31-7-23

$\max z = c^T x$

$x \in S = \{x \in R^n : Ax \le b, x \ge 0\}$

Assume

1) The objective $f(x)$ is non continuous

2) LP has a soln say $x^* \in S$.

Claim :- $x^* \notin$ interior of S.

If since $x^* \in S$ is an optimal soln of LP →
for some $\delta > 0$: $c^T x^* \ge c^T x \ \forall x \in N_\delta(x^*) \cap S$.
(nghd of x^*)

Let us assume contrary that $x^* \in$ interior S
$\exists \ \delta_1 > 0$ s.t. $N_{\delta_1}(x^*) \in S$
choose $\delta = \min(\delta_1, \delta) > 0$.

$\rightarrow c^T x^* \ge c^T x \quad \forall x \in N_\delta(x^*)$ — (1).

$\hat{x} = x^* + \delta/2 \dfrac{c}{\|c\|}$

$\|\hat{x} - x^*\| = \delta/2 < \delta$.

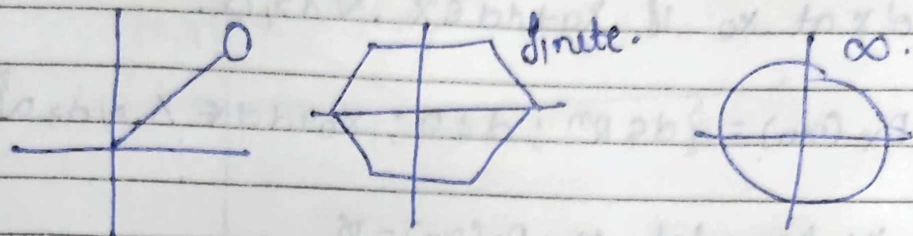
$\hat{x} \in N_\delta(x^*)$

$c^T \hat{x} = c^T x^* + \delta/2 \dfrac{c^T c}{\|c\|} \qquad (c^T c = \|c\|^2)$

$c^T \hat{x} = c^T x^* + \delta/2 \|c\| > c^T x^*$
(contradiction)

Let $X \in \mathbb{R}^n$ be a Convex set. A point $x_0 \in X$ is called an extreme point (EP) of X if $\nexists$ $x_1, x_2 \in X$, $x_1 \neq x_2$ & a scalar $\alpha \in (0,1)$ s.t.

$$x_0 = (1-\alpha)x_1 + \alpha x_2$$



Result: An EP of X is a boundry point (converse Not True)

If x_0 is an EP of convex set X then $x_0 \in X \ni$ $x_0 \in$ interior or $x_0 \in$ boundry X.

To Show: $x_0 \notin$ Interior X

Suppose $x_0 \in X$ s.t. $N_\delta(x_0) \subset X$

$$x_1 = x_0 - \frac{\delta}{2} e \qquad e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n.$$

$$x_2 = x_0 + \frac{\delta}{2} e.$$

$$\|x_1 - x_0\| = \frac{\delta}{2} < \delta.$$

$$\|x_2 - x_0\| = \frac{\delta}{2} < \delta.$$

$\left.\right\}$ $x_1, x_2 \in X$. & $x_1 \neq x_2$

$$\frac{x_1 + x_2}{2} = x_0$$

Hence x_0 is Not EP.

Contradiction: x_0 is Not in interior of X.

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X - Convex set in R^n

Let $x_0 \in X$ be arbitrary.
A vector $d \in R^n$; $d \neq 0$ is a direction vector of X at x_0 if $x_0 + \lambda d \in X$, $\forall \lambda \geq 0$.

$D_X(x_0) = \{d \in R^n ; d \neq 0 ; x_0 + \lambda d \in X, \forall \lambda \geq 0\}$.

X: bounded $\Leftrightarrow$ $D_X(x_0) = \emptyset$

Extreme Direction

A vector $d \neq 0$, $d \in D_X(x_0)$ is called an extreme direction if $\nexists$ $d', d'' \in D_X(x_0)$ $d' \neq N d''$, $N \geq 0$ s.t. $d = \lambda_1 d' + \lambda_2 d''$ for some $\lambda_1, \lambda_2 \geq 0$.

$S = \{x \in R^n, Ax \leq b, x \geq 0\}$.

(**) — $d \in D_X(x_0)$; $d \neq 0$, $x_0 + \lambda d \in S$, $\forall \lambda \geq 0$.
$Ax_0 + \lambda Ad \leq b$, $x_0 + \lambda d \geq 0$, $\forall \lambda \geq 0$.
$\Downarrow$

(*) — $\underset{\geq 0}{\lambda Ad} \leq \underset{\geq 0.}{b - Ax_0}$

If $Ad \leq 0$; then (*) is always True. $\forall \lambda \geq 0$.

$\Rightarrow$ Also if any component of $Ad > 0$, then (*) doesnt hold $\forall \lambda > 0$.
Note (**) hold $\forall \lambda \geq 0$.
$D_S = \{d \in R^n, d \neq 0 ; Ad \leq 0, d \geq 0\}$.
$\hookrightarrow$ Recession direction.

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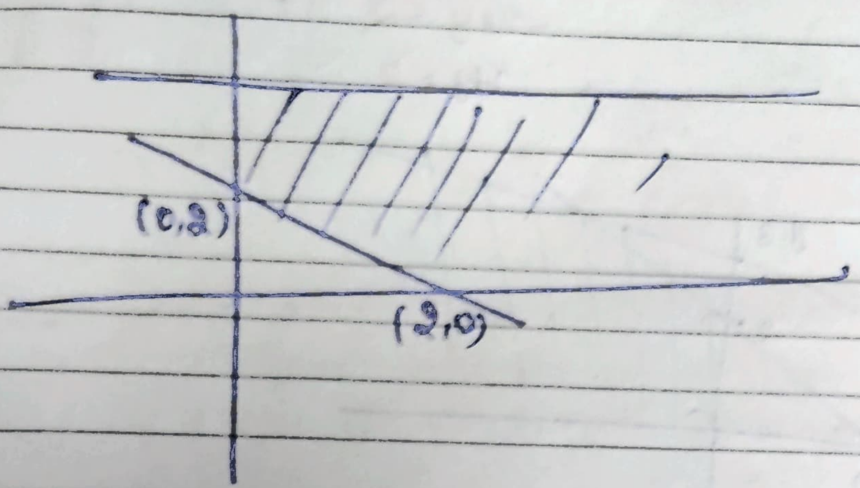
Q. $S = \{(x,y) : x+y \geq 2, y \leq 4, x,y \geq 0\}$.



$$A = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

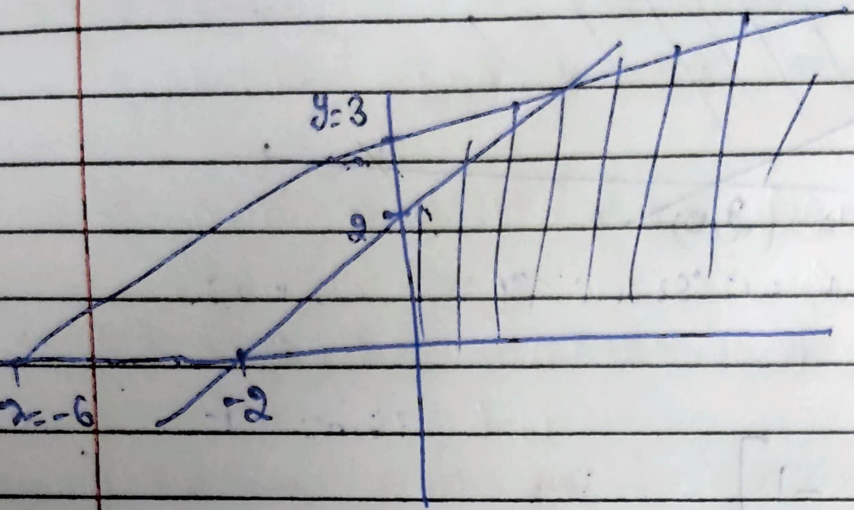
$$A \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$\Rightarrow Ad \leq 0. \Rightarrow \left.\begin{array}{l} -d_1 - d_2 \leq 0 \\ d_2 \leq 0 \end{array}\right\} \begin{array}{l} d_1 + d_2 \geq 0. \\ d_2 \leq 0. \end{array}$

$D_S = \{d \in \mathbb{R}^2, d \neq 0 : d_1 + d_2 \geq 0, d_2 \leq 0\}$.

$\rightarrow d_1 = \{ d \in \mathbb{R}^2, d \neq 0, d_1 \geq 0, d_2 = 0\}$.

Q $S = \{(x,y) : -x+2y \le 6, \; -x+y \le 2, \; x,y \ge 0\}$



$D_S = \{ d \in R^2, \; d \neq 0,$

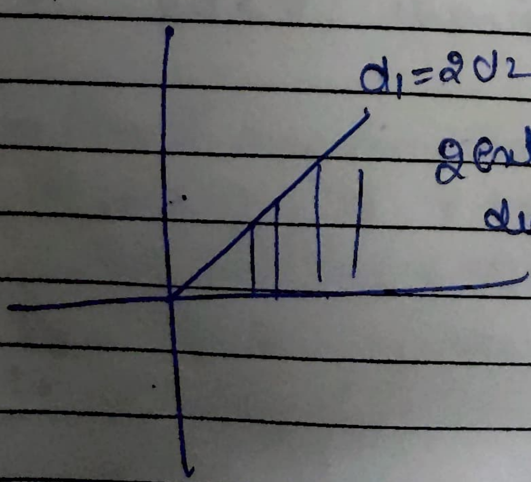
$$\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} d \le 0.$$

$-d_1 + 2d_2 \le 0.$

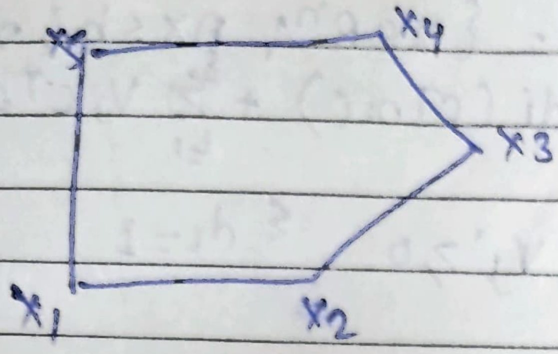
$-d_1 + d_2 \le 0.$

$\left.\begin{array}{l} d_1 \ge 2d_2 \\ d_1 \ge d_2 \end{array}\right\} d_1, d_2 \ge 0.$

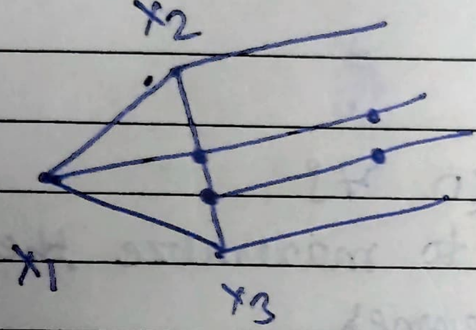
$D_S = \{ d \in R^2, \; d \neq 0, \; d_1 \ge 2d_2 \ge 0 \}.$



Representation Theorem of polyhedron



$$\bar{x} = \{\Sigma d_i x_i,\ d_i \geq 0,\ \Sigma d_i = 1\}.$$



Combination
Converse ~~Combinations~~ of extreme pts.

Let X be a polyhedron in R^n

Then any point $\bar{x} \in X$ can be represented by.

$$\bar{x} = \sum_{i=1}^{p} d_i x^i + \sum_{j=1}^{s} N_j d^j$$

$$d_i \geq 0 \ \forall i \qquad \sum_{i=1}^{p} d_i = 1$$

$$N_j \geq 0 \ \forall j$$

where $(x_1, x_2 --)$ are extreme pts of X.
& $(d^1, d^2 -- d^S)$ are extreme direc^n of S.

$\max z = c^T x$

Sub. to $x \in S = \{x \in R^n;\ Ax \le b',\ x \ge 0\}$

$\max c^T \xi = \sum_{i=1}^{p} d_i (c^T x^i) + \sum_{j=1}^{q} v_j c^T d^j$.

Sub. to: $d_i \ge 0,\ v_j \ge 0$ $\quad \sum d_i = 1$

Suppose $c^T d^j > 0$ for atleast one j
we can change corresponding $v_j \to \infty$.
& all others $v_j = 0$. $d_1 = 1,\ d_i = 1\ \forall i \ne 1$
$\Rightarrow$ unbound LP

Now, Suppose $c^T d^j \le 0\ \forall j$
Take all $v_j = 0\ \forall j$ to maximize the objective function becomes
$\sum_{i=1}^{p} d_i (c^T x^i),\ d_i \ge 0,\ \sum d_i = 1$.

Convex Combination of extreme pts.

$\max \{c^T x^1, c^T x^2 \ldots\} = c^T x^k$ (Say)
we set $d_k = 1$ & all other $d_i = 0$.
$\max z = c^T x^k$. obtained at x^k which is EP of S.

$X \to$ polyhedron in R^n

X is bounded $\Leftrightarrow D_X = \emptyset$

$$\bar{x} = \sum_{i=1}^{p} \alpha_i x^i + \sum_{j=1}^{s} \mu_j d^j \to \text{ED of } X.$$

$\hookrightarrow$ EP of X.

$\alpha_i, \mu_i \geq 0.$

$\sum_{i=1}^{p} \alpha_i = 1$

$\max c^T x.$

s.to $\left.\begin{array}{l} Ax \leq b \\ x \geq 0 \end{array}\right\} S.$

Q

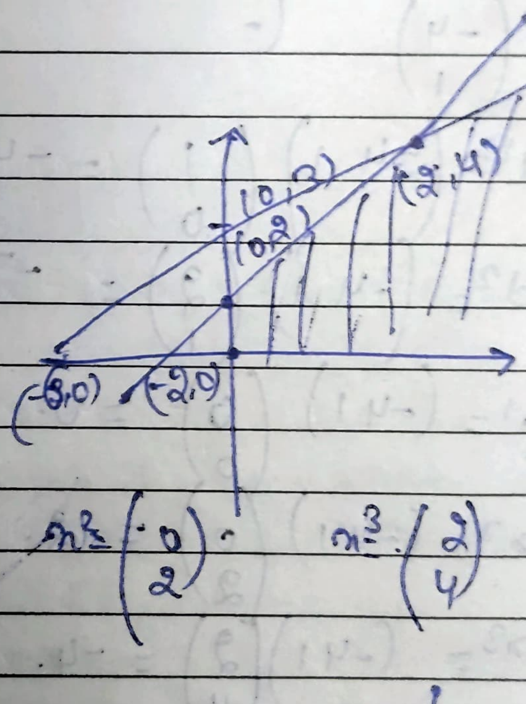
$\max \; -x_1 + 3x_2$

S.to.

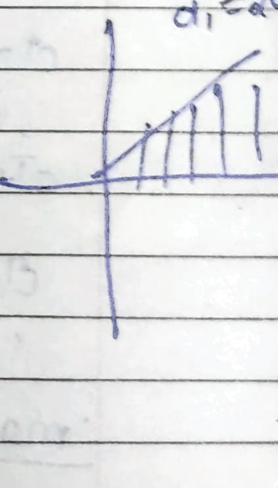
$-x_1 + x_2 \leq 2.$

$-x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0.$



$x^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad x^2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. $\quad x^3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$



$D_S = \{d \in R^2, \; Ad \leq 0, \; d \geq 0, \; d \neq 0\}.$

$d^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad A = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}$ $\qquad -d_1 + d_2 \leq 0.$

$\boxed{d_2 \leq d_1}$

$d^2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\qquad d_1, d_2 \geq 0.$

$-d_1 + 2d_2 \leq 0.$

$c^T d^1 = -1$ $\quad C = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ $\quad d_2 \leq \frac{d_1}{2}$

$c^T d^2 = 1$

~~$c^T d = (-1 \;\; 3)$~~ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\qquad c^T x^1 = 0.$ $\quad c^T x^3 = 10.$

$c^T x^2 = 6$

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Equivalent LP

$$\max \quad \overset{c^T x^1}{0 d_1} + \overset{c^T x^2}{6 d_2} + \overset{c^T x^3}{10 d_3} + \overset{c^T d^1}{\mu_1 (-1)} + \overset{c^T d^2}{\mu_2 (1)}$$

$d_1, d_2, d_3 \geq 0$

$\mu_1 \geq 0$

$\mu_2 \geq 0$

$\sum d_i = 1$

μ_2 is unbounded therefore the problem is unbounded.

eg objective f^n $-4x_1 + x_2$

$c = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ (-

$c^T d^1 = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -4.$

$c^T d^2 = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -7$

$c^T x^1 = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$

$c^T x^2 = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2.$

$c^T x^3 = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = -4.$

$$\max \quad \cancel{0} d_1 + 2 d_2 + (-4) d_3 - 4\mu_1 - 7\mu_2.$$

$d_i, \mu_i \geq 0.$

$\sum d_i = 1.$

$\mu_1, \mu_2 = 0$; $d_3, d_1 = 0$, $\boxed{d_2 = 1}$

Linear Algebra

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eg. $Ax \le b,\ -x \le 0$

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} A_{m\times n} \\ -I_{n\times n} \end{bmatrix} x_{n\times 1} \le \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$(m+n)\times n$

G (Submatrix of $\begin{bmatrix} A_{m\times n} \\ -I_{n\times n} \end{bmatrix}$) $r\times n$

$$Gx = g$$
$$(r\times n)(n\times 1) = r\times 1$$

(i) $r = n$.

$Gx = g$,

if G is full rank $= n$.

& G^{-1} exist then x (unique sol^n) is in EP of S

(ii) let $rank(G) < n$.

$\Rightarrow \exists d \neq 0,\ Gd = 0$. Then given any $\bar{x}$ of S we can find $\varepsilon > 0$, small enough s.t.

$x_1 = \bar{x} - \varepsilon d \in S$

$x_2 = \bar{x} + \varepsilon d \in S$

& $\bar{x} = \dfrac{x_1 + x_2}{2}$

$Gx_1 = G\bar{x} + \varepsilon Gd = G\bar{x} = g.$

$Gx_2 = G\bar{x} - \varepsilon Gd = G\bar{x} = g.$

$\begin{bmatrix} A \\ -I \end{bmatrix} \rightarrow H \begin{Bmatrix} \text{remaining} \\ \text{inequalities} \end{Bmatrix}$ $\quad H\bar{x} \le rhs.$

$G \begin{Bmatrix} \text{already} \\ \text{satisfies the} \end{Bmatrix}$

$Hx_1 = H\bar{x} - H\varepsilon d \le rhs.$

Basic Feasible Solution

$Ax \leq b$

$x \geq 0 \rightarrow -x \leq 0$.

$\begin{bmatrix} A \\ -I \end{bmatrix} x \leq b$.

$n \times m$.

$G_{n \times n} \rightarrow Gx = g$.

$\hat{x}$

Consider

$Ax = b$

$x \geq 0$.

→ Any linear inequality can be converted into equality.

$3x_1 + x_2 - 4x_3 \leq 5$!

add ≥ 0.

$3x_1 + x_2 - 4x_3 + x_4 = 5$

$x_4 \geq 0$.

(Slack variable.)

additional Non negative Variable

$2x_1 + 7x_2 - x_3 \geq 4$.

$-2x_1 - 7x_2 + x_3 \leq -4$.

$-2x_1 - 7x_2 + x_3 + x_4 = -4$

$\Leftrightarrow 2x_1 + 7x_2 - x_3 - x_4 = 4$

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Assume. $A: m \times n$; $m < n$.

Rank $A = m$.

$\boxed{\cancel{x \geq 0}}$

(1) — $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$

$Ax = b.$
$m \times n$
full row Rank $= m$

$\exists$ atleast one Submatrix $B_{m \times m}$ of A s.t. B^{-1} exists.

$\Rightarrow$ There are $n-m$ columns of A which are not part of B

$A = [B . R]$

B: $m \times m$

R: $m \times (n-m)$ — for those Columns in R, put $x_i = 0$.

$x : n \times 1$

→ out of which $n-m$'s are 0.

→ Remaining $m \times 1$ call it by x_B. $\underline{m \times 1}$

$Ax = b$

$\Leftrightarrow Bx_B = b.$

$\Leftrightarrow x_B = B^{-1} b.$

~~If Op~~

If $x_B \geq 0$

then $x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$ is called a BFS of (1).

& B is called basis matrix.

else.

$x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$ is a soln of $Ax = b$ but not a feasible soln of (1).

Any Real matrix

$M_{m \times n}$.

rank $M = m$.

Soln of $M_{n \times 1} x = u_{m \times 1}$.

$x \in \mathbb{R}^n$

$(n-m)$ free variables.

↳ assign any value & then get remaining (m).

For Convenience we put other = 0

BFS: $x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$, $x_B = B^{-1} b \geq 0$.

Degenerate BFS

If atleast one of the Component of $x_B : m \times 1$ is 0.

Non-Degenerate BFS.

$\underset{m \times 1}{x_B} = \underset{m \times m}{B^{-1}} \underset{m \times 1}{b}$.

> 0.

all m-components > 0.

If any polyhedron $\{ Ax = b, x \geq 0 \}$ has atleast one degenerate BFS then polyhedron is called a degenerate polyhedron.

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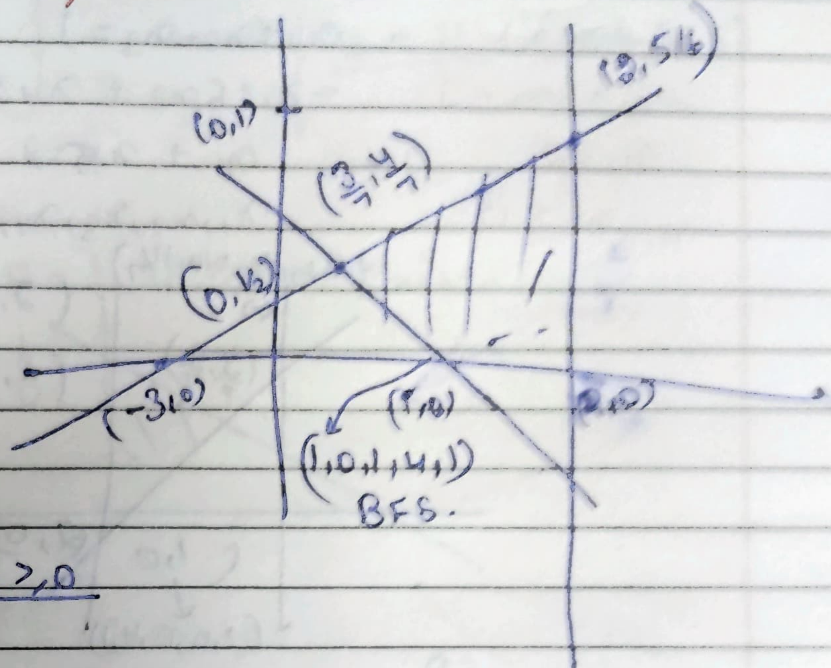
Let us consider a system

$$\begin{cases} x_1 + x_2 \geq 1 \\ -x_1 + 6x_2 \leq 3 \\ x_1 \leq 2. \\ x_1, x_2 \geq 0 \end{cases}$$

Convert into equality.

$\Downarrow$



$$x_1 + x_2 - x_3 = 1$$
$$-x_1 + 6x_2 + x_4 = 3.$$
$$x_1 + x_5 = 2.$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$A_{3\times5} = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 6 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}_{3\times1}$$

$\boxed{\begin{matrix} m = 3 \\ n = 5 \end{matrix}}$ Rank $A = 3$ # of choices of $B = {}^nC_m = {}^5C_3 = 10$. (which are invertible)

$$\text{let } B = \begin{array}{c} \begin{matrix} 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

put $x_1 = x_2 = 0$.

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$B x_B = b$

$-x_3 = 1 \Rightarrow x_3 = -1$

$x_4 = 3$ (Not a BFS)

$x_5 = 2$. (as ≥ 0 X)

$$B = \begin{array}{c} \begin{matrix} 1 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \text{(BFS)}$$

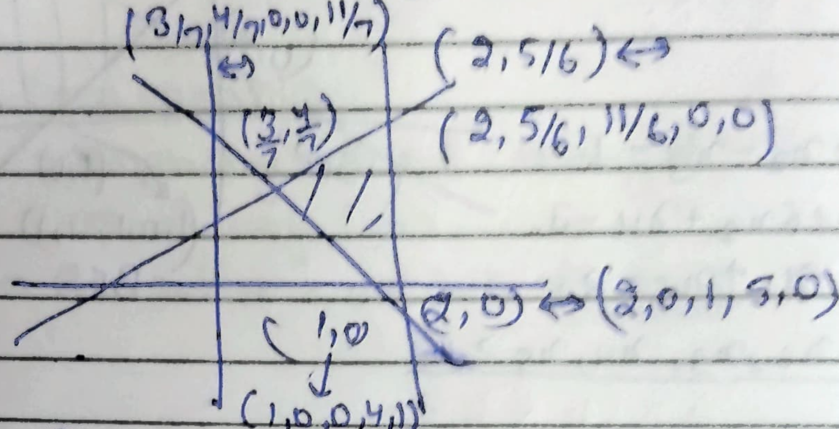
put $x_2 = x_3 = 0$. (Non degenerate)

$x_1 = 1$. $x_5 = 1$.

$x_4 = 4$

BFS:

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ -x_1 + 6x_2 &\leq 3 \\ x_1 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 - x_3 &= 1 \\ -x_1 + 6x_2 + x_4 &= 3 \\ x_1 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$



$A = 3 \times 5$

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 6 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1 = 1$
$x_2, x_3 = 0$
$x_4 = 4$, $x_5 = 1$
Nondegenerate

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_3 & x_4 \end{array} \\ B = & \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$x_2 = x_5 = 0$
$x_1 = 2$
$x_3 = 1$, $x_4 = 5$

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ B = & \begin{bmatrix} 1 & 1 & -1 \\ -1 & 6 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$x_4 = x_5 = 0$
$x_1 = 2$
$x_2 = 5/6$, $x_3 = 11/6$

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_5 \end{array} \\ B = & \begin{bmatrix} 1 & 1 & 0 \\ -1 & 6 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$x_3 = x_4 = 0$
$x_1 = 3/7$, $x_2 = 4/7$
$x_5 = 11/7$

$$B = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ -1 & 6 & P \\ 1 & P & 0 \end{bmatrix} \end{array}$$

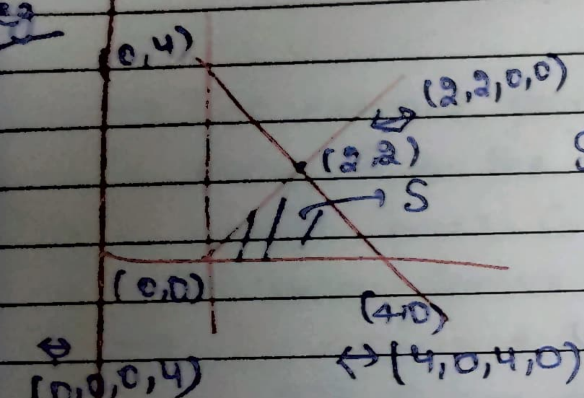
$x_3 = x_5 = 0.$

$x_1 = 2.$ | $x_4 = 11$ (Not BFS)

X $x_2 = -1$

$$B = \begin{array}{c} \begin{matrix} x_2 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & -6 & 0 \\ 6 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

B^{-1} doesn't exist.



Consider.

$S = \{ x_1 - x_2 \geq 0, \; x_1 + x_2 \leq 4, \; x_1, x_2 \geq 0 \}$

$x_1 - x_2 - x_3 = 0.$

$x_1 + x_2 + x_4 = 4$

$x_1, x_2, x_3, x_4 \geq 0.$

$$A_{2\times 4} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$ Rank $A = 2.$

max No. of BFS ${}^4C_2 = 6.$

$$B_1 = \begin{array}{c} \begin{matrix} x_1 & x_2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$x_3 = x_4 = 0.$ Non degenerate BFS.

$x_1 = 2, x_2 = 2.$

$$B_2 = \begin{array}{c} \begin{matrix} x_1 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$x_2 = x_4 = 0.$

$x_1 = 4, \; x_3 = 4.$ Non degenerate

$$B_3 = \begin{array}{c} \begin{matrix} x_1 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

$x_2 = x_3 = 0$

$x_1 = 0, \; x_4 = 4$

degenerate BFS.

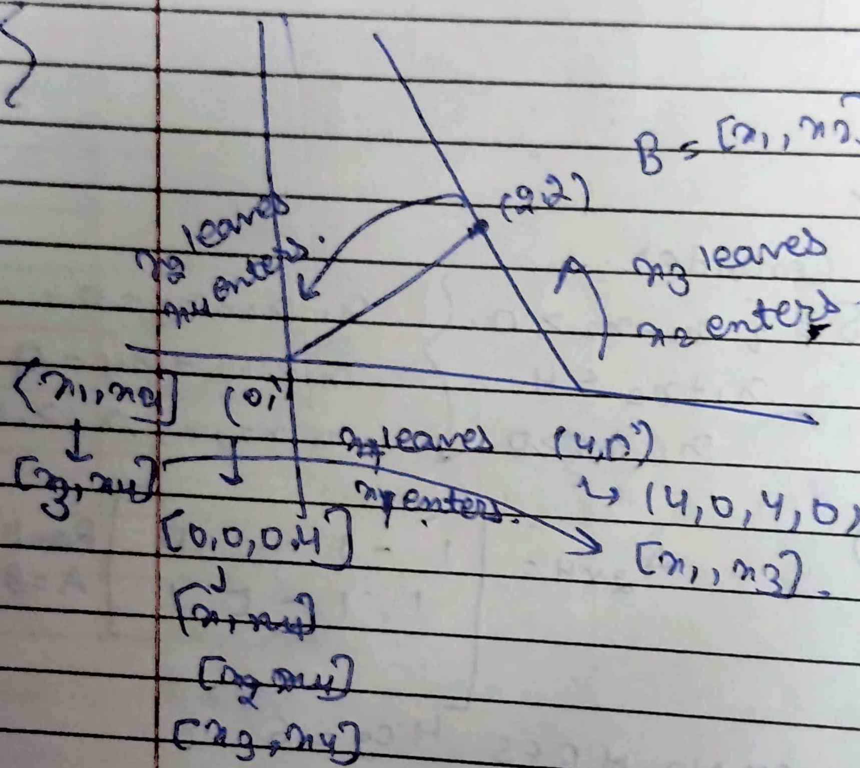
$$B_4 = \begin{array}{c} \begin{matrix} x_2 & x_3 \end{matrix} \\ \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$x_1 = x_4 = 0.$ X

$x_2 = 4, \; x_3 = -4$

$$B_5 = \begin{array}{cc} x_2 & x_4 \end{array} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$ $x_1 = x_3 = 0$. $\boxed{x_2 = 0.\ x_4 = 4}$ Degenerate BFS

$$B_6 = \begin{array}{cc} x_3 & x_4 \end{array} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$ $x_1 = 0\ x_2 = 0$. $x_3 = 0,\ x_4 = 4$. Degenerate BFS.



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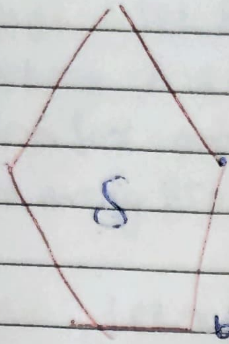


$[x_1, x_5, x_3, x_4]$.

$EP_2 \leftrightarrow BFS \leftrightarrow B_2$ $m \times m$ invertible.

[x_i - only 1 Change].

$EP_1 \leftrightarrow BFS \leftrightarrow B_1$ $m \times m$ invertible.

$[x_1, x_2, x_3, x_4]$

$S = \left\{ x : Ax = b, \; x \geq 0 \right\}$.

x^1, x^2

→ Two extreme points of S are adjacent iff every $\hat{x}$ on the line segment joining x^1 & x^2 has a unique Convex Combination in terms of x^1, x^2 or a unique Convex Combination of EP's of S.

→ In other words, in terms of BFS,

x^1 & x^2 are adjacent EP of S, iff the set of Column vectors $\{a_j : \text{either } x^1_j > 0 \text{ or } x^2_j > 0 \text{ or both}\}$ is of cardinality $n-1$.

→ or in terms of Basis variables, x^1_B & x^2_B one see only one swap (or exchange) of basic variable.

$\underset{m \times n}{A} \; \underset{n \times 1}{x} = \underset{m \times 1}{b}$

$A = [a_1 \; --- \; a_n]$.

a_1 - 1st col of A $\{a_i\}$ - $m \times 1$.

a_n - nth Col. of A

x^1, x^2 Adjacent EP of S.

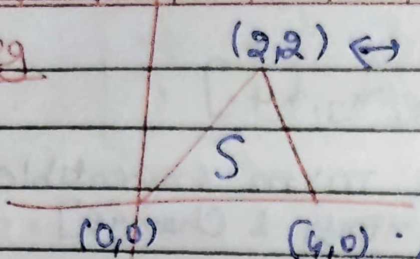
$n \times 1$ $n \times 1$.

$[a_1 \; a_2 \; a_4]$ $n-1$

(>0)

$x_B^2 = \{x_1, x_2\}$.

eg.



$(2,2) \leftrightarrow (2,2,0,0)$

$(4,0) \Leftrightarrow (4,0,4,0)$

$x_B^1 = [x_1, x_3]$.

$x_1 - x_2 \geq 0$.

$x_1 + x_2 \geq 4$.

$x_1, x_2 \geq 0$.

$\Downarrow$

$x_1 - x_2 - x_3 = 0$.

$x_1 + x_2 - x_4 = 4$.

$x_1, x_2, x_3, x_4 = 0$.

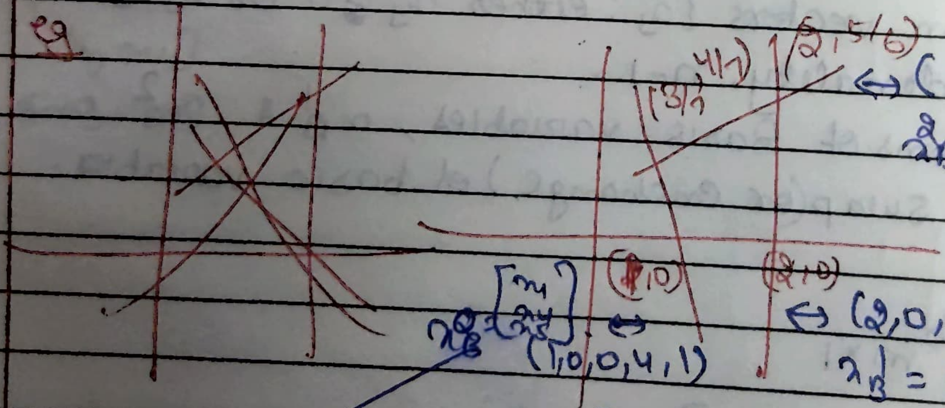
$A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}_{2\times 4}$. $b = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

cardinality

$\{a_1, a_2, a_3\} \rightarrow$ a_1 & $a_2 > 0$ (for 2nd) $\;(n-1)$

a_1 & $a_3 > 0$. (for 1st)

eg.



$\leftrightarrow (2, 5/6, 11/6, 0, 0)$

$x_B^2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$x_B^0 = \begin{bmatrix} x_1 \\ x_4 \\ x_5 \end{bmatrix} \leftrightarrow (1,0,0,4,1)$

$\leftrightarrow (2,0,1,5,0)$

$x_B^1 = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix}$.

$A = 3\times 5$.

~~m=3, n=5~~

$\hookrightarrow \{a_1, a_2, a_3, a_4, a_5\}$

$\{a_1, a_2, a_3, a_4\}$ (n-1).

let take x^0 & x^2. (Not adjacent)

$\{a_1, a_2, a_3, a_4, a_5\}$. = Not (n-1)

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Result:- Let the feasible set S of LP be $\{x \in R^n : Ax = b, x \geq 0\}$
where A is $m \times n$, $m < n$, rank $A = m$.
Then for every extreme point of S, we have a BFS.
& for every BFS of system we have an extreme pt of S.

proof Let x be a BFS of the system. $Ax = b$, $x \geq 0$.
To prove: x is an EP of S.
proving by contradiction.
Let x be not an EP of S.
$\Rightarrow \exists$ two distinct points. x^1 & x^2 in S, and a scalar $d \in (0,1)$
S.t. $x = (1-d)x^1 + (d)x^2 \to n \times 1$
Also x is a BFS, so.
$\exists$ a basis matrix $B_{m \times m}$ invertible S.t.

$$x = \begin{bmatrix} x_B = B^{-1}b \\ x_R = 0 \end{bmatrix} \begin{matrix} \to m \times 1 \\ \to (n-m) \times 1. \end{matrix}$$

(This is like partitioning x into 2 parts, basic (on top) & non-basic variables on bottom).

A:
$x = (1-d)x^1 + dx^2$

$$\Rightarrow \begin{bmatrix} x_B \\ x_R \end{bmatrix} = (1-d) \begin{bmatrix} x_B^1 \\ x_R^1 \end{bmatrix} + d \begin{bmatrix} x_B^2 \\ x_R^2 \end{bmatrix}_{n \times 1}$$

where $x_B^1 = m \times 1$, $x_B^2 = m \times 1$
Subvectors. $x^1 x^2$
Corresponding to the indices of x_B in B. &
In the same order Component.
(Indices & the order in which we are the LHS x^1 & RHS x^1, x^2 are maintained)

$$\Rightarrow x_B = (1-d)x_B^1 + dx_B^2$$

$$0 = \underbrace{(1-d)}_{>0} \underbrace{x_R^1}_{\geq 0} + \underbrace{d}_{>0} \underbrace{x_R^2}_{\geq 0} \Rightarrow \underline{x_R^1 = 0 \text{ \& } x_R^2 = 0}$$

$\cancel{Bx_B = (1-\lambda) Bx_B^1 + \lambda Bx_B^2}$

Also.

$Ax^1 = b.$ $\quad [B\ R]\begin{bmatrix} x_B^1 \\ x_R^1 \end{bmatrix} = b.$

$Ax^2 = b.$

$[B\ R]\begin{bmatrix} x_B^2 \\ x_R^2 \end{bmatrix} = b.$

$x_R^1 \,\&\, x_R^2 = 0 \Rightarrow.$

Now.

$Bx_B^1 = b.$ $\Rightarrow x_B = B^{-1}b.$

$Bx_B^2 = b$ $\quad x = x_B^2$

$\Rightarrow \boxed{x_B^1 = x_B^2}$

$\Rightarrow \boxed{x^1 = x^2} \Rightarrow$ Contradiction (as we assumed x^1 & x^2 to be distinct)

$\Rightarrow$ x must be an extreme point of S.

proved BFS $\Rightarrow$ EP.

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Conversely EP $\Rightarrow$ BFS

$$S = \{x \in R^n : Ax \geq b;\ x \geq 0\}.$$

let $x \in S$ be an EP of S.

$$x = (x_1, x_2 \ldots x_n)^T \quad x_i \geq 0 \quad \forall\ 1 \leq i \leq n$$

let k be index s.t. (WLOG) $x_1 \ldots x_k > 0$ & $x_{k+1} \ldots x_n = 0$ where $0 \leq k \leq n$

$$x = (x_1 \ldots x_k, 0 \ldots 0)^T$$

Let the corresponding columns of A ($x_i > 0$) be $\{a_1, a_2 \ldots a_k\}$. $a_i \sim m \times 1$

↳ these are L.I vectors in R^m. (~~k≤m~~)

Suppose $\{a_1 \ldots a_k\}$ be LD.

$\Rightarrow \exists$ some scalars d_i ~~≠0~~ d_i's, not all 0's, s.t.

$$\sum_{i=1}^{k} d_i a_i = 0. \quad -(1)$$

set $\mu = \min \left\{ \frac{x_i}{|d_i|},\ 1 \leq i \leq k,\ d_i \neq 0 \right\}$.

$\Rightarrow \mu > 0$

Take any No ε, $0 \leq \varepsilon \leq \mu$.

$x_i' = x_i + \varepsilon d_i \quad 1 \leq i \leq k \quad$ & $x_{i}' = x_i = 0 \quad \forall\ k+1 \leq i \leq n$

$x_i'' = x_i - \varepsilon d_i \quad 1 \leq i \leq k. \quad$ & $x_i'' = x_i = 0$ ''

Note.

(1) $x_i' \neq x_i''$ $\because$ atleast one of the d_i is not zero.

(2) $x = \frac{1}{2}(x' + x'')$

(3). x' & $x'' \geq 0$. by choice of $\varepsilon > 0$.

As $0 < \varepsilon < \mu = \min\left\{ \frac{x_i}{|d_i|} : d_i \neq 0 \right\}$.

$$0 < \varepsilon < \frac{x_i}{|d_i|} \quad \forall\, i,\ 1 \le i \le k \text{ for which } d_i \neq 0.$$

$d_i > 0$

$$0 < \varepsilon < \frac{x_i}{d_i}$$

$$\Rightarrow 0 < \varepsilon d_i < x_i$$

$$\therefore x_i - \varepsilon d_i > 0.$$

$$\Downarrow$$

$$\boxed{x_i'' \geq 0.}$$

$d_i < 0.$

$$0 < \varepsilon < \frac{x_i}{-d_i}$$

$$\cancel{0 > \varepsilon d_i >}$$

$$\Downarrow \quad \cancel{0 < -\varepsilon d_i < x_i}$$

$$\Rightarrow x_i + \varepsilon d_i > 0$$

$$\Downarrow$$

$$\boxed{x_i' > 0}$$

$$\Downarrow$$

$$x_i',\ x_i'' \geq 0$$

(4). $$Ax' = \sum_{i=1}^{n} a_i x_i' = \sum_{i=1}^{k} a_i x_i'$$

$$= \sum_{i=1}^{k} \left(a_i (x + \varepsilon d_i) \right)$$

$$= \sum_{i=1}^{k} a_i x_i + \varepsilon \underbrace{\sum_{i=1}^{k} d_i a_i}_{=0 \text{ by } (2)}$$

$$= \sum_{i=1}^{n} a_i x_i$$

$$= Ax = b \quad (\because x \in S)$$

Similarly $\boxed{Ax'' = b}$

by (3) & (4).

& $x', x'' \in S$

By (1).

$x' \neq x''$

By (2)

$x = \frac{1}{2}(x' + x'')$

which contradicts x is an e.p. of S.

$\Rightarrow$ (1) must be implies all $d_i = 0$

$\Rightarrow \{a_1 \ldots a_k\}$ are LI's. in $\mathbb{R}^m$.

$\Rightarrow k \le m \to k = m$ — (i')

$k < m$. — (ii')

Case (1) - $k = m$.

$B = [a_1, a_2 \ldots a_m]_{m \times m}$ LI

B must be invertible.

& $Ax = b$.

$\Rightarrow \sum_{i=1}^{n} a_i x_i = b \Rightarrow \sum_{i=1}^{m} a_i x_i = b. \Rightarrow Bx = b.$

$x \geq 0.$

x is a BFS of system $Ax = b$, $x \geq 0$.

& it is non-degenerate. as $x_i > 0 \ \forall \ 1 \le i \le k = m$.

Case 2 $k < m$.

$\{a_1, a_2 \ldots a_k\}$ is LI in $\mathbb{R}^m$, $k < m$.

$B = [\underbrace{a_1 \ldots a_k}_{m \times 1} \underbrace{— — —}_{m-k}]_{m \times m}$

$m-k$ (adding $m-k$ vectors)

This set can be extended to form a basis of $\mathbb{R}^n$.

Also Rank(A) = col. Rank(A) = row Rank(A) = m -

we can find $m-k$ columns in A which together with $\{a_1 \ldots a_k\}$ forms a basis of $\mathbb{R}^m$.

Note $Ax = Bx = b$ as $x_i = 0 \ \forall \ k+1 \le i \le n$, $x > 0 \ \forall \ 1 \le i \le k$.

$\Rightarrow x$ is a BFS of the system (degenerate)