# Discrete-Time Fourier Series

Lecture 28

# Discrete-time Fourier Transform

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

$$h[n] = \frac{1}{2\pi}\int_{0}^{2\pi} H(e^{j\Omega})e^{jn\Omega}d\Omega$$

# $h[n]$ is periodic

$$h[n] = \frac{1}{2\pi}\int_0^{2\pi} H(e^{j\Omega})e^{jn\Omega}d\Omega$$

$$h[n+N] = \frac{1}{2\pi}\int_0^{2\pi} H(e^{j\Omega})e^{j(n+N)\Omega}d\Omega$$

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$$h[n] = h[n+N] \qquad e^{jN\Omega} = e^{j2\pi k} \qquad \Omega = \frac{2\pi k}{N}$$

# $h[n]$ is periodic

$$h[n] = \frac{1}{2\pi}\int_{0}^{2\pi} H(e^{j\Omega})e^{jn\Omega}d\Omega$$

$$\Omega = \Omega_o k$$

$$h[n] = \frac{1}{2\pi}\sum_{k=0}^{N-1} H(e^{jk\Omega_o})e^{jk\Omega_o n}\Omega_o$$

$$h[n] = \sum_{k=0}^{N-1}\frac{1}{N} H(e^{jk\Omega_o})e^{jk\Omega_o n}$$

# $h[n]$ is periodic

$$h[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_o n}$$

$$a_k = \frac{1}{N} H(e^{jk\Omega_o})$$

# $a_k$ from $h[n]$

$$h[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_o n}$$

$$\sum_{n=0}^{N-1} h[n] e^{-jl\Omega_o n} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} a_k e^{jk\Omega_o n} e^{-jl\Omega_o n}$$

$$= \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{j(k-l)\Omega_o n}$$

# $a_k$ *from* $h[n]$

$$\sum_{n=0}^{N-1} e^{j(k-l)\Omega_o n} = \begin{cases} N & \textbf{if } k = l \\ \dfrac{1-e^{j(k-l)2\pi}}{1-e^{j(k-l)\Omega_o}} = 0 & \textbf{if } k \neq l \end{cases}$$

$$\sum_{n=0}^{N-1} e^{j(k-l)\Omega_o n} = N\delta[k-l]$$

# $a_k$ from $h[n]$

$$\sum_{n=0}^{N-1} h[n]e^{-jl\Omega_o n} = \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{j(k-l)\Omega_o n}$$

$$\sum_{n=0}^{N-1} h[n]e^{-jl\Omega_o n} = \sum_{k=0}^{N-1} a_k N\delta[k-l]$$

$$a_l = \frac{1}{N}\sum_{n=0}^{N-1} h[n]e^{-jl\Omega_o n}$$

# Discrete-time Fourier Series

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_o n}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}$$

# Think

| **Consider the following statements:** |
|---|
| a) $a_k$ are periodic with period N |
| b) There are only N orthogonal vectors (complex exponential) required to express a periodic discrete time signal. |

| | |
|---|---|
| 1) Both a and b are correct | 2) a is true, b is false |
| 3) a is false, b is true | 4) Both a and b are false |

# Think

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| | |
|---|---|
| 1) **Both a and b are correct** | 2) a is true, b is false |
| 3) a is false, b is true | 4) Both a and b are false |

# Discrete-time Fourier Series

$$a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(k+N)\Omega_o n}$$

$$a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n}e^{-jN\Omega_o n}$$

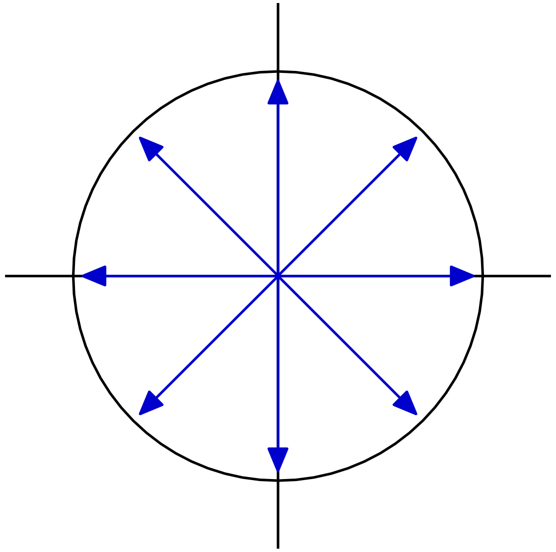
$$a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n}e^{-j2\pi n} = a_k$$

# Discrete-time Fourier Series

$$a_{k+N} = a_k$$

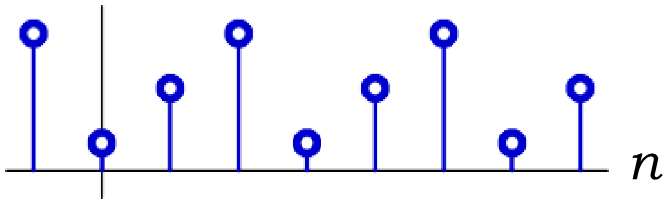
$$e^{-j(k+N)\Omega_o n} = e^{-jk\Omega_o n}$$

Number of distinct complex exponentials

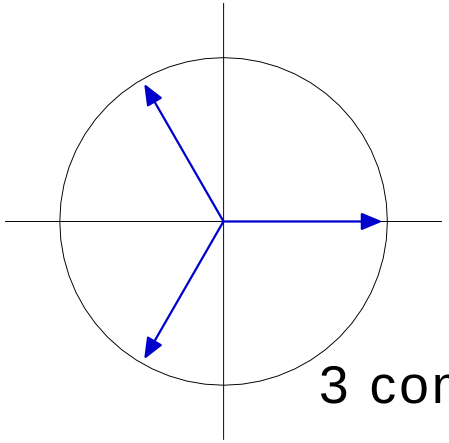


$$\frac{2\pi}{N} m = 2\pi \qquad m = N$$

# Discrete-time Fourier Series

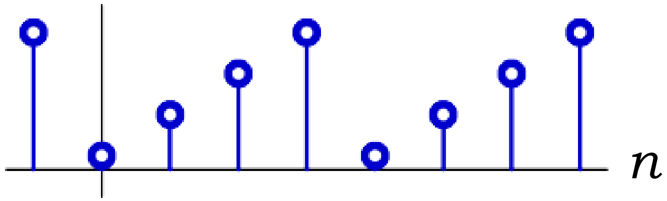



3 samples repeated in time

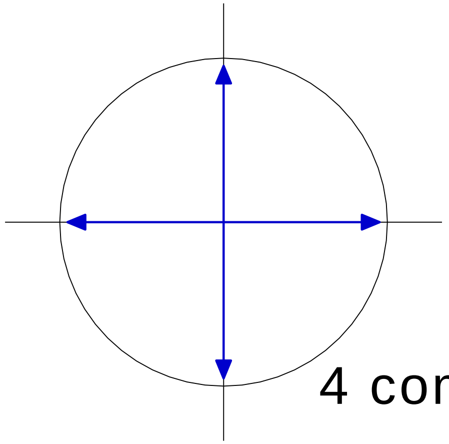


3 complex exponentials

Example: periodic in N = 4




4 samples repeated in time



4 complex exponentials

# Discrete-time Fourier Series

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_o n}$$

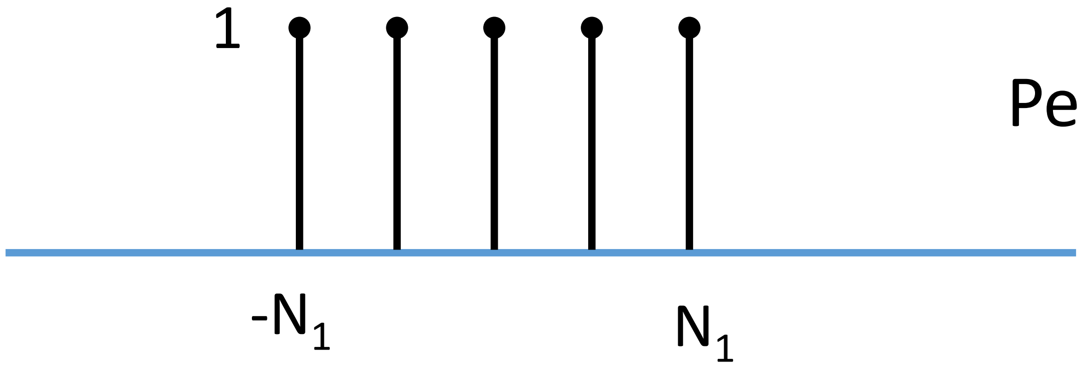
Strong duality

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}$$

Finite number of equations and finite number of unknowns

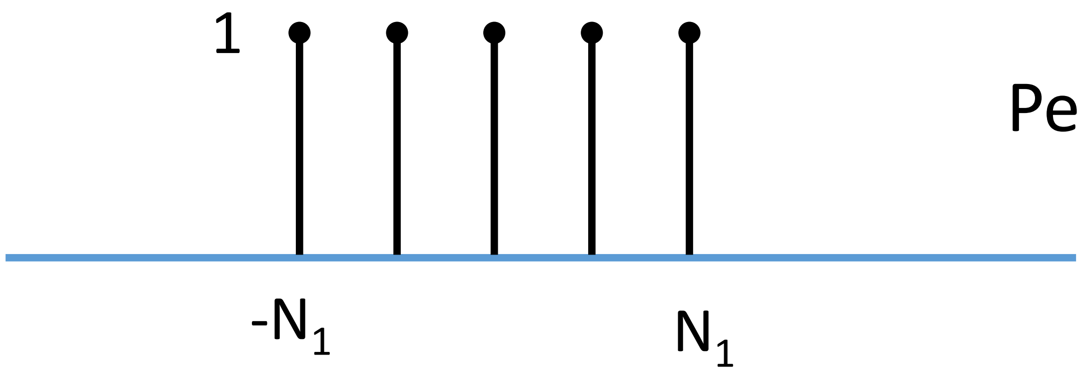
No convergence issues

# Example 1




Period = N

# Example 1




Period = N

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_O n}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\Omega_0 n} = \frac{1}{N}\left[\frac{\sin\left(\frac{k\Omega_0(2N_1+1)}{2}\right)}{\sin\left(\frac{k\Omega_0}{2}\right)}\right]$$

DT $\quad \dfrac{\sin(nx)}{\sin(x)} = \dfrac{\sin(x)}{x}$ CT

# Properties of DTFS

**Duality** $x[n] \rightarrow a_k$ $a_n \rightarrow \dfrac{x[-k]}{N}$

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Parsevals theorem

$$\frac{1}{N}\sum_{n=<N>} x[n]\overline{y[n]} = \sum_{l=<N>} a_l \overline{b_l}$$

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$$x[n] - x[n-1] \rightarrow \left(1 - e^{-jk\Omega_0}\right)a_k$$

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$$x[n] - x[n-1] \rightarrow \left(1 - e^{-jk\Omega_0}\right) a_k$$

Running sum

$$\sum_{k=-\infty}^{n} x[k] \rightarrow \frac{a_k}{1-e^{-jk\Omega_0}} \text{ If } a_0 = 0$$

# DTFS (represented by N complex exponentials)

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_o n}$$

# DTFS by Matrix

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} e^{\frac{j2\pi}{N}0.0} & e^{\frac{j2\pi}{N}1.0} & e^{\frac{j2\pi}{N}2.0} & e^{\frac{j2\pi}{N}3.0} \\ e^{\frac{j2\pi}{N}0.1} & e^{\frac{j2\pi}{N}1.1} & e^{\frac{j2\pi}{N}2.1} & e^{\frac{j2\pi}{N}3.1} \\ e^{\frac{j2\pi}{N}0.2} & e^{\frac{j2\pi}{N}1.2} & e^{\frac{j2\pi}{N}2.2} & e^{\frac{j2\pi}{N}3.2} \\ e^{\frac{j2\pi}{N}0.3} & e^{\frac{j2\pi}{N}1.3} & e^{\frac{j2\pi}{N}2.3} & e^{\frac{j2\pi}{N}3.3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

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Matrices are inverse of each other

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Number of operations increases as $N^2$

# Fast Fourier "Transform"

Divide FS of length 2N into two of length N (divide and conquer)

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^1 & W_4{}^2 & W_4{}^3 \\ W_4{}^0 & W_4{}^2 & W_4{}^0 & W_4{}^2 \\ W_4{}^0 & W_4{}^3 & W_4{}^2 & W_4{}^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

where $W_4^m = \frac{1}{4} e^{-j\frac{2\pi}{4}m}$

Number of Multiplications = 16
Number of Additions = 4×3=12
Total=28