MULTIVARIABLE CALCULUS
LECTURE 24

1. Mean Value Theorem

Mean Value Theorem : We will present the MVT for functions of several variables which is a consequence of MVT for functions of one variable.

Theorem 1.1. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Let $X_0 = (x_0, y_0) \in \mathbb{R}^2$ and $X = (x_0 + h, y_0 + k)$. Then there exists $C := (x_0 + ch, y_0 + ck)$, with $0 < c < 1$ such that*

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = hf_x(x_0 + ch, y_0 + ck) + kf_y(x_0 + ch, y_0 + ck).$$

Proof. Consider a function $\phi : [0, 1] \to \mathbb{R}$ by

$$\begin{aligned} \phi(t) &= f(x_0 + th, y_0 + tk) \\ &:= f(x(t), y(t)), \end{aligned}$$

where $x(t) = x_0 + t\,h, y(t) = y_0 + t\,k$.

By Chain Rule

$$\frac{d\phi}{dt} = f_x(x(t), y(t))\frac{dx}{dt} + f_y(x(t), y(t))\frac{dy}{dt} = hf_x(x(t), y(t)) + kf_y(x(t), y(t))$$

So by Lagrange Mean Value Theorem, there exists $c \in (0, 1)$ such that

$$\phi(1) - \phi(0) = \phi'(c),$$

i.e.,

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = hf_x(x_0 + ch, y_0 + ck) + kf_y(x_0 + ch, y_0 + ck).$$

This proves the result. □

Remark 1.1. If $f(x, y)$ is constant if and only if $f_x = 0$ and $f_y = 0$.

2. Mixed Derivatives

Now we move to a very important concept of mixed derivative. The first-order partial derivatives $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ of $z = f(x, y)$ can be differentiated (if possible ?) with respect to x and y to obtain the second x-derivative and the second y-derivative

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \tag{1}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \tag{2}$$

and the mixed second derivatives

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \tag{3}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \tag{4}$$

But $f_{xy} \neq f_{yx}$ is not always true. Consider the following example

Example 2.1. $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x,y) = \{ \begin{array}{cc} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & x = y = 0 \end{array}$$

This function is continuous everywhere. We now compute the first order partial derivatives. Now $f_y(0,0) = 0$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{h(h^2-k^2)}{h^2+k^2} = h$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = -k$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Similarly, $f_{yx}(0,0) = -1$.

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{-k}{k} = -1$$

So $f_{yx}(0,0) \neq f_{xy}(0,0)$.

We can impose some assumptions on the partial derivatives so that the mixed derivatives become equal. Next Theorem deals with that

Theorem 2.1. Clairaut's theorem *If* $f(x,y)$ *and its partial derivatives* f_x, f_y, f_{xy}, f_{yx} *are defined in a neighborhood of* (a,b) *and all are continuous at* (a,b) *then*

$$\frac{\partial^2 f}{\partial x\,\partial y}(a,b) = \frac{\partial^2 f}{\partial y\,\partial x}(a,b).$$

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Proof. Define a function :

$$S(x,y) := f(x,y) - f(x,b) - f(a,y) + f(a,b),$$

also define $A(x,y) := f(x,y) - f(a,y)$ and $B(x,y) := f(x,y) - f(x,b)$. Then clearly

$$S(x,y) = A(x,y) - A(x,b).$$

Applying LMVT on the function A keeping x variable fixed, we obtain

$$\begin{aligned}
A(x,y)-A(x,b) &= \frac{\partial A}{\partial y}(x,\eta_A)(y-b), \quad \text{where } \eta_A \in (y,b) \text{ or } (b,y)\\
&= \underbrace{\left[\frac{\partial f}{\partial y}(x,\eta_A)-\frac{\partial f}{\partial y}(a,\eta_A)\right]}_{\text{Apply again LMVT in x}}(y-b)\\
&= \frac{\partial^2 f}{\partial x\,\partial y}(\xi_A,\eta_A)(x-a)(y-b) \quad \text{where } \xi_A \in (x,a) \text{ or } (a,x).
\end{aligned}$$

Therefore we obtain

$$\frac{S(x,y)}{(x-a)(y-b)} = \frac{\partial^2 f}{\partial x\,\partial y}(\xi_A,\eta_A).$$

Further using continuity of $\frac{\partial^2 f}{\partial x\,\partial y}(a,b)$ we obtain : as $(x,y)\to(a,b)$ and $(\xi_A,\eta_A)\to(a,b)$,

$$\lim_{(x,y)\to(a,b)} \frac{S(x,y)}{(x-a)(y-b)} = \frac{\partial^2 f}{\partial x\,\partial y}(a,b).$$

Similarly,

$$S(x,y) = B(x,y)-B(a,y).$$

$$\begin{aligned}
B(x,y)-B(a,y) &= \frac{\partial B}{\partial x}(\xi_B,y)(x-a), \quad \text{where } \xi_B \in (x,a) \text{ or } (a,x)\\
&= \underbrace{\left[\frac{\partial f}{\partial x}(\xi_B,y)-\frac{\partial f}{\partial x}(\xi_B,y)\right]}_{\text{Apply again LMVT in y}}(x-a)\\
&= \frac{\partial^2 f}{\partial y\,\partial x}(\xi_B,\eta_B)(x-a)(y-b) \quad \text{where } \eta_B \in (y,b) \text{ or } (b,y).
\end{aligned}$$

Therefore we obtain

$$\frac{S(x,y)}{(x-a)(y-b)} = \frac{\partial^2 f}{\partial y\,\partial x}(\xi_B,\eta_B).$$

Further using continuity of $\frac{\partial^2 f}{\partial y\,\partial x}(a,b)$ we obtain : as $(x,y)\to(a,b)$ and $(\xi_B,\eta_B)\to(a,b)$,

$$\lim_{(x,y)\to(a,b)} \frac{S(x,y)}{(x-a)(y-b)} = \frac{\partial^2 f}{\partial x\,\partial y}(a,b).$$

Hence

$$\frac{\partial^2 f}{\partial y\,\partial x}(a,b) = \frac{\partial^2 f}{\partial x\,\partial y}(a,b).$$

This completes the proof. $\square$

3. Taylor's Theorem

We have studied extensively Taylor's theorem for one variable calculus. Now we shall discuss the Taylor's theorem for more than one variable functions. Essentially following the same idea we can state and proof the theorem. We state the following theorem :

Theorem 3.1. *(Taylor's Theorem) Suppose $f(x, y)$ and its partial derivatives up to $(n+1)th$ are continuous throughout an open rectangular region R centred at a point (a, b). Then for all points in R we have*

$$f(a+h, b+k) = f(a, b) + (hf_x + kf_y)\big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)} + \\ \dots + \frac{1}{n!}(hf_x + kf_y)^n\big|_{(a,b)} + \frac{1}{(n+1)!}(hf_x + kf_y)^{n+1}\big|_{(a+ch, b+ck)},$$

where $(a + ch, b + ck)$ is a point on the line segment joining (a, b) and $(a + h, b + k)$.

Proof. The proof follows immediately from one variable calculus. Apply Taylor's theorem (one variable) and Chain rule to the function

$$\phi(t) = f(a + ht, b + kt) \quad \text{at } t = 0.$$

This completes the proof. □