Laplace Transforms

Pierre-Simon Laplace (1749-1827)

1. He was an astronomer
2. He has significant contributions in calculus, Bayesian estimations, black holes, Laplace equation and Laplace transforms .
3. Central limit theorem and absurd theories like rule of succession

$$\Pr(\textit{sun will rise tomorrow}) = \frac{d+1}{d+2}$$

4. He is known as the "Newton of France."
5. D'Alembert interaction.

Pierre-Simon Laplace

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Continuous-Time Fourier Transform

- Representing signals as linear combination of basic signals $e^{j\omega t}$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Synthesis equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Analysis equation

LTI systems

- Impulse response h(t)

$$e^{j\omega t} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(\omega)e^{j\omega t}$$

LTI systems

- Impulse response h(t)

$$e^{st} \longrightarrow \boxed{\text{LTI}}$$

$$s = \sigma + j\omega$$

LTI systems

- Impulse response h(t)

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$s = \sigma + j\omega$$

LTI systems

- Impulse response h(t)

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LTI systems

- Impulse response h(t)

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$$s = \sigma + j\omega$$

$$\boxed{H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau}$$

LTI systems

- Impulse response h(t)

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow e^{st}\,H(s)$$

$$s = \sigma + j\omega$$

$$\boxed{H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau}$$

Laplace Transform

Bilateral

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

Connection between Laplace and Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$s = \sigma + j\omega \qquad X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(j\omega) = X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\} = X(\omega)$$

$$\boxed{X(j\omega) = \mathcal{F}\{x(t)\}}$$

New notation

Connection between Laplace and Fourier Transform

$$\boxed{X(s)\big|_{s=j\omega} = X(j\omega)}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

$$\boxed{X(s) = \mathcal{F}\{x(t)e^{-\sigma t}\}}$$

$\mathcal{L}$ may converge when $\mathcal{F}$ does not

Inverse Laplace Transform

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(s)\}$$

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{j\omega t}\,d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{(\sigma + j\omega)t}\,d\omega$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds$$

Inverse Laplace Transform

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Example 9.1

$$x(t) = e^{-at}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt$$

$$X(s) = \int_{0}^{\infty} e^{-at}e^{-st}dt$$

$$X(s) = \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$X(s) = \frac{-1}{s+a}[e^{-(s+a)t}]_0^{\infty}$$

$$= \frac{-1}{s+a}[0-1] = \frac{1}{s+a}$$

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$$=\frac{-1}{s+a}[0-1]=\frac{1}{s+a}$$

$$X(s)=\int_{0}^{\infty}e^{-(\sigma+\mathrm{j}\omega+a)t}dt$$

$$=\int_{0}^{\infty}e^{-(\sigma+a)t}e^{-\mathrm{j}\omega t}dt$$

$$\mathrm{Re}\{s\}>-a$$

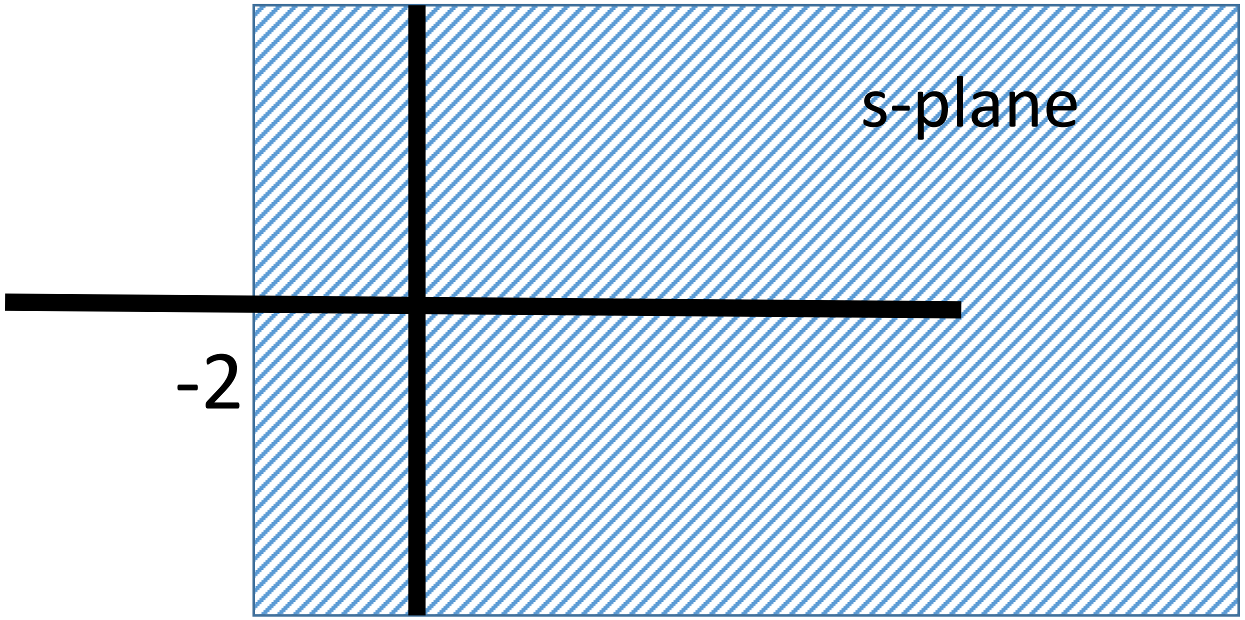
Example 9.1

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

Example 9.1

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

If a = 2



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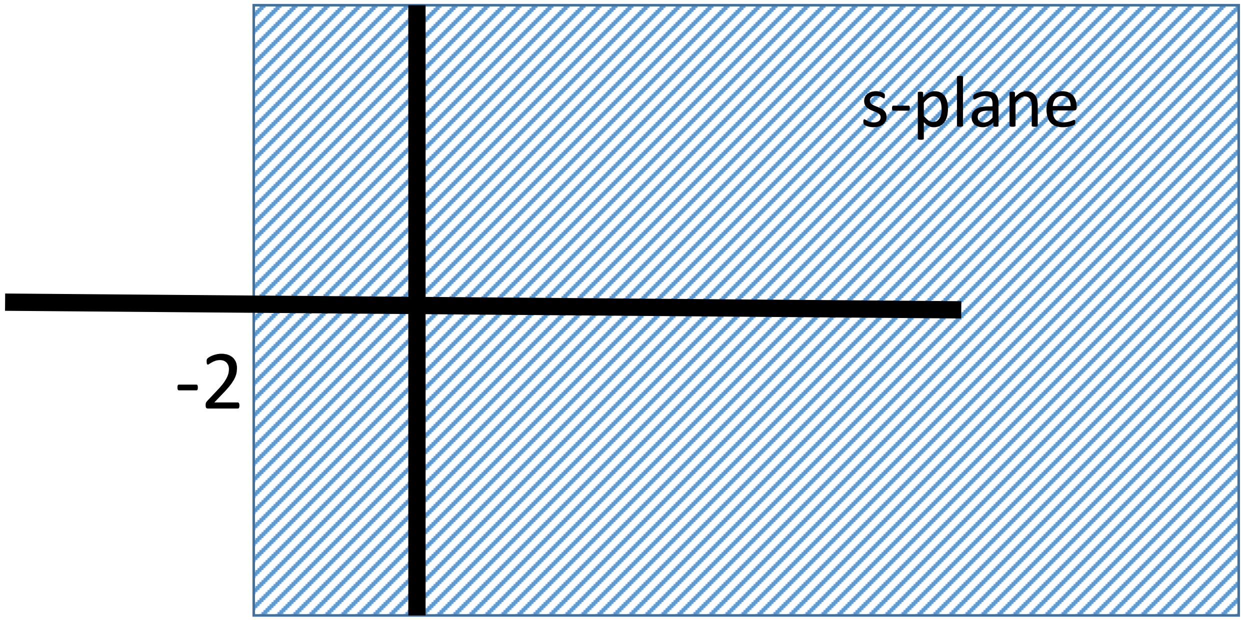

ROC (Region of Convergence)

Example 9.2

$$x(t) = -e^{-at}u(-t)$$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt$$

$$X(s) = \int_{0}^{-\infty} e^{-at}e^{-st}dt$$

$$X(s) = \int_{0}^{-\infty} e^{-(s+a)t}dt$$

$$X(s) = \frac{-1}{s+a}[e^{-(s+a)t}]_0^{-\infty}$$

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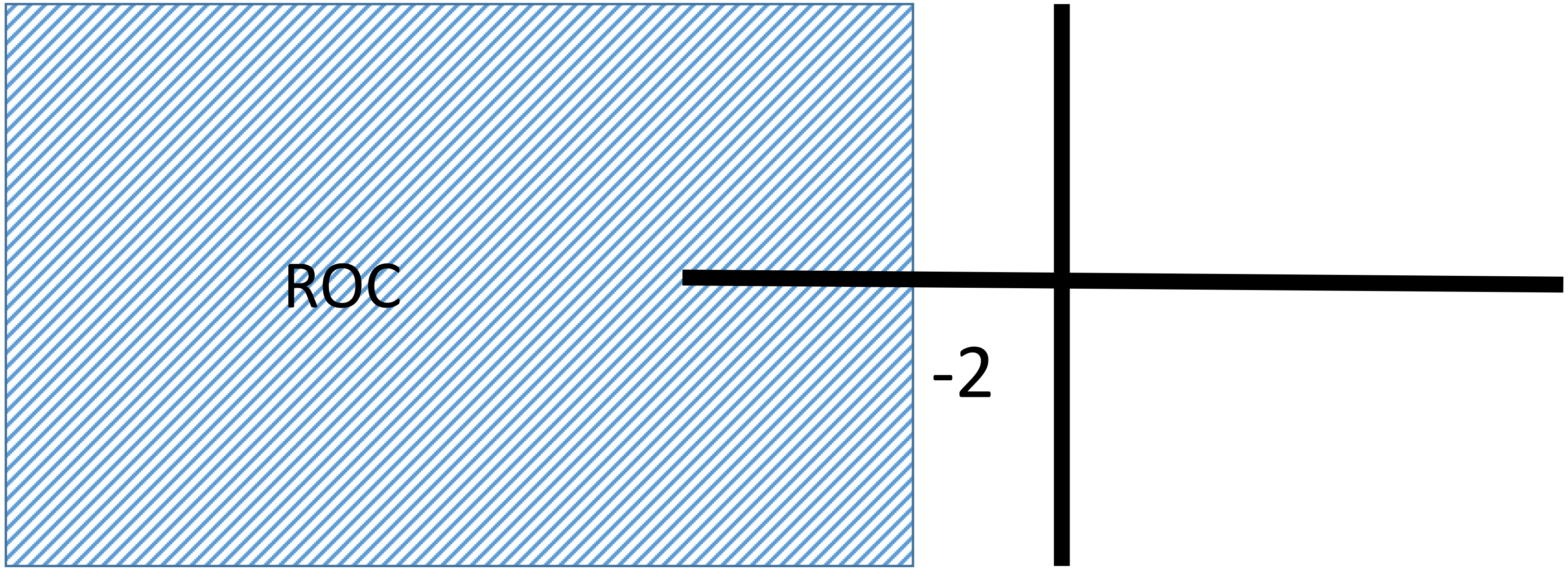
Example 9.2

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$$

$$a+\sigma<0$$

$$\sigma<-a$$

$$\mathrm{Re}\{s\}<-a$$



Importance of ROC

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} < -a$$

Importance of ROC

$$\underbrace{e^{-at}u(t)}_{\text{right}} \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+a} \qquad \underbrace{\mathrm{Re}\{s\} > -a}_{\text{right}}$$

$$\underbrace{-e^{-at}u(-t)}_{\text{left}} \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+a} \qquad \underbrace{\mathrm{Re}\{s\} < -a}_{\text{left}}$$

Example 9.3

$$e^{-t}u(t)+e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}$$

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$$\mathrm{Re}\{s\} > -1$$

Example 9.3

$$e^{-t}u(t)+e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}+$$

$$\mathrm{Re}\{s\}>-1$$

Example 9.3

$$e^{-t}u(t)+e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}+\frac{1}{s+2}$$

$$\mathrm{Re}\{s\}>-1 \quad \& \quad \mathrm{Re}\{s\}>-2$$

Example 9.3

$$e^{-t}u(t) + e^{-2t}u(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{2s+3}{(s+1)(s+2)}$$

$$\mathrm{Re}\{s\} > -1$$

Example 9.3

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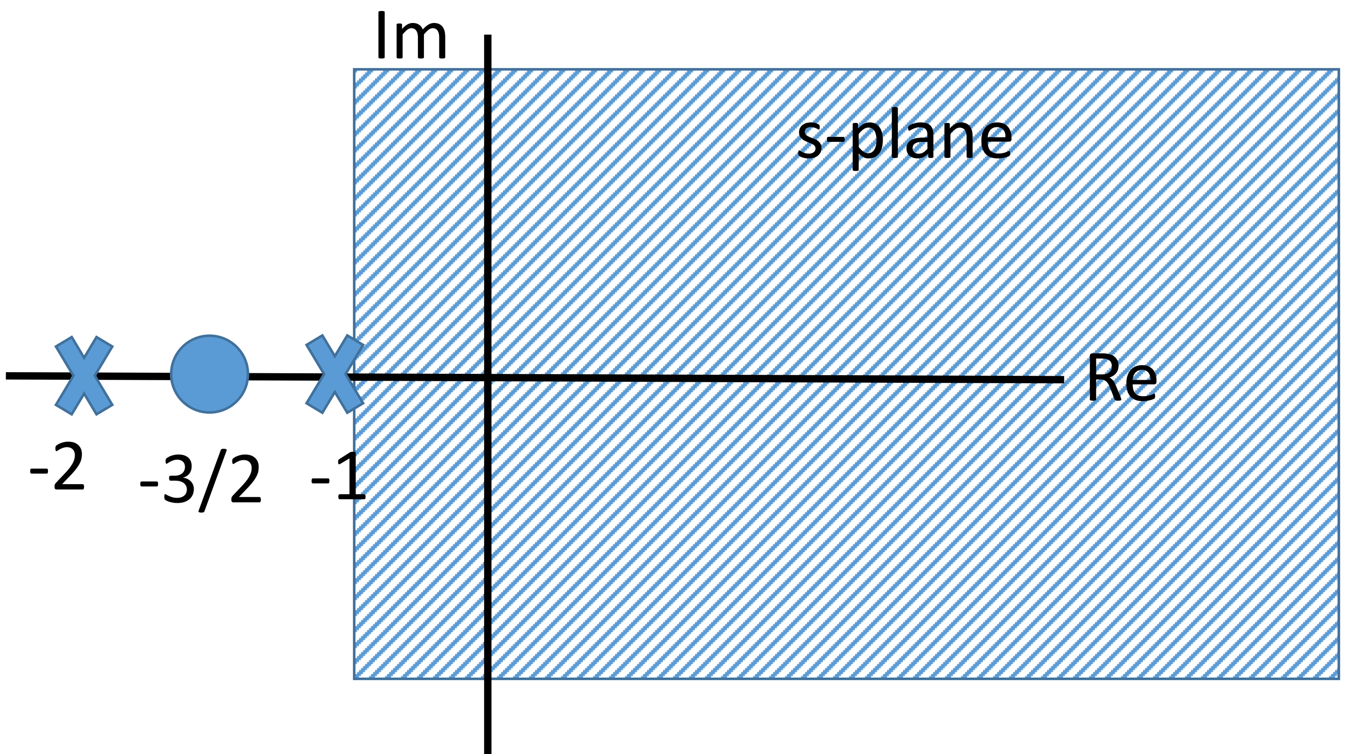

Laplace transform as a rational function

$$X(s) = \frac{N(s)}{D(s)}$$

Describe linear-constant coefficient differential equation

$$N(s) = 0$$

Zeros of X(s)

$$D(s) = 0$$

Poles of X(s)

ROC does not contain poles

Properties of the Region of Convergence

- The ROC contains no poles

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Properties of the Region of Convergence

- The ROC contains no poles

$$X(s) = \frac{N(s)}{D(s)}$$

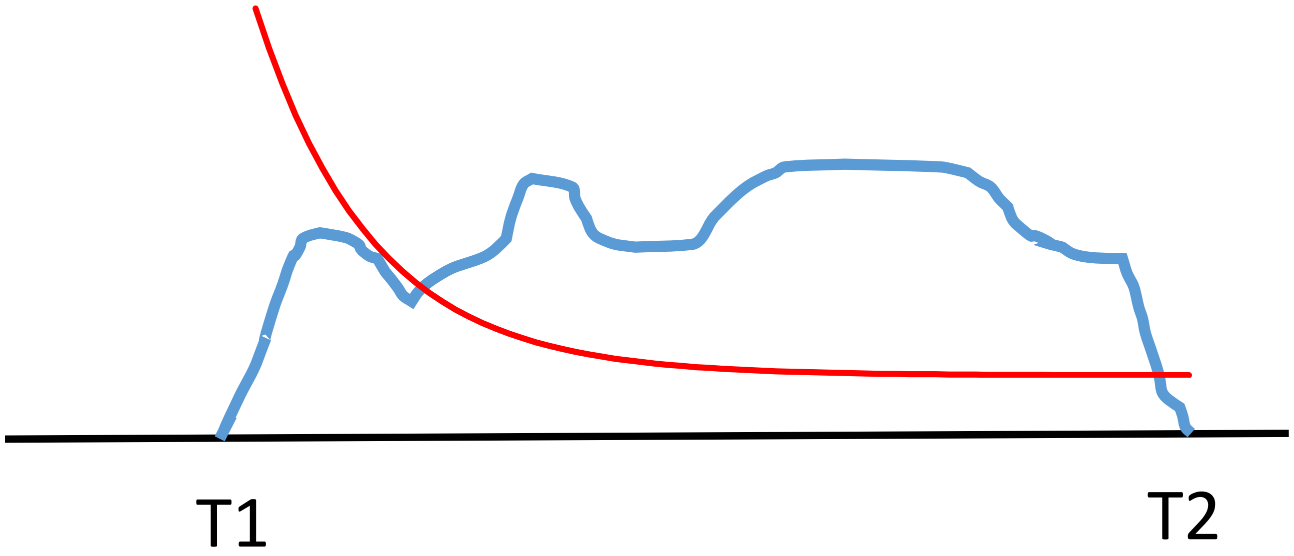
Poles of $X(s)$ are where $D(s) = 0$

- The ROC of $X(s)$ consists of a strip parallel to the $j\omega$-axis in the s-plane

- $\mathcal{F}\{x(t)\}$ converges implies ROC includes the $j\omega$-axis in the s-plane

Properties of the Region of Convergence

- If x(t) is of finite duration
  - -ROC is entire s-plane



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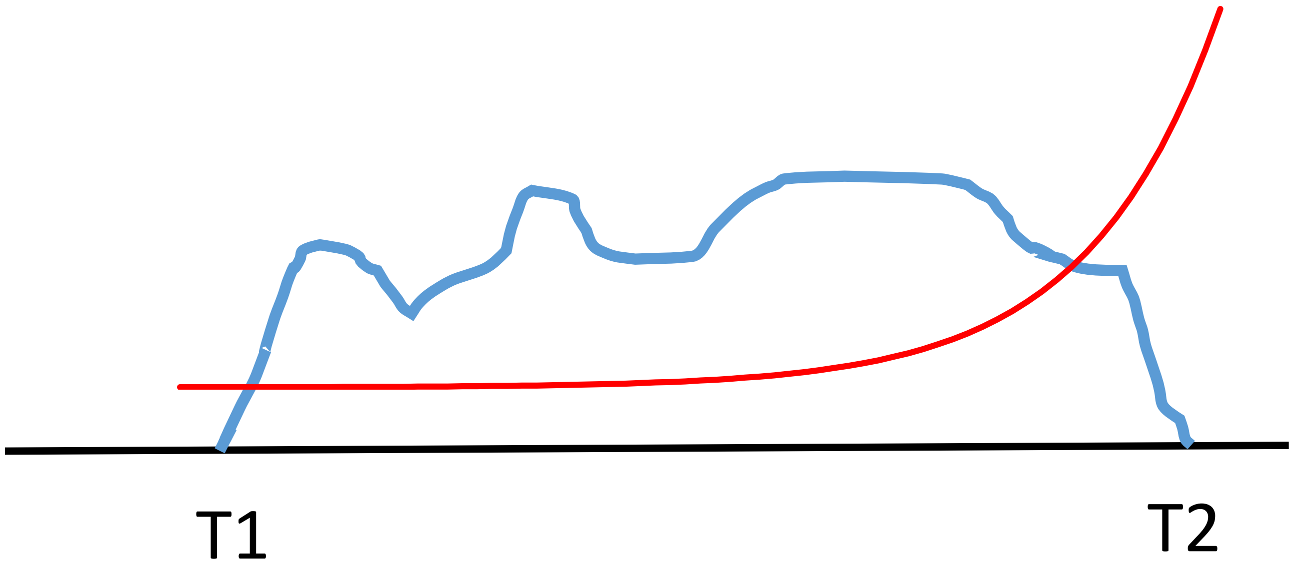
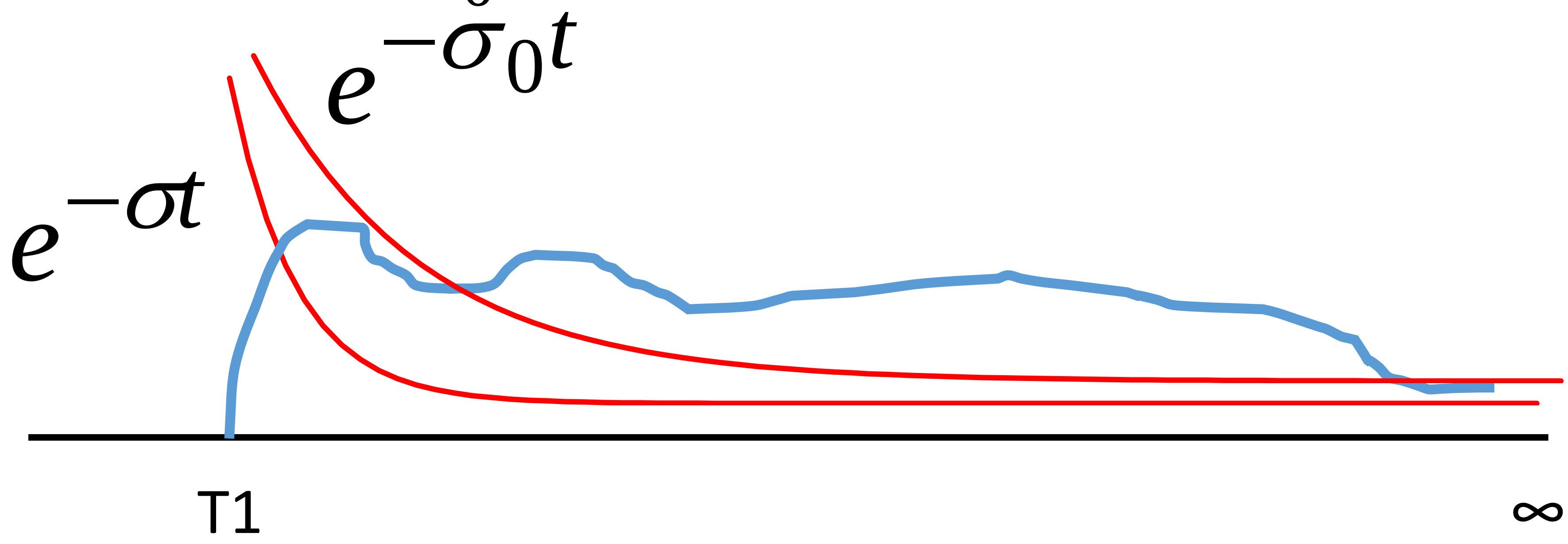


Properties of the Region of Convergence

- If x(t) is right-sided
  - -If σ_0 is in ROC, then $\sigma > \sigma_0$ is also in ROC



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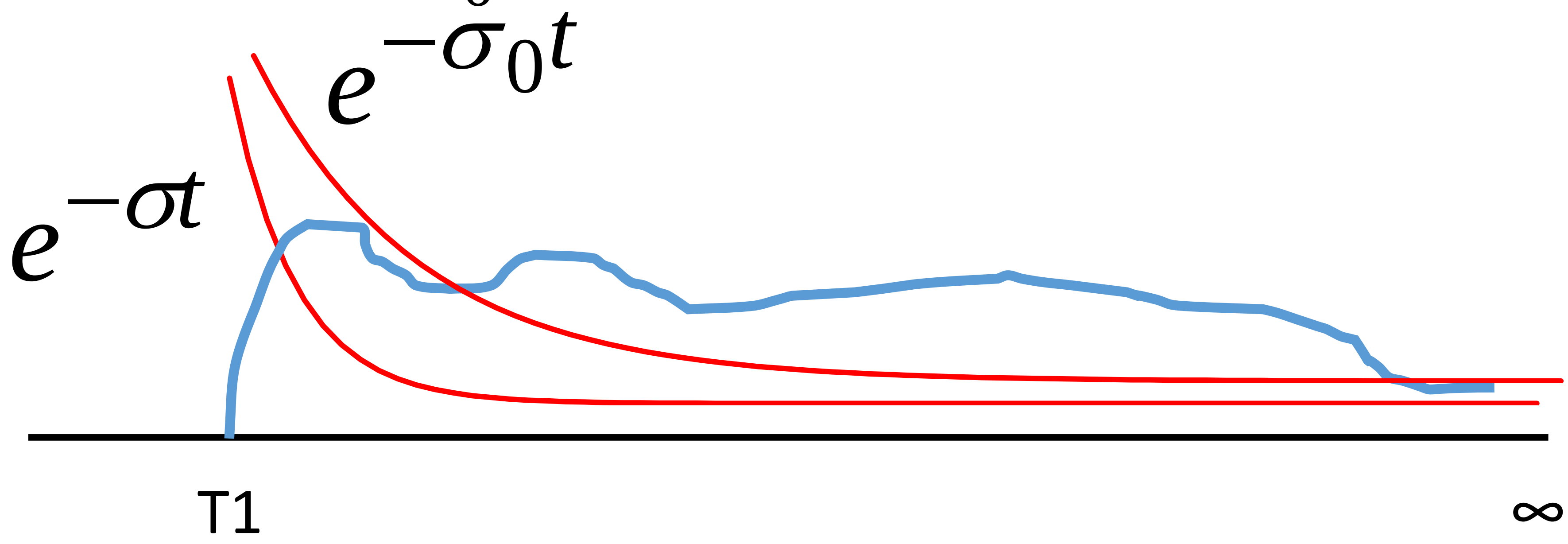

- If x(t) is right-sided and X(s) is rational
  - -ROC lies to the right of the rightmost pole

Properties of the Region of Convergence

- If x(t) is left-sided and Re{s} = σ_0 is in ROC
  - -all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC

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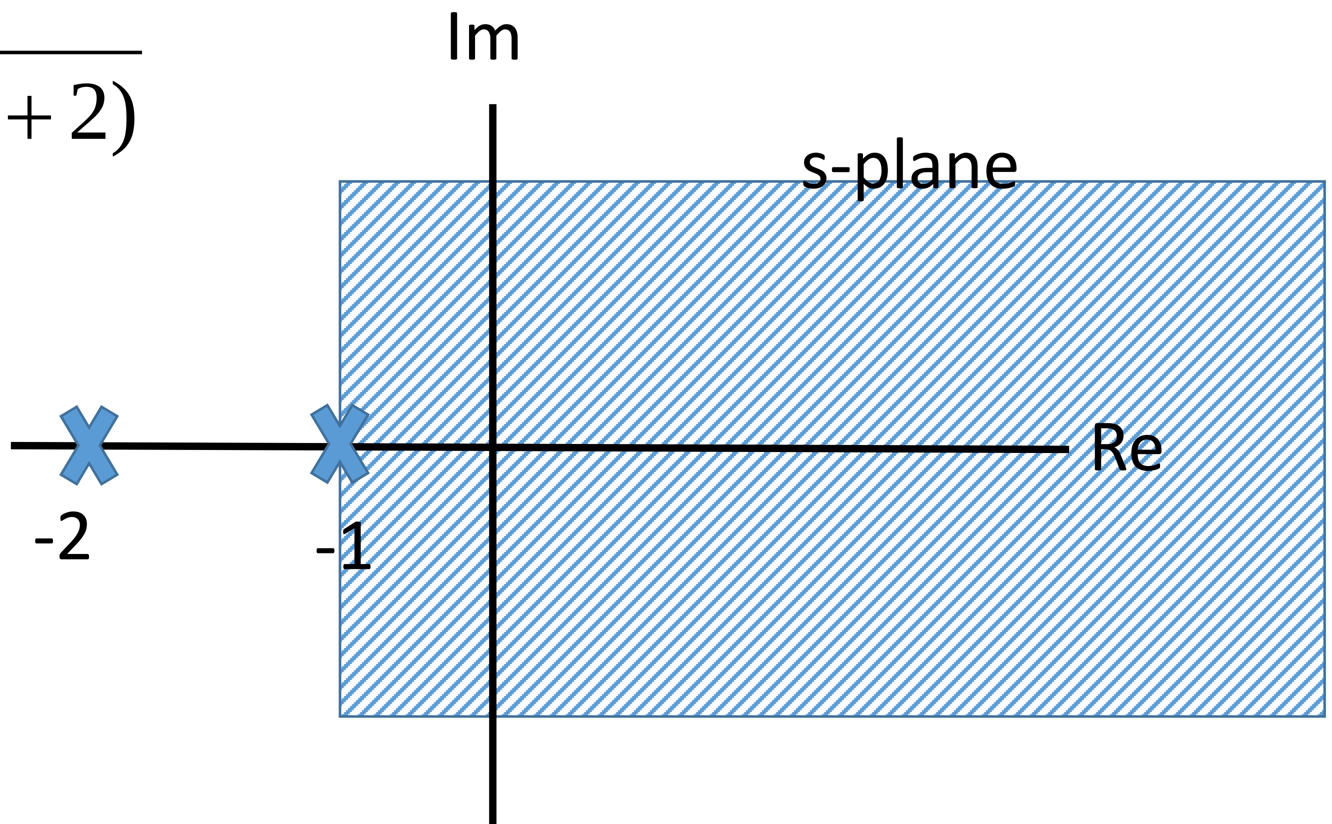
  - -all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC
- If x(t) is left-sided and X(s) is rational
  - -ROC lies to the left of the leftmost pole
- If x(t) is two-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC
  - -ROC is a strip in the s-plane

Properties of the Region of Convergence

- The ROC is a connected region
  - It cannot have multiple strips

$$X(s) = \frac{1}{(s+1)(s+2)}$$



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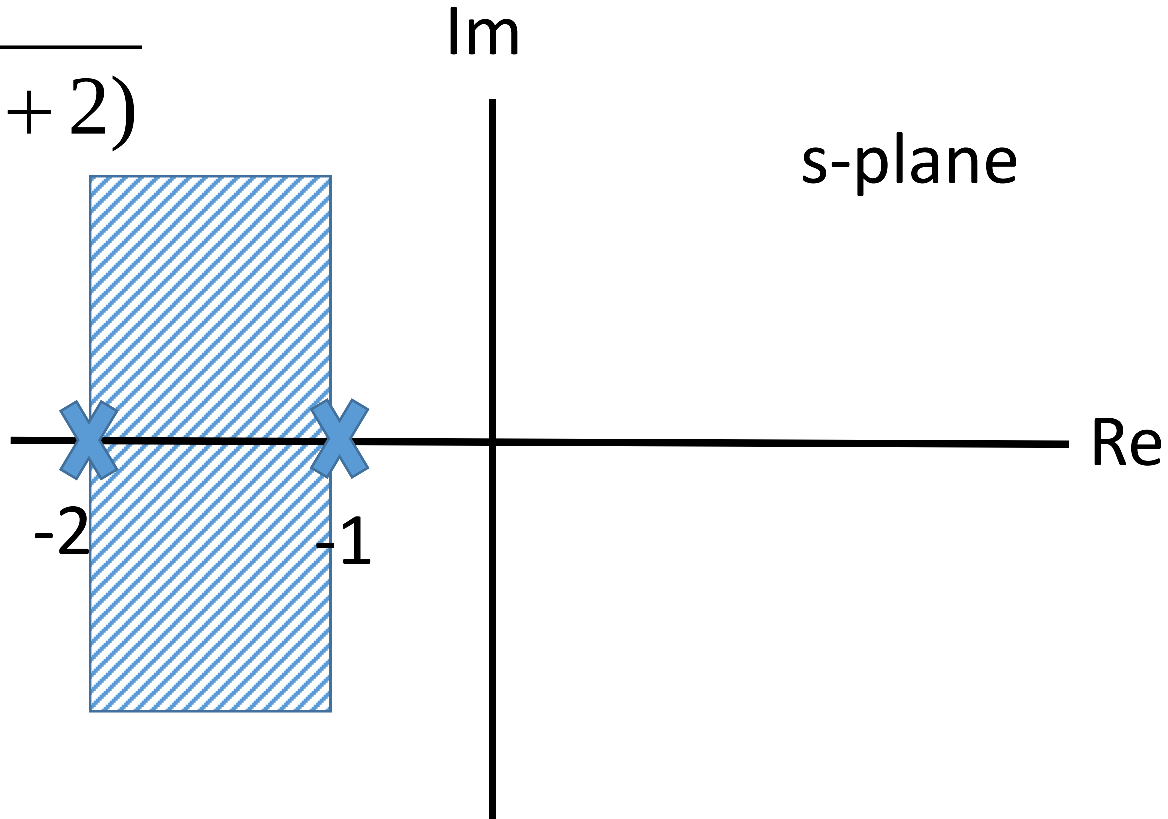

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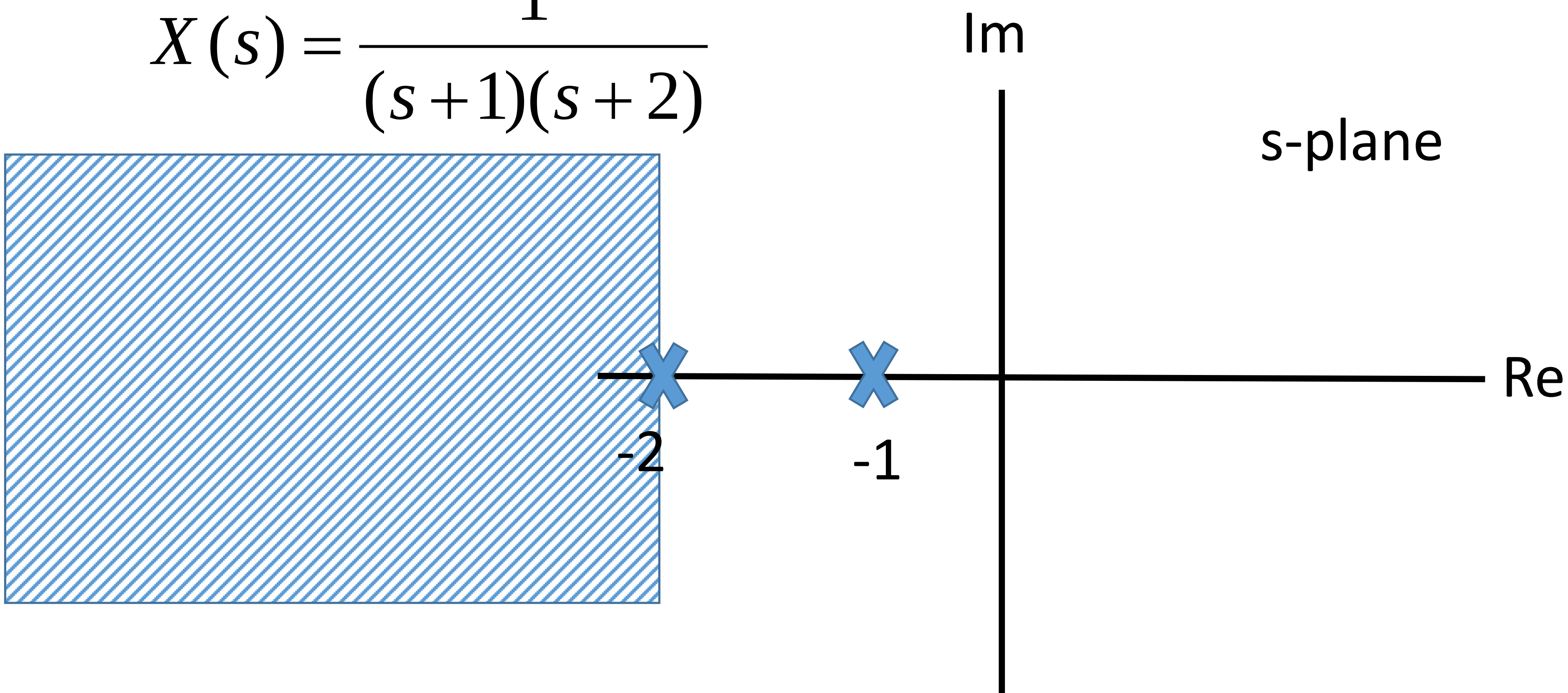

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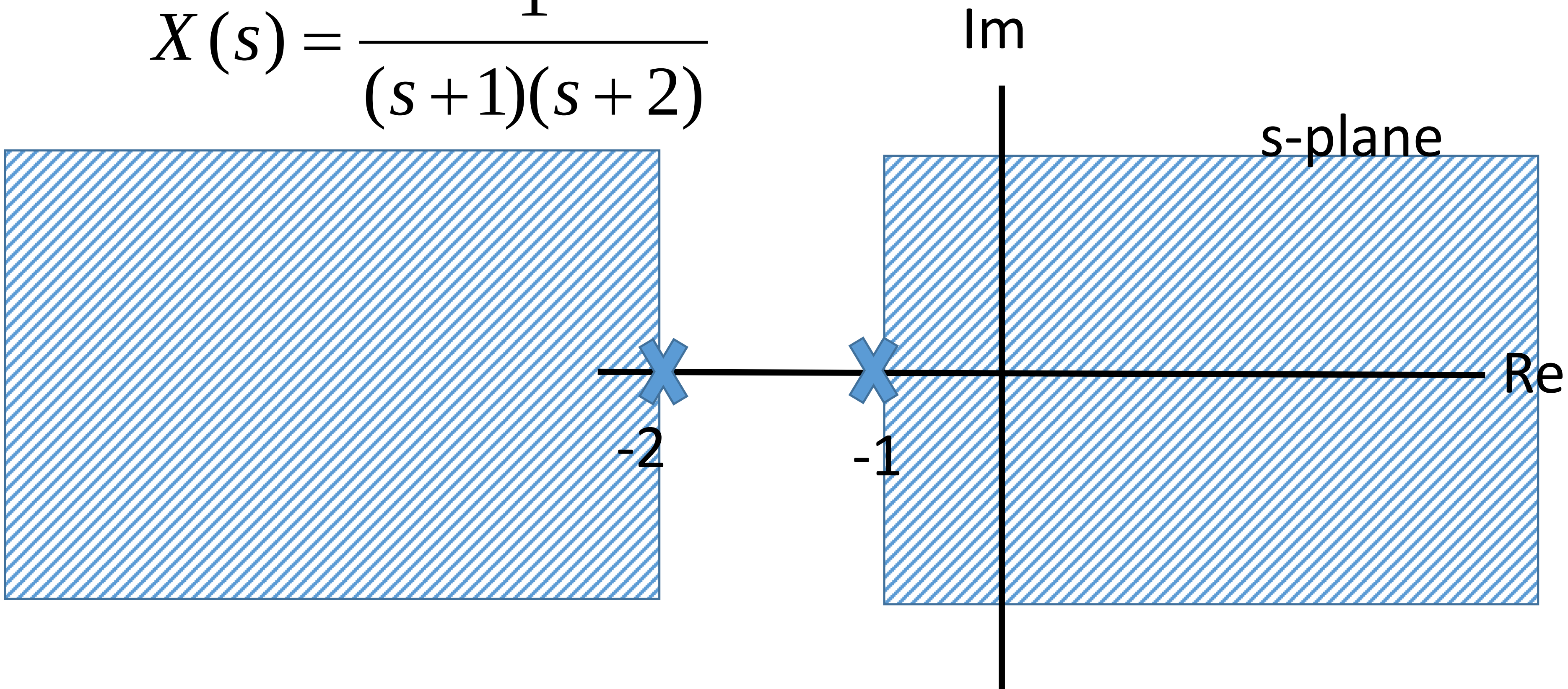

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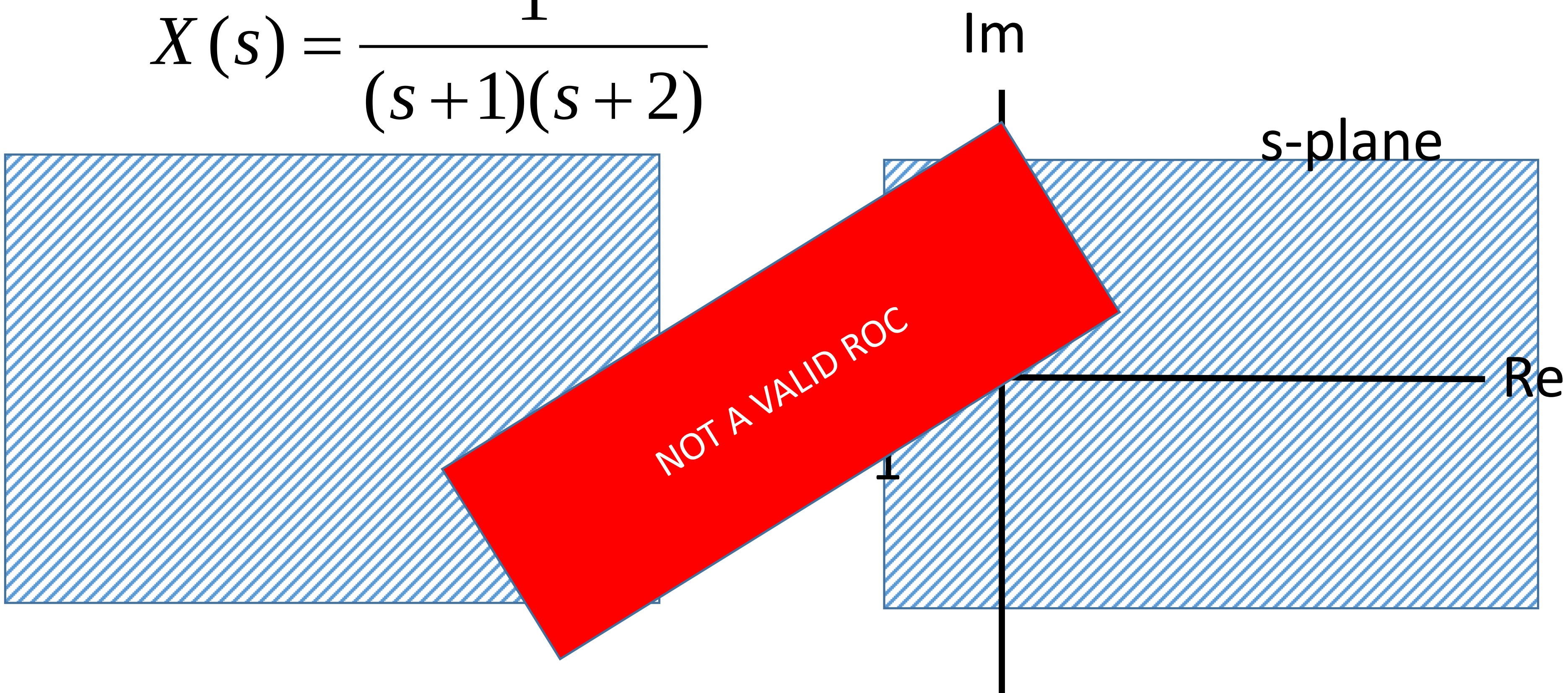
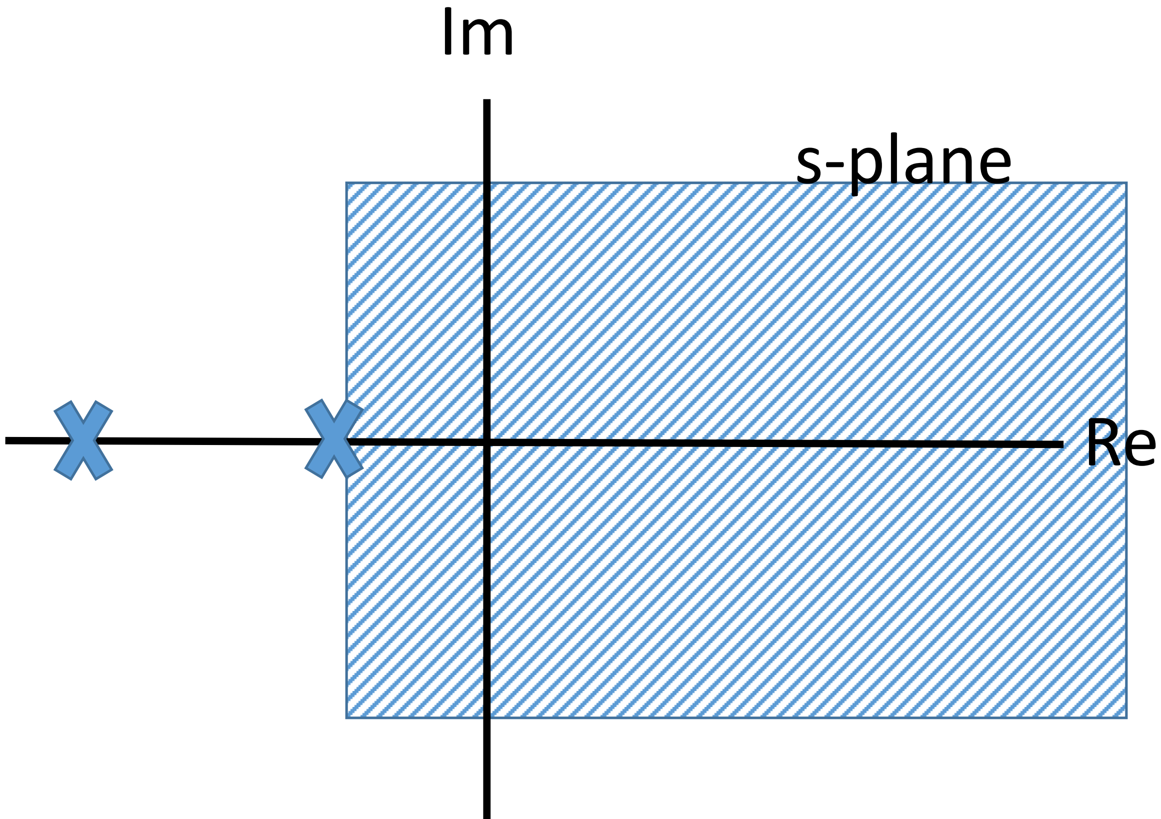


Inverse Laplace Transform

$$X(s) = \frac{1}{(s+1)(s+2)}$$



I	$x(t) = e^{-t}u(t) - e^{-2t}u(t)$
II	$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$
III	$x(t) = e^{-t}u(t) - e^{-2t}u(-t)$
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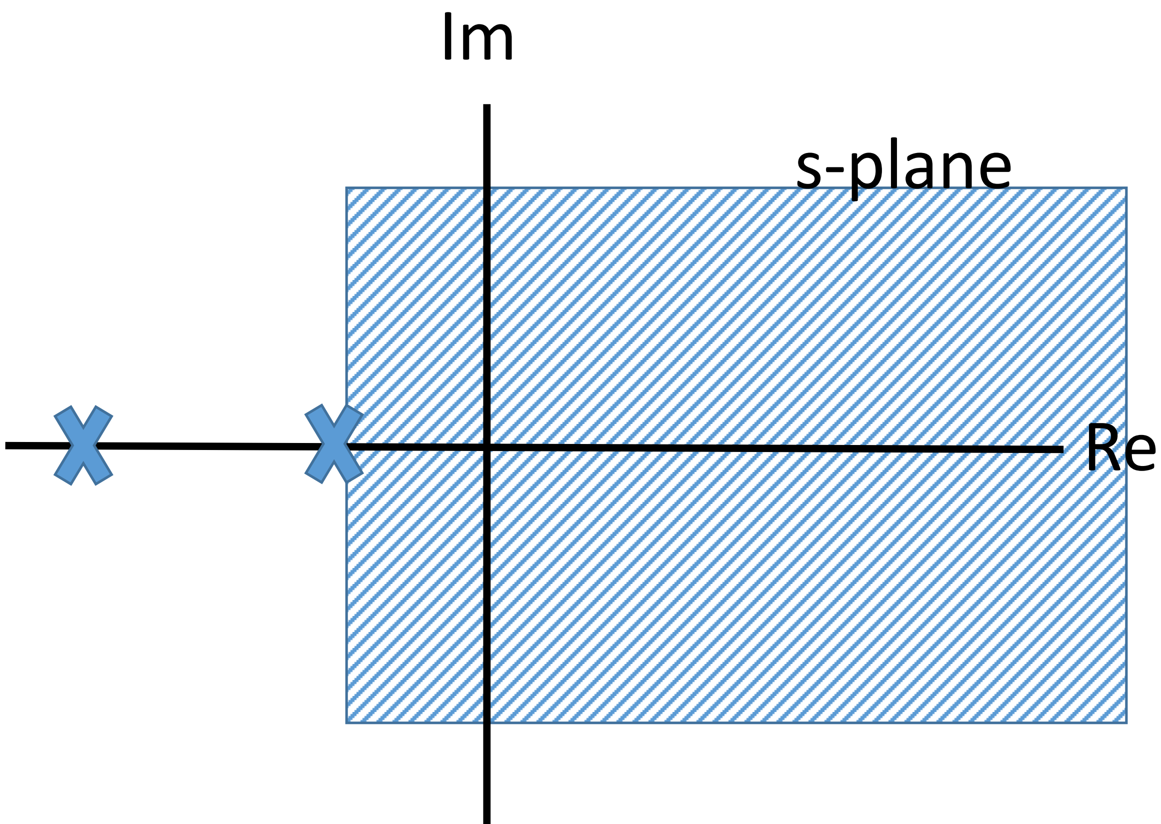

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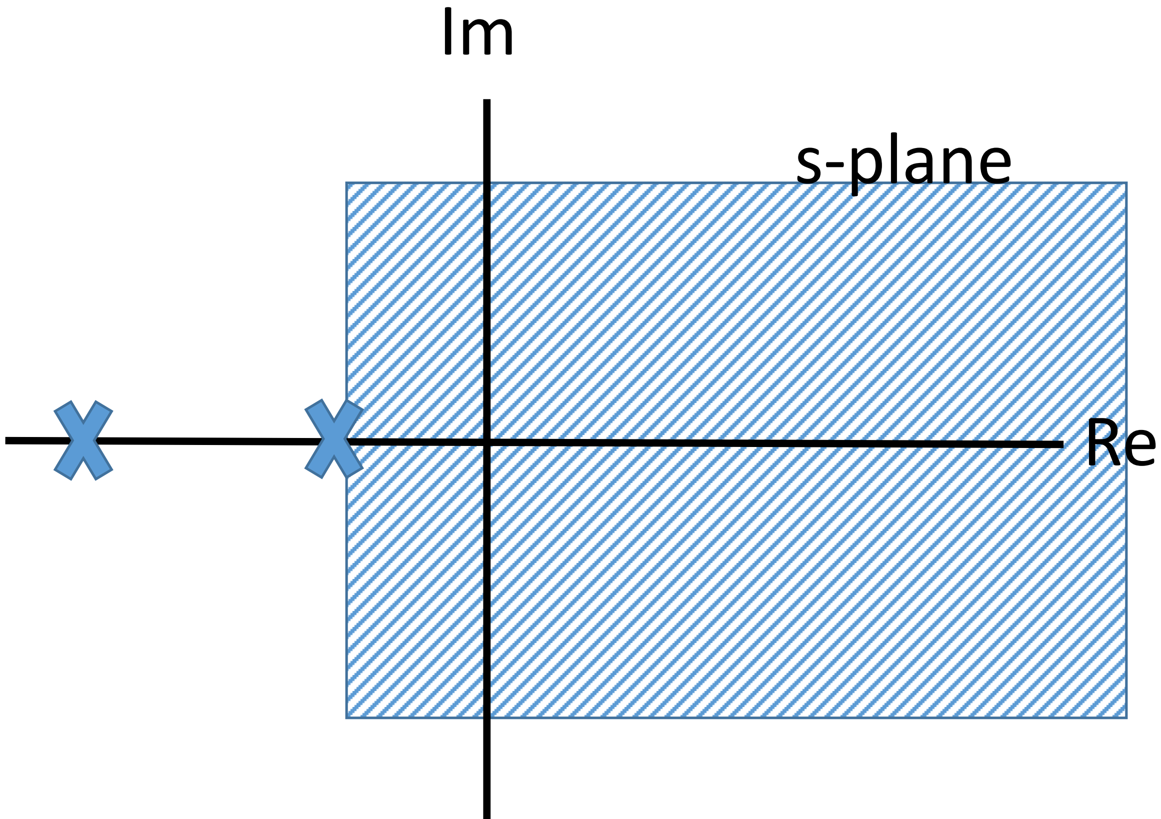


$$X(s)=\frac{A}{s+1}+\frac{B}{s+2}$$

$$(s+2)A+(s+1)B=1$$

$$s(A+B)+2A+B=1$$

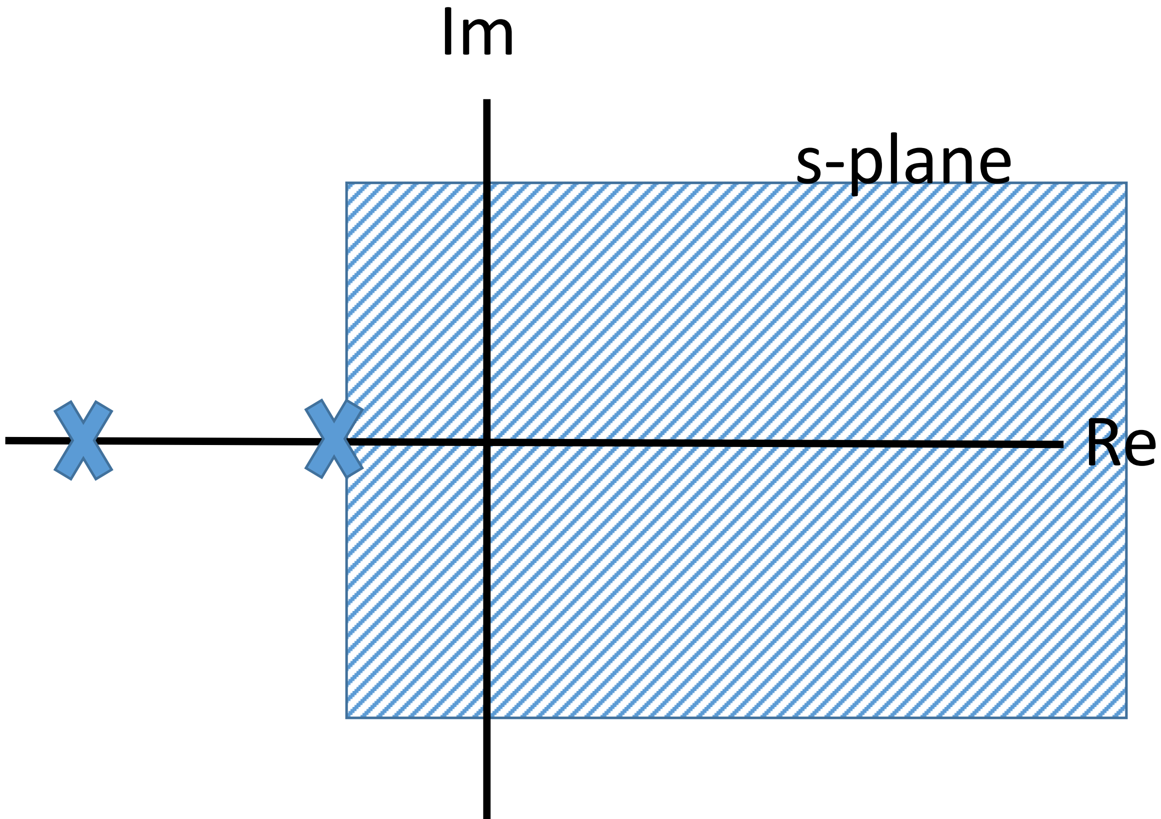
$$A+B=0 \qquad 2A+B=1$$

$$A=1 \qquad B=-1$$

Inverse Laplace Transform

$$X(s) = \frac{1}{(s+1)(s+2)}$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$



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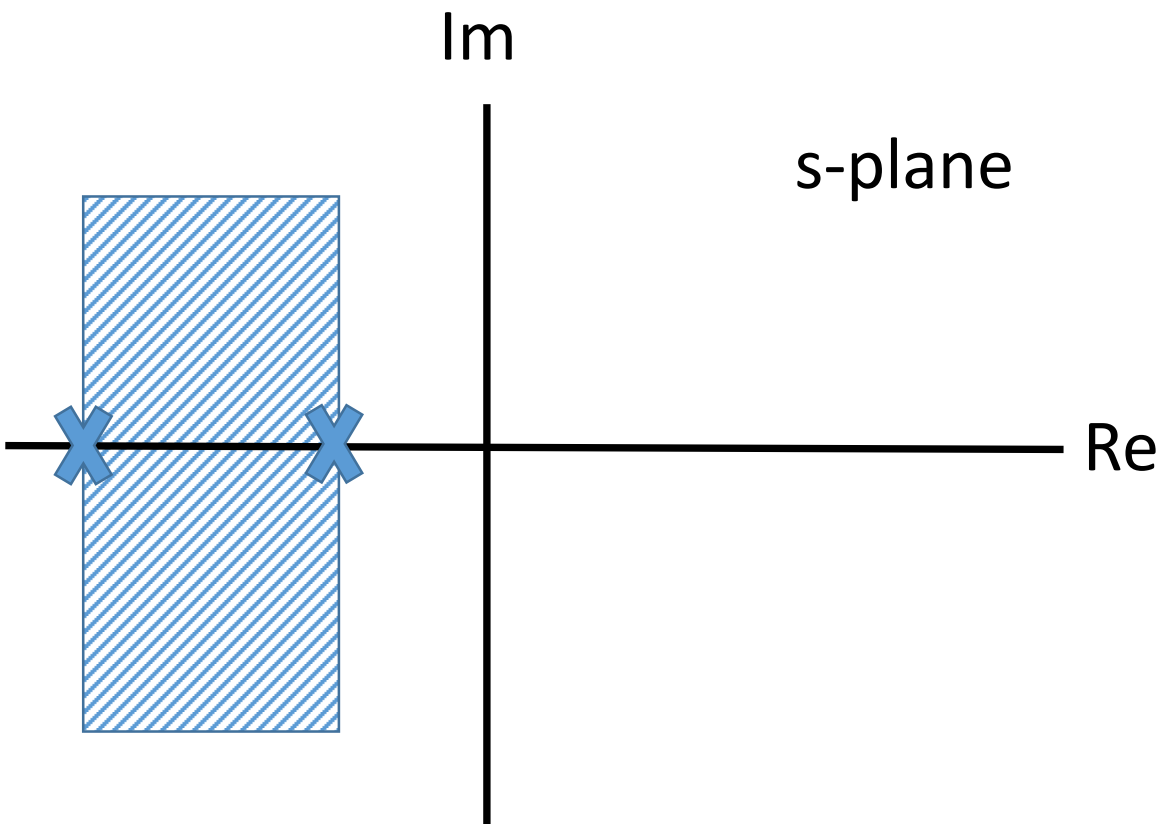

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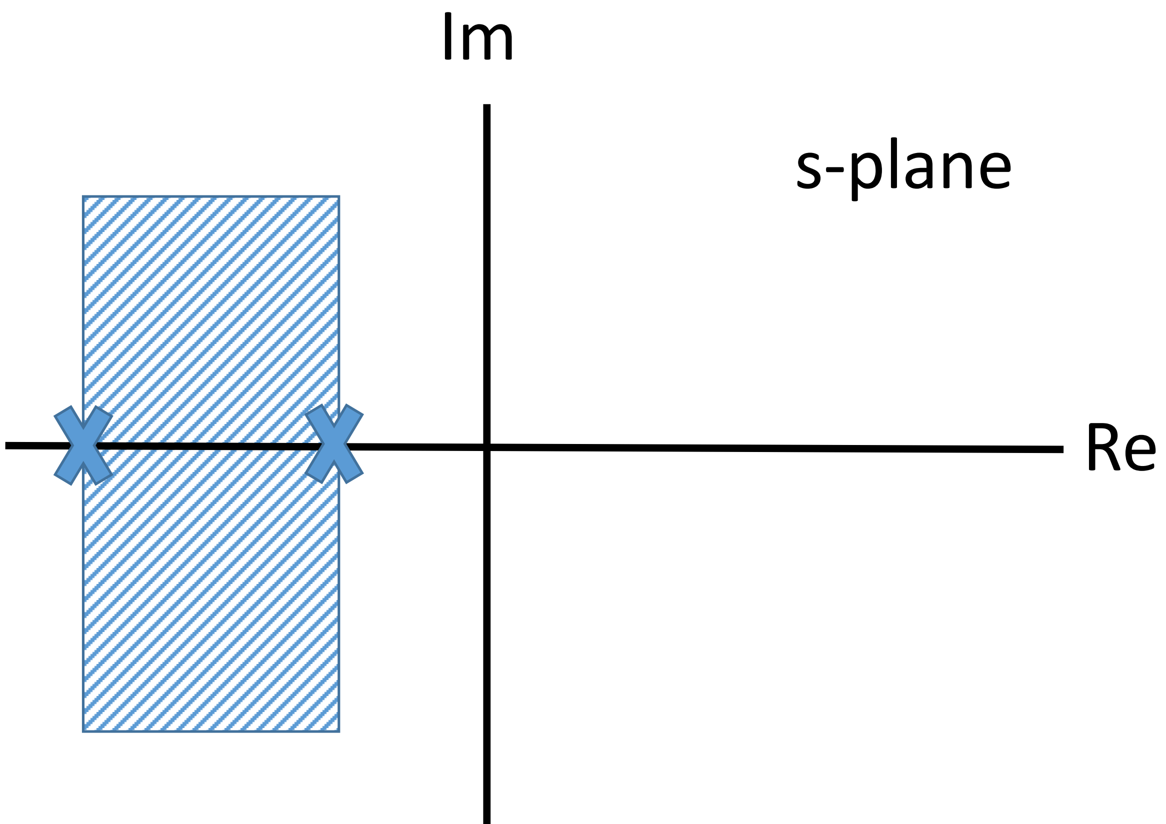

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Properties of Laplace Transforms

- **Linearity** $\quad ax(t)+by(t) \leftrightarrow aX(s)+bY(s) \quad \supset (R_1 \cap R_2)$
- **Time shifting** $\quad x(t-T) \leftrightarrow e^{-sT}X(s) \quad R$
- **Time scaling** $\quad x(at) \leftrightarrow \dfrac{1}{|a|}X\left(\dfrac{s}{a}\right) \quad aR$

Properties of Laplace Transforms

- **Multiply by t** $$tx(t) \leftrightarrow -\frac{dX(s)}{ds}$$ R

- **Differentiation in time** $$\frac{dx(t)}{dt} \leftrightarrow sX(s)$$ $\supset R$

Properties of Laplace Transforms

- **Multiply by $e^{-\alpha t}$** $\quad x(t)e^{-\alpha t} \leftrightarrow X(s+\alpha) \quad$ $shift\ R\ by\ -\alpha$
- **Convolution** $\quad \int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,d\tau \leftrightarrow X(s)Y(s) \quad$ $\supset (R_1 \cap R_2)$
- **Integrate in time** $\quad \int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{X(s)}{s} \quad$ $\supset (R \cap (Re(s) > 0))$

Properties of Laplace Transforms

If $x(t) = 0\ t < 0+$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0+}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{0+}^{\infty} x(t)e^{-st}dt = [x(t)e^{-st}/(-s)]_{0+}^{\infty} + \int_{0+}^{\infty} \frac{dx(t)}{dt}\frac{e^{-st}}{s}dt$$

$$X(s) = \frac{x(0+)}{s} - \frac{x(\infty)e^{-s\infty}}{s} + \frac{\int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st}dt}{s}$$

Properties of Laplace Transforms

If $x(t) = 0 \; t < 0+$

$$sX(s) = x(0+) - x(\infty)e^{-s\infty} + \int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$\lim\limits_{s\to\infty} sX(s) = x(0+)$ [Initial value theorem]

$\lim\limits_{s\to 0} sX(s) = x(0+) + x(\infty) - x(0+) = x(\infty)$ [Final value theorem]

Properties of Laplace Transforms

- Initial-value theorem

$$x(0+) = \lim_{s \to \infty} sX(s)$$

If $X(s)$ is rational, then for non-trivial output degree of denominator should be 1 plus degree of numerator.

$X(s) = \frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}$ implies $x(0+) = \frac{a_{n-1}}{b_n}$

Properties of Laplace Transforms

- Final-value theorem

$$x(\infty) = \lim_{s \to 0} sX(s)$$

If $X(s)$ is rational, then for non-trivial output denominator should contain a factor of s.

$X(s) = \frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s}$ implies $x(\infty) = \frac{a_0}{b_1}$

What is the physical meaning of $x(0)$ and $x(\infty)$?

- They corresponds to $x(t)$, which is continuous at these values.

Properties of Laplace Transforms

- **Output of an LTI system**

$$Y(s) \leftrightarrow X(s)H(s)$$

Causal and Stable LTI system

Choose the right option
I) All poles lie in right-half plane
II) All poles lie in left-half plane
III) Poles can lie anywhere
IV) There are no poles at all
V) I do not care

Causal and Stable LTI system

Choose the right option
I) All poles lie in right-half plane
II) **All poles lie in left-half plane**
III) Poles can lie anywhere
IV) There are no poles at all
V) I do not care

Unilateral Laplace Transform

$$\chi(s) = \int_{0-}^{\infty} x(t)e^{-st}dt$$

$$\chi(s) = \int_{0-}^{\infty} x(t)e^{-st}dt = \frac{x(0\,-)}{s} + \frac{1}{s}\int_{0-}^{\infty} \frac{dx(t)}{dt}e^{-st}dt$$

$$\int_{0-}^{\infty} \frac{dx(t)}{dt}e^{-st}dt = s\chi(s) - x(0-)$$

Solving DE

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2\mathrm{y}(\mathrm{t}) = \mathrm{x}(\mathrm{t})$$

$$y(0-) = \beta, y'(0-) = \alpha, x(t) = \delta(t)$$

$$s(sY(s) - \beta) - \alpha + 3(sY(s) - \beta) + 2Y(s) = X(s)$$

$$Y(s) = \frac{1 + \beta(s+3) + \alpha}{s^2 + 3s + 2}$$