Lecture 21
Signals and Systems (ELL205)

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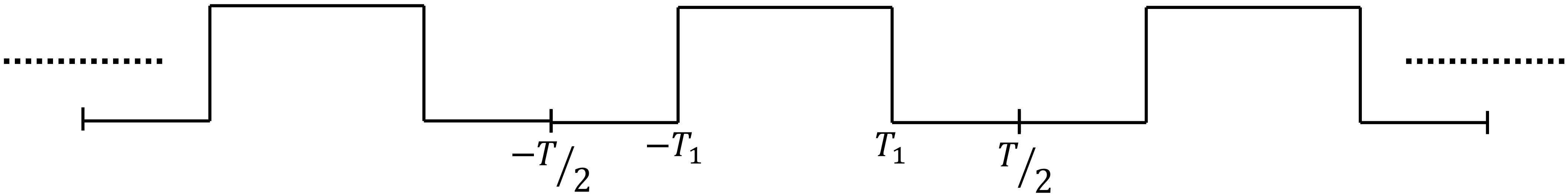
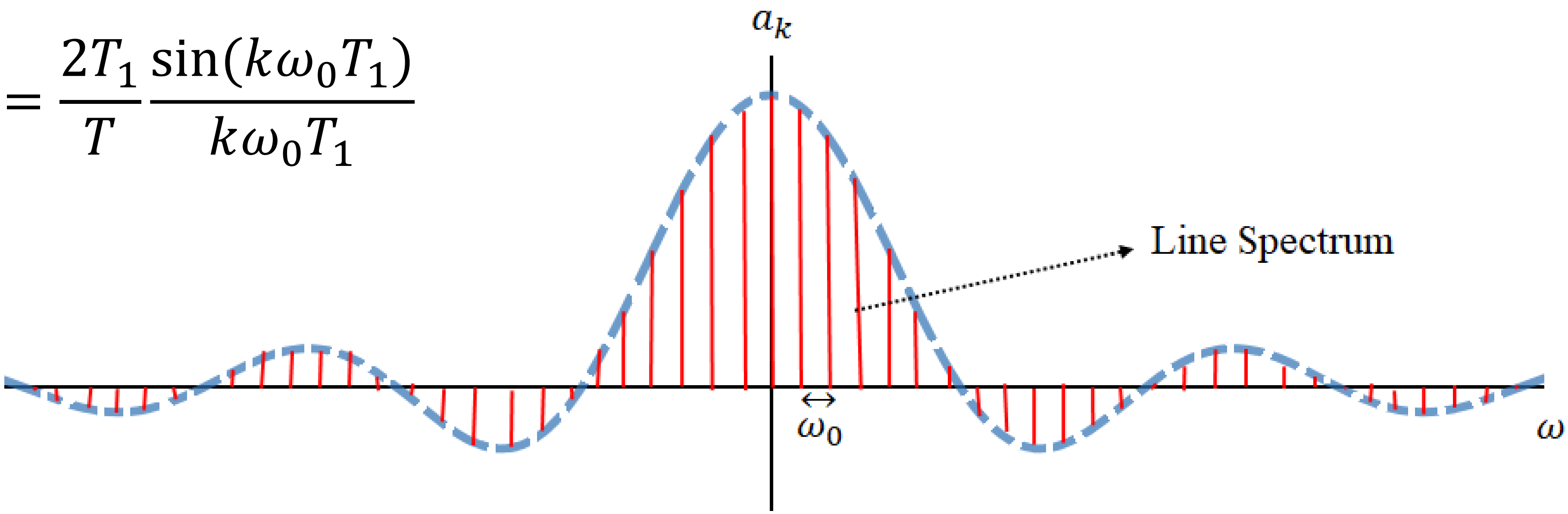
Lecture 21: Introduction to Fourier Transforms

Continuous-Time Fourier Series



$$a_k = \frac{2T_1}{T}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}$$



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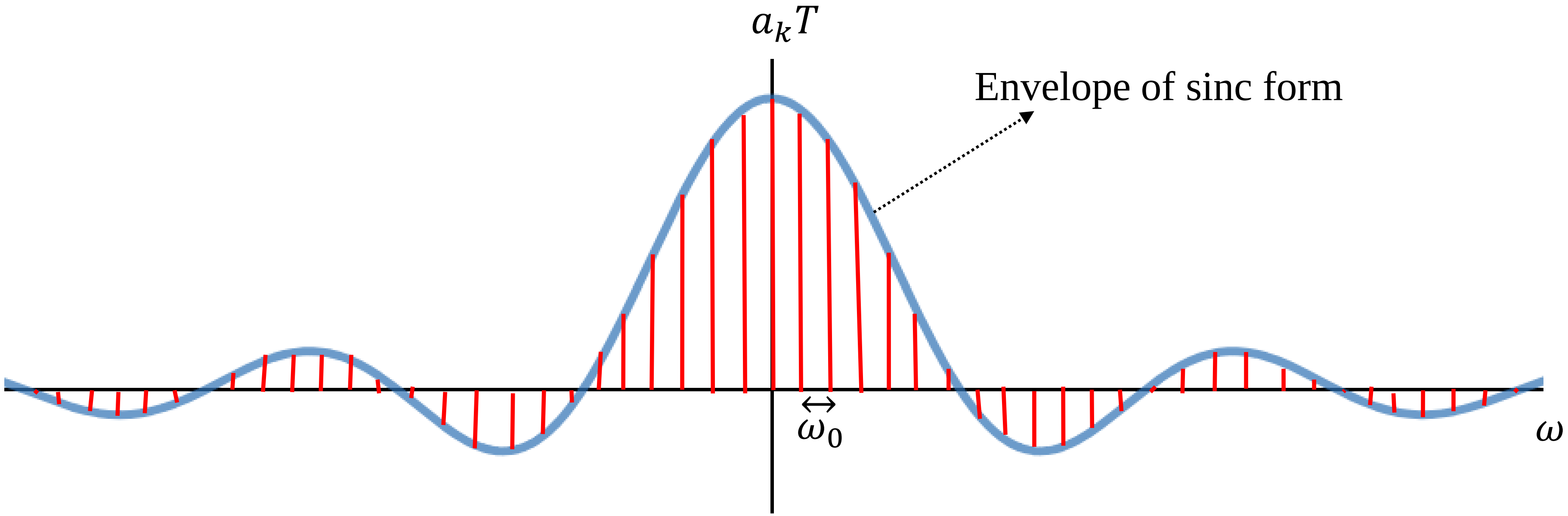
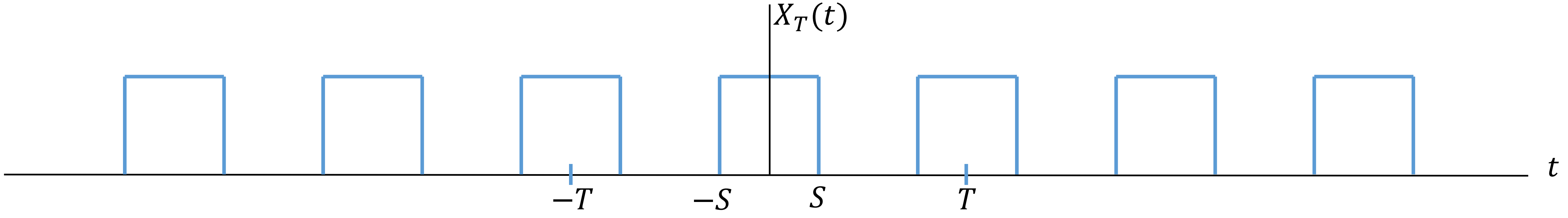
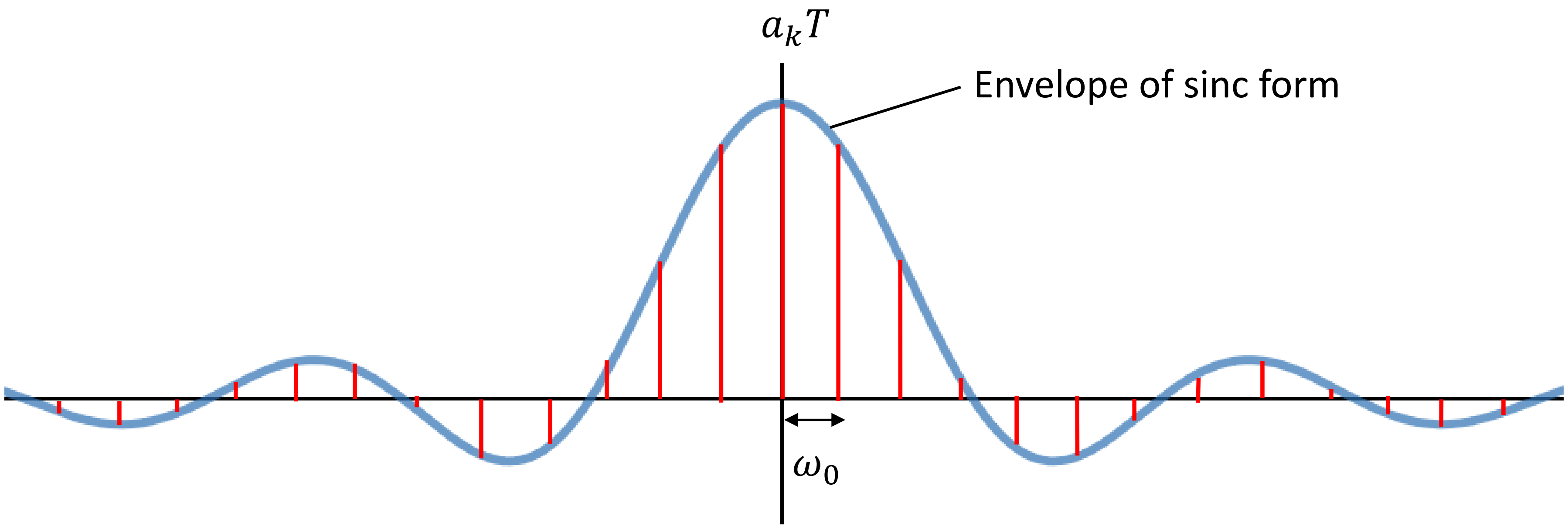


Effect of changing period





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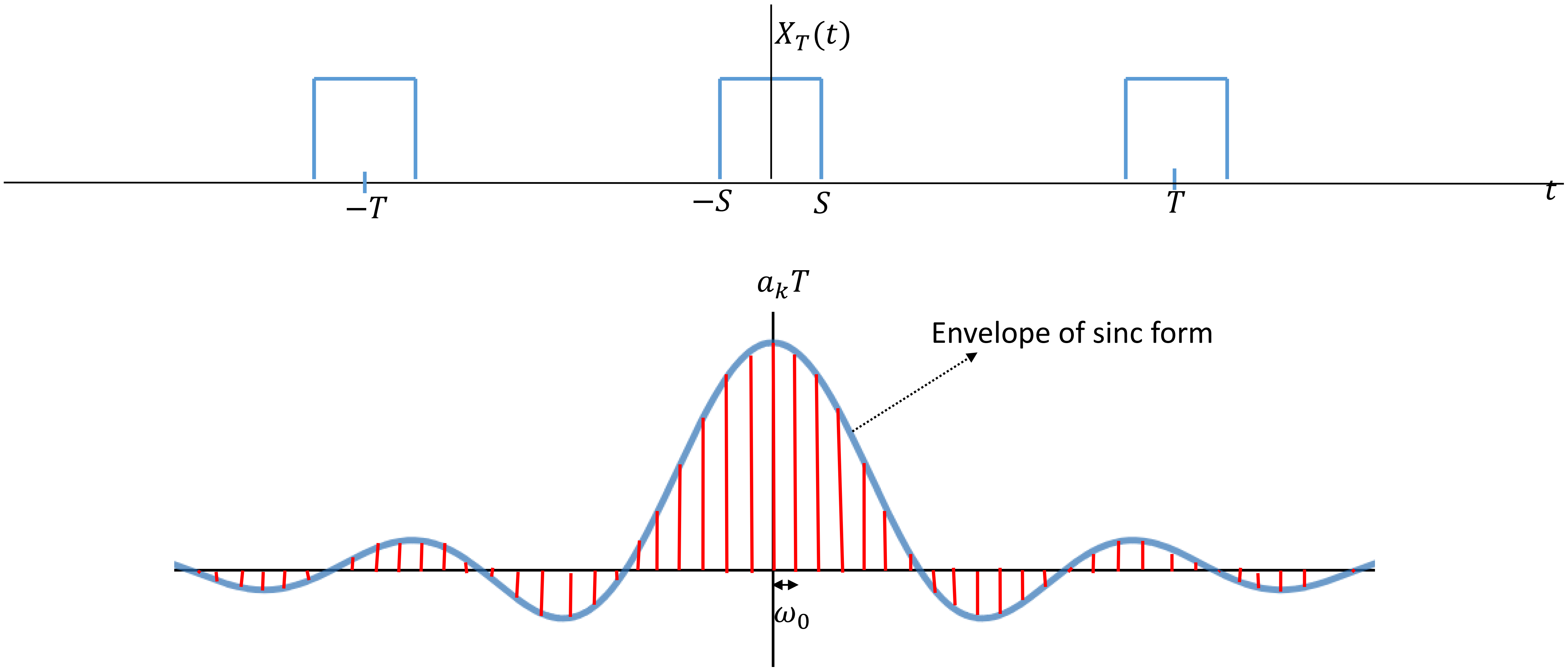

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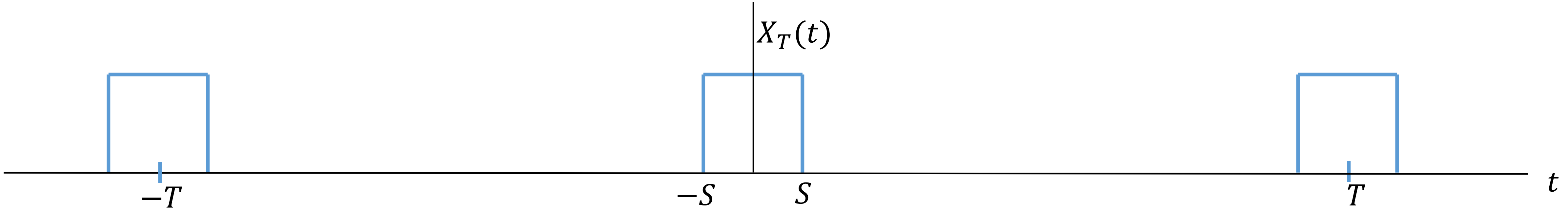
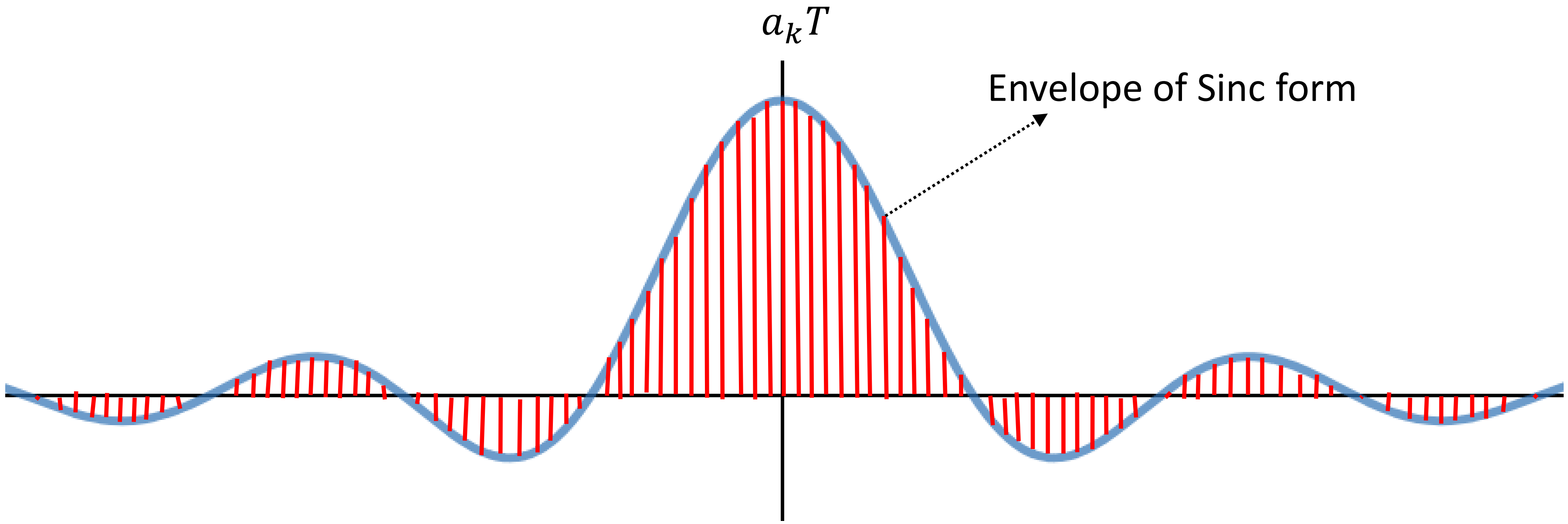
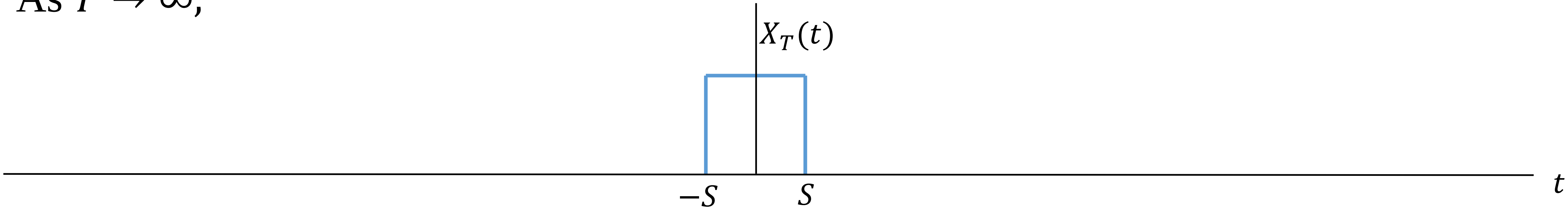


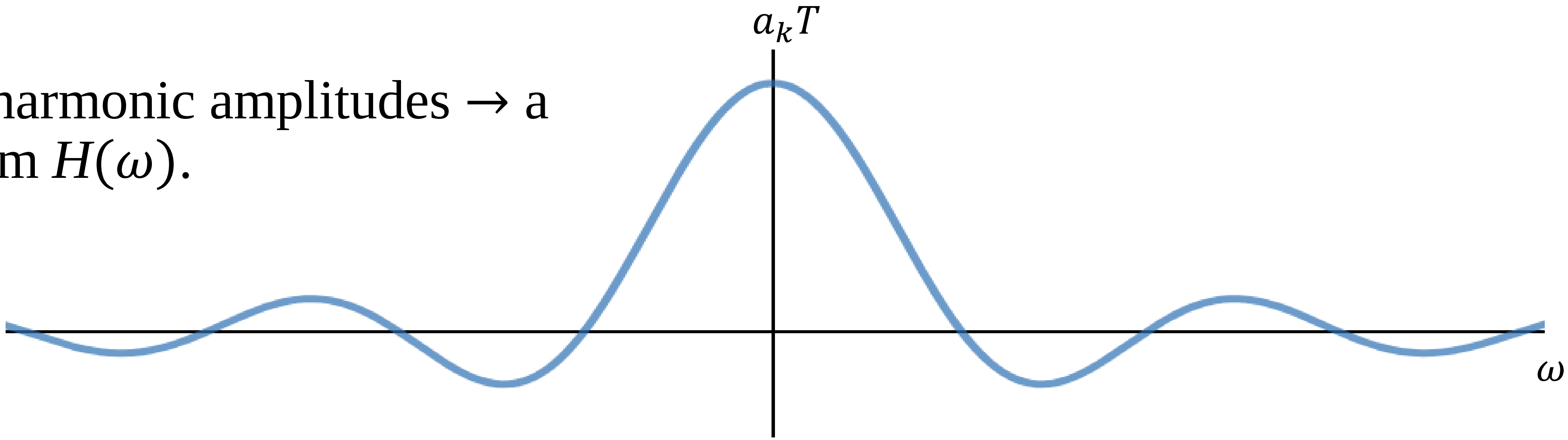


Effect of changing period

As $T \to \infty$,



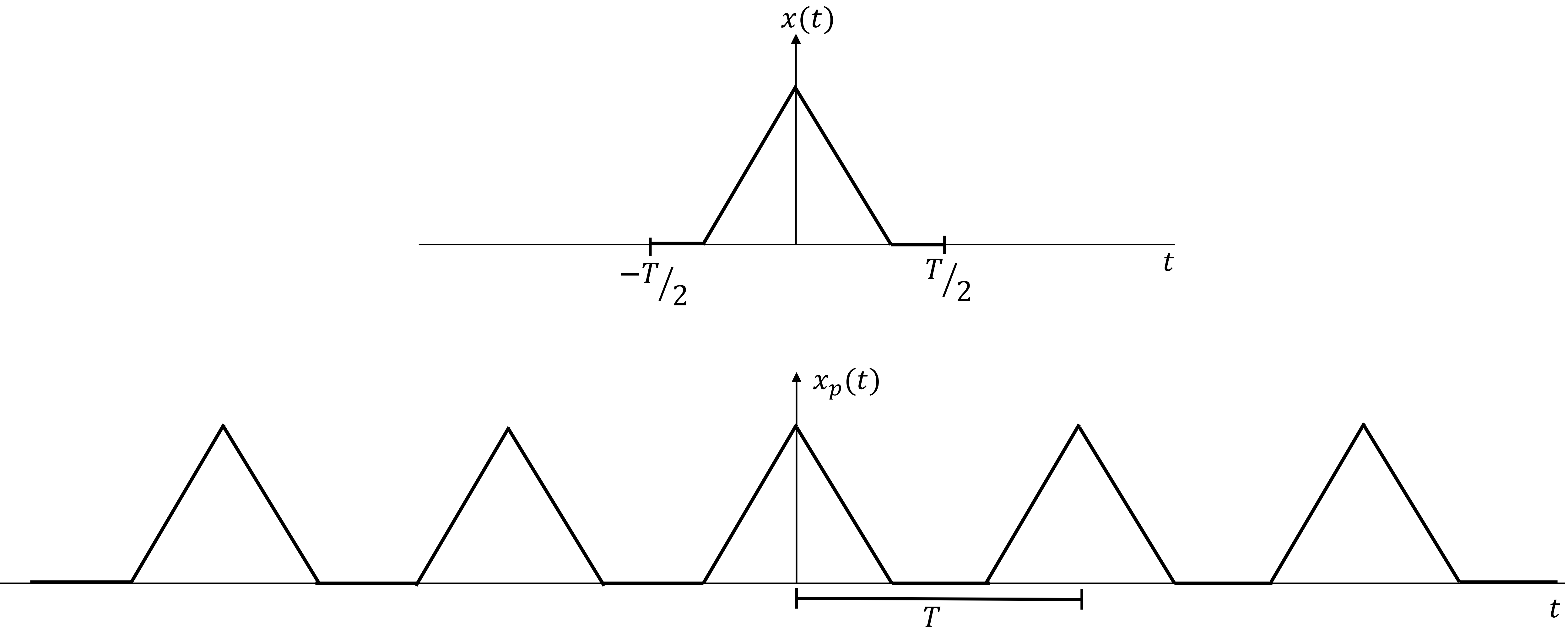
discrete harmonic amplitudes $\to$ a continuum $H(\omega)$.



Summary

Period	Spectrum	Analysis tool
Aperiodic	Continuous	Fourier Transforms
Periodic	Discrete (line)	Fourier Series

Analysis and Synthesis Eq. of Fourier Transform



$x_p(t) \rightarrow$ periodic version of $x(t)$ with period T (periodic extension)

Analysis & Synthesis

$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x_p(t) e^{-jk\omega_0 t} dt$$

Analysis & Synthesis

Step 1: $a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_p(t)e^{-jk\omega_0 t}dt$

Step 2: Since $x(t)$ and $x_p(t)$ are same in $-\frac{T}{2}$ to $+\frac{T}{2}$ so $x(t)$ replaces $x_p(t)$

$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\omega_0 t}dt$

Step 3: Since $x(t)$ exists only in $-\frac{T}{2}$ to $+\frac{T}{2}$ so a_k is:

$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}dt$

Analysis & Synthesis

Step 4: $a_k = \frac{1}{T} \int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t} dt$ $\Rightarrow$ $a_k = \frac{1}{T} X(k\omega_o)$

$$H(\omega) \triangleq \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau\,,$$

and thus $X(\omega) \triangleq \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$

Analysis & Synthesis

Step 5: $$x_p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(k\omega_0) e^{jk\omega_0 t}$$ using $$a_k = \frac{1}{T} X(k\omega_o)$$

Step 6: Substituting $T = \frac{2\pi}{\omega_0}$, $x_p(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0) e^{jk\omega_0 t} \omega_0$

Step 7: As $T \to \infty$, $x_p(t) = x(t)$, $k\omega_0 \to \omega$, $\omega_0 \to d\omega$, $\sum \to \int$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Analysis & Synthesis

Synthesis:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$\omega \to t$ Inverse Fourier Transform

Analysis:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$t \to \omega$ Fourier Transform

Fourier Series

Fourier Transform

Physical meaning of $X(\omega)$

$$a_k = \frac{1}{T} X(k\omega_0)$$

$$a_k = \frac{\omega_0}{2\pi} X(k\omega_0)$$

When $T \to \infty$,

$$a_\omega = \frac{X(\omega)d\omega}{2\pi}$$

$$2\pi a_\omega = X(\omega)d\omega$$

$X(\omega)$ is amplitude spectral density

Convergence of Fourier Transforms

$$\mathrm{X}(\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-W}^{W} X(\omega) e^{j\omega t} d\omega$$

$$e(t) = x(t) - \tilde{x}(t)$$

- $x(\tau)$ square integrable,

  If $\int_{-\infty}^{\infty} |x(\tau)|^2 d\tau < \infty$ Then $\int_{-\infty}^{\infty} \lim_{W \to \infty} |e(t)|^2 \, dt \to 0$

Convergence of Fourier Transforms

- Dirichlet conditions

  If $\int_{-\infty}^{\infty}|x(\tau)|d\tau < \infty$ and $x(t)$ is "well-behaved"

  Then $\lim_{W\to\infty} e(t) \to 0$ except at discontinuities

Well-behaved $\triangleq$

1) Finite no. of maxima and minima in a finite time period
2) Finite no. of finite discontinuities in a finite time period

These are sufficient conditions not necessary conditions. Ex. $u(t)$ is neither absolutely integrable nor an energy signal but has a Fourier Transform.

Try yourself:

Fourier Transform of $e^{-at}u(t)$ is $X(\omega)$ where $a > 0$

How many statements are correct?

1) $X(\omega) = \frac{1}{a+j\omega}$	2) $\|X(\omega)\|$ is an even function
3) $\angle X(\omega)$ is an odd function	4) $X(\omega)$ is a low pass filter

Try yourself:

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} e^{-j\omega t} dt\, u(t)$$

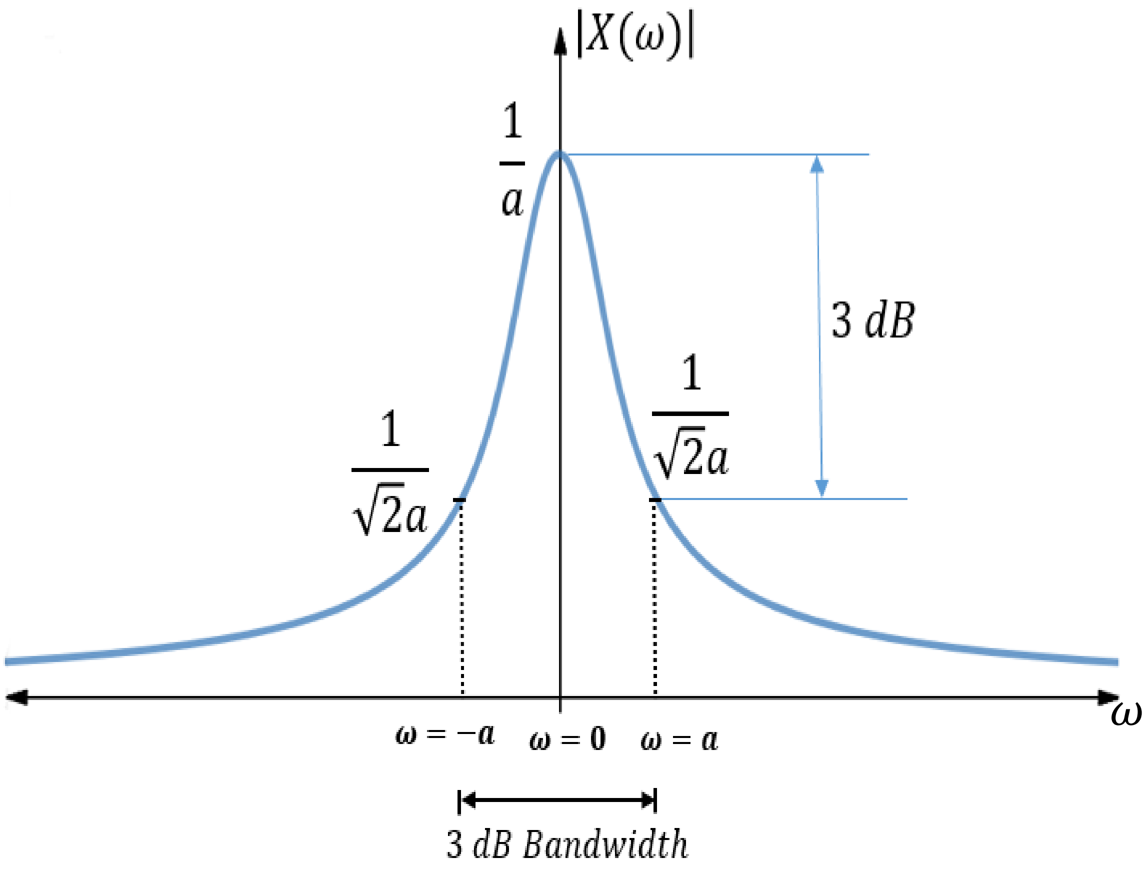
$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$X(\omega) = \frac{1}{a + j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$\angle X(\omega) = -tan^{-1}\left(\frac{\omega}{a}\right)$$

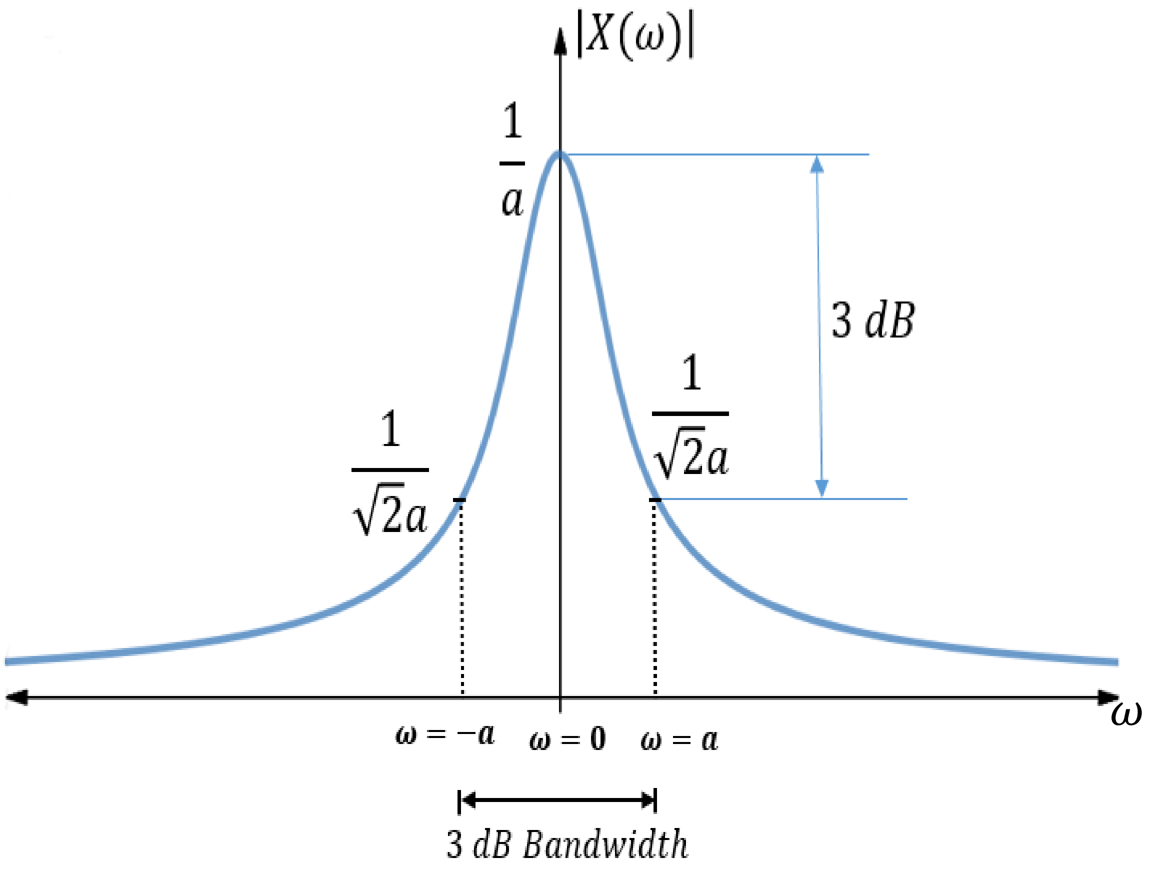
Try yourself:



$$dB = 20\log_{10} r$$

Ratio	dB
1	
$\sqrt{2}$	
2	
10	

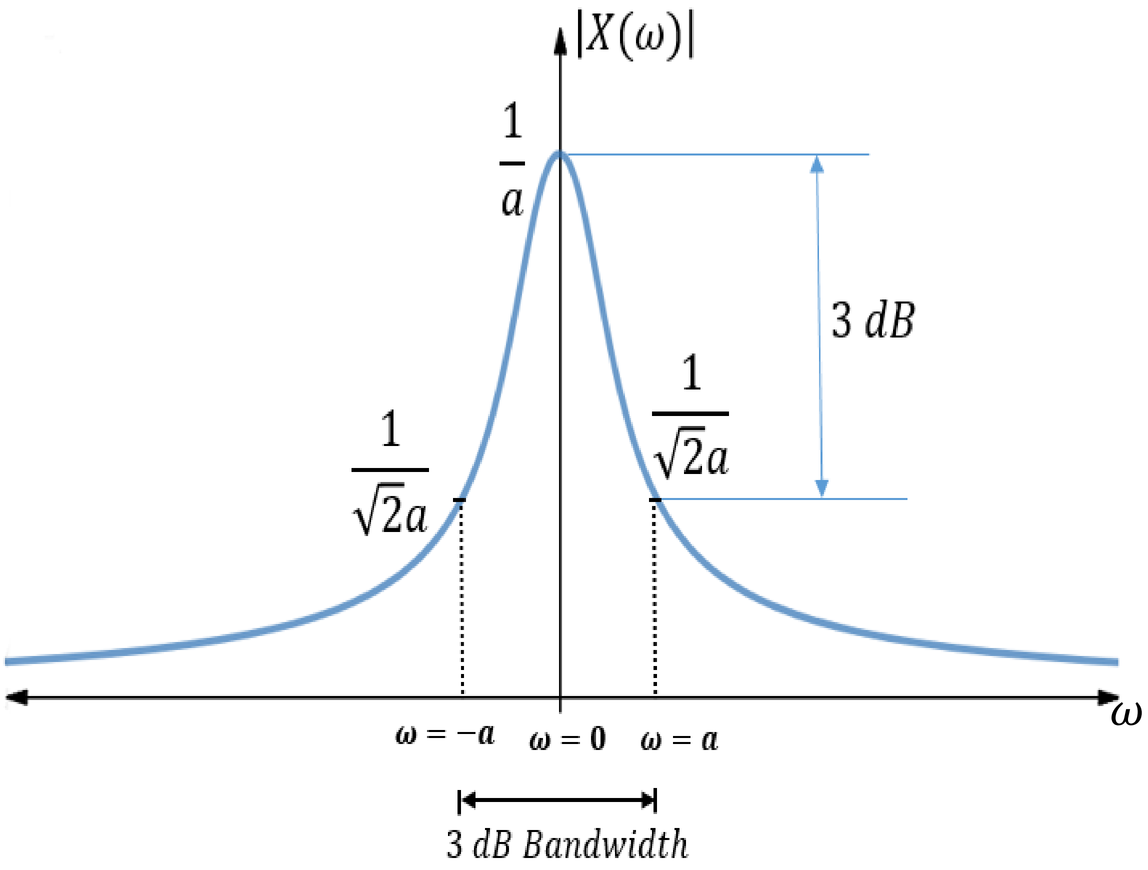
Try yourself:



$$dB = 20\log_{10} r$$

Ratio	dB
1	0
$\sqrt{2}$	
2	
10	

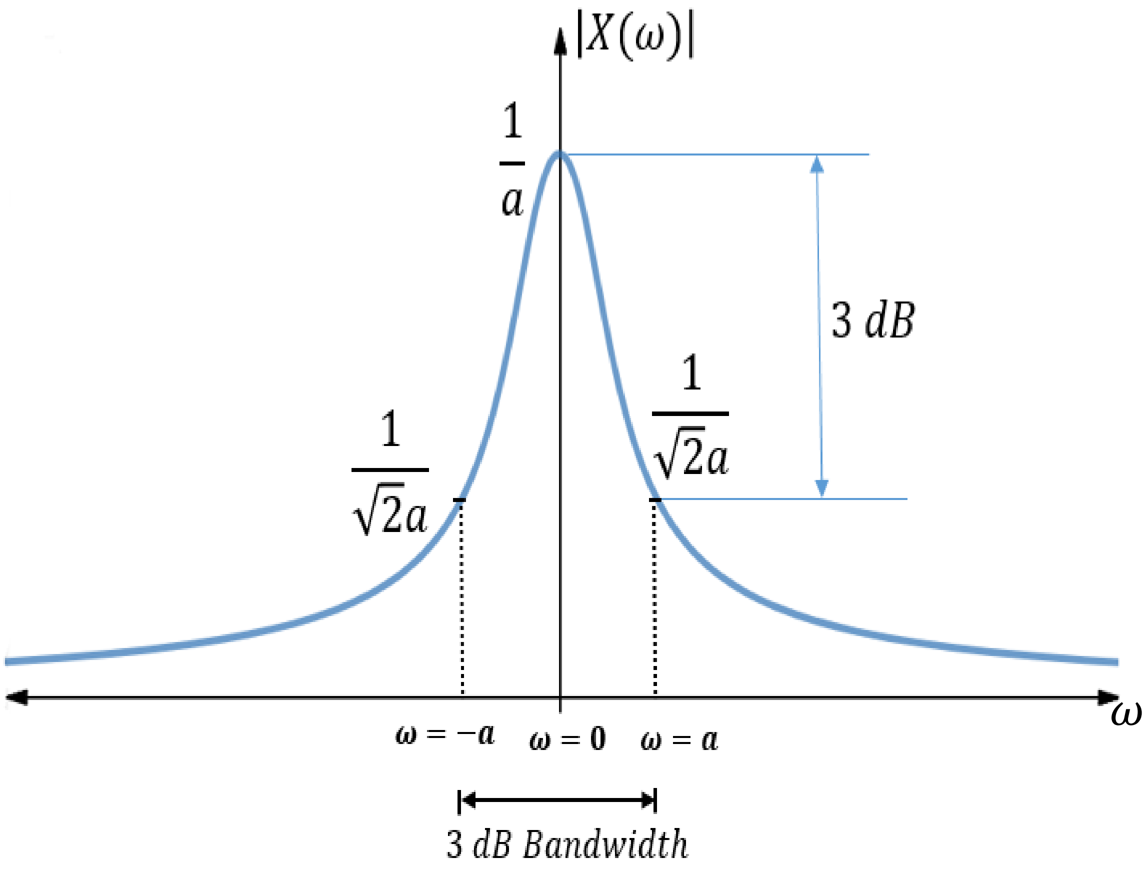
Try yourself:



$$dB = 20 \log_{10} r$$

Ratio	dB
1	0
$\sqrt{2}$	3
2	
10	

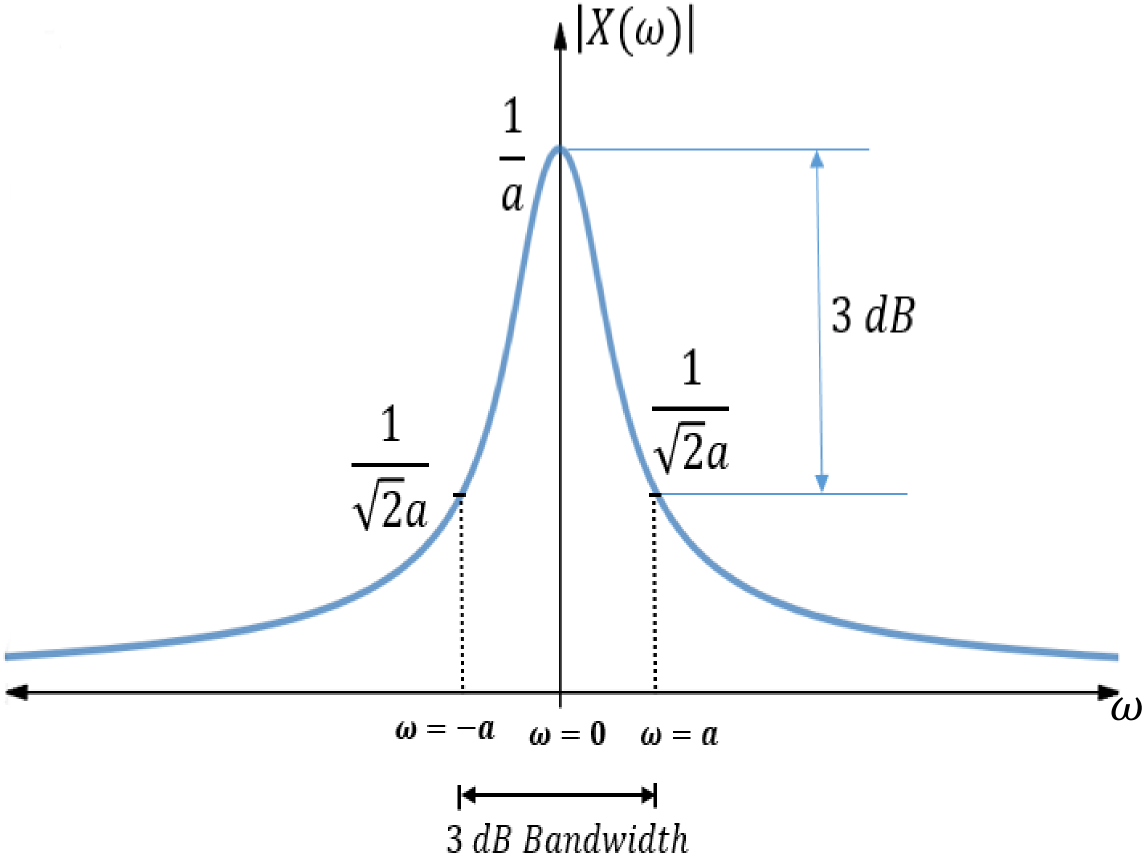
Try yourself:



$$dB = 20 \log_{10} r$$

Ratio	dB
1	0
$\sqrt{2}$	3
2	6
10	

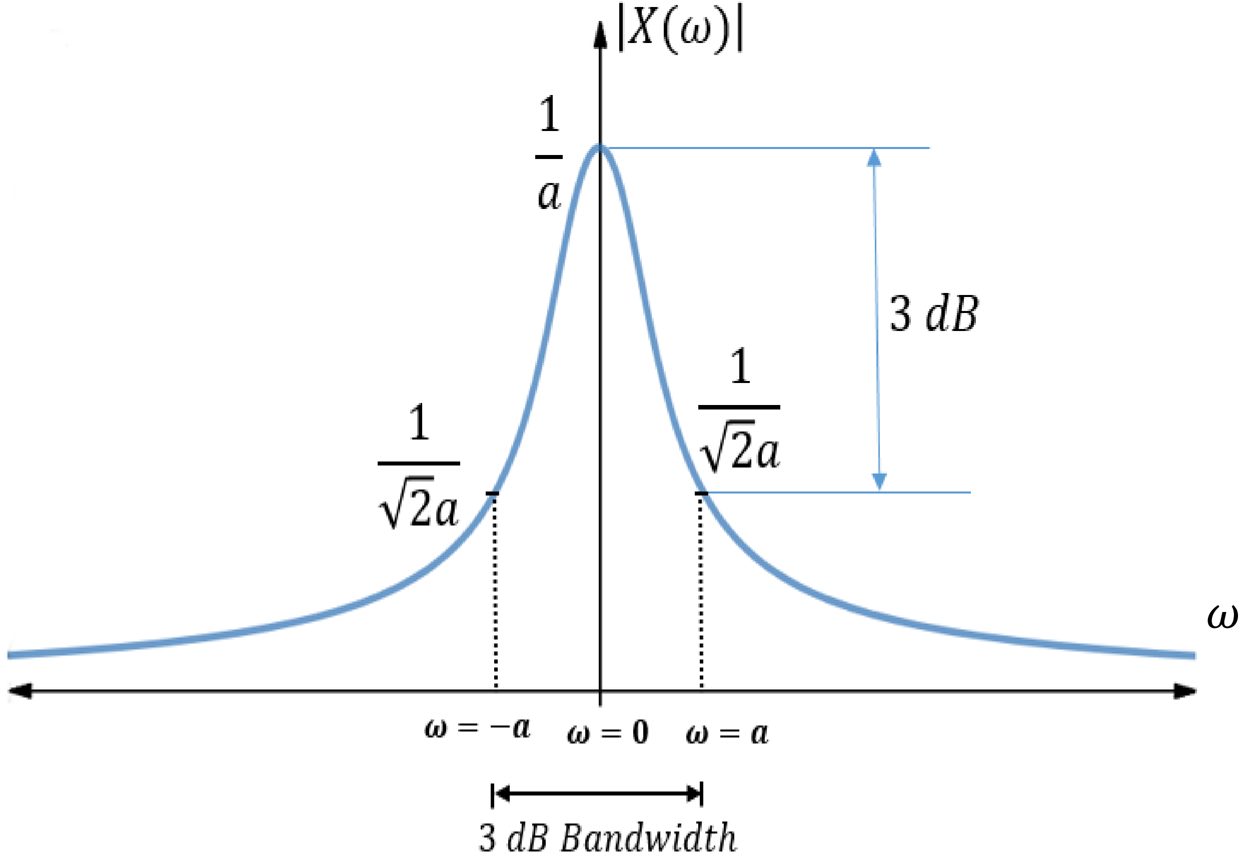
Try yourself:



$$dB = 20 \log_{10} r$$

Ratio	dB
1	0
$\sqrt{2}$	3
2	6
10	20

Try yourself:



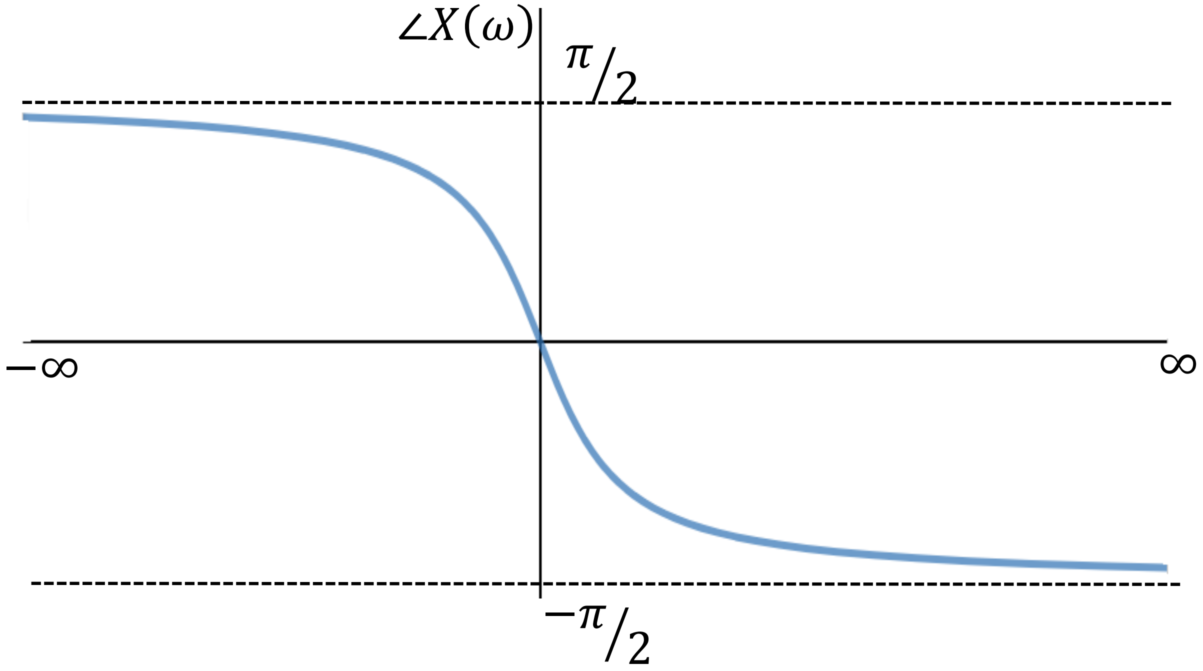
dB(decibel)= $20\log\left(\frac{1}{a} \times \frac{\sqrt{2}a}{1}\right) \approx 3dB$

One sided BW = a

Two sided BW = $2a$

By convention we normally use one-sided BW.

Try yourself:



$$\angle X(\omega) = -tan^{-1}\left(\frac{\omega}{a}\right)$$