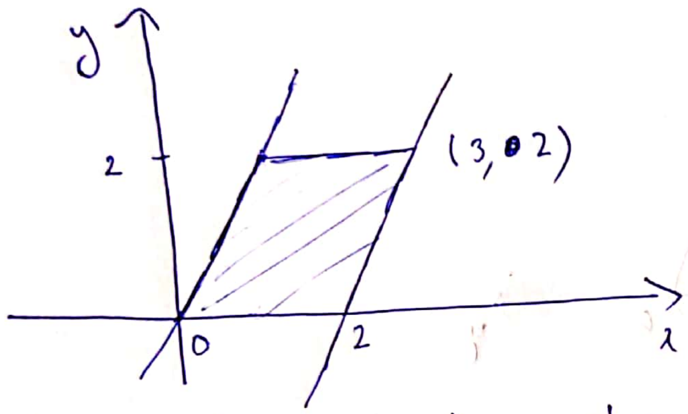
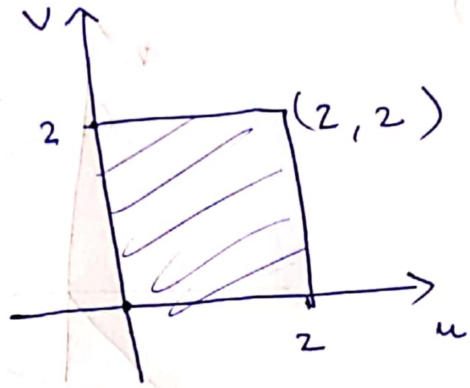
Harshit Mawandia 2020CS10348

Q1)





$$J = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$$

$x = u + v/2$

$$\cancel{\int_0^2 \int_0^2 v^3 (2u)^2 e^{(2u)^3} \times 1 \, du \, dv}$$

$y = v$

When at $(x, y) = (0, 0)$, $(u, v) = (0, 0)$

at $(x, y) = (2, 0)$, $(u, v) = (2, 0)$

at $(x, y) = (0, 2)$, $(u, v) = (0, 2)$

at $(x, y) = (3, 2)$, $(u, v) = (2, 2)$

$$\int_0^2 \int_0^2 v^3 (2u)^2 e^{(2u)^3} \times 1 \times du \, dv$$

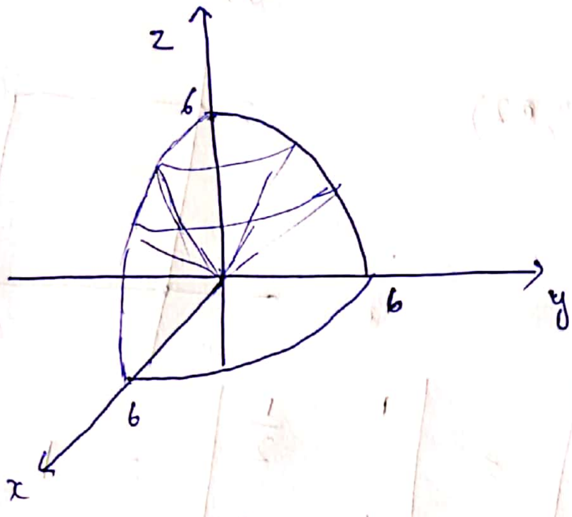
$$\int_0^2 4 \times (2u)^2 e^{(2u)^3} \times du \times$$

Let $(2u)^3 \longrightarrow t$, $3(2u)^2 \times 2 \, du \longrightarrow dt$

$$\int_0^{64} \frac{2}{3} \times e^t \, dt = \frac{2}{3}\left(e^{64} - 1\right)$$

Harshit Mawandia 2020CS10348

Q2)



$x^2 + y^2 + z^2 = 36$ (1) , $z^2 = 3(x^2 + y^2)$ (2) , $z^2 = x^2 + y^2$ (3)

Solving (1) and (2)

$\frac{4z^2}{3} = 36$ $z^2 = 27$, $z = 3\sqrt{3}$

$\cos\phi = \frac{3\sqrt{3}}{6} \cdot 2$ $\phi = 30° = \pi/6$

Solving (1) & (3)

$2z^2 = 36$

$z^2 = 18$, $z = 3\sqrt{2}$

$\cos\phi = \frac{3\sqrt{2}}{6} = \frac{1}{\sqrt{2}} \Rightarrow \phi = \pi/4$

In spherical co-ordinates r is from 0 to 6
θ is from 0 to $\pi/2$
ϕ is from $\pi/6$ to $\pi/4$

$$\int_{\pi/6}^{\pi/4}\int_0^{\pi/2}\int_0^6 r^2 \sin\phi \, dr \, d\theta \, d\phi$$

$$= \int_{\pi/6}^{\pi/4}\int_0^{\pi/2} \frac{6\times6\times6}{3} \sin\phi \, d\theta \, d\phi$$

Harshit Mawandia 2020CS10348

$$\Rightarrow \quad 72 \int_{\pi/6}^{\pi/4} \int_{0}^{\pi/2} \sin\phi \, d\theta \, d\phi$$

$$\Rightarrow \quad 36\pi \int_{\pi/6}^{\pi/4} \sin\phi \, d\phi$$

$$\Rightarrow \quad 36\pi \left[-\cos\phi \right]_{\pi/6}^{\pi/4}$$

$$\Rightarrow \quad 36\pi \left[\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \right)$$

$$\Rightarrow \quad 36\pi \times \left(\frac{\sqrt{3} - \sqrt{2}}{2} \right)$$

$$\Rightarrow \quad 18\pi \left(\sqrt{3} - \sqrt{2} \right)$$

Harshit Mawandia 2020CS10348

Q3)

$$I = \int_0^\infty \frac{1}{\sqrt{x^5 + x}}\, dx$$

$$I = \underbrace{\int_0^1 \frac{1}{\sqrt{x^5+x}}\, dx}_{I_1} + \underbrace{\int_1^\infty \frac{dx}{\sqrt{x^5+x}}}_{I_2}$$

For I_1

$$I_1 = \int_0^1 \frac{1}{\sqrt{x^5+x}}\, dx = \int_0^1 \frac{dx}{\sqrt{x}\,\sqrt{x^4+1}} < \int_0^1 \frac{dx}{\sqrt{x}}$$

$$I_1 < \int_0^1 \frac{dx}{\sqrt{x}} = 2$$

(since $\sqrt{x^4+1} > 1$ $\therefore \frac{1}{\sqrt{x^4+1}} < 1$)

$\therefore$ I_1 converges

For I_2

$$I_2 = \int_1^\infty \frac{1}{\sqrt{x^5+x}}\, dx \quad \cancel{< \int_1^\infty \frac{dx}{2x^{5/2}} \quad (2x^{5/2} > 2\sqrt{x^5+x})}$$

Let $\frac{1}{\sqrt{x^5+x}} = f(x)$ and $\frac{1}{\sqrt{x^5}} = g(x)$

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{5/2}}{\sqrt{x^5+x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^{3/2}}}} = 1$$

Harshit Mawandia 2020CS10348

Now for $g(x) = \frac{1}{x^{5/2}}$

$$\int_1^\infty g(x)\,dx = \left[-\frac{2}{3}\frac{1}{x^{3/2}}\right]_1^\infty = \frac{2}{3}$$

Now since $g(x)$ converges

and $L = \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ exists and is not 0

we know that $\int_1^\infty f(x)\,dx$ also converges

$\therefore$ $I = I_1 + I_2$ converges

since both I_1 and I_2 converge

and $0 < I = I_1 + I_2 < 2 + \frac{2}{3} = \frac{8}{3}$

Scanned with CamScanner

Harshit Mawandia 2020CS10348

Q4) Since $f(x)$ is twice differentiable and $f''(x)$ is continuous we know $f(x)$ and $f'(x)$ is continuous and differentiable (and limit exist at all $x \in (a,b)$)

$$L = \lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} \quad \left(\text{of the form } \frac{\sim 0}{\sim 0} \right)$$

applying L-Hopital rule

$$L = \lim_{h \to 0} \frac{f'(c+h) - f'(c-h)}{2h} \quad \left(\text{of the form } \frac{\sim 0}{\sim 0} \right)$$

again applying L-Hopital rule

$$L = \lim_{h \to 0} \frac{f''(c+h) + f''(c-h)}{2}$$

$$\Rightarrow L = \frac{2f''(c)}{2} = f''(c)$$

Scanned with CamScanner

Harshit Mawandia 2020CS10348

Q5) $f(x) = \frac{1}{(1+x)^2}$

$f'(x) = -\frac{1}{3(1+x)^3}$

$f''(x) = +\frac{1}{3\times 4\times(1+x)^4}$

⋮

$f^n(x) = \frac{(-1)^n}{3\times 4\times\cdots\times n(1+x)^n}$

~~Now we check the radius of convergence of this~~

$P_n(x) = \frac{1}{2^2} - \frac{(x-1)}{3\times 2^3} + \frac{(x-1)^2}{3\times 4\times 2^4\times 2!} + \cdots + \frac{(-1)^n}{3\times 4\times\cdots\times n\times 2^n}\times\frac{(x-1)^n}{n!}$

Now we check the radius of convergence about 1

$$R = \lim_{n\to\infty} \frac{\cancel{(-1)^n}}{\cancel{3\times 4\times\cdots\times n}\times \cancel{2^n}\times \cancel{n!}} \times \frac{\cancel{3\times 4\times\cdots}(n+1)\times 2^{n+1}\times(n+1)!}{\cancel{(-1)^{n+1}}}$$

$$= \lim_{n\to\infty} 2\times(n+1)^2 = \infty$$

Harshit Mawandia 2020CS10348

$\lim_{n \to \infty}$

$f(x) - P_n(x) =$ Remainder of Polynomial series

$$= R_N = \frac{(-1)^{n+1}}{3 \times 4 \times \cdots \times (n+1)} \times \frac{(x-1)^{n+1}}{(1+\delta)^{n+1} \times \lfloor n+1}$$

for some $\delta \in \left(\frac{1}{2}, 2\right)$

$$\lim_{n \to \infty} |f(x) - P_n(x)| = \lim_{n \to \infty} R_N$$

$$= \lim_{n \to \infty} \frac{2 \times (x-1)^{n+1}}{\left((n+1)!\right)^2 \times (1+\delta)^{n+1}}$$

for $\forall x \in \left(\frac{1}{2}, 2\right)$, $(x-1) < 1$

$\therefore \lim_{n \to \infty} (x-1)^n \longrightarrow 0$

for $\forall \delta \in \left(\frac{1}{2}, 2\right)$, $(1+\delta) > 1$

$\therefore \lim_{n \to \infty} (1+\delta)^n \longrightarrow \infty$

$\therefore \lim_{n \to \infty} R_n \longrightarrow 0$

$\therefore \lim_{n \to \infty} |f(x) - P_n(x)| = 0 \quad \forall x \in \left(\frac{1}{2}, 2\right)$

Harshit Mawandia 2020CS10348

Q6)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^3}{|x|^3 + |y|^\alpha} \text{ should exist}$$

~~for any value of α~~

~~$f_x(0,0) = 0$ (since $y=0$ in numerator)~~

~~$f_y(0,0) = 0$ (since $x=0$ in numerator)~~

~~therefore if limit exists~~

If we move along the path $y=0$, as $x \to 0$

$$\lim_{h \to 0} \frac{h^2 \times 0}{|h|^3} = 0$$

Therefore for the limit of the function to exist it has to be 0

Now $x < |x| < \sqrt{x^2+y^2}$

similarly for y: $y < |y| < \sqrt{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^3}{|x|^3 + |y|^\alpha} < \frac{(x^2+y^2)^{5/2}}{|x|^3+|y|^\alpha} = g(x)$$

at $\alpha \geq 5$, along the line $x=0$, $\lim_{y\to 0} \frac{y^5}{|y|^\alpha}$ does not exist

Harshit Marwandia 2020CS10348 HMr

Therefore $\alpha < 5$

Now

$$\lim_{(x,y)\to(0,0)} \frac{\left(\sqrt{x^2+y^2}\right)^5}{|x|^3+|y|^\alpha}$$

let $x = r\cos\theta$, $y = r\sin\theta$)

$$\lim_{r\to 0} \frac{r^5}{r^3|\cos^3\theta| + r^\alpha |\sin^\alpha\theta|}$$

exists for all $\alpha < 5$ and is equal to 0 (since both $|\cos\theta|$ and $|\sin\theta|$ are never 0 simultaneously)

$\therefore \boxed{\alpha < 5}$

Scanned with CamScanner

Harshit Mawandia 2020CS10348

Q7)

$f(x,y) = x^3y - xy^2 + cx^2$

$\nabla(f) = (3x^2y - y^2 + 2cx)\hat{i} + (x^3 - 2xy)\hat{j}$

$\boxed{\nabla f(x,y) = \hat{i}\,\frac{\partial}{\partial x} f(x,y) + \hat{j}\,\frac{\partial}{\partial y} f(x,y)}$

now for $f(x,y)$ to be increasing fastest at the point $P_0 = (3,2)$ in the direction $A = 2\hat{i} + 5\hat{j}$, ∇f should be in the direction of $\bar{A}$

$\nabla f_{(3,2)} = (54 - 4 + 6c)\hat{i} + 15\hat{j} = \lambda(2\hat{i} + 5\hat{j})$

$= (50 + 6c)\hat{i} + 15\hat{j} = \lambda(2\hat{i} + 5\hat{j})$

equating $15\hat{j} = \lambda \times 5\hat{j}$

we get $\lambda = 3$

$\therefore 50 + 6c = 6$

$6c = -44$

$\boxed{c = \frac{-22}{3}} \approx -7.333$

Harshit Mawandia 2020CS10348

Q 8)

$$f(x,y,z) = xyz$$

We have to find max and min value of $f(x,y,z)$ subject to constraint

$$x^2+2y^2+3z^2=1 \Rightarrow x^2+2y^2+3z^2-1=0$$

let $g(x,y,z) = x^2+2y^2+3z^2-1 \rightarrow (4)$

By Lagrange multipliers:

We know for max and min value

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

$$(yz, xz, xy) = \lambda(2x, 4y, 6z)$$

$yz = 2\lambda x \rightarrow (1)$, $xz = 4\lambda y \rightarrow (2)$, $xy = 6\lambda z \rightarrow (3)$

dividing (1) and (2) we get

$$\frac{y}{x} = \frac{1}{2}\frac{x}{y} \Rightarrow x^2 = 2y^2$$

By dividing (1) and (3) we get

$$\frac{z}{x} = \frac{1}{3}\frac{x}{z} \Rightarrow x^2 = 3z^2$$

Harshit Marwandia 2020CS10348

By dividing (2) and (3) we get

$$\frac{z}{y} = \frac{2}{3}\frac{y}{z} \Rightarrow 2y^2 = 3z^2$$

We know $x^2 = 2y^2 = 3z^2 = k^2$

By Putting k^2 in the constraint we get

$$3k^2 = 1$$

$$k^2 = \frac{1}{3} \qquad k = \pm\frac{1}{\sqrt{3}}$$

$$\therefore \quad x = \pm\frac{1}{\sqrt{3}}, \quad y = \pm\frac{1}{\sqrt{6}}, \quad z = \pm\frac{1}{3}$$

Putting in $f(x,y,z)$

We get $f(x,y,z) = \pm\left(\frac{1}{9\sqrt{2}}\right)$

$\therefore$ Max of $f(x,y,z) = \frac{1}{9\sqrt{2}}$

and Min of $f(x,y,z) = -\frac{1}{9\sqrt{2}}$

Harshit Mawandia 2020CS10348

Q9)
$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n\int_0^1 f(x)\,dx\right)$$

Since $f(x)$ is continuous in $[0,1]$,

we know it is riemann integrable.

i.e $f(x) \in R[0,1]$

Now if we

take $P_n = \{ \left(\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n} \dots \frac{x_i}{n} \dots \frac{n}{n}\right) \}$

$\lim_{n\to\infty} \|P\| = \lim_{n\to\infty} \frac{1}{n} = 0$

Then if $m_i = \inf (f(x)$, for $x \in [x_{i-1}, x_i]$

and $M_i = \sup (f(x)$, for $x \in [x_{i-1}, x_i]$

we know that

$$\sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i$$

for $\Delta x_i \longrightarrow 0$

$$\sum m_i \Delta x_i = L^{(P_n,f)} = U^{(P_n,f)} = \sum M_i \Delta x_i$$

for Riemann integrable function

Harshit Mawandia 2020CS10348

$$\therefore \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = L = U = \int_0^1 f(x)\,dx \quad \text{—} \; (1)$$

$\therefore$ for $L = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n\int_0^1 f(x)\,dx$

$$L = \lim_{n\to\infty} \left(n \left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n}\right) - n\int_0^1 f(x)\,dx\right.$$

$$= \lim_{n\to\infty} n \left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} - \int_0^1 f(x)\,dx\right)$$

Now (I) by using eq^n (1)

We know for $n \to \infty$

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$$

$\therefore$ (I) $L = \lim_{n\to\infty} n(0) = 0$