INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS
MTL 100 (CALCULUS): SEMESTER I 2020 − 21
Major Examination

DATE: 11/02/2021 Total Marks: 40 Time: 9.30 am – 12:00 pm

MARKS WILL BE AWARDED ONLY FOR THOSE ANSWERS WITH PROPER JUSTIFICATION

Question 1: Using the transformation $x = u + \frac{v}{2}$ and $y = v$, evaluate

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)^2 e^{(2x-y)^3} dxdy. \qquad [5]$$

Question 2: Using spherical coordinates, find the volume of the solid in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 36$ and the cones $z^2 = 3(x^2 + y^2)$ and $z^2 = x^2 + y^2$. [5]

Question 3: Discuss whether the following improper integral converges or diverges:

$$\int_0^\infty \frac{1}{\sqrt{x^5 + x}} dx. \qquad [4]$$

Question 4: Let $f : (a, b) \to \mathbb{R}$ be a twice differentiable function such that $f''(x)$ is continuous on (a, b). Given $c \in (a, b)$, show that

$$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c). \qquad [4]$$

Question 5: Let $f(x) = \dfrac{1}{(1+x)^2}$, $x \in (0, 2)$. Find the nth degree Taylor polynomial $P_n(x)$ of f **about the point** $x = 1$. Further, show that

$$\lim_{n \to \infty} |f(x) - P_n(x)| = 0, \quad \forall x \in \left(\frac{1}{2}, 2\right). \qquad [5]$$

Question 6: Determine **all values** of the constant $\alpha > 0$ for which the limit

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y^3}{|x|^3 + |y|^\alpha} \text{ exists.} \qquad [5]$$

Question 7: Let $f(x, y) = x^3 y - xy^2 + cx^2$, where c is a constant. Find **the value of** c if the function f increases fastest at the point $p_0 = (3, 2)$ in the direction of the vector $A = 2\hat{i} + 5\hat{j}$. [4]

Question 8: Using Lagrange multipliers, find the maximum and minimum values of $f(x, y) = xyz$ subject to the constraint $x^2 + 2y^2 + 3z^2 = 1$. [4]

Question 9: Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that f is differentiable and f' is continuous on $(0, 1)$. Compute the following limit :

$$\lim_{n \to \infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n \int_0^1 f(x)\, \mathrm{d}x. \qquad [4]$$

—–ALL THE BEST—–