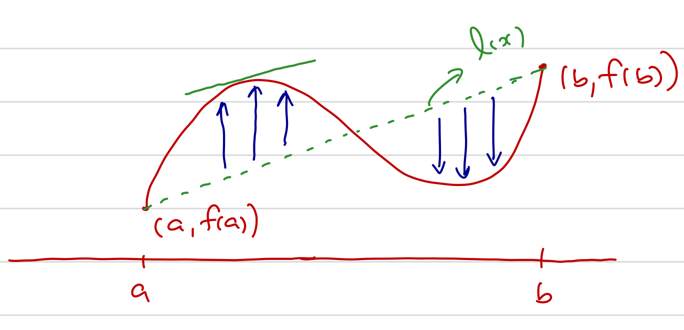
Lecture 2 - Differentiation

Lagrange's Mean value theorem (MVT):

If f is a cts on $[a,b]$, and
f is diff on (a,b)
then $\exists\ c \in (a,b)$ such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



Proof: Define
$$H(x) = f(x) - l(x)$$
$$= f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a).$$

Observe that (i) $H(a) = 0 = H(b)$
(ii) H is cts on $[a,b]$
(iii) H is diff on (a,b).

By Rolle's thm, $\exists\ c \in (a,b)$ s.t.
$$H'(c) = 0$$
$$f'(c) - l'(c) = 0$$
$$\therefore\ f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Cor: Suppose f is diff on I and $f'(x)=0\ \forall x \in I$.
Then f is constant.

Proof: Fix $a \in I$. claim: $f(x) = f(a),\ \forall x \in I$.
let $x \in I$, $x \neq a$.
Using MVT, $\exists\ \xi$ between 'a' & 'x' s.t.
$$\frac{f(x)-f(a)}{x-a} = f'(\xi) = 0$$
$$\Rightarrow f(x) = f(a).$$

Cor: Suppose f & g are diff on I and
$f'(x) = g'(x)\ \forall x \in I$.
Then $f = g + c$, where c is a constant.
Pf: Exc.

Some Inequalities

(1) Show that $e^x > 1+x$, $x \in \mathbb{R}$.
Consider $f(x) = e^x$, $x \in \mathbb{R}$.
let $x > 0$, Using MVT, we get
$$\frac{e^x - e^0}{x-0} = e^{\xi}, \text{ where } \xi \in (0,x).$$
Use $e^{\xi} > 1$, we get
$$\frac{e^x - 1}{x} > 1$$
$$\Rightarrow e^x > 1+x,\quad x > 0 \quad \rightarrow (*)$$

Try: (i) S.T. $\log(1+x) < x$, $\forall x > 0$.
(Don't use $(*)$)
(ii) S.T. $\tan x > x$, $\forall x \in (0, \pi/2)$

(2) $f(x) = \cos x$, $x \in \mathbb{R}$
For $x, y \in \mathbb{R}$, $x < y$, use MVT,
$$\frac{\cos y - \cos x}{y - x} = (-\sin \xi), \text{ for some } \xi \in (x,y).$$
$$|\cos y - \cos x| \le 1 \cdot |y - x|,$$

Cor: Suppose f is diff on I & f' is bdd on I.
Then f is Lipschitz cts on I.
Pf: For $x, y \in I$, $x < y$, using MVT,
$$|f(x) - f(y)| \le \sup_{s \in I} |f'(s)|\, |x-y|.$$

Rmk: Lipschitz cts $\Rightarrow$ uniformly cts
* If f is diff on I & f' is bdd on I
then f is uniformly cts on I.

Warning: If f' is not bdd on I
then f is not uniformly cts on I.
Ans: No. It is a false statement.
eg: $f(x) = \sqrt{x}$, $x > 0$.

Monotone functions & derivatives.

We say that f is said to increasing on I
if for every $x, y \in I$, $x < y$,
we get $f(x) < f(y)$.

Thm Suppose f is diff on I and
$f'(x) > 0\ \forall x \in I$.
Then f is increasing on I.

Pf: For $x, y \in I$, $x < y$, use MVT.
$$\frac{f(y)-f(x)}{y-x} = \underbrace{f'(\xi)}_{>0}, \text{ where } \xi \in (x,y)$$
$$\Rightarrow f(y) - f(x) > 0.$$
$$\therefore f(x) < f(y).$$

L'Hôpital's Rule:

Suppose f and g are diff. on I. & $a \in I$.
Suppose $f(a) = 0 = g(a)$, and f' & g' are
cts at a, $g'(a) \neq 0$.
Then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

proof. Since $g'(a) \neq 0$ and g' is cts at 'a',
$\exists\ \delta > 0$ s.t.
$$g'(x) \neq 0 \quad \forall\ 0 < |x-a| < \delta$$
$$\Rightarrow g(x) \neq 0 \quad \forall\ 0 < |x-a| < \delta$$
(Exc.).

claim $\forall (x_n)$, $x_n \neq a$, $|x_n - a| < \delta$, $x_n \to a$
WST $\displaystyle\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f'(a)}{g'(a)}.$

Using MVT, $\exists\ \xi_n$ between a & x_n,
η_n between a & x_n
such that
$$f(x_n) - f(a) = f'(\xi_n)(x_n - a)$$
$$g(x_n) - g(a) = g'(\eta_n)(x_n - a)$$
$$\Rightarrow \frac{f(x_n)}{g(x_n)} = \frac{f'(\xi_n)}{g'(\eta_n)}$$
$\because x_n \to a$, we get $\xi_n \to a$ & $\eta_n \to a$
as $n \to \infty$.
$\because f'$ & g' are cts at a & $g'(a) \neq 0$,
we get
$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \lim_{n\to\infty} \frac{f'(\xi_n)}{g'(\eta_n)} = \frac{f'(a)}{g'(a)}.$$

Rmk The above result can be derived with mild assumptions.
Suppose f & g are diff on I, $a \in I$.
Suppose $f(a) = 0 = g(a)$, and
if $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ exists.
Then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = L = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

eg: $h(x) = \dfrac{\sin x}{x}$, $0 \neq x \in \mathbb{R}$.
$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\cos 0}{1} = 1.$$

Inverse function & derivative.

$f(x) = x^2$, $x \in (0, \infty)$.
$f^{-1}(y) = \sqrt{y}$, $y \in (0, \infty)$ $\quad f^{-1} \circ f(x) = x.$
$$(f^{-1})'(y_0) = \frac{1}{2\sqrt{y_0}}.$$

Qn: Can we use the knowledge of f'
to compute $(f^{-1})'$?
Yes.

Thm Suppose $f: I \to J$ is a bijective fn, where
I & J are open intervals. Suppose f
is diff at $x_0 \in I$ & f^{-1} is diff at
$y_0 = f(x_0)$.
Then
$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \text{ if } f'(x_0) \neq 0.$$

pf:
$$f^{-1} \circ f(x) = x$$
$$(f^{-1} \circ f)'(x_0) = 1.$$
Use chain rule,
$$(f^{-1})'(f(x_0))\, f'(x_0) = 1$$
$$\Rightarrow (f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Inverse function theorem.

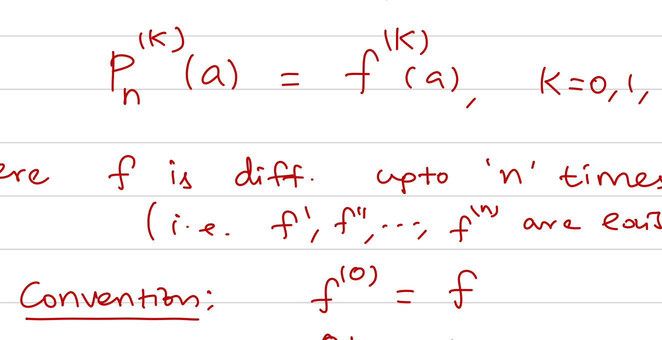
Suppose $f: I \to J$ be a bijective,
f is diff at x_0, $f'(x_0) \neq 0$.
Suppose f^{-1} is cts at $y_0 = f(x_0)$. Then
$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Pf: WST
$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$
$\Leftrightarrow \forall (y_n)$, $y_n \neq y_0$, $y_n \to y_0$, WST
$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}.$$
Let (y_n) be a seq. s.t. $y_n \neq y_0$, & $y_n \to y_0$.
There exists (x_n) s.t. $f(x_n) = y_n$.
$\because f^{-1}$ is cts at y_0, $x_n = f^{-1}(y_n) \longrightarrow f^{-1}(y_0) = x_0$
as $n \to \infty$.
$$\Rightarrow \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0)$$
$\because f'(x_0) \neq 0$, we get
$$\lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}. \quad \text{(why?)}$$
i.e.
$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}$$

Exc: Consider $f(x) = x^{1/n}$, $n \ge 1$, $x > 0$.
Show that $f'(x) = \frac{1}{n} x^{\frac{1}{n} - 1}$.

Approximation of "smooth" functions using polynomials:

Consider a tangent line of f:
At 'a', we have
$$P_1(x) = f(a) + f'(a)(x-a)$$
It is a natural linear approximation of
f at 'a'.



$$P_1(0) = 0 = \sin(0)$$
$$P_1'(0) = 1 = \cos(0).$$
$$P_1(a) = f(a);\quad P_1'(a) = f'(a).$$

Find a 'n' degree polynomial $P_n(x)$ s.t.
$$P_n^{(k)}(a) = f^{(k)}(a),\quad k = 0, 1, \dots, n.$$
Here f is diff. upto 'n' times.
(i.e. $f', f'', \dots, f^{(n)}$ are exists)

Convention: $f^{(0)} = f$
$0! = 1.$

Taylor Polynomial

Let $P_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$.

$$P_n^{(k)}(a) = k!\, c_k = f^{(k)}(a)$$

$$\Rightarrow \quad c_k = \frac{f^{(k)}(a)}{k!}, \quad k = 0, \ldots, n.$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called n^{th} degree Taylor polynomial of f at 'a'.

Qn: What is the approximation error for $f(x) - P_n(x)$?

Taylor's Theorem:

Suppose f is diff upto $(n+1)$ times on I & $a \in I$. Then for given $x \in I, (x \neq a)$ there exists a $\xi \in I$ which is between a & x s.t.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

$$= P_n(x) + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$ is called the n^{th} remainder term.