MTL-100 Lecture-4

Example: Prove that the sequence $(a_n)_{n=1}^{\infty}$, where $a_n = \left(1+\frac{1}{n}\right)^n$, is convergent.

Solution:

Claim 1: $a_n < a_{n+1} \quad \forall n \in \mathbb{N}$.

$$\iff \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\iff \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} < 1+\frac{1}{n+1} = \frac{n+2}{n+1} \quad -(*)$$

L.H.S of $(*)$ = G.M. of $x_1 = 1, x_2 = x_3 = \cdots = x_{n+1} = 1+\frac{1}{n}$

Also, A.M. of $x_1, x_2, \ldots, x_{n+1}$

$$= \frac{1+n\left(1+\frac{1}{n}\right)}{n+1} = \frac{n+2}{n+1} = \text{R.H.S. of } (*)$$

By the A.M.-G.M. inequality, $(*)$ holds.

Hence $a_n < a_{n+1}$.

Claim 2: $a_n < 3 \quad \forall n \in \mathbb{N}$.

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k \quad \text{(By Binomial theorem)}$$

$$= 1 + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}\cdot\frac{1}{n^k}$$

$$= 1 + \sum_{k=1}^{n} \frac{1}{k!}\, 1\cdot\underbrace{\left(1-\frac{1}{n}\right)}_{<1}\underbrace{\left(1-\frac{2}{n}\right)}_{<1}\cdots\underbrace{\left(1-\frac{k-1}{n}\right)}_{<1}$$

$$< 1 + \sum_{k=1}^{n}\frac{1}{k!} \le 1 + \sum_{k=1}^{n}\frac{1}{2^{k-1}} = 1 + \left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right)$$

$$\left(\because\ k! \ge 2^{k-1} \quad \forall k \in \mathbb{N}\right)$$

$\therefore \quad a_n < 1 + \dfrac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1 + 2\left(1-\frac{1}{2^n}\right)$

$< 1+2 = 3.$

$\therefore \quad a_n < 3 \quad \forall n \in \mathbb{N}.$

By Claim 1 & 2, we see that the sequence $(a_n)_{n=1}^{\infty}$ is a bounded monotone sequence and hence it is convergent.

Remark: We denote by e the limit of the above sequence.

So, $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

$\boxed{2 < e < 3}$

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Lemma: Every sequence has a monotone subsequence.

Proof: Let $(a_n)_{n=1}^{\infty}$ be any sequence.

We shall say that the nth term a_n is a "peak" if $a_n > a_k \quad \forall k > n$.

Case I: Suppose the sequence has infinitely many peaks, say $a_{m_1}, a_{m_2}, a_{m_3}, \cdots$.

Then $a_{m_1} > a_{m_2} > a_{m_3} > \cdots$

Hence the subseq. $(a_{m_k})_{k=1}^{\infty}$ is a decreasing sequence.

Case II: Suppose the sequence $(a_n)_{n=1}^{\infty}$ has only finitely many peaks.
(Note that this includes the no peaks case also).
Let a_{n_1} be beyond all the peaks.
Then a_{n_1} is not a peak.
$\Rightarrow \exists\ n_2 > n_1$ s.t. $a_{n_2} \geq a_{n_1}$
Again, a_{n_2} is not a peak.
$\Rightarrow \exists\ n_3 > n_2$ s.t. $a_{n_3} \geq a_{n_2}$.
Proceeding this way we get a subseq. $(a_{n_k})_{k=1}^{\infty}$ which is a nondecreasing seq.

Proof of Bolzano-Weierstrass theorem:
Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence.
By the previous lemma, let $(a_{n_k})_{k=1}^{\infty}$ be a monotone subsequence.
Then $(a_{n_k})_{k=1}^{\infty}$ is a bounded monotone seq. and hence is convergent.
Thus (a_n) has a convergent subseq.

Application: The sequence $(\sin(n))_{n=1}^{\infty}$ has a convergent subsequence.

Nested Interval Theorem:

Let $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ be nonempty closed and bounded intervals such that

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

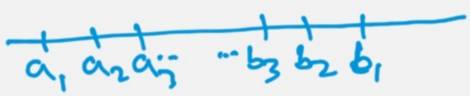
and $\lim_{n\to\infty}(b_n - a_n) = 0$.

Then $\bigcap_{n=1}^{\infty} I_n$ contains precisely one point.

Proof: $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences such that $a_1 \le a_n \le b_n \le b_1$.

$\therefore$ $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are bounded sequences.

Also, $I_n \supseteq I_{n+1}$ $\Rightarrow$ $(a_n)_{n=1}^{\infty}$ is nondecr. & $(b_n)_{n=1}^{\infty}$ is nonincr.



Thus (a_n) & (b_n) are both bounded monotone sequences.

Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$.

Then $0 = \lim_{n\to\infty}(b_n - a_n) = b - a$

$\Rightarrow$ $b = a$.

Also, $a_n \le a$ $\forall n$ & $b_n \ge b$ $\forall n$.

So, $a_n \le a = b \le b_n$ $\forall n \in \mathbb{N}$.

$\Rightarrow$ $a \in [a_n, b_n] = I_n$ $\forall n \in \mathbb{N}$.

$\therefore$ $a \in \bigcap_{n=1}^{\infty} I_n$.

It is easy to show $\bigcap_{n=1}^{\infty} I_n$ cannot contain any other point. (Exercise).

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Remark: If we replace the closed intervals by open intervals, then the result is not true.

e.g. Take $I_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$.

$I_1 \supset I_2 \supset I_3 \supset \cdots$

Also, $\text{length}(I_n) = \frac{1}{n} \to 0$ as $n \to \infty$.

But $\bigcap_{n=1}^{\infty} I_n = \phi$.

(If $x \in \bigcap_{n=1}^{\infty} I_n$, then $0 < x < \frac{1}{n}$ $\forall n$, which is not possible).

Cauchy Sequences

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is said to be a Cauchy sequence if given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - a_m| < \varepsilon$ $\forall n, m \geq N$.

Theorem: Every convergent sequence is Cauchy.

Proof: Let (a_n) be a sequence that converges to L.

Let $\varepsilon > 0$ be given.

Then $\exists N \in \mathbb{N}$ s.t. $|a_n - L| < \frac{\varepsilon}{2}$ $\forall n \geq N$

Now, if $n, m \geq N$, then

$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Hence, (a_n) is a Cauchy seq.

Lemma: Every Cauchy seq. is bounded.

Proof: Let (a_n) be a Cauchy seq.
Then taking $\varepsilon = 1$, we get $N \in \mathbb{N}$ s.t.
$$|a_n - a_m| < 1 \quad \forall\, n, m \geq N.$$
In particular, $|a_n - a_N| < 1 \quad \forall n \geq N$.
$$\Rightarrow \quad |a_n| \leq |a_n - a_N| + |a_N|$$
$$< 1 + |a_N| \quad \forall n \geq N.$$
Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_N|\}$
Then $|a_n| \leq M \quad \forall n \in \mathbb{N}$.
Hence, $(a_n)_{n=1}^{\infty}$ is a bounded seq.

Theorem: Every Cauchy sequence of real numbers is convergent.

Proof: Let $(a_n)_{n=1}^{\infty}$ be a Cauchy seq.
Then by the prev. lemma, $(a_n)_{n=1}^{\infty}$ is bounded.
By the Bolzano Weierstrass theorem, $(a_n)_{n=1}^{\infty}$ has a convergent subseq. $(a_{n_k})_{k=1}^{\infty}$.
Let $a_{n_k} \to L$.
Claim: $\lim_{n\to\infty} a_n = L$.
Let $\varepsilon > 0$ be given.
Since (a_n) is Cauchy, $\exists\, N_1 \in \mathbb{N}$ s.t.
$$|a_n - a_m| < \frac{\varepsilon}{2} \quad \forall\, n, m \geq N_1. \quad \text{—(i)}$$
Also, since $\lim_{k\to\infty} a_{n_k} = L$, $\exists\, N_2 \in \mathbb{N}$ s.t.

$$|a_{n_k} - L| < \frac{\varepsilon}{2} \quad \forall\, k \geq N_2 \text{ --- (ii)}$$

Now choose $K = \max\{N_1, N_2\}$.

If $n \geq K$, then

$$|a_n - L| \leq \underbrace{|a_n - a_{n_K}|}_{< \frac{\varepsilon}{2} \text{ [by (i)]}} + \underbrace{|a_{n_K} - L|}_{< \frac{\varepsilon}{2} \text{ [by (ii)]}}$$

$(K \geq N_1 \Rightarrow n_K \geq K \geq N_1)$.

$\therefore \quad |a_n - L| < \varepsilon \quad \forall\, n \geq K$.

Hence, $\lim_{n\to\infty} a_n = L$.

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Cauchy's criterion for convergence:

A sequence $(a_n)_{n=1}^{\infty}$ of real numbers is convergent if and only if it is Cauchy.

Application:

Let (a_n) be a sequence satisfying

$$|a_{n+1} - a_n| \leq \alpha\, |a_n - a_{n-1}| \quad \forall\, n \geq 2 \text{ and for some } 0 < \alpha < 1$$

Then (a_n) is a Cauchy sequence and hence convergent.

Proof: First, by induction we can prove that

$$|a_{n+1} - a_n| \leq \alpha^{n-1} |a_2 - a_1| \quad \forall\, n \in \mathbb{N}.$$

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Now, for $n \geq m$,

$$|a_n - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots + |a_{m+1} - a_m|$$
$$\leq \alpha^{n-2}|a_2 - a_1| + \alpha^{n-3}|a_2 - a_1| + \dots + \alpha^{m-1}|a_2 - a_1|$$
$$= |a_2 - a_1| \left(\alpha^{m-1} + \alpha^{m} + \dots + \alpha^{n-2}\right)$$
$$= |a_2 - a_1| \alpha^{m-1} \left(1 + \alpha + \alpha^2 + \dots + \alpha^{n-m-1}\right)$$
$$= |a_2 - a_1| \alpha^{m-1} \left(\frac{1 - \alpha^{n-m}}{1 - \alpha}\right)$$
$$< |a_2 - a_1| \frac{\alpha^{m-1}}{1 - \alpha}.$$

For $\varepsilon > 0$ given, we can choose $N \in \mathbb{N}$ s.t.
$|a_2 - a_1| \frac{\alpha^{N-1}}{1-\alpha} < \varepsilon$ (This can be done because $\lim_{k \to \infty} \alpha^k = 0$, as $0 < \alpha < 1$)

Then $n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon$.
$\therefore$ (a_n) is a Cauchy seq. & hence convergent.

Example: Let (a_n) be defined by
$a_1 = 1$, $a_{n+1} = 1 + \frac{1}{a_n}$ for $n \geq 1$.
Then (a_n) is Cauchy.

Proof: Note that $|a_{n+1} - a_n|$

$$= \left|\left(1 + \frac{1}{a_n}\right) - \left(1 + \frac{1}{a_{n-1}}\right)\right|$$
$$= \frac{|a_n - a_{n-1}|}{|a_n a_{n-1}|} \quad \text{for } n \geq 2.$$

Also $n \geq 2 \Rightarrow$ $a_n a_{n-1} = \left(1+\frac{1}{a_{n-1}}\right) a_{n-1}$

$= a_{n-1} + 1 \geq 2$

$(\because a_n \geq 1 \ \forall n)$

$\therefore$ $|a_{n+1} - a_n| = \dfrac{|a_n - a_{n-1}|}{|a_n a_{n-1}|} \leq \frac{1}{2} |a_n - a_{n-1}|$

$\therefore$ The previous result is applicable with $\alpha = \frac{1}{2}$.

Thus (a_n) is Cauchy.