Vector - Subspace - Lecture 7

Recall :- $W \subseteq V$ is subspace of V if $\alpha u + \beta v \in W$ for all $\alpha, \beta \in \mathbb{F}$ and $u, v \in W$.

Examples

1. $W = \{0\}$ (Zero vector)

2. $W = V(\mathbb{F})$ (Entire vector space)

These two subspaces are called trivial subspaces.

3. $V = \mathbb{R}^2(\mathbb{R})$

$$W = \{ (x, y) \in \mathbb{R}^2 \mid \underbrace{x + 2y = 0}_{\text{eq}^n \text{ of line}} \} \neq \phi$$

Let $u, v \in W$ and $\alpha, \beta \in \mathbb{R}$.

$u = (x_1, y_1)$ with $x_1 + 2y_1 = 0$

$v = (x_2, y_2)$ with $x_2 + 2y_2 = 0$

$$\alpha u + \beta v = (\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)$$

$$= (\underbrace{\alpha x_1 + \beta x_2}_{x}, \underbrace{\alpha y_1 + \beta y_2}_{y}) \in \mathbb{R}^2$$

We need to prove $x + 2y = 0$

$$\Rightarrow (\alpha x_1 + \beta x_2) + 2(\alpha y_1 + \beta y_2)$$

$$= \alpha(x_1 + 2y_1) + \beta(x_2 + 2y_2)$$

$$= \alpha \cdot 0 + \beta \cdot 0 = 0$$

$\alpha u + \beta v \in W$

$\Rightarrow W$ is a subspace.

(4) $V = \mathbb{R}^3(\mathbb{R})$ and

$$W = \{ (x, y, z) \in \mathbb{R}^3 \mid \underbrace{x+y-z=0}_{\text{eq}^n \text{ of plane}} \}$$

Verify that $\alpha u + \beta v \in W$ $\forall$ $\alpha, \beta \in \mathbb{R}$ & $u, v \in W$.

(5) $V = \mathbb{R}^3(\mathbb{R})$ and

$$W = \{ (x, y, z) \in \mathbb{R}^3 \mid \underbrace{x+y+z=5}_{\text{eq}^n \text{ of plane}} \}$$

W is not a subspace of V.

Note that $(0,0,0) \notin W$

(6) $V = M_2(\mathbb{R})$ and $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a = c+d \\ b = c-d \end{array} \right\}$

(7) $V = M_2(\mathbb{C})$ and

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a = \bar{d} \right\}.$$ Then

(a) W is not a subspace of complex vector space $M_2(\mathbb{C})$

(b) W is a vector subspace of real vector space $M_2(\mathbb{C})$.

<u>Theorem</u> $\div$ If W_1 and W_2 are two subspaces of V. Then

(a) W_1+W_2 is a subspace of V.

(b) $W_1 \cap W_2$ is a subspace of V.

<u>Proof</u> $\div$ (a) $W_1+W_2 = \{ w_1+w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2 \}$

let $u = u_1+u_2$ $\qquad u_1, v_1 \in W_1$ –

$v = v_1+v_2$ $\qquad u_2, v_2 \in W_2$ *

$\alpha, \beta \in \mathbb{F}$

$$\alpha u+\beta v = \alpha(u_1+u_2) + \beta(v_1+v_2)$$

$$= \underbrace{(\alpha u_1+\beta v_1)}_{\in W_1} + \underbrace{(\alpha u_2+\beta v_2)}_{\in W_2}$$

$\alpha u+\beta v \in W_1+W_2$

$\Rightarrow W_1+W_2$ is a subspace.

(b) If $u, v \in W_1 \cap W_2$

$\Rightarrow u, v \in W_1$ and $u, v \in W_2$

$\Rightarrow \alpha u + \beta v \in W_1$ and $\alpha u + \beta v \in W_2$ $\quad \alpha, \beta \in \mathbb{F}$

$\Rightarrow \alpha u + \beta v \in W_1 \cap W_2$ $\quad \alpha, \beta \in \mathbb{F}$.

Remark (1) If $\{ W_i \mid i \geq 1 \}$ is a family of subspaces of V. Then $\bigcap_{i \geq 1} W_i$ is also a subspace.

(2) $W_1 \cup W_2$ need not be a subspace.

$W_1 = \{ (x, 0) \mid x \in \mathbb{R} \}$

$W_2 = \{ (0, y) \mid y \in \mathbb{R} \}$

$W_1 \cup W_2$ is not a subspace.

Subspace generated by a set

Let $S \subseteq V$ be a non empty subset in V. Then the subspace generated by S, usually denoted by $\langle S \rangle$, is the intersection of all the subspaces of V containing S.

$$\langle S \rangle = \bigcap_{S \subseteq W \leq V} W$$

(clearly it is a subspace of V)

By definition, it is the smallest subspace containing S.

Linear span of a set $\rightarrow$ Let $V(\mathbb{F})$ be a vector space and $\phi \neq S \subseteq V(\mathbb{F})$. Then the Linear span of

S, denoted by L(S) is given by

$$L(S) = \{ \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \mid \alpha_i \in \mathbb{F}, u_i \in S \}$$

If $S = \phi$, then $L(S) = \{0\}$.

The expression $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$ is called linear combination of vectors $u_1, u_2, \ldots, u_n$.

$S \subseteq L(S)$ (span (S))

easy to see.

Exercise :- L(S) is a subspace of $V(\mathbb{F})$.

<u>Hint</u> :→ $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

$v = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$ (w.l.o.g)

Then it is easy to very $\alpha u + \beta v \in L(S)$

<u>Theorem</u> :→ $L(S) = \langle S \rangle$ (subspace generated by S)

<u>Proof</u> :→ $L(S)$ is a subspace of V containing S.

Hence by the defn of $\langle S \rangle$, we have

$$\langle S \rangle \subseteq L(S) \quad \text{— eq}^n \ (1)$$

On the other hand, if $S \subseteq W$

$$L(S) \subseteq W$$

$L(S) \subseteq W \quad \forall\ W$ with $S \subseteq W$

$$\Rightarrow L(S) \subseteq \bigcap_{S \subseteq W \leq V} W = \langle S \rangle$$

$$\Rightarrow L(S) \subseteq \langle S \rangle \quad \text{--- eq}^n \text{ (2)}$$

Hence $L(S) = \langle S \rangle$. by eq^n (1) and (2)

<u>Example</u> (1) $S = \{(1,1,1), (1,2,3)\}$. Find the subspace generated by S.

<u>Proof</u> :- $\langle S \rangle = L(S) = \{\alpha(1,1,1) + \beta(1,2,3) \mid \alpha, \beta \in \mathbb{R}\}$

$$L(S) = \{ (\alpha+\beta, \alpha+2\beta, \alpha+3\beta) \mid \alpha, \beta \in \mathbb{R} \}$$

$$\Updownarrow$$

$$L(S) = \{ (x, y, z) \in \mathbb{R}^3 \mid x = \alpha+\beta, \; y = \alpha+2\beta, \; z = \alpha+3\beta \}$$

$$\Rightarrow \quad \left.\begin{aligned} x &= \alpha+\beta \\ y &= \alpha+2\beta \\ z &= \alpha+3\beta \end{aligned}\right\} \quad \text{system of linear eq}^{n}$$

$$\left|\begin{array}{ccc} 1 & 1 & x \\ 1 & 2 & y \\ 1 & 3 & z \end{array}\right| \rightarrow \left|\begin{array}{ccc} 1 & 1 & x \\ 0 & 1 & y-x \\ 0 & 2 & z-x \end{array}\right| \rightarrow \left|\begin{array}{ccc} 1 & 1 & x \\ 1 & 1 & y-x \\ 0 & 0 & z-2y+x \end{array}\right|$$

$$\Longleftrightarrow \quad z - 2y + x = 0$$

In other words

$$L(S) = \{ (x, y, z) \in \mathbb{R}^2 \mid z - 2y + x = 0 \}$$

Example 2 $\Rightarrow$ $S = \{ (1, 2, 1), (1, 0, -1), (1, 1, 0) \} \subseteq \mathbb{R}^3(\mathbb{R})$.

What is $\langle S \rangle = L(S)$?

Sol^n $\Rightarrow$

$$L(S) = \{ \alpha (1, 2, 1) + \beta (1, 0, -1) + \gamma (1, 1, 0) \mid \alpha, \beta, \gamma \in \mathbb{R} \}$$

$$= \{ (\alpha + \beta + \gamma, 2\alpha + \gamma, \alpha - \beta) \mid \alpha, \beta, \gamma \in \mathbb{R} \}$$

$\Longrightarrow$

$$x = \alpha + \beta + \gamma$$

$$y = 2\alpha + \gamma \qquad \qquad \alpha, \beta, \gamma \in \mathbb{R}.$$

$$z = \alpha - \beta$$

matrix for system of eqn

$$\begin{bmatrix} 1 & 1 & 1 & x \\ 2 & 0 & 1 & y \\ 1 & -1 & 0 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & x \\ 0 & -1 & -\frac{1}{2} & \frac{y-2x}{2} \\ 0 & 0 & 0 & x-y+z \end{bmatrix}$$

$$\Longrightarrow \quad x-y+z=0$$

$$L(S) = \{ (x, y, z) \in \mathbb{R}^3 \mid x-y+z=0 \}$$

Example $\rightarrow$ If we take $e_i = (0, 0, \ldots, \underset{i\text{th place}}{1}, \ldots 0) \in \mathbb{R}^n$

$1 \le i \le n$

Then $L(\{e_1, e_2, \ldots, e_n\}) = \mathbb{R}^n(\mathbb{R})$

Finite dimensional Vector space

A vector space $V(\mathbb{F})$ is said to be f.d if there exists a finite set $S \subseteq V$ such that $L(S) = V(\mathbb{F})$.

$\mathbb{R}^n(\mathbb{R})$ is finite dim

$\mathbb{C}^n(\mathbb{C})$ is finite dim (Is $\mathbb{C}^n(\mathbb{R})$ finite dim ?)

$P_n(\mathbb{R})$ is finite dim

$P_n(\mathbb{R}) = \{ f(x) \in \mathbb{R}[x] \mid \deg(f(x)) \leq n \}$

$S = \{1, x, x^2, \dots, x^n\}$. Then

$L(S) = P_n(\mathbb{R}$

Such finite S is not unique.

In $\mathbb{R}^2$, if we take

$$S_1 = \{e_1, e_2\} = \{(1,0), (0,1)\} \quad *$$

$$S_2 = \{(1,0), (2,1)\} \quad *$$

$$L(S_1) = L(S_2) = \mathbb{R}^2$$

Linear Independence and dependence

Let $V(\mathbb{F})$ be a vector space and $\phi \neq S \subseteq V$. Then S is said to be linearly dependent (L.D) if $\exists$ vectors $u_1, \ldots, u_n \in S$ and scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ (at least one nonzero)

s.t $\quad \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$

Example $\Rightarrow$ $S = \{ \underset{u}{(2,2,2)}, \underset{v}{(3,3,3)} \} \subseteq \mathbb{R}^3(\mathbb{R})$

$$3u - 2v = 0$$

Hence S is L.D.

Def^n $\Rightarrow$ A subset $S \subseteq V(\mathbb{F})$ is said to be linearly independent (L.I) if it is not linearly dependent (L.D)

In particular if S is a finite set given by
$S = \{ u_1, u_2, \ldots, u_n \}$. then

$$S \text{ is L.I} \iff \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$$
$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 \cdots = \alpha_n = 0$$

Example ① $S = \{ e_i \mid 1 \le i \le n \} \subseteq \mathbb{R}^n(\mathbb{R})$

Then S is L.I.

② $S = \{ (1,2,1), (2,1,4), (3,3,5) \}$. Find if S is L.I. or not.

③ $S = \{ (1,2,3), (2,3,4), (1,1,2) \}$. Find if S is L.I. or not.

Propositions ① Subset of a L.I. set is L.I.

② Superset of a L.D set is L.D.

③ Zero vector can not belong to a L.I. set.