COL 202 HOMEWORK 1

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Solution 1.
The number of matches that would be played amongst the teams $= \binom{n}{2} = \frac{n(n-1)}{2}$
Now to prove the second claim, we will use induction.

$f(k) =$: In a tournament of k teams, there always exists a consistent subset S such that $|S| \geq \log_2(k+1)$

Using Induction:

Base Case: For n = 2, there are only 2 teams say T_1 and T_2. Lets say that T_1 wins, the T_1, T_2 form a consistent set and $|S| = 2 \geq log_2 3$. So $f(2)$ is true.

Induction Hypothesis: Let $f(k)$ be true for all k < n.

Induction Step: Suppose there are n teams in a tournament. Lets choose any team T_i. Considering 2 cases.

CASE 1: Let $n = 2m$. T_i will play $n - 1 = 2m - 1$ matches. . By Pigeon-Hole principle, either T_i wins against at least m teams or loses at least m teams. When T_i wins against m teams, by Induction Hypothesis, we can say that there would be a consistent subset, say S_1 within the m teams where $|S_1| \geq \log_2(m+1)$. Since, T_i wins against every $T_j \in S_1, S_2 = S_1 \cup T_i$ forms a consistent set. Therefore,
$|S_2| = \log_2(m+1) + 1$.

$$|S_2| \geq \log_2(2m+2) \geq \log_2(n+1) \tag{1}$$

In case T_i loses m matches atleast, we get a consistent set S_1 amongst the m teams where T_i loses to every $T_j \in S_1$ and then we proceed as earlier.

CASE 2: $n = 2m+1$. By Pigeon Hole Principle, T_i wins against atleast m teams or loses against atleast m teams, so by the argument we made earlier we can form an S_2 such that,

$$|S_2| \geq \log_2(2m+2) \geq \log_2(n+1) \tag{2}$$

Therefore, in both the cases, there is a consistent set S_2 such that,
$|S_2| \geq \log_2(n+1) \geq \log_2 n$

Solution 2.

We will make this proof by construction. We have to find an x such that
$S = \{ax | a \in \mathbb{Z}\}$. We can use the result of problem 8 in tutorial

$$\forall\, x, y \in \mathbb{Z}\, \exists\, a, b \in \mathbb{Z} : gcd(x, y) = ax + by$$

By Construct of S, $ax + by \in S\, \forall\, x, y \in S\ and\, a, b \in \mathbb{Z}$. Thus we get,
$gcd(x, y) \in S\, \forall\, x, y \in S$. Further, considering the set T $=\{|s| : s \in S\}$. Note that $T \subseteq S$. Also because $gcd(x, y) \geq 0\, \forall\, x, y$, there also must be $gcd(x, y) \in T\, \forall\, x, y \in S$.

Now, $T \subset \mathbb{N}$, therefore by well-Ordering Principle, there must be a minimum element (say t_0) in T. Consider $\gcd(t_0, s)$ for any $s \in S$. Now,

$$gcd(t_0, s) \leq t_0$$

But since $gcd(t_0, s) \in T$ by minimality of t_0 we must have $gcd(t_0, s) = t_0 \, \forall \, s \in S$. we may write $s = at_0$ for some $a \in \mathbb{Z}$, or in other words, $S = \{at_0 | \, a \in \mathbb{Z}\}$

Solution 3.

($1 \implies 2$) Let there be any b $\in$ B'. By defination, suppose $a = f^{-1}(b)$, or $f(a) = b$. Therefore by the defination of B', we can say that $b = (f \circ g)^k(b^*)$ for some $k \in \mathbb{N} \cup 0$ and $b^* \in B \diagdown Im(f)$. Therefore we can say that,

$$f(a) = (f \circ g)^k(b^*)$$

Again by definition of $B', b \neq b^*$ since $b \in Im(f)$ and so $k \geq 1$. Also, by injectivity of f we must have,

$$a = g((f \circ g)^{k-1}(b^*)) = g(\bar{b}), \bar{b} \in B'$$

Therefore, $a \in A'$ ($2 \implies 3$) If $f^{-1}(b)$ exists and is in A', then we hace $a = g(\bar{b})$ for some $b \in B'$. Write $\bar{b} = (f \circ g)^k(b^*)$ for some $k \in \mathbb{N} \cup 0$ and $b^* \in B \diagdown Im(f)$, and so we can say that,

$$a = g((f \circ g)^{k-1}(b^*)) = (f \circ g)^{k+1}(b^*)$$

And therefore, $b \in B'$ and $g(b) \in A'$. ($3 \implies 1$) Since $g(b) \in A'$,by defination of A' it follows that $b \in B'$

Solution 4.

Let us consider a string s_T over a set T and l_s be the length of the string s. Now, Let the set A_i be defined as:

$$A_i = \{s_A \, | \, l_s = i\}$$

Now,

$$A^* = \bigcup_{i=1}^{\infty} A_i$$

Now, since A is a finite set, let if it's cardinality in n. Therefore the cardinality of each A_i will be $|A_i| = n^i$. So each A_i is finite.

We know that if we have a countably infinite collection of sets, every one of which is finite and countable, then $\bigcup_{i=1}^{\infty} A_i$ is countable. Hence, A^* is countable.

If A is a set of countably infinite elements, we can define an injection from A to N (as discussed in class). Furthermore, we have proved in class that the set of finite subsets of N is countable. This concludes that the set of finite subsets over A is also countable.

Solution 5.

We prove the given statement using Principle of Mathematical Induction.

Proposition P(k) =: Every graph (of degree k) has atleast two vertices having equal degree.

Base case: We start with k = 2 as we need atleast 2 nodes. For k = 2, we have one of the two cases necessarily:

Case 1 : The two nodes are not connected. In this case, the degree of each node will be 0 and so, both have equal degree.

Case 2 : The two nodes are connected with each other. In this case, the degree of each node will be 1, and again, both have equal degree.

Both these cases are mutually exclusive and exhaustive. So, we can say that P(2) is true.

Induction hypothesis : We assume that P(k) is true $\forall k < n$. Mathematically, we assume that for every graph that has less than n nodes, we have atleast two nodes having equal degree.

Induction Step : Consider a graph with n nodes. Consider a specific node, say N_1, and the set S of all other remaining nodes. By the Induction hypothesis, there exist atleast 2 nodes in set S which have equal degree (since the number of nodes in S is n-1). Let the number of such nodes be $m(m \geq 2)$. Consider following 2 cases:

Case 1 : When m > 2 : Let us use PHP to understand this case - By PHP, only one of the 2 can be true at a time : Either N_1 is connected to ≥ 2 nodes out of these m nodes, or it is connected to at most 1 node of these m nodes. In the former case, we can consider any 2 such nodes to which N_1 is connected. These 2 nodes, had equal degree before (by definition) and now again, their degree are equal. In the latter case, consider any node out of the remaining m-1 nodes. Since m>2, we can select atleast 2 nodes from this set. These too have equal degree (by definition). Thus, the proposition holds true for case 1.

Case 2 : When m = 2 : Let these 2 nodes having equal degree be N_p and N_q. To tackle this case, we can consider the following 3 sub-cases :
SubCase 1 : N_1 is connected to 2 both N_p and N_q . In this case, the degree of N_p and N_q is still the same.
SubCase 2 : N_1 is connected to none of N_p and N_q. Again, the degree of N_p and N_q is equal.
SubCase 3 : N_1 is connected to exactly 1 of the N_p and N_q. In this case, we will use PHP.

Consider the n nodes. Consider a node, say N_r. The degree of N_r can range from 0 to (n-1). (Since there are n-1 nodes other than N_r, it can at minimum be connected to 0 nodes and at max n-1 nodes). Consider the following 2 scenarios :

A) If degree of N_r is 0, then the degree of remaining all nodes can at max be (n-2) (As there are only n-2 nodes remaining). Thus degree of any node can take values only from set $S_1 = \{0,1,2, ... , n\text{-}2\}$. $|S_1| =$ n-1, and the total number of nodes are n. By PHP, there will be atleast 2 nodes, which have same degree.

B) If degree of N_r is n-1, then the degree of remaining all nodes has to be greater than 0. (As N_r is connected to each node, degree of any node can't be 0). Thus degree of any node can take values only from set $S_2 = \{1,2,3, ... , n\text{-}1\}$. $|S_2| =$ n-1, and the total number of nodes are n. By PHP, there will be atleast 2 nodes, which have same degree. Thus we have considered all possible degrees of all nodes and thus our cases are exhaustive. Thus, our proposition is true.