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Q1). a) given, $AB = 5I_{n \times n}$ — (1)

post-multiply both sides by A

$ABA = 5A$

Taking det both sides in (1)

$\det(AB) = \det(5I_{n \times n})$

$\det(A)\det(B) = 5^n$

clearly $\det(A) \neq 0$ and $\det(B) \neq 0$

$\therefore$ A and B are invertible matrices

let $C = A^{-1} \Rightarrow CA = I$

multiplying C both sides as a pre-matrix

$\underbrace{CA}_{I}BA = C(5A)$

$BA = 5I$

$\therefore \quad BA = 5I_{n \times n}$

Hence proved

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b) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Rank = ~~4~~ 3

$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$

B will also be Rank 3 at max

$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix} =$

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Q2) a) $B = \{1-x,\ 1+x^2,\ 1-x^3,\ 1+x-x^3\}$

For B to be a basis

$$\alpha_1(1-x) + \alpha_2(1+x^2) + \alpha_3(1-x^3) + \alpha_4(1+x-x^3) = 0$$

means all $\alpha_i = 0$

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_4) + (\alpha_4-\alpha_1)x + \alpha_2 x^2 + (-\alpha_3-\alpha_4)x^3 = 0$$

$\boxed{\alpha_2 = 0}$ [coeff of x^2]

$\alpha_4 = \alpha_1$ [for coeff of x]

$\alpha_3 = -\alpha_4$ [for coeff of x^3]

$$\alpha_1 + \underbrace{\alpha_2}_{0} + \underbrace{\alpha_3 + \alpha_4}_{0} = 0 \quad \text{[constant]}$$

$\Rightarrow \alpha_1 = 0 \Rightarrow \alpha_4 = 0 \Rightarrow \alpha_3 = 0$

Hence all $\alpha_i = 0$ for the polynomial to be 0. So the set is linearly independent since $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0,0,0,0)$

Now B has 4 elements which are linearly independant with coeff corresponding to $1, x, x^2, x^3$ in atleast one of the elements of B and dimension of $P_3 = 4$ Hence B is a basis of P_3.

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5) $\alpha(1-x) + \beta(1+x^2) + \gamma(1-x^3) + \delta(1+x-x^3)$

$= 1 + x + x^2 + x^3$

$1 = \alpha + \beta + \gamma + \delta$ — (1)

$1 = -\alpha + \delta$ — (2)

$1 = \beta$ — (3)

$1 = -\delta - \gamma$ — (4)

(2) + (4)

$2 = -\alpha - \gamma$

Now in (1)

$1 = \underbrace{(\alpha + \gamma)}_{-2} + 1 + \delta$

$\delta = 2$

so $\gamma = -3$

$\alpha = 1$

so coordinates $\Rightarrow \begin{bmatrix} 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}$

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Q3) For $T : \mathbb{C}^2 \to \mathbb{C}^2$, if T is linear transformation then its form is always like

$T(z_1, z_2) = (az_1 + bz_2, cz_1 + dz_2)$

$T \circ T(z_1, z_2) = T(az_1 + bz_2, cz_1 + dz_2)$

$= (a(az_1 + bz_2) + b(cz_1 + dz_2), c(az_1 + bz_2) + d(cz_1 + dz_2))$

$= ((a^2 + bc)z_1 + (ab + bd)z_2, (ac + cd)z_1 + (bc + d^2)z_2)$

Given, $T^2(z_1, z_2) = (-z_1 + 2z_2, -z_2)$

$\Rightarrow a^2 + bc = -1$, $ab + bd = 2$,
$ac + cd = 0$, $bc + d^2 = -1$

$ac + cd = 0 \Rightarrow c = 0$ or $a = -d$

Case I : $c = 0$

$a^2 = -1 \Rightarrow a = \pm i$

$d^2 = -1 \Rightarrow d = \pm i$

$ab + bd = 2 \Rightarrow b(a + d) = 2$

a, d cannot be of same sign

$\Rightarrow b(\pm 2i) = 2 \Rightarrow b = \mp i$

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$\Rightarrow$ In case - I, $(a, b, c, d) = (\pm i, \mp i, 0, \pm i)$

$(a, b, c, d) = (i, -i, 0, i)$ or $(-i, i, 0, -i)$

Case - II $a = -d$

$ab + bd = 2 \Rightarrow (a+d) b = 2$

$\Rightarrow 0 = 2$

$\Rightarrow a \neq -d$

This case has no sol^n

The possible linear transformation are

① $T(z_1, z_2) = (iz_1 - iz_2, iz_2)$

② $T(z_1, z_2) = (-iz_1 + iz_2, -iz_2)$

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Q4)

b) $W_1 = \text{span}\{(4,3,2,1),(1,1,1,2),(3,2,1,-1)\}$

$W_2 = \text{span}\{(1,0,3,2),(4,3,2,1)\}$

$W_1 + W_2 = \text{span}\{\{(4,3,2,1),(1,1,1,2),(3,2,1,-1)\} \cup \{(1,0,3,2),(4,3,2,1)\}\}$

To know the dimension of this span, ($\text{span}(S_1 \cup S_2)$) we find the row reduced form of the matrix given below

$$\begin{vmatrix} 1 & 1 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix} \Rightarrow R_5 \Rightarrow R_5 - R_2$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -5 \\ 0 & -1 & -2 & -5 \\ 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$R_3 \to R_3 - R_2$

~~$R_4 \to R_4 - R_2$~~

We can switch R_4 & R_3 & then $R_3 \to R_3 - R_2$

$$\begin{vmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -5 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Now we can see that there are 3 non zero rows in the row reduced form of the matrix

which means $\dim(W_1 + W_2) = 3$

Q 5) Let $\sum_{i=1}^{4} x_i = X$

$T(x_1, x_2, x_3, x_4) = (X, X, X, X)$

If λ is an eigen value then $\to$

$(X, X, X, X) = \lambda(x_1, x_2, x_3, x_4)$

$\Rightarrow \lambda x_1 = \lambda x_2 = \lambda x_3 = \lambda x_4 = X$

case $1 \Rightarrow \lambda = 0$

Then $0 = 0 = 0 = 0 = X = x_1 + x_2 + x_3 + x_4$

Hence $x_1 + x_2 + x_3 + x_4 = 0$

Now, here we choose x_1, x_2, x_3 arbitrarily such that $x_4 = -(x_1 + x_2 + x_3)$

So $V_0 = (x_1, x_2, x_3, (-x_1 + x_2 + x_3))$

which clearly has dim $= 3$

Case 2 $\lambda \neq 0$

$\lambda x_1 = \lambda x_2 = \lambda x_3 = \lambda x_4 = X$

since $\lambda \neq 0$

$x_1 = x_2 = x_3 = x_4 = \frac{X}{\lambda}$

Let $x_1 = a$

Then $a = \frac{x}{\lambda} = \frac{(4a)}{\lambda}$

$\therefore$ $\lambda = 4$ since $(a \neq 0)$

so $V_4 = (a, a, a, a)$ where $a \in R$

which has basis as $\rightarrow (1, 1, 1, 1)$

and hence dimension $= 1$

Now dim of $V = R^4 = 4$

sum of dimension of eigenspace $= 3 + 1$

$= 4 = \dim(R^4)$

Since these both are equal

Therefore T is diagonalizable and hence there exists a basis β such that $[T]_\beta$ is diagonal

that basis is

$$\left((1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (1, 1, 1, 1) \right)$$