Lecture 4 - Differentiation

Recall,
the Taylor's Series of f (which is infinitely diff at 'a') at the point 'a' is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \longrightarrow (*)$$

Sufficient Condition: Suppose all derivatives of f are bdd by some fixed constant on I. ($a \in I$), i.e., $\exists M \geq 0$ s.t. $|f^{(n)}(x)| \leq M \ \forall x \in I, \forall n \geq 1$.
Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \ \forall x \in I.$$

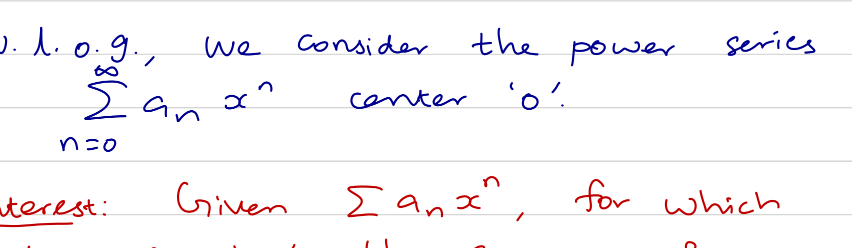
Power Series

For a given seq (a_n) and a point 'c', we define

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

it's called the power series centered at 'c'.

First Observation: at the center $x = c$, the series converges.



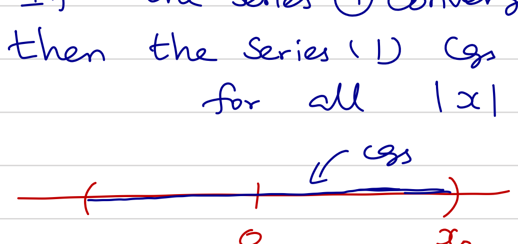
$$\sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} a_n X^n$$

W.l.o.g., we consider the power series $\sum_{n=0}^{\infty} a_n x^n$ center '0'.

Interest: Given $\sum a_n x^n$, for which values of 'x' the series cgs?

properties: Consider $\sum_{n=0}^{\infty} a_n x^n \longrightarrow (1)$.

1. If the series (1) converges at 'x_0' ($x_0 \neq 0$), then the series (1) cgs absolutely for all $|x| < |x_0|$.



Pf: For $|x| < |x_0|$, WPT $\sum_{n=0}^{\infty} |a_n||x|^n$ cgs.

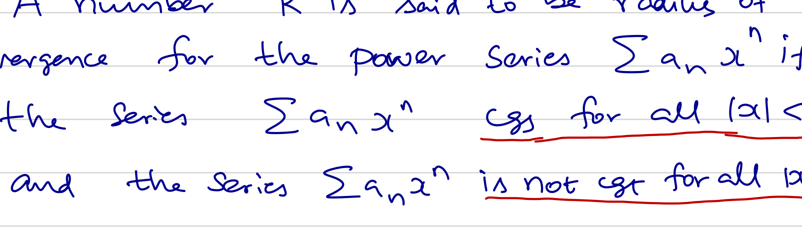
It is enough to show that $\left\{ \sum_{n=0}^{N} |a_n||x|^n \right\}_{N \geq 1}$ bdd.

Consider $\sum_{n=0}^{N} |a_n||x|^n = \sum_{n=0}^{N} |a_n||x_0|^n \left|\frac{x}{x_0}\right|^n \longrightarrow (2)$

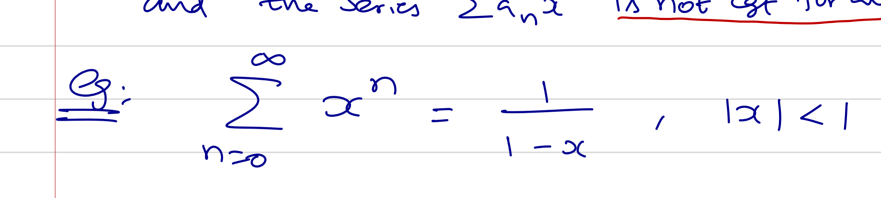
$\because \sum a_n x_0^n$ cgs, $\lim_{n\to\infty} a_n x_0^n = 0$
$\Rightarrow \exists M > 0 \ni |a_n x_0^n| \leq M \ \forall n \geq 1$.

$\therefore$, $\sum_{n=0}^{N} |a_n||x|^n \leq M \sum_{n=0}^{N} \left|\frac{x}{x_0}\right|^n \leq M \sum_{n=0}^{\infty} \left|\frac{x}{x_0}\right|^n < \infty$.

2. If the series (1) is not cgt at 'x_0', then the series (1) is not cgt for all $|x| > |x_0|$.



Pf: Exc. Hint: Use property 1.



Radius of convergence

A number R is said to be radius of convergence for the power series $\sum a_n x^n$ if the series $\sum a_n x^n$ cgs for all $|x| < R$ and the series $\sum a_n x^n$ is not cgt for all $|x| > R$.

Eg: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x| < 1$

$a_n = 1$, $\sum x^n$ is not cgt for all $|x| > 1$.

$R = 1$.

Thm Consider $\sum_{n=0}^{\infty} a_n x^n \longrightarrow (*)$

Then $\boxed{R = \frac{1}{\limsup |a_n|^{1/n}}}$ is the radius of convergence for (*).

Pf: Put $b_n = a_n x^n$.

Consider $\limsup_{n\to\infty} |b_n|^{1/n} = |x| \limsup_{n\to\infty} |a_n|^{1/n}$

Suppose $\limsup |b_n|^{1/n} < 1 \Rightarrow \sum b_n$ cgs.
$\Rightarrow |x| \limsup |a_n|^{1/n} < 1 \Rightarrow \sum a_n x^n$ cgs.
$\therefore$ $|x| < \frac{1}{\limsup |a_n|^{1/n}} \Rightarrow \sum a_n x^n$ cgs. (By root test)

If $|x| > \frac{1}{\limsup |a_n|^{1/n}} \Rightarrow \sum a_n x^n$ is not cgt.

Rmk:
(i). Suppose $\limsup |a_n|^{1/n} = 0$. Then $R = \infty$.
(ii) Suppose $\limsup |a_n|^{1/n} = \infty$. Then $R = 0$.
(iii) Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$ exists.

Then $R = \frac{1}{l} = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$.

$$\boxed{R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}$$

Examples:

1. $\sum_{n=1}^{\infty} \frac{x^n}{n}$, $a_n = \frac{1}{n}$; $\frac{a_n}{a_{n+1}} = \frac{n+1}{n} \to 1$.
The radius of convergence $R = 1$.

2. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $a_n = \frac{1}{n!}$; $\frac{a_n}{a_{n+1}} = \frac{(n+1)!}{n!} = n+1 \to \infty$
$R = \infty$.

3. $\sum_{n=0}^{\infty} n!\, x^n$, $a_n = n!$
$R = 0$.

4. $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$, $a_n = \frac{n^n}{n!}$

$\frac{a_n}{a_{n+1}} = \frac{n^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}} = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e}$ as $n\to\infty$.

$R = \frac{1}{e}$.

Exc: 1. $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^{2n}$

2. $\frac{x}{2} + x^2 + \frac{x^3}{2^2} + \frac{x^4}{2} + \frac{x^5}{2^3} + \cdots$

Find the radius of convergence.

Some interesting Properties (without proof)

1. Uniqueness: Suppose $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for all x in some interval. Then $a_n = b_n$ for all $n = 0, 1, \ldots$

Define $f(x) := \sum_{n=0}^{\infty} a_n x^n$, $|x| < R$, R - radius of cgs.

Term-by-term differentiation:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^n, \quad |x| < R$$

$\left[\lim_{m\to\infty}\left(\lim_{n\to\infty} \frac{n}{n+m}\right) = 1 \ ; \ \lim_{n\to\infty}\left(\lim_{m\to\infty} \frac{n}{n+m}\right) = 0\right]$

$*$ $f(x) := \sum_{n=0}^{\infty} a_n x^n$ is infinitely diff on $(-R, R)$.
The Taylor's Series of f about '0'

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} a_n x^n = f(x)}$$

3. Term-by-term integration:
Consider $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < R$.

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \quad |x| < R.$$

Eg: Consider $1 - x + x^2 - \cdots = \frac{1}{1+x}$, $|x| < 1$

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x 1\,dt - \int_0^x t\,dt + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad |x| < 1$$

For $x = 1$, $1 - \frac{1}{2} + \frac{1}{3} - \cdots = \log(1+1)$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad x \in (-1, 1].$$

Maximum interval

2. Binomial Series
Let $F(x) = (1+x)^p$, $p \in \mathbb{Q}$.
Show that

$$(1+x)^p = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n, \quad |x| < 1.$$

Pf: $F(x) = (1+x)^p$.
$F^{(n)}(x) = p(p-1)\ldots(p-(n-1))(1+x)^{p-n}$

Define $f(x) := \sum \frac{F^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n$.

$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1$. (why?).

$\therefore$ $f(x)$ is well-defined for $|x| < 1$.

Claim $\left[(1+x)^{-p} f(x)\right]' = 0$, $\forall |x| < 1$

$$f'(x) = \sum_{n=1}^{\infty} \frac{p(p-1)\ldots(p-(n-1))}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n)}{n!} x^n$$

$$(1+x) f'(x) = p + \sum_{n=1}^{\infty} \frac{p(p-1)\ldots(p-n)}{n!} x^n + \sum_{n=1}^{\infty} \frac{p(p-1)\ldots(p-(n-1))}{(n-1)!} x^n$$

$(1+x) f'(x) = p f(x)$

$$\left[(1+x)^{-p} f(x)\right]' = -p(1+x)^{-p-1} f(x) + (1+x)^{-p} f'(x) = 0$$

$\Rightarrow$ $f(x) = c(1+x)^p$
$\because f(0) = 1. \Rightarrow c = 1$.
$\therefore f(x) = (1+x)^p$, $|x| < 1$.

$$(1+x)^p = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n, \quad |x| < 1.$$

4. Product & division of Power Series.

Suppose $f(x) = \sum a_n x^n$, $|x| < R_1$ &
$g(x) = \sum b_n x^n$, $|x| < R_2$.

Then $fg(x) = \sum c_n x^n$, $|x| < \min\{R_1, R_2\}$.
where, $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

In addition, $g(x) \neq 0$, $\forall |x| < R_2$, we have

$$\left(\frac{f}{g}\right)(x) = \sum c_n x^n, \quad |x| < \min\{R_1, R_2\}.$$

Exc Find the Taylor's series expansion about '0':
(a) $e^{-x^2}\cos x$, (b) $\tan x$.

Applications: * power series is used in solving ODEs.

$y' = y$, $y(0) = 1$.
$y(x) = \sum_{n=0}^{\infty} a_n x^n$ $a_0 = 1$, $a_n = \frac{1}{n!}$.

* Integration: $\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \cdots\right) dt = x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots$