Vector Spaces – lecture 6

let A be an $m \times n$ matrix and consider the following hom system $AX = 0$.

$$S = \{ X \in \mathbb{C}^n \mid AX = 0 \} \neq \phi$$

ⓐ $0 \in S$

ⓑ If $X_1, X_2 \in S \Rightarrow AX_1 = 0$ & $AX_2 = 0$

$$\Longrightarrow \quad A(X_1 + X_2) = AX_1 + AX_2 = 0 + 0 = 0$$

$$\Longrightarrow \quad X_1 + X_2 \in S$$

ⓒ $\alpha \in \mathbb{C}$, $\quad A(\alpha X) = \alpha AX = \alpha \cdot 0 = 0 \Longrightarrow \alpha X \in S$.

Vector Space A vector space V over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), denoted by $V(\mathbb{F})$, is a non empty set, satisfying the following properties:

① **Vector addition** ⇒ Every pair $u, v \in V$, corresponds to a unique element $u \oplus v \in V$. such that
(Binary operation: $V \times V \to V$ or also called as vector addition)

ⓐ $u \oplus v = v \oplus u$ (commutative law)

ⓑ $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ (Associative law)

(c) There exist a unique vector $0 \in V$ such that

$$u \oplus 0 = 0 \oplus u = u \quad \forall \; u \in V$$

(This is also called as additive identity)

(d) For every $u \in V$, there exist a unique $(-u) \in V$ such that

$$u \oplus (-u) = (-u) \oplus u = 0$$

($-u$ is called additive inverse of u)

(2) Scalar multiplication :- For each $u \in V$ and $\alpha \in \mathbb{F}$ there corresponds a unique element $\alpha \otimes u \in V$ such that

ⓐ $\alpha \otimes (\beta \otimes u) = \alpha\beta \otimes u \quad \forall\ \alpha, \beta \in \mathbb{F}\ \&\ u \in V$

ⓑ $1 \otimes u = u \quad \forall u \in \mathbb{F}$

($\mathbb{F} \times V \longrightarrow V$ such that $(\alpha, u) \longrightarrow \alpha \otimes u$)

③ Distributive law :- For given $\alpha, \beta \in \mathbb{F}$ and $u, v \in V$

ⓐ $\alpha \otimes (u \oplus v) = \alpha \otimes u \oplus \alpha \otimes v$

ⓑ $(\alpha + \beta) \otimes u = \alpha \otimes u \oplus \beta \otimes u$

Elements of $\mathbb{F}$ are called scalars and that of V are called vectors.

If $\mathbb{F} = \mathbb{R}$, then $V(\mathbb{F})$ is called real vector space.

If $\mathbb{F} = \mathbb{C}$, then $V(\mathbb{F})$ is called complex vector space.

<u>Theorem</u> :- Let $V(\mathbb{F})$ be a vector space. Then

(1) $u \oplus v = u \iff v = 0$

(2) $\alpha \otimes u = 0 \iff$ Either $\alpha = 0$ or $u = 0$

(3) $(-1) \otimes u = -u$

<u>Proof</u> :- (1) If $v = 0$, then $u \oplus v = u \oplus 0 = u$

$\Leftarrow$ If $u \oplus v = u$

$$\Rightarrow (-u) \oplus (u \oplus v) = (-u) \oplus u = 0$$

$(-u \oplus u) \oplus v = 0 \Rightarrow v = 0$

② Suppose $\alpha = 0$, then

$$0 \otimes u = (0 \oplus 0) \otimes u = 0 \otimes u \oplus 0 \otimes u$$

$$\Rightarrow 0 \otimes u = 0$$

// If $u = 0$, then $\alpha \otimes u = 0$

$\Leftarrow$ Suppose $\alpha \otimes u = 0$

If $\alpha = 0$, then we are done.

If not, then $\frac{1}{\alpha} \in \mathbb{F}$

$$\frac{1}{\alpha} \otimes 0 = \frac{1}{\alpha} \otimes (\alpha \otimes u) = \left(\frac{1}{\alpha} \cdot \alpha\right) \otimes u = 1 \otimes u = u$$

$$\Rightarrow u = 0$$

③ $(-1) \otimes u = -u$

$0 = 0 \otimes u = (1-1) \otimes u = 1 \otimes u \oplus (-1) \otimes u$

$\Rightarrow \quad 0 = u \oplus (-1) \otimes u$

$\Rightarrow \quad (-1) \otimes u = -u$

Examples of vector space

① $V = \mathbb{R}$ and $\mathbb{F} = \mathbb{R}$

$\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$(a, b) \longrightarrow a+b$ (vector addition)

(F) $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ (scalar multiplication)

$(a, b) \Rightarrow a \cdot b$

Verify that $\mathbb{R}(\mathbb{R})$ is a vector space.

$V = \mathbb{R}^+$ is not a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

$$\left(\begin{array}{l} \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \\ (-1, a) \rightarrow -a \notin \mathbb{R}^+ \end{array} \right)$$

We can make $\mathbb{R}^+$ a vector space over $\mathbb{R}$ with some diff operations.

② $V = \mathbb{R}^n = \{ (a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R} \}$

If $u, v \in V$

$u = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$

$v = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$

Vector addition

$u \oplus v = (a_1+b_1, a_2+b_2, \dots, a_n+b_n) \in \mathbb{R}^n$

Scalar multi $\alpha \in \mathbb{R}$ and $u \in \mathbb{R}^n$

$\alpha \otimes u = \alpha (a_1, a_2, \dots a_n) = (\alpha a_1, \alpha a_2, \dots \alpha a_n) \in \mathbb{R}^n$

Easy to verify that $\mathbb{R}^n(\mathbb{R})$ is a real vector space with $0 = (0, \dots 0)$ as zero vector.

③ $V = M_{m\times n}(\mathbb{R}) = \{ A = (a_{ij})_{m\times n} \mid a_{ij} \in \mathbb{R} \}$

$A, B \in V$, $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$

$A+B = (a_{ij} + b_{ij})_{m\times n} \in M_{m\times n}(\mathbb{R})$

and given $\alpha \in \mathbb{R}$

$\alpha \cdot A = \alpha \cdot (a_{ij}) = (\alpha a_{ij}) \in M_{m\times n}(\mathbb{R})$

Verify $M_{m\times n}(\mathbb{R})$ is a real vector space.

④ $\mathbb{C}^n = \{ (x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{C} \}$

$u = (x_1, x_2, \dots, x_n)$ & $v = (y_1, y_2, \dots, y_n)$

$u \oplus v = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) \in \mathbb{C}^n$

|| $\alpha \in \mathbb{C}$, $\alpha \otimes u = \alpha(x_1, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots \alpha x_n) \in \mathbb{C}^n$

$\mathbb{C}^n(\mathbb{C})$ is a complex vector space

$\mathbb{C}^n$ can be made as real vector space

$$\mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$$

$$(\alpha, (x_1, x_2, \dots x_n)) \to (\alpha x_1, \alpha x_2, \dots \alpha x_n) \in \mathbb{C}^n$$

Remark:- $i(1, 0, \dots 0) = (i, 0 \dots 0)$ in $\mathbb{C}^n(\mathbb{C})$

but $i(1, 0 \dots) \neq (i, 0 \dots)$ in $\mathbb{C}^n(\mathbb{R})$

(5) $C[0,1]$ = Set of cont^n real valued f^n on $[0,1]$

$f, g \in C[0,1]$

$(f+g)(x) = f(x) + g(x) \in C[0,1]$

|| $\alpha \in \mathbb{R}$ and $f \in C[0,1]$

$(\alpha \cdot f)(x) = \alpha \cdot f(x) \quad \forall x \in [0,1]$

Verify that $C[0,1]$ is a real vector space.

(6) $\mathbb{R}[x] = \{ f(x) \mid f(x) = a_0 + a_1 x + \cdots + a_n x^n, n \geq 0 \}$

Verify $\mathbb{R}[x]$ is a real vector space.

(7) $P_n(x) = \{ f(x) \in \mathbb{R}[x] \mid \deg(f(x)) \leq n \}$.

Subspaces

Definition :→ Let $V(\mathbb{F})$ be a vector space. A non empty subset $W \subseteq V$ is said to be a subspace of V if W is a vector space with respect to the binary operation coming from V.

$$\Updownarrow$$

A non empty subset W is a subspace of V if $\underline{\alpha \cdot u + \beta \cdot v} \in W \quad \forall\ \alpha, \beta \in \mathbb{F}\ (\mathbb{R} \text{ or } \mathbb{C})$ & $u, v \in W$

$(+ = \oplus\ ,\ \alpha \cdot u = \alpha \otimes u)$

$\Rightarrow$ Note that $u, v \in W$

Then $\alpha u, \beta v \in W$ (Scalar mult)

$\Rightarrow \alpha u + \beta v \in W$ (by vector addition)

$\Leftarrow$ we are given $\alpha u + \beta v \in W$ $\forall\ \alpha, \beta \in \mathbb{F}\ (\mathbb{R}$ or $\mathbb{C})$

& $u, v \in W$

<u>Vector addition</u>: take $\alpha = \beta = 1$

then $1 \cdot u + 1 \cdot v = u + v \in W$ $\forall\ u, v \in W$

<u>Scalar multi</u>: take $\beta = 0$

then $\alpha \cdot u + 0 \cdot v = \alpha u \in W$.

Note that $u + v = v + u$ $(\because u, v \in W \subseteq V)$

$(u+v)+w = u + (v+w)$ $(\because u, v, w \in W \subseteq V)$

If we take $\alpha = -1$ and $\beta = 0 \implies (-1)u + 0 \cdot v = -u \in W$

$\|$ $u + (-u) = 0 \in W$ (by vector addition)

Distributive law : Holds true because they are the elements of V