COL 352 Introduction to Automata and Theory of Computation

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January 29, 2023

Lecture 8: Regular Expressions

Recap

Definition (Pattern)

A pattern α is a string of symbols of a certain form representing a (possibly infinite) set of strings in Σ^*.

$$L(\alpha) = \{x \in \Sigma^* \mid x \text{ matches } \alpha\}$$

Recap: Atomic and Compound Patterns

1. $a \in \Sigma$, $L(a) = \{a\}$
2. ε, $L(\varepsilon) = \{\varepsilon\}$
3. $\varnothing$, $L(\varnothing) = \varnothing$
4. Σ, matching any alphabet
5. Σ^*, matching any finite string
6. x matches $\alpha + \beta$ if $L(\alpha + \beta) = L(\alpha) \cup L(\beta)$
7. x matches $\alpha \cap \beta$ if $L(\alpha \cap \beta) = L(\alpha) \cap L(\beta)$
8. x matches $\alpha\beta$ if $x = yz$ where $L(\alpha\beta) = L(\alpha)L(\beta)$
9. x matches $\overline{\alpha}$ if $L(\overline{\alpha}) = \overline{L(\alpha)} = \Sigma^* \setminus L(\alpha)$
10. x matches α^* if x can be expressed as zero or more of strings that match α, i.e., $L(\alpha^*) = L(\alpha)^*$
11. x matches α^+ if x can be expressed as one or more of strings that match α, i.e., $L(\alpha^+) = L(\alpha)^+$

Recap: DFA to regular expression

Lemma

Any regular language can be specified by a regular expression

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Want: Given any DFA, convert it into a regular expression.

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Given any DFA A, we can obtain a regular expression, say R_A, such that $L(A) = L(R_A)$.

Recap: Computing with labelled graphs

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Regular expressions

For a regular expression E we write $L(E)$ for its language. The set of valid regular expressions RegEx can be defined recursively as the following:

	Syntax	Semantics
Empty String	ϵ	$L(\epsilon) = \{\epsilon\}$
Empty Set	$\emptyset$	$L(\emptyset) = \emptyset$
Single Letter	a	$L(a) = \{a\}$
Union	$E + F$	$L(E + F) = L(E) \cup L(F)$
Concatenation	$E.F$	$L(E.F) = L(E) \circ L(F)$
Kleene Star	E^*	$L(E)^*$

NFA to regular expressions

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Theorem

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA, then there is an RE R such that $L(R) = L(A)$.

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Let us assign states in $Q = \{q_1, \ldots, q_n\}$ an arbitrary order, where $q_0 = q_1$ and $q_1 < \cdots < q_n$.

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Each path on a DFA corresponds to a word.

We will incrementally consider longer and longer paths.

NFA to regular expressions (contd)

Proof. (contd.)

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$p(i, j, k) :=$ the set of paths from q_i to q_j that do not have intermediate states that are greater than q_k.

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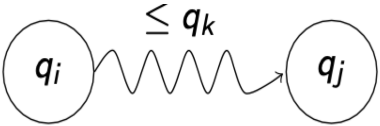

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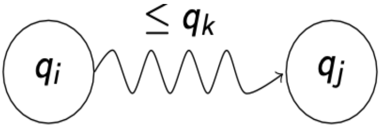

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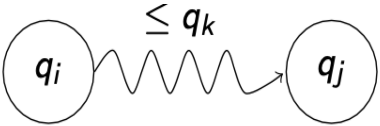

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Let $R(i, j, k)$ be the regular expression that defines the set of words along the paths in $p(i, j, k)$.

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Let us define $R(i, j, k)$ by induction over k.

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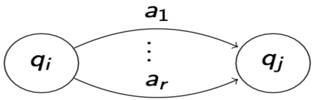
Base case: Let $\{a_1,\dots,a_r\} = \{a \mid \delta(q_i,a) = q_j\}$, i. e., letters that take q_i to q_j.

$$R(i,j,0) := \begin{cases} a_1 + \cdots + a_r, i \neq j \\ a_1 + \cdots + a_r + \varepsilon \quad \text{Otherwise} \end{cases}$$



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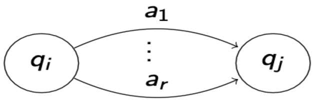

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Induction step: By induction hypothesis, we have regular expressions for the paths upto $k-1$. Let us consider the paths that also go via state q_k.

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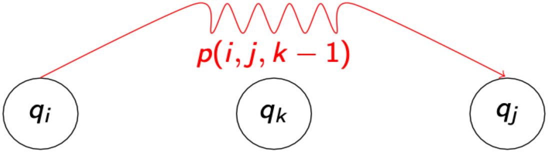

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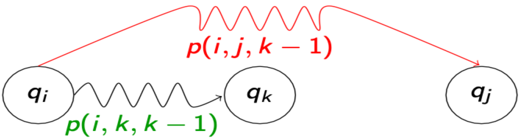

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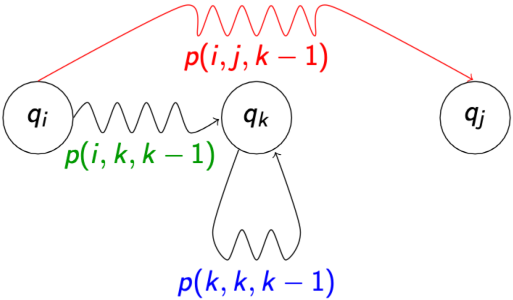

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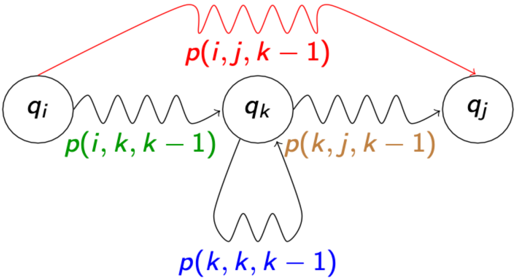

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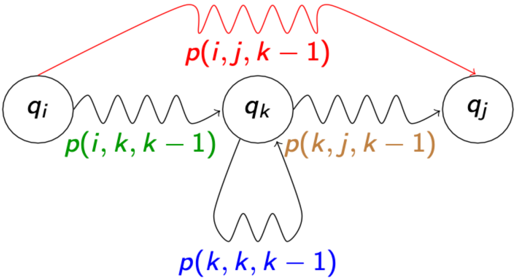

$$R(i,j,k) := R(i,j,k-1) + R(i,k,k-1)R(k,k,k-1)^{*}R(k,j,k-1)$$

NFA to regular expressions (contd)

Let $F = \{q_{j1}, \ldots, q_{js}\}$

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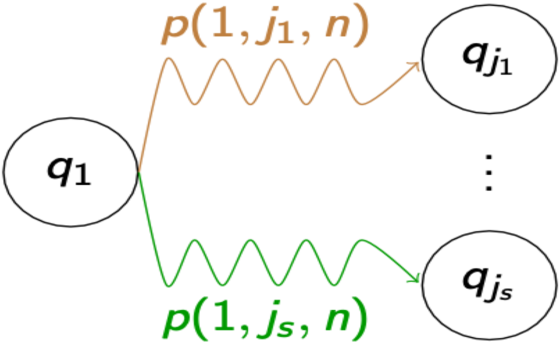

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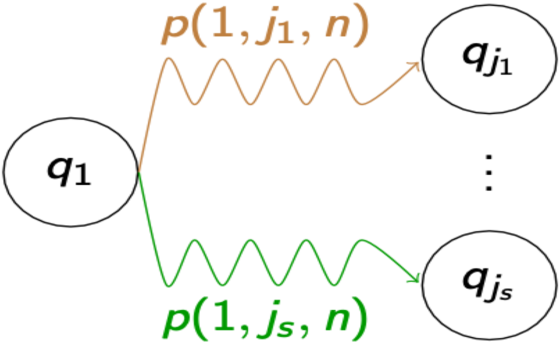
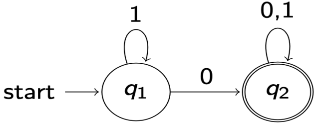


The following regular expression will recognize $L(A)$

$$R(1, j_1, n) + \cdots + R(1, j_s, n)$$

Examples



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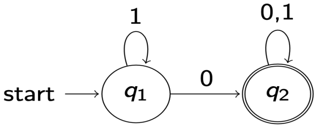
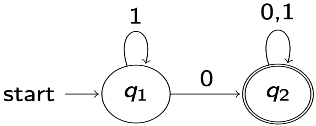


Base Cases:

$$R(1,1,0) = \varepsilon + 1$$

Examples



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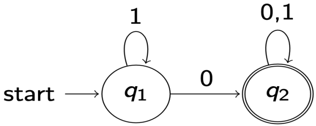

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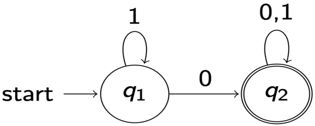

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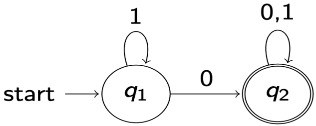

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$$R(1,2,1) = R(1,2,0) +$$

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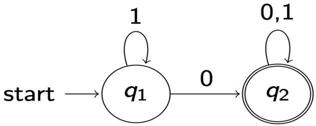

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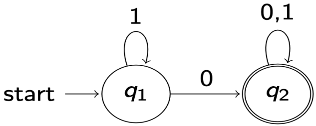

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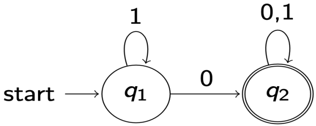

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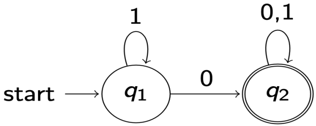

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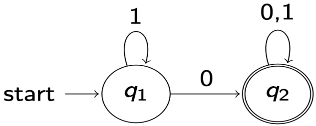

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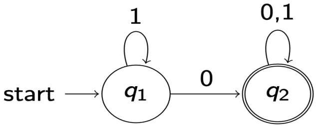

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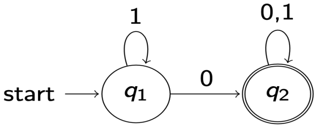

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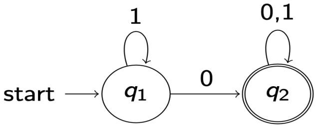

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Examples (Contd.)

$$
\begin{aligned}
L(A) &= R(1,2,2) \\
&= R(1,2,1) + R(1,2,1)R(2,2,1)^* R(2,2,1) \\
&= 1^*0 + 1^*0(\varepsilon + 0 + 1)^*(\varepsilon + 0 + 1) \\
&= 1^*0(0+1)^*
\end{aligned}
$$

Limitations of Finite Automata

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$$L_{0,1} = \{0^n 1^n \mid n \geq 0\}$$

00000000000000000000000000000 1111111111111111111111111111111

Proving that $L_{0,1}$ is not a regular language

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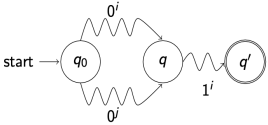
□



Proving that PAL is not a regular language

Lemma

$\forall n \in \mathbb{N}$ let $PAL_n = \{w \cdot w^R \mid w \in \Sigma^*, |w| = n\}$. Any automaton accepting PAL_n must have $|\Sigma|^n$ states.

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Proof.

By Pigeon Hole Principle.

Proving that PAL is not a regular language

Lemma

$\forall n \in \mathbb{N}$ let $PAL_n = \{w \cdot w^R \mid w \in \Sigma^*, |w| = n\}$. Any automaton accepting PAL_n must have $|\Sigma|^n$ states.

Proof.

By Pigeon Hole Principle.

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□

Corollary

Let $PAL = \cup_{n \geq 0} PAL_n$. PAL *is not regular.*