COL 352 Introduction to Automata and Theory of Computation

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Lecture 7: Pattern Matching and Regular Expressions

Recap

- DFA = NFA = ε-NFA.
- All of them recognize (compute/decide) exactly regular languages
- Regular languages are closed under union, intersection, complement, concatenation, Kleene star, ...

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- Certain Algebraic connection (acceptability via finite semi-group): Eilenberg'76
- Regular expressions (Kleene'50s).
- Rational languages.

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Kleene star

For a language L, its Kleene closure, denoted L^* is the set of all strings obtained by taking any number of strings from L with possible repetitions and concatenating all of them.

$$L^* = \cup_{i \geq 0} L^i$$

$$L^0 = \{\epsilon\}, L^i = L \circ L^{i-1}$$

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$$L(\alpha) = \{x \in \Sigma^* \mid x \text{ matches } \alpha\}$$

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7. x matches $\alpha \cap \beta$ if $L(\alpha \cap \beta) = L(\alpha) \cap L(\beta)$
8. x matches $\alpha\beta$ if $x = yz$ where $L(\alpha\beta) = L(\alpha)L(\beta)$
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9. x matches $\overline{\alpha}$ if $L(\overline{\alpha}) = \overline{L(\alpha)} = \Sigma^* \setminus L(\alpha)$
10. x matches α^* if x can be expressed as zero or more of strings that match α, i.e., $L(\alpha^*) = L(\alpha)^*$
11. x matches α^+ if x can be expressed as one or more of strings that match α, i.e., $L(\alpha^+) = L(\alpha)^+$

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Express each of these languages as a pattern.

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- Can you get rid of complementation?

Regular expressions

For a regular expression E we write $L(E)$ for its language. The set of valid regular expressions RegEx can be defined recursively as the following:

	Syntax	Semantics
Empty String	ϵ	$L(\epsilon) = \{\epsilon\}$
Empty Set	$\emptyset$	$L(\emptyset) = \emptyset$
Single Letter	a	$L(a) = \{a\}$
Union	$E + F$	$L(E + F) = L(E) \cup L(F)$
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Precedence rules: $* > \circ > +$

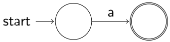
Language defined by regular expression

Lemma

The language defined by any regular expression is regular.

Example

$$(a+b)^*$$



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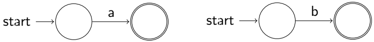

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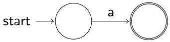

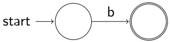

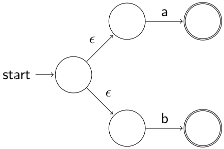

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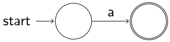

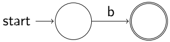

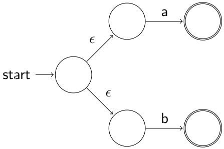

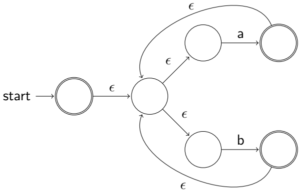

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What about the converse?

Example

- $(aaa)^* + (aaaaa)^*$
- $(11 + 0)^*(00 + 1)^*$

A few axioms we can use for simplification

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	$\beta + \gamma\alpha \leq \gamma \implies \beta\alpha^* \leq \gamma$

where

$$\begin{aligned} \alpha \leq \beta &\iff L(\alpha) \subseteq L(\beta) \\ &\iff L(\alpha + \beta) = L(\beta) \\ &\iff \alpha + \beta = \beta \end{aligned}$$

A few consequences that follow

Exercise!

$$(\alpha\beta)^*\alpha \equiv \alpha(\beta\alpha)^*$$
$$(\alpha^*\beta)^*\alpha^* \equiv (\alpha+\beta)^*$$
$$\alpha^*(\beta\alpha^*)^* \equiv (\alpha+\beta)^*$$
$$(\epsilon+\alpha)^* \equiv \alpha^*$$
$$\alpha\alpha^* \equiv \alpha^*\alpha$$

Example

- $(aaa)^* + (aaaaa)^*$
- $(11+0)^*(00+1)^*$
- $(1+01+001)^*(\varepsilon+0+00)$

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Example

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$(1+01+001)^*(\varepsilon+0+00)$ = all strings over $\{0,1\}$ with no substring of more than two adjacent 0's.

DFA to regular expression

Lemma

Any regular language can be specified by a regular expression

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Want: Given any DFA, convert it into a regular expression.

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Given any DFA A, we can obtain a regular expression, say R_A, such that $L(A) = L(R_A)$.

Computing with labelled graphs

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Lemma

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