Lecture 19
Signals and Systems (ELL205)

By Dr. Abhishek Dixit

Dept. of Electrical Engineering

IIT Delhi

Inner Product Space

$v = \{v_1, v_2, \dots . v_n\}$ and
$u = \{u_1, u_2, \dots . u_n\}$

$$\langle v, u \rangle \stackrel{\text{def}}{=} \sum_{i=1}^{n} v_i {u_i}^*$$

Direction

- $\cos(\angle(v,u)) = \frac{\langle v,u\rangle}{\|v\|\|u\|}$ (If v and u are real)
- Two vectors are orthogonal if $\langle v, u\rangle = 0$
- Cauchy-Schwarz inequality

$$|\langle v, u\rangle| \leq \|v\|\|u\|$$

Equality is satisfied when $v = \alpha u$

Proof of Cauchy-Schwarz inequality

- $|v|^2 = |v_{\perp u}|^2 + |v_{|u}|^2$
- $|v|^2 \geq |v_{|u}|^2$

Proof of Cauchy-Schwarz inequality

- $|v|^2 = |v_{\perp u}|^2 + \left|v_{|u}\right|^2$
- $|v|^2 \geq \left|v_{|u}\right|^2 = \left|\frac{\langle v,u\rangle u}{|u|^2}\right|^2 = \frac{|\langle v,u\rangle|^2}{|u|^2}$

Proof of Cauchy-Schwarz inequality

- $|v|^2 = |v_{\perp u}|^2 + \left|v_{|u}\right|^2$
- $|v|^2 \geq \left|v_{|u}\right|^2 = \left|\frac{\langle v,u\rangle u}{|u|^2}\right|^2 = \frac{|\langle v,u\rangle|^2}{|u|^2}$
- $|v|^2|u|^2 \geq |\langle v, u\rangle|^2$
- $|\langle v, u\rangle| \leq |v||u|$

Outline

- Introduction to vectors
- Introduction to inner product and projection theorem
- Introduction to signals as vectors

$\mathcal{L}_2$ space is a vector space

- $\mathcal{L}_2$ space is a vector space

Is $\mathcal{L}_2$ space a vector space?

Addition: $\mathrm{u}(t) + v(t) \stackrel{\text{def}}{=} u(t) + v(t)$

Scalar multiplication: $\alpha u(t) \stackrel{\text{def}}{=} \alpha u(t)$

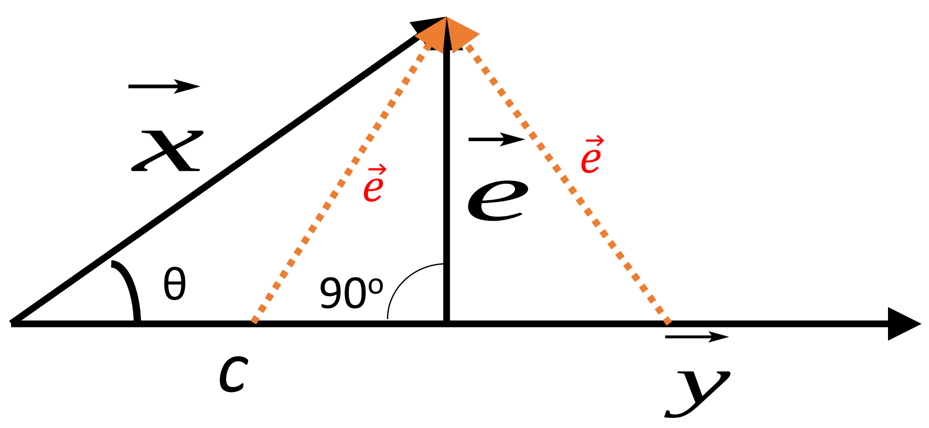
Is $u + v$ also a finite energy signal?

$$\int_{-\infty}^{\infty} |u(t) + v(t)|^2 dt \leq \int_{-\infty}^{\infty} 2|u(t)|^2 dt + \int_{-\infty}^{\infty} 2|v(t)|^2 dt < \infty$$

Is αu also a finite energy signal?

$$\int_{-\infty}^{\infty} |\alpha u(t)|^2 dt = \int_{-\infty}^{\infty} \alpha^2 |u(t)|^2 dt < \infty$$

Minimum error in approximation



$$\vec{x} \approx c\vec{y} \qquad \vec{e} = \vec{x} - c\vec{y}$$

$$c\left\|\vec{y}\right\| = \left\|\vec{x}\right\|\cos\theta$$

$$c\left\|\vec{y}\right\|^2 = \left\|\vec{x}\right\|\left\|\vec{y}\right\|\cos\theta$$

$$c\left\|\vec{y}\right\|^2 = \vec{x}.\vec{y} \qquad c = \frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}$$

Minimum error in approximation in real signals

Estimating *x* in terms of *y* $x(t) \approx cy(t)$

Defining error signal $e(t) = x(t) - cy(t)$ and its energy $E_e(c) = \int_{t_1}^{t_2} (x(t) - cy(t))^2 dt$

Differentiating energy $\frac{dE_e(c)}{dc} = 0$ to get $\int_{t_1}^{t_2} 2\big(x(t) - cy(t)\big)y(t)dt = 0$

From above we calculate *c* as: $c = \dfrac{\int_{t_1}^{t_2} x(t)y(t)dt}{\int_{t_1}^{t_2} y^2(t)dt}$

Equivalence of vectors and real signals

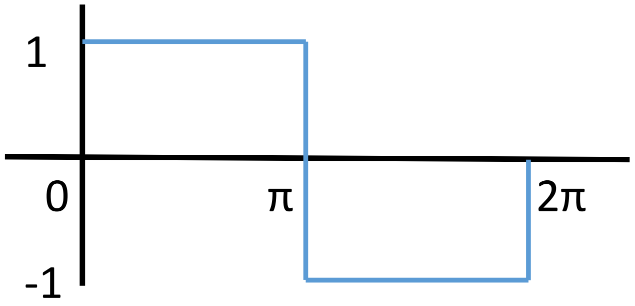
$$c = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}$$

$$c = \frac{\int_{t_1}^{t_2} x(t)y(t)dt}{\int_{t_1}^{t_2} y^2(t)dt}$$

$$\langle \vec{x}, \vec{y} \rangle = \int_{t_1}^{t_2} x(t)y(t)dt$$

$$\|\vec{y}\|^2 = \int_{t_1}^{t_2} y^2(t)dt$$

Try it yourself !!



$$x(t) = csint$$

What is *c* for minimum error?

1) $c = \frac{4}{\pi}$	**2)** $c = \frac{2}{\pi}$
3) $c = \frac{1}{\pi}$	**4)** $c = \frac{8}{\pi}$

Finding *c*

$$y(t) = sint$$

$$E_y = \int_0^{2\pi} \sin^2 t\, dt = \pi$$

$$c = \frac{1}{E_y}\int_0^{2\pi} x(t) sin\, t\, dt = \frac{1}{\pi}\left[\int_0^{\pi} sintdt + \int_{\pi}^{2\pi} -sintdt\right] = \frac{4}{\pi}$$

$$x(t) \cong \frac{4}{\pi} sint$$

Minimum error in approximation in complex signals

Estimating *x* in terms of *y* $x(t) \approx cy(t)$

Defining error signal $e(t) = x(t) - cy(t)$ and its energy $E_e(c) = \int_{t_1}^{t_2} | x(t) - cy(t) |^2 \, dt$

Algebraic workout:

$$E_e(c) = \int_{t_1}^{t_2} \big(x(t) - cy(t)\big)\big(x^*(t) - c^* y^*(t)\big) dt$$

$$E_e(c) = \int_{t_1}^{t_2} \big(x(t)x^*(t) - cy(t)x^*(t) - c^* x(t)y^*(t) + |c|^2 y(t)y^*(t)\big) dt$$

Minimum error in approximation in complex signals

$$E_e(c) = \int_{t_1}^{t_2} \left(x(t)x^*(t) - cy(t)x^*(t) - c^*x(t)y^*(t) + |c|^2 y(t)y^*(t) \right) dt$$

$$E_e(c) = \int_{t_1}^{t_2} |x(t)|^2 dt + \left| c\sqrt{E_y} - \frac{1}{\sqrt{E_y}} \int_{t_1}^{t_2} x(t)y^*(t)dt \right|^2$$

Minimum error in approximation in complex signals

$$E_e(c) = \int_{t_1}^{t_2} \left(x(t)x^*(t) - cy(t)x^*(t) - c^*x(t)y^*(t) + |c|^2 y(t)y^*(t) \right) dt$$

$$|a - b|^2 = |a|^2 + |b|^2 - ab^* - a^*\, b$$

$$E_e(c) = \int_{t_1}^{t_2} |x(t)|^2 dt + \left| \underbrace{c\sqrt{E_y}}_{a} - \frac{1}{\sqrt{E_y}} \underbrace{\int_{t_1}^{t_2} x(t)y^*(t)dt}_{b} \right|^2$$

$$E_e(c) = \int_{t_1}^{t_2} |x(t)|^2 dt + \underbrace{|c|^2 E_y}_{|a|^2} + \underbrace{\left| \frac{1}{\sqrt{E_y}} \int_{t_1}^{t_2} x(t)y^*(t)dt \right|^2}_{|b|^2}$$

$$- \underbrace{c \int_{t_1}^{t_2} x^*(t)y(t)dt}_{ab^*} - \underbrace{c^* \int_{t_1}^{t_2} x(t)y^*(t)dt}_{a^*b}$$

Minimum error in approximation in complex signals

$$E_e(c) = \int_{t_1}^{t_2} \left(x(t)x^*(t) - cy(t)x^*(t) - c^*x(t)y^*(t) + |c|^2 y(t)y^*(t)\right)dt$$

$$E_e(c) = \int_{t_1}^{t_2} |x(t)|^2 dt + \left|c\sqrt{E_y} - \frac{1}{\sqrt{E_y}}\int_{t_1}^{t_2} x(t)y^*(t)dt\right|^2 - \left|\frac{1}{\sqrt{E_y}}\int_{t_1}^{t_2} x(t)y^*(t)dt\right|^2$$

$$E_e(c) = \int_{t_1}^{t_2} |x(t)|^2 dt + |c|^2 E_y + \left|\frac{1}{\sqrt{E_y}}\int_{t_1}^{t_2} x(t)y^*(t)dt\right|^2$$
$$- c\int_{t_1}^{t_2} x^*(t)y(t)dt - c^*\int_{t_1}^{t_2} x(t)y^*(t)dt - \left|\frac{1}{\sqrt{E_y}}\int_{t_1}^{t_2} x(t)y^*(t)dt\right|^2$$

Minimum error in approximation in complex signals

$$E_e(c) = \int_{t_1}^{t_2} |x(t)|^2 dt - \left| \frac{1}{\sqrt{E_y}} \int_{t_1}^{t_2} x(t) y^*(t) dt \right|^2 + \left| c\sqrt{E_y} - \frac{1}{\sqrt{E_y}} \int_{t_1}^{t_2} x(t) y^*(t) dt \right|^2$$

$$c = \frac{\int_{t1}^{t2} x(t) y^*(t) dt}{\int_{t_1}^{t_2} |y(t)|^2 dt}$$

Equivalence of vectors and complex signals

$$c = \frac{\langle \vec{x}, \vec{y} \rangle}{||\vec{y}||^2}$$

$$c = \frac{\int_{t1}^{t2} x(t) y^*(t) dt}{\int_{t_1}^{t_2} |y(t)|^2 dt}$$

$$\langle \vec{x}, \vec{y} \rangle = \int_{t_1}^{t_2} x(t) y^*(t) dt$$

$$||\vec{y}||^2 = \int_{t_1}^{t_2} |y(t)|^2 dt$$

The Inner Product space of $\mathcal{L}_2$ waveforms

- $< u, v > \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} u(t)v^*(t)dt$
- $< u, u > \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} u(t)u^*(t)dt$

  $= \int_{-\infty}^{\infty} |u(t)|^2 dt$ (Energy of the signal)

Properties of Inner Product

a) Hermitian symmetry: $\langle v, u\rangle = \langle u, v\rangle^*$

Proof: $\langle u, v\rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} u(t)v^*(t)dt$

$$\langle u, v\rangle^* = \left(\int_{-\infty}^{\infty} u(t)v^*(t)dt\right)^* = \int_{-\infty}^{\infty} u^*(t)v(t)dt = \langle v, u\rangle$$

b) Hermitian bilinearity: $\langle \alpha v + \beta u, w\rangle = \alpha\langle v, w\rangle + \beta\langle u, w\rangle$

$$\langle v, \alpha u + \beta w\rangle = \alpha^*\langle v, u\rangle + \beta^*\langle v, w\rangle$$

Properties of Inner Product

c) Strict positivity: $\langle v, v\rangle \geq 0$ with equality if and only if $v = 0$

$\int_{-\infty}^{\infty} |v(t)|^2 dt = 0$ if $v \neq 0$ Axiom is not satisfied !!

Properties of Inner Product

c) Strict positivity: $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$

$\int_{-\infty}^{\infty} |v(t)|^2 dt = 0$ if $v \neq 0$ Axiom is not satisfied !!

Vectors in $\mathcal{L}_2$ space are not functions but are equivalence class (all indistinguishable functions belong to same equivalence class)

Outline

- Introduction to vectors
- Introduction to inner product and projection theorem
- Introduction to signals as vectors
- Signal spaces

Signal Orthogonal Vector Space

$$x(t) = \sum_{n=1}^{N} c_n x_n(t)$$

$$\int_{t_1}^{t_2} x_m(t)\overline{x_n(t)}dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

If $E_n = 1$, orthonormal basis set

Error in approximation

$$e(t) = x(t) - \sum_{n=1}^{N} c_n x_n(t)$$

$$E_e = \int_{t_1}^{t_2} | x(t) - \sum_{n=1}^{N} c_n x_n(t) |^2 \, dt$$

$$\frac{\partial E_e}{\partial c_n} = 0$$

For minimum error c_n is

$$c_n = \frac{\int_{t_1}^{t_2} x(t)\overline{x_n(t)}dt}{\int_{t_1}^{t_2} \left|x_n(t)\right|^2 dt}$$

What can we choose as basis vectors?

$$\int_{t_1}^{t_2} x_m(t)\overline{x_n(t)}dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

1. Complex Exponentials
2. Walsh functions, e.g., CDMA applications
3. Legendre Polynomials, e.g., spherical geometries
4. Laugerre functions, e.g., data compression
5. Hermite Polynomials, e.g., interpolation
6. Bessel functions, optical fiber communication (cylindrical geometries)
7. Chebyshev Polynomials, e.g., filter designing
8. Jacobi Polynomials, e.g., data compression and filter designing

Complex Exponentials

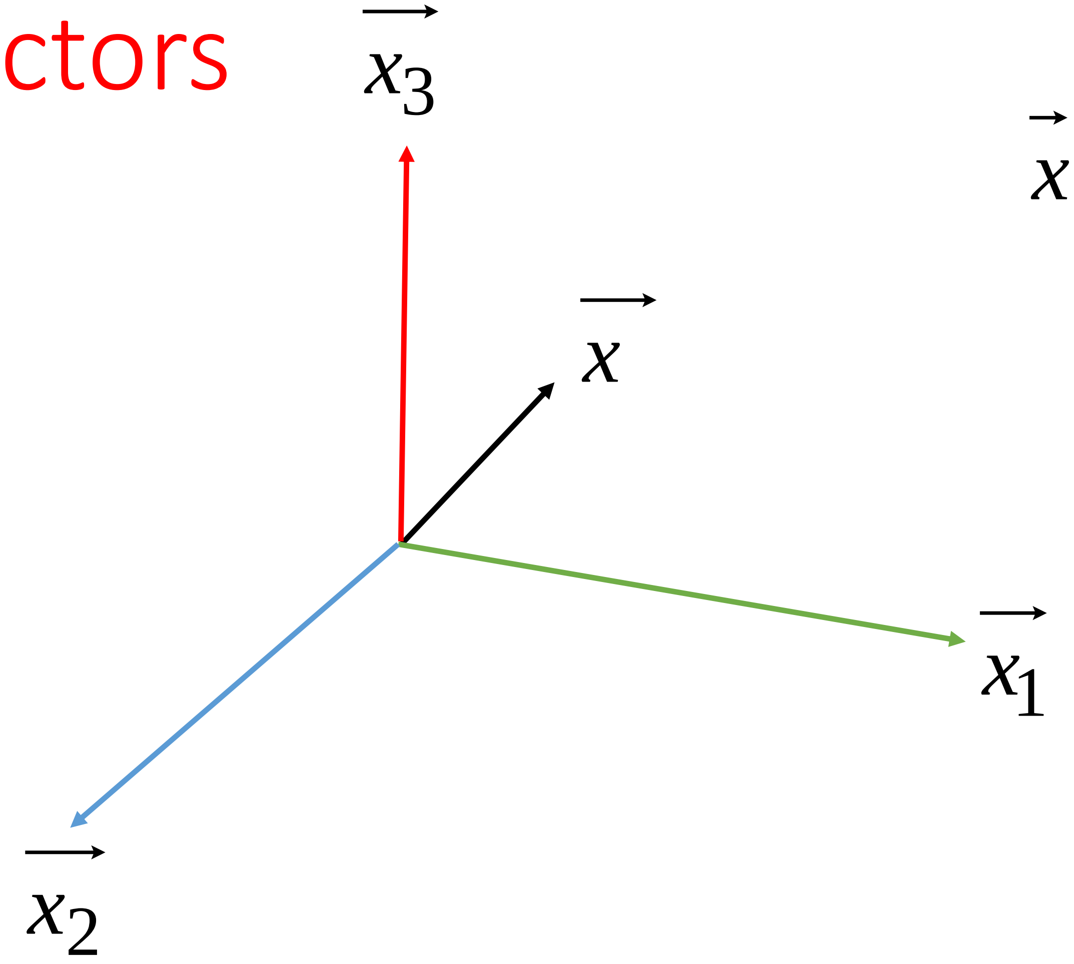
$$\int_0^T e^{jk\omega_o t} e^{-jn\omega_o t} dt = \begin{cases} 0 & k \neq n \\ T & k = n \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$c_k = \frac{\int_{t_1}^{t_2} x(t)\overline{x_k(t)}dt}{\int_{t_1}^{t_2} |x_k(t)|^2 dt}$$

$$a_k = \frac{\int_0^T x(t) e^{-jk\omega_o t} dt}{T}$$

3D Vectors



$$\vec{x} = c_1 \vec{x_1} + c_2 \vec{x_2} + c_3 \vec{x_3}$$

$$c_1 = \frac{\vec{x}.\vec{x_1}}{\left|\vec{x_1}\right|^2}$$

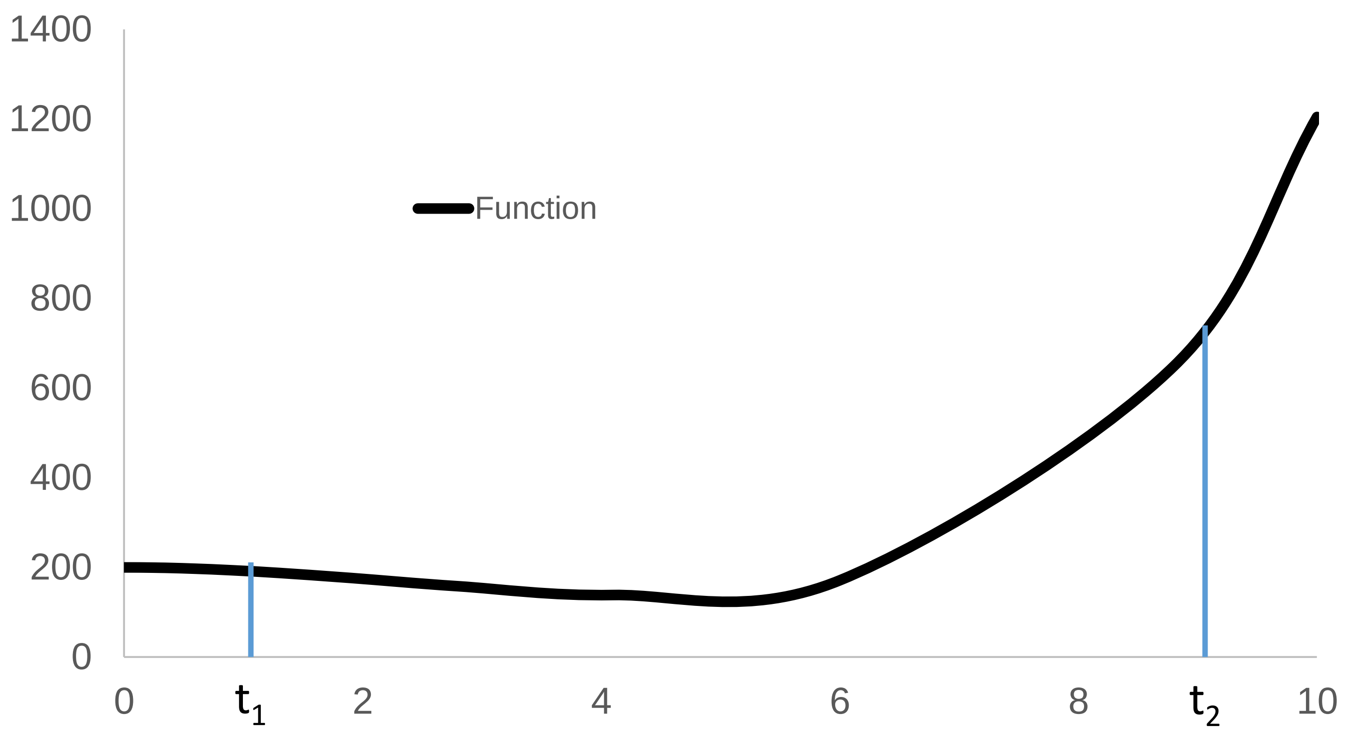
$$c_2 = \frac{\vec{x}.\vec{x_2}}{\left|\vec{x_2}\right|^2}$$

Finality Property

$$x(t) \approx c_0 x_0(t) + c_1 x_1(t) + c_2 x_2(t)$$

$$x(t) \approx c_0 x_0(t) + c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t) + c_4 x_4(t)$$

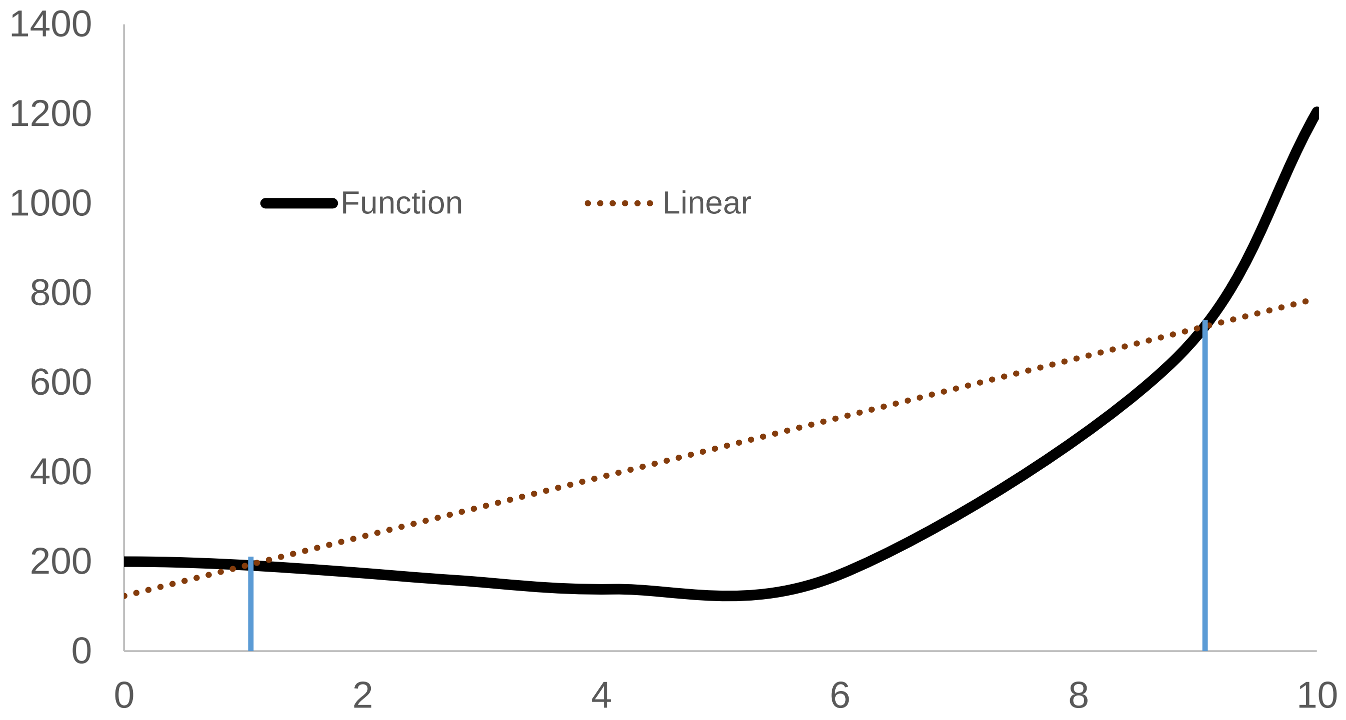
Contrast it with Polynomial approximation



$$x(t_1) = a_o + a_1 t_1$$

$$x(t_2) = a_o + a_1 t_2$$

Contrast it with Polynomial approximation

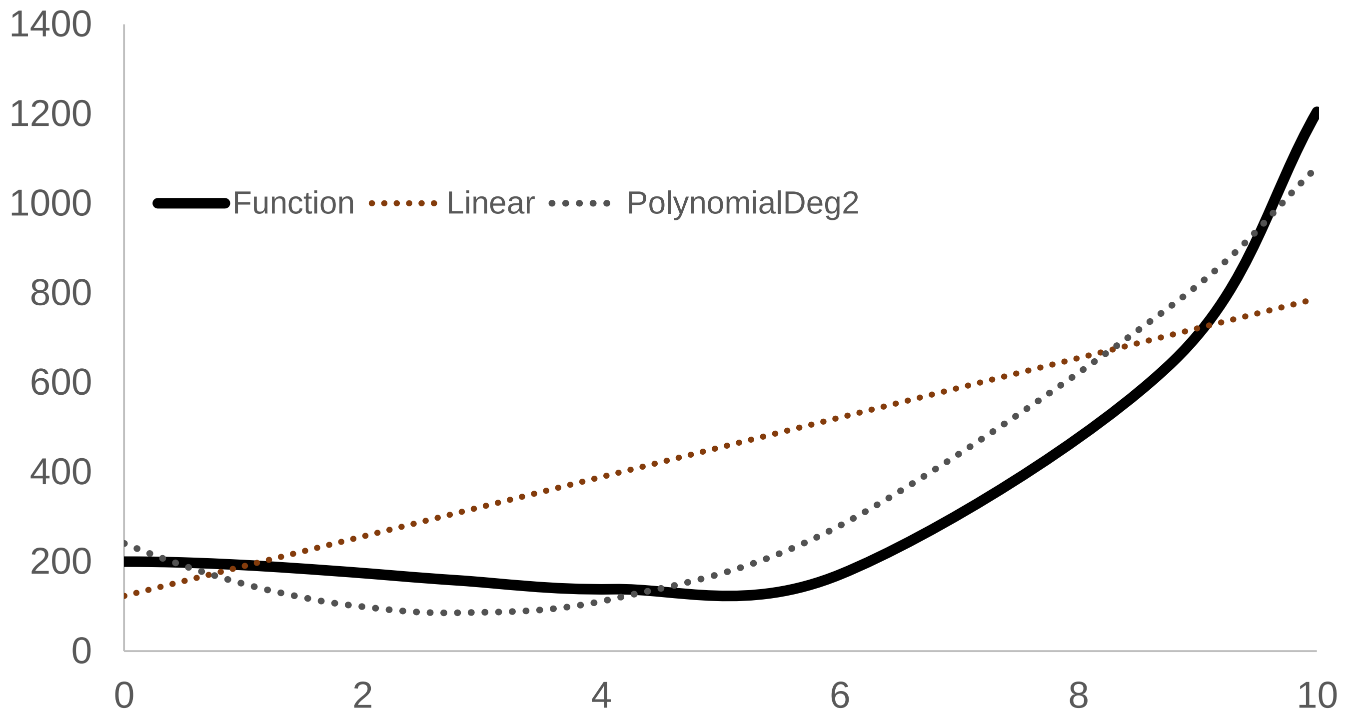


$$x(t_1) = a_o + a_1 t_1$$

$$x(t_2) = a_o + a_1 t_2$$

$$x(t) = 123.56 + 66.418t$$

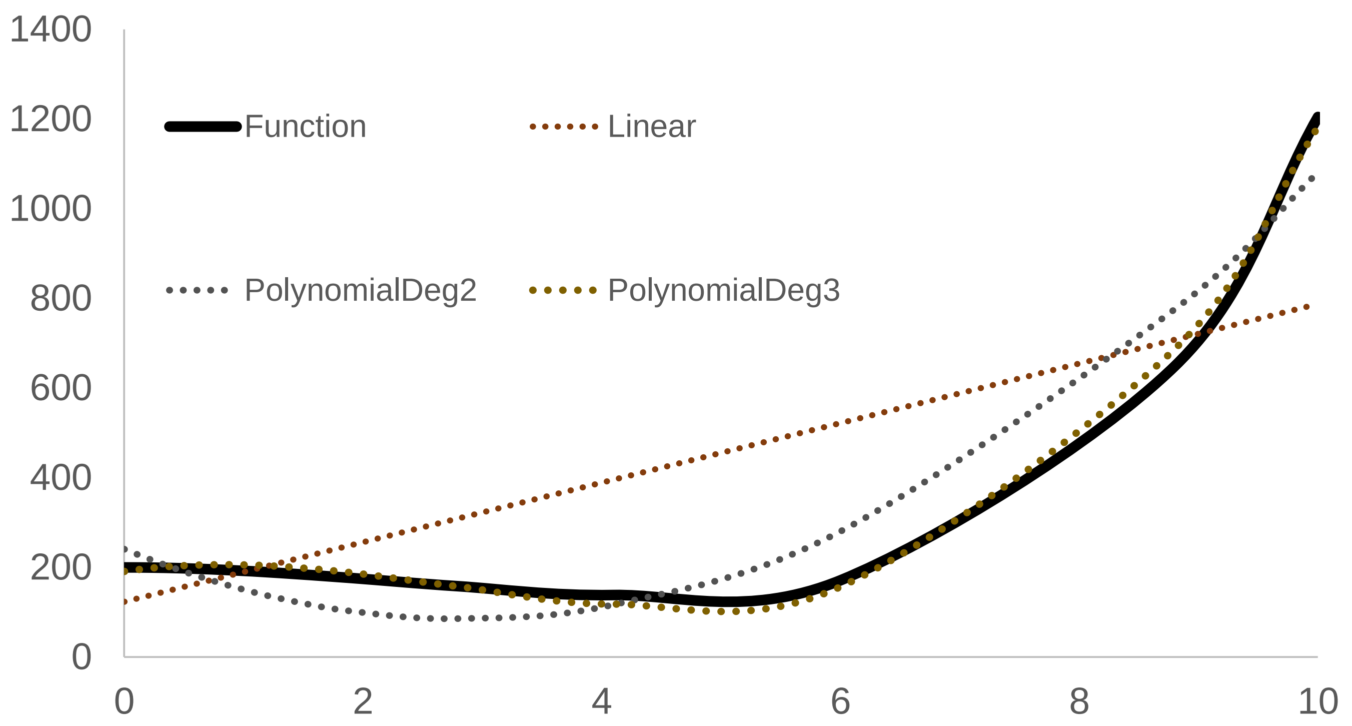
Contrast it with Polynomial approximation



$$x(t) = 123.56 + 66.418t$$

$$x(t) = 240.81 - 109.48t + 19.348t^2$$

Contrast it with Polynomial approximation

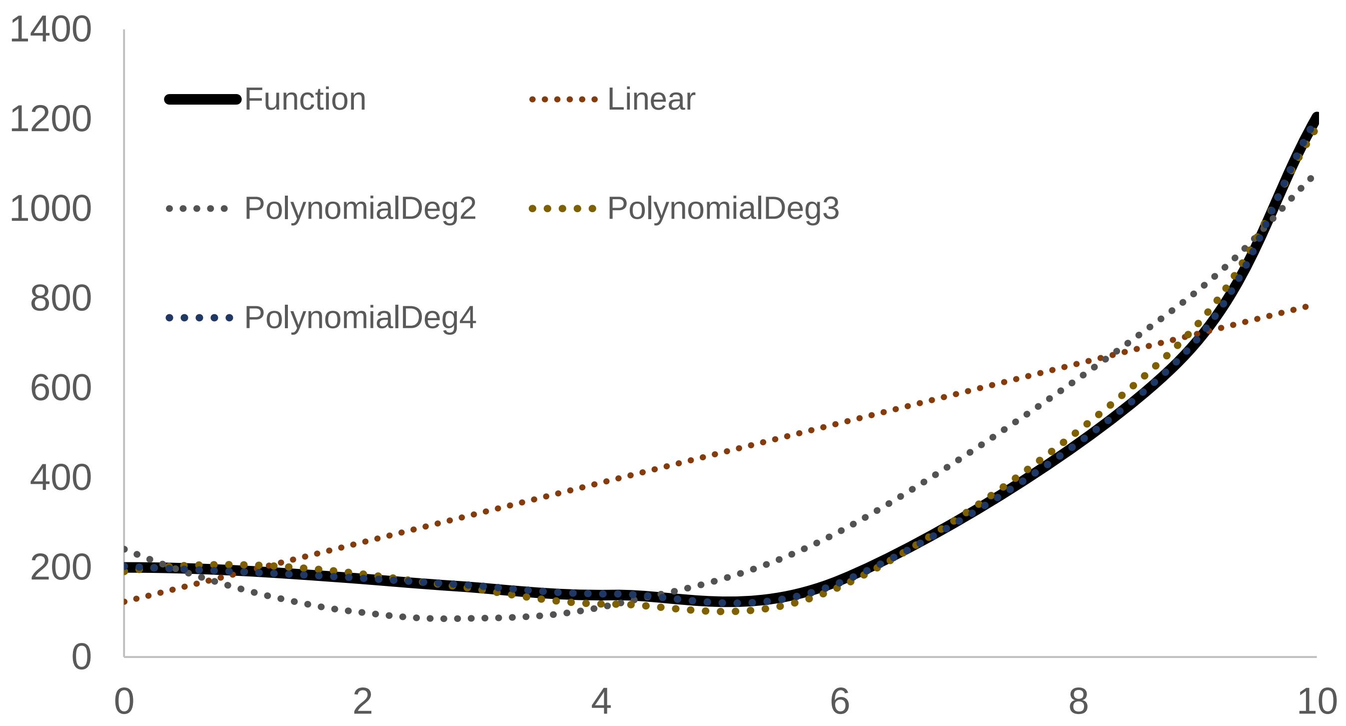


$$x(t) = 123.56 + 66.418t$$

$$x(t) = 240.81 - 109.48t + 19.348t^2$$

$$x(t) = 190.53 + 39.011t - 27.653t^2 + 3.367t^3$$

Contrast it with Polynomial approximation



$$x(t) = 123.56 + 66.418t$$

$$x(t) = 240.81 - 109.48t + 19.348t^2$$

$$x(t) = 190.53 + 39.011t - 27.653t^2 + 3.367t^3$$

$$x(t) = 202.44 - 14.653t + 3.7952t^2 - 2.1398t^3 + 0.2908t^4$$

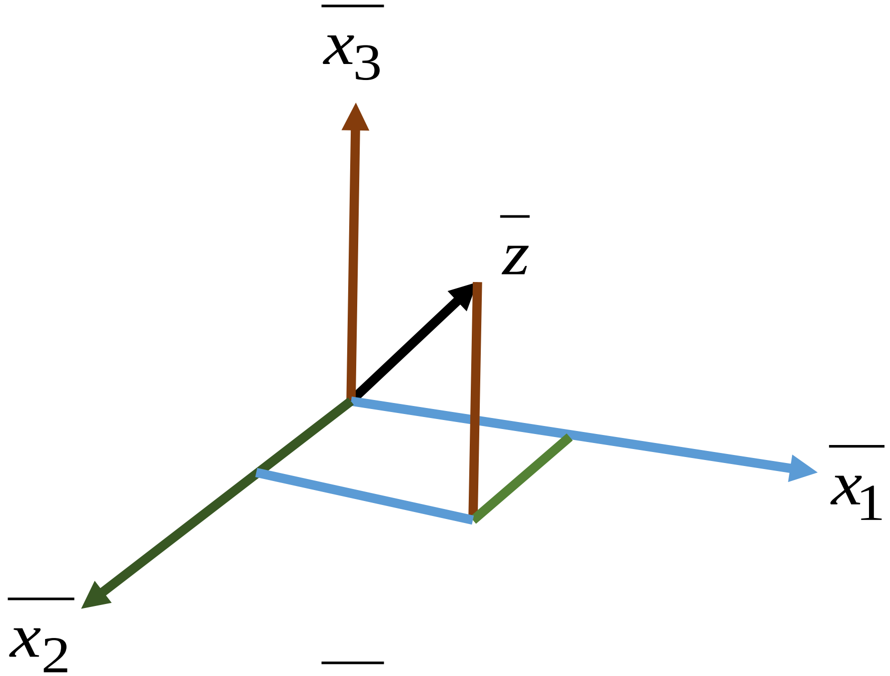
Equivalence theorem

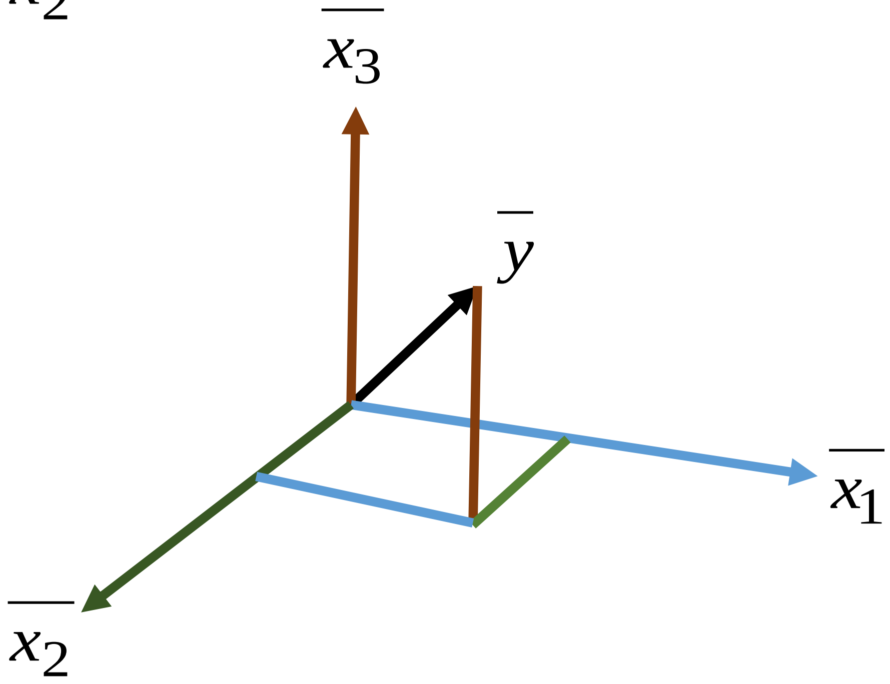
$$\vec{z} = \sum_{i=1}^{\infty} z_i \overrightarrow{x_i} \qquad \vec{y} = \sum_{i=1}^{\infty} y_i \overrightarrow{x_i}$$

For

$$\vec{z} = \vec{y}$$

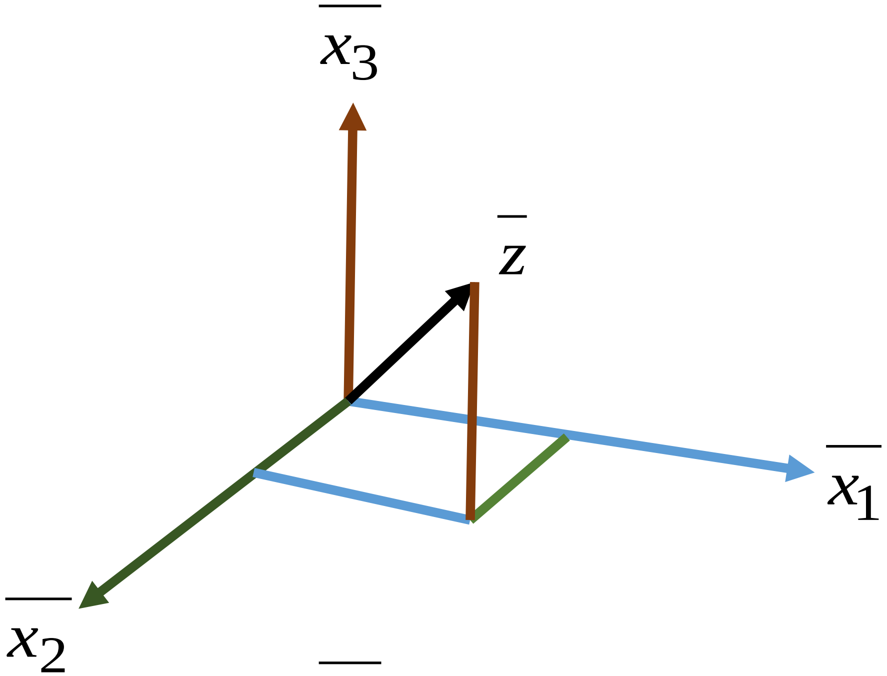
$$z_i = y_i \; \forall \; i$$





Equivalence theorem

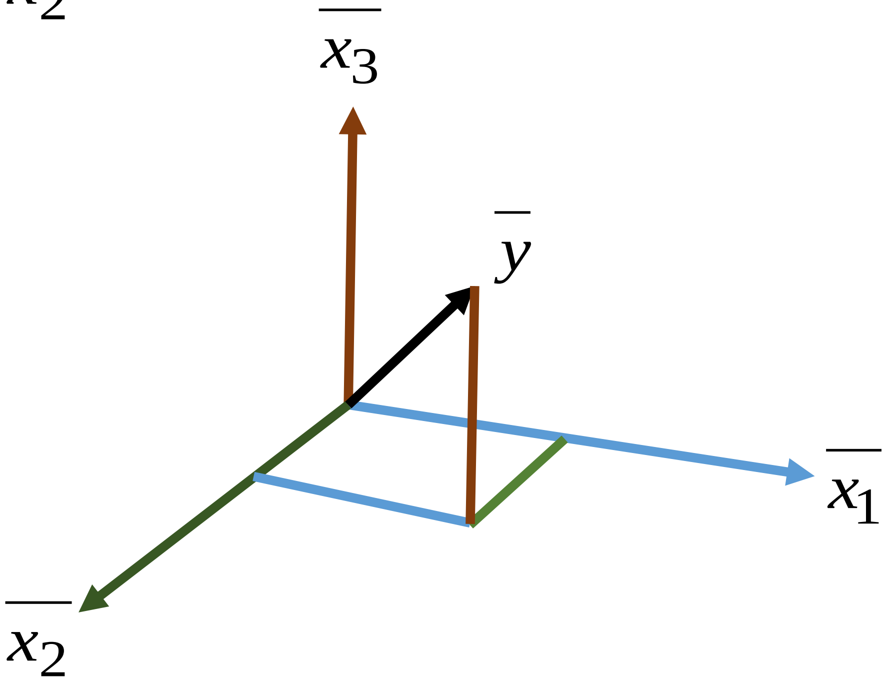
$$x(t) = \sum_{n=1}^{N} x_n \lambda_n(t) \qquad y(t) = \sum_{n=1}^{N} y_n \lambda_n(t)$$



For

$$x(t) = y(t)$$

$$x_n = y_n \; \forall \, n$$



Parseval's theorem

$$\vec{z} = \vec{x} + \vec{y} \quad \& \quad \vec{x}.\vec{y} = 0 \qquad |\vec{z}|^2 = |\vec{x}|^2 + |\vec{y}|^2$$

Applying this in Fourier Series

$$\int_0^T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} \int_0^T |a_k e^{jk\omega_o t}|^2 \, dt = T \sum_{k=-\infty}^{\infty} |a_k|^2$$

Energy in the error signal

$$x(t) = \sum_{n=1}^{N} c_n x_n(t) \qquad e(t) = x(t) - \sum_{n=1}^{N} c_n x_n(t)$$

$$E_e = \int_{t_1}^{t_2} \left(x(t) - \sum_{n=1}^{N} c_n x_n(t) \right)^2 dt$$

$$= \int_{t_1}^{t_2} x^2(t)dt + \sum_{n=1}^{N} \int_{t_1}^{t_2} c_n{}^2 x_n{}^2(t)dt - 2\sum_{n=1}^{N} \int_{t_1}^{t_2} c_n x(t) x_n(t)dt$$

Energy in the error signal

$$E_e = \int_{t_1}^{t_2} x^2(t)dt + \sum_{n=1}^{N} \int_{t_1}^{t_2} c_n{}^2 x_n{}^2(t)dt - 2\sum_{n=1}^{N} \int_{t_1}^{t_2} c_n x(t) x_n(t)dt$$

Energy in the error signal

$$E_e = \int_{t_1}^{t_2} x^2(t)dt + \sum_{n=1}^{N} c_n{}^2 \int_{t_1}^{t_2} x_n{}^2(t)dt - 2\sum_{n=1}^{N} c_n \int_{t_1}^{t_2} x(t)x_n(t)dt$$

$$c_n \boxed{\int_{t_1}^{t_2} x_n{}^2(t)dt} = \int_{t_1}^{t_2} x(t)x_n(t)dt$$

Energy in the error signal

$$E_e = \int_{t_1}^{t_2} x^2(t)dt + \sum_{n=1}^{N} c_n^2 \int_{t_1}^{t_2} x_n^2(t)dt - 2\sum_{n=1}^{N} c_n \int_{t_1}^{t_2} x(t)x_n(t)dt$$

$$c_n E_n = \int_{t_1}^{t_2} x(t)x_n(t)dt$$

Energy in the error signal

$$E_e = \int_{t_1}^{t_2} x^2(t)dt + \sum_{n=1}^{N} c_n{}^2 \boxed{\int_{t_1}^{t_2} x_n{}^2(t)dt} - 2\sum_{n=1}^{N} c_n{}^2 E_n$$

Energy in the error signal

$$E_e = \int_{t_1}^{t_2} x^2(t)dt + \sum_{n=1}^{N} c_n{}^2 E_n - 2\sum_{n=1}^{N} c_n{}^2 E_n$$

Energy in the error signal

$$E_e = \int_{t_1}^{t_2} x^2(t)dt - \sum_{n=1}^{N} c_n{}^2 E_n$$

As N tends to ∞, $E_e \rightarrow 0$

Fourier Series equivalence to Vector Decomposition

Synthesis

$$\hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$x(t) = x(t+T) = \sum_{k} a_k e^{jk\omega_o t}$$

Analysis

$$x = \hat{r}.\hat{x}$$

$$y = \hat{r}.\hat{y}$$

$$z = \hat{r}.\hat{z}$$

$$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\omega_o t}dt$$