Lecture 4 Mon 31 Jul 2023

S — feasible set, convex polyhedron. Obj. func^n

(LP) $\max \; z = c^T x$

s.t. $x \in S = \{ x \in \mathbb{R}^n : \underbrace{Ax \leq b, \; x \geq 0}_{\text{constraints}} \}$

Assume: (i) The obj. func^n is not constant.

(ii) (LP) has a sol^n, say $x^* \in S$.

Claim: $x^* \notin$ interior of S.

Pf: Since $x^* \in S$ is an optimal sol^n of LP

$\Rightarrow$ for some $\delta > 0$,

$$c^T x^* \geq c^T x, \quad \forall \; x \in N_\delta(x^*) \cap S$$

Let us assume contrary that $x^* \in$ interior S

$\Rightarrow \exists \; \delta_1 > 0$ s.t. $N_{\delta_1}(x^*) \subset S$.

Choose $r = \min\{\delta, \delta_1\} > 0$

$$c^T x^* \geq c^T x \quad \forall \; x \in N_r(x^*) \subset S.$$

Construct $\hat{x} = x^* + \frac{r}{2} \frac{c}{\|c\|}$

$\Rightarrow \hat{x} - x^* = \left(\frac{r}{2}\right)\left(\frac{c}{\|c\|}\right)$

$\Rightarrow \|\hat{x} - x^*\| \leq \left|\frac{r}{2}\right| < r$

$\Rightarrow \hat{x} \in N_r(x^*)$

$\Rightarrow c^T \hat{x} = c^T x^* + \left(\frac{r}{2}\right) \|c\|. > c^T x^*.$

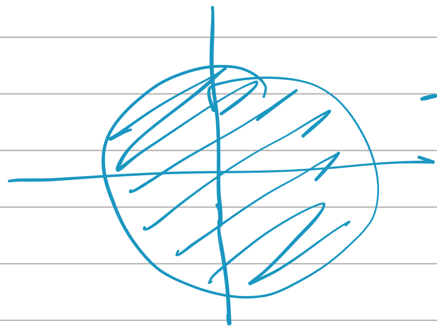
* Extreme points of a convex set.

Let $X \subset \mathbb{R}^n$ be a convex set.
A point $x_0 \in X$ is called an extreme point (EP) of X if $\exists\, x_1, x_2 \in X$, $x_1 \neq x_2$ and a scalar $\lambda \in (0,1)$ such that

$$x_0 = \lambda x_2 + (1-\lambda) x_1$$

* Finite no. of extreme points are there in polyhedron.

Eg of infinite extreme pts
(cannot be a polyhedron)



$\rightarrow$ All boundary pts are extreme pts (uncountably extreme pts).

* Result: An EP of X is a boundary pt. of X. Converse is not true.

If x_0 is an EP of convex set X, then $x_0 \in X \Rightarrow x_0 \in$ interior X or $x_0 \in$ boundary X.

To show: $x_0 \notin$ interior X.

Suppose $x_0 \in$ interior X

$\Rightarrow \exists\, \delta > 0$ s.t. $\quad N_\delta(x_0) \subset X$

$$x_1 = x_0 - \frac{\delta}{2}\frac{e}{\|e\|} \qquad e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$x_2 = x_0 - \frac{\delta}{2}\frac{e}{\|e\|}$$

$\|x_1 - x_0\| = \frac{\delta}{2} = \|x_2 - x_0\|$ and $x_1 \neq x_2$.

$\Rightarrow x_1, x_2 \in N_\delta(x_0)$

and $x_0 = \frac{1}{2}x_1 + \frac{1}{2}x_2$

Hence, x_0 cannot be EP

$\Rightarrow$ contradiction

Hence, $x_0 \notin$ interior X.

$X \rightarrow$ convex set in $\mathbb{R}^n$.

Let $x_0 \in X$ be arbitrary.

A vector $d \in \mathbb{R}^n$, $d \neq 0$, a dir^n vector of X at x_0 if $x_0 + \lambda d \in X$, $\forall \lambda \geq 0$.

$D_X(x_0) = \{d \in \mathbb{R}^n : d \neq 0 : x_0 + \lambda d \in X \; \forall \lambda \geq 0\}$