MTL-100 Lecture-2

Sequences and Limits

Defn (Sequence): A sequence of real numbers is a function from $\mathbb{N}$ to $\mathbb{R}$.

Notation: We write a sequence as $(a_n)_{n=1}^{\infty}$ or $(a_n)_{n\in\mathbb{N}}$ or $(a_1, a_2, a_3, \cdots)$ rather than the functional notation $a(n)$.

· We say a_n is the nth term of the sequence.

Examples:

① $(c)_{n=1}^{\infty}$, where $c \in \mathbb{R}$ (constant seq.)

② $\left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right)$

③ $(\sqrt{n})_{n=1}^{\infty}$

④ $((-1)^n)_{n=1}^{\infty} = (-1, 1, -1, 1, \cdots)$

⑤ $(n^{1/n})_{n\in\mathbb{N}}$ (nth root of n)

⑥ $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$

⑦ $(\sin(n))_{n\in\mathbb{N}} = (\sin 1, \sin 2, \sin 3, \cdots)$

⑧ $\left(\frac{\sin(n)}{n}\right)_{n\in\mathbb{N}}$

Remark: It is important to distinguish between a sequence and its set of values (or the range).

e.g. For the seq. $((-1)^n)_{n\in\mathbb{N}}$, the set of values attained by the seq. is $\{-1, 1\}$.

Limit of a sequence

Informal Defn: A sequence is said to converge to a limit L, where $L \in \mathbb{R}$ if the terms of the seq are "eventually" "arbitrarily close" to L.

Formal Defn:

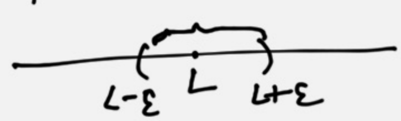
A sequence $(a_n)_{n=1}^{\infty}$ is said to converge to a real number L if for any $\varepsilon > 0$ (given) there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \text{for all } n \geq N.$$

Notation: $L = \lim_{n\to\infty} a_n$ or $a_n \to L$ or (a_n) converges to L.

Remarks:

① $|a_n - L| < \varepsilon \iff L-\varepsilon < a_n < L+\varepsilon$



② N depends on ε.
Normally if we choose ε smaller, then N will be bigger.

Theorem (Uniqueness of Limits):

If $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$, then $L_1 = L_2$.

Proof: Let $\varepsilon > 0$ be given.
Since $\lim a_n = L_1$, $\exists\, N_1 \in \mathbb{N}$ s.t.
$|a_n - L_1| < \frac{\varepsilon}{2}$ $\forall\, n \geq N_1$.
||ly, $\exists\, N_2 \in \mathbb{N}$ s.t. $|a_n - L_2| < \frac{\varepsilon}{2}$ $\forall\, n \geq N_2$.

Let $N = \max\{N_1, N_2\}$.

Then
$$
\begin{aligned}
|L_2 - L_1| &= |(a_N - L_1) - (a_N - L_2)| \\
&\leq |a_N - L_1| + |a_N - L_2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

So, $0 \leq |L_2 - L_1| < \varepsilon$ for every $\varepsilon > 0$.

$\Rightarrow$ $|L_2 - L_1| = 0 \Rightarrow L_2 = L_1$.

Examples:

① $(c)_{n\in\mathbb{N}}$ converges to c.

Pf: $a_n = c$ $\forall\, n \in \mathbb{N}$.
$|a_n - c| = 0 < \varepsilon$ $\forall\, n \geq 1$ for any $\varepsilon > 0$.

② $\lim_{n\to\infty} \frac{1}{n} = 0$

Proof: Let $\varepsilon > 0$.
By the Archimedean property,
$\exists\, N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$.
So, $n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \varepsilon$
$\Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$
$\therefore \lim_{n\to\infty} \frac{1}{n} = 0$.

③ The seq. $((-1)^n)_{n\in\mathbb{N}}$ does not converge.

Pf: Assume $L = \lim_{n\to\infty} (-1)^n$.
Take $\varepsilon = 1$. Then $\exists\, N \in \mathbb{N}$ s.t.
$|(-1)^n - L| < 1 \quad \forall\, n \geq N$.
Taking $n = 2N \geq N$, we get
$|1 - L| < 1 \Rightarrow 0 < L < 2$ — (i)
Taking $n = 2N+1 > N$, we get
$|-1 - L| < 1 \Rightarrow -2 < L < 0$ — (ii)
From (i) & (ii), we get a contradiction.

Theorem: Every convergent sequence must be bounded.

Proof: Let (a_n) be a sequence & $L = \lim_{n\to\infty} a_n$.

Taking $\varepsilon = 1$, we get $N \in \mathbb{N}$ s.t.

$$|a_n - L| < 1 \quad \forall n \geq N.$$

$\therefore$ For $n \geq N$, $|a_n| \leq |a_n - L| + |L| < 1 + |L|$

If we take $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1+|L|\}$,

then $|a_n| \leq M \quad \forall n \in \mathbb{N}$.

Operations on Limits

Theorem: Let (a_n) & (b_n) be two convergent sequences with $\lim_{n\to\infty} a_n = L$ & $\lim_{n\to\infty} b_n = M$.

Then (i) $\lim_{n\to\infty}(c\,a_n) = cL$ for any $c \in \mathbb{R}$.

(ii) $\lim_{n\to\infty}(a_n + b_n) = L + M$

(iii) $\lim_{n\to\infty} a_n b_n = LM$

(iv) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ if $M \neq 0$

Proof: (i) If $c=0$, then $ca_n = 0 \;\forall n$.

$\therefore \lim_{n\to\infty} ca_n = 0 = cL$.

Assume $c \neq 0$. Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} a_n = L$, $\exists\, N \in \mathbb{N}$ s.t.

$$|a_n - L| < \frac{\varepsilon}{|c|} \quad \forall n \geq N.$$

$$\Rightarrow |ca_n - cL| = |c|\,|a_n - L| < |c|\frac{\varepsilon}{|c|} \quad \forall n \geq N$$
$$= \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} ca_n = cL.$$

(ii) $\left|(a_n+b_n) - (L+M)\right| \leq |a_n - L| + |b_n - M|$.

Let $\varepsilon > 0$. Then $\exists\, N_1 \in \mathbb{N}$ s.t.

$$|a_n - L| < \frac{\varepsilon}{2} \quad \forall n \geq N_1$$

Also, $\exists\, N_2 \in \mathbb{N}$ s.t. $|b_n - M| < \frac{\varepsilon}{2} \;\forall n \geq N_2$.

Now, if $n \geq N = \max\{N_1, N_2\}$, then

$$|(a_n+b_n) - (L+M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\therefore \lim_{n\to\infty} (a_n + b_n) = L + M.$$

(iii) To prove: $\lim_{n\to\infty}(a_n b_n) = LM$.

$$|a_n b_n - LM| = |a_n b_n - a_n M + a_n M - LM|$$
$$= |a_n(b_n - M) + (a_n - L)M|$$
$$\leq |a_n||b_n - M| + |a_n - L||M|.$$

Since (a_n) is convergent, $\exists\, K > 0$ s.t.
$$|a_n| \leq K \quad \forall n.$$

Now if $\varepsilon > 0$ is given, we can choose $N_1 \in \mathbb{N}$ s.t.
$$|b_n - M| < \frac{\varepsilon}{2K} \quad \forall n \geq N_1.$$

Since $\lim_{n\to\infty} a_n = L$, $\exists\, N_2 \in \mathbb{N}$ st.

$$|a_n - L| < \frac{\varepsilon}{2(|M|+1)} \quad \forall n \geq N_2.$$

(Note that we chose $|M|+1$ instead of $|M|$ in the denominator as M may be equal to zero).

Then $n \geq N_2 \Rightarrow |a_n - L||M| < \frac{\varepsilon}{2(|M|+1)}|M| < \frac{\varepsilon}{2}$.

So, $n \geq N = \max\{N_1, N_2\}$
$$\Rightarrow |a_n b_n - LM| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\lim_{n\to\infty}(a_n b_n) = LM$.

(iv) To prove: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$.

It is enough to show that if $M \neq 0$, then $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$ (then use (iii)).

Since $\lim_{n\to\infty} b_n = M \neq 0$, by taking $\varepsilon = \frac{|M|}{2} > 0$,
$\exists N_1 \in \mathbb{N}$ s.t. $|b_n - M| < \frac{|M|}{2}$ $\forall n \geq N_1$.

$\Rightarrow$ if $n \geq N_1$, $|b_n| \geq |M| - |b_n - M| > \frac{|M|}{2}$.

So, for $n \geq N_1$,

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|b_n - M|}{|b_n|\,|M|} < \frac{2}{|M|^2}|b_n - M|$$

$\left(\because |b_n| > \frac{|M|}{2}\right)$

$\therefore$ If $\varepsilon > 0$, we can choose $N_2 \in \mathbb{N}$ s.t.
$|b_n - M| < \varepsilon \frac{|M|^2}{2}$ $\forall n \geq N_2$.

Take $N = \max\{N_1, N_2\}$.

$$n \geq N \Rightarrow \left|\frac{1}{b_n} - \frac{1}{M}\right| < \frac{2}{|M|^2}|b_n - M| < \frac{2}{|M|^2}\cdot \varepsilon\frac{|M|^2}{2} = \varepsilon.$$

$\therefore \lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$.

Applications :

① $\lim_{n\to\infty} \frac{2}{n^2} = 2\lim_{n\to\infty}\frac{1}{n^2}$ (by (i))

$= 2\lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right)$

$= 2\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{n}\right)$ (by (iii))

$= 2.0.0 = 0$.

② $\lim_{n\to\infty}\frac{3n^2-2n}{5n^2+3} = \lim_{n\to\infty}\left(\frac{3-\frac{2}{n}}{5+\frac{3}{n^2}}\right)$

$a_n = 3-\frac{2}{n} \to 3-0=3$

$b_n = 5+\frac{3}{n^2} \to 5+0=5$

$\therefore$ Required limit $= \frac{3}{5}$.

Theorem : If $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty}|a_n| = |a|$

Pf: $\big||a_n|-|a|\big| \le |a_n-a|$

If $\varepsilon>0$ is given, we can choose $N\in\mathbb{N}$ s.t. $|a_n-a|<\varepsilon \quad \forall n\geq N$

$\therefore \big||a_n|-|a|\big| < \varepsilon \quad \forall n\geq N$.

$\Rightarrow \lim_{n\to\infty}|a_n| = |a|$.

Theorem: Let (a_n) be a sequence such that $a_n \geq 0$ and $a_n \to a$. Then $\sqrt{a_n} \to \sqrt{a}$.

Pf: (Exercise) : $a \geq 0$.

Case I: If $a = 0$, then we have to show $a_n \to 0 \Rightarrow \sqrt{a_n} \to 0$.

Let $\varepsilon > 0$ be given.

Since $a_n \to 0$, $\exists\, N \in \mathbb{N}$ s.t.

$$|a_n - 0| < \varepsilon^2 \quad \forall\, n \geq N.$$

i.e. $0 \leq a_n < \varepsilon^2 \quad \forall\, n \geq N$.

$\Rightarrow \quad 0 \leq \sqrt{a_n} < \varepsilon \quad \forall\, n \geq N$.

$\Rightarrow \quad |\sqrt{a_n} - 0| < \varepsilon$

$\therefore \quad \lim_{n\to\infty} \sqrt{a_n} = 0$.

Case II: $a > 0$.

$$\left|\sqrt{a_n} - \sqrt{a}\right| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}}$$

Since $\lim_{n\to\infty} a_n = a$, given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon\sqrt{a} \quad \forall\, n \geq N$

$$\therefore \quad \left|\sqrt{a_n} - \sqrt{a}\right| < \frac{\varepsilon\sqrt{a}}{\sqrt{a}} = \varepsilon \quad \forall\, n \geq N.$$

$\Rightarrow \quad \lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

Question: If $a_n^2 \to a$, then is it necessary that $a_n \to \sqrt{a}$?