Lecture on 10/12/2020

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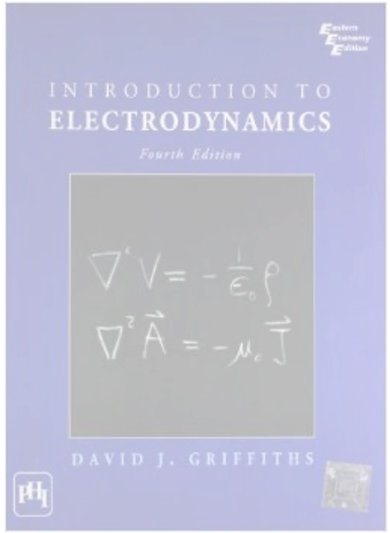
PYL101 course:

Electromagnetics & Quantum Mechanics

- *Next few classes, we will discuss the following topics:*
- *Magnetostatics (ch5)*
- *Magnetic fields in matter (ch6)*
- *Electrodynamics (ch7)*
- *Continuity equation and Poynting's theorem (ch8)*



Maxwell's equations *plus* the Lorentz force law,

$$\vec{F} = e\overrightarrow{(E} + \vec{v}\times\vec{B})$$

summarize the entire theoretical content of classical electrodynamics.

✧ Maxwell's equations tell 'how charges produce fields'.................

stationary charges -> E
moving charges -> B

changing fields also...

❖ Force law tells 'how fields affect charges' and their motion……

Maxwell's Equations in Matter:

- Maxwell's equations are complete and correct, however, when we are working with materials that are subject to electric and magnetic **polarization**……….

- there is a more convenient way to write these equations: **(since there will be bound charges/ currents)**

Static case
- electric polarization **P** produces bound charge density

$$\rho_b = -\nabla . P$$

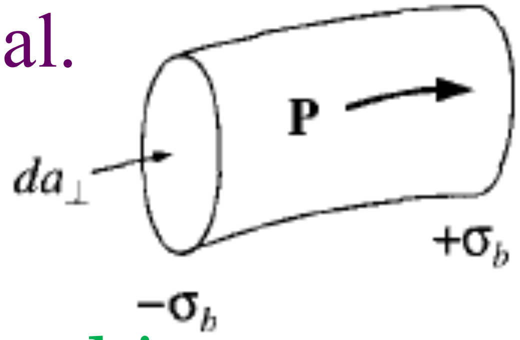
- & magnetic polarization (magnetization) **M** produces $\mathbf{J_b}$

$$J_b = \nabla \times M$$

✧ In **non-static case**, we have to consider that any change in electric polarization involves a flow of (bound) paired charges (means a current, & let's call it $\mathbf{J_p}$)…...

✧ so this current also should be included in the total **J**.

Let's examine a tiny chunk of polarized material.



Charge density on the surface is:

$$\sigma_b = P$$

Let's assume that **P** points to the right, and is increasing, then each +ve charge moves a bit to the right and each -ve charge to the left…

When the **P** increases a bit, the charge on each end increases accordingly, giving a net current

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp$$

The cumulative effect is the polarization current $\mathbf{J_p}$.

Note that this **polarization current ($\mathbf{J_p}$)** has nothing to do with the *bound* current $\mathbf{J_b}$……..

$\mathbf{J_b}$ is associated with *magnetization* of the material, and involves the spin and orbital motion of electrons; called magnetization current…

$\mathbf{J_p}$ is the result of the linear motion of charges when the electric polarization changes, called – polarization current….

This polarization current density is: $\mathbf{J}_p = \frac{\partial \sigma_b}{\partial t} = \frac{\partial \mathbf{P}}{\partial t}$

Let's check whether above equation is consistent with the continuity equation:

$$\nabla \cdot \mathbf{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}$$

The continuity equation is satisfied; in fact, $\mathbf{J_p}$ is essential to ensure the conservation of bound charge.

A changing magnetization does not lead to any analogous accumulation of charge or current.

The bound current $J_b = \nabla \times M$ varies in response to changes in M, to be sure, but that's about it…

The total charge density has ***two*** parts:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$$

and the current density has ***three*** parts:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

Gauss's law can now be written as:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \mathbf{P})$$

$\boxed{\nabla \cdot \mathbf{D} = \rho_f}$ where, **D** can be written as:

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

For, linear media $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$

Ampere's law (with Maxwell's correction):

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$= \mu_0 \left(\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Now, let's use

$$\mathbf{H} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad \& \quad \mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}}$$

Note that **Faraday's law** and $\nabla . B = 0$ are not affected, as they do not involve ρ or J.

Maxwell's equations:
in terms of free charges & currents

$$\text{(i)}\quad \nabla \cdot \mathbf{D} = \rho_f, \qquad \text{(iii)}\quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)}\quad \nabla \cdot \mathbf{B} = 0, \qquad \text{(iv)}\quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

Maxwell's equations in matter now contain both **E** & **D**, also **B** & **H**.

Hence, we also mention material equations along with them.

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \text{and} \quad \mathbf{M} = \chi_m \mathbf{H},$$

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

$$\epsilon \equiv \epsilon_0(1+\chi_e) \text{ and } \mu \equiv \mu_0(1+\chi_m)$$

Maxwell's equations:

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho_f && \Rightarrow (i) \quad \oint_S \vec{D}\cdot d\vec{a} = Q_{f_{end}} \\ \vec{\nabla}\cdot\vec{B} &= 0 && \Rightarrow (ii) \quad \oint_S \vec{B}\cdot d\vec{a} = 0 \end{aligned}\right\}$$

Over any closed surface S

$$\left.\begin{aligned} \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} && \Rightarrow (iii) \quad \oint_P \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B}\cdot d\vec{a}\right) \\ \vec{\nabla}\times\vec{H} &= \vec{J}_f + \frac{\partial\vec{D}}{\partial t} && \Rightarrow (iv) \quad \oint_P \vec{H}\cdot d\vec{l} = I_{f_{encl.}} + \frac{d}{dt}\left(\int_S \vec{D}\cdot d\vec{a}\right) \end{aligned}\right\}$$

For any surface S bounded by the closed loop P

- Now let's discuss the boundary conditions,

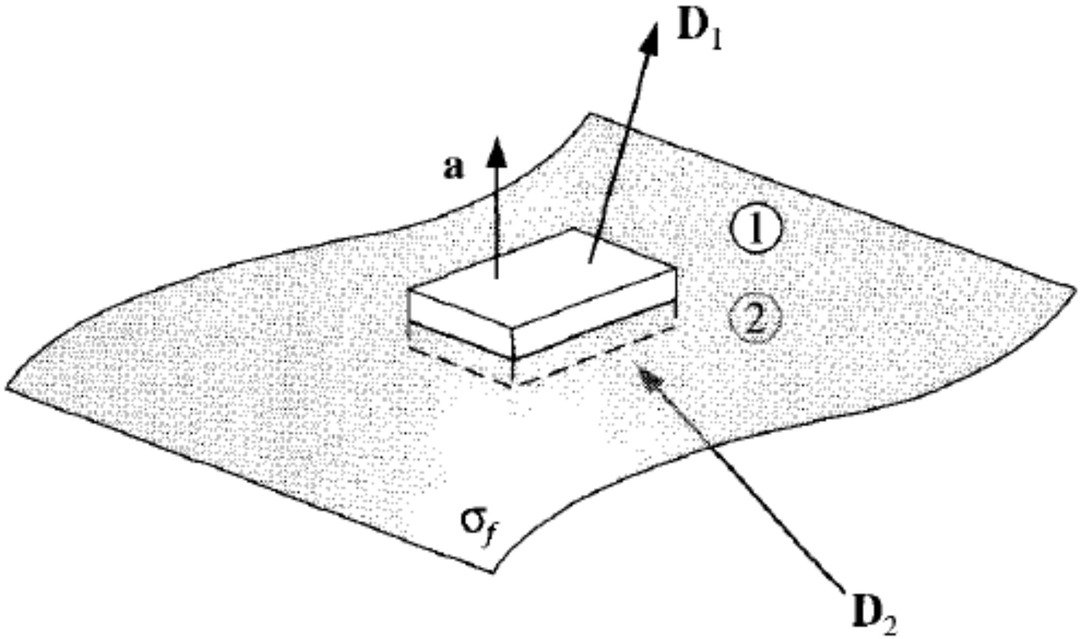
Generally, the fields E, B, D, & H are discontinuous at a boundary between two different media or at a surface that carries charge density or current density….

- The discontinuities can be deduced from (the integral form of) Maxwell's equations…..

Boundary Conditions:

$$(i) \oint_S \vec{D} \cdot d\vec{a} = Q_{f_{end}}$$

Applying (i) to a wafer thin Gaussian pillbox existing just slightly into the material on either side of the boundary..



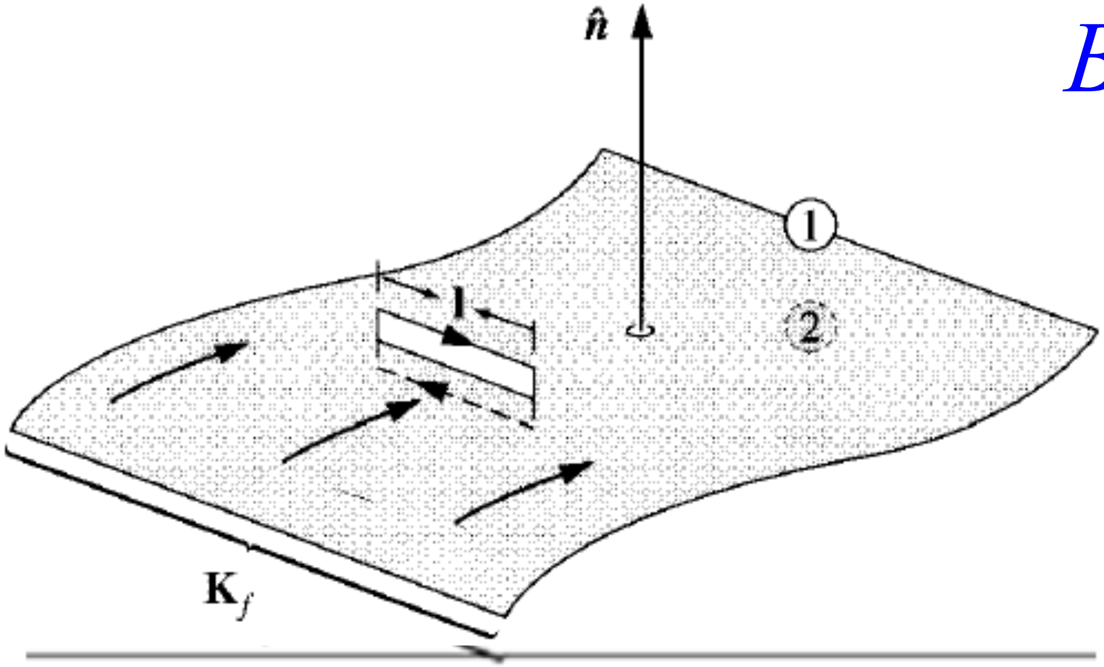
$$\boxed{\vec{D}_1 \cdot \vec{a} - \vec{D}_2 \cdot \vec{a} = \sigma_f a}$$

- The edge of wafer contributes nothing in the limit of small thickness

Therefore, $$\boxed{\Rightarrow D_1^{\perp} - D_2^{\perp} = \sigma_f}$$

- Identical reasoning, applied to (ii) gives $\left[(ii) \oint_S \vec{B} \cdot d\vec{a} = 0\right]$

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0}$$



Boundary Conditions:

$$(iii) \quad \oint_P \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{a}\right)$$

- For (iii) and (iv); consider a very thin Amperian Loop straddling the surface; Ampere's Law give

$$\boxed{\vec{E}_1 \cdot \vec{l} - \vec{E}_2 \cdot \vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a}}$$ (no contribution from the 2 ends)

- In the limit as the <u>width</u> of the loop → 0, the flux through the loop <u>vanishes</u>

$$\boxed{\Rightarrow \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0}$$

Components of E parallel to the interface are continuous across the boundary

$$(iv)\quad \oint_P \vec{H}\cdot d\vec{l} = I_{f_{encl.}} + \frac{d}{dt}\left(\int_S \vec{D}\cdot d\vec{a}\right)$$

$$I_{f_{\text{enc}}} = \mathbf{K}_f \cdot (\hat{\mathbf{n}} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l}$$

As flux **D**.da →0 in limit, loop width →0

- By same token as in (iii), (iv) gives

$$\boxed{\Rightarrow \vec{H}_1\cdot\vec{l} - \vec{H}_2\cdot\vec{l} = I_{f_{encl.}} + 0 \Rightarrow \vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n}}$$

discontinuous by free surface current density

- For a linear medium, the boundary conditions can be modified to:

$(\because \vec{D} = \varepsilon\vec{E} \quad \& \quad \vec{B} = \mu\vec{H})$

$$\left\{\begin{array}{ll} (i)\ \ \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f, & (iii)\ \ \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0 \\ (ii)\ \ \vec{B}_1^{\perp} - \vec{B}_2^{\perp} = 0, & (iv)\ \dfrac{\vec{B}_1^{\parallel}}{\mu_1} - \dfrac{\vec{B}_2^{\parallel}}{\mu_2} = \vec{K}_f \times \hat{n} \end{array}\right\}$$

Consider, if there are no free charges or free currents at the interface, then rewrite…..

$$\begin{array}{ll} (i)\ \ \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0 & (iii)\ \ \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0 \\ (ii)\ \ \vec{B}_1^{\perp} - \vec{B}_2^{\perp} = 0 & (iv)\ \ \dfrac{\vec{B}_1^{\parallel}}{\mu_1} - \dfrac{\vec{B}_2^{\parallel}}{\mu_2} = 0 \end{array}$$

We shall see again,.. that these equations are the basis for the theory of reflection and refraction.

Electromagnetic (EM) waves?

- ✧ **Faraday**'s law told us that "time-varying magnetic fields generate electric fields"

- ✧ **Maxwell** found that "time-varying electric fields generate magnetic fields"

If we take both together, then
electric field → magnetic field → electric field → …

- ➢ with this we can have disturbances in the electric and magnetic fields that propagate across space

⇨ **ELECTROMAGNETIC WAVES**

Note: No medium is required, E and B can exist in a vacuum

Energy in electromagnetic waves/review:

Results from Ch.2:

Work necessary to assemble static charge distribution against the Coulomb's repulsion is (eq.2.45)

- Energy content of electric field: $$W_e = \frac{\varepsilon_0}{2}\int E^2\, d\boldsymbol{\tau}$$

Results from Ch.7:

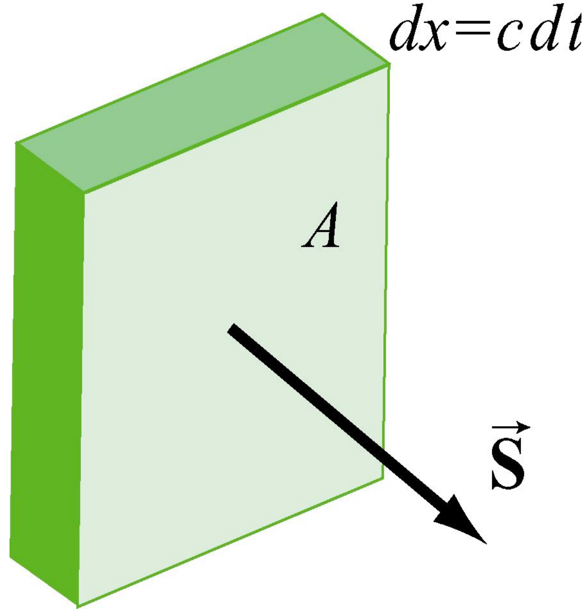
Work required to get currents going (against back *emf*) is, eq.7.34/7.35

- Energy in magnetic field: $$W_m = \frac{1}{2\mu_0}\int B^2 d\tau$$

➢ This suggests that the total electromagnetic energy stored in the volume V (energy density):

$$U_{em} = \frac{1}{2}\int (\varepsilon_0 E^2 + \frac{B^2}{\mu_0}) d\boldsymbol{\tau}$$

The Poynting Vector



- Electric and magnetic fields store energy.
- Energy can also be transported by electromagnetic waves that consist of both E & B fields…..

✧ Consider a plane electromagnetic wave passing through a small volume element of area A and thickness dx…

✧ The total energy stored in the electromagnetic fields in the volume element is given by

$$dU = uA\,dx = (u_E + u_B)A\,dx = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)A\,dx$$

✧ Because the EM wave propagates with the speed of light c, the amount of time it takes for the wave to move through the volume element is $dt = dx/c$ …….

The Poynting Vector

✧ the rate of change of energy per unit area, denoted by the symbol ***S***, as

$$S = \frac{dU}{A\,dt} = \frac{c}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$ The SI unit of S is $[\mathrm{W \cdot m^{-2}}]$.

Recall that the magnitude of the fields satisfy $E = cB$ and $c = 1/\sqrt{\mu_0 \varepsilon_0}$

$$S = \frac{c}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) = \frac{cB^2}{\mu_0} = c\varepsilon_0 E^2 = \frac{EB}{\mu_0}$$

➢ The rate of energy flow per unit area is called the "Poynting vector" and defined by the vector product

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$$

$$\left|\vec{\mathbf{S}}\right| = \frac{\left|\vec{\mathbf{E}} \times \vec{\mathbf{B}}\right|}{\mu_0} = \frac{EB}{\mu_0} = S$$

For our plane transverse electromagnetic waves, the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are perpendicular, and the magnitude of $\vec{\mathbf{S}}$ is

The Poynting Vector

EM waves are able to transport energy from transmitter to receiver……..
(example: from the Sun to our skin).

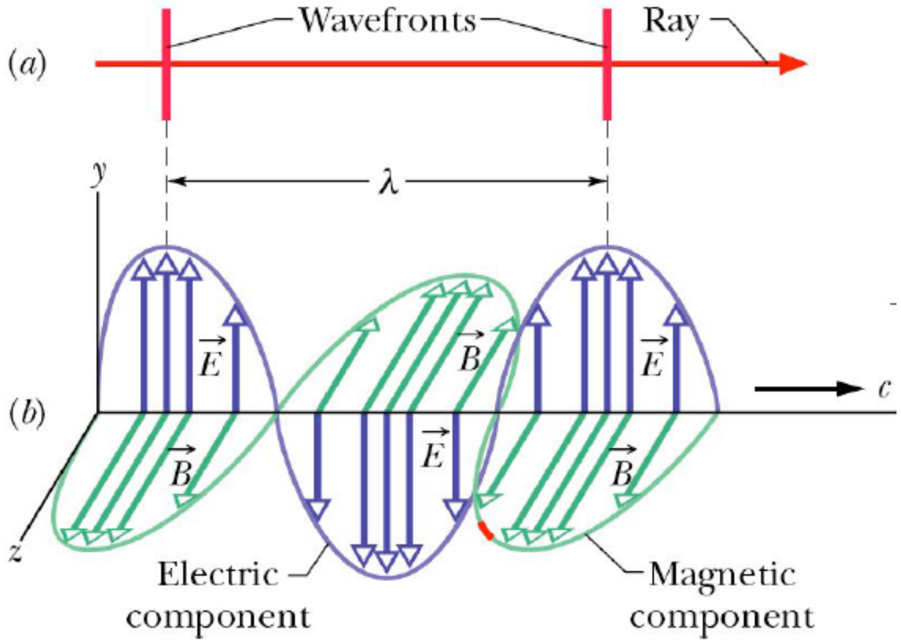
J. Henry Poynting

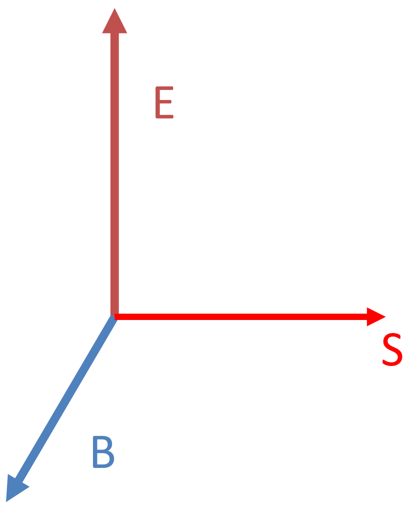
(1852-1914)

The power transported by the wave and its direction is quantified by the Poynting vector.

For EM wave, since
E is perpendicular to **B**:

$$\boxed{\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}}$$





In a wave, the fields change with time..

Therefore, the Poynting vector changes too!!

The direction is constant, but the magnitude changes from 0 to a max value……

Derivation of Poynting's theorem

✧ Let **E** and **B** are fields at time t to due to some ρ & J distribution. In the next instant dt, the charges move a bit...

Q: What is the work done *dW* by EM forces acting on theses charges in the interval *dt*?

- The work done on a charge q (using Lorentz force law) is,

$$dW = \vec{F}\cdot\vec{dl} = q(\vec{E} + \underbrace{\vec{v}\times\vec{B}}_{\text{zero}})\cdot\vec{v}dt$$

$$= q\vec{E}\cdot\vec{v}dt$$

$\because q = \rho d\tau$ & $\vec{J} = \rho\vec{v}$, the rate at which work is done on all charges in a volume V is,

$$\frac{dW}{dt} = \int_V (\vec{E}\cdot\vec{J})d\tau$$

E.**J** ≡ work done per unit time, per unit volume

(i.e. power delivered/vol.)

Derivation of Poynting's theorem

$$\boxed{\vec{E}\cdot\vec{J} = \left(\frac{1}{\mu_0}\right)\vec{E}\cdot\left(\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)}$$

(using **J** from Maxwell's 4th eqn.)

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}$$

$$= \frac{1}{\mu_0}\underbrace{\vec{E}\cdot(\vec{\nabla}\times\vec{B})} - \varepsilon_0\underbrace{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}}$$

$$\boxed{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial(E^2)}{\partial t}}$$

Product rule 6: $\boxed{\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})}$

$$\boxed{\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{\nabla}\cdot(\vec{E}\times\vec{B})}$$

(1st term on RHS $\boxed{\vec{B}\cdot(\vec{\nabla}\times\vec{E}) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}}$ using $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$)

$$\boxed{\vec{E}\cdot\vec{J} = -\frac{1}{\mu_0}\left[\frac{1}{2}\frac{\partial B^2}{\partial t} + \vec{\nabla}\cdot(\vec{E}\times\vec{B})\right] - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t}}$$

(only in terms of ***E*** & ***B***)

Derivation of Poynting's theorem

$$\boxed{\vec{E}\cdot\vec{J} = -\frac{1}{\mu_0}\left[\frac{1}{2}\frac{\partial B^2}{\partial t} + \vec{\nabla}\cdot(\vec{E}\times\vec{B})\right] - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t}}$$

(only in terms of ***E*** & ***B***)

$$\boxed{\Rightarrow \vec{E}\cdot\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{B^2}{\mu_0} + \varepsilon_0 E^2\right] - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})}$$

$$\therefore \frac{dW}{dt} = \int_V (\vec{E}\cdot\vec{J})\,d\tau = -\frac{d}{dt}\int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)d\tau - \underbrace{\frac{1}{\mu_0}\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{B})\,d\tau}$$

Using Divergence theorem on 2nd term,

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)d\tau - \frac{1}{\mu_0}\oint_S (\vec{E}\times\vec{B})\cdot d\vec{a}}$$

(I) (II)

This is "Poynting's Theorem"

"work-energy theorem" of electrodynamics

Derivation of Poynting's theorem

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)d\tau - \frac{1}{\mu_0}\oint_S (\vec{E}\times\vec{B})\cdot d\vec{a}}$$

If we define $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$, then $\boxed{\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint \vec{S}\cdot d\vec{a}}$

LHS $\equiv$ Rate of work done **<u>on</u>** charges by EM force

RHS, I^st term $\equiv$ <u>Rate of decrease</u> of total energy stored in the fields

RHS, II^nd term $\equiv$ Energy flowing per unit time out of the surface

$$\Rightarrow \boldsymbol{S} \equiv \textit{energy flux density}$$

The energy per unit time, per unit area, transported by the EM fields is called the "Poynting Vector (S)"….

$\vec{S}\cdot d\vec{a} =$ energy per unit time crossing the infinitesimal surface ***da***

$\boxed{\oint \vec{S}\cdot d\vec{a}}$ gives the total power passing through the closed surface.