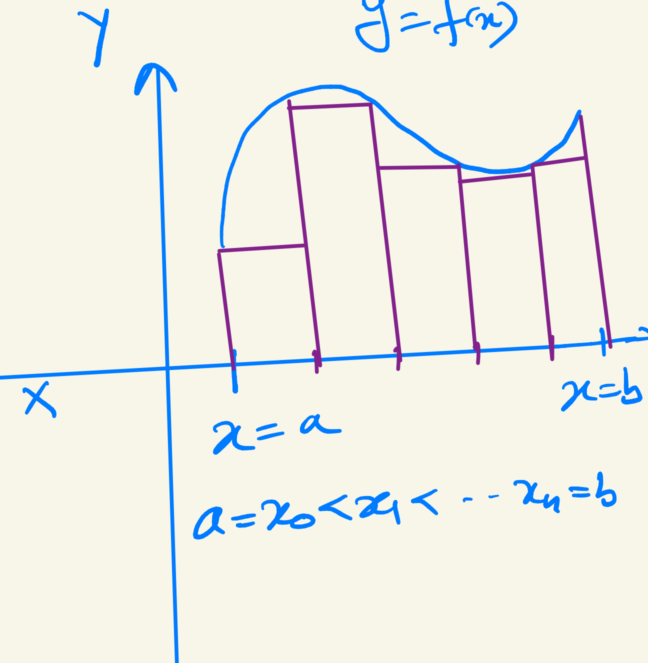
Multiple Integrals - Lecture 1

1 Double Integral

$I \subseteq \mathbb{R}$, $I = [a, b]$ and $f : I \to \mathbb{R}$, $f(x) \geq 0$

and f is continuous.



$a = x_0 < x_1 < x_2 < \cdots < x_n = b$

$M_i = \sup \{ f(x) \mid x \in [x_{i-1}, x_i] \}$

$m_i = \inf \{ f(x) \mid x \in [x_{i-1}, x_i] \}$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \quad \& \quad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$\boxed{L(P,f) \leq \text{Area} \leq U(P,f)} \quad \text{———} \quad \circledast$$

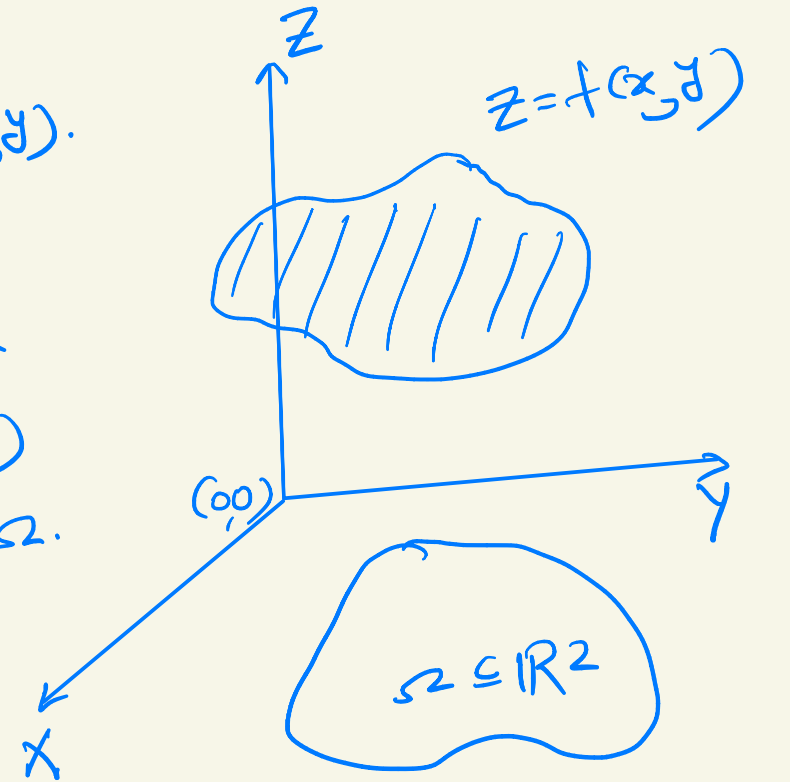
$$\underline{\int_a^b} f(x)\,dx = \sup \{ L(P,f) \mid P \}$$

$$\overline{\int_a^b} f(x)\,dx = \inf \{ U(P,f) \mid P \}$$

$$\boxed{\text{Area} = \int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx}$$

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain and f be a cont^n f^n with $f(x,y) \geq 0$.

Take the surface $z = f(x,y)$.



Problem :→ Find the volume of solid below $z = f(x,y)$ and above the domain Ω.

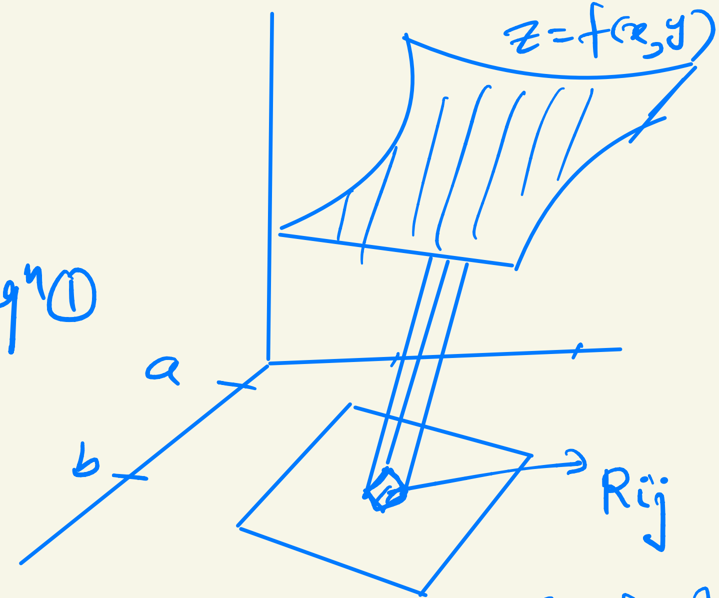
Start with simplest domain

$$\Omega = [a,b] \times [c,d].$$

$$a = x_1 < x_2 \cdots < x_n = b$$

$$c = y_1 < y_2 \cdots < y_m = d$$

$$R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \text{ — eq}^n \text{ (1)}$$



$$M_{ij} = \sup\{ f(x,y) \mid (x,y) \in R_{ij} \}$$

$$R = \Omega = [a,b] \times [c,d]$$

$$m_{ij} = \inf\{ f(x,y) \mid (x,y) \in R_{ij} \}$$

$$\boxed{U(P,f) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} \, |R_{ij}| \; , \quad L(P,f) = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} \, |R_{ij}|}$$

Define $\underline{\iint_{\Omega}} f(x,y)\,dA = \sup\{L(P,f) \mid P\}$ and

$\overline{\iint_{\Omega}} f(x,y)\,dA = \inf\{U(P,f) \mid P\}$. Then

$$\boxed{\text{Volume} = \iint_{\Omega} f(x,y)\,dA = \underline{\iint_{\Omega}} f(x,y)\,dA = \overline{\iint_{\Omega}} f(x,y)\,dA}$$

Problem :- $\boxed{\text{How do we compute double integral?}}$

In one-variable case we have $\Rightarrow$ | $\int_a^b f(x)\,dx = F(b) - F(a) \longrightarrow (*)$

<u>Theorem</u> $\Rightarrow$ (Fubini's theorem for rectangle)

If f is continuous on domain $\Omega = [a,b] \times [c,d]$.

Then
$$\iint_\Omega f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

Example $\div$ Compute $\iint_{\Omega} 6xy^2\,dA$, where $\Omega = [2,4] \times [1,2]$

Soln $\div$

$$\iint_{\Omega} 6xy^2\,dA = \int_2^4 \left(\int_1^2 6xy^2\,dy \right) dx$$

$$= \int_2^4 6x \left[\frac{y^3}{3} \right]_1^2 dx$$

$$= \int_2^4 6x \left[\frac{8}{3} - \frac{1}{3} \right] dx$$

$$= 14 \left[\frac{x^2}{2} \right]_2^4 = ?$$

Exercise $\div$ $\int_1^2 \left(\int_2^4 6xy^2\,dx \right) dy = \iint_{\Omega} 6xy^2\,dA.$

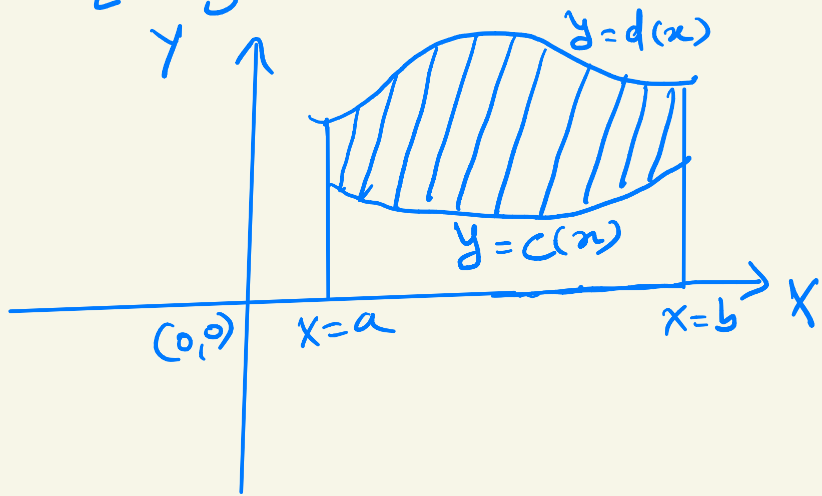
Regular domains :→

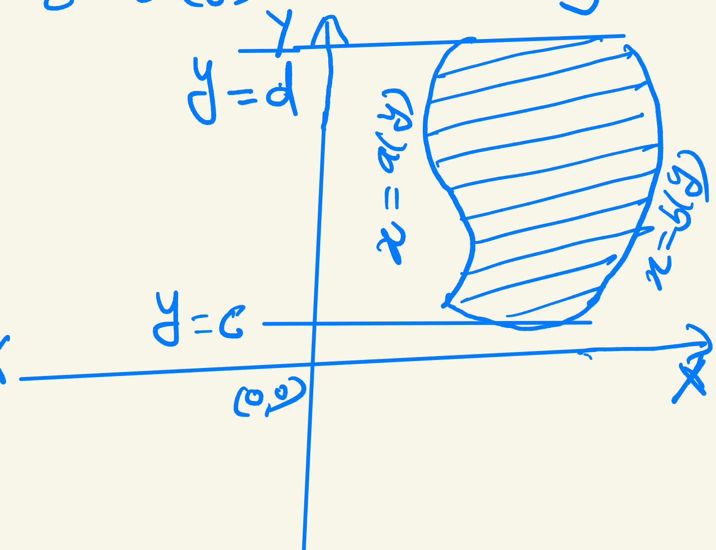
(1) A domain Ω is said to be Y- regular if

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \ \& \ c(x) \leq y \leq d(x)\}$$

(2) A domain Ω is said to be X- regular if

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid c \leq y \leq d \ \& \ a(y) \leq x \leq b(y)\}$$





Theorem :- (Fubini's theorem for regular domains)

let f be continuous on Ω (integrable on Ω). Then

① If Ω is Y- regular, we have

$$\iint_\Omega f(x,y)\,dA = \int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)\,dy \right) dx$$

② If Ω is X- regular, we have

$$\iint_\Omega f(x,y)\,dA = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\,dx \right) dy.$$

Example-1 Compute $\iint_\Omega (x+y+xy)\,dA$, where Ω is triangle bounded by $y=0$, $y=x$ and $x=1$.

Sol^n :-

Y-regular

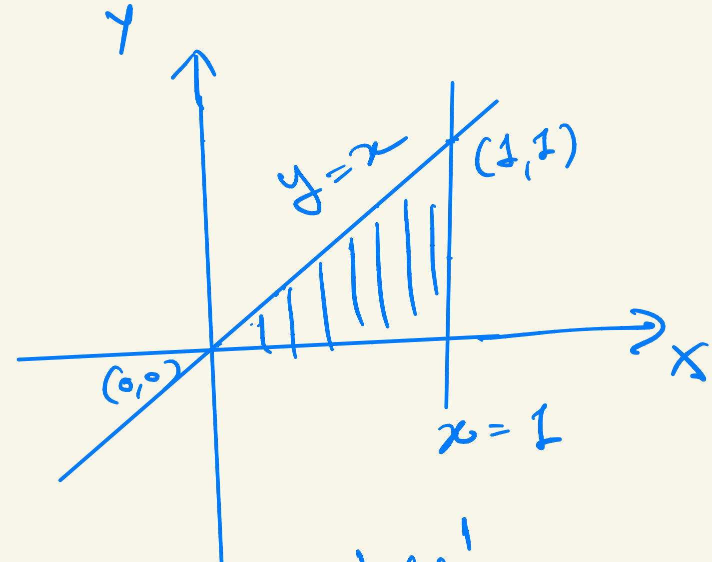
$0 \le x \le 1$

$0 \le y \le x$

X-regula

$0 \le y \le 1$

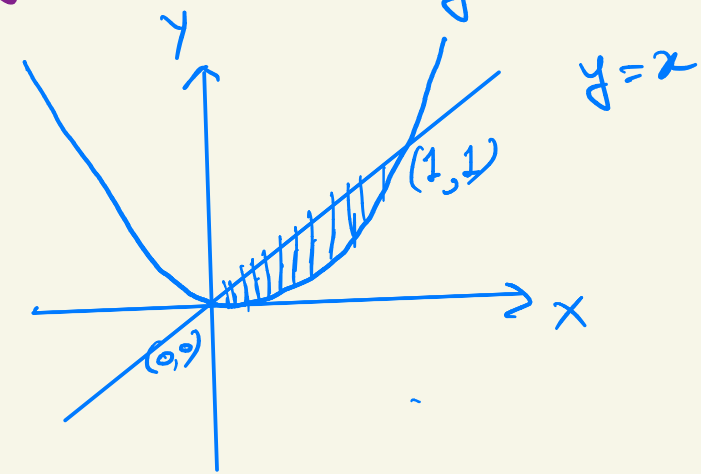
$y \le x \le 1$



$$\iint_\Omega (x+y+xy)\,dA = \int_0^1 \left(\int_0^x (x+y+xy)\,dy\right)dx = \int_0^1 \left(\int_y^1 (x+y+xy)\,dx\right)dy$$

Example ② Compute $\iint_{\Omega} (2+4x)$, where Ω is domain bounded by $y=x$ and $y=x^2$.

Sol^n.



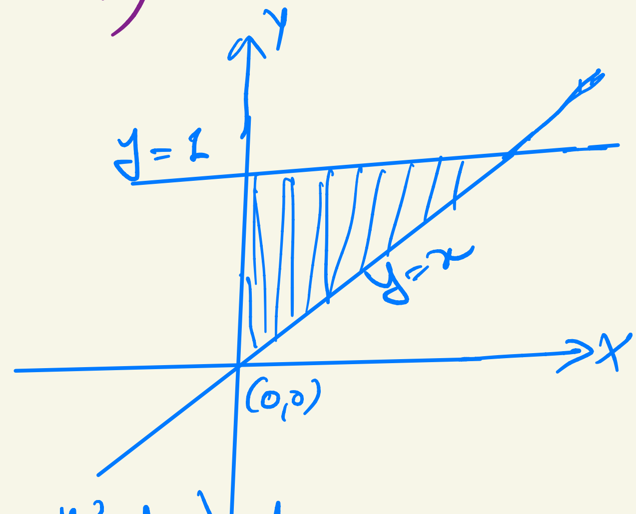
Y- regular	X- regular
$0 \le x \le 1$	$0 \le y \le 1$
$x^2 \le y \le x$	$y \le x \le \sqrt{y}$

$$\iint_{\Omega} (2+4x)\,dA = \int_0^1 \left(\int_{x^2}^{x} (2+4x)\,dy \right) dx$$

Example ③ Compute $\int_0^1 \left(\int_x^1 e^{y^2}\, dy \right) dx$.

Sol^n

Y-regular	x-regular
$0 \le x \le 1$	$0 \le y \le 1$
$x \le y \le 1$	$0 \le x \le y$



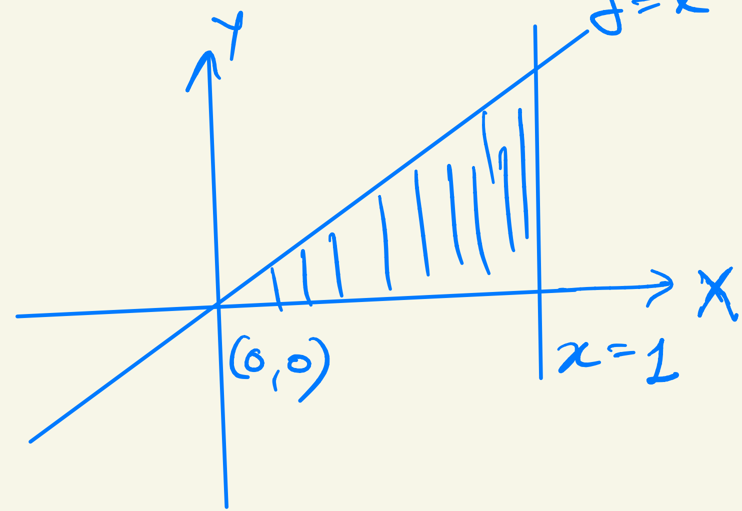
$$\int_0^1 \left(\int_x^1 e^{y^2}\, dy \right) dx = \int_0^1 \left(\int_0^y e^{y^2}\, dx \right) dy$$

$$= \int_0^1 e^{y^2} [x]_0^y \, dy = \int_0^1 y e^{y^2}\, dy$$

$$= \frac{1}{2} \left[e^{y^2} \right]_0^1 = \frac{1}{2} e.$$

Example ④ Compute $\iint_\Omega \frac{\sin x}{x}$, where Ω is the domain bounded by $y=0$, $y=x$ and $x=1$

Sol^n.



Y- regular	X- regular
$0 \le x \le 1$	$0 \le y \le 1$
$0 \le y \le x$	$y \le x \le 1$

Note that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\iint_{\Omega} \frac{\sin x}{x} = \int_0^1 \left(\int_y^1 \frac{\sin x}{x}\, dx \right) dy$$

$$= \int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx$$

$$= \int_0^1 \frac{\sin x}{x} \left[y \right]_0^x dx = \int_0^1 \sin x\, dx$$

<u>Remark</u> $:\Rightarrow$ We can have any bounded domain Ω.

① Divide Ω into small sub-domain Ω_k.

② Define integration on Ω by using Ω_k's.