Lecture 5

§ Invertible Matrices.

Def: An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix B s.t $AB = I$ and $BA = I$

- The matrix B is unique!

[Suppose there exists B and C s.t $AB = BA = AC = CA = I$

$$B(AB) = B(AC) \Rightarrow (BA)B = (BA)C \Rightarrow B = C$$]

Def: If an $n \times n$ matrix A is invertible then A^{-1} is called the inverse of A & denote the unique $n \times n$ matrix by A^{-1} s.t $AA^{-1} = A^{-1}A = I$

- An invertible matrix A is also called non-singular. If A does not have an inverse then its called singular.

If A is invertible matrix ($n \times n$)

Properties: (1) $(A^{-1})^{-1} = A$

(2) If B is also invertible ($n \times n$) then AB is also invertible and

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

③ If C & D are $n \times m$ matrices.

If $AC = AD \nRightarrow C = D$.

④ If $AC = \underline{0}$ then $C = \underline{0}$

zero matrix.

§ Determinants.

Suppose 1×1 matrix $[a]$ then $[a]$ is invertible if $a \neq 0$

If A is 2×2 matrix. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$AA^{-1} = \underline{1} = A^{-1}A.$$

A^{-1} makes sense when $\overbrace{ad-bc}^{\text{determinant of } A} \neq 0$.

- Let A be a 2×2 matrix, then $\det A = ad - bc$.

- A is a 3×3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & (a_{11}a_{22} - a_{12}a_{21}) & (a_{11}a_{23} - a_{13}a_{21}) \\ 0 & (a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) \end{bmatrix}$$

Assume at least $a_{11}, a_{21}, a_{31} \neq 0$.

after rearrangement $a_{11} \neq 0$

the matrix is invertible when

$$\begin{bmatrix} (a_{11}a_{22}-a_{12}a_{21}) & (a_{11}a_{23}-a_{13}a_{21}) \\ (a_{11}a_{32}-a_{12}a_{31}) & (a_{11}a_{33}-a_{13}a_{31}) \end{bmatrix}$$ is invertible.

A is invertible when determinant of ↑ is non zero.

- determinant of 3×3 matrix. $a_{11}\left[a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}\right]$

Def: Let A be a n×n matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The determinant of A is given by

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$

where $C_{11}, \ldots, C_{1n}$ are the cofactor matrices of $a_{11} \ldots a_{1n}$.

$$C_{ij} = (-1)^{i+j}\det(M_{ij})$$

M_{ij} are called minors of a_{ij}

where M_{ij} denote the $(n-1)\times(n-1)$ matrix that we get from A by deleting the row & column containing a_{ij}

(delete the i^{th} row & j^{th} column)

Ex: $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 0 \\ 3 & 1 & 2 \end{bmatrix}$

$$C_{11} = (-1)^{1+1} \det \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = 8$$

$$C_{12} = (-1)^{1+2} \det \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = -2$$

$$C_{13} = (-1)^{1+3} \det \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} = (-1)^4 (1-12)$$
$$= -11$$

$$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$
$$= 2(8) + (-1)(-2) + 3(-11) = -15.$$

Theorem: $\det(I_{n \times n}) = 1$

$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ $\qquad \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

Theorem: Let A be an $n \times n$ matrix.

(a) $\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$ (expansion along row i)

(b) $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$ (expansion along col j)

(original definition is expansion along row 1)

Ex: $A = \begin{bmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix}$

Expand along 2^{nd} col or the 4^{th} row.

Expand along 2^{nd} col: $\det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} + a_{42}C_{42}$

$a_{22} = a_{42} = 0$

$\Rightarrow \det A = a_{12}C_{12} + a_{32}C_{32}$.

$$C_{12} = (-1)^{1+2} \det \begin{bmatrix} 1 & -1 & 2 \\ 5 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} = -20.$$

$$C_{32} = (-1)^{3+2} \det \begin{bmatrix} -2 & 4 & -1 \\ 1 & -1 & 2 \\ 0 & 3 & -1 \end{bmatrix} = -11$$

$\det A = 1(-20) + (-1)(-11) = -9$.

Ex:

$$\begin{bmatrix} -1 & 2 & 7 & -5 & 8 \\ 0 & 3 & 4 & 1 & -9 \\ 0 & 0 & -2 & 4 & 11 \\ 0 & 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det = (-1) \det \begin{vmatrix} 3 & 4 & 1 & -9 \\ 0 & -2 & 4 & 11 \\ 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-1)(3) \det \begin{pmatrix} -2 & 4 & 11 \\ 0 & -4 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$= (-1)(3)(-2) \det \begin{vmatrix} -4 & 5 \\ 0 & 1 \end{vmatrix} = (-1)(3)(-2)(-4)(1)$$

Def: If A is diagonal, upper triangular or lower triangular then $\det A$ is the product of the terms along the diagonal
$(\det A = a_{11} a_{22} \cdots a_{nn})$.

Theorem: A is $n \times n$ matrix $\det(A^T) = \det A$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \det = a_{11}a_{22} - a_{12}a_{21}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \quad \det = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Theorem: A is an $n \times n$ matrix.

(a) If A has a row or column of zeros then $\det A = 0$

(b) If A has two identical rows or columns then $\det A = 0$.

§ Properties of determinants.

Theorem: A is an $n \times n$ matrix.

(a) Suppose that B is produced by interchanging two rows of A. $\det A = -\det B$.

(b) Suppose that B is produced by multiplying a row of A by a constant c. Then

$$\det A = \frac{1}{c} \det B.$$

(c) Suppose B is produced by adding a multiple of one row of A to another then $\det A = \det B$.

Ex:

$$A = \begin{bmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$A_1 = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 4 & -1 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

$\det A = -\det A_1$

$$A_2 = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 9 & -6 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 5R_1$

$R_3 \to R_2 + R_3$.

$\det A_1 = \det A_2$

$\det A = -\det A_2$.

$$A_3 = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

$R_3 \to -\frac{1}{3} R_3$.

$\det A_2 = -3 \det A_3$.

$\det A = 3 \det A_3$.

$$A_4 = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$\det A_3 = \det A_4$

$\det A = 3 \det A_4$.

$= 3(1)(1)(-3)(1)$

$= -9$.

Theorem: Let A be an $n\times n$ matrix. Then A is invertible if and only if $\det A \neq 0$.

Theorem: If A & B are $n\times n$ matrices then
$$\det AB = \det A \cdot \det B = \det(BA)$$

(AB may not equal BA but their determinants are equal!)

$$\det(A+B) \neq \det A + \det B.$$

Theorem: If E & B are $n\times n$ matrices & E is an elementary matrix then $\det EB = \det E \det B$.

[Suppose E is an elementary matrix obtained by interchanging two rows of identity matrix $I_{n\times n}$.

$\det I_{n\times n} = 1 \Rightarrow \det E = -1$

EB is obtained by performing the same interchange of two rows on $B \Rightarrow \det(EB) = -\det B$.

$\det(EB) = -1 \cdot \det B = \det E \cdot \det B$]

Theorem: A be an $n\times n$ matrix. Then
$$\det A^{-1} = \frac{1}{\det A}.$$

[$AA^{-1} = I_{n\times n} \Rightarrow \det AA^{-1} = \det I_{n\times n} = 1$

$\qquad\qquad\qquad\qquad \det A \cdot \det A^{-1}$

§ Cramer's Rule.

Theorem: Suppose $A\underline{x} = \underline{b}$ is a linear system of equations, & A is an $n \times n$ invertible matrix. Then the components of the unique solution to $A\underline{x} = \underline{b}$ is given by

$$x_i = \frac{\det A_i}{\det A}$$

where A_i is the matrix obtained from the matrix A after replacing ith column by $\underline{b}$.

Ex:

$$\begin{aligned} 3x_1 + x_2 \qquad\quad &= 5 \\ -x_1 + 2x_2 + x_3 &= -2 \\ -x_2 + 2x_3 &= -1 \end{aligned}$$

$A\underline{x} = \underline{b}$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

Check $\det A \neq 0$

$$A_1 = \begin{bmatrix} 5 & 1 & 0 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 3 & 5 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 3 & 1 & 5 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$x_1 = \frac{\det A_1}{\det A} = \frac{28}{17}$ $\quad x_2 = \frac{\det A_2}{\det A} \Rightarrow \frac{1}{17}$ $\quad x_3 = \frac{\det A_3}{\det A} = \frac{-8}{17}$.

§ Inverses from Determinants.

$C_{ij} = (-1)^{i+j} M_{ij}$ ← minor corresponding to a_{ij}

Cofactor matrix: $C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & & & \\ C_{n1} & \cdots & \cdots & C_{nn} \end{bmatrix}$

Adjoint matrix: $\operatorname{adj}(A) = C^T$.

Theorem: If A is an invertible matrix then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

$$\left[A(\operatorname{adj} A) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \ddots \\ C_{1n} & & C_{nn} \end{bmatrix} \right.$$

$$= \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \ddots \det A \end{bmatrix}$$

$$\left. = \det A \, I_{n\times n} \right]$$

Ex: $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

$C_{11} = (-1)^2 \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$ $\qquad C_{21} = (-1)^3 \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = -2$

$C_{31} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$ $\qquad C_{12} = (-1)^3 \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} = 2$

$C_{22} = 6 \quad C_{32} = -3 \quad C_{13} = 1, \ C_{23} = 3 \quad C_{33} = 7.$

$C = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 6 & 3 \\ 1 & -3 & 7 \end{bmatrix}$ $\quad \operatorname{adj} A = C^T = \begin{bmatrix} 5 & -2 & 1 \\ 2 & 6 & -3 \\ 1 & 3 & 7 \end{bmatrix}$

$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = 17.$

$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{17} \begin{bmatrix} 5 & -2 & 1 \\ 2 & 6 & -3 \\ 1 & 3 & 7 \end{bmatrix}$