PYL101

Electromagnetic Waves and Quantum Mechanics

Tutorial Sheet 4 (L7-L8)

Problem 1:

A particle of mass m is confined in a box of unit length in one dimension. It is described by the wavefunction

$\psi(x) = \sqrt{\frac{8}{5}}$ sin πx (1 + cos πx), for $0 \leq x \leq 1$ and zero outside this interval. Find the expectation value of energy.

Solution: We can write

$$\psi(x) = \sqrt{\frac{8}{5}}\,\sin\pi x\,(1+\cos\pi x)$$

$$= \frac{2}{\sqrt{5}}\left(\sqrt{2}\sin(\pi x)\right) + \frac{1}{\sqrt{5}}\left(\sqrt{2}\sin(2\pi x)\right)$$

$$= \frac{2}{\sqrt{5}}\phi_1(x) + \frac{1}{\sqrt{5}}\phi_2(x) \qquad \because \phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

In bra-ket notation we can write,

$$|\psi\rangle = \frac{2}{\sqrt{5}}|\phi_1(x)\rangle + \frac{1}{\sqrt{5}}|\phi_2(x)\rangle$$

$$\langle E\rangle = \langle\psi|\hat{H}|\psi\rangle$$

$$= \left(\frac{2}{\sqrt{5}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2|\right)\hat{H}\left(\frac{2}{\sqrt{5}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle\right)$$

Solution of problem 1 continued…

$$= \left(\frac{2}{\sqrt{5}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2|\right)\hat{H}\left(\frac{2}{\sqrt{5}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle\right)$$

$$= \left(\frac{2}{\sqrt{5}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2|\right)\left(\frac{2}{\sqrt{5}}\frac{\pi^2\hbar^2}{2m}|\phi_1\rangle + \frac{1}{\sqrt{5}}\frac{2^2\pi^2\hbar^2}{2m}|\phi_2\rangle\right) \quad \because \hat{H}|\phi_n\rangle = \frac{n^2\pi^2\hbar^2}{2ma^2}|\phi_n\rangle$$

$$\langle E\rangle = \frac{4\pi^2\hbar^2}{5m}$$

Problem 2:

The ground state energy of a particle of mass m in an infinite square well potential of width a is E. If the width of the well is reduced to $a/2$, find the new ground state energy.

Solution:

Initially, the ground state energy of the particle in an infinite square well potential of width a is

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$$

If the width of the wall is reduced to $a/2$ then the new ground state energy

$$E_1' = \frac{\pi^2\hbar^2}{2m({}^{a}/_{2})^2}$$

$$E_1' = 4\,\frac{\pi^2\hbar^2}{2ma^2}$$

$$E_1' = 4E$$

Problem 3:

The state of a particle of mass m in a one-dimensional infinite potential well in the interval 0 to L is given by a normalized wave function $\psi(x) = \sqrt{\frac{2}{L}}\left(\frac{3}{5}\,sin\left(\frac{2\pi x}{L}\right) + \frac{4}{5}\,sin\left(\frac{4\pi x}{L}\right)\right)$. If its energy is measured, what are the possible outcomes. Also, calculate the average value of energy.

Solution: We have

$$\psi(x) = \sqrt{\frac{2}{L}}\left(\frac{3}{5}sin\left(\frac{2\pi x}{L}\right) + \frac{4}{5}sin\left(\frac{4\pi x}{L}\right)\right)$$

We can write it in bra-ket notation as,

$$|\psi(x)\rangle = \frac{3}{5}|\psi_2(x)\rangle + \frac{4}{5}|\psi_4(x)\rangle$$

Measurement will give us the energy

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

Therefore, possible energy outcomes corresponding to give wave functions are

$$E_2 = \frac{4\pi^2\hbar^2}{2mL^2} \text{ and } E_4 = \frac{16\pi^2\hbar^2}{2mL^2}$$

Solution of problem 3 continued

The average value of energy is given by the expectation

$$\langle\psi(x)|\hat{H}|\psi(x)\rangle = \left(\frac{3}{5}\langle\psi_2(x)| + \frac{4}{5}\langle\psi_4(x)|\right)\hat{H}\left(\frac{3}{5}|\psi_2(x)\rangle + \frac{4}{5}|\psi_4(x)\rangle\right)$$

$$= \left(\frac{3}{5}\langle\psi_2(x)| + \frac{4}{5}\langle\psi_4(x)|\right)\left(\frac{3}{5}\frac{4\pi^2\hbar^2}{2mL^2}|\psi_2(x)\rangle + \frac{4}{5}\frac{16\pi^2\hbar^2}{2mL^2}|\psi_4(x)\rangle\right)$$

$$= \frac{146\pi^2\hbar^2}{25mL^2}$$

Problem 4:

A free particle of mass m moves along the x-direction at $t = 0$. The state of the particle is given by,

$\psi(x,0) = \frac{1}{(2\pi\alpha)^{\frac{1}{4}}}\, exp\left[-\frac{x^2}{4\alpha^2} + ix\right]$, where α is a real constant. Find the expectation value of momentum in this state.

Solution:

$$\psi(x,0) = \frac{1}{(2\pi\alpha)^{\frac{1}{4}}} exp\left[-\frac{x^2}{4\alpha^2} + ix\right]$$

$$\langle\hat{p}\rangle = \int_{-\infty}^{+\infty} \psi^\star(x)\left(-i\hbar\frac{\partial}{\partial x}\psi(x)\right)dx$$

$$= \frac{1}{(2\pi\alpha)^{1/2}}\int_{-\infty}^{+\infty} e^{\left[-\frac{x^2}{4\alpha^2}-ix\right]}\left(-i\hbar\frac{\partial}{\partial x}e^{\left[-\frac{x^2}{4\alpha^2}+ix\right]}\right)dx$$

$$= \frac{(-i\hbar)}{(2\pi\alpha)^{1/2}}\int_{-\infty}^{+\infty} e^{\left[-\frac{x^2}{2\alpha^2}\right]}\left(-\frac{x}{2\alpha^2}+i\right)dx$$

$$= \frac{(i\hbar)}{(2\pi\alpha)^{1/2}}\int_{-\infty}^{+\infty} e^{\left[-\frac{x^2}{2\alpha^2}\right]}\left(\frac{x}{2\alpha^2}\right)dx \;+\frac{\hbar}{(2\pi\alpha)^{1/2}}\int_{-\infty}^{+\infty} e^{\left[-\frac{x^2}{2\alpha^2}\right]}dx$$

$$= 0 + \frac{\hbar}{(2\pi\alpha)^{1/2}}\sqrt{2\pi}\alpha = \hbar\sqrt{\alpha} \qquad \because \int_{-\infty}^{+\infty} e^{-\alpha x^2}x^n\,dx = \frac{\Gamma(n+1/2)}{\alpha^{n+1/2}}$$

Problem 5:

A wavefunction represents the normalized state of a free particle, $\psi(x,0) = N\, e^{\frac{-x^2}{2a^2}+ik_0x}$. Find the normalization constant.

Solution:

The state $\psi(x,0)$ is normalized,

$$\int_{-\infty}^{+\infty} \psi^{\star}(x,0)\psi(x,0)\,dx = 1$$

$$|N|^2 \int_{-\infty}^{+\infty} e^{\frac{-x^2}{a^2}}\,dx = 1$$

$$2|N|^2 \int_{0}^{\infty} e^{\frac{-x^2}{a^2}}\,dx = 1$$

$$2|N|^2 \frac{\Gamma(1/2)}{2\left(\frac{1}{a^2}\right)^{\frac{1}{2}}} = 1 \qquad \because \int_0^{\infty} e^{-\alpha x^2} x^n\,dx = \frac{\Gamma\left({}^{n+1}/_{2}\right)}{2\alpha^{{}^{n+1}/_{2}}}$$

$$|N|^2 \frac{\sqrt{\pi}}{{}^{1}/_{a}} = 1 \qquad \because \Gamma(1/2) = \sqrt{\pi}$$

$$N = \frac{1}{\pi^{1/4} a^{1/2}}$$