Lecture 2

Note Title 04-03-2021

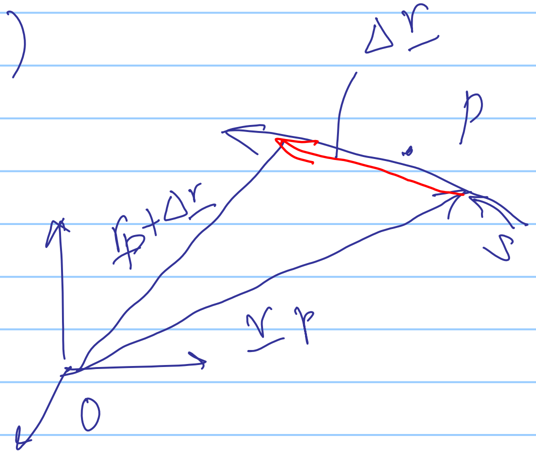
$\dot{v}_\phi \neq \dot{r}_\phi$; $\dot{v}_r = \dot{r}$

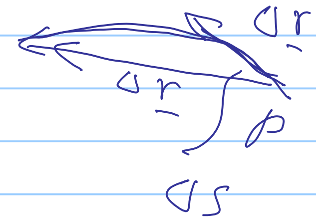
$\nearrow r\dot{\phi}$

$a_r = \ddot{r} - r\dot{\phi}^2 \neq \dot{v}_r$

$a_\phi = 2\dot{r}\dot{\phi} + r\ddot{\phi} \neq \dot{v}_\phi$

$$a_\phi = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)$$





Path Coordinates

$\lim\limits_{\Delta s \to 0} \frac{\Delta \underline{r}}{\Delta s} \longrightarrow$ tends to a tangent at P.

magnitude $\rightarrow 1$.

$$\lim_{\Delta s \to 0} \frac{\Delta \underline{r}}{\Delta s} = \frac{d\underline{r}}{ds} = \text{unit vector tangent to the path at } P$$

$$= \underline{e}_t$$

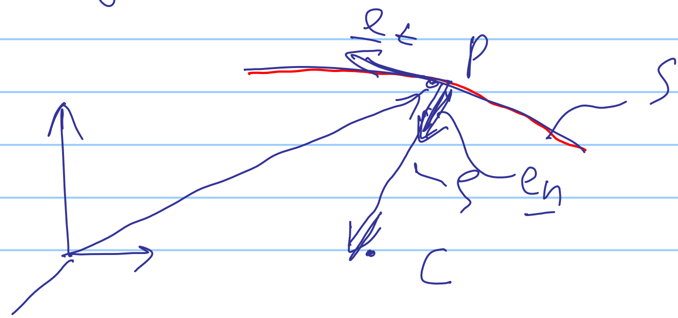
$$\frac{d\underline{e}_t}{ds} = \frac{d^2\underline{r}}{ds^2} = \frac{1}{\rho}\underline{e}_n \qquad \underline{e}_n \rightarrow \text{unit vector}$$

consider $\frac{d}{ds}(\underline{e}_t \cdot \underline{e}_t) = 0 = \underline{e}_t \cdot \frac{d\underline{e}_t}{ds} + \frac{d\underline{e}_t}{ds} \cdot \underline{e}_t = 2\frac{d\underline{e}_t}{ds} \cdot \underline{e}_t$

$\Rightarrow \underline{e}_t \perp \frac{d\underline{e}_t}{ds} \rightarrow$ fixes the plane in which $\underline{e}_n$ lies.

$\underline{e}_n$ such that $\underline{r}_p + \rho\,\underline{e}_n = \underline{r}_c$

$\underline{r}_c \rightarrow$ the position vector of the centre of curvature C



Third vector in the rt handed sense $\perp$r to both $\underline{e}_t$ and $\underline{e}_n$

$$\underline{e}_b = \underline{e}_t \times \underline{e}_n$$

$$\underline{e}_t \times \underline{e}_n = \underline{e}_b \qquad \underline{e}_b \times \underline{e}_t = \underline{e}_n$$

$$\underline{e}_n \times \underline{e}_b = \underline{e}_t$$

ρ ? $\rightarrow$ radius of curvature

$$\left| \underline{e}_t \times \frac{1}{\rho} \underline{e}_n \right| = \frac{1}{\rho} = \left| \frac{d\underline{r}}{ds} \times \frac{d^2 \underline{r}}{ds^2} \right|$$

Osculating plane — plane containing $\underline{e}_t$ and $\underline{e}_n$

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d\underline{r}}{ds}\frac{ds}{dt} = \dot{s}\,\underline{e}_t$$

$$\underline{a} = \frac{d^2\underline{r}}{dt^2} = \frac{d}{dt}\left[\frac{d\underline{r}}{ds}\frac{ds}{dt}\right] = \frac{d\underline{r}}{ds}\frac{d^2 s}{dt^2} + \frac{d^2\underline{r}}{ds^2}\left(\frac{ds}{dt}\right)^2$$

$$= \ddot{s}\,\underline{e}_t + \frac{\dot{s}^2}{\rho}\,\underline{e}_n \quad \Rightarrow \quad \begin{array}{ll} v_t = \dot{s}, & v_n = 0 \\ a_t = \ddot{s} & a_n = \dfrac{\dot{s}^2}{\rho} \end{array}$$

$a_n \rightarrow$ directed towards the centre of curvature.

$$\Rightarrow \quad \underline{v} \times \underline{a} = \frac{\dot{s}^3}{\rho}\,\underline{e}_b \quad \Rightarrow \quad \frac{1}{\rho} = \frac{|\underline{v} \times \underline{a}|}{|v|^3}$$

Parametric representation of the path $\underline{r} = \underline{r}(\tau)$

$$\frac{d\underline{r}}{ds} = \frac{d\underline{r}}{d\tau} \cdot \frac{d\tau}{ds}$$

note $\dfrac{d\underline{r}}{ds} = \underline{e}_t = \left|\dfrac{d\underline{r}}{ds}\right| = 1$

$\Rightarrow \left|\dfrac{d\tau}{ds}\right| = \dfrac{1}{\left|\dfrac{d\underline{r}}{d\tau}\right|}$

$$\frac{d^2\underline{r}}{ds^2} = \frac{d}{ds}\left[\frac{d\underline{r}}{d\tau} \cdot \frac{d\tau}{ds}\right] = \frac{d\underline{r}}{d\tau}\,\frac{d^2\tau}{ds^2} + \frac{d^2\underline{r}}{d\tau^2}\left(\frac{d\tau}{ds}\right)^2$$

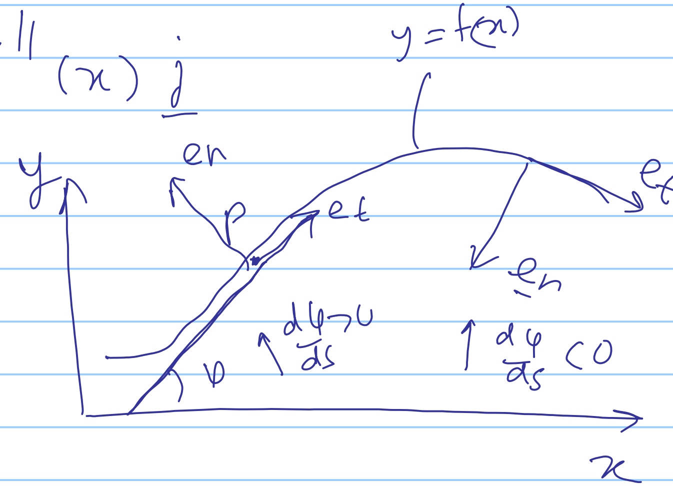
$$\frac{1}{\rho} = \left| \frac{d\underline{r}}{d\tau} \times \frac{d^2\underline{r}}{d\tau^2} \right| \left| \frac{d\tau}{ds} \right|^3 = \left| \frac{d\underline{r}}{d\tau} \times \frac{d^2\underline{r}}{d\tau^2} \right| \Big/ \left| \frac{d\underline{r}}{d\tau} \right|^3$$

put $\tau = t$ $\quad \frac{1}{\rho} = |\underline{v} \times \underline{a}| \big/ |\underline{v}|^3$

eg: plane curve $\quad \underline{r}(x) = x\underline{i} + f(x)\underline{j} \quad (y = f(x))$

$$\frac{d\underline{r}}{dx} = \underline{i} + f'(x)\underline{j} \qquad \frac{d^2\underline{r}}{dx^2} = 0 + f''(x)\underline{j}$$

$$\frac{1}{\rho} = \frac{|f''(x)|}{|1 + f'(x)^2|^{3/2}}$$



$$\underline{e}_t = \cos\varphi\,\underline{i} + \sin\varphi\,\underline{j} \qquad \frac{\underline{e}_n}{\rho} = (-\sin\varphi\,\underline{i} + \cos\varphi\,\underline{j})\frac{d\varphi}{ds}$$

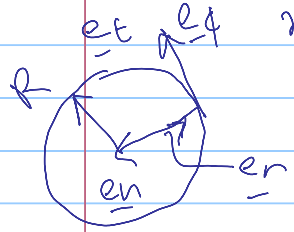
$$|\underline{e}_t| = 1 = \left|\frac{d\varphi}{ds}\right| \Rightarrow \rho^{-1} = \left|\frac{d\varphi}{ds}\right|$$

$$\underline{e}_n = \left(-\sin\varphi\, \underline{i} + \cos\varphi\, \underline{j}\right) \frac{\frac{d\varphi}{ds}}{\left|\frac{d\varphi}{ds}\right|} = \pm\left(-\sin\varphi\, \underline{i} + \cos\varphi\, \underline{j}\right)$$

$+$ sign where $\frac{d\varphi}{ds} > 0$ $\quad$ $-$ sign where $\frac{d\varphi}{ds} < 0$

Similarly $\rho = \left|\frac{d^2x}{dy^2}\right| \Big/ \left|1 + \left(\frac{dx}{dy}\right)^2\right|^{3/2}$ if $x = f(y)$.

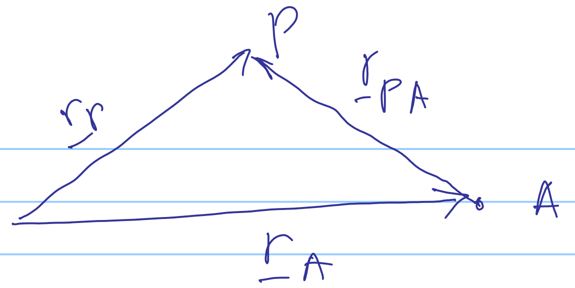
eg Circular trajectory. (Radius R)



$r = R, \quad z = 0, \quad \dot{r} = \dot{z} = 0, \quad \ddot{r} = \ddot{z} = 0, \quad \rho = R \quad \underline{e}_t = \underline{e}_\phi, \quad \underline{e}_r = -\underline{e}_n$

$v = \dot{s}\,\underline{e}_t = R\dot{\phi}\,\underline{e}_\phi, \quad \underline{a} = \ddot{s}\,\underline{e}_t + \frac{\dot{s}^2}{\rho}\,\underline{e}_n = -R\dot{\phi}^2\,\underline{e}_r + R\ddot{\phi}\,\underline{e}_\phi$

Relative velocity & accn.



$$\underline{v}_{PA}|_F = \frac{d}{dt}\left(\underline{r}_P - \underline{r}_A\right)\Big|_F = \dot{\underline{r}}_P|_F - \dot{\underline{r}}_A|_F = \underline{v}_P|_F - \underline{v}_A|_F$$

$$\underline{a}_{PA}|_F = \frac{d}{dt}\left(\underline{v}_{PA}\right)\Big|_F = \dot{\underline{v}}_P|_F - \dot{\underline{v}}_A|_F = \underline{a}_P|_F - \underline{a}_A|_F$$

→ Degrees of freedom - no. of independent coordinates required to specify the system. For a rigid body — 6 degrees of freedom

Index notation

coordinates $x, y, z \Rightarrow x_1, x_2, x_3$

Components of $\underline{a}$ $a_x, a_y, a_z \Rightarrow a_1, a_2, a_3$

unit vectors $\underline{i}, \underline{j}, \underline{k} \Rightarrow \underline{e}_1, \underline{e}_2, \underline{e}_3$

eg: $\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$

$= a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = \sum_{i=1}^{3} a_i \underline{e}_i$

Summation sign → redundant? $\Rightarrow a_i \underline{e}_i$ ← summation convention

i → repeated (dummy) index

replace with any other index. $\sum_{i=1}^{3} a_i \underline{e}_i = \sum_{k=1}^{3} a_k e_k$

$\Rightarrow a_i \underline{e}_i = a_k \underline{e}_k = a_p \underline{e}_p \ldots$

non repeated indices are called free indices.

Change the free index $\Rightarrow$ different equations in diff. directions (free index $\rightarrow$ represents the direction)

$$a_i + b_k c_k d_i + T_{ij} c_j = 0$$

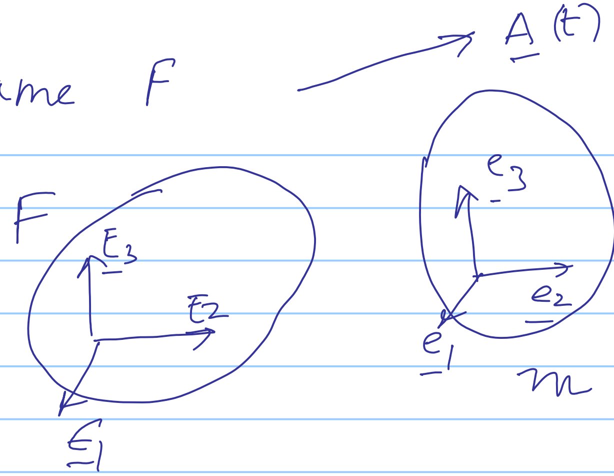
$$\longrightarrow \quad \underline{a} + (\underline{b} \cdot \underline{c}) \underline{d} + \underline{\underline{T}} \, \underline{c} = 0$$

1 component $i=1$

$$a_1 + (b_1 c_1 + b_2 c_2 + b_3 c_3) d_1 + T_{11} c_1 + T_{12} c_2 + T_{13} c_3 = 0$$

change $i=2$ & $i=3$ $\rightarrow$ 2 other equations.

Angular velocity of a frame m wrt a frame F



Rt handed triad $\underline{E}_i$ embedded in F

— " — $\underline{e}_i$ — " — in m

The rate of change of m relative to F in time t

is given by $\overset{\circ}{\underline{e}}_i|_F(t)$ (since $\underline{e}_i(t)$)

Say

$$\overset{\circ}{e}_1|_F(t) = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3$$

$$\overset{\circ}{e}_2|_F(t) = b_1 \underline{e}_1 + b_2 \underline{e}_2 + b_3 \underline{e}_3$$

$$\overset{\circ}{e}_3|_F(t) = c_1 \underline{e}_1 + c_2 \underline{e}_2 + c_3 \underline{e}_3$$

9 components but all independent

consider $(\underline{e}_1 \cdot \underline{e}_1) = 1 \Rightarrow (\underline{e}_1 \overset{\circ}{\cdot} \underline{e}_1)|_F = 2\, \overset{\circ}{\underline{e}}_1|_F \cdot \underline{e}_1 = 0$

$\Rightarrow \quad a_1 = 0$ similarly $b_2 = 0$ and $c_3 = 0$

$(\underline{e}_1 \cdot \underline{e}_2) = 0 \Rightarrow (e_1 \overset{\circ}{\cdot} e_2)|_F = \overset{\circ}{\underline{e}}_1|_F \cdot \underline{e}_2 + \underline{e}_1 \cdot \overset{\circ}{\underline{e}}_2|_F = 0$

$a_2 + b_1 = 0 \Rightarrow \underline{b_1 = -a_2}$

Similarly $c_2 = -b_3$ and $a_3 = -c_1$

cyclic changes $a \to b \to c \to a \quad 1 \to 2 \to 3 \to 1$

$\Rightarrow$ only 3 ind. components

$$\left.\mathring{\underline{e}}_1\right|_F = a_2 \underline{e}_2 - c_1 \underline{e}_3 = [b_3 \underline{e}_1 + c_1 \underline{e}_2 + a_2 \underline{e}_3] \times \underline{e}_1$$

$$\left.\mathring{\underline{e}}_2\right|_F = -a_2 \underline{e}_1 + b_3 \underline{e}_3 = [b_3 \underline{e}_1 + c_1 \underline{e}_2 + a_2 \underline{e}_3] \times \underline{e}_2$$

$$\left.\mathring{\underline{e}}_3\right|_F = c_1 \underline{e}_1 - b_3 \underline{e}_2 = [b_3 \underline{e}_1 + c_1 \underline{e}_2 + a_2 \underline{e}_3] \times \underline{e}_3$$

Call vector $\underline{\omega} = b_3 \underline{e}_1 + c_1 \underline{e}_2 + a_2 \underline{e}_3 \Big\}$ $\left.\mathring{\underline{e}}_i\right|_F = \underline{\omega} \times \underline{e}_i$

$\underline{\omega}$? Solve for $\underline{\omega}$ by taking $e_i \times (\ )$

$$\underline{e}_i \times \left.\mathring{\underline{e}}_i\right|_F = \underline{e}_i \times (\underline{\omega} \times \underline{e}_i) = (\underline{e}_i \cdot \underline{e}_i)\underline{\omega} - (\underline{e}_i \cdot \underline{\omega})\underline{e}_i$$

$$\underline{e}_i \cdot \underline{e}_i = \underline{e}_1 \cdot \underline{e}_1 + \underline{e}_2 \cdot \underline{e}_2 + \underline{e}_3 \cdot \underline{e}_3 = 1+1+1 = 3$$

$$\underline{e}_i \cdot \underline{\omega} = \omega_i, \qquad \omega_i e_i = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3 = \underline{\omega}$$

$$\Rightarrow \quad \underline{e}_i \times \dot{\underline{e}}_i|_F = 3\underline{\omega} - \underline{\omega} = 2\underline{\omega}$$

$$\underline{\omega} = \frac{1}{2} \underline{e}_i \times \dot{\underline{e}}_i|_F$$

$\underline{\omega}$ is defined as the angular velocity $\underline{\omega}_{m|F}$ of frame m wrt frame F