Some Continuous Distributions :

$\theta \to$ Associated with density function.

$X \to$ RV. , $f_\theta(x)$ - PDF of X

$\underline{\theta} \in \Theta \longrightarrow$ space of Parameter

eg i) $\Theta = [0,\infty)$

ii) $\Theta = \mathbb{R}$

$\theta \to$ vector also, $\theta = (\theta_1, \theta_2, \ldots \theta_n)$

$$\theta \in \Theta \subseteq \mathbb{R}^n$$

what are the properties of these Parameters ?

$X \to$ RV , $f_\theta(x) \to$ PDF

$\theta \in \Theta \subseteq \mathbb{R}$

θ is called a location parameter if PDF of RV, $Y = \underline{X - \theta}$ does not depend on θ.

- θ is called a scale parameter if PDF of $\frac{X}{\theta}$ does not depend on θ.

- The graph of $f_\theta(x)$ stretches out for higher values of θ.

- The graph of $f_\theta(x)$ contracts if we decrease the value of θ.

- Suppose we have two parameter family of distribution.

$\theta = (\theta_1, \theta_2)$

we call it as location-scale

parameter family if PDF of

$\frac{X-\theta_1}{\theta_2}$ does not depend on θ_1, θ_2

| θ_1 — location Parameter
| θ_2 — scale

→ Some PDFs also have shape parameter, i.e., shape of PDF $f_\theta(x)$ changes as we change θ.

e.g. Parameter λ in Poisson distribution is shape parameter.

i) <u>Uniform Distribution</u>: An RV X is said to follow a uniform distribution on interval $[a, b]$ if

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0 & \text{o.w.} \end{cases}$$

$E[X] = \frac{a+b}{2}$ $\quad$ $var(X) = \frac{(b-a)^2}{12}$

(exercise)

Gamma Distribution :-

An RV X is said to follow a gamma distribution with parameters $\alpha, \beta > 0$ if its PDF is given by

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma\alpha \, \beta^{\alpha}} & x > 0 \\ 0 & o.w. \end{cases}$$

$$\Gamma\alpha = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$$

$\Gamma\frac{1}{2} = \sqrt{\pi}$ $\quad$ $\Gamma(z+1) = z\Gamma z$

$\Gamma n+1 = n!$

$n \to$ integer

$\beta \to$ scale parameter

$\alpha \to$ shape parameter.

$\boxed{\frac{X}{\beta} \to \text{PDF is free from } \beta}$

exercise.

$$M_X(t) = E[e^{tX}]$$

$$= \frac{1}{\Gamma\alpha\ \beta^\alpha} \int_0^\infty x^{\alpha-1} e^{tx}\ e^{-x/\beta}\ dx$$

$$= \frac{1}{\Gamma\alpha\ \beta^\alpha} \int_0^\infty x^{\alpha-1}\ e^{-x(1/\beta - t)}\ dx$$

$$x\left(\frac{1}{\beta} - t\right) = y$$

$$dx = \frac{\beta}{1-\beta t}\ dy$$

$$= \frac{1}{\Gamma\alpha\ \cancel{\beta^\alpha}} \int_0^\infty \left(\frac{\cancel{\beta} y}{1-\beta t}\right)^{\alpha-1} e^{-y}\ \frac{\cancel{\beta}}{1-\beta t}\ dy$$

let $t < \frac{1}{\beta}$

$$= \frac{1}{\Gamma\alpha} \frac{1}{(1-\beta t)^\alpha} \underline{\int_0^\infty y^{\alpha-1}\ e^{-y}\ dy}$$

$$= \frac{1}{(1-\beta t)^\alpha}$$

$$\boxed{M_X(t) = (1-\beta t)^{-\alpha}}$$ if $t < 1/\beta$

$E[X] = \alpha\beta$, $\quad var(X) = \alpha\beta^2$

(exercise)

<u>Special Case</u>:

If $\alpha = 1$, we say

that X follows exponential distribution with mean parameter β and PDF of X is given by

$$f(x) = \begin{cases} \frac{e^{-x/\beta}}{\beta} & x > 0 \\ 0 & o.w \end{cases}$$

$$\boxed{M_X(t) = (1-\beta t)^{-1}} \qquad t < 1/\beta$$

Another form of exponential distribution is defined by PDF

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \; x > 0 \\ 0 & o.w. \end{cases}$$

(λ — Rate parameter.)

$$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

\# Let $X_1, X_2, \ldots X_n$ be independent RVs where $X_i \sim G(\alpha_i, \beta)$

Then $S_n = \sum_{i=1}^{n} X_i \sim G\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$

Proof:

$$M_{S_n}(t) = E\left[e^{t(X_1+\cdots+X_n)}\right]$$

$$= E\left[\prod_{i=1}^{n} e^{tX_i}\right]$$

$$= \prod_{i=1}^{n} E\left[e^{tX_i}\right].$$

$$= \prod_{i=1}^{n} (1-\beta t)^{-\alpha_i}$$

$$M_{S_t}(t) = (1-\beta t)^{-\sum_{i=1}^{n}\alpha_i}$$

$\searrow$ MGF of $G(\sum_i \alpha_i, \beta)$

$$\Rightarrow \quad S_n \sim G\left(\sum_{i=1}^{n}\alpha_i, \beta\right)$$

If $X_1, X_2, \ldots X_n$ are independent exponential RVs with mean parameter β

Then $\displaystyle\sum_{i=1}^{n} X_i \sim G(n, \beta)$

Theorem: Let X be an exponential distribution with mean parameter β. Then, X has memoryless property, i.e.,

$$P\{X > r+s \mid X > s\} = P\{X > r\}.$$

Proof:

$$P\{X > r+s \mid X > s\} = \frac{P\{X > r+s,\ X > s\}}{P\{X > s\}}$$

$$= \frac{P\{X > r+s\}}{P\{X > s\}} = \frac{e^{-\frac{(r+s)}{\beta}}}{e^{-s/\beta}}$$

$$= e^{-r/\beta} = P\{X > r\}$$

Example:

$X \to$ length of time that a certain item functions before failing.

$P\{X > r+s \mid X > s\} \to$ Probability that an item that is still functioning at age s and will continue to function for at least additional time r

$\|$

$P\{X > r\} \to$

Exercise: If $X_1, X_2, \ldots X_n$ are independent exponential RVs with mean parameters $\beta_1, \beta_2, \ldots \beta_n$, respectively.

Then, $\underline{\text{Min}(X_1, \ldots X_n)}$ follows exponential distribution with Parameter $\sum_{k=1}^{n} \frac{1}{\beta_k}$

Normal distribution:

An RV X is said to follow a normal distribution with parameters $\underline{\mu}$ & $\underline{\sigma}^2$ if its PDF is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in \mathbb{R}$$

we denote it as $X \sim N(\underline{\mu}, \sigma^2)$

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

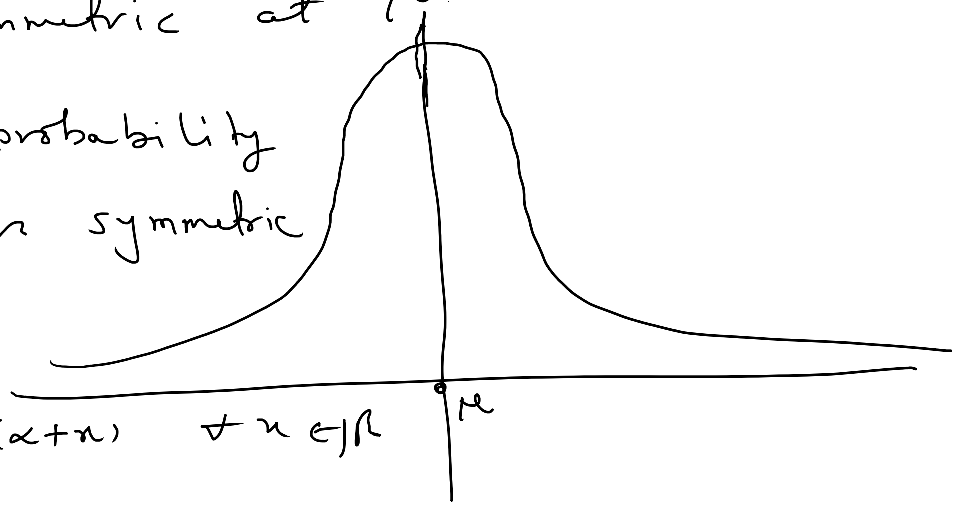
$$E[X] = \mu \qquad \text{var}(X) = \sigma^2$$

If $\mu = 0$, $\sigma^2 = 1$, Then X is said to follow a standard normal distribution.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad x \in \mathbb{R}$$

If $\quad X \sim N(\mu, \sigma^2)$

X is symmetric at μ.



Def^n: A probability distribution is symmetric at α if

$$f(\alpha - x) = f(\alpha + x) \quad \forall x \in \mathbb{R}$$

For $N(\mu, \sigma^2)$ $\quad \boxed{\alpha = \mu}$

\# If $X \sim N(\mu, \sigma^2)$

$aX + b \sim N(a\mu + b, a^2\sigma^2)$

$$M_{aX+b}(t) = E\left[e^{t(aX+b)}\right]$$

$$= E\left[e^{(at)X} \cdot e^{tb}\right]$$

$$= e^{tb} E\left[e^{(at)X}\right] = e^{tb} \times M_X(at)$$

$$= e^{tb} \, e^{\mu(at) + \frac{\sigma^2_x}{2}(at)^2}$$

$$= e^{\underline{(a\mu+b)}t + a^2\sigma^2_x \frac{t^2}{2}}$$

$\downarrow$

MGF of $N(a\mu+b, a^2\sigma^2)$

$\Rightarrow \quad aX+b \sim N(a\mu+b, a^2t^2)$

Let $\quad a = \frac{1}{\sigma} \qquad b = -\frac{\mu}{\sigma}$

$$\boxed{\frac{X-\mu}{\sigma} \sim N(0,1)}$$

$\rightarrow$ does not depend on μ & σ

$\Rightarrow \quad \mu \rightarrow$ location parameter $\quad$ | $\sigma \rightarrow$ standard deviation

$\sigma \rightarrow$ scale parameter

Normal distribution belong to location-scale parameter family.

CDF of N(0,1)

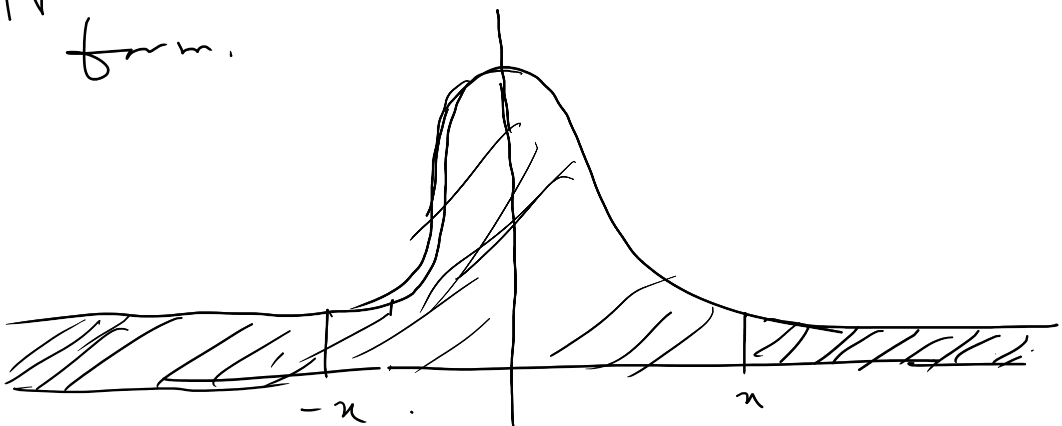
$$\Phi(x) = \frac{1}{\,} \int^{x} e^{-y^2/2} \, dy$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

$\downarrow$

does not have analytical form.

But approximations are available in tabular form.



$$\boxed{\Phi(-x) = 1 - \Phi(x)}$$ because $P\{Z > x\}$

$= P\{Z < -x\}$

$Z \sim N(0,1)$

Example: $X \sim N(3,4)$

$$P\{2 < X \le 5\} = P\left\{\frac{2-3}{2} \le \frac{X-3}{2} \le \frac{5-3}{2}\right\}$$

$$= P\{-1/2 \le Z \le 1\}$$

$$= \Phi(1) - \Phi(-1/2)$$

$\downarrow$

$\boxed{F(5) - F(2)}$

CDF of $N(3,4)$

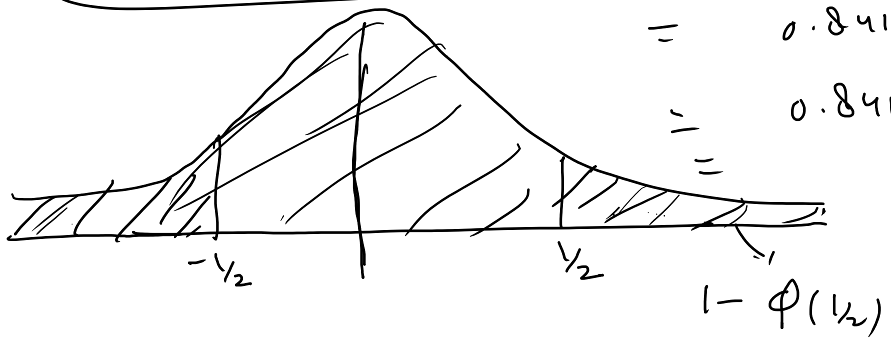
CDF of $N(3,4)$

$$= 0.8413 - \phi(-1/2)$$

$$= 0.8413 - 1 + \phi(1/2)$$

$$= 0.8413 - 1 + 0.6915$$

$$= 0.5328.$$



$$\boxed{\begin{aligned} \phi(0) &= 1/2 \\ &\parallel \\ P\{X &\le 0\} \end{aligned}}$$

\# Let $X_1, X_2, \ldots X_n$ be independent Random variables, where $X_i \sim N(\mu_i, \sigma_i^2)$

Then $\underline{S_n = \sum_{k=1}^{n} X_k} \sim N\left(\sum_1^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$

$$M_{S_n}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$= \prod_1^n e^{\mu_i t + \sigma_i^2 \frac{t^2}{2}}$$

$$= e^{\underline{\sum \mu_i} t + \underline{\sum_{i=1}^n \sigma_i^2} \frac{t^2}{2}}$$

$$\Rightarrow S_n \sim N\left(\sum_1^n \mu_i, \sum_1^n \sigma_i^2\right)$$

$a_1 X_1 + \cdots + a_n X_n \sim N\left(\sum_1^n a_i \mu_i , \sum_1^n a_i^2 \sigma_i^2 \right)$

<u>CF.</u> $\boxed{\phi_X(t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}}$