COL 352 Introduction to Automata and Theory of Computation

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Lecture 19: CFGs to PDAs and back

Equivalence of NPDA and CFG

Theorem

$L = L(G)$ for some context-free grammar G if and only if it is accepted by some NPDA.

NPDA

Definition

A non-deterministic pushdown automaton (NPDA)
$A = (Q, \Sigma, \Gamma, \delta, q_0, \perp, F)$, where

Q:	set of states	Σ:	input alphabet
Γ:	stack alphabet	q_0:	start state
$\perp$:	start symbol	F:	set of final states

$\delta \subseteq Q \times \Sigma \times \Gamma \times Q \times \Gamma^*$.

Understanding δ

For $q \in Q, a \in \Sigma$ and $X \in \Gamma$, if $\delta(q, a, X) = (p, \gamma)$,
then p is the new state and γ replaces X in the stack.

if $\gamma = \epsilon$ then X is popped.
if $\gamma = X$ then X stays unchanges on the top of the stack.
if $\gamma = \gamma_1\gamma_2 \dots \gamma_k$ then X is replaced by γ_k
and $\gamma_1\gamma_2 \dots \gamma_{k-1}$ are pushed on top of that.

Example

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P \to \varepsilon & P \to 1
\end{array}
$$

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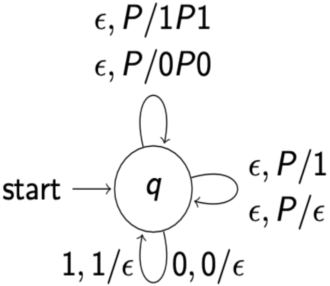

$$A_G = (\{q\}, \{0,1\}, \{0,1,P\}, \delta, q, P, \varnothing)$$

Equivalence of NPDAs and CFGs

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CFG to NPDA

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Proof.

- Assume CFG is in the Chomsky normal form.
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Repeat the above procedure. $\square$

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- $\perp = S$ (stack bottom is start symbol of CFG)
- $F = \varnothing$ (Acceptance by empty stack)

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Guess production rule and push on to the stack and verify guess while popping.

NPDA to CFG

Theorem

$L = L(M)$ *for some NPDA* M *then there is some grammar* G *such that* $L(G) = L(M)$.

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- Idea: Design a CFG that for each pair of states p, q in P, have a variable $A_{p,q}$ which generates all strings that can take P from p (with empty stack) to q (with empty stack).

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- Idea: Design a CFG that for each pair of states p, q in P, have a variable $A_{p,q}$ which generates all strings that can take P from p (with empty stack) to q (with empty stack).
- Modify P so that
  - It has single accept state.
  - It empties its stack before accepting.
  - Each transition either pushes a symbol or pops a symbol (not both).

NPDA to CFG

Given a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, \perp, q_F)$ with restriction that every transition is either pushes a symbol or pops a symbol form the stack, i.e. $\delta(q, a, X)$ contains either (q_0, YX) or (q_0, ϵ).

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Lemma

$L(G_p) = L(P)$.

PDA to CFG

Lemma

If $A_{p,q} \Longrightarrow^* x$ *then* x *can bring the PDA* P *from state* p *on empty stack to state* q *on empty stack.*

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  - If it is $A_{p,q} \to A_{p,r} A_{r,q}$, then the string x can be broken into two parts $x_1 x_2$ such that $A_{p,r} \Longrightarrow^* x_1$ and $A_{r,q} \Longrightarrow^* x_2$ in at most n steps. The claim easily follows in this case.
  - If it is $A_{p,q} \to a A_{r,s} b$, then the string x can be broken as ayb such that $A_{r,s} \Longrightarrow^* y$ in n steps. Notice that from p on reading a the PDA pushes a symbol X to stack, while it pops X in state s and goes to q.

$\square$