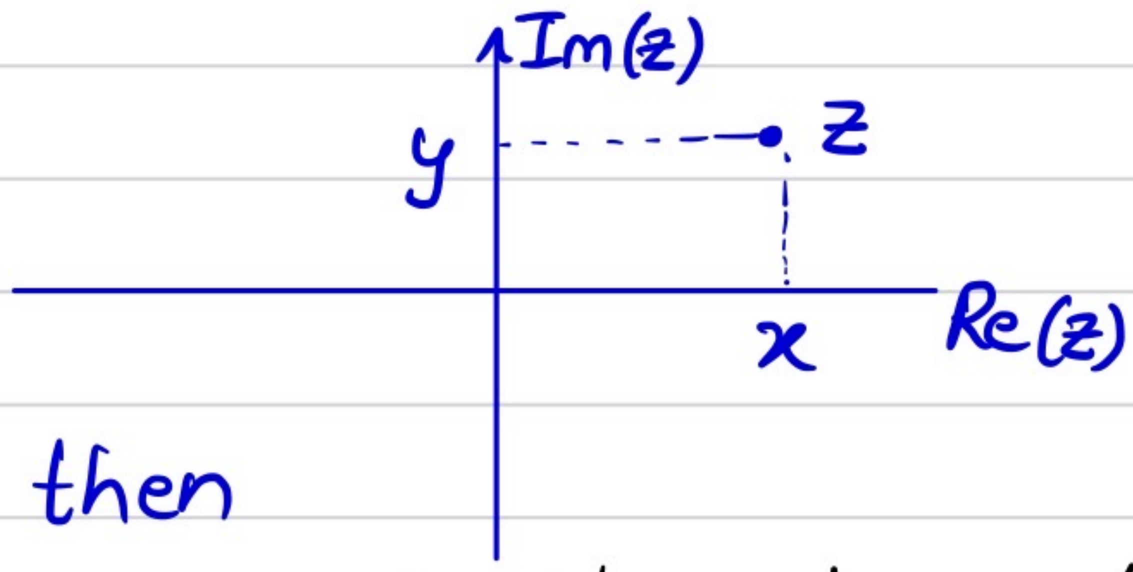
Complex Analysis

Complex numbers : $z = x + iy$ with $i = \sqrt{-1}$ & $x, y \in \mathbb{R}$

$x = \mathrm{Re}(z)$ Real part of z.
$y = \mathrm{Im}(z)$ Imaginary part of z.



Complex plane $\angle z$

Equality : If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ then

$z_1 = z_2$ means $x_1 = x_2$ & $y_1 = y_2$

$z = 0 \equiv \mathrm{Re}(z) = 0 = \mathrm{Im}(z)$

Inequalities among complex numbers are meaningless in general.

Addition : $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Additive identity : $0 \equiv 0 + i0$
Additive inverse : $-z = (-1).z = -x + i(-y)$

Multiplication : $z_1 . z_2 = (x_1 + iy_1).(x_2 + iy_2)$
$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$ $\{$ using $i^2 = -1$

Addition & multiplication defined above are associative as well as commutative

- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$; $z_1 + z_2 = z_2 + z_1$
- $z_1.(z_2.z_3) = (z_1.z_2).z_3$; $z_1.z_2 = z_2.z_1$

Multiplication is distributive over Addition

- $z_1.(z_2 + z_3) = z_1.z_2 + z_1.z_3$

Multiplicative Identity : $1 \equiv 1 + i0$

Multiplicative Inverse : $z^{-1}.z = z.z^{-1} = 1 \equiv (1 + i0)$

$z = x + iy \Rightarrow$ $\boxed{z^{-1} = \dfrac{x - iy}{x^2 + y^2}}$

Exists $\forall$ complex numbers except Additive identity $0 \equiv 0 + i0$.

- $(z_1.z_2)^{-1} = z_1^{-1}.z_2^{-1}$

Subtraction : $z_1 - z_2 \equiv z_1 + (-z_2)$

Division : $\dfrac{z_1}{z_2} \equiv z_1 \cdot z_2^{-1} = z_2^{-1} \cdot z_1$

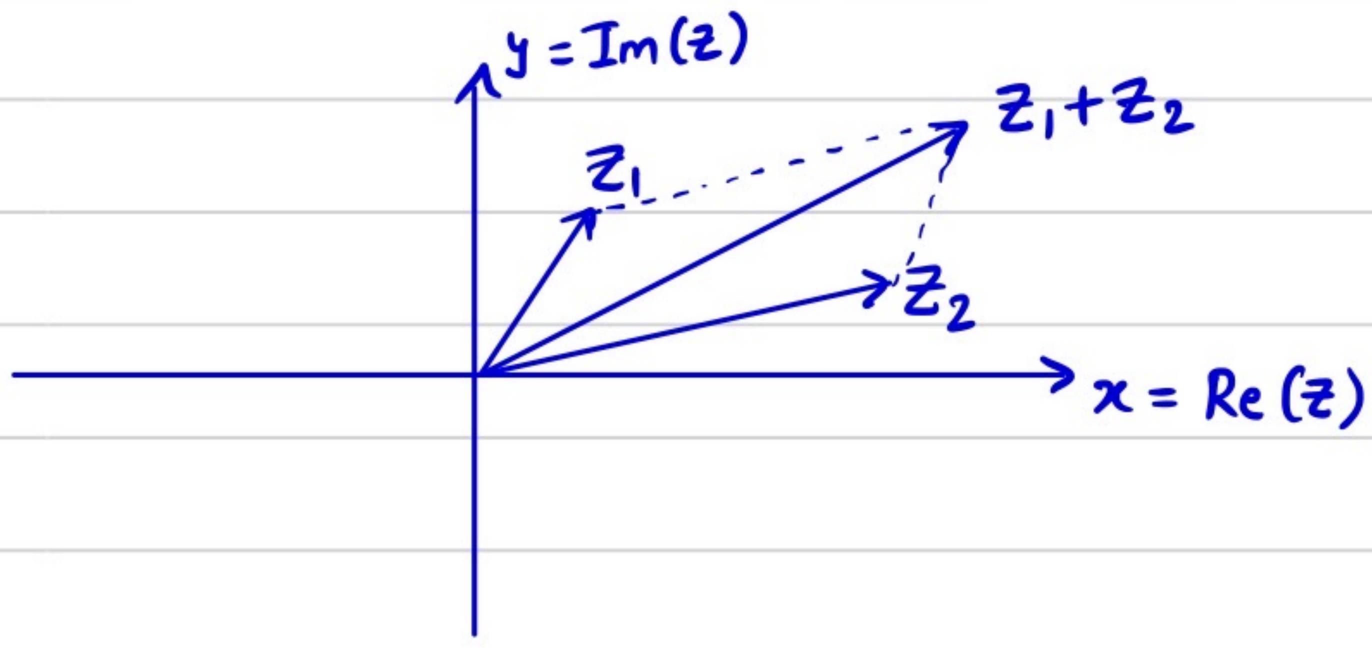
- $z^n \equiv \underbrace{z \cdot z \cdot \ldots z}_{n \text{ times}}$

e.g. $z = x + iy$ the $z^2 = (x+iy)\cdot(x+iy)$
$$= (x^2 - y^2) + i\, 2xy$$

z^n can computed using Binomial expansion & $i^2 = -1$.
$$z^n = (x+iy)^n = \sum_{m=0}^{n} {}^nC_m \, x^n (iy)^{n-m}$$

Now use $i^2 = -1 \Rightarrow i^3 = -i, \quad i^4 = +1$.

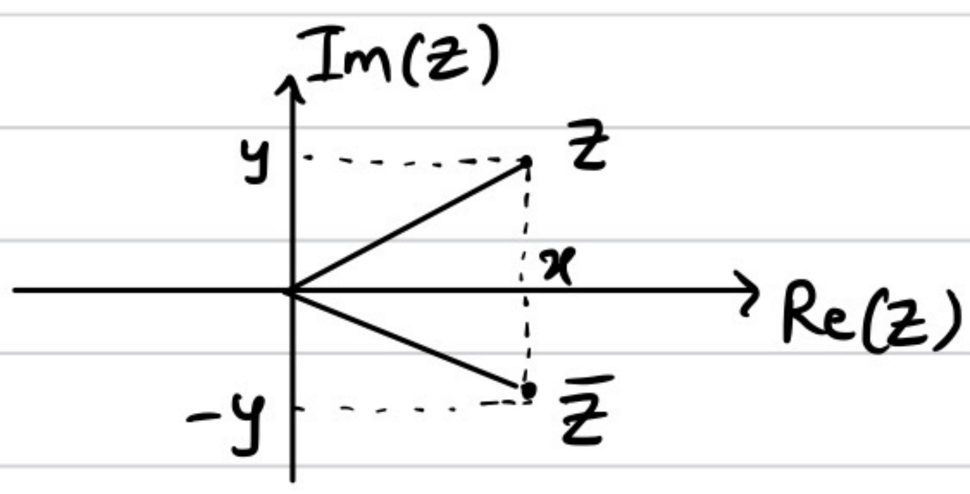
★ Complex number can naturally be associated with vectors in the two dimensional plane.



- The addition of complex numbers is the same as vector addition
- The length/Norm of the vector associated with complex number $z = x + iy$ is referred to as the Modulus or Absolute value of z, denoted by
$$|z| = \sqrt{x^2 + y^2}$$

Triangle inequality : $|z_1 + z_2| \leq |z_1| + |z_2|$
→ follows from triangle inequality for vectors.

Also implies : $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$



Complex conjugation : $\forall \; z = x + iy$
$$z^* = \bar{z} \equiv x - iy$$

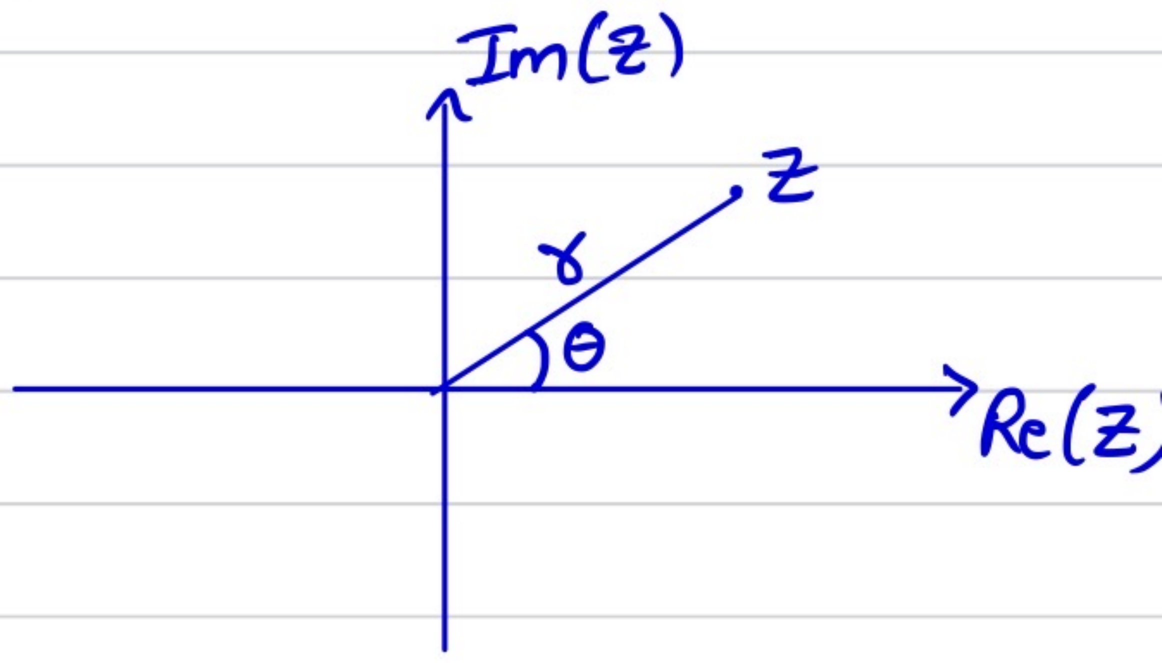
$\Rightarrow$
- $\bar{\bar{z}} = z$
- $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$
- $\overline{(z_1 \cdot z_2)} = \bar{z}_1 \cdot \bar{z}_2$

- $\overline{(z^{-1})} = (\bar{z})^{-1}$

Further, we can write $\quad Re(z) = \frac{1}{2}(z+\bar{z})$

$$Im(z) = \frac{1}{2i}(z-\bar{z}) = -\frac{i}{2}(z-\bar{z})$$

$$|z| = |\bar{z}| = \sqrt{z\bar{z}}$$



Polar representation & Exponential form:

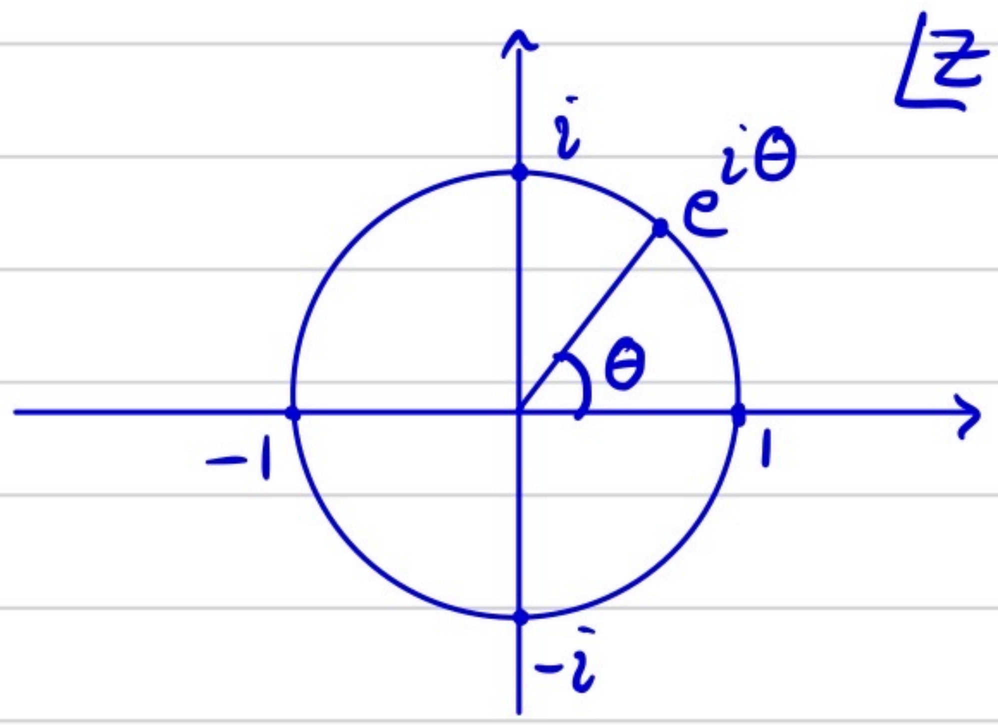
$$z = x + iy$$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta} \longrightarrow \text{Euler's formula } \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

with $\quad r = |z| = \sqrt{x^2+y^2} \quad \in (0,\infty)$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$: argument of z ; $\arg(z)$

→ defined upto $2n\pi$ addition, since $\tan\theta$ is 2π periodic.

$\theta \in (-\pi, \pi)$ is referred to as the principal value of the argument of z.



Note that: $\quad e^{i\pi} = -1 \quad , \quad e^{i2\pi} = +1$

$e^{i\pi/2} = i \quad , \quad e^{-i\pi/2} = -i$

If $\quad z_1 = r_1 e^{i\theta_1} \quad , \quad z_2 = r_2 e^{i\theta_2} \quad$ then

- $z_1 . z_2 = (r_1 r_2) . e^{i(\theta_1+\theta_2)}$
- $z_1^{-1} = \left(\frac{1}{r_1}\right) e^{-i\theta_1}$
- $\frac{z_1}{z_2} = z_1 . z_2^{-1} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}$
- $z_1^n = (r_1^n) . e^{in\theta_1} \quad \forall \quad n \in \mathbb{Z}$

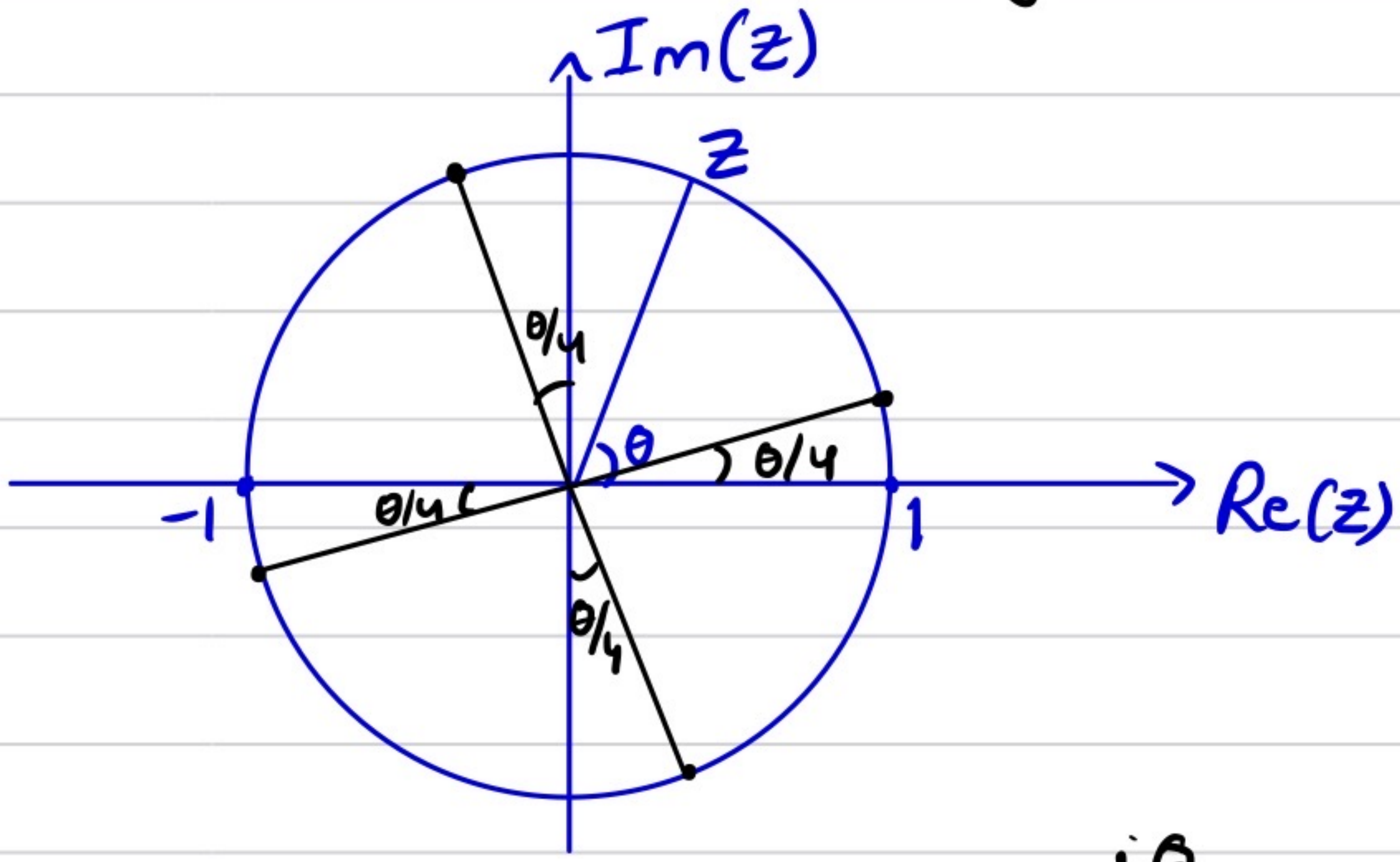
Roots of Complex numbers:

Let $z = re^{i\theta} \equiv re^{i(\theta + 2m\pi)}$ $\forall\ m \in \mathbb{Z}$

then their are exactly n n-th roots of z

$$(z)^{\frac{1}{n}} \equiv (r)^{\frac{1}{n}} e^{\frac{i(\theta+2m\pi)}{n}} \quad \forall\ m \in (0, 1, 2, \cdots (n-1))$$

Note that for $m = n, n+1, n+2, \cdots$ the roots just repeat again.



4-th roots of $e^{i\theta}$

Lets look at the n-th roots of $z = +1 = e^{i\,2m\pi}$

$$z^{1/n} = e^{i\,\frac{2m}{n}\pi} \qquad m = (0, 1, 2 \cdots (n-1))$$

Denote

$\omega_n = e^{i2\pi/n}$, then the n-roots are

$$\{1, \omega_n, \omega_n^2, \omega_n^3, \cdots \omega_n^{n-1}\} \quad ; \quad \omega_n^n = 1$$

Note that all the roots satisfy (as they must)

$$(\omega_n^m)^n = +1.$$

A useful property of these roots is

$\star$ $$\boxed{1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} = 0}$$ Ex: prove this.