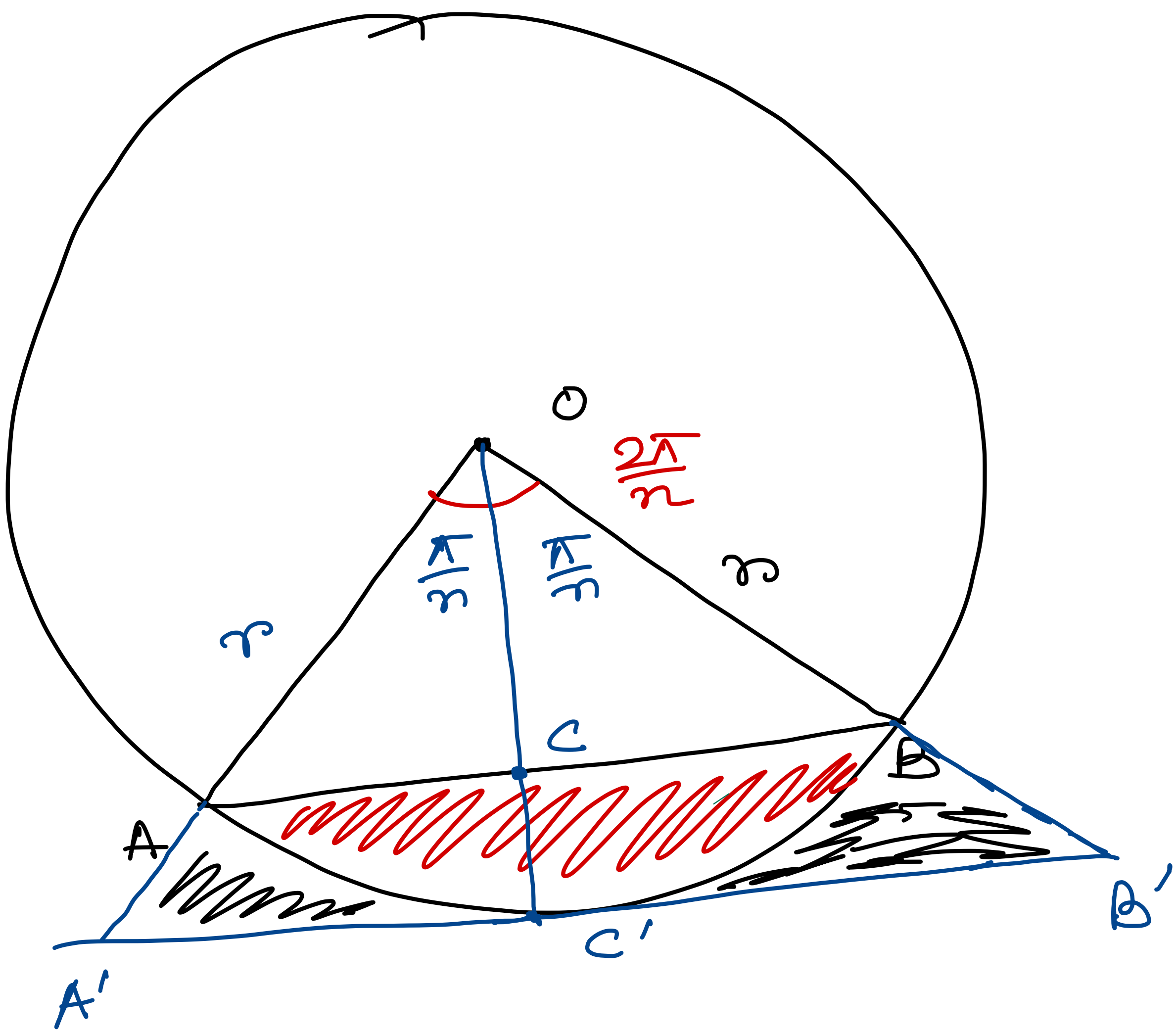
Lecture 1

– Riemann Integration.

Aparajita Dasgupta

Riemann Integration

Disc = D



Area of $OAB = 2\ OAC$

$$= 2 \cdot \frac{1}{2} r \sin\frac{\pi}{n} \cdot r\cos\frac{\pi}{n}.$$

$$= r^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n}.$$

$$\boxed{\text{Total area of inscribed } \Delta\text{'s} = n r^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n}.}$$

Area of $OA'B'$

$$= 2 \cdot \frac{1}{2} r \cdot r \tan\frac{\pi}{n}$$

$$= r^2 \tan\frac{\pi}{n}.$$

Total area of superscribed Δ's $= n r^2 \tan\frac{\pi}{n}$.

Let us assume the area of the disc $D = \alpha$.

$$n r^2 \sin\frac{\pi}{n} \cos\frac{\pi}{n} \leq \alpha \leq n r^2 \tan\frac{\pi}{n}$$

Taking $n \to \infty$ we get

$$\lim_{n\to\infty} n r^2 \sin\frac{\pi}{n} \cos\frac{\pi}{n}$$

$$= \lim_{n\to\infty} r^2 \underbrace{\boxed{\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}}}_{1} \cdot \pi \underbrace{\boxed{\cos\frac{\pi}{n}}}_{1}.$$

$= \pi r^2$

Then we get,

$$\boxed{\pi r^2 \leq \alpha.} \quad \text{— } 1$$

Calculate the limit of

$$\lim_{n \to \infty} n r^2 \tan \frac{\pi}{n}$$

$= \pi r^2$

So again we get,

$$\boxed{\alpha \leq \pi r^2} \quad \text{— } 2$$

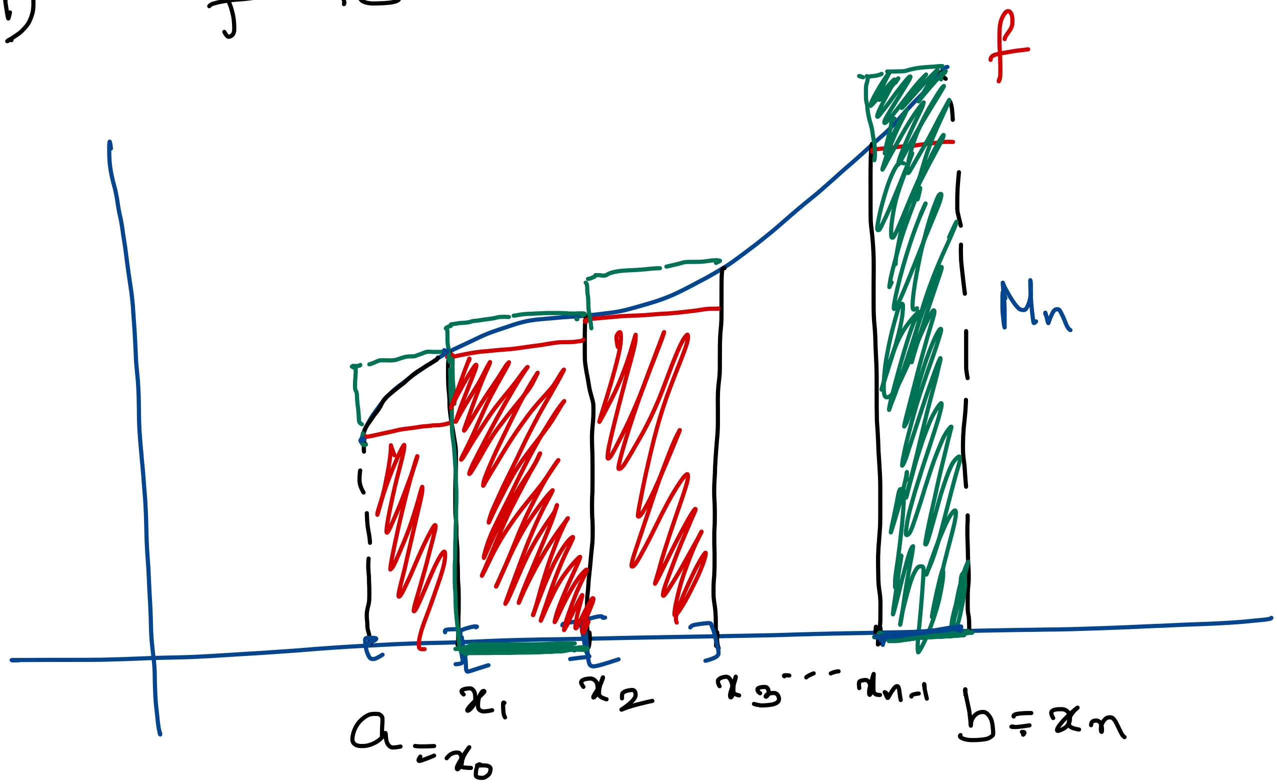
From 1 & 2 we get

$$\alpha = \pi r^2$$

Let $f : [a, b] \longrightarrow \mathbb{R}$.

Assume that.

1) f is bdd & non-negative.



We define

1) Partitions : It is just collection of points.

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$$

between a & b.

How to draw the rectangle?

So I define.

$$M_i = \sup \left\{ f(x) \mid x \in [x_{i-1}, x_i] \right\}$$

$$m_i = \inf \left\{ f(x) \mid x \in [x_{i-1}, x_i] \right\}$$

Next we again define

- $$\underbrace{U(P, f)}_{\text{Upper sum}} = \sum_{i=1}^{n} M_i \Delta x_i \ , \quad \Delta x_i = |x_i - x_{i-1}|$$

  = Total area of superscribed rectangles.

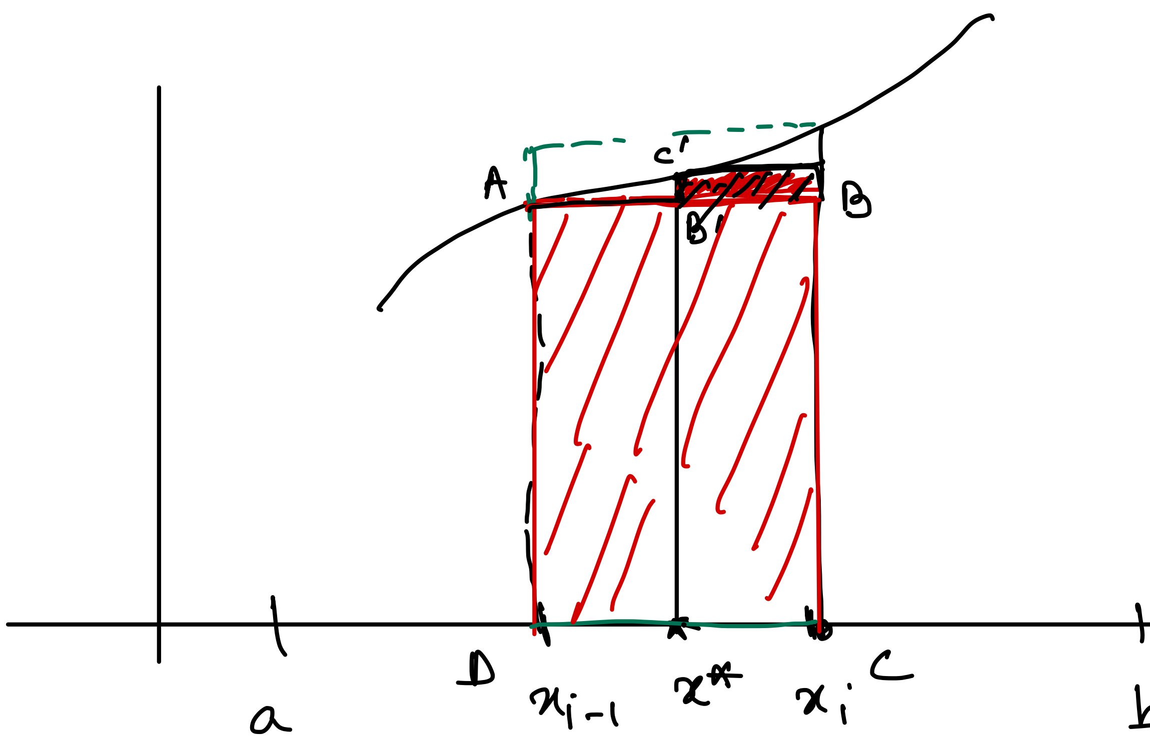
- $$\underbrace{L(P, f)}_{\text{Lower sum}} = \sum_{i=1}^{n} m_i \Delta x_i$$

If we check the picture.

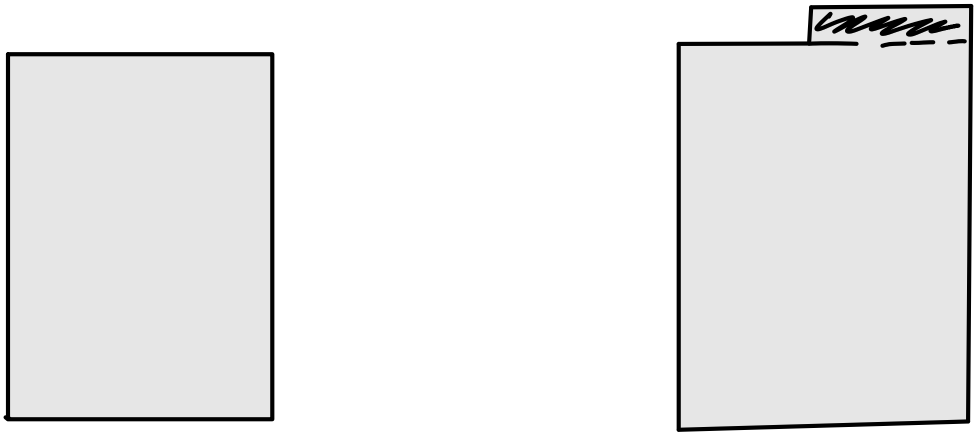
then

sum of Red rectangles $\leq$ sum of green rectangles

$$\boxed{L(P,f) \leq U(P,f)}$$



Earlier we had



For that first I will define,

Refinement of the partition.

Defn. Let P_1 & P_2 be partition of $[a, b]$

Then P_2 is called the refinement of P_1 if

$$P_1 \subseteq P_2$$

Q: What is the connection between $L(P_1, f)$, $L(P_2, f)$?

<u>Theo:</u> If P_1 and P_2 are partitions of $[a,b]$ s.t, $P_2 \supseteq P_1$ then

$$L(P_2, f) \geq L(P_1, f) \ \&$$

$$U(P_2, f) \leq U(P_1, f).$$

<u>Proof:</u> Assume P_2 is the refinement of P_1 with 'one' extra point.

Let us say,

$$P_1 = \{\underset{\substack{\shortparallel \\ a}}{x_0}, x_1, \ldots x_{i-1}, x_i, \ldots \underset{\substack{\shortparallel \\ b}}{x_n}\}$$

&

$$P_2 = \{x_0, x_1, \ldots, x_{i-1}, \underline{x^*}\ x_i, \ldots x_n\}$$

So then we can write

$$P_2 = P_1 \cup \{x^*\}$$

Let us denote

$$w_1 = \inf\{f(x) \mid x \in [x_{i-1}, x^*]\}$$

(say)

$$w_2 = \inf\{f(x) \mid x \in [x^*, x_i]\}$$

We observe that,

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

then

$$\left.\begin{array}{c} w_1 \geqslant m_i \\ \& \\ w_2 \geqslant m_i \end{array}\right\} \text{Why?}$$

[Infimum of a smaller set $\geqslant$ infimum of the bigger set] !

&

$$\{ f(x) \mid x \in [x_{i-1}, x_i] \}$$

is bigger that

$$\{ f(x) \mid x \in [x_{i-1}, x^*] \}$$

&

$$\{ f(x) \mid x \in [x^*, x_i] \}$$

Now we look at,

$$L(P_2, f) - L(P_1, f)$$

$$= \sum_{j=1}^{i-1} m_j \Delta x_j + w_1 (x^* - x_{i-1}) + w_2 (x_i - x^*) + \sum_{j=i+1}^{n} m_j \Delta x_j - \sum_{j=1}^{i-1} m_j \Delta x_j - m_i (x_i - x_{i-1}) - \sum_{j=i+1}^{n} m_j \Delta x_j$$

$$= w_1 (x^* - x_{i-1}) + w_2 (x_i - x^*) - m_i (x_i - x_{i-1})$$

$$= \underline{w_1 (x^* - x_{i-1})} + w_2 \underline{(x_i - x^*)} - m_i \underline{(x_i - x^*)} - \underline{m_i (x^* - x_{i-1})}$$

$$= (\underbrace{w_1 - m_i}_{\geq 0}) (x^* - x_{i-1}) + (\underbrace{w_2 - m_i}_{\geq 0}) (x_i - x^*)$$

$$\geq 0$$

This gives,

$$\boxed{L(P_1, f) \leq L(P_2, f)}$$

Exercise Check pictorially the case of $U(P, f)$?

We denote here.

$$v_1 = \sup\{f(x) \mid x \in [x_{i-1}, x^*]\}$$

&

$$v_2 = \sup\{f(x) \mid x \in [x^*, x_i]\}$$

and we know.

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

Observe:

$$v_1 \leq M_i$$

&

$$v_2 \leq M_i$$

Then this will give.

$$\boxed{U(P_2, f) \leq U(P_1, f)}$$

(Proceeding similarly)

We assumed earlier.

$$\{f(x) \mid x \in [a,b]\} \text{ is bdd}$$

So sup & inf exists.

Let $$M = \sup\{f(x) \mid x \in [a,b]\}$$

$$m = \inf\{f(x) \mid x \in [a,b]\}$$

Notice:

$$M(b-a) \geq U(P,f)$$

$$(\because M_i \leq M)$$

&

$$m(b-a) \leq U(P,f)$$

$$(\because m \leq M_i)$$

Combining

$$m(b-a) \leq U(P,f) \leq M(b-a) \quad \text{--- } (1)$$

Similarly,

$$m(b-a) \leq L(P,f) \leq M(b-a) \quad -(2)$$

Together we get,

$$\boxed{m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)}$$

Upper Riemann Sum

$$\overline{\int_a^b} f(x)\,dx = \inf_P U(P,f)$$

Lower Riemann Sum

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(P,f)$$

Riemann Integrable.

A function 'f' in [a, b] is called Riemann Integrable over [a, b] if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

and in this case we define,

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

$$= \underline{\int_a^b} f(x)\,dx .$$

Lemma (In general)

$$\underline{\int_{a}^{b}} f(x)\,dx \le \overline{\int_{a}^{b}} f(x)\,dx$$

Pf: Let P_1, P_2 be any two partition of $[a,b]$.

Let 'P' be the common refinement of P_1 & P_2.

$P = P_1 \cup P_2$.

We know,

$$L(P_1, f) \le L(P, f) \le U(P, f) \le U(P_2, f)$$

Let us fix P_2.

Then,

$$\sup_{P_1} L(P_1, f) \leq U(P_2, f)$$

$$\Rightarrow \quad \underline{\int_a^b} f(x)\,dx \leq U(P_2, f) \quad \forall P_2$$

Then we get,

$$\underline{\int_a^b} f(x)\,dx \leq \inf_{P_2} U(P_2, f)$$

$$= \overline{\int_a^b} f(x)\,dx$$

Hence,

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Ex. A bounded fn that is not Riemann integrable.

Consider, $f : [0,1] \to \mathbb{R}$,

$$f(x) = 1, \quad x \in \mathbb{Q}$$
$$= 0, \quad x \in \mathbb{R} \setminus \mathbb{Q}$$

Soln. Let 'P' be any partition of $[0,1]$.

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1$$

[If we take any $[x_{i-1}, x_i]$ then it contains both rationals & irrationals.
so $M_i = 1$ & $m_i = 0 \ \forall i$]

Similarly we get

$$L(P, f) = \sum_i m_i \Delta x_i = 0$$

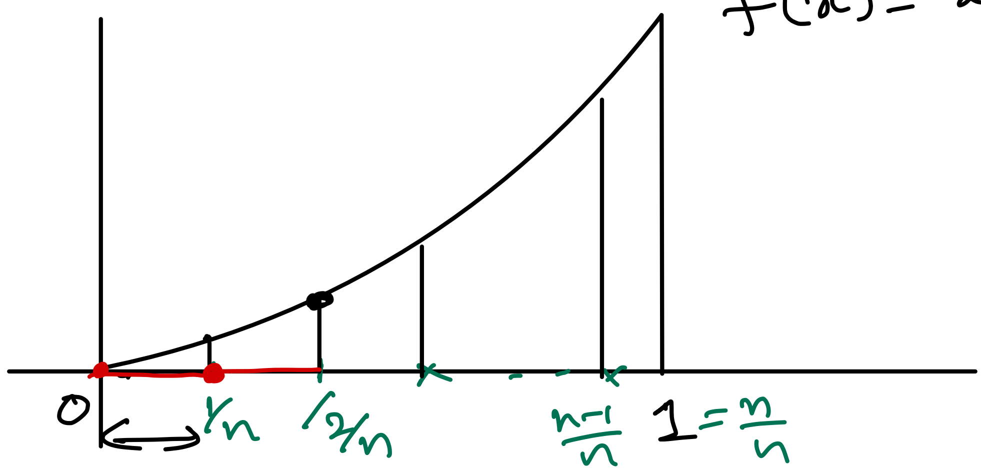
Now 'p' was arbitrary.

So we get,

$$\left.\begin{array}{l} \overline{\int_a^b} f(x)\, dx = 1 \\[2em] \underline{\int_a^b} f(x)\, dx = 0 \end{array}\right\} \neq 0$$

$\Rightarrow$ $\boxed{\text{f is not Riemann integrable.}}$

Ex: $f(x) = x^2$, on $[0,1]$



Soln: Consider

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots \frac{n-1}{n}, \frac{n}{n}\right\}$$

Look into

$$U(P,f) = \sum_i M_i \Delta x_i$$

Now,

$$M_1 = \left(\frac{1}{n}\right)^2 ; \quad \Delta x_1 = \frac{1}{n}.$$

$$M_2 = \left(\frac{2}{n}\right)^2 ; \quad \Delta x_2 = \frac{1}{n}$$

and so on . . .

So we get,

$$U(P_n, f) = \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n}$$

$$+ \cdots + \left(\frac{n-1}{n}\right)^2 \cdot \frac{1}{n} + \frac{n^2}{n^2} \cdot \frac{1}{n}$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right)$$

$$= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

Now from the previous definition we get,

$$U(P, f) \geq \overline{\int_a^b} f(x)\, dx$$

In particular,

$$U(P_n, f) \geq \overline{\int_a^b} f(x)\, dx$$

$$\frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \geq \overline{\int_a^b} f(x)\,dx$$

Taking $n \to \infty$ we get,

$$\overline{\int_a^b} f(x)\,dx \leq \frac{1}{3}$$

Lets look into,

$$L(P_n, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

Then

$$m_1 = 0$$

$$m_2 = \frac{1}{n}, \ldots\ldots \quad m_n = \frac{n-1}{n}.$$

So we get,

$$L(P_n, f) = \frac{1}{n^3} \frac{n(n-1)(2n-1)}{6}$$

Again we know.

$$L(P_n, f) \leq \underline{\int_a^b} f(x)\,dx$$

Then taking $n \to \infty$ we get

$$\frac{1}{3} \leq \underline{\int_a^b} f(x)\,dx.$$

Together we get,

$$\frac{1}{3} \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$$

$$\leq \frac{1}{3}$$

Hence,

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

$$= \frac{1}{3}$$

So f is Riemann-integrable & $\int_a^b f(x)\,dx = \frac{1}{3}$.

Q 1 What are good examples of R - int fn s?

2) Is there any NASC for Riemann integrability?

3) Do we have larger class of fn s which are R- int. ?