LIMIT & CONTINUITY

LECTURE 1 : Limit of functions

LECTURE 2 : Limit of functions

LECTURE 3 : Continuity

LECTURE 4 : Continuity

LECTURE 5 : Uniform Continuity.

Theorem:– Let f be a real valued function defined on (a,b) and $c \in (a,b)$. Then $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c+} f(x) = L$ & $\lim_{x\to c-} f(x) = L$.

Proof:– Let $\lim_{x\to c} f(x) = L$.

Let (x_n) be any arbitrary sequence in (a,c) & $x_n \to c$. Then $f(x_n) \to L$ as $n \to \infty$ [since $\lim_{x\to c} f(x) = L$]

$\therefore \lim_{x\to c-} f(x) = L$.

Similarly, $\lim_{x\to c+} f(x) = L$.

Conversely, Let $\lim_{x\to c-} f(x) = L$ & $\lim_{x\to c+} f(x) = L$

For any sequence (x_n) in (a,c) or in (c,b), $x_n \to c$ as $n\to\infty$, we have $f(x_n) \to L$ as $n \to \infty$.

Now let (x_n) be any arbitrary sequence in (a,b) s.t. $x_n \neq c$ & $x_n \to c$. We have to prove that $f(x_n) \to L$ as $n \to \infty$.

<u>Case 1</u> :- If $x_n > c$ for only finite values of n then $f(x_n) \to L$ as $n \to \infty$ $\left[\text{as } \lim_{x \to c-} f(x) = L\right]$

<u>Case 2</u> :- If $x_n < c$ for only finite values of n then $f(x_n) \to L$ as $n \to \infty$ $\left[\text{as } \lim_{x \to c+} f(x) = L\right]$

<u>Case 3</u> :- If $x_n > c$ & $x_n < c$ for infinite values of n then assume $\{r_1, r_2, r_3, \dots\}$ s.t. $x_{r_n} < c$ $\forall n$, and $\{m_1, m_2, m_3, \dots\}$ s.t. $x_{m_n} > c$ $\forall n$.

$\{x_1, x_2, x_3, \dots\} = \{x_{r_1}, x_{r_2}, \dots\} \cup \{x_{m_1}, x_{m_2}, \dots\}$

(x_{r_n}) & (x_{m_n}) are subsequences of (x_n). Therefore

$\cdot\ x_{r_n} \to c$ & $x_{m_n} \to c$ as $n \to \infty$

Since $\lim_{x \to c-} f(x) = L$, $f(x_{r_n}) \to L$ as $n \to \infty$

$\lim_{x \to c+} f(x) = L$, $f(x_{m_n}) \to L$ as $n \to \infty$ ~~w~~

This implies $f(x_n) \to L$ as $n \to \infty$

$\therefore \lim_{x \to c} f(x) = L.$

Choose arbitrary $\varepsilon > 0$. Since $f(x_{r_n}) \to L$ as $n \to \infty$

$\exists N_1 \in \mathbb{N}$ s.t. $|f(x_{r_n}) - L| < \varepsilon \quad \forall n \geq N_1$

Since $f(x_{m_n}) \to L$ as $n \to \infty$, $\exists N_2 \in \mathbb{N}$ s.t.

$|f(x_{m_n}) - L| < \varepsilon \quad \forall n \geq N_2$.

Choose, $N = \max\{r_{N_1}, m_{N_2}\}$. Then $|f(x_n) - L| < \varepsilon$

$\forall n \geq N$. Therefore $f(x_n) \to L$ as $n \to \infty$.

Theorem :- Let f be a real valued function such that $\lim_{x \to c} f(x) = L$ where $L > 0$. Then $\exists$ an open interval (a, b) containing c such that $f(x) > 0 \; \forall \, x \in (a, b) \setminus \{c\}$

Proof :- We claim that, $\exists \, N \in \mathbb{N}$ such that $f(x) > 0 \; \forall$ $x \in (c - \frac{1}{N}, c + \frac{1}{N}) \setminus \{c\}$.

If possible, there is no such $N \in \mathbb{N}$ such that the above condition holds.

Then $\forall \, n \in \mathbb{N}$, $\exists \, x_n \in (c - \frac{1}{n}, c + \frac{1}{n}) \setminus \{c\}$ s.t.

$$f(x_n) \leq 0.$$

Then $x_n \neq c$ & $x_n \to c$. But $f(x_n) \leq 0$

Since $f(x_n) \to L$ & $f(x_n) \leq 0 \; \forall \, n$,

$$L \leq 0.$$

This is a contradiction.

Similarly, if $L < 0$ then $\exists$ an open interval (a, b) containing c s.t. $f(x) < 0 \; \forall \, x \in (a, b) \setminus \{c\}$.

$$f(x) = x \qquad x \in (0,1) \cup (1,2)$$
$$= 0 \qquad x = 1$$

$f(x)$ is defined for $(0,2)$. $f : (0,2) \to \mathbb{R}$.

$\lim_{x \to 1} f(x) = 1 > 0$. Here $f(x) > 0 \;\forall\; x \in (0,2) \setminus \{1\}$

But here $f(1) = 0$

Theorem:- Let f, g, h be real valued functions such that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$.

Then (i) $\lim_{x \to c} \left(f(x) \pm g(x) \right) = L \pm M$.

(ii) $\lim_{x \to c} f(x) g(x) = LM$

(iii) If $M \neq 0$ then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

(iv) If $f(x) \le g(x)$ $\forall$ $x \in (c-\delta, c+\delta) \setminus \{c\}$ then $L \le M$.

(v) If $f(x) \le h(x) \le g(x)$ $\forall$ $x \in (c-\delta, c+\delta) \setminus \{c\}$ for some $\delta > 0$, and $L = M$ then $\lim\limits_{x \to c} h(x) = L$.

Proof :- (i) Let (x_n) be an arbitrary sequence such that $x_n \neq c$ and $x_n \to c$. Since $\lim\limits_{x \to c} f(x) = L$ & $\lim\limits_{x \to c} g(x) = M$, $f(x_n) \to L$ & $g(x_n) \to M$ as $n \to \infty$.

Therefore, $f(x_n) \pm g(x_n) \to L \pm M$ as $n \to \infty$.

$\therefore$ $\lim\limits_{x \to c} f(x) \pm g(x) = L \pm M$.

(ii) Similar Proof.

(iii) Since $M \neq 0$, $\exists\ \delta > 0$ s.t. $g(x) \neq 0\ \forall$ $x \in (c-\delta, c+\delta) \setminus \{c\}$

Thus, $\dfrac{f(x)}{g(x)}$ is well defined on $(c-\delta, c+\delta) \setminus \{c\}$.

Let (x_n) be a arbitrary sequence in $(c-\delta, c+\delta) \setminus \{c\}$ such that $x_n \to c$ as $n \to \infty$.

Since $\lim\limits_{x \to c} f(x) = L$ & $\lim\limits_{x \to c} g(x) = M$

$$\Rightarrow f(x_n) \to L \quad \& \quad g(x_n) \to M \text{ as } n \to \infty$$

$$\Rightarrow \frac{f(x_n)}{g(x_n)} \to \frac{L}{M}$$

$$\therefore \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

(v) Let (x_n) be an arbitrary sequence in $(c-\delta, c+\delta) \setminus \{c\}$ s.t. $x_n \to c$ as $n \to \infty$.

$\Rightarrow f(x_n) \to L$ & $g(x_n) \to L$ as $n \to \infty$.

$\therefore f(x_n) \leq h(x_n) \leq g(x_n)$

Since $f(x_n) \to L$ & $g(x_n) \to L$, by Sandwich theorem for sequence we have, $h(x_n) \to L$ as $n \to \infty$.

$\therefore$ Since $h(x_n) \to L$ as $n \to \infty$ for any arbitrary sequence (x_n) in $(c-\delta, c+\delta) \setminus \{c\}$ with $x_n \to c$, $\lim_{x \to c} h(x) = L$

(Sandwich theorem for limit).

Horizontal Asymptote :- Let $f: A \to \mathbb{R}$ be a function. Then a line $y = b$ is said to be a horizontal asymptote of $y = f(x)$ if either $\lim\limits_{x \to \infty} f(x) = b$ or $\lim\limits_{x \to -\infty} f(x) = b$.

Vertical Asymptote :- Let $f: A \to \mathbb{R}$ be a function. Then a line $x = a$ is said to be a vertical asymptote of $y = f(x)$ if either $\lim\limits_{x \to a+} f(x) = \pm\infty$ or

$\lim\limits_{x \to a-} f(x) = \pm\infty$.

Example :- Let $f : \mathbb{R} \setminus \{-2\} \to \mathbb{R}$ be a function defined by $f(x) = \dfrac{x+3}{x+2}$. Find horizontal & vertical asymptote for the function.

Here
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+3}{x+2} = \lim_{x\to\infty} \frac{1+\frac{3}{x}}{1+\frac{2}{x}} = \frac{1+0}{1+0} = 1$$

Also,
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x+3}{x+2}$$
$$= \lim_{x\to-\infty} \frac{1+\frac{3}{x}}{1+\frac{2}{x}} = 1$$

$\therefore$ $y = 1$ is a horizontal asymptote for $y = f(x)$

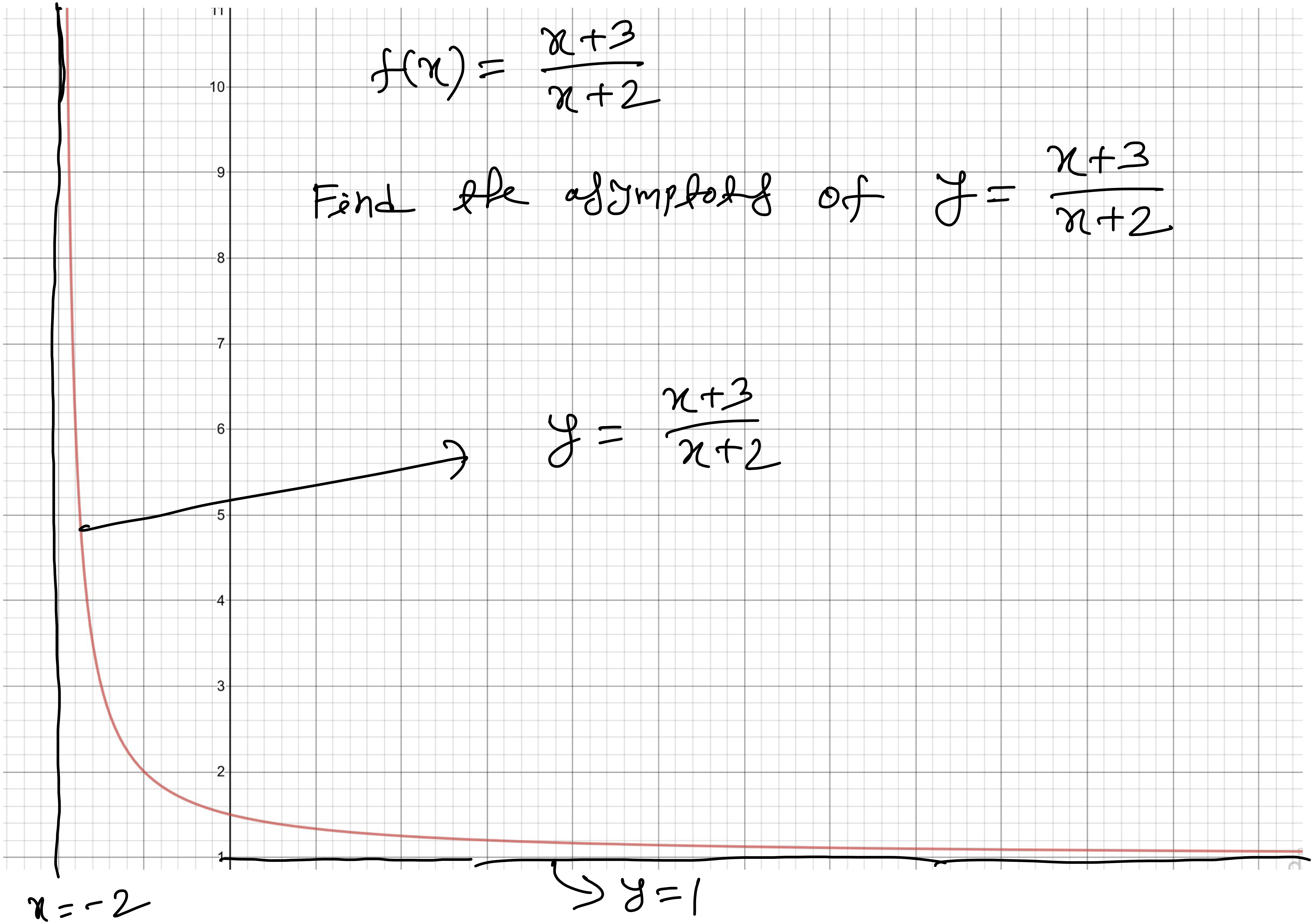
$\therefore$
$$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} \frac{x+3}{x+2} = \infty$$
$$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} \frac{x+3}{x+2} = -\infty$$

— $x = -2$ is a vertical asymptote for $y = f(x)$

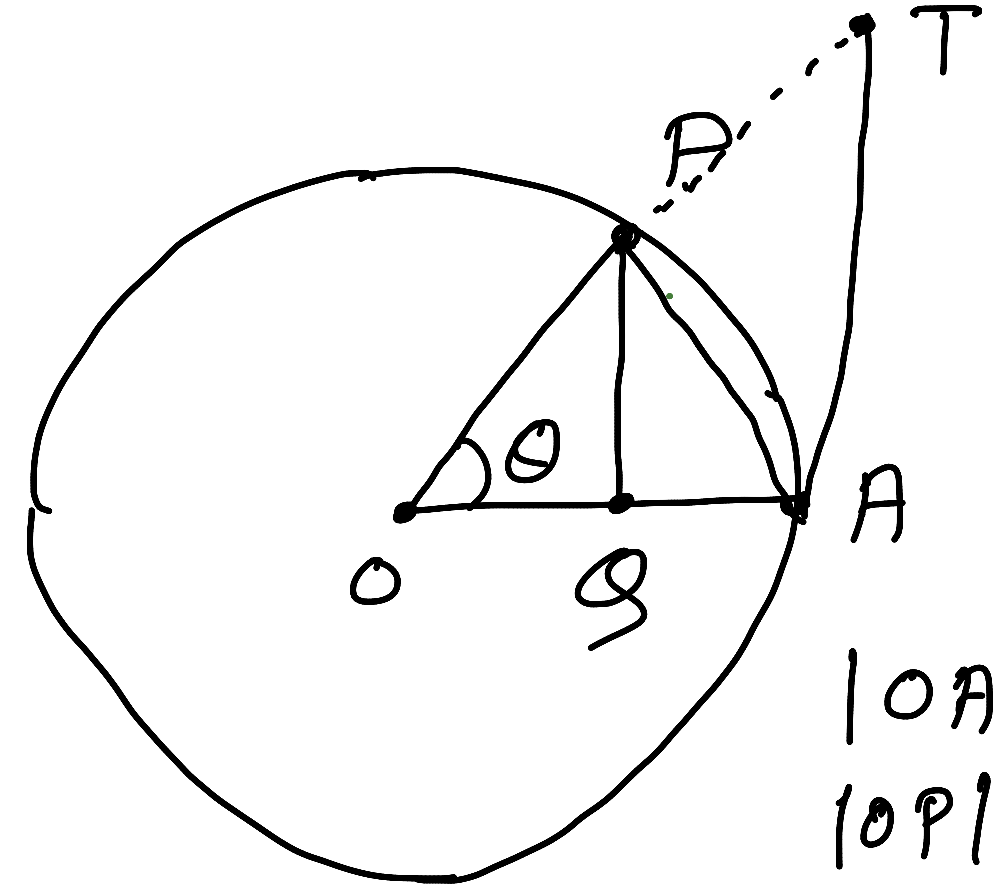
$$f(x) = \frac{x+3}{x+2}$$

Find the asymptotes of $y = \frac{x+3}{x+2}$

$$y = \frac{x+3}{x+2}$$



Show that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.



$|OA| = r$
$|OP| = r$

Proof:- Assume $0 < \theta < \frac{\pi}{2}$

$\angle POQ = \theta$

$\angle PQO = \frac{\pi}{2}$

$\angle OAT = \frac{\pi}{2}$

Area of $\triangle OAP = \frac{1}{2} \times |OA| \times |PQ|$ $\left[\frac{|PQ|}{|OP|} = \sin\theta \right.$

$= \frac{1}{2} \times |OA| \times |OP| \sin\theta$

$= \frac{1}{2} r^2 \sin\theta$

Area of the sector $OAP = \pi r^2 \times \frac{\theta}{2\pi} = \frac{r^2}{2}\theta$

Area of $\triangle OAT = \frac{1}{2} \times |OA| \times |AT|$ $\left[\frac{|AT|}{|OA|} = \tan\theta \right]$

$= \frac{1}{2} \times |OA| \times |OA| \times \tan\theta = \frac{1}{2} \times r^2 \times \tan\theta$

Thus, $\frac{r^2}{2}\sin\theta \le \frac{r^2\theta}{2} \le \frac{r^2}{2}\tan\theta$

Here $r > 0$ & $\sin\theta > 0$ as $0 < \theta < \frac{\pi}{2}$

$\therefore \quad 1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$

$\Rightarrow \quad \cos\theta \le \frac{\sin\theta}{\theta} \le 1$ [This is true for $\theta \in (0, \frac{\pi}{2})$]

$$\therefore \underbrace{\lim_{\theta\to 0+}\cos\theta}_{=1} \le \lim_{\theta\to 0+}\frac{\sin\theta}{\theta} \le 1$$

$$\therefore \lim_{x\to 0+}\frac{\sin\theta}{\theta} = 1$$

Let $f(\theta) = \frac{\sin\theta}{\theta}$. Then $f(-\theta) = f(\theta)$. Thus f is an even function. $\lim_{\theta\to 0-}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0+}\frac{\sin\theta}{\theta} = 1$

Thus, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Theorem :- Let $f: A \to \mathbb{R}$ be a function and a be a limit point of A. Then $\lim_{x \to a} f(x) = L$ if and only if for any $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that $0 < |x-a| < \delta$ & $x \in A$

$\Rightarrow |f(x) - L| < \varepsilon$. —— (*)

Proof :- Assume (*) holds. We prove that $\lim_{x \to a} f(x) = L$

Let (x_n) be a arbitrary sequence in A such that $x_n \neq a$ $\forall n \in \mathbb{N}$ & $x_n \longrightarrow a$ as $n \to \infty$.

Choose arbitrary $\varepsilon > 0$.

By (*), $\exists$ a $\delta > 0$ s.t. $0 < |x-a| < \delta$ & $x \in A$

$\Rightarrow |f(x) - L| < \varepsilon$.

Since $x_n \to a$ as $n \to \infty$, $\exists\ N \in \mathbb{N}$ s.t.

$$0 < |x_n - a| < \delta \quad \forall\ n \geq N.$$

Therefore, by (*), $|f(x_n) - L| < \varepsilon \quad \forall\ n \geq N$

$$\therefore\ f(x_n) \to L \text{ as } n \to \infty.$$

$$\therefore\ \lim_{x \to a} f(x) = L.$$

Converse, Assume $\lim_{x \to a} f(x) = L$. We prove the condition (*) holds.

If possible the condition (*) does not hold.

(*) says, $\forall \varepsilon\ \exists \delta\ \forall\ x \in (a-\delta, a+\delta) \cap A \setminus \{a\}$ we have $|f(x) - L| < \varepsilon$

Negation is, $\exists \varepsilon\ \forall \delta\ \exists\ x \in (a-\delta, a+\delta) \cap A \setminus \{a\}$ we have $|f(x) - L| \geq \varepsilon$.

If the condition (*) does not hold then

$\exists\ \varepsilon$ such that for all δ, there is $x \in (a-\delta, a+\delta) \setminus \{a\}$

we have $|f(x) - 2| \geq \varepsilon$.

Let $\delta = \frac{1}{n}$. Let $x_n \in (a - \frac{1}{n}, a + \frac{1}{n}) \cap A \setminus \{a\}$ be

such that $|f(x_n) - 2| \geq \varepsilon$ → fixed real number.

Here $x_n \to a$ as $n \to \infty$ but $f(x_n) \not\to L$ as $n \to \infty$

Thus $\lim_{x \to a} f(x) \neq L$.

This is a contradiction.

Therefore, the condition (*) must hold.

We are done.

If the condition (*) does not hold then $\exists\, \varepsilon > 0$ such that for any $\delta > 0$, there is an $x \in (a-\delta, a+\delta) \cap A \setminus \{a\}$ we have $|f(x) - L| \geq \varepsilon$.

Let $\delta = \frac{1}{n}$. Let $x_n \in (a - \frac{1}{n}, a + \frac{1}{n}) \cap A \setminus \{a\}$ be such that

$$|f(x_n) - L| \geq \varepsilon.$$

Since (x_n) is a sequence in A & $x_n \longrightarrow a$ as $n \to \infty$

but $f(x_n) \not\longrightarrow L$

$\therefore \lim_{x \to a} f(x) \neq L$. This is a contradiction.

$\therefore$ The condition (*) must hold.