Cauchy Riemann conditions:

Let $f(z) = u(x,y) + i\, v(x,y)$, $z = x+iy$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Let $z = z_0 + \Delta z = (x_0 + \epsilon_x) + i(y_0 + \epsilon_y)$ then

$$f'(z_0) = \lim_{\substack{\epsilon_x \to 0 \\ \epsilon_y \to 0}} \frac{u(x_0+\epsilon_x, y_0+\epsilon_y) + i\, v(x_0+\epsilon_x, y_0+\epsilon_y) - u(x_0,y_0) - i\, v(x_0,y_0)}{\epsilon_x + i\,\epsilon_y}$$

Note that this limit should exist and be independent of ways of taking ϵ_x & ϵ_y to 0. Let us consider two different orderings, namely

1. first $\epsilon_x \to 0$ & then $\epsilon_y \to 0$.
2. " $\epsilon_y \to 0$ & " $\epsilon_x \to 0$.

① $$f'(z_0) = \lim_{\epsilon_y \to 0} \left[\frac{u(x_0, y_0+\epsilon_y) - u(x_0,y_0)}{i\,\epsilon_y} + \frac{v(x_0, y_0+\epsilon_y) - v(x_0,y_0)}{\epsilon_y} \right]$$

$$= -i \frac{\partial u(x_0,y_0)}{\partial y} + \frac{\partial}{\partial y} v(x_0,y_0)$$

② $$f'(z_0) = \lim_{\epsilon_x \to 0} \left[\frac{u(x_0+\epsilon_x, y_0) - u(x_0,y_0)}{\epsilon_x} + i\, \frac{v(x_0+\epsilon_x, y_0) - v(x_0,y_0)}{\epsilon_x} \right]$$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i\, \frac{\partial v}{\partial x}(x_0,y_0)$$

① $\equiv$ ② gives

$$\boxed{\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad ; \quad \frac{\partial v}{\partial x}(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0)}$$

Even though we have just used two of infinitely many possible ways of taking the limit, it turns out that these condition as enough for the existence of $f'(z_0)$. These conditions are known as the <u>Cauchy-Riemann equations</u>.

Lets see some examples

- $f(z) = |z|^2 = z\bar{z} = x^2+y^2 + i\,0$

$$u = x^2+y^2 \quad , \quad v = 0$$

$$\partial_x u = 2x \quad , \quad \partial_x v = 0$$

$$\partial_y u = 2y \quad , \quad \partial_y v = 0$$

Clearly the CR eqns are not satisfied. As we saw earlier $f(z) = |z|^2$ is not complex differentiable

- $f(z) = z^2 = \underbrace{x^2-y^2}_{u} + i\,\underbrace{2xy}_{v}$

$$\partial_x u = 2x = \partial_y v$$

$$\partial_y u = -2y = -\partial_x v$$

- $f(z) = e^z = e^{x+iy} = \underbrace{e^x \cos y}_{u} + i\,\underbrace{e^x \sin y}_{v}$

$$\partial_x u = e^x \cos y = \partial_y v$$

$$\partial_y u = -e^x \sin y = -\partial_x v$$

Analytic functions : A function $f(z)$ is analytic at some point z_0 if the derivative $f'(z)$ exists in some open neighborhood of z_0.

Analytic in some open set in $\mathbb{C}$ plane $\rightarrow$ analytic at every point in the set.

Entire function : Analytic in whole complex plane
e.g. polynomial functions, e^z

Singular point : z_0 is a singular point/singularity of $f(z)$ if $f(z)$ is not analytic at z_0.

If $f(z)$ is an analytic in open neighbourhood of z_0 but not at z_0 then z_0 is called isolated singularity of $f(z)$

- If $f(z)$ and $g(z)$ are analytic in some domain then
  - $f(z)+g(z)$, $f(z)\cdot g(z)$, $f(g(z))$ & $g(f(z))$ are also analytic.
  - $\dfrac{f(z)}{g(z)}$ is analytic except at the zeros of $g(z)$.

★ If $f'(z) = 0$ in some domain $D \subset \mathbb{C}$, then $f(z) = \text{const}$ in D.

CR equations
$\oplus$ $f'(z) = 0$ $\Rightarrow$
$$\partial_x u = \partial_y v = 0$$
$$\partial_y u = -\partial_x v = 0$$

$\Rightarrow$ $u = \text{constant}$ $\Rightarrow$ $f(z) = \text{Const.}$
& $v = \text{constant}$

★ If $f(z)$ & $\overline{f(z)}$ are both analytic in domain D then $f(z) = \text{constant}$.

$$f(z) = u + iv \xrightarrow{\text{CR eqn.}} \partial_x u = \partial_y v, \quad \partial_x v = -\partial_y u$$
$$\overline{f(z)} = u - iv \xrightarrow{\text{CR eqn.}} \partial_x u = -\partial_y v, \quad \partial_x v = \partial_y u$$

$$\Rightarrow \quad \partial_x u = 0 = \partial_y v \quad \& \quad \partial_x v = 0 = \partial_y u$$

$$\Rightarrow \quad u = \text{Const.}, \quad v = \text{const.}$$

$$\Rightarrow \quad f(z) = \text{const.}$$

- The real & imaginary part of a complex analytic function are <u>Harmonic functions</u> of x & y.

$$f(z) = u(x,y) + i\, v(x,y)$$

$$\text{CR eqns} \Rightarrow \quad \partial_x u = \partial_y v \;; \quad \partial_y u = -\partial_x v$$

$$\Rightarrow \quad \partial_x^2 u = \partial_x \partial_y v \;; \quad \partial_y^2 u = -\partial_y \partial_x v$$

Adding together gives

$$\partial_x^2 u + \partial_y^2 u = \partial_x \partial_y v - \partial_y \partial_x v = 0$$

Similarily $\partial_x^2 v + \partial_y^2 v = -\partial_x \partial_y u + \partial_y \partial_x u = 0$

i.e. $u(x,y)$ & $v(x,y)$ are Harmonic functions.

$$0 = \partial_x^2 u + \partial_y^2 u = \vec{\nabla} \cdot \vec{\nabla} u \equiv \nabla^2 u \qquad \{\text{Laplace equations}\}$$

- u & v are often referred to as <u>Harmonic conjugates</u> of each other.

* Given a Harmonic function u, one can construct a complex analytic function by determining its Harmonic conjugate v

e.g. Let $u = y^3 - 3x^2 y \Rightarrow (\partial_x^2 + \partial_y^2) u = -6y + 3\times 2 \cdot y = 0$

$$\partial_y v = \partial_x u = -6xy \qquad \Rightarrow v = -3xy^2 + c(x)$$

$$\partial_x v = -\partial_y u = -3(y^2 - x^2) \qquad \hookrightarrow \partial_x v = -3y^2 + \partial_x c(x) = -3y^2 + 3x^2$$

$$\Rightarrow \quad \partial_x c(x) = 3x^2 \Rightarrow c(x) = x^3 + c_0$$

$$v = -3xy^2 + x^3 + c_0$$

$$\left\{ \begin{array}{l} x = \frac{1}{2}(z + \bar{z}) \\ y = \frac{1}{2i}(z - \bar{z}) \end{array} \right.$$

$$\Rightarrow \quad f(z) = u + iv = (-3x^2 y + y^3) + i(-3xy^2 + x^3) + i c_0$$

$$= i\left[x^3 + 3x(iy)^2 + 3x^2(iy) + (iy)^3\right] + i c_0$$

$$= i z^3 + i c_0$$

Some elementary functions & their properties

- $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$ $\quad \left\{ P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n \right.$

$\Rightarrow \quad |f(z)| = e^x \quad , \quad \arg[f(z)] = y$

- $f(z + i2\pi n) = f(z)$ $\quad$ {Periodic in $Im(z)$}

- $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2} \quad ; \quad e^0 = 1$

- $\overline{(e^z)} = e^{\bar{z}}$

- Differentiable everywhere in complex plane w/ $\frac{d}{dz} e^z = e^z$

- $f(z) = Log\, z$

$\boxed{\begin{matrix} y = Log\, x \\ x = e^y \end{matrix}}$ $\quad \boxed{\begin{matrix} e^{Ln\, x} = x \\ Ln(e^x) = x \end{matrix}}$

- $e^w = z \iff w = \log z$

- Multivalued : $z = re^{i(\theta + 2n\pi)} \Rightarrow Log\, z = \log r + i(\theta + 2n\pi)$ where $n \in \mathbb{Z}$

Principal value of $Log(z)$: $\log(r) + i\theta$ i.e. set $n = 0$

- Different choices of n above defines different Branches of $Log(z)$, each being single valued.
  $n = 0$ Branch is referred to as Principal branch.

- On any fixed branch, $Log(z)$ is single valued as well as continuous & differentiable with

$$\frac{d}{dz}(Log(z)) = \frac{1}{z}$$

- Verify that CR equations are satisfied for $\log(z)$.

- $Log(z_1 z_2) = Log(z_1) + Log(z_2)$

$Log(z^m) = m\, Log(z)$

$Log(z_1/z_2) = Log(z_1) - Log(z_2)$

$\exists$ a branch of Log function on which these hold.

$$z^a = \exp[\underbrace{\mathrm{Log}(z^a)}] = e^{a\ln z}$$

- $f(z) = z^a$ where $a \in \mathbb{C}$
$$\equiv e^{a\,\mathrm{Log}(z)}$$

E.g. Consider $i^i = e^{i\,\mathrm{Log}(i)}$
$$= e^{i\,\mathrm{Log}[e^{i(\frac{\pi}{2}+2n\pi)}]}$$
$$= e^{i[i(\frac{\pi}{2}+2n\pi)]} = e^{-(\frac{\pi}{2}+2n\pi)} \qquad n \in \mathbb{Z}$$

- $$\frac{d}{dz}(z^a) = \frac{d}{dz}\left[e^{a\,\mathrm{Log}(z)}\right] = e^{a\,\mathrm{Log}(z)} \cdot a\frac{d}{dz}(\mathrm{Log}(z))$$
$$= \frac{a}{z}\cdot e^{a\,\mathrm{Log}(z)} = \frac{a}{z}\cdot z^a = a z^{a-1}$$

- $$(z_1 z_2)^a = z_1^a z_2^a$$

- Trignometric functions:

Use Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
$$\Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

to define
$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}, \qquad \cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$
$$\Rightarrow \sin(z+2\pi n) = \sin(z), \quad \cos(z+2\pi n) = \cos(z)$$
$$\frac{d}{dz}(\sin z) = \cos z \quad \& \quad \frac{d}{dz}(\cos z) = -\sin z$$

Others defined in standard way from Sin & Cos.

Most trignometric identities carry over with a bit of care.

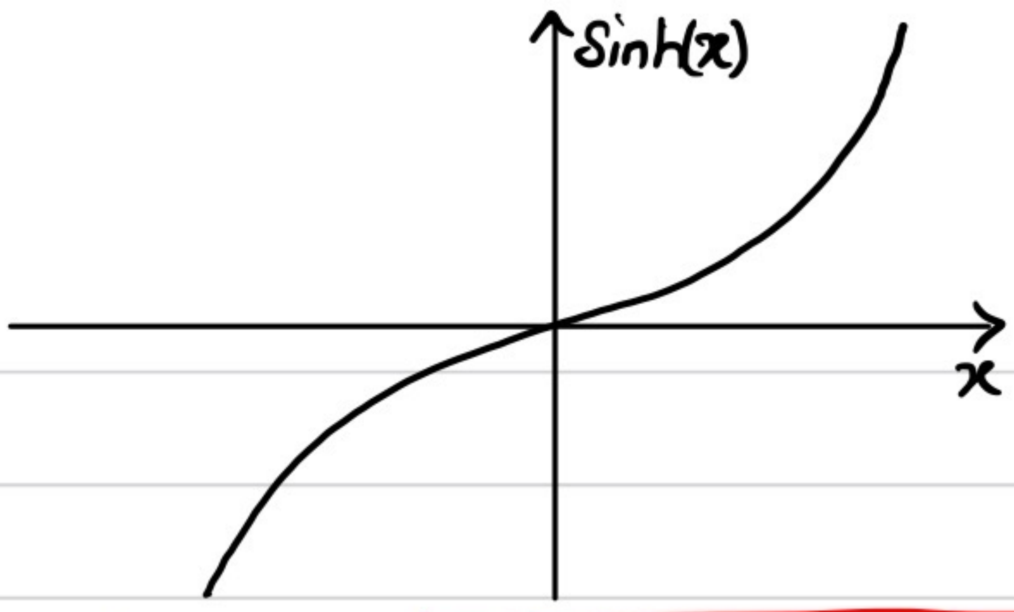
$$\sin^2(z) + \cos^2(z) = 1$$
$$\sin\left(z \pm \frac{\pi}{2}\right) = \pm\cos(z)$$
$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$
$$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

- Show that $\sin(z)$ & $\cos(z)$ are unbounded in the complex plane (unlike their real counterparts)

Verify $|\sin z|^2 = \sin^2 x + \sinh^2 y$ w/ $z = x+iy$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$

$$\boxed{\sinh y = \frac{e^y - e^{-y}}{2}}$$



$\sin z = 0 \Rightarrow z = n\pi \quad \forall\ n \in \mathbb{Z}$

$\cos z = 0 \Rightarrow z = \frac{\pi}{2} + n\pi \quad \forall\ n \in \mathbb{Z}$

$$\boxed{\cosh y = \frac{e^y + e^{-y}}{2}}$$

Hyperbolic functions : Again defined in standard way

$$\sinh(z) = \frac{e^z - e^{-z}}{2}, \qquad \cosh(z) = \frac{e^z + e^{-z}}{2}$$

$\Rightarrow \sinh(iz) = i \sin z \quad ; \quad \cosh(iz) = \cos z$

$\cosh^2(z) - \sinh^2(z) = 1$

$\frac{d}{dz} \sinh(z) = \cosh(z) \quad , \quad \frac{d}{dz} \cosh(z) = \sinh(z)$

- Zeros of $\sinh(z)$ & $\cosh(z)$ lie along imaginary axis.

$0 = \sinh(z) = -i \sin(iz) \Rightarrow z = in\pi \quad n \in \mathbb{Z}$

$0 = \cosh(z) = \cos(iz) \Rightarrow z = i\left(\frac{\pi}{2} + n\pi\right) \quad n \in \mathbb{Z}$