Any solⁿ of
2^{nd} order homogeneous linear ODE

$$x'' + a(t)x' + b(t)x = 0 \quad \text{——} \ (1)$$

is a linear combination of two L.I. solⁿ of DE (1)

$\{x_1(t), x_2(t)\} \rightarrow$ fundamental set of solⁿs

then general solⁿ is

$$x(t) = c_1 x_1(t) + c_2 x_2(t). \quad \text{——} \ (2)$$

IVP (3)
$$\begin{cases} x'' + a(t)x' + b(t)x = 0 \\ x(t_0) = \alpha, \ x'(t_0) = \beta, \quad t_0 \in I \end{cases}$$

$a(t), b(t) \rightarrow$ cts & bdd on I.

- The unique solⁿ of IVP (3) can be obtained by determining the constants c_1 & c_2 in eqⁿ (2) using initial conditions.

2^{nd} Order linear ODE with constant coefficients

Consider

$$x'' + ax' + bx = 0, \quad a, b \in \mathbb{R}$$

— (4)

<u>Recall</u>

$$x' + Kx = 0$$

sol^n $\quad x = c\,e^{-Kt}$

we try $\quad x(t) = e^{mt} \quad$ as sol^n of (4).

Substitute $\quad x(t) = e^{mt} \quad$ in (4), we have

$$m^2 e^{mt} + a m e^{mt} + b e^{mt} = 0$$

$$\underbrace{(m^2 + am + b)}\, \underset{\text{never zero}}{e^{mt}} = 0$$

$m^2 + am + b = 0 \quad \longleftrightarrow \quad$ characteristic polynomial (char. eq^n)

has two roots

$$m_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \qquad m_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

<u>Case 1</u> (Distinct real roots) $\quad a^2 - 4b > 0$

$$x_1(t) = e^{m_1 t}, \quad x_2(t) = e^{m_2 t}$$

are sol^n of DE (4) & they are L.I. since

$$W(x_1,x_2)(0) = \begin{vmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ m_1 & m_2 \end{vmatrix}$$

$$= m_2 - m_1 \neq 0$$

The general is

$$\boxed{x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}}$$

<u>Case 2</u> $(a^2 - 4b < 0)$ (complex roots)

$$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta$$

The sol^{n}s of DE (4) are

$$e^{m_1 t} = e^{(\alpha+i\beta)t} = e^{\alpha t} e^{i\beta t}$$
$$= e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

$$e^{m_2 t} = e^{\alpha t} e^{-i\beta t} = e^{\alpha t}(\cos\beta t - i\sin\beta t)$$

(Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$)

$$x_1(t) = \frac{e^{m_1 t} + e^{m_2 t}}{2} = e^{\alpha t}\cos\beta t$$

$$x_2(t) = \frac{e^{m_1 t} - e^{m_2 t}}{2i} = e^{\alpha t}\sin\beta t$$

real valued functions

$x_1(t)$, $x_2(t)$ $\longrightarrow$ sol^{n}s of DE ④

$\downarrow$ L.I. since

$$W(x_1,x_2)(0) = \begin{vmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \alpha & \beta \end{vmatrix} = \beta \neq 0$$

The general sol^n is

$$\boxed{x(t) = c_1 e^{\alpha t} \cos\beta t + c_1 e^{\alpha t} \sin\beta t}.$$

Case 3 $(a^2 - 4b = 0)$ (Equal real roots)

$$m_1 = m_2 = m = \frac{-a}{2}$$

$$x_1 = e^{mt} = e^{-(a/2)t} \longrightarrow \text{sol}^n \text{ of DE ④}$$

- We have to find another sol^n $x_2(t)$ so that $x_1(t)$ & $x_2(t)$ are L.I.

$\downarrow$ to find

method of reduction of order

Method of reduction of order (if a sol^n of DE is known)

$$x''(t) + a(t)\, x'(t) + b(t)\, x(t) = 0 \quad — ⑤$$

- Determine the soln of ⑤ say $x_2(t)$ when a non-zero soln of ⑤ say $x_1(t)$ is known.

Determinde $v(t)$ s.t. $x_2(t) = v(t)\, x_1(t)$ $\underline{\underline{\neq}}$ $c\, x_1(t)$

is a soln of ⑤.

$$x_2' = v' x_1 + v x_1'$$

$$x_2'' = v'' x_1 + 2v' x_1' + v x_1''$$

Since x_2 satisfies ⑤

$$\left(v'' x_1 + 2v' x_1' + v x_1''\right) + a(t)\left(v' x_1 + v x_1'\right) + b(t)\, v x_1 = 0$$

$$v''(x_1) + v'(2x_1' + a x_1) + v\underbrace{(x_1'' + a x_1' + b x_1)}_{\substack{\parallel \\ 0}} = 0$$

$$v'' x_1 + v'(2x_1' + a x_1) = 0$$

Set $w(t) = v'(t)$ then

$$w' x_1 + w(2x_1' + a x_1) = 0$$

$w' + (2x_1' \ldots a)\, w = 0$ → linear

$$w' + \left(2\frac{x_1'}{x_1} + a\right) w =$$ first order ODE

$$v'(t) = w(t) = c\, \frac{e^{-\int a(t)\,dt}}{x_1^2}$$

Upon integrating

$$v(t) = c \int \frac{e^{-\int a(t)\,dt}}{x_1^2}\, dt$$

By choosing $c = 1$, we get

$$v(t) = \int \frac{e^{-\int a(t)\,dt}}{x_1^2}$$

$$\boxed{\begin{aligned} x_2(t) &= v(t)\, x_1(t) \\ &= x_1 \int \frac{e^{-\int a(t)\,dt}}{x_1^2} \end{aligned}}$$

is a soln of DE (5)

Claim x_1 & x_2 are L.I.

If x_1 & x_2 are L.D. then there exists a constant γ s.t.

$$x_2(t) = \gamma\, x_1(t)$$

$$\int e^{-\int a(t)\,dt} \qquad \gamma\, x_1(t)$$

$$x_1 \int \frac{\cdot}{x_1^2} = \cdot$$

$$\Rightarrow \quad r = \int \frac{e^{-\int a(t)\,dt}}{x_1^2}$$

upon differentiating w.r.t. t,

$$0 = \frac{e^{-\int a(t)\,dt}}{x_1^2}$$

$$\Rightarrow \quad e^{-\int a(t)\,dt} = 0 \qquad \Rightarrow \Leftarrow$$

$\Rightarrow \quad x_1$ & x_2 are L.I.

<u>Case 3</u> (Equal roots)

$$x_1 \doteq e^{mt} = e^{-a/2\, t}$$

$$v(t) = \int \frac{e^{-\int a(t)\,dt}}{x_1^2} = \int \frac{e^{-at}}{e^{-at}} = t$$

Using reduction of order method

$$\therefore \quad x_2(t) = v(t)\, x_1(t)$$

$$= t\, e^{mt}$$

sol^n of DE (4)

The general sol^n of (4) is

$$\boxed{x(t) = c_1 e^{mt} + c_2 t e^{mt}}$$

Exp Solve $x''(t) + x'(t) = 0$

$x(0) = 0$

$x'(0) = 1$

$$x'' + x' = 0$$

$x = e^{mt}$

$$m^2 + m = 0$$

$\Rightarrow$ $m = \underset{m_1}{0}, \underset{m_2}{-1}$

Sol^{n}s are

$$x_1(t) = e^{m_1 t} = 1$$

$$x_2(t) = e^{m_2 t} = e^{-t}$$

The general sol^n is

$$\boxed{x(t) = c_1 + c_2 e^{-t}}$$

$x'(t) = -c_2 e^{-t}$

$x(0) = 0 \quad \Rightarrow \quad c_1 + c_2 = 0$

$x'(0) = 1 \quad \Rightarrow \quad 1 = x'(0) = -c_2$

$\Rightarrow \quad c_1 = 1, \quad c_2 = -1$

The unique sol^n of IVP is

$$\boxed{x(t) = 1 - e^{-t}}$$

Exer Solve the following DE's

1. $x''(t) + 4x(t) = 0$

2. $x''(t) - 2x'(t) + x(t) = 0$

3. $x'' - 6x' + 9x = 0, \quad x(0) = 0, \quad x'(0) = 5.$

The Euler Equation

$$a t^2 \underbrace{x''}_{= \frac{d^2x}{dt^2}} + b t \underbrace{x'}_{= \frac{dx}{dt}} + c x = \cancel{f(t)}^{\,0} \qquad (6)$$

$\downarrow$

homogeneous Euler eq^n

- change independent variable from t to s using $\boxed{t = e^s}$, this will reduce (6) into a ODE with constant coefficients.

$$t = e^s \quad , \quad s = \ln t$$

$$\underline{\underline{\frac{dx}{ds}}} = \frac{dx}{dt}\,\frac{dt}{ds} = x' e^s = x' t$$

$$\frac{d^2x}{ds^2} = \frac{d^2x}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{dx}{dt}\,\frac{d^2t}{ds^2}$$

$$= x'' e^{2s} + x' e^s$$

$$= x'' t^2 + \underline{\underline{x' t}}$$

$$\therefore \quad t^2 x'' = \frac{d^2x}{ds^2} - \frac{dx}{ds}$$

We can rewrite eqn (6) as,

$$a\left(\frac{d^2x}{ds^2} - \frac{dx}{ds}\right) + b\left(\frac{dx}{ds}\right) + cx = 0$$

$$a\,\frac{d^2x}{ds^2} + (b-a)\,\frac{dx}{ds} + cx = 0$$

$\uparrow$

2nd order linear DE with constant

coefficients.

$t = e^s$ or $s = \ln t$

* $x(s) \longrightarrow x(t)$.

3^{rd} order linear ODE with constant coefficients

Consider

$$x''' + a_1 x'' + a_2 x' + a_3 x = 0 \quad \text{—— (1)}$$

$a_1, a_2, a_3 \in \mathbb{R}$

We try $x = e^{mt}$ as a sol^n of (1).

Substituting $x = e^{mt}$ in (1), we get

$$(m^3 + a_1 m^2 + a_2 m + a_3)\underbrace{e^{mt}}_{\text{Never zero}} = 0$$

$$\underbrace{m^3 + a_1 m^2 + a_2 m + a_3}_{\text{cubic eq}^n} = 0$$

Three roots

Let m_1, m_2, m_3 be the roots of this char. eq^n..

Case 1 $\quad m_1, m_2, m_3$ as distinct real roots

$x_1(t) = e^{m_1 t}$, $x_2(t) = e^{m_2 t}$ & $x_3(t) = e^{m_3 t}$

are sol^n of DE ① & they are L.I. since

$$W(x_1, x_2, x_3)(0) = \begin{vmatrix} x_1(0) & x_2(0) & x_3(0) \\ x_1'(0) & x_2'(0) & x_3'(0) \\ x_1''(0) & x_2''(0) & x_3''(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \\ m_1^2 & m_2^2 & m_3^2 \end{vmatrix}$$

$$= (m_3 - m_2)(m_2 - m_1)(m_3 - m_1)$$

$$\neq 0$$

The general sol^n is

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} + c_3 e^{m_3 t}.$$

Case 2 $\quad m_1 \neq m_2 = m_3$

$x_1 = e^{m_1 t}$, $x_2 = e^{m_2 t}$, $x_3(t) = t e^{m_2 t}$

are sol^n of (1) & they are L.I. since

can be found by using method of reduction of order

$$W(x_1,x_2,x_3)(0) = \begin{vmatrix} 1 & 1 & 1 \\ m_1 & m_2 & 1 \\ m_1^2 & m_2^2 & 2m_2 \end{vmatrix}$$

$\neq 0$ (compute)

General sol^n

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} + c_3 t e^{m_2 t}$$

<u>Case 3</u> $m_1 = m_2 = m_3 = m$

Sol^ns are

$$x_1 = e^{mt}, \quad x_2(t) = t e^{mt}, \quad x_3(t) = t^2 e^{mt}$$

(check: Applying method of reduction of order)

$$W(x_1,x_2,x_3)(0) = \begin{vmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ m^2 & 2m & 2 \end{vmatrix} = 2 \neq 0$$

$\therefore$ $x_1, x_2, x_3 \rightarrow$ L.I.

The general sol^n is

The general sol^n is

$$\boxed{x(t) = c_1 e^{mt} + c_2 t e^{mt} + c_3 t^2 e^{mt}}.$$

<u>Case 4</u> $m_1, m_2 \to$ complex roots

$m_3 \to$ real root

say $m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta$

The sol^n are

$$x_1(t) = e^{\alpha t} \cos\beta t$$

$$x_2(t) = e^{\alpha t} \sin\beta t$$

$$x_3(t) = e^{m_3 t}$$

Check : $x_1, x_2, \& x_3$ are L.I.

The general sol^n is

$$\boxed{x(t) = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t) + c_3 e^{m_3 t}}.$$

<u>Exer</u>

<u>Solve</u>

① $t^2 x'' + t x' - x = 0$

② $x''' - x'' + 100x' - 100x = 0$

$x(0) = 4$

$x'(0) = 11$

$x''(0) = -299$