Assignment-1 (PYL121)
Vector Spaces and Linear algbera

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September 14, 2022

1. Total Marks : 40
2. Submission Deadline : 20:00 hrs on 22/09/2022 (Thursday) on **Moodle**. No late submission!
3. Copying is not acceptable and will lead to zero marks.

Q1 Using the following expresssion of the determinant of an $n \times n$ matrix A

$$\begin{aligned} \text{Det(A)} &= \frac{1}{n!}\epsilon_{i_1 i_2 \ldots i_n}\ \epsilon_{j_1 j_2 \ldots j_n}\ A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n} \\ &\equiv \epsilon_{j_1 j_2 \ldots j_n}\ A_{1 j_1} A_{2 j_2} \ldots A_{n j_n} \\ &\equiv \epsilon_{i_1 i_2 \ldots i_n}\ A_{i_1 1} A_{i_2 2} \ldots A_{i_n n} \end{aligned} \tag{1}$$

Show that

(a) Det(A) remains invariant if we add to any row an arbitrary linear combination of all other rows i.e.

$$A_{ij} \longrightarrow A_{ij} + \sum_{k=1 (k \neq i)}^{n} A_{kj}$$

(b) Show the same for column replacements.

(Comment : Note that (a) and (b) imply that if the rows (or columns) of the matrix are not all linearly independent then the determinant vanishes.)

(c) Show that if the rows of the matrix A are multiplied by numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively, then the determinant of the new matrix is given by

$$\alpha_1 \alpha_2 \ldots \alpha_n Det(A)$$

(Hint/Suggestion : Though the above statements can be shown in general, if you find it challenging then it might be useful to play with n=2 and 3 cases to get some intuition and then generalize to arbitrary n.)

(Marks : 2+2+2)

Q2 Consider a two level system (2 dimensional complex vector space) whose Hamiltonian (a special linear operators on this vector space) is given by

$$\hat{H} = \alpha(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|)$$

where α is a real number having the dimensions of energy and $|\phi_1\rangle$, $|\phi_2\rangle$ are normalized eigenvectors (also referred to as eigenstates) of a Hermitian operator $\hat{A}$ with different eigenvalues.

(a) Determine if $\hat{H}$ is a projection operator or not. Further determine if $\hat{H}$ is Hermition or not.

(b) Compute $\alpha^{-2}\hat{H}^2$ and show that it is a projection operator.

(c) Compute the commutators $[\hat{H}, |\phi_1\rangle\langle\phi_1|]$ and $[\hat{H}, |\phi_2\rangle\langle\phi_2|]$ and show that they are negative of each other. Is this a coincidence?

(d) Find the matrix representation of H in the $\Phi \equiv (|\phi_1\rangle, |\phi_2\rangle)$ basis. Using this, compute the eigenvalues and corresponding normalized eigenvectors of $\hat{H}$. Write these eigenvectors in Φ basis.

(Marks : 2+2+2+3)

Q3 Given two matrices U and H, which are related as:-

$$U = e^{iaH} \;\; ; \;\; a \in \mathbb{R} \tag{2}$$

Prove the following:-

1. H is hermitian $\iff$ U is unitary.
2. $\mathrm{tr}(H) = 0 \iff \det(U) = 1$

(Marks : 2+2)

Q4 Prove the following:-

1. Any 2×2 unitary matrix can be written as

$$\hat{U} = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{bmatrix} \;\; ; \;\; |\alpha|^2 + |\beta|^2 = 1 \;\; ; \;\; \alpha, \beta \in \mathbb{C} \tag{3}$$

2. All eigenvalues of a unitary operator lie on the unit circle in the complex plane.

(Marks : 2+2)

Q5 Let $\sigma_x, \sigma_y, \sigma_z$ be the Pauli spin matrices given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

(a) Explicitly compute the 2×2 matrix defined by the following exponential

$$U_y(\theta) = \exp\left(\frac{i\theta\sigma_y}{2}\right)$$

by using the Taylor series expansion of the exponential. Here $\theta \in (0, 2\pi)$.

(b) Let (x, y, z) and (x', y', z') represent coordinates of two points in real 3d space which are related by the equation

$$\left(x'\sigma_x + y'\sigma_y + z'\sigma_z\right) = \exp\left(\frac{i\theta\sigma_y}{2}\right).\left(x\sigma_x + y\sigma_y + z\sigma_z\right).\exp\left(\frac{-i\theta\sigma_y}{2}\right) \tag{5}$$

Using the result of part (a), compute the RHS of the above equation and compare with LHS to determine (x', y', z') in terms of (x, y, z).

(c) Write down the 3×3 matrix $R(\theta)$ relating the column vectors $X' = (x', y', z')^T$ and $X = (x, y, z)^T$ via

$$X' = R(\theta).X$$

Do you recognize the matrix $R(\theta)$? What is its interpretation in 3d real space?

(Marks : 2+3+2)

Q6 (a) Let x and p be the coordinate and the conjugate momentum in one a dimensional classical system. Evaluate the classical Poisson bracket

$$\{x, F(p)\}_{\text{PB}}$$

where $F(p)$ is some sufficiently smooth and differentiable function of p.

(b) In Canonical quantization of the classical systems, the classical phase space is replaced with a linear vector space (complex) of quantum states of the system, the coordinates and conjugate momentum are lifted to linear operators $\hat{x}$ and $\hat{p}$ and the classical Poisson brackets (PB) are replaced by quantum commutator as

$$[\hat{A}, \hat{B}] = i\hbar\{A, B\}_{\text{PB}}. \tag{6}$$

e.g. in quantum mechanics, we get $[\hat{x}, \hat{p}] = i\hbar\{x, p\}_{\text{PB}} = i\hbar$. Using this evaluate the commutator

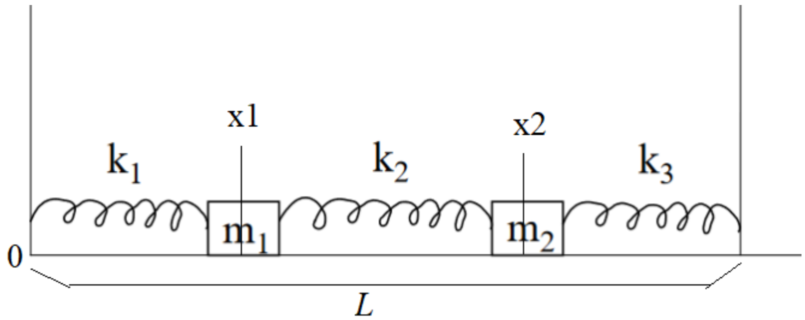
$$\left[\hat{x}, \exp\left(\frac{ia\hat{p}}{\hbar}\right)\right]$$

where a is a real number.

(c) If $|x'\rangle$ is an eigenstate of $\hat{x}$ with eigenvalue x' then prove that $\exp\left(\frac{ia\hat{p}}{\hbar}\right)|x'\rangle$ is also an eigenstate of $\hat{x}$. Further, determine the eigenvalue. (Hint : Use the commutation relation obtained in (b))

(Marks : 1+1+2)

Q7 Consider the motion of masses given in the figure below. Let the displacement of the masses m_1 and m_2 from their equilibrium position be $x_1(t)$ and $x_2(t)$, at any time t. Using Classical Mechanics and Linear Algebra, answer the following.



1. Write down the equations of motion for the above system and convert them into the matrix form i.e.,

$$\frac{d^2}{dt^2}X(t) = M.X(t)$$
$$\text{where} \quad \text{X(t)} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \text{and M is a } 2 \times 2 \text{ matrix.} \tag{7}$$

   Compute M explicitly.

2. Assuming the solution to be of the form

$$X(t) = X_0 e^{\lambda t} \Rightarrow M.X_0 = \lambda^2 X_0 \tag{8}$$

   Find the eigenvalues λ^2 and their corresponding eigenvectors for the matrix M. And thus find out the "normal mode" solutions.

3. Write the general solution to the differential equation, assuming (for simplicity)

$$x_1(0) = 0 \;\; ; \;\; x_2(0) = \delta \;\; ; \;\; \dot{x}_1(0) = 0 \;\; ; \;\; \dot{x}_2(0) = 0 \tag{9}$$

$$k_1 = k_3 = k \;\; ; \;\; k_2 = 2K \;\; ; \;\; m_1 = m_2 = m \tag{10}$$

(Marks : 2+2+2)