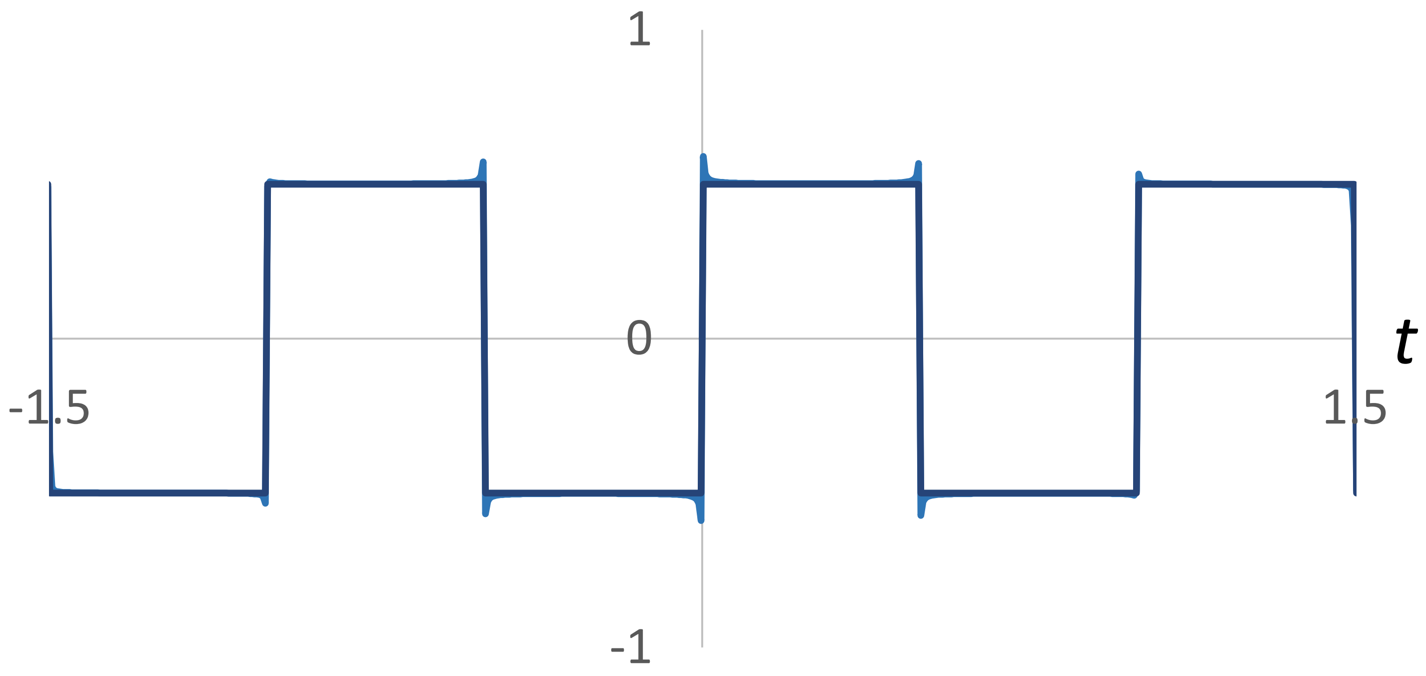


Convergence of Fourier Series

Convergence of the Fourier Series by incrementally adding terms.

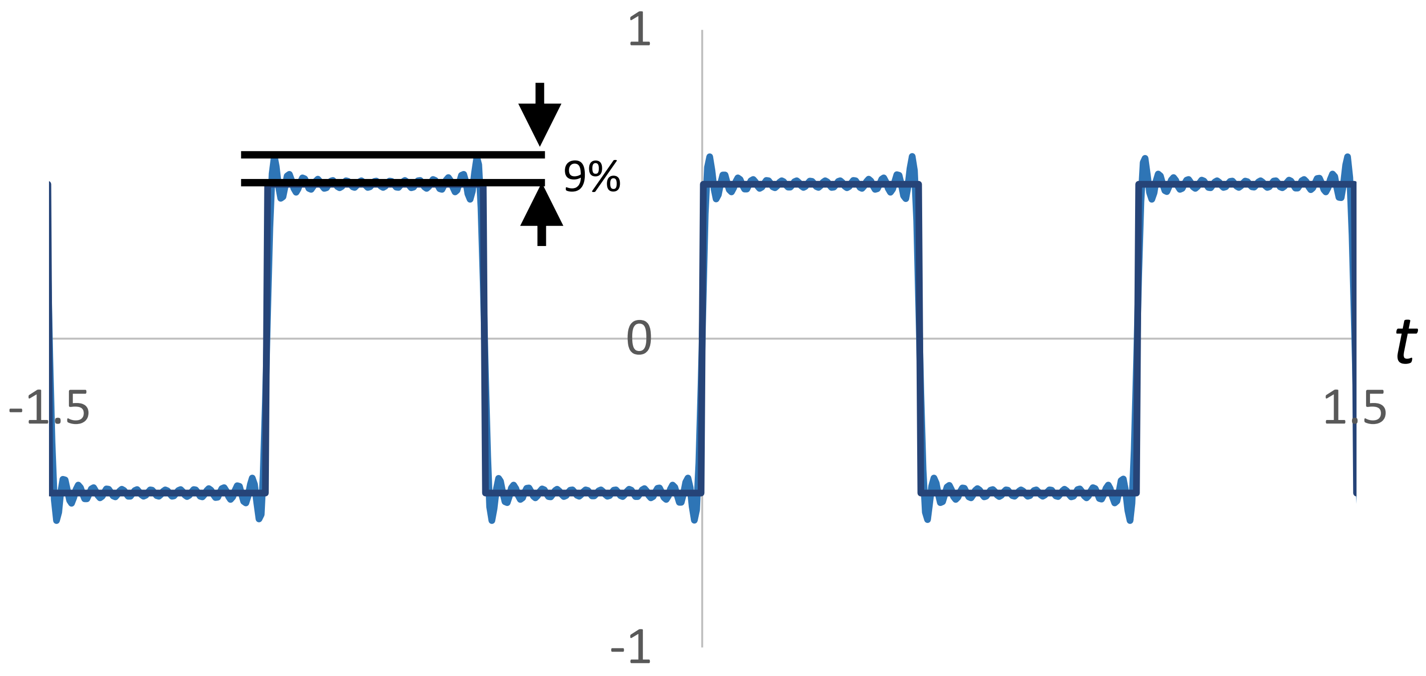
Example: square wave

$$x(t) = \sum_{k=-199}^{199} \frac{1}{j\pi k} e^{jk2\pi t}$$



Convergence of Fourier Series

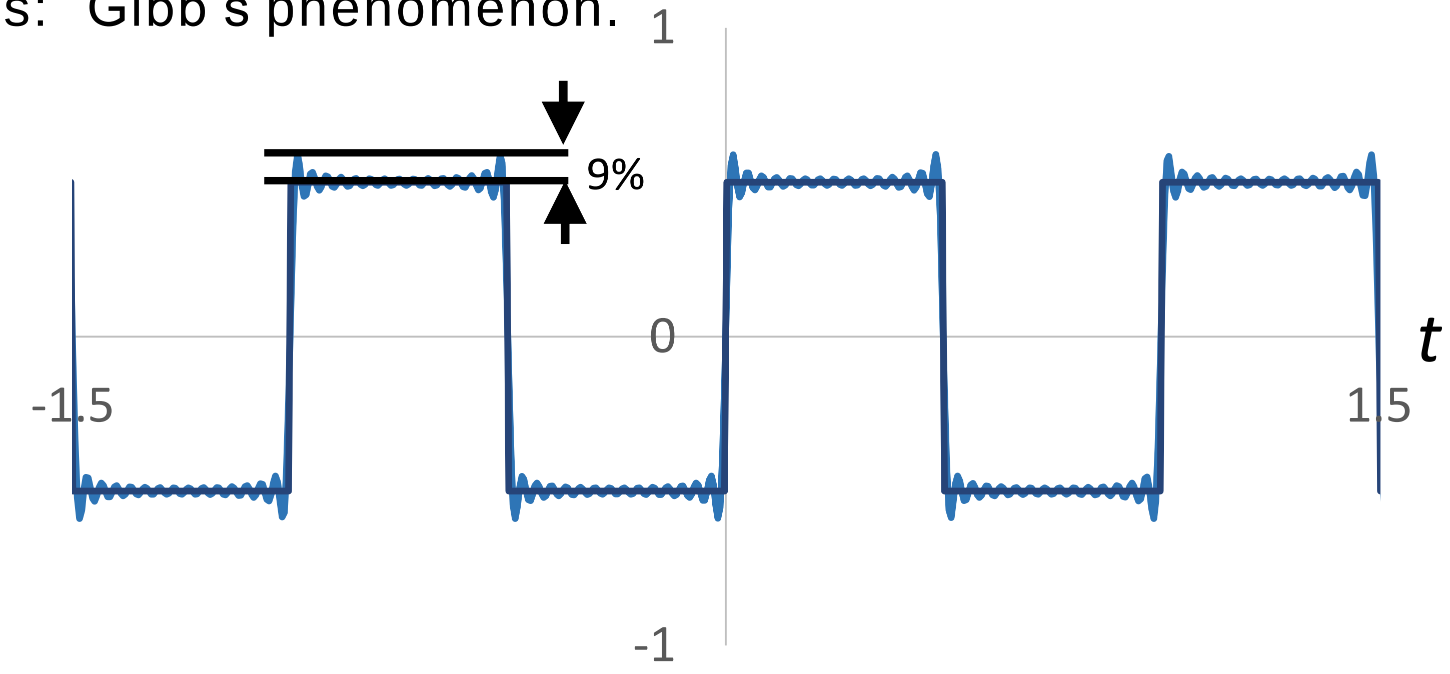
Albert Michelson horror



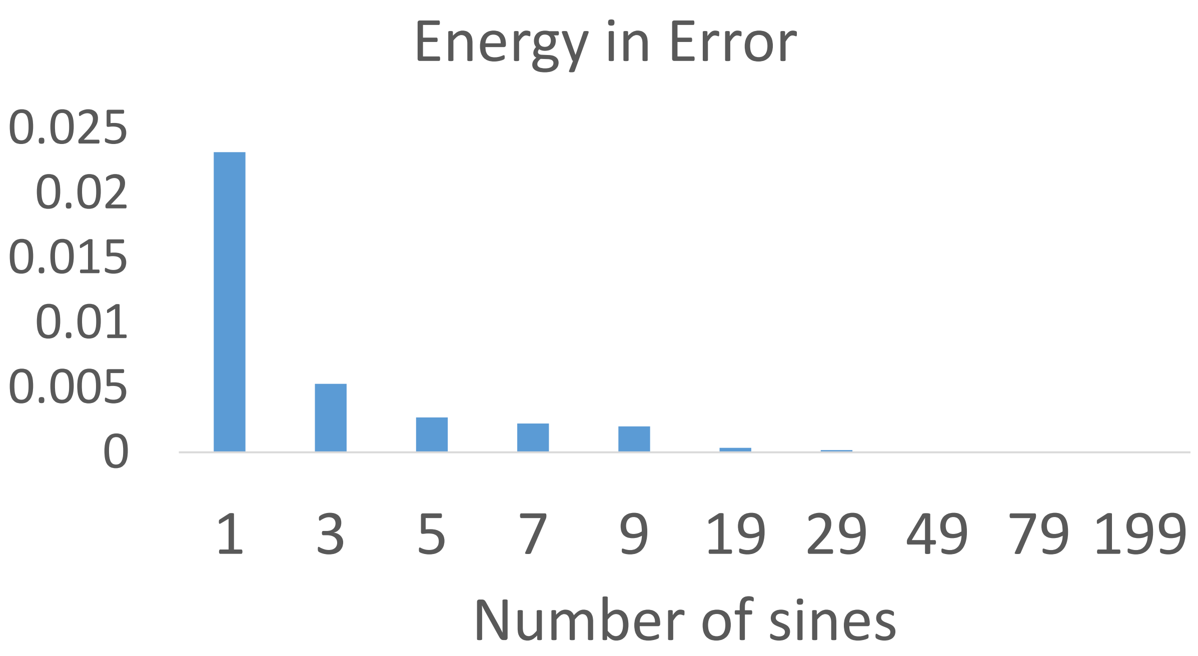
Convergence of Fourier Series

Albert Michelson horror

Partial sums of Fourier series of discontinuous functions “ring” near discontinuities: Gibb’s phenomenon.



Energy in Error



Convergence of Fourier Series

- $x(\tau)$ square integrable,

  If $\int_T |x(\tau)|^2 d\tau < \infty$

  Then $\int_T |e_N(t)|^2 dt \to 0$ as $N \to \infty$

- Dirichlet conditions

  If $\int_T |x(\tau)| d\tau < \infty$ and $x(t)$ is "well-behaved"

  Then $e_N(t) \to 0$ as $N \to \infty$ except at discontinuities

Convergence of Fourier Series

- $x(\tau)$ square integrable,

If $\int_T |x(\tau)|^2 d\tau < \infty$ Then $\int_T |e_N(t)|^2 dt \to 0$ as $N \to \infty$

$$\lim_{N\to\infty}\left(\int_T \left|x(\mathrm{t}) - \sum_{k=-N}^{N} a_k e^{jk\omega_o t}\right|^2 dt\right) \to 0$$

$$x(\mathrm{t}) \triangleq l.i.m \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Properties of FS

1. Linearity If $x(t) \leftrightarrow a_k$ & $y(t) \leftrightarrow b_k$ then

$$\alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$$

Properties of FS

1. Linearity

If $x(t) \leftrightarrow a_k$ & $y(t) \leftrightarrow b_k$ then

$$\alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$$

Proof:

$$c_k = \frac{1}{T}\int_T (\alpha x(t) + \beta y(t))e^{-jk\omega_o t}dt$$

$$c_k = \frac{\alpha}{T}\int_T x(t)e^{-jk\omega_o t}dt + \frac{\beta}{T}\int_T y(t)e^{-jk\omega_o t}dt$$

$$c_k = \alpha a_k + \beta b_k$$

Properties of FS

2. Time-scaling

If $x(t) \leftrightarrow a_k$ then $x(\alpha t) \leftrightarrow a_k$

Properties of FS

2. Time-scaling

If $x(t) \leftrightarrow a_k$ then $x(\alpha t) \leftrightarrow a_k$

Proof:

Starting with def. $b_k = \frac{\alpha}{T}\int_{T/\alpha} x(\alpha t)e^{-jk\alpha\omega_o t}\, dt$

Substituting, $\alpha t = \lambda$

we get, $b_k = \frac{1}{T}\int_{T} x(\lambda)e^{-jk\omega_o \lambda}\, d\lambda$

Properties of FS

3. Flipping

Properties of FS

4. Differentiation

If $x(t) \leftrightarrow a_k$ then $\dfrac{dx(t)}{dt} \leftrightarrow jk\omega_o a_k$

Properties of FS

4. Differentiation

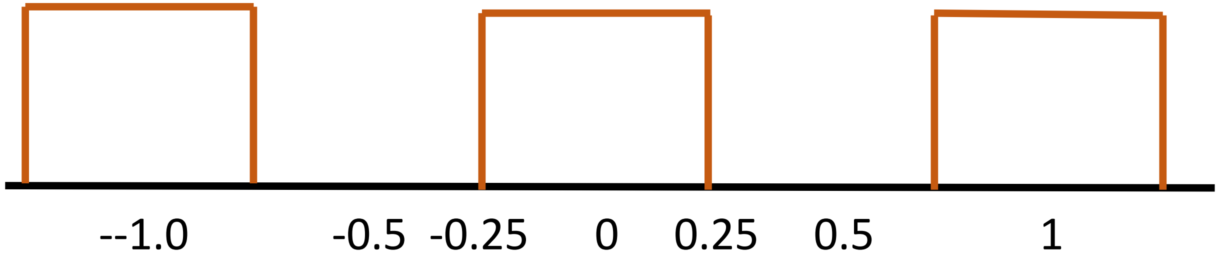
If $x(t) \leftrightarrow a_k$ then $\dfrac{dx(t)}{dt} \leftrightarrow jk\omega_o a_k$

Proof:

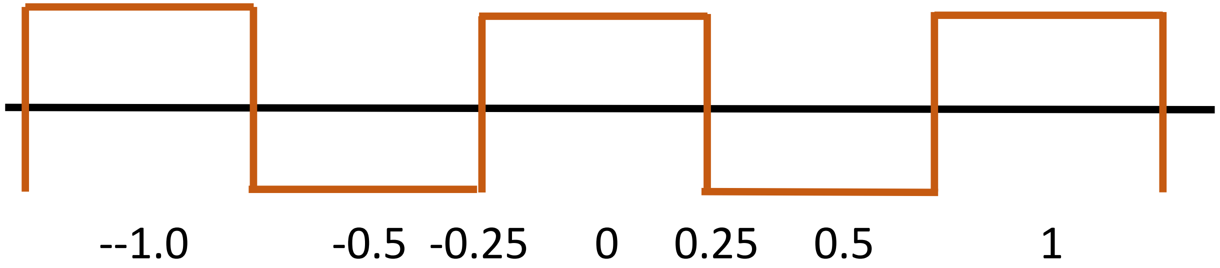
$$x(t) = \sum_k a_k e^{jk\omega_o t}$$

$$\frac{dx(t)}{dt} = \sum_k jk\omega_o a_k e^{jk\omega_o t}$$

Use properties

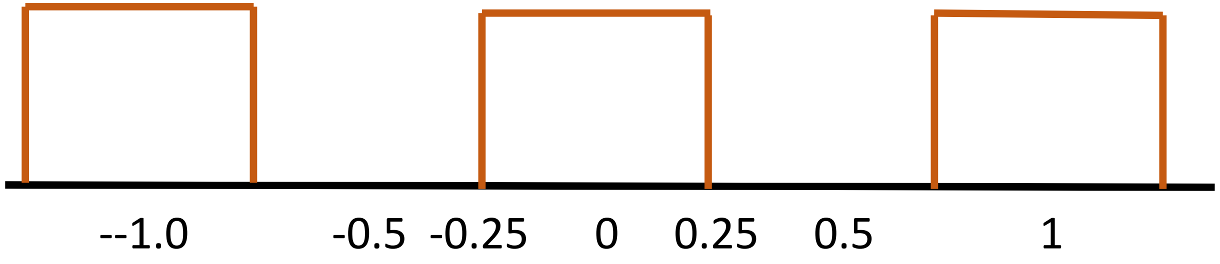


a_k

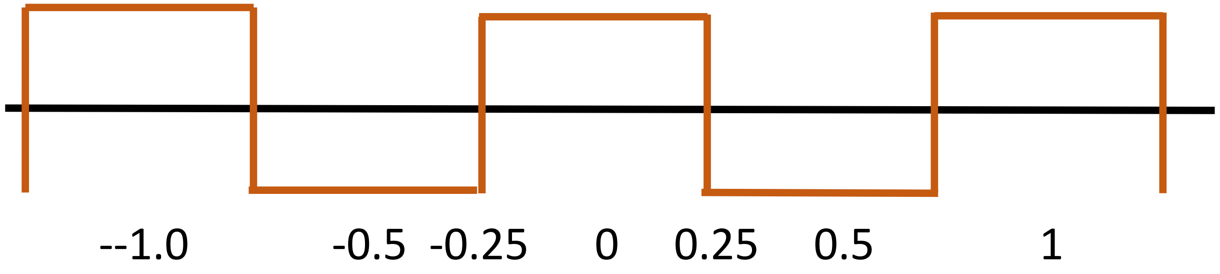


b_k =?

Use properties



a_k



$b_k = ?$

$b_k = a_k$ for $k \neq 0$

$b_o = 0$

Use properties

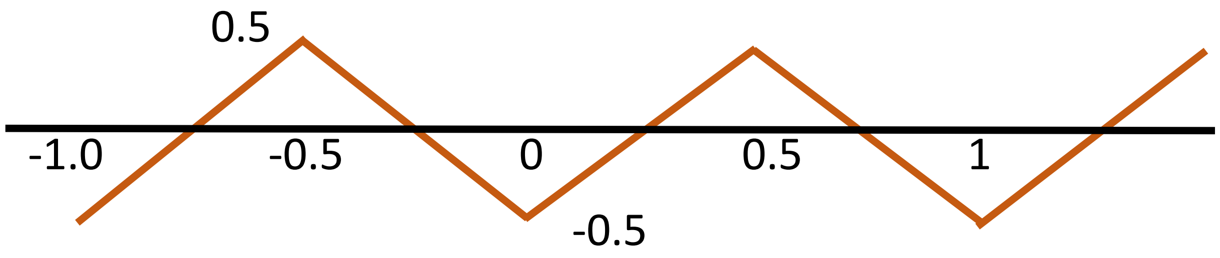
$$x(t) = m + \widetilde{x(t)}$$

$\widetilde{x(t)}$ has mean value 0

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{T}\int_T m dt + \frac{1}{T}\int_T \widetilde{x(t)} dt = m$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{m}{T}\int_T e^{-jk\omega_0 t}dt + \frac{1}{T}\int_T \widetilde{x(t)}e^{-jk\omega_0 t}dt = 0 + b_k \qquad \text{For } k \neq 0$$

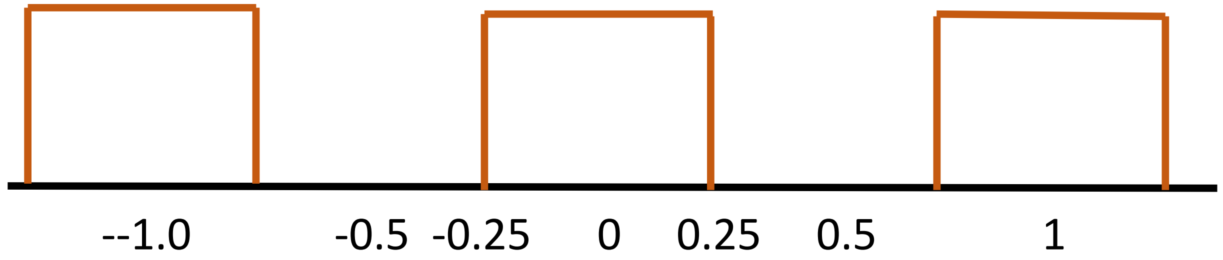
Use properties



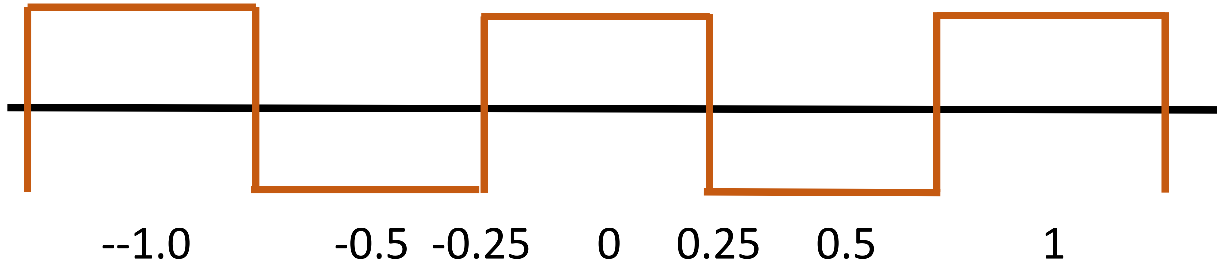
How many of the following terms are correct?

1) Odd harmonics decreases as k^2	2) Even harmonics are zero
3) a_0 is 0	4) It requires a fewer terms for signal construction

Step 1



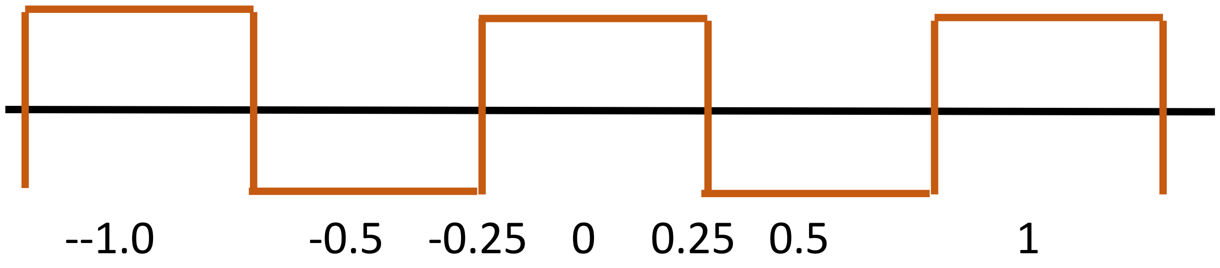
$$a_k = \frac{\sin(\frac{k\pi}{2})}{k\pi}$$



$$a_k = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} \quad \text{for } k \neq 0$$

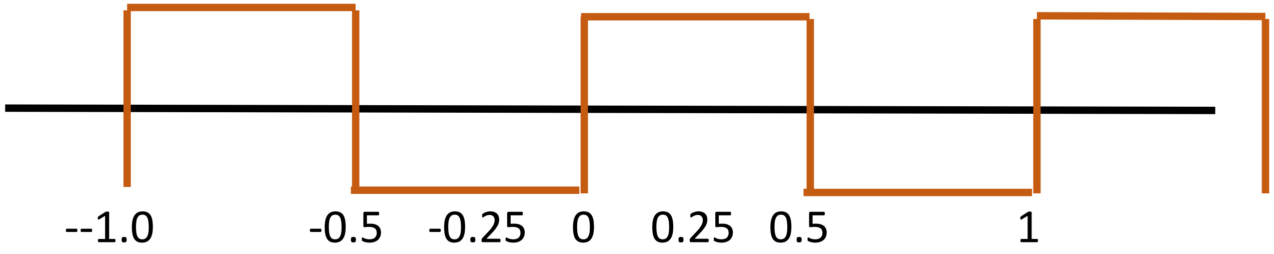
$$a_o = 0$$

Step 2



$$a_k = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi}$$

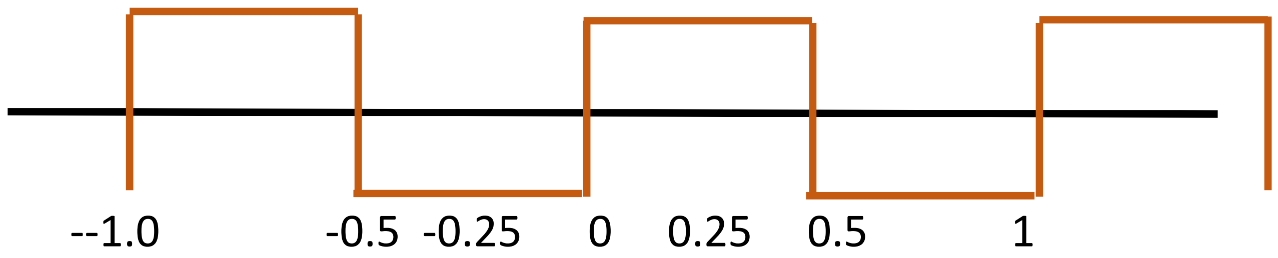
$$a_o = 0$$



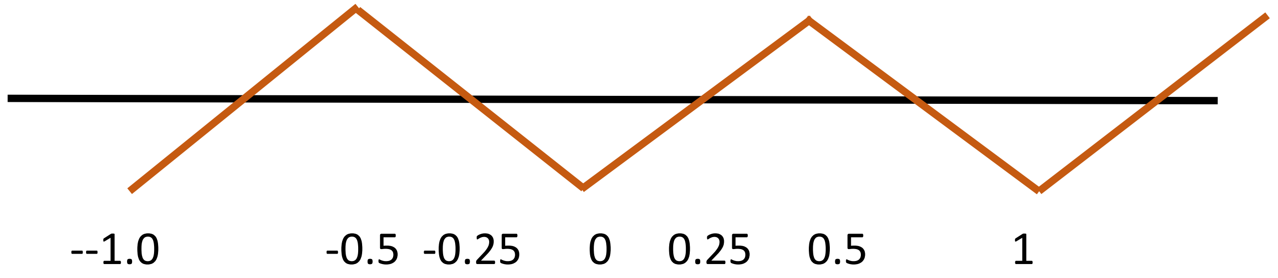
$$a_k = \frac{\sin\left(\frac{k\pi}{2}\right) e^{-jk\pi/2}}{k\pi} \qquad a_0 = 0$$

$$a_k = \begin{cases} \dfrac{1}{jk\pi} & k\ odd \\ 0 & k\ even \end{cases}$$

Step 3

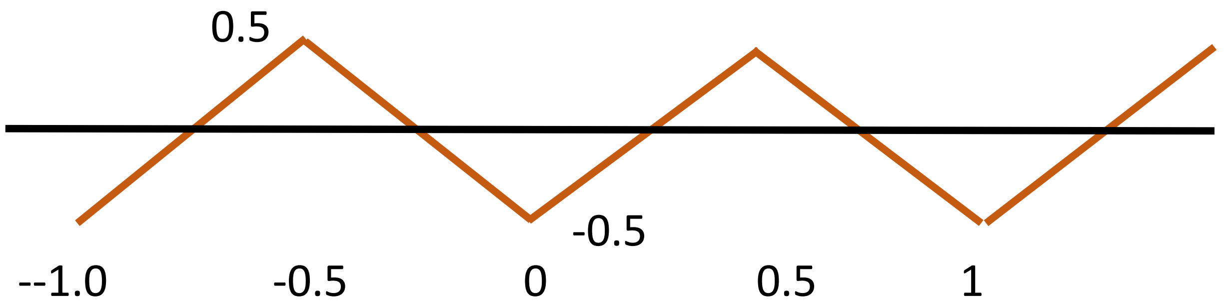


$$a_k = \begin{cases} \dfrac{1}{jk\pi} & k\ odd \\ 0 & k\ even \end{cases}$$



$$b_k = \begin{cases} \dfrac{-1}{2\pi^2 k^2} & k\ odd \\ 0 & k\ even \end{cases}$$

Use properties



How many of the following terms are correct?

1) Odd harmonics decreases as k^2	2) Even harmonics are zero
3) a_0 is 0	4) It requires a fewer terms for signal construction

Properties of FS

5. Integration

If $x(t) \leftrightarrow a_k$ then $\displaystyle\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{a_k}{jk\omega_o}$

Caution: $\displaystyle\int_{-\infty}^{t} x(\tau)d\tau$ is periodic & finite only if $x(\tau)$ has zero DC value

Properties of FS

6. Multiplication

If $x(t) \leftrightarrow a_k$ & $y(t) \leftrightarrow b_k$ then

$$x(t)y(t) \leftrightarrow \sum_{\lambda} a_{\lambda} b_{k-\lambda} = a_k * b_k$$

Properties of FS

6. Multiplication

If $x(t) \leftrightarrow a_k$ & $y(t) \leftrightarrow b_k$ then

$$x(t)y(t) \leftrightarrow \sum_\lambda a_\lambda b_{k-\lambda} = a_k * b_k$$

Proof:

$$x(t)y(t) = \sum_\lambda a_\lambda e^{j\lambda\omega_o t} \sum_l b_l e^{jl\omega_o t} = \sum_k c_k e^{jk\omega_o t}$$

$$\lambda\omega_o + (k-\lambda)\,\omega_o = k\omega_o$$

$$a_\lambda \times b_{k-\lambda} + \cdots . = c_k$$

$$a_\lambda \times b_{k-\lambda} + a_{\lambda-1} \times b_{k-\lambda+1} + \cdots = c_k$$

$$\sum_\lambda a_\lambda \times b_{k-\lambda} = c_k$$

Properties of FS

6. Multiplication

If $x(t) \leftrightarrow a_k$ & $y(t) \leftrightarrow b_k$ then

$$x(t)y(t) \leftrightarrow \sum_{\lambda} a_{\lambda} b_{k-\lambda} = a_k * b_k$$

Multiplication in time domain leads to convolution in frequency domain

Properties of FS

7. Parseval's theorem

$$x(t) \leftrightarrow a_k$$

$$\frac{1}{T}\int_{T} |x(t)|^2\, dt = \sum_{k} |a_k|^2$$

One more issue

$x(t) \leftrightarrow a_k$ $\qquad$ $x(-t) \leftrightarrow a_{-k}$

If $x(t) = x(-t)$

$$\sum_k a_k e^{jk\omega_o t} = \sum_k a_{-k} e^{jk\omega_o t}$$

Can $\sum_k a_k e^{jk\omega_o t} = \sum_k a_{-k} e^{jk\omega_o t}$ without $a_k = a_{-k}$?