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By submitting this homework you state that you have understood what counts as academic dishonesty and the academic dishonesty policy of the course, and you agree to it.

1. Recall that we defined an intersecting family to be a collection of subsets of a given set S such that no two sets in the collection are disjoint. We also proved that when S is a finite set with $|S| = n$, the size of the largets intersecting family of subsets of S is 2^{n-1}. What if we want an intersecting family in which every set has the same given size, say k (a.k.a. a k-uniform intersecting family)? Let us find an answer to this question. Observe that when $k > n/2$, the answer is trivial, so let us assume $k \leq n/2$.

    1. [**1 point**] Prove that there exists a k-uniform intersecting family containing $C(n-1, k-1)$ sets ($C(n-1, k-1)$ denotes "$(n-1)$ choose $(k-1)$").
    2. [**1 point**] Suppose $A \subseteq S$ and $|A| = k$. Imagine that the elements of S are to be assigned to n distinct places on the circumference of a circle. How many ways are there to do so in such a way that elements of A appear consecutively?

       More formally, given a bijection $f : S \longrightarrow \{0, \ldots, n-1\}$, we say that the elements of a size-k set A "appear consecutively under f" if $\{f(x) \mid x \in A\} = \{m, (m+1) \bmod n, \ldots, (m+k-1) \bmod n\}$, for some $m \in \{0, \ldots, n-1\}$. We are interested in finding the number of such f's.
    3. [**2 points**] Suppose $\mathcal{F}$ is a family of subsets of S, each of size k, and $|\mathcal{F}| > C(n-1, k-1)$. Prove that the elements of S can be arranged on the circle in such a way that the elements of more than k of the sets in $\mathcal{F}$ appear consecutively. (Hint: Double counting + pigeon-hole.)

       More formally, we need to prove that there exists a bijection $f : S \longrightarrow \{0, \ldots, n-1\}$ under which more than k of the sets in $\mathcal{F}$ appear consecutively.
    4. [**2 points**] Hence argue that if $|\mathcal{F}| > C(n-1, k-1)$, then $\mathcal{F}$ cannot be a k-uniform intersecting family.

2. Consider the poset $(2^S, \subseteq)$, where $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. A non-empty chain $\{A_1, A_2, \ldots, A_k\}$ of this poset, where $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k$, is said to be a *symmetric chain* if $|A_1| + |A_k| = n$ and $|A_{i+1}| = |A_i| + 1$ for each $i = 1, \ldots, k-1$.

    1. [**2 points**] Prove that the set 2^S can be partitioned into symmetric chains. (Hint: Induction on n.)
    2. [**2 points**] Using the above result, find the size of the largest antichain in 2^S as a function of n, and prove your answer.