COL 352 Introduction to Automata and Theory of Computation

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Lecture 10: Pumping Lemma: Examples

Converse of Pumping Lemma

Converse of Pumping Lemma

Example

Consider

$$L = \overbrace{\{ca^n b^n\}}^{L_1} \cup \overbrace{\{c^k w \mid k \neq 1, w \text{ starts with } a \text{ or } b\ \}}^{L_2}$$

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- For L_1, the long enough word has to be ca^n.
- But then, consider the partition of $w = xyz$ where $x = \varepsilon, y = c$
- Then if you pump up,i.e., $c^k a^n b^n \in L_2 \subseteq L$
- And if you pump down, i.e., $c^0 a^n b^n = a^n b^n \in L_2 \subseteq L$

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for each n
there exist words x, y, z such that $xyz \in L$ and $|y| \geq n$,
for each breakup of y into three words uvw such that $v \neq \epsilon$, then
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In our earlier version of pumping lemma, $x, z = \epsilon$
Exercise: Does this help avoid the example in the previous slide?

Pumping Lemma for regular languages

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$$\begin{aligned} xuv^2wz &= a^n b^j b^{2r} b^e \\ &= a^n b^{j+2r+e} \\ &= a^n b^{n+r} \end{aligned}$$

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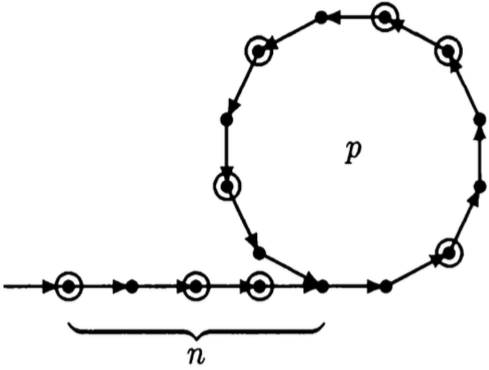

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Consider the homomorphism $h : \Sigma^* \to \{a\}$ by $h(b) = a$ for every $b \in \Sigma$

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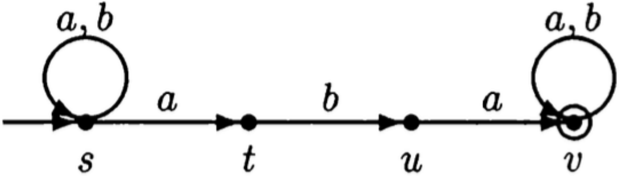

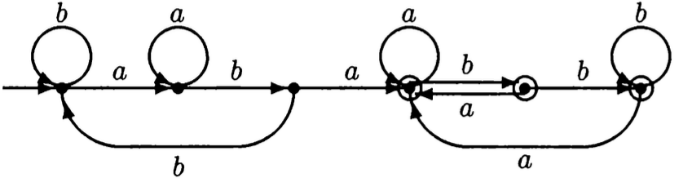

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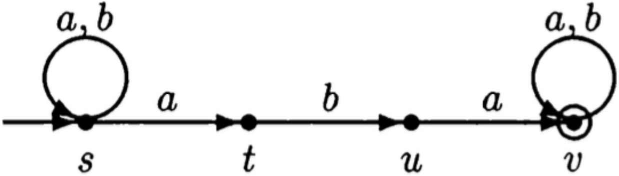

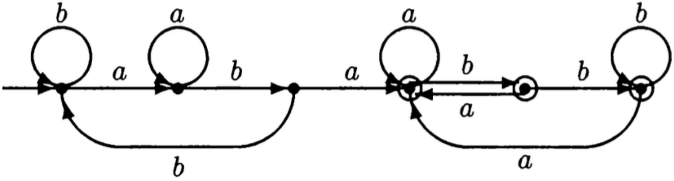

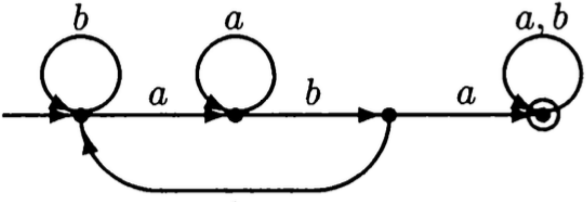

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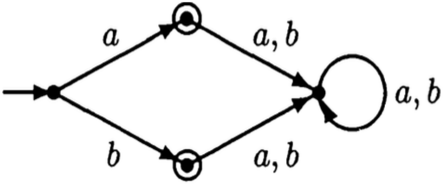
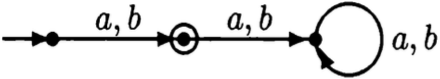
  - Collapse "equivalent" states.

Minimization : Example 2





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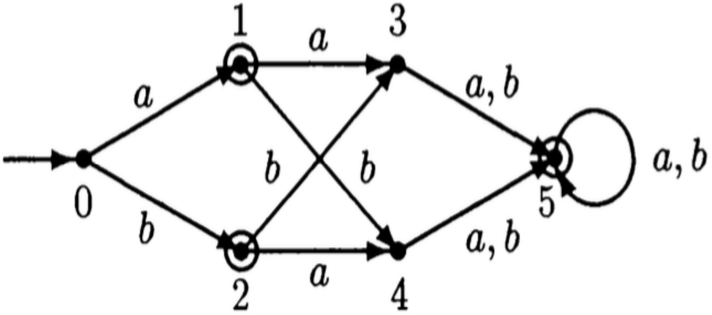
Minimization Example 3



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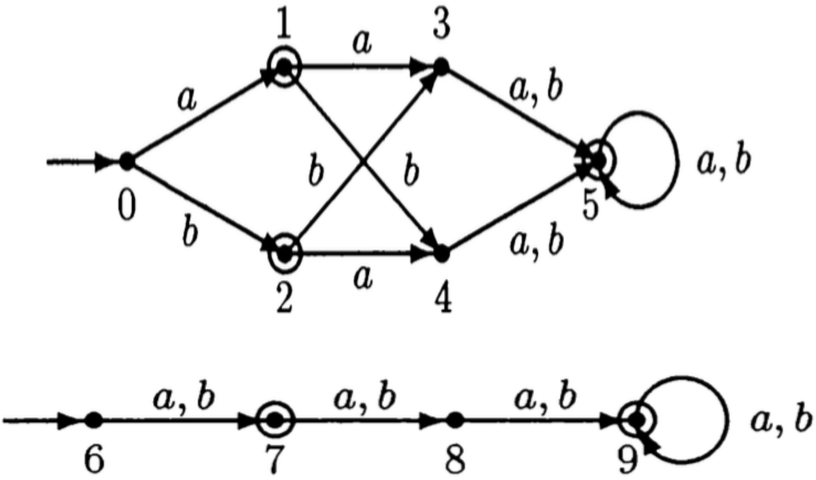

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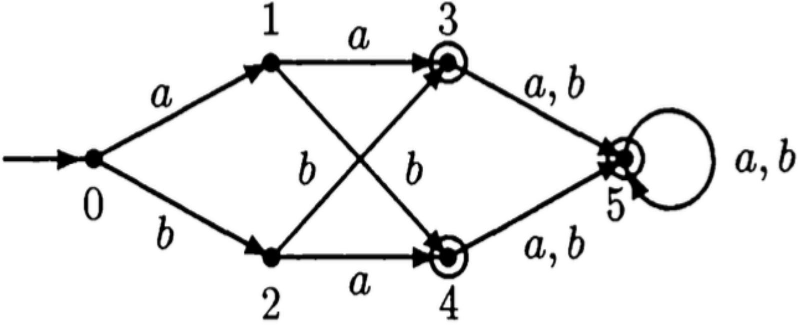

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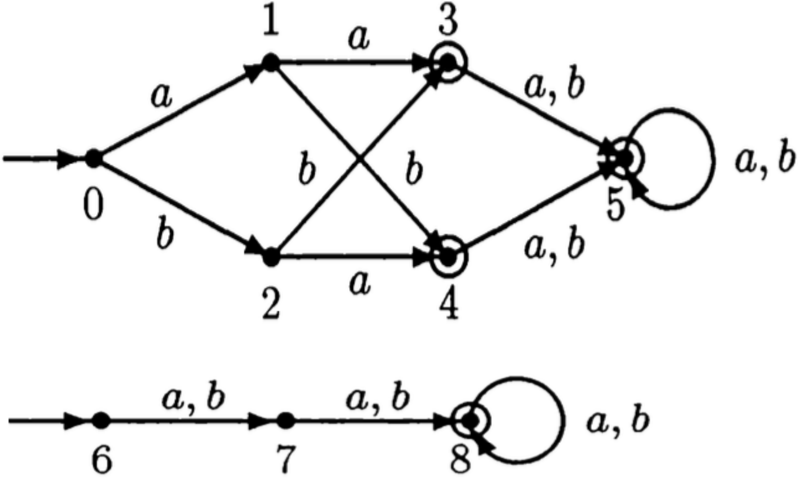

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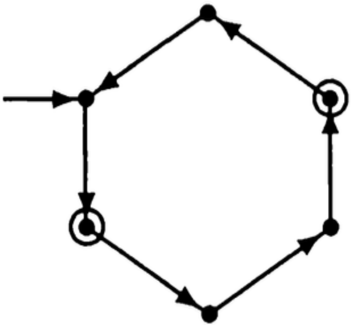

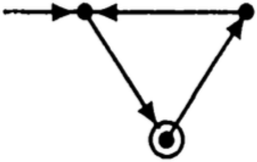

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