CHAPTER

2

AXIOMS AND FORCE SYSTEMS

In this chapter, force and mass are introduced. Moment of a force, work, kinetic energy, momentum and moment of momentum are defined. Euler's axioms are presented. Equivalent force systems and resultant of a force system are defined. The axioms of Coulomb friction, rolling friction, electromagnetic and gravitational forces are presented. The principles of impulse–momentum and angular impulse–moment of momentum are derived. The work–energy relation for centre of mass is established. Free body diagrams and reactions are discussed. Conservative forces and their potential energies are presented. The dynamics of the centre of mass of any system is covered.

2.1 Force and Moment

The concept of force (a primitive) arises from muscular effort. Force is a *quantitative measure of the mechanical interaction of material bodies in contact or at a distance.* It is represented by an arrow, whose length gives its magnitude, direction gives its line of action, arrow-head gives its sense and the tail usually represents its point of application P (Fig. 2.1a). It is denoted by $\vec{F}$ with magnitude $F = |\vec{F}|$. Its SI unit is newton (N). The forces acting on a body (Fig. 2.1b) may be modelled as *discrete* forces, $\vec{F}_i$ N acting at point i with position vector $\vec{r}_i$, or *distributed* forces distributed over a volume, a surface or a line. These may be functions of time t. The force distributed over a volume is called *body force* and is described by a *body force density* $\vec{f}_b\,\mathrm{N/m^3}$, for example, gravitational force. The force distributed over a surface is called *surface force* and is described by a *surface force density* $\vec{f}_s\,\mathrm{N/m^2}$, for example, pressure. The force distributed over a line is called *line force* and is described by a *line force density* $\vec{f}_l\,\mathrm{N/m}$, for example, load per unit length on a beam. A discrete force is an idealisation of a physical force distributed over a 'small' contact area or over a 'small' volume.

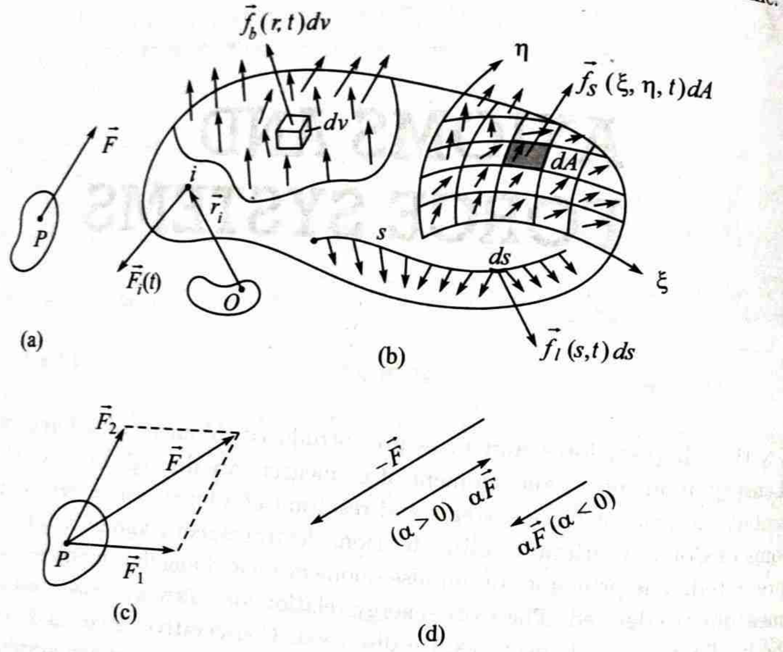


Fig. 2.1

Forces are frame-indifferent axiomatically, i.e., the interaction between material bodies is independent of the reference frame of observation. All *forces are governed by the parallelogram law (axiom) of forces*: 'the effect of forces $\vec{F}_1$ and $\vec{F}_2$ acting at the same point P of a body is the same as due to a single force $\vec{F}$ acting at the same point, which is represented by the diagonal arrow of a parallelogram formed by the two force arrows $\vec{F}_1, \vec{F}_2$, emanating at point P as adjacent sides (Fig. 2.1c). We write this addition as $\vec{F} = \vec{F}_1 + \vec{F}_2$. The product of a force $\vec{F}$ with a scalar α, denoted by $\alpha\vec{F}$, is a force of magnitude $|\alpha||\vec{F}|$ with the same point of application and line of action as $\vec{F}$ and sense the same as or opposite to that of $\vec{F}$ according as α is positive or negative (Fig. 2.1d). We define $-\vec{F} = (-1)\vec{F}$.

Four fundamental forces: gravitational, electromagnetic, weak nuclear and strong nuclear forces arise due to interaction between bodies. Electromagnetic forces of attraction and repulsion cancel each other to a large extent in large systems and gravity is predominant. Electromagnetic forces are much stronger than gravitational forces on the atomic scale and play a significant role in most chemical and biological processes. Nuclear forces are stronger compared to others for subatomic particles at distances of the order of 10^{-15} m and are negligible when the separation of the interacting parti-

cles is greater than 10^{-10} m. Frictional force, spring force, rope tension, hydrostatic pressure and muscular force are macroscopic manifestations of electromagnetic forces.

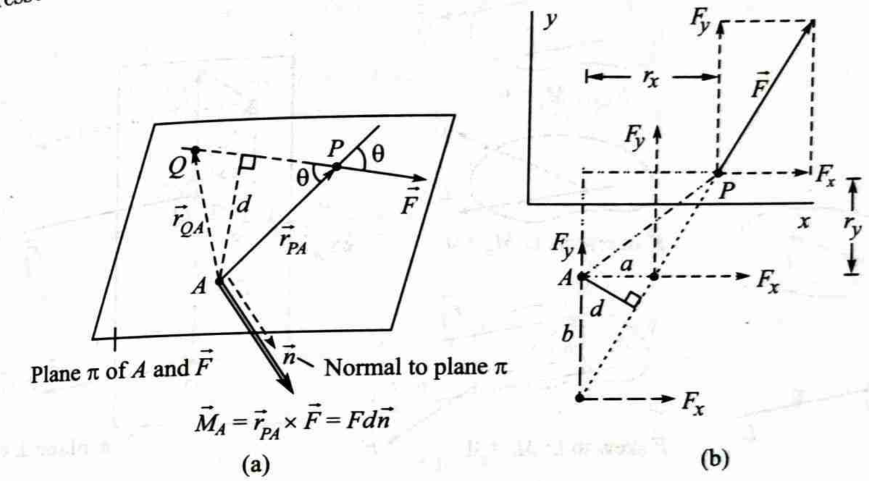


Fig. 2.2

Another type of interaction between bodies is called *moment* (a primitive). It tends to rotate the body. It is modelled as an arrow vector acting at a point with dimension $[ML^2T^{-2}]$ and with the operations of addition and scalar multiplication defined just as for forces. This concept finds application in polar media and dislocation theory. Closely related to moment is the entity of *moment* $\vec{M}_A$ of a force $\vec{F}$ acting at point P about a point A (Fig. 2.2a) with units of newton metre (N.m). It is defined by

$$\vec{M}_A = \vec{r}_{PA} \times \vec{F} = \overrightarrow{AP} \times \vec{F} = (AP)\,F \sin\theta\, \vec{n} = Fd\,\vec{n} \qquad (2.1)$$

$$= \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

where $\vec{F} = F_x\vec{\mathrm{i}} + F_y\vec{\mathrm{j}} + F_z\vec{\mathrm{k}}$, $\vec{r}_{PA} = r_x\vec{\mathrm{i}} + r_y\vec{\mathrm{j}} + r_z\vec{\mathrm{k}}$ and $\vec{n}$ is the unit vector normal to the plane of $\vec{F}$ and A, given by the cross product. Note that $\vec{M}_A = \overrightarrow{AQ} \times \vec{F}$, where $\overrightarrow{AQ}$ is the relative position vector of *any* convenient point Q on the line of action of $\vec{F}$, as

$$\overrightarrow{AQ} \times \vec{F} = (\overrightarrow{AP} + \overrightarrow{PQ}) \times \vec{F} = \overrightarrow{AP} \times \vec{F} = \vec{M}_A \qquad (\overrightarrow{PQ} \parallel \vec{F})$$

$\vec{M}_A$ is zero if $\vec{F}$ passes through A. For a force $\vec{F}$ coplanar with point A in the xy-plane (Fig. 2.2b), $\vec{n} = \pm\vec{\mathrm{k}}$ and eq. (2.1) yields

$$\vec{M}_A = (r_xF_y - r_yF_x)\,\vec{\mathrm{k}} = \pm|F_ya|\,\vec{\mathrm{k}} = \pm|F_xb|\,\vec{\mathrm{k}} = \pm Fd\,\vec{\mathrm{k}} \qquad (2.2)$$

with +ve/−ve according as $\vec F$ tends to rotate the body about A in direction $\vec k/-\vec k$.

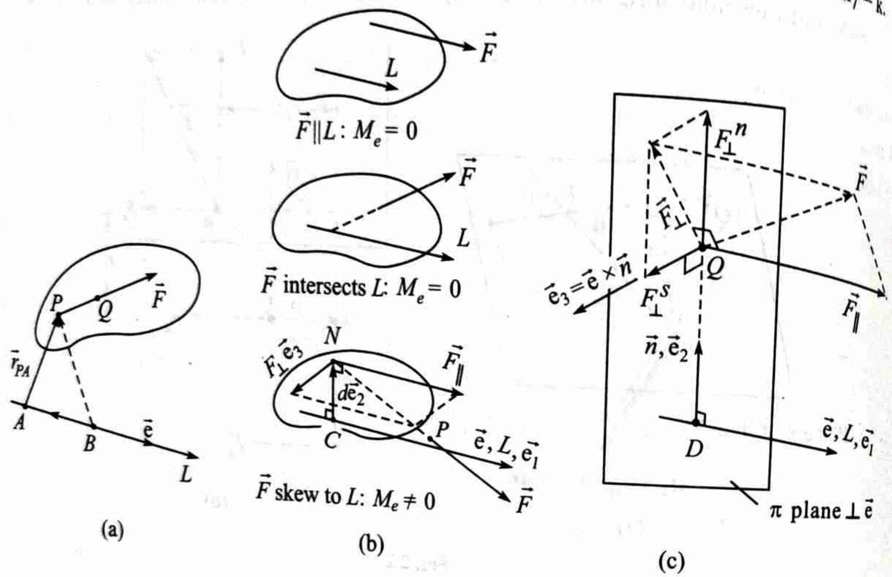


Fig. 2.3

The *moment* $\vec M_e$ *of the force* $\vec F$ *about a directed line* L (Fig. 2.3a) with unit vector $\vec e$ along it is defined as the component along $\vec e$ of its moment $\vec M_A$ about *any* point A on L:

$$M_e = \vec M_A \cdot \vec e = \overrightarrow{AP} \times \vec F \cdot \vec e = \begin{vmatrix} r_x & r_y & r_z \\ F_x & F_y & F_z \\ e_x & e_y & e_z \end{vmatrix}, \qquad \vec M_e = M_e \vec e \qquad (2.3)$$

where $\vec e = e_x \vec i + e_y \vec j + e_z \vec k$. $\vec M_e$ is well-defined since for any other point B on L,

$$\vec M_B \cdot \vec e = \overrightarrow{BP} \times \vec F \cdot \vec e = (\overrightarrow{BA} + \overrightarrow{AP}) \times \vec F \cdot \vec e$$
$$= \overrightarrow{BA} \times \vec F \cdot \vec e + M_e = M_e \qquad (\overrightarrow{BA} \parallel \vec e)$$

Equation (2.3) $\Rightarrow$ $\vec M_e = \vec 0$ iff $\vec F$ intersects L or is parallel to L (Fig. 2.3b). $\vec M_e \ne \vec 0$ only if $\vec F$ and L are skew lines. Note that $\vec M_A$ is the vector sum of the moments of the force about three orthogonal axes through A.

Consider the plane π through *any* point Q on the line of action of $\vec F$ which is normal to $\vec e$ (Fig. 2.3c). It intersects line L at D. The unit vector $\vec n = \vec e_2$ along DQ is normal to $\vec e$ and the unit vector $\vec e_3 = \vec e \times \vec n$ at Q is skew to vector $\vec e$ through D. Resolve $\vec F$ into components: $\vec F_\parallel$, $\vec F_\perp$ which are parallel and normal to $\vec e$. Resolve $\vec F_\perp$ into components $\vec F_\perp^n$, $\vec F_\perp^s$ along $\vec n$ and $\vec e_3$, where $\vec F_\perp^n$ intersects line L and $\vec F_\perp^s$ is skew to line L.

$$\vec M_e = [(\vec r_{QD} \times \vec F) \cdot \vec e]\,\vec e = [\{DQ\,\vec e_2 \times (F_\parallel \vec e_1 + F_\perp^n \vec e_2 + F_\perp^s \vec e_3)\} \cdot \vec e_1]\,\vec e = (DQ) F_\perp^s \vec e$$

Thus, the moment of a force about a line is the moment of its transverse skew component $F_\perp^s$ about the line. Let $d = CN$ be the minimum distance between L and $\vec F$ (Fig. 2.3b). As $CN \perp \vec F$, in the resolution of $\vec F = \vec F_\parallel + \vec F_\perp$, $\vec F_\perp = F_\perp \vec e_3$ is perpendicular to both L and CN. Hence, $M_e = \vec M_C \cdot \vec e = d\vec e_2 \times (F_\parallel \vec e_1 + F_\perp \vec e_3) \cdot \vec e_1 = F_\perp d$.

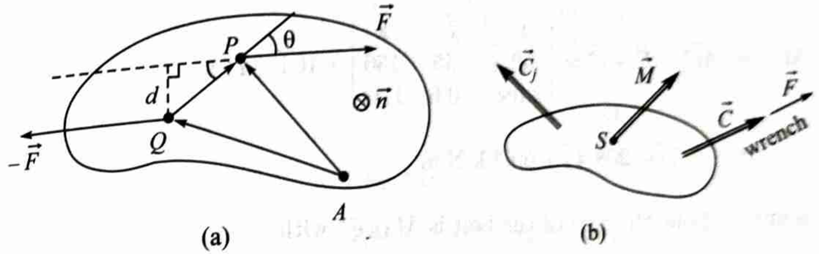


Fig. 2.4

A *couple* is a set of forces $\vec F$ and $-\vec F$ (Fig. 2.4a) with different lines of action. Its moment about every point is the same and is called the *moment $\vec C$ of the couple*:

$$\vec M_A = \overrightarrow{AP} \times \vec F + \overrightarrow{AQ} \times (-\vec F) = (\overrightarrow{AP} - \overrightarrow{AQ}) \times \vec F = \overrightarrow{QP} \times \vec F$$
$$= F d \vec n = \vec C$$

which is independent of A. The force sum of a couple is zero. Operationally, a couple will not merely signify a pair of forces $\vec F$ and $-\vec F$, but also a moment (a primitive independent of force). The moment $\vec C_1$ of a couple or a moment $\vec M$ is denoted by an arrow with double lines whose length gives its magnitude, direction gives its line of action, arrow-head gives its sense and the tail its point of application S (Fig. 2.4b).

A wrench is a system of force $\vec F$ and a moment (or moment of a couple) $\vec C$ whose direction is the same as or opposite to $\vec F$ (Fig. 2.4b). The force systems acting on a drill bit during drilling and on a screw being tightened or removed are wrenches.

Example 2.1: Find $\vec M_A$ and M_{AB} for the force system shown in Fig. E2.1.

Solution: Let $\vec e$, $\vec e^*$, $\vec e_1$ be the unit vectors along RC, AB, $\vec F$:

$$\vec e = (-5\vec j + 12\vec k)/(5^2 + 12^2)^{1/2} = (-5\vec j + 12\vec k)/13$$
$$\vec e^* = (-4\vec i + 3\vec k)/(4^2 + 3^2)^{1/2} = -0.8\vec i + 0.6\vec k$$
$$\vec F = 2600\,\vec e_1 = 2600(3\vec e + 4\vec i)/(3^2 + 4^2)^{1/2} = 1560\,\vec e + 2080\,\vec i$$

$$= 1560(-5\vec{j}+12\vec{k})/13+2080\vec{i} = (2.08\vec{i}-0.6\vec{j}+1.44\vec{k})10^3\ \text{N}$$

$$\overrightarrow{AC} = \overrightarrow{AP}+\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RC} = -30\vec{j}+50\vec{i}+70\vec{k}+65\vec{e}\ \text{mm}$$

$$= 50\vec{i}-55\vec{j}+130\vec{k}\ \text{mm} = (50\vec{i}-55\vec{j}+130\vec{k})10^{-3}\,\text{m}$$

$$\vec{C} = -26\vec{e} = -26(-5\vec{j}+12\vec{k})/13 = 10\vec{j}-24\vec{k}\ \text{N.m}$$

$$\vec{M}_A = \overrightarrow{AC}\times\vec{F}+\vec{C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 50 & -55 & 130 \\ 2.08 & -0.6 & 1.44 \end{vmatrix} + 10\vec{j}-24\vec{k}$$

$$= -1.2\vec{i}+208.4\vec{j}+60.4\vec{k}\ \text{N.m}$$

The moment about the axis of the bolt is $M_{AB}\vec{e}^*$ with

$$M_{AB} = M_{e^*} = \vec{M}_A\cdot\vec{e}^* = (-1.2)(-0.8)+60.4\times 0.6 = 37.2\,\text{N.m}$$

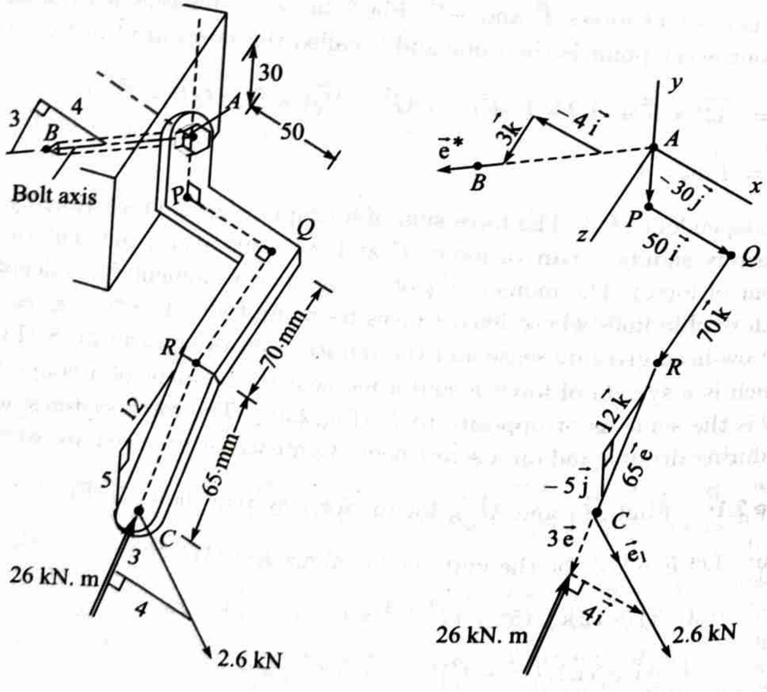


Fig. E2.1

Example 2.2: Find $\vec{M}_A$ of the force system shown in Fig. E2.2a. $AD = 19.5\,\text{cm}$.

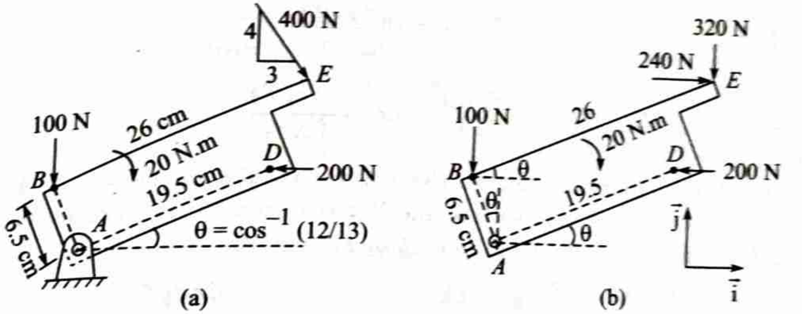


Fig. E2.2

Solution: For the coplanar force system, we resolve the forces and position vectors in the $\vec{i}$ and $\vec{j}$ components (Fig. E2.2b) and take their moment about A using the product of force and distance, with the sense x to y being $\vec{k}$. As $\cos\theta = 12/13 \Rightarrow \sin\theta = 5/13$.

$$\vec{F} = 400(-4\vec{j}+3\vec{i})/(4^2+3^2)^{1/2} = 240\vec{i}-320\vec{j}\ \text{N}$$

$$\vec{M}_A = [200(19.5\sin\theta)+100(6.5\sin\theta)-240(6.5\cos\theta+26\sin\theta)$$
$$-320(26\cos\theta-6.5\sin\theta)]10^{-2}\vec{k}-20\vec{k}$$
$$= [200(7.5)+100(2.5)-240(16)-320(21.5)]10^{-2}\vec{k}-20\vec{k} = -109.7\vec{k}\ \text{N.m}$$

2.2 Mass and Centre of Mass

The property of a body due to which an external force is required to accelerate it is called inertia and its measure is called its mass (inertial mass). Mass (gravitational mass) also appears in the gravitational law of attraction. Experience within the domain of classical mechanics has shown that inertial mass and gravitational mass of a body are equal. The mass $m(B)$ of a body B is taken axiomatically as a positive real scalar, which remains invariant with time and equals the sum of the masses $m(P_i)$ of all its parts P_i (Fig. 2.5a)[1]. By the *continuum hypothesis*, mass is modelled to be continuously distributed over a region of space (Fig. 2.5a,b) and the *mass density* at a point exists and is taken as $\rho\,\text{kg/m}^3$, $\sigma\,\text{kg/m}^2$ or $\lambda\,\text{kg/m}$ for idealised distribution of mass over a volume, an area (over mid-surface for thin plates, membranes and shells) or a line (over centroidal axis for thin bars, beams, flexible cables and strings).

[1]We use the same symbol 'm' for mass and a moving frame. The meaning is usually clear from the context. In some situations (especially Chapter 3), we have used $\textcircled{m}$ for the frame for greater clarity.

The *centre of mass* C of a body (Fig. 2.5a,b), in a given configuration, is defined by

$$\int_m \vec{r}_P\, dm \equiv m\vec{r}_C \quad \Rightarrow \quad \vec{r}_C \equiv \frac{\int_m \vec{r}_P\, dm}{\int_m dm} \tag{2.4}$$

$$\vec{r}_C = \frac{\int \rho\, \vec{r}_P\, dv}{\int \rho\, dv}, \quad \text{or} \quad \frac{\int \sigma\, \vec{r}_P\, dA}{\int \sigma\, dA}, \quad \text{or} \quad \frac{\int \lambda\, \vec{r}_P\, ds}{\int \lambda\, ds}$$

where P is a typical point in the mass element dm and dm equals ρdv, σdA, λds for mass distribution over volume, surface and line respectively. The expressions for $\vec{r}_P$ and dv in Cartesian and cylindrical coordinate systems are:

(a) $x, y, z:\quad \vec{r}_P = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}, \qquad dv = dx\, dy\, dz$

(b) $r, \phi, z:\quad \vec{r}_P = r\cos\phi\,\vec{\imath} + r\sin\phi\,\vec{\jmath} + z\,\vec{k}, \qquad dv = dr(r\, d\phi)dz$

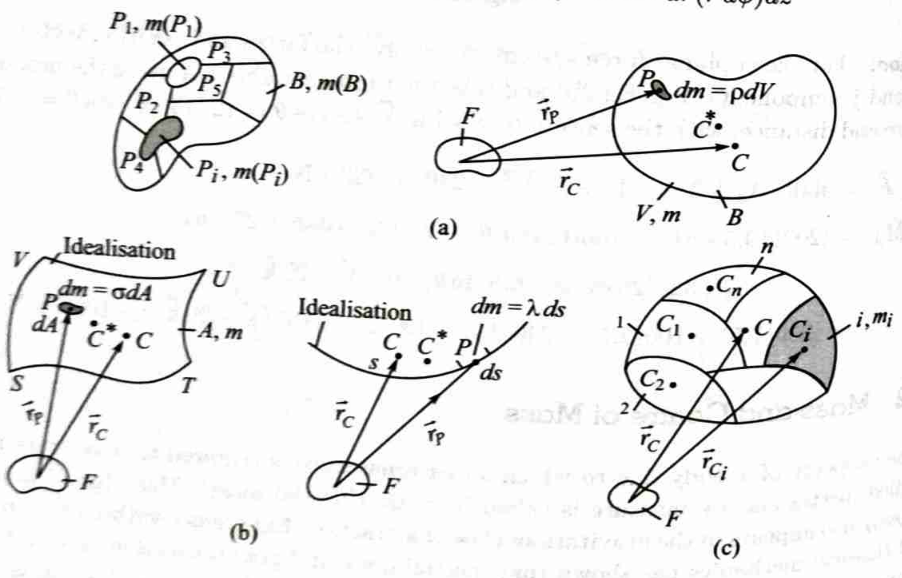


Fig. 2.5

The *centroids* C^* of a volume, a surface and a curve (Fig. 2.5a,b) are defined by

$$\vec{r}_{C^*} = \frac{\int \vec{r}_P\, dv}{\int dv}, \qquad \vec{r}_{C^*} = \frac{\int \vec{r}_P\, dA}{\int dA}, \qquad \vec{r}_{C^*} = \frac{\int \vec{r}_P\, ds}{\int ds} \tag{2.5}$$

C coincides with C^* for homogeneous bodies. C does not coincide with a specific material point for a deformable body.

It is easy to prove that:

1. For a body with a *centre of mass symmetry* O $[\rho(-\vec{r}) = \rho(\vec{r})]$, $C \equiv O$. (Fig. 2.6-1)

2. For a body with a *plane of mass symmetry*, say xy-plane $[\rho(x, y, -z) = \rho(x, y, z)]$, C lies on this plane. (Fig. 2.6-2)
3. For two planes of mass symmetry, C lies on their line of intersection. (Fig. 2.6-3)
4. A body is called a *body of revolution* if its volume is obtained by rotating an area about an axis (called *axis of revolution*), say z-axis, and its density $\rho = \rho(r, z)$. Then, C lies on the axis of revolution of the body. (Fig. 2.6-4)
5. For a body with 3 planes of mass symmetry not through one line, C lies at their point of intersection. (Fig. 2.6-5)

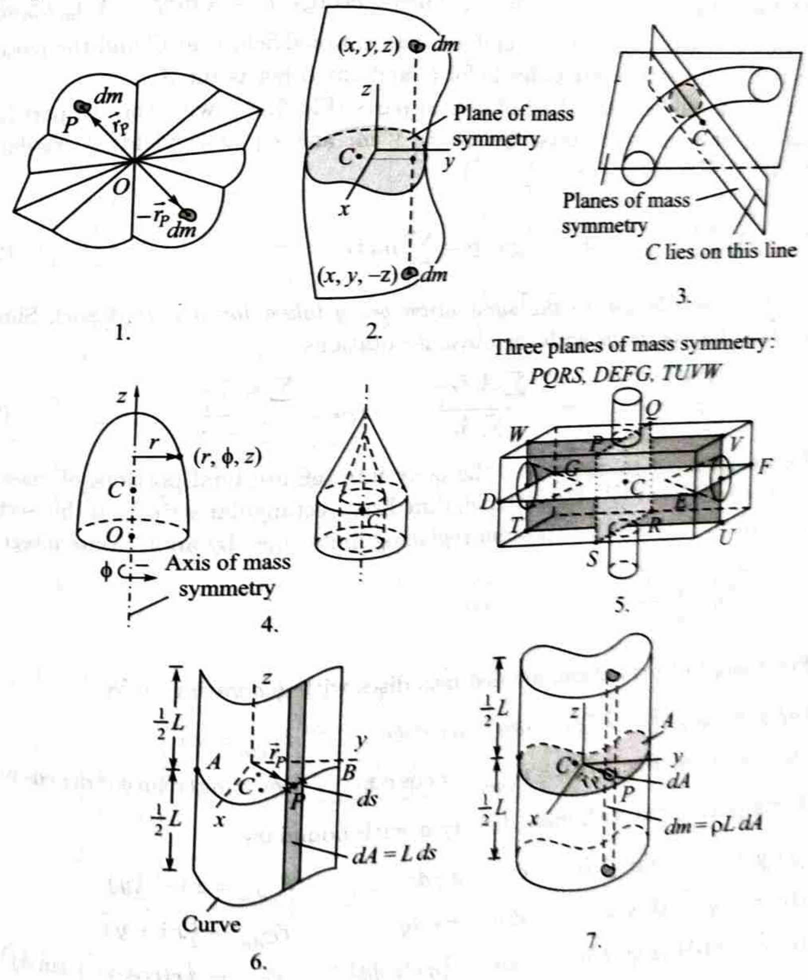


Fig. 2.6

6. For a cylindrical surface, C^* is at the centroid of its curve of intersection with its mid-plane of symmetry. (Fig. 2.6-6)
7. For a homogeneous cylindrical body, C is at the centroid of its cross-section in the mid-plane of symmetry. (Fig. 2.6-7)

The centroid is useful for finding the resultant force due to hydrostatic pressure on a plane area and stresses in thin bars. We shall find later that C is a point of vital importance in mechanics, for example, momentum $\vec{p} = m\vec{v}_C$, $\vec{F} = m\vec{a}_C$, moment of momentum $\vec{H}_A = \vec{H}_C + \vec{r}_{CA} \times m\vec{v}_{CA}$, kinetic energy $T = \frac{1}{2}mv_C^2 + \frac{1}{2}\int_m v_{PC}^2 dm$, $\vec{M}_C = \dot{\vec{H}}_C$, the centre of gravity for uniform gravitational field is at C and the parallel axis theorem for inertia matrices holds for C and any other point A.

For composite bodies consisting of several parts (Fig. 2.5c), with the ith part having mass m_i and centre of mass C_i, $m = \sum m_i$ and eq. (2.4) yields $\int_m \vec{r}_P\, dm = \sum \int_{m_i} \vec{r}_P\, dm = \sum m_i \vec{r}_{C_i} \quad \Rightarrow$

$$\vec{r}_C = \frac{\sum m_i \vec{r}_{C_i}}{\sum m_i} \quad \Rightarrow \quad m\vec{r}_C = \sum m_i \vec{r}_{C_i} \tag{2.6}$$

with negative contribution in the summation being taken for a cutout part. Similar relations hold for centroids with the obvious notations:

$$\vec{r}_{C^*} = \frac{\sum V_i \vec{r}_{C_i^*}}{\sum V_i}, \qquad \vec{r}_{C^*} = \frac{\sum A_i \vec{r}_{C_i^*}}{\sum A_i}, \qquad \vec{r}_{C^*} = \frac{\sum s_i \vec{r}_{C_i^*}}{\sum s_i} \tag{2.7}$$

It is often convenient to decompose the body into infinitesimal portions of mass dm with centre of mass at C_{dm}, for example, thin discs, rectangular strips and thin sectors, and apply the decomposition theorem replacing finite sums by appropriate integrals:

$$\vec{r}_C = \frac{\int_m \vec{r}_{C_{dm}}\, dm}{\int_m dm}$$

1. For a solid of revolution, divided into discs with generating curve
   (a) $y = f(x)$ (Fig. 2.7a): $\quad dm = \rho\pi y^2\, dx, \qquad \vec{r}_{C_{dm}} = x\,\vec{i}$
   (b) $r = g(\phi)$ (Fig. 2.7b): $\quad \vec{r}_{C_{dm}} = r\cos\phi\,\vec{i}, \qquad dm = \rho\pi(r\sin\phi)^2\, d(r\cos\phi)$
2. For a plane area with mass density σ with boundary
   (a) $y = f(x)$ (Fig. 2.7c): $\quad dm = \sigma y\, dx, \qquad \vec{r}_{C_{dm}} = x\,\vec{i} + \frac{1}{2}y\,\vec{j}$
   (b) $x = g(y)$ (Fig. 2.7d): $\quad dm = \sigma x\, dy, \qquad \vec{r}_{C_{dm}} = \frac{1}{2}x\,\vec{i} + y\,\vec{j}$
   (c) $r = h(\phi)$ (Fig. 2.7e): $\quad dm = \frac{1}{2}\sigma r(r\, d\phi), \qquad \vec{r}_{C_{dm}} = \frac{2}{3}r(\cos\phi\,\vec{i} + \sin\phi\,\vec{j})$

3. For a plane curve with mass density λ, which is described as
   (a) $y = f(x)$ (Fig. 2.7f): $\quad \vec{r}_P = x\,\vec{i} + y\,\vec{j}, \qquad dm = \lambda\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$
   (b) $x = g(y)$ (Fig. 2.7f): $\quad \vec{r}_P = x\,\vec{i} + y\,\vec{j}, \qquad dm = \lambda\left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{1/2} dy$
   (c) $r = h(\phi)$ (Fig. 2.7g): $\quad \vec{r}_P = r\cos\phi\,\vec{i} + r\sin\phi\,\vec{j}, \quad dm = \lambda\left[\left(\frac{dr}{d\phi}\right)^2 + r^2\right]^{1/2} d\phi$
   (d) $\phi = l(r)$ (Fig. 2.7g): $\quad \vec{r}_P = r\cos\phi\,\vec{i} + r\sin\phi\,\vec{j}, \quad dm = \lambda\left[1 + r^2\left(\frac{d\phi}{dr}\right)^2\right]^{1/2} dr$

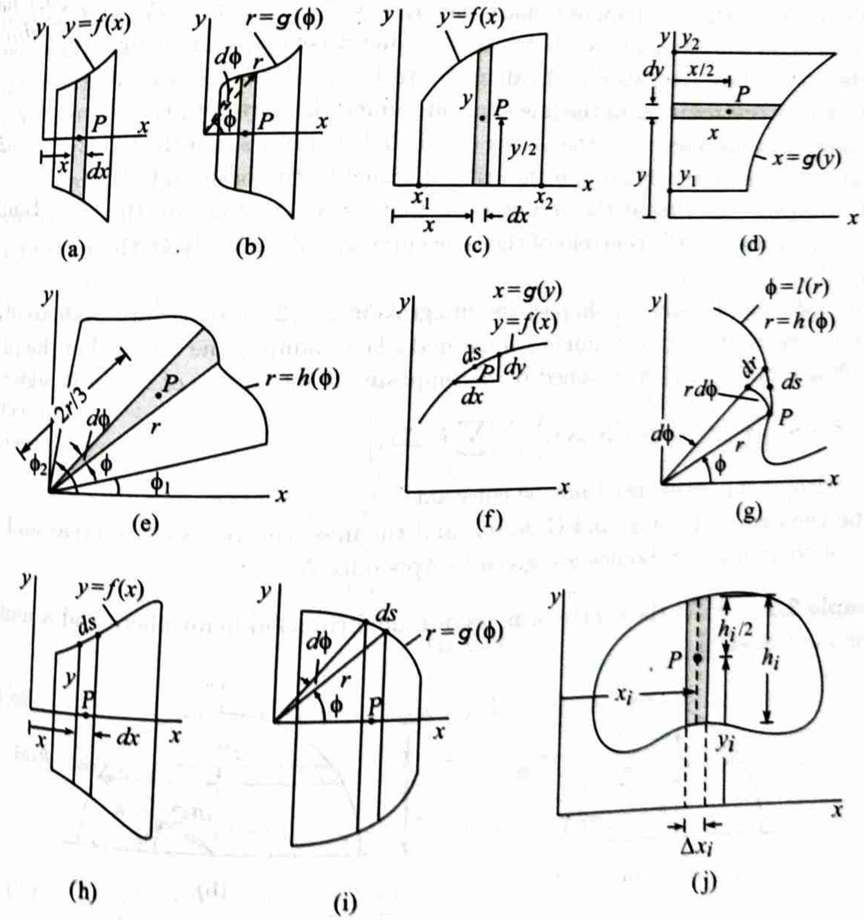


Fig. 2.7

4. A surface of revolution (σ) divided into rings, with generating curve

(a) $y = f(x)$ (Fig. 2.7h): $dm = 2\pi y\, ds\, \sigma = 2\pi\sigma y \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{1/2} dx, \ \vec{r}_{C_{dm}} = x\vec{i}$

(b) $r = g(\phi)$ (Fig. 2.7i): $dm = 2\pi\sigma r \sin\phi \left[\left(\dfrac{dr}{d\phi}\right)^2 + r^2\right]^{1/2} d\phi, \ \vec{r}_{C_{dm}} = r\cos\phi\,\vec{i}$

§ **Exercise 2.1:** (a) By dividing a triangle ABC into strips parallel to its sides, show that its centroid lies at the point of intersection of its medians, i.e., $\vec{r}_{C^*} = (\vec{r}_A + \vec{r}_B + \vec{r}_C)/3$. (b) By dividing a tetrahedron $ABDE$ into thin plates parallel to a face, show that its centroid lies at the point of concurrence of its transversals connecting vertices to the centroids of opposite faces, i.e., $\vec{r}_{C^*} = (\vec{r}_A + \vec{r}_B + \vec{r}_D + \vec{r}_E)/4$. C^* divides the transversals in the ratio 3 : 1. (c) By dividing a cone / pyramid into thin plates parallel to the base and also by dividing it into tetrahedrons meeting at the vertex, show that its centroid lies on the line segment joining the vertex to the centroid of the base area at a distance from the vertex equal to 3/4th of its length. (d) By dividing a conical / pyramidal surface into thin rings parallel to the base and also by dividing it into triangles meeting at the vertex, show that its centroid lies on the line segment joining the vertex to the centroid of the base curve at a distance from the vertex equal to 2/3rd of its length.

For bodies with complex shapes, the integrals in eq. (2.4) for $\vec{r}_C$ are evaluated approximately by graphical or numerical methods. For example, the centroid of the plane area shown in Fig. 2.7j is obtained by decomposing it into strips of width Δx_i:

$$\vec{r}_{C^*} = \left[\sum (x_i\vec{i} + y_i\vec{j}) h_i \Delta x_i\right] \Big/ \left[\sum h_i \Delta x_i\right]$$

The width of the strips need not be constant.

The theorem of Pappus and Guldinus and the mass centres, surface areas and volumes of some uniform bodies are given in Appendix A.

Example 2.3: Find the centre of mass of a uniform solid hemisphere and a uniform hemispherical shell.

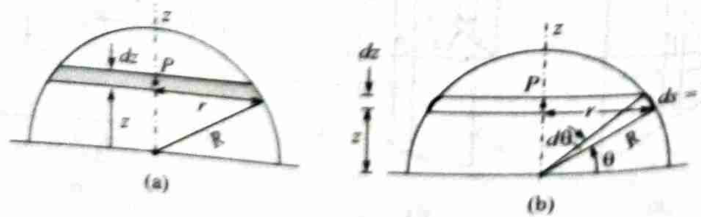


Fig. E2.3

Solution: Decompose the solid hemisphere into thin circular plates (Fig. E2.3a) with $dV = \pi r^2 dz = \pi(R^2 - z^2)dz$, $\vec{r}_P = z\vec{k}$,

$$\vec{r}_C = \frac{\int_V \vec{r}_P\, dV}{V} = \frac{\int_0^R \pi(R^2 - z^2) z\, dz\, \vec{k}}{\int_0^R \pi(R^2 - z^2)\, dz} = \frac{3}{8} R\vec{k}$$

Decompose the shell into thin rings (Fig. E2.3b) with $dA = 2\pi r ds = 2\pi(R\cos\theta)(R\,d\theta)$.

$$\vec{r}_C = \frac{\int_A \vec{r}_P\, dA}{A} = \frac{\int_0^{\pi/2} (R\sin\theta\,\vec{k}) 2\pi R^2 \cos\theta\, d\theta\, \vec{k}}{\int_0^{\pi/2} 2\pi R^2 \cos\theta\, d\theta} = \tfrac{1}{2} R\vec{k}$$

Example 2.4: Find the centre of mass of (a) a thin right circular conical shell of base radius R, height h and density varying linearly with distance from the base, with σ_1 at the base to σ_0 at the vertex, and (b) a solid right circular cone of base radius R, height h and density varying linearly with distance from the axis of symmetry, with ρ_0 at the axis to ρ_1 at radius R. (c) Rework parts (a) and (b) for one-half of the body obtained by splitting with a plane through the axis of symmetry.

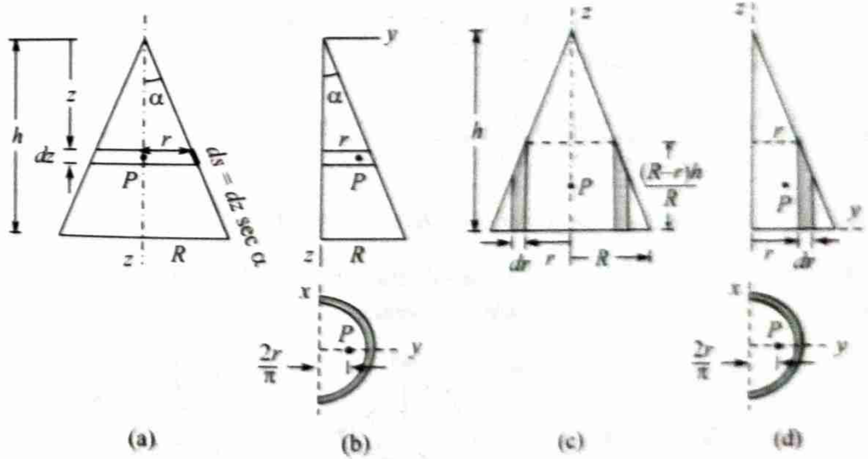


Fig. E2.4

Solution: (a) For the ring element shown in Fig. E2.4a,

$$ds = dz\sec\alpha, \qquad r = Rz/h, \qquad \vec{r}_P = z\vec{k}, \qquad \sigma = \sigma_0 + (\sigma_1 - \sigma_0)z/h$$

$$dm = 2\pi r\, ds\, \sigma = 2\pi R[\sigma_0 z + (\sigma_1 - \sigma_0)z^2/h] \sec\alpha\, dz\, /h$$

$$\vec{r}_C = \frac{\int_m \vec{r}_P\, dm}{\int_m dm} = \frac{\int_0^h [\sigma_0 z^2 + (\sigma_1 - \sigma_0)z^3/h]\, dz}{\int_0^h [\sigma_0 z + (\sigma_1 - \sigma_0)z^2/h]\, dz}\vec{k} = \frac{\sigma_0 + 3\sigma_1}{2(\sigma_0 + 2\sigma_1)} h\vec{k}$$

For a uniform shell, $\sigma_0 = \sigma_1$ and $\vec{r}_C = (2h/3)\,\vec{k}$. For one-half shell (Fig. E2.4b), $\vec{r}_P = z\,\vec{k} + (2r/\pi)\,\vec{j} = z(2\tan\alpha\,\vec{j}/\pi + \vec{k})$ and hence

$$\vec{r}_C = \frac{\sigma_0 + 3\sigma_1}{2(\sigma_0 + 2\sigma_1)}\,h(2\tan\alpha\,\vec{j}/\pi + \vec{k})$$

$$\Rightarrow \text{ for } \sigma_0 = \sigma_1: \quad \vec{r}_C = \frac{2}{3}(2R\,\vec{j}/\pi + h\,\vec{k})$$

(b) For an element of radius r, thickness dr and height $(R-r)h/R$ (Fig. E2.4c),

$$\vec{r}_P = [(R-r)h/2R]\,\vec{k}, \qquad dm = [\rho_0 + (\rho_1 - \rho_0)r/R](2\pi r)\,dr\,(R-r)h/R$$

$$\vec{r}_C = \frac{\int_m \vec{r}_P\,dm}{\int_m dm} = \frac{h\int_0^R (R-r)[\rho_0 Rr + (\rho_1 - 2\rho_0)r^2 - (\rho_1 - \rho_0)r^3/R]\,dr}{2R\int_0^R[\rho_0 Rr + (\rho_1 - 2\rho_0)r^2 - (\rho_1 - \rho_0)r^3/R]\,dr}\,\vec{k}$$

$$= \frac{3\rho_0 + 2\rho_1}{\rho_0 + \rho_1}\left(\frac{h}{10}\right)\vec{k}$$

For a uniform cone, $\rho_0 = \rho_1$ and $\vec{r}_C = \frac{1}{4}h\,\vec{k}$. For one-half cone (Fig. E2.4d), $\vec{r}_P = \frac{1}{2}[(R-r)h/R]\,\vec{k} + (2r/\pi)\,\vec{j}$ and

$$\vec{r}_C = \frac{\int_m \vec{r}_P\,dm}{\int_m dm} = \frac{3\rho_0 + 2\rho_1}{\rho_0 + \rho_1}\left(\frac{h}{10}\right)\vec{k} + \frac{2\rho_0 + 3\rho_1}{\rho_0 + \rho_1}\left(\frac{2R}{5\pi}\right)\vec{j}$$

For a uniform half-cone, $\rho_0 = \rho_1$ and $\vec{r}_C = (R/\pi)\,\vec{j} + (h/4)\,\vec{k}$.

Example 2.5: Find the centre of mass of uniform (a) thin circular arc, (b) thin circular sector, and (c) thin circular annular sector.

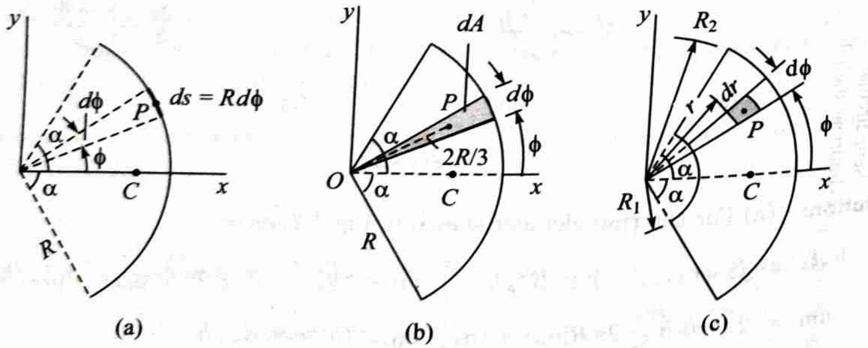


Fig. E2.5

Solution: We use polar coordinates since the boundaries are: r = constant and ϕ = constant. The elements for integration are shown in Fig. E2.5.

$$\text{(a)} \quad \vec{r}_C = \frac{\int \vec{r}_P\,ds}{\int ds} = \frac{\int_{-\alpha}^{\alpha} R(\cos\phi\,\vec{i} + \sin\phi\,\vec{j})R\,d\phi}{\int_{-\alpha}^{\alpha} R\,d\phi} = \frac{R\sin\alpha}{\alpha}\,\vec{i}$$

$$\text{(b)} \quad \vec{r}_C = \frac{\int \vec{r}_P\,dA}{\int dA} = \frac{\int_{-\alpha}^{\alpha} \frac{2}{3}R(\cos\phi\,\vec{i} + \sin\phi\,\vec{j})(\frac{1}{2}R^2\,d\phi)}{\int_{-\alpha}^{\alpha}\frac{1}{2}R^2\,d\phi} = \frac{2R\sin\alpha}{3\alpha}\,\vec{i}$$

$$\text{(c)} \quad \vec{r}_C = \frac{\int \vec{r}_P\,dA}{\int dA} = \frac{\int_{-\alpha}^{\alpha}\int_{R_1}^{R_2} r(\cos\phi\,\vec{i} + \sin\phi\,\vec{j})(r\,dr\,d\phi)}{\int_{-\alpha}^{\alpha}\int_{R_1}^{R_2} r\,dr\,d\phi} = \frac{2(R_2^3 - R_1^3)\sin\alpha}{3(R_2^2 - R_1^2)\alpha}\,\vec{i}$$

For a semi-circular thin plate, $R_1 = 0, \alpha = \frac{1}{2}\pi \Rightarrow \vec{r}_C = (4R_2/3\pi)\,\vec{i}$. For a semi-circular thin annular plate of mean radius R and thickness t, $\alpha = \frac{1}{2}\pi$, $R_1 = R - \frac{1}{2}t$, $R_2 = R + \frac{1}{2}t$, $\vec{r}_C = (2R/\pi)[1 + t^2/12R^2]\,\vec{i}$. The centroid of a semi-circular arc ($t = 0$) of radius R is at $(2R/\pi)\,\vec{i}$. The error e in approximating a semi-circular annular plate to a semi-circular arc is $e \simeq (100t^2/12R^2)\% \Rightarrow e < 0.1\%$ for $t/R < 0.1$ and $e < 1\%$ even for $t/R = 0.3$.

Example 2.6: Find the centre of mass of the body given in Fig. E2.6. The thin semi-circular ring, thin semi-circular cylindrical shell, and the solid top and base have mass densities λ, σ and ρ, respectively.

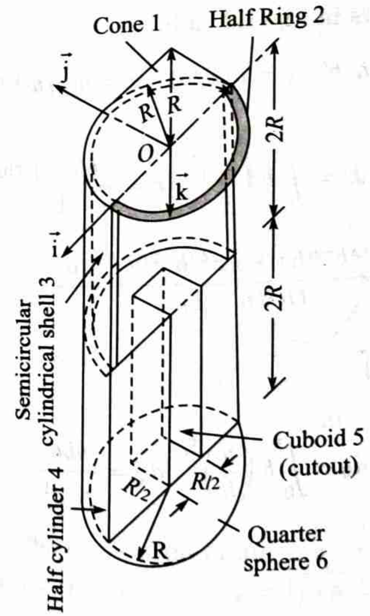


Fig. E2.6

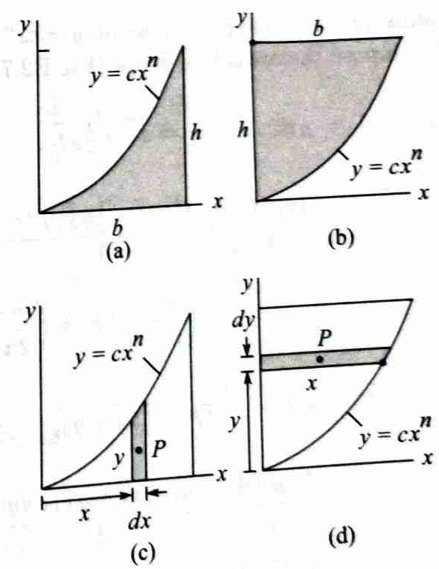


Fig. E2.7

Solution: We decompose the body into a cone 1, a thin semi-circular ring 2, a thin semi-circular cylindrical shell 3, a semi-circular solid cylinder 4, a deletion of a cuboid 5 of size $\frac{1}{2}R \times \frac{1}{2}R \times 2R$ and a quarter sphere 6. The contribution of the deleted cuboid is taken as negative in the summations. The mass and centre of mass of each part are:

$$m_1 = (\pi R^2/3)R\rho = \pi R^3\rho/3, \qquad \vec{r}_{C_1} = -(R/4)\,\vec{k}$$

$$m_2 = \pi R\lambda, \qquad \vec{r}_{C_2} = -(2R/\pi)\,\vec{j}$$

$$m_3 = (\pi R)(2R)\sigma = 2\pi R^2\sigma, \qquad \vec{r}_{C_3} = (2R/\pi)\,\vec{j} + R\,\vec{k}$$

$$m_4 = (\tfrac{1}{2}\pi R^2)(2R)\rho = \pi R^3\rho, \qquad \vec{r}_{C_4} = (4R/3\pi)\,\vec{j} + 3R\,\vec{k}$$

$$m_5 = (\tfrac{1}{2}R)(\tfrac{1}{2}R)(2R)\rho = R^3\rho/2, \qquad \vec{r}_{C_5} = -(R/4)\,\vec{i} + (R/4)\,\vec{j} + 3R\,\vec{k}$$

$$m_6 = [\tfrac{1}{4}(4\pi R^3/3)]\rho = \pi R^3\rho/3, \qquad \vec{r}_{C_6} = (3R/8)\,\vec{j} + (4R + 3R/8)\,\vec{k}$$

$$\vec{r}_C = \frac{\sum m_i \vec{r}_{C_i}}{\sum m_i} = [(R^4\rho/8)\,\vec{i} - \{2R^2\lambda - 4R^3\sigma - (29 + 3\pi)R^4\rho/24\}\,\vec{j}$$

$$+\{2\pi R^3\sigma + (35\pi - 12)R^4\rho/8\}\,\vec{k}] / [(5\pi/3 - 0.5)R^3\rho + 2\pi R^2\sigma + \pi R\lambda]$$

Example 2.7: Find the area and centroid of figures in Figs E2.7a,b.

Solution: The point (b, h) lies on $y = cx^n \Rightarrow c = h/b^n$, $y = h(x/b)^n$, $x = b(y/h)^{1/n}$. Decompose the areas into strips (Fig. E2.7c,d).

(a) $dA = y\,dx$, $\quad \vec{r}_P = x\,\vec{i} + \frac{1}{2}y\,\vec{j}$, $\quad A = \int_0^b y\,dx = \int_0^b h\left(\frac{x}{b}\right)^n dx = \frac{bh}{n+1}$

$$\vec{r}_C = \frac{\int \vec{r}_P\,dA}{A} = \frac{\int_0^b (x\,\vec{i} + \frac{1}{2}y\,\vec{j})y\,dx}{A} = \frac{\int_0^b (x^{n+1}b^{-n}h\,\vec{i} + \frac{1}{2}x^{2n}b^{-2n}h^2\,\vec{j})\,dx}{bh/(n+1)}$$

$$= \left(\frac{n+1}{n+2}\right) b\,\vec{i} + \left(\frac{n+1}{2n+1}\right)\frac{h}{2}\,\vec{j}$$

(b) $dA = x\,dy$, $\quad \vec{r}_P = \frac{1}{2}x\,\vec{i} + y\,\vec{j}$, $\quad A = \int_0^h x\,dy = \int_0^h b\left(\frac{y}{h}\right)^{1/n} dy = \frac{nbh}{n+1}$

$$\vec{r}_C = \frac{\int \vec{r}_P\,dA}{A} = \frac{\int_0^h (\frac{1}{2}x\,\vec{i} + y\,\vec{j})x\,dy}{A}$$

$$= \frac{\int_0^h (\frac{1}{2}b^2h^{-2/n}y^{2/n}\,\vec{i} + bh^{-1/n}y^{1/n+1}\,\vec{j})\,dy}{nbh/(n+1)} = \left(\frac{n+1}{n+2}\right)\frac{b}{2}\,\vec{i} + \left(\frac{n+1}{2n+1}\right)h\,\vec{j}$$

2.3 Euler's Axioms

The *momentum* $\vec{p}_{|F}$ and the *moment of momentum* $\vec{H}_{A|F}$ of a body B (a system) about a point A (which could be a material point of the body, a fixed location of frame F or a point of another body in motion with respect to F) with respect to frame F (Fig. 2.8a) are defined by

$$\vec{p}_{|F} = \int_m \vec{v}_{P|F}\,dm, \qquad \vec{H}_{A|F} = \int_m \vec{r}_{PA} \times \vec{v}_{PA|F}\,dm \tag{2.8}$$

The concept of $\vec{H}_A$ became more significant when its natural unit was discovered in quantum mechanics to be equal to Planck's constant h divided by 2π. It is a tiny unit, $h/2\pi = 1.054 \times 10^{-34}$ kg.m^2/s, but it implies huge rotation speeds of the order of 10^{11} Hertz in systems of atomic size.

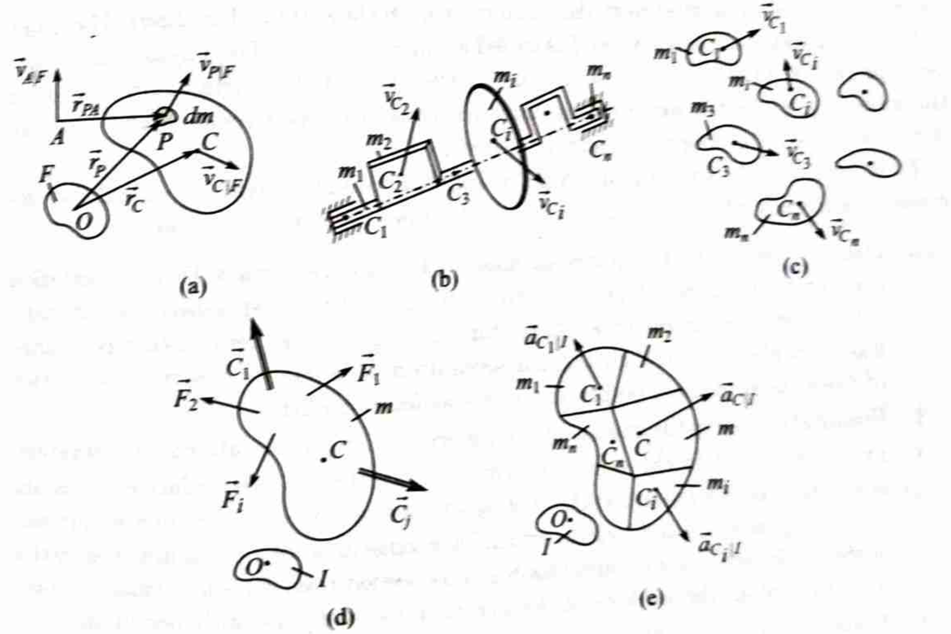


Fig. 2.8

The momentum of any system (Fig. 2.8a,b,c) can be expressed in terms of the velocity of C, since $\vec{p}_{|F} = \overline{[\int \vec{r}_P\,dm]}_{|F} = \overline{(m\vec{r}_C)}_{|F} \Rightarrow$

$$\vec{p}_{|F} = m\vec{v}_{C|F} = \sum m_i \vec{v}_{C_i|F} \tag{2.9}$$

i.e., momentum of a body equals the momentum evaluated as if the total mass m of the body were concentrated at C and moving at its velocity $\vec{v}_{C|F}$.

§ **Exercise 2.2:** Frame T translates arbitrarily with respect to frame F. Using $\vec{v}_{P|F} = \vec{v}_{P|T} + \vec{v}_{T|F}$, prove that $\vec{p}_{|F} = \vec{p}_{|T} + m\vec{v}_{T|F}$ and $\vec{H}_{A|F} = \vec{H}_{A|T}$.

Euler's axioms relate $\dot{\vec{p}}$ and $\dot{\vec{H}}_O$ with the external forces and moments.

Euler's Axioms: There exists a frame I such that for *any system*

$$\dot{\vec{p}}_{|I} = \vec{F} \tag{2.10}$$

$$\dot{\vec{H}}_{O|I} = \vec{M}_O \tag{2.11}$$

where O **is a point fixed in** I, $\vec{F}$ is the sum of all the *external* forces from the surroundings on the system and $\vec{M}_O$ is the sum of the moments about O of all *external* loads (forces and moments) from the surroundings on the system (Fig. 2.8d). The frame I in which Euler's axioms are valid is called an *inertial frame.* The physical laws have been presented in vector form, which are independent of the origin or orientation of the axes in I, since the experiments on which these are based remain valid when we shift and rotate the coordinate axes attached to I.

It is impossible to find an exactly inertial physical frame. In order of decreasing accuracy, the following four physical frames are commonly used.

1. Frame attached to the centre of mass of the solar system with its orientation (i.e., orientation of a triad fixed to it) fixed relative to specified *distant* stars. These distant stars are called *fixed* stars since they are so remote that no change has been observed in their angular separation. It is used to describe the motion of tides, astronomical bodies and inter-planetary rockets.
2. Frame attached to the sun's centre with orientation fixed relative to distant stars.
3. Frame attached to the earth's centre with orientation fixed relative to distant stars. It is used to describe the motion of trade winds, cyclones, ocean currents, long-range missiles and rockets, which have long-distance motion near the earth's surface. However, sometimes this frame is needed even for small range motion, e.g., to explain the working of the gyrocompass and Foucault's pendulum.
4. Frame attached to the earth and rotating with it at constant angular velocity relative to frame in item 3. It yields excellent results for almost all dynamic problems in engineering, where the range of motion is small compared to the earth's radius, such as turbines, machine dynamics, automobiles and channel flows.

The six scalar equations corresponding to eqs (2.10) and (2.11), when applied to the whole system, are *not sufficient* for determining the motion of a *general system* which is deformable and hence possesses an infinite number of dof. However, these six scalar

equations are *necessary and sufficient* for the complete determination of the motion of a *rigid body* since it has precisely six degrees of freedom.[2]

For the purpose of application of Euler's axioms to a system, it is necessary that

1. the system should be *well-identified* and its sketch be drawn *in isolation* from its surroundings, and
2. the *external forces and moments* exerted by the surroundings on the system should be drawn on it.

Such a diagram is called a **free body diagram** (FBD). The forces exerted by one part of the system on another part of the system are called *internal forces*. These should not be shown in an FBD since they do not appear in Euler's Axioms.

The **equation of motion of the centre of mass** C of *any system* is obtained using $\vec{p}_{|I}$ from eq. (2.9) in eq. (2.10) (Fig. 2.8e):

$$\vec{F} = m\vec{a}_{C|I} = \sum m_i \vec{a}_{C_i|I} \tag{2.12}$$

$$F_x = m\ddot{x}_C, \qquad F_y = m\ddot{y}_C, \qquad F_z = m\ddot{z}_C$$

$$F_r = m(\ddot{r}_C - r_C\dot{\phi}_C^2), \quad F_\phi = m(2\dot{r}_C\dot{\phi}_C + r_C\ddot{\phi}_C), \quad F_z = m\ddot{z}_C \tag{2.13}$$

$$F_t = m\ddot{s}_C, \qquad F_n = ma_{C_n} = m\dot{s}_C^2 / \rho_C, \qquad F_b = 0$$

where the position of C with respect to frame I has Cartesian coordinates (x_C, y_C, z_C), cylindrical polar coordinates (r_C, ϕ_C, z_C), path coordinate s_C, and the radius of curvature of its path is ρ_C. The system can be a rigid body, a deformable body, a system of rigid bodies, a finite part or an infinitesimal part of a body. Equation (2.12) is the real content of Newton's second law.

If $\vec{F}(t) \equiv \vec{0}$, then $\vec{F} = m\vec{a}_{C|I} \Rightarrow$

$$\begin{aligned} \vec{a}_{C|I} &= \vec{0}, & \sum m_i \vec{a}_{C_i|I} &= \vec{0} \quad \forall\, t \\ \vec{v}_{C|I}(t) &= \vec{v}_{C|I}(0), & \sum m_i \vec{v}_{C_i|I}(t) &= \sum m_i \vec{v}_{C_i|I}(0) \end{aligned} \tag{2.14}$$

i.e., the centre of mass C of any system moves with uniform velocity relative to frame I if the total external force on it is zero. The other material points of the system may not have uniform velocity and the body can in fact deform, split or rotate. This is the precise content of Newton's first law. If in addition $\vec{p}(0) = \vec{0}$, then eq. (2.14) $\Rightarrow$

$$\begin{aligned} \vec{v}_C(t) &\equiv \vec{0}, & \sum m_i \vec{v}_{C_i}(t) &\equiv \vec{0} \\ \vec{r}_C(t) &= \vec{r}_C(0), & \sum m_i \vec{r}_{C_i}(t) &= \sum m_i \vec{r}_{C_i}(0) \end{aligned} \tag{2.15}$$

[2] Note, however, that if one wants to predict the motion of systems of rigid bodies, especially when friction is present, one could encounter *deterministic chaos*, i.e., although the system may be in principle deterministic, in practice prediction may not be possible beyond a short initial period because of the extreme sensitivity of the system to infinitesimal changes in initial and boundary conditions.

i.e., the centre of mass C of any system remains at rest relative to frame I, though parts of the system may move. If only a particular Cartesian component of $\vec{F}(t)$ is zero for all t, then eqs (2.14), (2.15) hold only for that component. However, these comments do not hold for $\vec{e}_r$ or $\vec{e}_\phi$ components if $F_r(t) \equiv 0$ or $F_\phi(t) \equiv 0$.

The kinematic variables $\vec{a}_{C|I}$ and $\vec{v}_{PA|I}$ in Euler's axioms have the same value in all inertial frames in uniform translation with respect to one another. Hence, the force $\vec{F}(\vec{r}, \dot{\vec{r}}, t)$ appearing in these axioms must be of the form $\vec{F}(\vec{r}_2 - \vec{r}_1, \dot{\vec{r}}_2 - \dot{\vec{r}}_1, t)$. All forces do in fact depend on the relative positions and/or velocities of interacting bodies.

§ **Exercise 2.3:** (a) Derive Euler's 2nd axiom from the 1st axiom for a mass-point subjected to a force $\vec{F}$ and no discrete moment. (b) In the theory of relativity, $m = m_0/[1-(v/c)^2)]^{1/2}$, where m_0 is the rest mass. Prove that the momentum principle for a mass-point $\vec{F} = d(m\vec{v})/dt$ becomes $\vec{F} = m\vec{a} + [(v/c)^2/\{1-(v/c)^2\}]m\dot{v}\,\vec{e}_t$. $\vec{F}$ differs from $m\vec{a}$ by less than 1% for $v/c < 0.1$.

2.4 Equivalent / Resultant Force Systems

To apply Euler's axioms to a system, we need to compute $\vec{F}$ and $\vec{M}_O$ of the external loads acting on it. We shall often refer to a load system consisting of forces and moments simply as a force system.

Two force systems are defined to be *equivalent* if they have the same total force $\vec{F}$ and the same total moment $\vec{M}_A$ about *one* point A. A *simpler force system* equivalent to a given force system is called its *resultant*.

The following inferences follow for equivalent force systems.

1. Equivalent force systems have equal total moment $\vec{M}_B$ about *any* point B.

   Proof: Consider a force system consisting of n discrete forces with the ith force $\vec{F}_i$ acting at point i, and m couples with moments $\vec{C}_j$, $j = 1, \ldots, m$ (Fig. 2.9a).

$$\vec{M}_B = \sum_i \vec{Bi} \times \vec{F}_i + \sum_j \vec{C}_j = \sum_i (\vec{BA} + \vec{Ai}) \times \vec{F}_i + \sum_j \vec{C}_j$$
$$= \vec{BA} \times \sum_i \vec{F}_i + \Big(\sum_i \vec{Ai} \times \vec{F}_i + \sum_j \vec{C}_j\Big) = \vec{BA} \times \vec{F} + \vec{M}_A$$

   $\Rightarrow$ The moment $\vec{M}_B$ for equivalent force systems is the same since $\vec{F}$ and $\vec{M}_A$ for them are equal. The proof can be extended for distributed forces and moments. Hence, *for finding the moment sum and force sum, a given force system can be replaced by its resultant.* For example, for applying Euler's axioms, a complex external force system can be replaced by its resultant.

2. Two equivalent force systems impart the same motion to a *single rigid body* since it is governed only by $\vec{F}$ and $\vec{M}_O$, which are the same for the two systems. But the *internal forces* induced are *different.* For the equivalent force systems in Fig. 2.9b,

the rigid rod has the same motion but the internal forces differ.

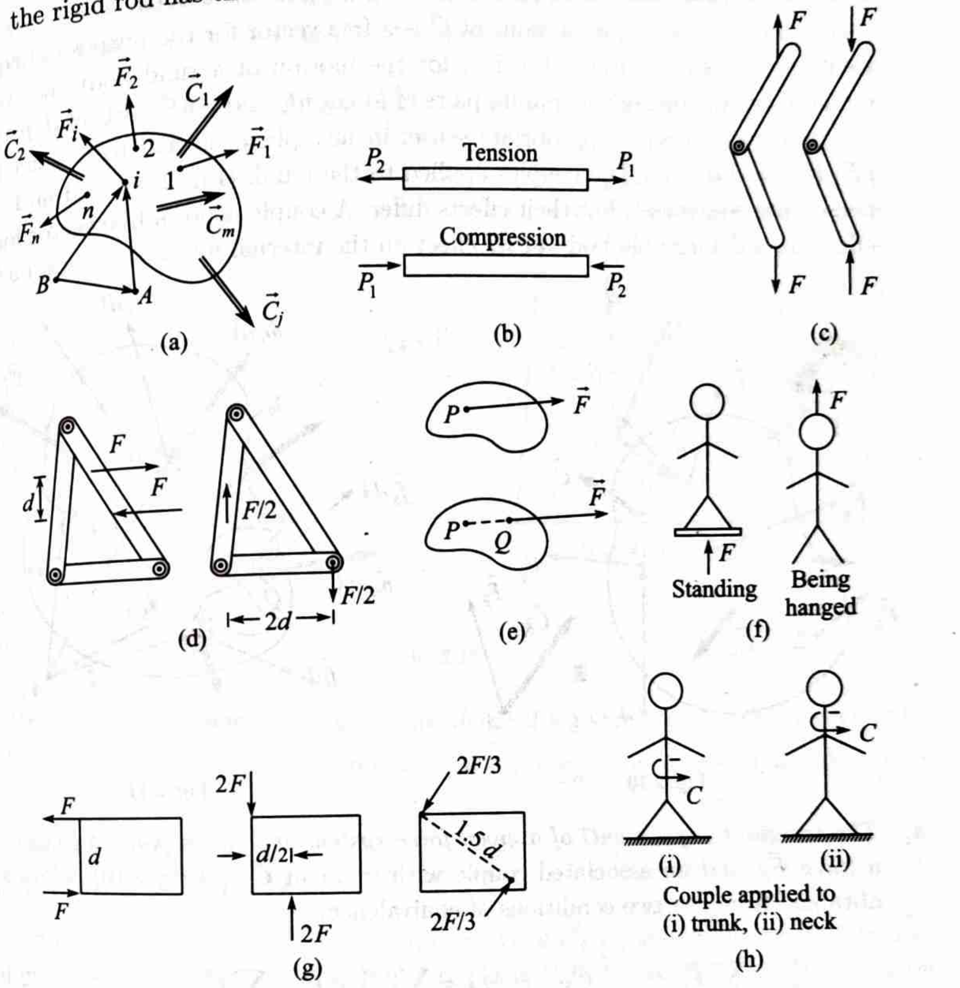


Fig. 2.9

3. Two equivalent force systems, in general, impart different motion to a deformable system and to a system of rigid bodies, as illustrated for equivalent force systems in Fig. 2.9c. However, if the system of rigid bodies together form a rigid system, then equivalent force systems acting on it produce the same motion (Fig. 2.9d).

   Force is a transmissible vector for the purpose of equivalent force systems (Fig. 2.9e). Thus, the magnitude, line of action and sense of a force are significant for its effect on the motion of a rigid body and its point of application is immaterial. The force on the feet of a person (a deformable body) is equivalent to the force on the neck (Fig. 2.9f) but their effect is completely different! Force is not a transmissible vector for its effect on a deformable body.

A moment or a couple of moment $\vec{C}$ is a free vector for the purpose of equivalent force systems (Fig. 2.9g), i.e., for the motion of a rigid body, it can be replaced by any one of the infinite pairs of forces $p\vec{F}$ and $-p\vec{F}$, distant d/p apart in any direction with appropriate sense, in any plane normal to $\vec{C}$ such that $pF(d/p) = Fd = C$. The couples applied to the trunk (Fig. 2.9h) and neck of a person are equivalent, but their effects differ. A couple is not a free vector for its effect on a deformable body or its effect on the internal forces in a rigid body.

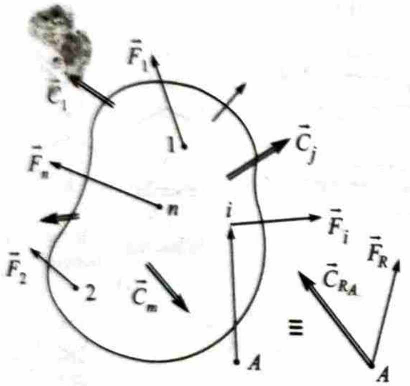


Fig. 2.10

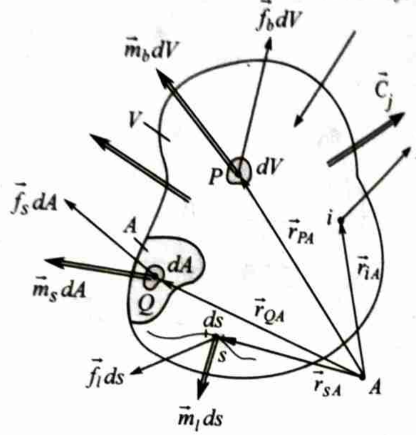


Fig. 2.11

4. *The resultant (equivalent) of a given force system at a given point A* consists of a force $\vec{F}_R$ and an associated couple with moment $\vec{C}_{R_A}$ (Fig. 2.10), which are obtained from the two conditions of equivalence:

$$\vec{F}_R = \sum_i \vec{F}_i, \qquad \vec{C}_{R_A} = \vec{M}_A = \sum_i \overrightarrow{Ai} \times \vec{F}_i + \sum_j \vec{C}_j \qquad (2.16)$$

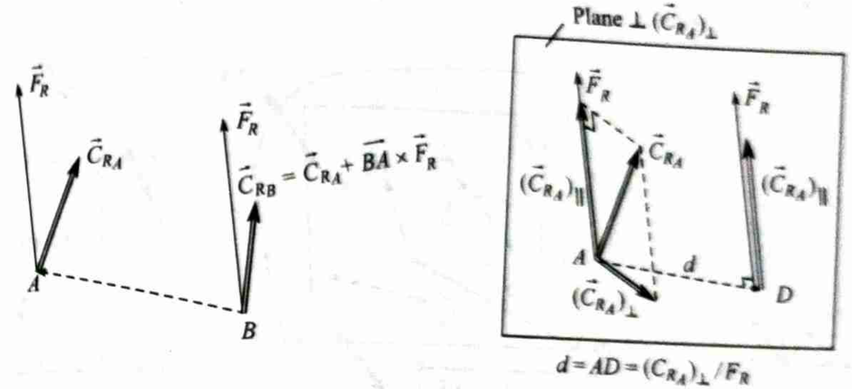
The subscript A has not been used in $\vec{F}_R$ since it is independent of A. $\vec{C}_{R_A}$ depends on A. The integrals $\int_s \vec{f}_l\, ds$, $\int_A \vec{f}_s\, dA$, $\int_V \vec{f}_b\, dV$ and $\int_s (\vec{r}_{SA} \times \vec{f}_l + \vec{m}_l) ds$, $\int_A (\vec{r}_{QA} \times \vec{f}_s + \vec{m}_s) dA$, $\int_V (\vec{r}_{PA} \times \vec{f}_b + \vec{m}_b) dV$ are added to the right hand side of eqs $(2.16)_1$ and $(2.16)_2$ respectively, if the given force system includes distributed forces and distributed moments (Fig. 2.11) $\vec{m}_l, \vec{m}_s, \vec{m}_b$ distributed over a line, a surface and a volume as N.m/m, N.m/m^2, N.m/m^3, respectively.

The direction cosines l, m, n of $\vec{F}_R$ can be obtained from

$$l = \cos\alpha = \frac{F_{R_x}}{F_R}, \qquad m = \cos\beta = \frac{F_{R_y}}{F_R}, \qquad n = \cos\gamma = \frac{F_{R_z}}{F_R}$$

The direction cosines of $\vec{C}_{R_A}$ are obtained similarly. For the case of $\vec{F}_R = F_{R_x}\vec{i} + F_{R_y}\vec{j}$, angle α that $\vec{F}_R$ makes with x-axis is given by

$$\cos\alpha = \frac{F_{R_x}}{F_R}, \qquad \sin\alpha = \frac{F_{R_y}}{F_R}, \qquad \tan\alpha = \frac{F_{R_y}}{F_{R_x}}$$



(a) (b)

Fig. 2.12

5. *The simplest resultant (simplest equivalent) of a given force system is a wrench.*

Proof: Let the resultant at point A be $\vec{F}_R$, $\vec{C}_{R_A}$. The resultant at point B is $\vec{F}_R$, $\vec{C}_{R_B}$ (Fig. 2.12a) with

$$\vec{M}_B = \vec{C}_{R_B} = \vec{C}_{R_A} + \overrightarrow{BA} \times \vec{F}_R = (\vec{C}_{R_A})_\parallel + (\vec{C}_{R_A})_\perp + \overrightarrow{BA} \times \vec{F}_R \qquad (a)$$

where $(\vec{C}_{R_A})_\parallel$, $(\vec{C}_{R_A})_\perp$ are the components of $\vec{C}_{R_A}$ parallel and normal to $\vec{F}_R$. It is possible to choose an appropriate point B such that the last two terms on the right hand side of eq. (a) cancel each other, since $\overrightarrow{BA} \times \vec{F}_R \perp \vec{F}_R$. For such a point, say D, the resultant is a wrench consisting of $\vec{F}_R$, $(\vec{C}_{R_A})_\parallel$ (Fig. 2.12b).

It follows that if $(\vec{C}_{R_A})_\parallel = \vec{0}$, i.e., *if* $\vec{C}_{R_A} \cdot \vec{F}_R = 0$, *then the simplest resultant is a single force* $\vec{F}_R$ *if* $\vec{F}_R \neq \vec{0}$, *or a single couple* $\vec{C}_{R_A}$ *if* $\vec{F}_R = \vec{0}$, *or a null system if* $\vec{F}_R = \vec{0}$, $\vec{C}_{R_A} = \vec{0}$. Such a simplest resultant exists for the coplanar and parallel force systems, defined next, since the condition $\vec{C}_{R_O} \cdot \vec{F}_R = 0$ holds.

The simplest resultant of a concurrent force system (Fig. 2.13a) is a single force through the point of concurrence. *A coplanar force system* (Fig. 2.13b) consists of forces in one plane (say xy-plane) and couples with moments along the normal to this plane (i.e., parallel to z). For the origin O, $\vec{C}_{R_O} \cdot \vec{F}_R = M_{O_z}\vec{k} \cdot (F_x\vec{i} + F_y\vec{j}) = 0$. For the case of $\vec{F}_R \neq \vec{0}$, the line of action of the simplest resultant $\vec{F}_R = F_x\vec{\mathrm{i}} + F_y\vec{\mathrm{j}}$ is at distance d from O and cuts the x and y axes at $A(a, 0, 0)$ and $B(0, -b, 0)$ such that

$$aF_y = bF_x = F_R d = M_{O_z} = \vec{C}_{R_O} \cdot \vec{\mathrm{k}}$$

$$\Rightarrow \quad d = \frac{M_{O_z}}{F_R}, \quad a = \frac{M_{O_z}}{F_y}, \quad b = \frac{M_{O_z}}{F_x}, \quad \tan\alpha = \frac{F_y}{F_x}$$

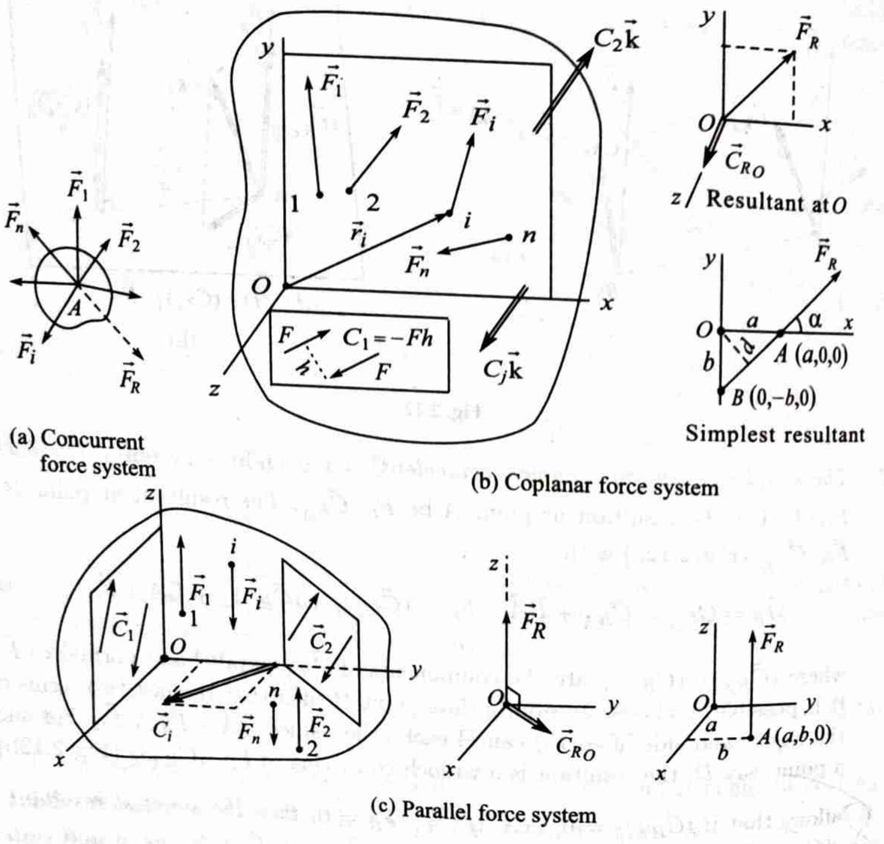


Fig. 2.13

A parallel force system (Fig. 2.13c) consists of forces parallel to a line (say z-axis) and couples with moments normal to this line (i.e., having components along $\vec{\mathrm{i}}$ and $\vec{\mathrm{j}}$): $\vec{C}_{R_O} \cdot \vec{F}_R = (M_{O_x}\vec{\mathrm{i}} + M_{O_y}\vec{\mathrm{j}}) \cdot F_R\vec{\mathrm{k}} = 0$. For $\vec{F}_R \neq \vec{0}$, the simplest resultant $\vec{F}_R = F_R\vec{\mathrm{k}}$ intersects the xy-plane at $(a, b, 0) \ni$

$$(a\vec{\mathrm{i}} + b\vec{\mathrm{j}}) \times F_R\vec{\mathrm{k}} = \vec{C}_{R_O} = M_{O_x}\vec{\mathrm{i}} + M_{O_y}\vec{\mathrm{j}}$$

$$\Rightarrow \quad a = -\frac{M_{O_y}}{F_R}, \qquad b = \frac{M_{O_x}}{F_R}$$

The resultant of a force system may be computed for finding $\vec{F}$ and $\vec{M}_O$. Although the simplest resultant is a wrench, we usually use the resultant force–couple system at a convenient point A since the latter is easier to determine.

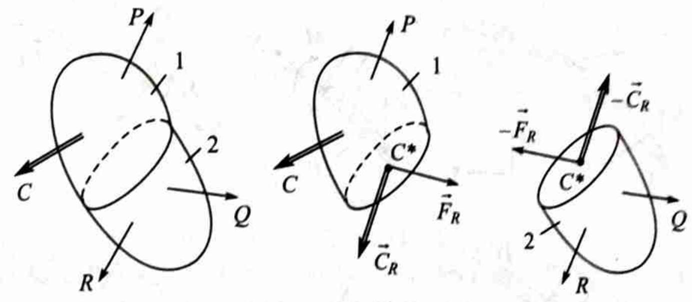


Fig. 2.14

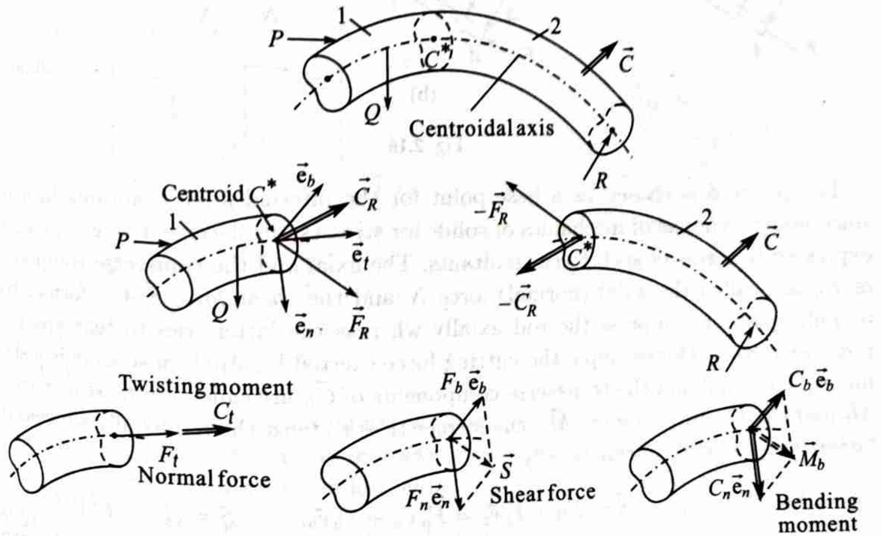


Fig. 2.15

Internal force resultants: For an internal section of a body, the distributed force across the section can be replaced by its resultant at the centroid C^* of the cross-section, consisting of a resultant force $\vec{F}_R$ and a resultant couple with moment $\vec{C}_R$ at C^* (Fig. 2.14). For the case of a *thin bar* (i.e., a body whose transverse dimensions are small compared to the axial length), these can be resolved (Fig. 2.15) in terms of the right-handed unit triad (tangent $\vec{e}_t$, normal $\vec{e}_n$, binormal $\vec{e}_b$) of its centroidal axis joining the centroids of its cross-sections:

$$\vec{F}_R = F_t\vec{e}_t + F_n\vec{e}_n + F_b\vec{e}_b, \qquad \vec{C}_R = C_t\vec{e}_t + C_n\vec{e}_n + C_b\vec{e}_b \tag{2.17}$$

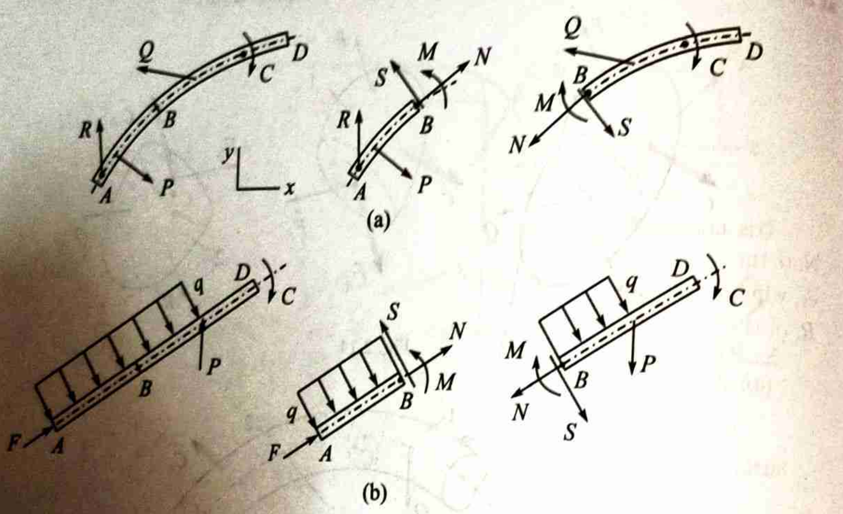


Fig. 2.16

The centroid is chosen as a base point for the internal force resultants in a bar, since many formulae of mechanics of solids for stresses, displacement, energy, etc., are expressed in terms of such force resultants. The axial and the transverse components of $\vec{F}_R$ are called the axial (normal) force N and the *shear force* $\vec{S}$, the former tries to pull apart or compress the rod axially whereas the latter tries to tear apart the rod transversely (for example the cutting force exerted by a pair of scissors is a shear force). The axial and the transverse components of $\vec{C}_R$ are called the *twisting moment* M_t and the *bending moment* $\vec{M}_b$, the former tries to twist the rod axially whereas the latter tries to bend it transversely.

$$\begin{aligned} N &= \vec{F}_R\cdot\vec{e}_t = F_t, \quad & \vec{S} &= \vec{F}_R - F_t\vec{e}_t = F_n\vec{e}_n + F_b\vec{e}_b, \quad & S &= (F_n^2+F_b^2)^{1/2} \\ M_t &= \vec{C}_R\cdot\vec{e}_t = C_t, \quad & \vec{M}_b &= \vec{C}_R - M_t\vec{e}_t = C_n\vec{e}_n + C_b\vec{e}_b, \quad & M_b &= (C_n^2+C_b^2)^{1/2} \end{aligned} \tag{2.18}$$

For a bar with a plane centroidal axis, say in the xy-plane, subjected to a coplanar force system in the plane of the centroidal axis, the components F_{Rz}, C_{Rx}, C_{Ry} of $\vec{F}_R, \vec{C}_R$ are zero, and the internal force resultants are often shown as a normal force N, a shear force S and a bending moment M (Fig. 2.16).

2.4.1 Centre of parallel forces

Consider a parallel force system (Fig. 2.17) consisting only of forces $F_i\vec{e}$ acting at $\vec{r}_i$, $i = 1, \ldots, n$ such that $\vec{F}_R \neq \vec{0}$. Its resultant $\vec{F}_R$ always passes through a point R called the *centre of parallel forces*, irrespective of the direction of $\vec{e}$, since

$$\vec{F}_R = \left(\sum_i F_i\right)\vec{e}, \qquad \vec{M}_O = \sum_i \vec{r}_i \times F_i\vec{e} = \vec{r}_R \times \vec{F}_R = \vec{r}_R \times \left(\sum_i F_i\right)\vec{e} \quad (1)$$

$$\Rightarrow \quad \left[\left(\sum_i F_i\right)\vec{r}_R - \left(\sum_i \vec{r}_i F_i\right)\right] \times \vec{e} = \vec{0}, \ \forall\, \vec{e}$$

$$\Rightarrow \quad \vec{r}_R = \frac{\sum_i \vec{r}_i F_i}{\sum_i F_i}, \ \forall\, \vec{e} \tag{2.19}$$

Note that for a specific direction $\vec{e} = \vec{e}_1$, eq. (1) yields the equation of a line, parallel to $\vec{e}_1$, which passes through R. Hence, eq. (2.19) yields the common point of intersection, R, of the various lines obtained by changing the direction of $\vec{e}$.

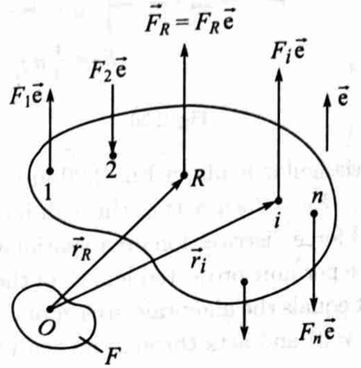


Fig. 2.17

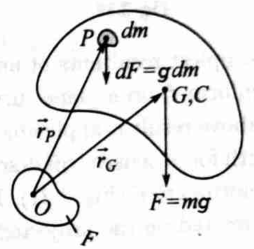


Fig. 2.18

For a distributed force, the summation in eq. (2.19) is replaced by integration. The centre of the distributed parallel *uniform gravitational force* $dF = g\,dm$ on mass element dm of a body of mass m, called *centre of gravity* G, coincides with C (Fig. 2.18):

$$\vec{r}_G = \vec{r}_R = \frac{\sum_i \vec{r}_i F_i}{\sum_i F_i} = \frac{\int_F \vec{r}_P\, dF}{\int_F dF} = \frac{\int_m \vec{r}_P\, g\, dm}{\int_m g\, dm} = \frac{\int_m \vec{r}_P\, dm}{\int_m dm} = \vec{r}_C$$

Consider a parallel distributed force $f_l\,\vec{n}$ (N/m) normal to a plane curve C_1. At every point of the curve, erect an ordinate parallel to $\vec{n}$ equal to f_l. The surface so generated is called a *loading surface* (Fig. 2.19). Its *simplest resultant* equals the algebraic *area A of the loading surface* (if $A \neq 0$) and *acts through its centroid* C^* :

$$dF = f_l\, ds = dA, \qquad \vec{F}_R = \int_F dF\, \vec{n} = \int_A dA\, \vec{n} = A\, \vec{n} \tag{1}$$

$$\vec{r}_{C^*} = \frac{\int_A \vec{r}_P\, dA}{A} = \frac{\int (\vec{r}_Q + 0.5 f_l \vec{n})\, dF}{F}$$

$$= \frac{\int \vec{r}_Q\, dF}{F} + (\ldots)\vec{n} = \vec{r}_R + (\ldots)\vec{n} \tag{2}$$

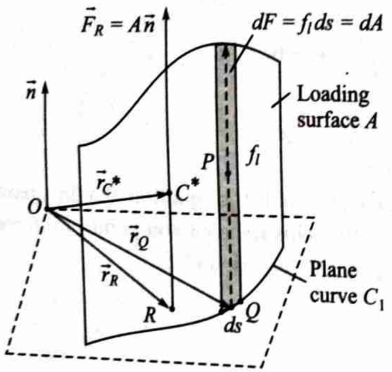


Fig. 2.19

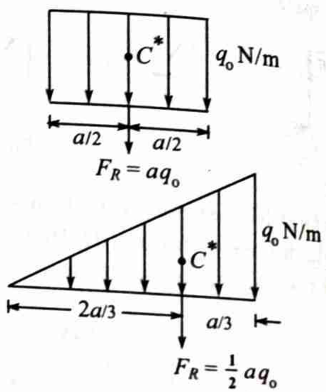


Fig. 2.20

The simplest resultants of uniform and triangular loads in Fig. 2.20 are as shown. For a uniform f_l on a plane curve of length s, $\vec{F}_R = f_l s\, \vec{n}$ acting through its centroid.

The above result is applicable to a parallel force distribution on a spatial or a plane curve with force density $f_l\vec{n}$ described as force per unit projected length of the curve in a plane normal to $\vec{n}$ (Fig. 2.21). The resultant equals the algebraic area A of the loading surface erected on the projected curve (if $A \neq 0$) and acts through its centroid C^*.

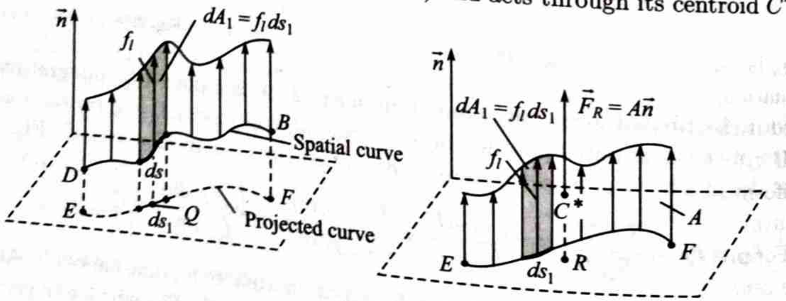


Fig. 2.21

Consider a parallel distributed force $f_s\vec{n}$ (N/m^2) normal to a plane area A. At every point of the area, erect an ordinate parallel to $\vec{n}$ equal to f_s. The solid so generated is

called a *pressure space* (Fig. 2.22a). Its *simplest resultant* equals the algebraic *volume V of the pressure space* (if $V \neq 0$) and *acts through its centroid* C^*.

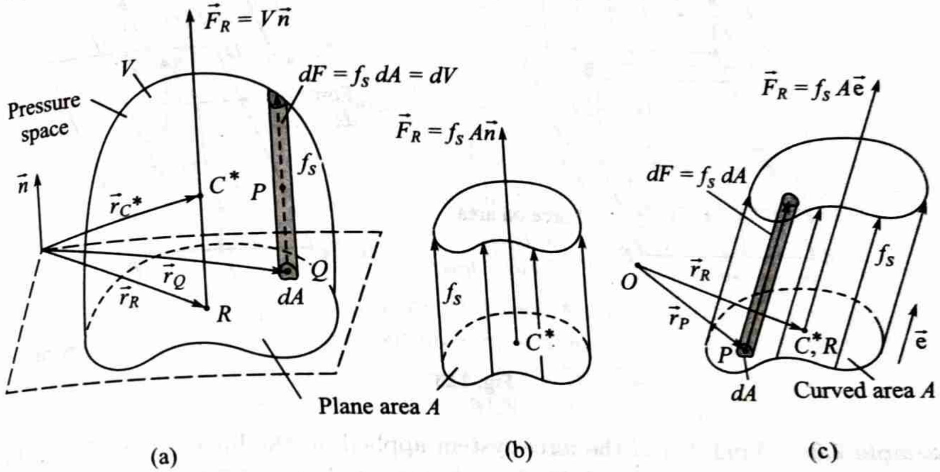


Fig. 2.22

$$dF = f_s\, dA = dV, \qquad \vec{F}_R = \int_F dF\, \vec{n} = \int_V dV\, \vec{n} = V\, \vec{n} \tag{3}$$

$$\vec{r}_{C^*} = \frac{\int_V \vec{r}_P\, dV}{V} = \frac{\int (\vec{r}_Q + 0.5 f_s \vec{n})\, dF}{F}$$

$$= \frac{\int \vec{r}_Q\, dF}{F} + (\ldots)\vec{n} = \vec{r}_R + (\ldots)\vec{n} \tag{4}$$

If f_s is uniform, then $\vec{F}_R = f_s A\vec{n}$ acts at the centroid of the plane area (Fig. 2.22b).[3] Equations (1) to (4) imply that the tabulated values of area, volume, centroids can be used if the loading surface/pressure space is composed of basic shapes.

If $f_s(x, y) = f_s(x)$ acts on a rectangular area in the xy-plane (Fig. 2.23), which is uniform along the width b, then the surface force distribution can be replaced by an equivalent line force distribution $f_l = f_s b$ on the centre line of the rectangle.

For pressure load on a plane area, the point R where the resultant force acts is called the *centre of pressure*. In general, the simplest resultant of a pressure distribution on a curved surface may be a wrench and the centre of pressure may not exist.

[3] Resultant of uniform distributed parallel force $f_s\vec{e}$ (N/m^2) (Fig. 2.22c) on area A (plane/curved) is $\vec{F}_R = \int_A f_s\, dA\, \vec{e} = f_s A\vec{e}$ through its centroid C^* as $\vec{r}_R = \int_A \vec{r}_P f_s\, dA / \int_A f_s\, dA = \int_A \vec{r}_P dA/A = \vec{r}_{C^*}$. In particular, this result applies to uniform tangential parallel force distribution.

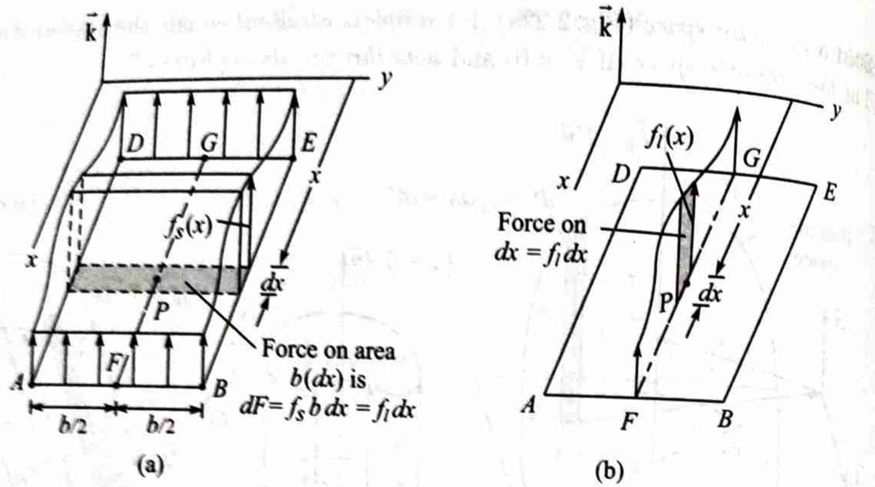


Fig. 2.23

Example 2.8: Find $\vec{M}_A$ of the force system applied on the hinged bar (Fig. E2.8a) and the resultant (equivalent) of this force system at A. $P = 5.773\,\text{kN}$.

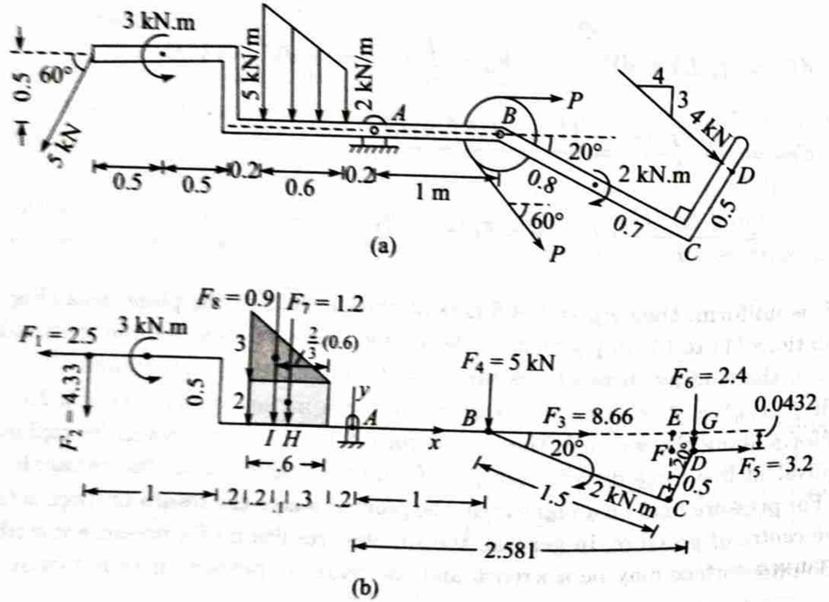


Fig. E2.8

Solution: The coplanar forces are resolved into suitable components (Fig. E2.8b). The two forces on the pulley are equivalent to F_3 and F_4 at B since their $\vec{M}_B = \vec{0}$.

$$F_1 = 5\cos 60° = 2.5\,\text{kN}, \qquad F_4 = P\sin 60° = 5\,\text{kN}$$

$$F_2 = 5\sin 60° = 4.330\,\text{kN}, \qquad F_5 = 4[4/(3^2+4^2)^{1/2}] = 3.2\,\text{kN}$$

$$F_3 = P + P\cos 60° = 8.660\,\text{kN}, \qquad F_6 = 4[3/(3^2+4^2)^{1/2}] = 2.4\,\text{kN}$$

$$BE = 1.5\cos 20° = 1.410\,\text{m}, \qquad FD = 0.5\sin 20° = 0.1710\,\text{m}$$

$$CE = 1.5\sin 20° = 0.5130\,\text{m}, \qquad DG = EF = CE - CF = 0.0432\,\text{m}$$

$$CF = 0.5\cos 20° = 0.4698\,\text{m}, \qquad AG = 1 + 1.410 + 0.1710 = 2.581\,\text{m}$$

The trapezoidal distributed force is decomposed into a rectangle and a triangle. The resultant of the rectangular distribution is $F_7 = \text{area} = 0.6(2) = 1.2\,\text{kN}$, and acts at H through its centroid with $AH = 0.2 + 0.6/2 = 0.5\,\text{m}$. The resultant of the triangular distribution is $F_8 = \text{area} = \frac{1}{2}(0.6)0.3 = 0.9\,\text{kN}$, and acts at I through its centroid with $AI = 0.2 + \frac{2}{3}(0.6) = 0.6\,\text{m}$.

For coplanar force system, instead of using cross product, it is easier to compute the moment of each force component about A as the product of its magnitude with its perpendicular distance from A and assign +ve or −ve sign according as it tends to rotate the bar from x ($\vec{i}$) to y ($\vec{j}$), or y ($\vec{i}$) to x ($\vec{j}$). The same applies to couples. Thus,

$$\begin{aligned}\vec{M}_A &= [2.5\times 0.5 + 4.330\times 2 + 8.660\times 0 - 5\times 1 + 3.2\times 0.0432 \\ &\quad -2.4\times 2.581 + 1.2\times 0.5 + 0.9\times 0.6 + 3 - 2]\,\vec{k} = 0.9938\,\vec{k}\ \text{kN.m}\end{aligned}$$

The resultant (equivalent) of the given force system at A is a force–couple system consisting of a force $\vec{F}_R$ and a couple $\vec{C}_R$ with $\vec{C}_R = \vec{M}_A = 0.9938\,\vec{k}$ kN.m and

$$\begin{aligned}\vec{F}_R &= \sum_i \vec{F}_i = -2.5\,\vec{i} - 4.330\,\vec{j} + 8.660\,\vec{i} - 5\,\vec{j} + 3.2\,\vec{i} - 2.4\,\vec{j} - 1.2\,\vec{j} - 0.9\,\vec{j} \\ &= 9.36\,\vec{i} - 13.83\,\vec{j}\ \text{kN}\end{aligned}$$

Example 2.9: A rectangular plate (Fig. E2.9) of weight 2 kN is hinged along AB. Find $\vec{M}_A$ and M_{AB} of the force system shown and its resultant (equivalent) at A. G lies in the xz-plane. $\alpha = 120°$, $\beta = 60°$, $\gamma = 45°$, $OG = 4\,\text{m}$, $DF = 3\,\text{m}$, weight $F_5 = 2\,\text{kN}$ and the arrow of moment 2 kN.m is in the plane $ABED$.

Solution: Let $\vec{n}$ be normal to the plane. The coordinates and position vectors are:

$$A(4,0,0),\quad B(0,0,3),\quad D(4,2,0),\quad E(0,2,3)$$

$$G(4\sin 30°, 0, 4\cos 30°) = (2, 0, 3.464),\quad H(0,1,3)$$

$$\vec{r}_A = 4\vec{i}, \quad \vec{r}_B = 3\vec{k}, \quad \vec{r}_D = 4\vec{i}+2\vec{j}, \quad \overrightarrow{AB} = -4\vec{i}+3\vec{k}$$

$$\vec{r}_E = 2\vec{j}+3\vec{k}, \quad \vec{r}_G = 2\vec{i}+3.464\vec{k}, \quad \vec{r}_H = \vec{j}+3\vec{k}\text{ m}$$

$$\vec{r}_C = (\vec{r}_A+\vec{r}_B+\vec{r}_D+\vec{r}_E)/4 = 2\vec{i}+\vec{j}+1.5\vec{k}\text{ m}$$

$$\vec{e} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{-4\vec{i}+3\vec{k}}{(4^2+3^2)^{1/2}} = -0.8\vec{i}+0.6\vec{k}$$

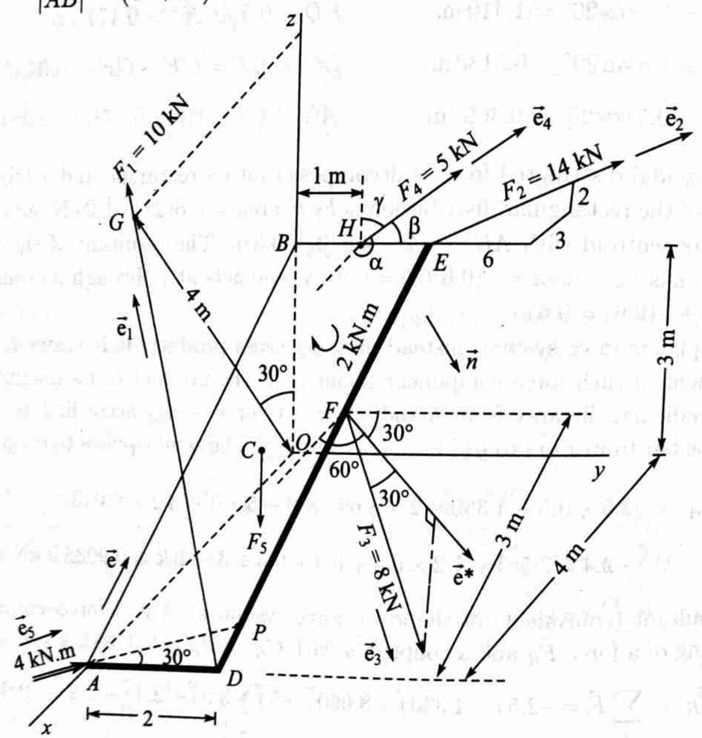


Fig. E2.9

$$\overrightarrow{AB}\times\overrightarrow{AD} = (-4\vec{i}+3\vec{k})\times 2\vec{j} = -6\vec{i}-8\vec{k}$$

$$\overrightarrow{DG} = \vec{r}_G - \vec{r}_D = -2\vec{i}-2\vec{j}+3.464\vec{k}\text{ m}$$

$$\vec{r}_F = \vec{r}_D + 3\vec{e} = 4\vec{i}+2\vec{j}+3(-0.8\vec{i}+0.6\vec{k}) = 0.6\vec{i}+2\vec{j}+1.8\vec{k}\text{ m}$$

$$\vec{n} = \frac{\overrightarrow{AB}\times\overrightarrow{AD}}{|\overrightarrow{AB}\times\overrightarrow{AD}|} = \frac{-6\vec{i}-8\vec{k}}{(6^2+8^2)^{1/2}} = -0.6\vec{i}-0.8\vec{k}$$

The two couples and the five forces are given by

$$\vec{C}_1 = 2\vec{n} = 2(-0.6\vec{i}-0.8\vec{k}) = -1.2\vec{i}-1.6\vec{k}\text{ kN.m}$$

$$\vec{C}_2 = 4\vec{e}_5 = 4(\cos 30^\circ\vec{j}+\sin 30^\circ\vec{e}) = 4[0.866\vec{j}+0.5(-0.8\vec{i}+0.6\vec{k})]$$

$$= -1.6\vec{i}+3.464\vec{j}+1.2\vec{k}\text{ kN.m}$$

$$\vec{F}_1 = 10\vec{e}_1 = 10\frac{\overrightarrow{DG}}{|\overrightarrow{DG}|} = 10\frac{-2\vec{i}-2\vec{j}+3.464\vec{k}}{(2^2+2^2+3.464^2)^{1/2}}$$

$$= -4.472\vec{i}-4.472\vec{j}+7.746\vec{k}\text{ kN}$$

$$\vec{F}_2 = 14\vec{e}_2 = 14\frac{6\vec{j}-3\vec{i}+2\vec{k}}{(6^2+3^2+2^2)^{1/2}} = -6\vec{i}+12\vec{j}+4\vec{k}\text{ kN}$$

$$\vec{F}_3 = 8\vec{e}_3 = 8[\cos 30^\circ\vec{e}^* - \sin 30^\circ\vec{k}]$$

$$= 8[0.866(\cos 60^\circ\vec{i}+\sin 60^\circ\vec{j}) - 0.5\vec{k}] = 3.464\vec{i}+6\vec{j}-4\vec{k}\text{ kN}$$

$$\vec{F}_4 = 5\vec{e}_4 = 5(\cos\alpha\vec{i}+\cos\beta\vec{j}+\cos\gamma\vec{k})$$

$$= 5(\cos 120^\circ\vec{i}+\cos 60^\circ\vec{j}+\cos 45^\circ\vec{k}) = -2.5\vec{i}+2.5\vec{j}+3.536\vec{k}\text{ kN}$$

$$\vec{F}_5 = -2\vec{k}\text{ kN}$$

The position vectors of the points of application of the forces with respect to A are:

$$\overrightarrow{AD} = \vec{r}_D - \vec{r}_A = 2\vec{j}\text{ m}, \qquad \overrightarrow{AE} = \vec{r}_E - \vec{r}_A = -4\vec{i}+2\vec{j}+3\vec{k}\text{ m}$$

$$\overrightarrow{AH} = \vec{r}_H - \vec{r}_A = -4\vec{i}+\vec{j}+3\vec{k}\text{ m}, \qquad \overrightarrow{AC} = \vec{r}_C - \vec{r}_A = -2\vec{i}+\vec{j}+1.5\vec{k}\text{ m}$$

$$\overrightarrow{AF} = \vec{r}_F - \vec{r}_A = -2.4\vec{i}+2\vec{j}+1.8\vec{k}\text{ m}$$

The moment about A of $\vec{F}_i$ acting at I, is computed by $\overrightarrow{AI}\times F_i$:

$$\vec{M}_A = \overrightarrow{AD}\times\vec{F}_1 + \overrightarrow{AE}\times\vec{F}_2 + \overrightarrow{AF}\times\vec{F}_3 + \overrightarrow{AH}\times\vec{F}_4 + \overrightarrow{AC}\times\vec{F}_5 + \vec{C}_1 + \vec{C}_2$$

$$= (15.49\vec{i}+8.944\vec{k})+(-28\vec{i}-2\vec{j}-36\vec{k})-18.8\vec{i}-3.365\vec{j}-21.33\vec{k}$$

$$+(-3.964\vec{i}+6.644\vec{j}-7.5\vec{k})+(-2\vec{i}-4\vec{j})+(-1.2\vec{i}-1.6\vec{k})$$

$$+(-1.6\vec{i}+3.464\vec{j}+1.2\vec{k}) = -40.07\vec{i}+0.743\vec{j}-56.29\vec{k}\text{ kN.m}$$

The moment M_e of the force system about line AB is given by

$$M_e = \vec{M}_A\cdot\vec{e} = (-40.07)(-0.8)+(-56.29)(0.6) = -1.718\text{ kN.m.}$$

The resultant (equivalent) at A is a force–couple system, consisting of a force $\vec{F}_R$ and a couple $\vec{C}_R$ given by $\vec{C}_R = \vec{M}_A = -40.07\vec{i} + 0.743\vec{j} - 56.29\vec{k}$ kN.m and

$$\vec{F}_R = \sum_i \vec{F}_i = -4.472\vec{i} - 4.472\vec{j} + 7.746\vec{k} - 6\vec{i} + 12\vec{j} + 4\vec{k}$$

$$+3.464\vec{i} + 6\vec{j} - 4\vec{k} - 2.5\vec{i} + 2.5\vec{j} + 3.536\vec{k} - 2\vec{k}$$

$$= -9.508\vec{i} + 16.03\vec{j} + 9.282\vec{k} \text{ kN}$$

Example 2.10: Find the simplest resultant of the line and surface loads shown in Fig. E2.10a–f. The axisymmetric pressure p in case (e) and the circumferential force τ N/m^2 in case (f), on circular area of radius R, have been shown as a function of r.

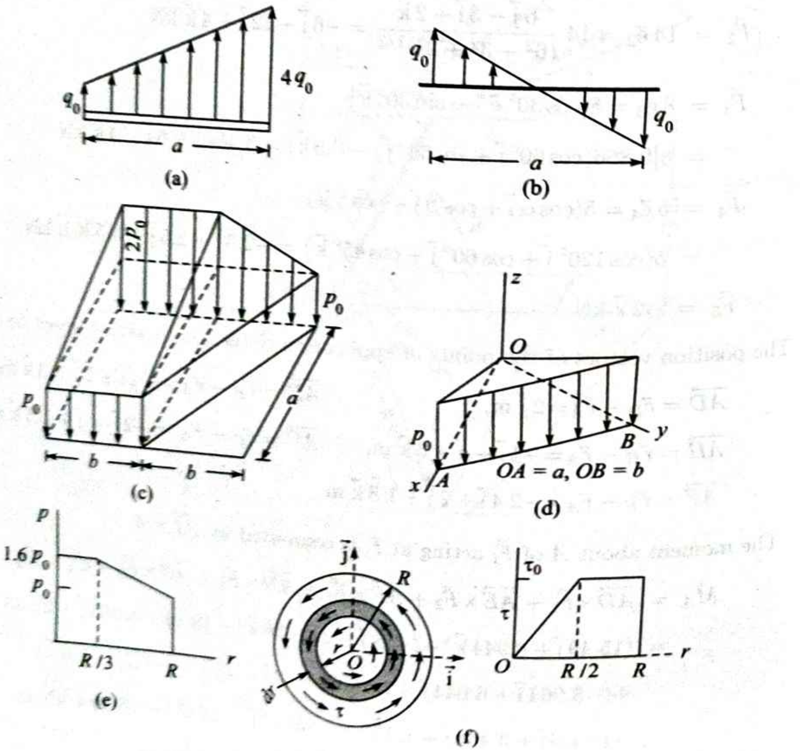


Fig. E2.10a-f

Solution: (a) Decompose the loading surface into a rectangular and a triangular area

(Fig. E2.10g). Their resultants are $F_1\vec{k}$ and $F_2\vec{k}$:

$$F_1 = (\text{area of rectangle}) = aq_0, \quad \text{through } C_1 \text{ at } \vec{r}_1 = \tfrac{1}{2}a\,\vec{i}$$

$$F_2 = (\text{area of triangle}) = \tfrac{1}{2}a(3q_0) = 1.5aq_0, \quad \text{through } C_2 \text{ at } \vec{r}_2 = \frac{2}{3}a\,\vec{i}$$

$$F_R = F_1 + F_2 = aq_0 + 1.5aq_0 = 2.5aq_0 \neq 0$$

The resultant is a force $\vec{F}_R = F_R\vec{k} = 2.5aq_0\vec{k}$ acting at point R with

$$\vec{r}_R = \frac{\vec{r}_1F_1 + \vec{r}_2F_2}{F_R} = [\tfrac{1}{2}a\,\vec{i}(aq_0) + \tfrac{2}{3}a\,\vec{i}(1.5aq_0)]/2.5aq_0 = 0.6a\,\vec{i}$$

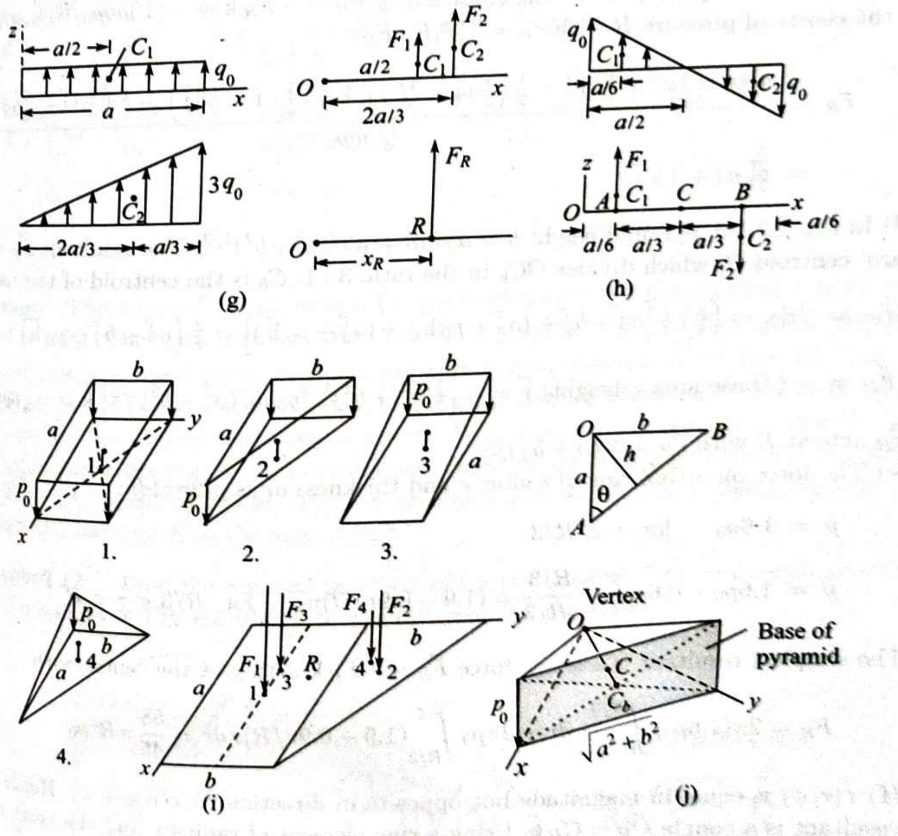


Fig. E2.10g-j

(b) The resultants of the two triangular loadings are $F_1 = F_2 = \frac{1}{2}(\frac{1}{2}a)q_0 = aq_0/4$, acting through their centroids at A and B. $AC = CB = \frac{2}{3}(\frac{1}{2}a) = \frac{1}{3}a$ (Fig. E2.10h). The simplest resultant is a single couple of moment $\vec{C}_R = F_1(AB)\,\vec{j} = \frac{1}{4}aq_0(\frac{2}{3}a)\,\vec{j} = \frac{1}{6}q_0a^2\vec{j}$.

(c) Resultant F_i = volume of pressure space (Fig. E2.10i) and acts through its centroid:

1. a cuboid $a\times b\times p_0$: $F_1 = V_1 = abp_0$ at $\vec{r}_1 = \frac{1}{2}a\vec{i} + \frac{1}{2}b\vec{j}$,
2. a triangular prism: $F_2 = V_2 = \frac{1}{2}abp_0$ at $\vec{r}_2 = \frac{1}{3}a\vec{i} + (b+\frac{1}{3}b)\,\vec{j} = \frac{1}{3}a\vec{i} + \frac{4}{3}b\vec{j}$,
3. a triangular prism: $F_3 = V_3 = \frac{1}{2}abp_0$ at $\vec{r}_3 = \frac{1}{3}a\vec{i} + \frac{1}{2}b\vec{j}$,
4. a tetrahedron: $F_4 = V_4 = \frac{1}{3}(\frac{1}{2}ab)p_0 = \frac{1}{6}abp_0$ through its centroid at 1/4th of its height from each base at $\vec{r}_4 = \frac{1}{4}a\vec{i} + (b+\frac{1}{4}b)\,\vec{j} = \frac{1}{4}a\vec{i} + \frac{5}{4}b\vec{j}$.

$F_R = \sum F_i = 13abp_0/6 \neq 0 \Rightarrow$ the resultant is $\vec{F}_R = -F_R\vec{k} = -(13abp_0/6)\,\vec{k}$ acting at the centre of pressure R with $\vec{r}_R = \sum \vec{r}_iF_i/F_R$:

$$\vec{r}_R = \frac{abp_0\left[\left(\frac{1}{2}a\vec{i}+\frac{1}{2}b\vec{j}\right)+\frac{1}{2}\left(\frac{1}{3}a\vec{i}+\frac{4}{3}b\vec{j}\right)+\frac{1}{2}\left(\frac{1}{3}a\vec{i}+\frac{1}{2}b\vec{j}\right)+\frac{1}{6}\left(\frac{1}{4}a\vec{i}+\frac{5}{4}b\vec{j}\right)\right]}{\frac{13}{6}abp_0}$$

$$= \frac{21}{52}a\vec{i} + \frac{3}{4}b\vec{j}$$

(d) In Fig. E2.10j, pyramid height $h = a\sin\theta = ab/(a^2+b^2)^{1/2}$. The resultant $\vec{F}_R$ acts thro' centroid C, which divides OC_b in the ratio $3:1$. C_b is the centroid of the base.

$$\vec{r}_C = \frac{3}{4}\vec{r}_{C_b} = \left(\frac{3}{4}\right)\frac{1}{4}\left[a\vec{i}+b\vec{j}+(a\vec{i}+p_0\vec{k})+(b\vec{j}+p_0\vec{k})\right] = \frac{3}{8}\left(a\vec{i}+b\vec{j}+p_0\vec{k}\right)$$

$$\vec{F}_R = -\frac{1}{3}(\text{base area}\times\text{height})\,\vec{k} = -\left[\frac{1}{3}(a^2+b^2)^{1/2}p_0ab/(a^2+b^2)^{1/2}\right]\vec{k} = -\frac{1}{3}p_0ab\,\vec{k}$$

$\vec{F}_R$ acts at R with $\vec{r}_R = 3(a\vec{i}+b\vec{j})/8$.

(e) The force on a thin ring of radius r and thickness dr is $p(2\pi r)dr$:

$$p = 1.6p_0 \quad \text{for } r < R/3$$

$$p = 1.6p_0 - 0.6p_0\frac{r-R/3}{R-R/3} = (1.9-0.9r/R)p_0 \quad \text{for } R/3 < r < R$$

The simplest resultant is a single force $\vec{F}_R = -F_R\vec{k}$ acting at the centre with

$$F_R = 2\pi(1.6p_0)\int_0^{R/3} r\,dr + 2\pi p_0\int_{R/3}^{R}(1.9-0.9r/R)r\,dr = \frac{58}{45}\pi R^2p_0$$

(f) $\tau(r,\phi)$ is equal in magnitude but opposite in direction to $\tau(r,\phi+\pi)$. Hence, the resultant is a couple $\vec{C}_R = C_R\vec{k}$. Using a ring element of radius r and thickness dr:

$$C_R = M_{Oz} = \int_0^{\frac{1}{2}R} r\left(\frac{2\tau_0 r}{R}\right)(2\pi r)dr + \int_{\frac{1}{2}R}^{R} r\,\tau_0(2\pi r)dr = \frac{31}{48}\pi\tau_0R^3$$

Example 2.11: Find the total force due to constant pressure p on an area A on the inside of a pressure vessel (Fig. E2.11a). Also, find the resultant force due to constant pressure p on the inside of spherical and cylindrical vessels with circular and rectangular cut-outs as shown in Fig. E2.11b,c.

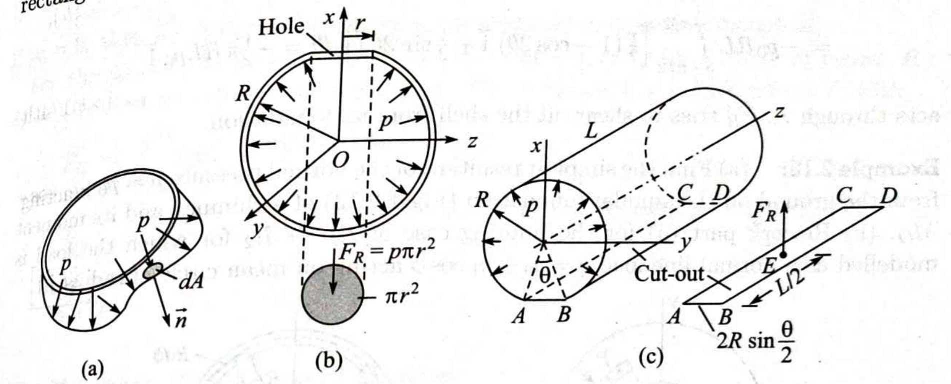


Fig. E2.11

Solution: The force $d\vec{F}$ on an area $d\vec{A} = dA\,\vec{n}$ with outward unit normal $\vec{n}$ is $d\vec{F} = pdA\vec{n} = pd\vec{A}$. The total force on area $\vec{A}$ is $\vec{F}_R = \int_{\vec{A}} p\,d\vec{A} = p\vec{A} \Rightarrow$ the total force on a closed pressure vessel under constant pressure is zero since its $\vec{A} = \vec{0}$ (see Ex. 1.2).

The projected areas of the spherical vessel on yz, zx, xy planes are $-\pi r^2, 0, 0$, i.e., $\vec{A} = -\pi r^2\vec{i}$. The simplest resultant is a force $\vec{F}_R = -p\pi r^2\vec{i}$ through the centre O.

The projected areas of the cylindrical vessel on yz, zx, xy planes are $2R\sin(\theta/2)L, 0, 0$, i.e., $\vec{A} = 2RL\sin(\theta/2)\,\vec{i}$. The simplest resultant is a force $\vec{F}_R = 2pRL\sin(\theta/2)\,\vec{i}$ through the centroid E of the area $ABDC$.

Example 2.12: Find the simplest resultant of the distributed force on the cylindrical roof of a hangar in Fig. E2.12 due to wind pressure $p = p_0\sin\theta$.

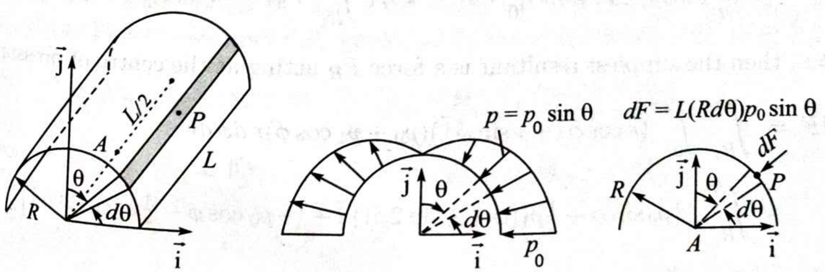


Fig. E2.12

Solution: The force on the area element $(R\,d\theta)L$ is a radial force $dF = p(R\,d\theta)L$ at P. These forces are concurrent at A at the middle of the axis. The resultant force $\vec{F}_R$:

$$\vec{F}_R = \int d\vec{F} = \int_{-\pi/2}^{\pi/2} RLp_0 \sin\theta(-\sin\theta\,\vec{\imath} - \cos\theta\,\vec{\jmath})\,d\theta$$

$$= -p_0RL\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2}(1-\cos 2\theta)\,\vec{\imath} + \tfrac{1}{2}\sin 2\theta\,\vec{\jmath}\right]d\theta = -\tfrac{1}{2}\pi RLp_0\,\vec{\imath}$$

acts through A. $\vec{F}_R$ tries to shear off the shell from its foundation.

Example 2.13: (a) Find the simplest resultant of the normal pressure $p = p_0 + p_1\cos\phi$ from the ground on the annular foundation (Fig. E2.13) of a chimney and its moment $\vec{M}_O$. (b) Rework part (a) for the limiting case of $R_1 \to R_2$ for which the load is modelled as a normal line load $q = q_0 + q_1\cos\phi$ acting on mean circle of radius R.

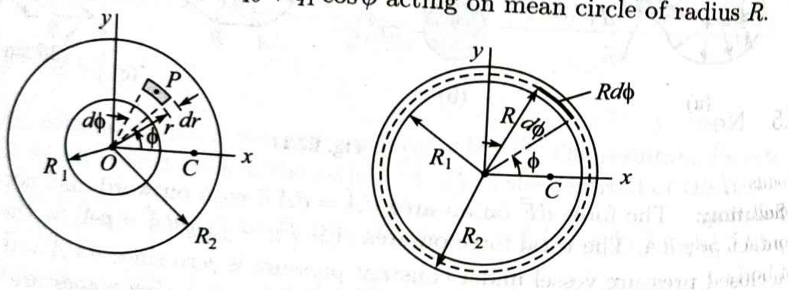


Fig. E2.13

Solution: The force on the foundation is a parallel force system with force $dF = p\,dA$ on an element of area $dA = r\,d\phi\,dr$. The total force $\vec{F}_R = F_R\,\vec{k}$ with

$$F_R = \int dF = \int_{R_1}^{R_2}\int_0^{2\pi}(p_0 + p_1\cos\phi)r\,d\phi\,dr$$

$$= \int_{R_1}^{R_2}(p_0\phi + p_1\sin\phi)\Big|_0^{2\pi} r\,dr = 2\pi p_0\int_{R_1}^{R_2} r\,dr = \pi p_0(R_2^2 - R_1^2)$$

If $p_0 \neq 0$, then the simplest resultant is a force $\vec{F}_R$ acting at the centre of pressure C:

$$\int \vec{r}_P\,dF = \int_{R_1}^{R_2}\int_0^{2\pi}(r\cos\phi\,\vec{\imath} + r\sin\phi\,\vec{\jmath})(p_0 + p_1\cos\phi)r\,d\phi\,dr$$

$$= \int_{R_1}^{R_2}\left[\{p_0\sin\phi + \tfrac{1}{2}p_1(\phi + \tfrac{1}{2}\sin 2\phi)\}\,\vec{\imath} + (-p_0\cos\phi - \tfrac{1}{4}p_1\cos 2\phi)\,\vec{\jmath}\right]\Big|_0^{2\pi} r^2\,dr$$

$$= p_1\pi\int_{R_1}^{R_2} r^2\,dr\,\vec{\imath}$$

$$\vec{r}_C = \frac{\int \vec{r}_P\,dF}{F_R} = \frac{p_1(R_2^3 - R_1^3)}{3p_0(R_2^2 - R_1^2)}\,\vec{\imath}$$

$$\vec{M}_O = \vec{r}_C \times \vec{F}_R = -\tfrac{1}{3}p_1\pi(R_2^3 - R_1^3)\,\vec{\jmath}$$

If $p_0 = 0$, then $\vec{F}_R = \vec{0}$ and the simplest resultant is a couple of moment $\vec{M}_O$.

For the parallel line force distribution $q = q_0 + q_1\cos\phi$ on a circle of radius R, $dF = qR\,d\phi$ on an element of length $R\,d\phi$. The resultant $\vec{F}_R = F_R\,\vec{k}$ acts at C with

$$F_R = \int_0^{2\pi}(q_0 + q_1\cos\phi)R\,d\phi = 2\pi q_0R$$

$$\vec{r}_C = \frac{\int \vec{r}_P\,dF}{F_R} = \frac{\int_0^{2\pi} R(\cos\phi\,\vec{\imath} + \sin\phi\,\vec{\jmath})(q_0 + q_1\cos\phi)R\,d\phi}{2\pi q_0R} = \frac{q_1R}{2q_0}\,\vec{\imath}$$

$$\vec{M}_O = \vec{r}_C \times \vec{F}_R = -\pi q_1R^2\,\vec{\jmath}$$

2.5 Newton's Third Law of Motion

Loads (forces and moments) of interaction between two bodies (or their parts) have equal magnitudes and opposite directions. Let body B_1 interact with body B_2 at a contact point or over a contact surface or at a distance (Fig. 2.24a,b,c). Let the action of body B_1 on body B_2 have force sum $\vec{F}_{21}$ and moment sum $\vec{M}_{O_{21}}$ about O (a fixed point of an inertial frame), and the action of B_2 on B_1 have force sum $\vec{F}_{12}$ and moment sum $\vec{M}_{O_{12}}$ about O (Fig. 2.24d,e). Then,

$$\vec{F}_{12} = -\vec{F}_{21}, \qquad \vec{M}_{O_{12}} = -\vec{M}_{O_{21}} \tag{2.20}$$

$$\Rightarrow \quad \vec{M}_{A_{12}} = -\vec{M}_{A_{21}}, \qquad \forall\,A$$

Proof: Let external load due to surroundings of B_1 and B_2 be a force sum $\vec{F}_1^*$ and moment sum $\vec{M}_{O_1}^*$ about O on B_1, and a force sum $\vec{F}_2^*$ and moment sum $\vec{M}_{O_2}^*$ about O on B_2. Apply Euler's axioms to B_1 alone, to B_2 alone and to $B_1 \cup B_2$ (Fig. 2.24f):

$$\dot{\vec{p}}_{1|I} = \vec{F}_{12} + \vec{F}_1^*, \qquad \dot{\vec{H}}_{O_1} = \vec{M}_{O_{12}} + \vec{M}_{O_1}^* \tag{a}$$

$$\dot{\vec{p}}_{2|I} = \vec{F}_{21} + \vec{F}_2^*, \qquad \dot{\vec{H}}_{O_2} = \vec{M}_{O_{21}} + \vec{M}_{O_2}^* \tag{b}$$

$$\dot{\vec{p}}_{(B_1\cup B_2)|I} = \dot{\vec{p}}_{1|I} + \dot{\vec{p}}_{2|I} = \vec{F}_1^* + \vec{F}_2^* \tag{c}$$

$$\dot{\vec{H}}_{O(B_1\cup B_2)|I} = \dot{\vec{H}}_{O_1} + \dot{\vec{H}}_{O_2} = \vec{M}_{O_1}^* + \vec{M}_{O_2}^* \tag{d}$$

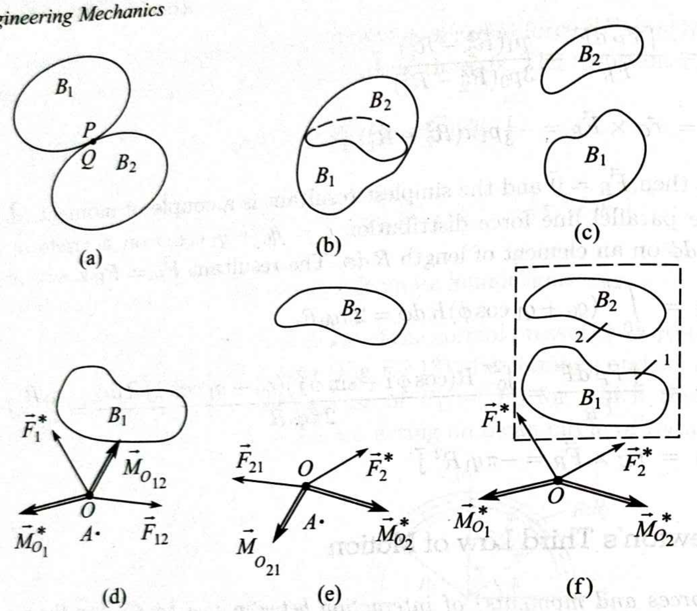


Fig. 2.24

The result is obtained by substituting from (a) and (b) in (c) and (d). Newton's 3rd law is not an axiom as it has been derived from Euler's axioms. B_1 and B_2 can be finite or infinitesimal parts of the same body. Hence, the loads of interaction have equal magnitudes and opposite directions both point-wise as well as globally in terms of the total force sum and the total moment sum of the loads of 'action' and 'reaction'. The terms 'action' and 'reaction' do not imply any cause and effect; rather, a mutual simultaneous interaction is implied. Thus, a single isolated force in nature cannot exist. In Fig. 2.14 the internal force resultants on part 1 are $\vec{F}_R$, $\vec{C}_R$ and on part 2 are $-\vec{F}_R$, $-\vec{C}_R$. The theorem of action and reaction is also valid for impulsive forces and moments.

2.6 Kinetic Energy, Power and Work

Energy is a primitive which can be transformed into various forms. Mass and energy are mutually inconvertible in classical mechanics.[4] The *kinetic energy* $T_{|F}$ of a body relative to frame F is defined by

$$T_{|F} \equiv \tfrac{1}{2}\int_m \vec{v}^{\,2}_{P|F}\, dm \tag{2.21}$$

$$= \tfrac{1}{2}\int_m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dm = \tfrac{1}{2}\int_m (\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)dm \tag{2.22}$$

[4]We simply define T here and avoid the usual circular definition of T as the *capacity to do work.*

(x, y, z) and (r, ϕ, z) are Cartesian and cylindrical coordinates of P (Fig. 2.25) in F.

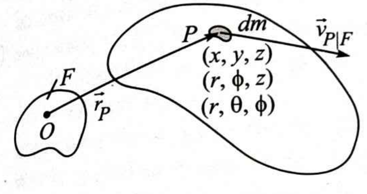


Fig. 2.25

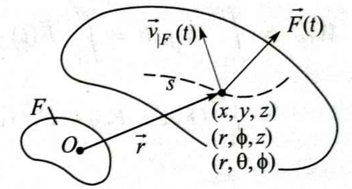


Fig. 2.26

A body with kinetic energy has the capacity to do work as in pile-driving. The *power* $\dot{W}_{|F}(t)$ with respect to frame F of a force $\vec{F}(t)$ acting on a material point with velocity $\vec{v}_{|F}(t)$ and coordinates (x, y, z), (r, ϕ, z), (r, θ, ϕ), s (Fig. 2.26), is defined by

$$\dot{W}_{|F}(t) \equiv \vec{F}(t)\cdot\vec{v}_{|F}(t) \tag{2.23}$$

$$\begin{aligned} &= F_x v_x + F_y v_y + F_z v_z = F_x\dot{x} + F_y\dot{y} + F_z\dot{z} \\ &= F_r v_r + F_\phi v_\phi + F_z v_z = F_r\dot{r} + F_\phi r\dot{\phi} + F_z\dot{z} \\ &= F_r v_r + F_\theta v_\theta + F_\phi v_\phi = F_r\dot{r} + F_\theta r\dot{\theta} + F_\phi r\dot{\phi}\sin\theta = F_t v_t = F_t\dot{s} \end{aligned} \tag{2.24}$$

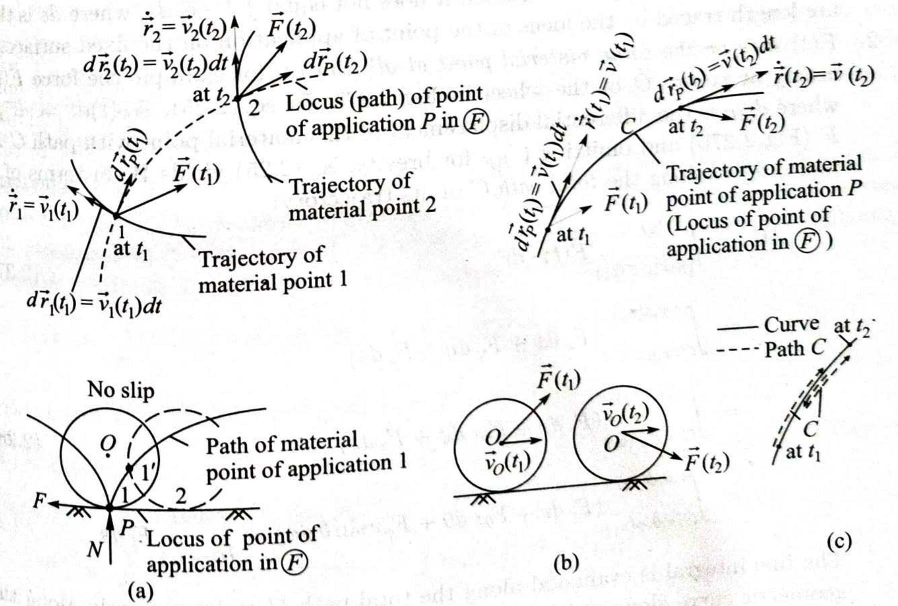


Fig. 2.27

The *work* done $W_{|F}$ with respect to frame F by a force $\vec{F}$ from t_1 to t_2 is defined by

$$W_{|F} \equiv \int_{t_1}^{t_2} \dot{W}_{|F}\, dt = \int_{t_1}^{t_2} \vec{F}(t)\cdot \vec{v}_{|F}(t)\, dt \tag{2.25}$$

$$= \int_{t_1}^{t_2} (F_x\dot{x} + F_y\dot{y} + F_z\dot{z})\, dt$$

$$= \int_{t_1}^{t_2} (F_r\dot{r} + F_\phi r\dot{\phi} + F_z\dot{z})\, dt \tag{2.26}$$

$$= \int_{t_1}^{t_2} (F_r\dot{r} + F_\theta r\dot{\theta} + F_\phi r\dot{\phi}\sin\theta)\, dt = \int_{t_1}^{t_2} F_t\dot{s}\, dt$$

We consider two cases:

1. $\vec{F}(t)$ *acts on different material points at different instants* (Fig. 2.27a). The locus of its point of application in F differs from the trajectories of the material points on which $\vec{F}$ acts at different instants. In eq. (2.25), $\vec{v}_{|F}(t)dt$ is *not equal to* $d\vec{r}$ of the same material point for all t or even $d\vec{r}_P$. For example, the contact point of a body rolling without slip on a fixed surface has zero velocity. Hence, the power of the friction force $\vec{F}$ acting at this material point is zero for all t (Fig. 2.27a) and the work done by $\vec{F}$ is zero and it does not equal $\int \vec{F}\cdot\vec{e}_t\, ds$, where ds is the arc length traced by the locus of the point of application on the fixed surface.
2. $\vec{F}(t)$ *acts on the same material point at all times* t, for example the force $\vec{F}(t)$ acting at centre O of the wheel in Fig. 2.27b. In eq. (2.25), $\vec{v}_{|F}(t)dt = d\vec{r}_{|F}$, where $d\vec{r}_{|F}$ is the differential displacement of this material point with path C in F (Fig. 2.27b) and omitting $(\ )_{|F}$ for brevity, eq. (2.25) yields W in terms of a *line integral* along the *total path* C on its trajectory:

$$W = \int_{\text{path}:\vec{r}(t_1)}^{\vec{r}(t_2)} \vec{F}(t)\cdot d\vec{r} \tag{2.27}$$

$$= \int_{C:x,y,z(t_1)}^{x,y,z(t_2)} (F_x\, dx + F_y\, dy + F_z\, dz)$$

$$= \int_{C:r,\phi,z(t_1)}^{r,\phi,z(t_2)} (F_r\, dr + F_\phi r\, d\phi + F_z\, dz) \tag{2.28}$$

$$= \int_{C:r,\theta,\phi(t_1)}^{r,\theta,\phi(t_2)} (F_r\, dr + F_\theta r\, d\theta + F_\phi r\sin\theta\, d\phi) = \int_{C:s(t_1)}^{s(t_2)} F_t\, ds$$

The line integral is evaluated along the total path C and not merely along the geometric curve along which it lies as a part of the curve may be traversed more

than once (Fig. 2.27c). The two are identical only if the material point moves in one direction on its trajectory. If the force is time-dependent, then the work expression (2.27) is not strictly a line integral.

If the force acts on a specific material point at all times and depends only on its position vector $\vec{r}$, i.e., it is of the form $\vec{F}(\vec{r})$, then eq. (2.27) is often used to compute work, provided the path C is known or the force is conservative, i.e., the work is independent of the particular path C followed between positions $\vec{r}(t_1)$ and $\vec{r}(t_2)$. The line integrals in eq. (2.28) can be evaluated by converting them to ordinary definite integrals using the equation of the path C. For example, if the path is described as $\vec{r}(\tau)$ in terms of a parameter τ, then $\vec{F}(\vec{r}) = \vec{F}(\tau)$, $dx = (dx/d\tau)d\tau$, etc., and eq. (2.27) yields

$$W = \int_{C:\vec{r}(\tau_1)}^{\vec{r}(\tau_2)} \vec{F}\cdot d\vec{r} = \int_{\tau_1}^{\tau_2} \left[F_x(\tau)\frac{dx}{d\tau} + F_y(\tau)\frac{dy}{d\tau} + F_z(\tau)\frac{dz}{d\tau}\right] d\tau$$

In particular, if $\tau = x$, with $F_x = f(x,y,z) = f[x, y(x), z(x)] = F_x(x)$, etc., then

$$W = \int_{x_1}^{x_2} \left[F_x(x) + F_y(x)\frac{dy}{dx} + F_z(x)\frac{dz}{dx}\right] dx$$

Example 2.14: Find the total work done on the disc (Fig. E2.14a,b,c) by force P and frictional force F acting at the point of contact, in the time interval in which C moves a distance d and the conveyor moves a distance b relative to the ground. In case (b), F acts on the disc in the direction of motion of the conveyor. In case (c), C moves in a circle of radius R with speed $v_C(t)$ and the point of contact at time t has velocity $3\vec{v}_C(t)$. Find also the work done by the cable tension on the disc in case (a).

Solution: (a) The instantaneous centre of rotation of the disc (Fig. E2.14d) is at I, since the velocity of the part of the cable not in contact with the disc is zero. Hence, the tension T in it does no work. $v_A = \omega(IA) = (v_C/r)(R+r)$ downward. The work done by P acting downward at C and F acting upward at A on the disc is

$$W = \int(\vec{P}\cdot\vec{v}_C + \vec{F}\cdot\vec{v}_A)\, dt = P\int v_C\, dt - F\int v_A\, dt$$

$$= [P - F(R+r)/r]\int v_C\, dt = [P - F(R+r)/r]d$$

(b) In Fig. E2.14e, $v_A = v$, $v_B = v_C + \omega R$, $v_A = v_C - \omega R \Rightarrow v_B = 2v_C - v_A = 2v_C - v$.

Work done by P and F on the disc is

$$W = \int(\vec{P}\cdot\vec{v}_B + \vec{F}\cdot\vec{v}_A)\, dt = P\int v_B\, dt + F\int v_A\, dt$$

$$= P\int(2v_C - v)\, dt + F\int v\, dt = P(2d - b) + Fb$$

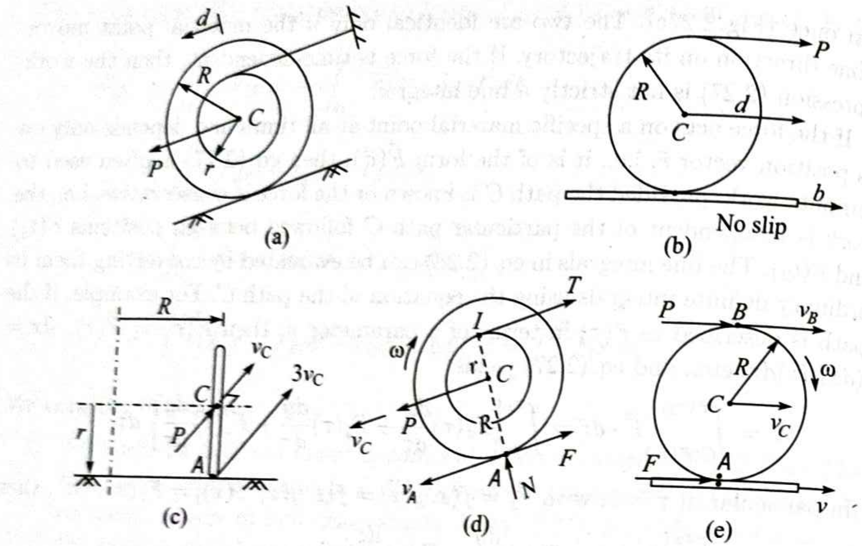


Fig. E2.14

(c) Work done by P and F (in the direction opposite to $\vec{v}_A$) on the disc is

$$W = \int (\vec{P} \cdot \vec{v}_C + \vec{F} \cdot \vec{v}_A)\, dt = \int [Pv_C - F(3v_C)]\, dt = (P - 3F)d$$

2.7 Expressions of some Entities in Terms of C

$\vec{p}_{|F}$, $\vec{H}_{A|F}$, $T_{|F}$, $\dot{W}_{|F}$ can be expressed in terms of the centre of mass C (Fig. 2.28) as:

$$\vec{p}_{|F} = m\vec{v}_{C|F} \tag{2.29}$$

$$\vec{H}_{A|F} = \vec{H}_{C|F} + \vec{r}_{CA} \times m\vec{v}_{CA|F} \tag{2.30}$$

$$T_{|F} = \tfrac{1}{2}mv^2_{C|F} + \tfrac{1}{2}\int_m v^2_{PC|F}\, dm \tag{2.31}$$

$$\dot{W}_{|F} = \vec{F} \cdot \vec{v}_{C|F} + \sum_i \vec{F}_i \cdot \vec{v}_{iC|F} \tag{2.32}$$

Thus, each of $\vec{H}_A$, T, and $\dot{W}$ is a sum of two contributions—the first one is as if the body were a hypothetical mass-point of mass m and moving with C and the second one is due to the motion relative to C.

For brevity, we omit $(\ )_{|F}$ in the proofs. Using $\int_m \vec{r}_{PC}\, dm = \vec{0}$, $\int_m \vec{v}_{PC}\, dm = \vec{0}$:

$$\int_m \vec{r}_{PC}\, dm = \int_m (\vec{r}_P - \vec{r}_C)\, dm = \vec{r}_C m - \vec{r}_C m = \vec{0} \quad \Rightarrow \quad \int_m \vec{v}_{PC}\, dm = \vec{0}$$

$$\vec{H}_A = \int_m \vec{r}_{PA} \times \vec{v}_{PA}\, dm = \int_m (\vec{r}_{PC} + \vec{r}_{CA}) \times (\vec{v}_{PC} + \vec{v}_{CA})\, dm$$

$$= \int_m \vec{r}_{PC} \times \vec{v}_{PC}\, dm + \vec{r}_{CA} \times \int_m \vec{v}_{PC}\, dm$$

$$+ \int_m \vec{r}_{PC}\, dm \times \vec{v}_{CA} + \vec{r}_{CA} \times \vec{v}_{CA} \int_m dm = \vec{H}_C + \vec{r}_{CA} \times m\vec{v}_{CA}$$

$$T = \tfrac{1}{2}\int_m \vec{v}_P^2\, dm = \tfrac{1}{2}\int_m \vec{v}_P \cdot \vec{v}_P\, dm = \tfrac{1}{2}\int_m (\vec{v}_{PC} + \vec{v}_C) \cdot (\vec{v}_{PC} + \vec{v}_C)\, dm$$

$$= \tfrac{1}{2}\int_m v_{PC}^2\, dm + \tfrac{1}{2}v_C^2 \int_m dm + \int_m \vec{v}_{PC}\, dm \cdot \vec{v}_C = \tfrac{1}{2}mv_C^2 + \tfrac{1}{2}\int_m v_{PC}^2\, dm$$

$$\dot{W} = \sum_i \vec{F}_i \cdot \vec{v}_i = \sum_i \vec{F}_i \cdot (\vec{v}_{iC} + \vec{v}_C) = \sum_i \vec{F}_i \cdot \vec{v}_C + \sum_i \vec{F}_i \cdot \vec{v}_{iC}$$

$$= \vec{F} \cdot \vec{v}_C + \sum_i \vec{F}_i \cdot \vec{v}_{iC}$$

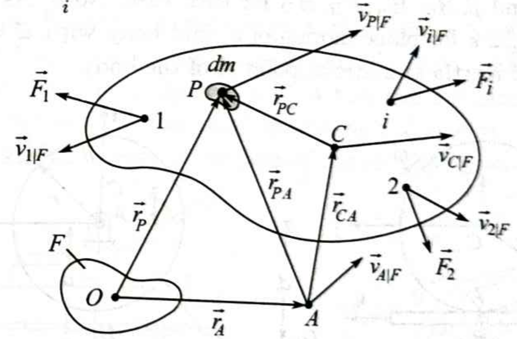


Fig. 2.28

The *rate of work $\dot{W}$ done by the external forces on a rigid body* (Fig. 2.29a) is:

$$\dot{W} = \sum_i \vec{F}_i \cdot \vec{v}_i = \sum_i \vec{F}_i \cdot (\vec{v}_C + \vec{\omega} \times \vec{r}_{iC}) = \sum_i \vec{F}_i \cdot \vec{v}_C + \vec{\omega} \cdot \sum_i \vec{r}_{iC} \times \vec{F}_i$$

$$\Rightarrow \quad \dot{W} = \vec{F} \cdot \vec{v}_C + \vec{M}_C \cdot \vec{\omega} \tag{2.33}$$

Hence, two equivalent force systems perform the same work on a rigid body, as their $\vec{F}$ and $\vec{M}_C$ are the same. Thus, the work done by the distributed uniform gravitational force on a rigid body is the same as that due to its resultant mg acting at C. The rate of work done by a moment $\vec{C}$ (or a couple–moment) acting at point Q of a rigid body (Fig. 2.29b) is obtained from eq. (2.33) or is defined as

$$\dot{W} = \vec{M}_C \cdot \vec{\omega} = \vec{C} \cdot \vec{\omega} \tag{2.34}$$

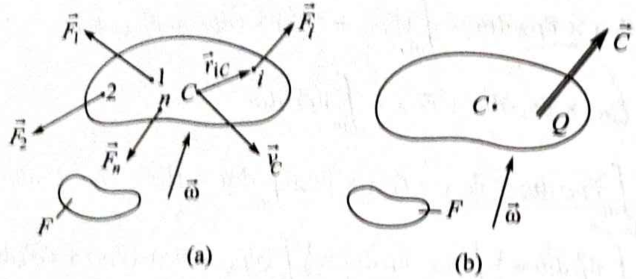


Fig. 2.29

§ Exercise 2.4: Prove that if A is a point of a rigid body, then $\dot{W} = \vec{F} \cdot \vec{v}_A + \vec{M}_A \cdot \vec{\omega}$.

Example 2.15: (a) A thin uniform disc of mass m moves with slip on a conveyor moving at velocity v_0 (Fig. E2.15a). (b) A thin non-uniform disc of mass m, centre of mass C and axial principal moment of inertia I, is rotating about a fixed axis at A (Fig. E2.15b). Find $\vec{p}, \vec{H}_C, \vec{H}_A, \vec{H}_B, \vec{H}_D$ for each case. Note: As will be shown in Chapter 3, $\vec{H}_A = I^A_{zz}\omega\vec{k}$ for plane motion of a rigid body with $\vec{\omega} = \omega\vec{k}$ and I^A_{zz} is a principal moment of inertia at material point A of the body.

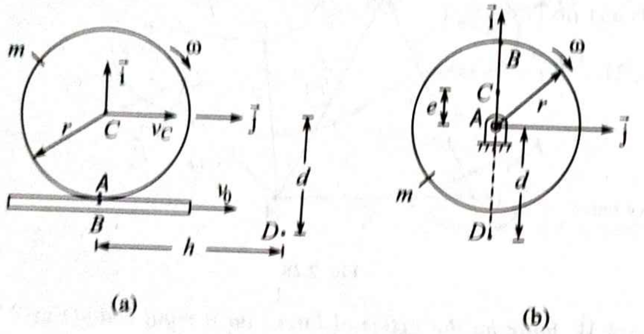


Fig. E2.15

Solution: (a) We use the parallel axes theorem for moment of inertia (see Chapter 3 for details). Note that A and C are material points of the disc, but B and D are not.

$$\vec{p} = m\vec{v}_C = mv_C\vec{j}, \qquad \vec{H}_C = I^C_{zz}\omega\vec{k} = \tfrac{1}{2}mr^2\omega\vec{k}$$

$$\vec{H}_A = \vec{H}_C + \vec{r}_{CA} \times m\vec{v}_{CA} = \tfrac{1}{2}mr^2\omega\vec{k} + r\vec{i} \times m\omega r\,\vec{j} = \tfrac{3}{2}mr^2\omega\vec{k} = I^A_{zz}\omega\vec{k}$$

$$\vec{H}_D = \vec{H}_C + \vec{r}_{CD} \times m\vec{v}_{CD} = \tfrac{1}{2}mr^2\omega\vec{k} + (d\vec{i} - h\vec{j}) \times mv_C\vec{j}$$

$$= (\tfrac{1}{2}mr^2\omega + mv_Cd)\,\vec{k} \neq I^D_{zz}\omega\vec{k}$$

$$\vec{H}_B = \vec{H}_C + \vec{r}_{CB} \times m\vec{v}_{CB} = \tfrac{1}{2}mr^2\omega\vec{k} + r\vec{i} \times m(v_C - v_0)\,\vec{j}$$

$$= [\tfrac{1}{2}mr^2\omega + m(v_C - v_0)r]\,\vec{k} \neq \vec{H}_A \tag{1}$$

$\Rightarrow$ For no slip, $v_C = v_0 + \omega r$, $\vec{H}_B = \frac{3}{2}mr^2\omega\vec{k} = \vec{H}_A$. Since $\vec{v}_B = \vec{v}_A$, B can be considered to be a point of the disc at that instant.

(b) A, B, C are material points of the disc, but D is not and $\vec{v}_C = e\omega\vec{j}$.

$$\vec{p} = m\vec{v}_C = me\omega\vec{j}, \qquad \vec{H}_A = I^A_{zz}\omega\vec{k} = I\omega\vec{k}$$

$$\vec{H}_C = I^C_{zz}\omega\vec{k} = (I^A_{zz} - me^2)\omega\vec{k} = (I - me^2)\omega\vec{k}$$

$$\vec{H}_B = I^B_{zz}\omega\vec{k} = [I^C_{zz} + m(r-e)^2]\omega\vec{k} = [I - me^2 + m(r-e)^2]\omega\vec{k}$$

$$\vec{H}_D = \vec{H}_C + \vec{r}_{CD} \times m\vec{v}_{CD} = (I - me^2)\omega\vec{k} + (d+e)\vec{i} \times me\omega\vec{j}$$

$$= (I + med)\omega\vec{k} \neq I^D_{zz}\omega\vec{k}$$

Example 2.16: (a) Find the kinetic energy of the chain of mass m of a tank relative to the tank frame 1 and the ground frame F (Fig. E2.16a). (b) Ring 1 rotates about a fixed axis (Fig. E2.16b) at ω relative to F. A thin rod 2 is pinned to the ring at C and rotates at Ω relative to 1. Two small sliders, each of mass m, slide relative to 2 at speed v. Find the kinetic energy of the pair of sliders with respect to frames 2, 1, F.

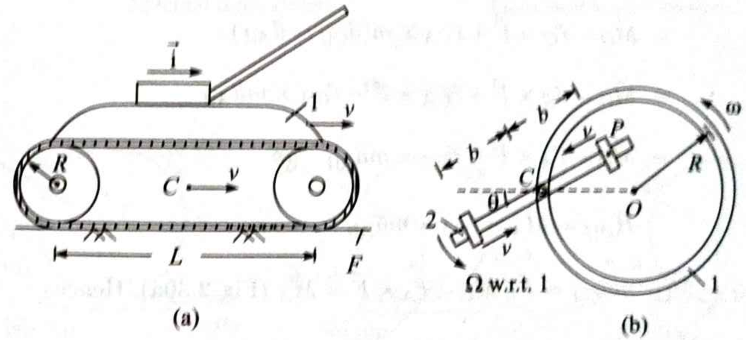


Fig. E2.16

Solution: (a) The centre of mass C of the chain is at rest relative to 1 and moves at v relative to F: $\vec{v}_{C|1} = \vec{0}$, $\vec{v}_{C|F} = v\vec{i}$. For any point P of the chain, $|\vec{v}_{PC|1}| = |\vec{v}_{PC|F}| = v$.

$$T_{|F} = \tfrac{1}{2}mv^2_{C|F} + \tfrac{1}{2}\int_m v^2_{PC|F}\,dm = \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv^2 = mv^2$$

$$T_{|1} = \tfrac{1}{2}mv^2_{C|1} + \tfrac{1}{2}\int_m v^2_{PC|1}\,dm = \tfrac{1}{2}mv^2$$

(b) The sliders have rectilinear motion in frame 2 with speed v: $T_{|2} = 2(\frac{1}{2}mv^2) = mv^2$. Using polar coordinates for sliders in frame 1 with origin at C, $\dot{\phi} = \Omega$, $r = b$, $\dot{r} = -v$,

$$T_{|1} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \times 2 = m(v^2 + b^2\Omega^2)$$

The centre of mass C of the pair of sliders has $v_{C|F} = \omega R$. $\vec{\omega}_{2|F} = (\Omega + \omega)\vec{k}$ and $\vec{r}_{PC} = b\,\vec{e}_r$, $\vec{v}_{P|2} = -v\,\vec{e}_r$. Hence, $\vec{v}_{PC|F} = \vec{\omega}_{2|F} \times \vec{r}_{PC} + \vec{v}_{P|2} = (\Omega+\omega)b\,\vec{e}_\phi - v\,\vec{e}_r$.

$$T_{|F} = \tfrac{1}{2}(2m)v_{C|F}^2 + \tfrac{1}{2}mv_{PC|F}^2 \times 2 = m[\omega^2R^2 + v^2 + (\Omega+\omega)^2b^2]$$

2.8 $\dot{\vec{H}}_A - \vec{M}_A$ Relation for Arbitrary Point A

Euler's second axiom takes the form $\dot{\vec{H}}_{A|I} = \vec{M}_A - \vec{r}_{CA} \times m\vec{a}_{A|I}$ for an arbitrary point A (Fig. 2.30) Using eq. (2.30): $\vec{H}_{A|I} = \vec{H}_{C|I} + \vec{r}_{CA} \times m\vec{v}_{CA|I}$,

$$\vec{H}_{O|I} = \vec{H}_{C|I} + \vec{r}_{CO} \times m\vec{v}_{CO|I} \quad \Rightarrow \quad \vec{H}_{C|I} = \vec{H}_{O|I} - \vec{r}_C \times m\vec{v}_{C|I}$$

$$\Rightarrow \vec{H}_{A|I} = \vec{H}_{O|I} - \vec{r}_C \times m\vec{v}_{C|I} + \vec{r}_{CA} \times m\vec{v}_{CA|I}$$

$$\begin{aligned}\dot{\vec{H}}_{A|I} &= \dot{\vec{H}}_{O|I} - \vec{v}_{C|I} \times m\vec{v}_{C|I} - \vec{r}_C \times m\vec{a}_{C|I} \\ &\quad + \vec{v}_{CA|I} \times m\vec{v}_{CA|I} + \vec{r}_{CA} \times m\vec{a}_{CA|I} \\ &= \vec{M}_O - \vec{r}_C \times \vec{F} + \vec{r}_{CA} \times m(\vec{a}_{C|I} - \vec{a}_{A|I}) \\ &= \vec{M}_O - \vec{r}_C \times \vec{F} + \vec{r}_{CA} \times \vec{F} - \vec{r}_{CA} \times m\vec{a}_{A|I} \\ &= \vec{M}_O - \vec{r}_A \times \vec{F} - \vec{r}_{CA} \times m\vec{a}_{A|I} \quad \Rightarrow\end{aligned}$$

$$\dot{\vec{H}}_{A|I} = \vec{M}_A - \vec{r}_{CA} \times m\vec{a}_{A|I} \tag{2.35}$$

since $\dot{\vec{H}}_{O|I} = \vec{M}_O$, $m\vec{a}_{C|I} = \vec{F}$, $\vec{M}_O - \vec{r}_A \times \vec{F} = \vec{M}_A$ (Fig. 2.30a). Hence,

$$\dot{\vec{H}}_{A|I} = \vec{M}_A \tag{2.36}$$

provided point A satisfies at least one of the following three conditions:

1. $\vec{a}_{A|I} = \vec{0}$, i.e., A has zero acceleration in I,
2. $\vec{a}_{A|I} \parallel \vec{r}_{CA}$, i.e., acceleration of A is along AC,
3. $\vec{r}_{CA} = \vec{0}$, i.e., A coincides with C. It yields the most useful form

$$\dot{\vec{H}}_{C|I} = \vec{M}_C \tag{2.37}$$

If $\vec{M}_A(t) \equiv \vec{0}$, then eq. (2.36) yields $\vec{H}_A(t) = \vec{H}_A(0)$.

Equation 2.36)/(2.37) is derived using $\vec{F} = m\vec{a}_{C|I}$, $\dot{\vec{H}}_{O|I} = \vec{M}_O$, and so is not independent of these. Hence, once $\vec{F} = m\vec{a}_{C|I}$ and a moment equation have been written for a system, writing another moment equation does not yield an independent equation.

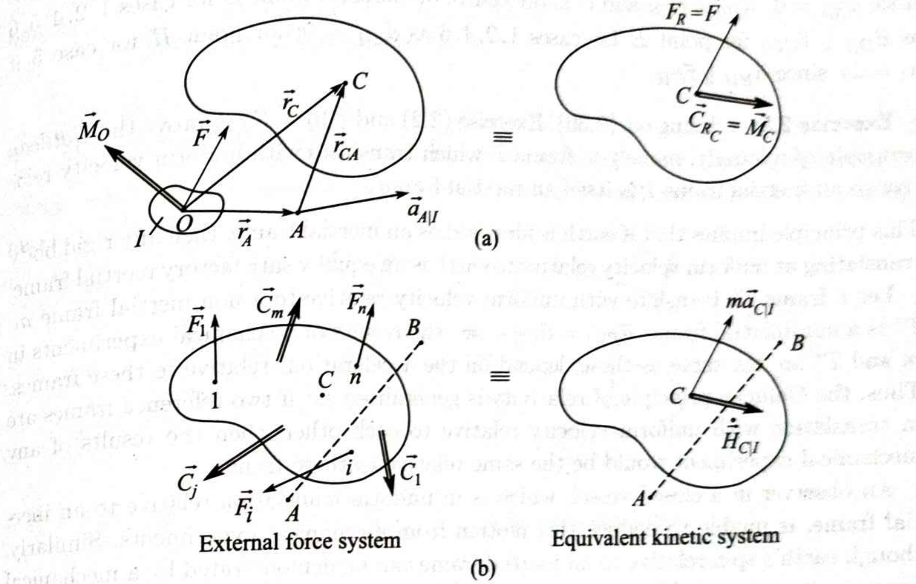


Fig. 2.30a,b

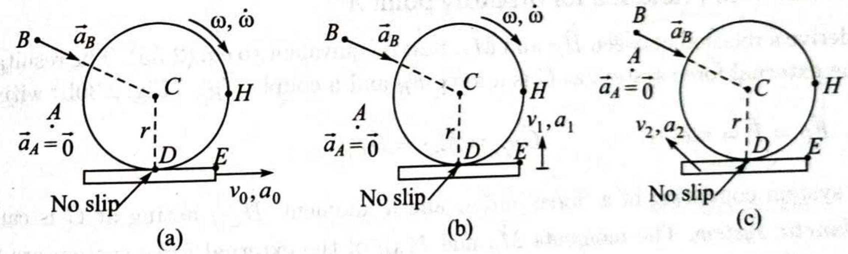


Fig. 2.31

Consider 7 types of motion of a uniform disc rolling without slip on

(a) a conveyor (Fig. 2.31a) with

1. $v_0 = 0$, $a_0 = 0$; 2. $v_0 \neq 0$, $a_0 = 0$; 3. $v_0 \neq 0$, $a_0 \neq 0$;

(b) an elevator (Fig. 2.31b) with

4. $v_1 \neq 0$, $a_1 = 0$; 5. $v_1 \neq 0$, $a_1 \neq 0$;

(b) The sliders have rectilinear motion in frame 2 with speed v: $T_{|2} = 2(\frac{1}{2}mv^2) = mv^2$. Using polar coordinates for sliders in frame 1 with origin at C, $\dot{\phi} = \Omega$, $r = b$, $\dot{r} = -v$,

$$T_{|1} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \times 2 = m(v^2 + b^2\Omega^2)$$

The centre of mass C of the pair of sliders has $v_{C|F} = \omega R$. $\vec{\omega}_{2|F} = (\Omega + \omega)\vec{k}$ and $\vec{r}_{PC} = b\vec{e}_r$, $\vec{v}_{P|2} = -v\vec{e}_r$. Hence, $\vec{v}_{PC|F} = \vec{\omega}_{2|F} \times \vec{r}_{PC} + \vec{v}_{P|2} = (\Omega+\omega)b\vec{e}_\phi - v\vec{e}_r$.

$$T_{|F} = \tfrac{1}{2}(2m)v_{C|F}^2 + \tfrac{1}{2}mv_{PC|F}^2 \times 2 = m[\omega^2R^2 + v^2 + (\Omega+\omega)^2b^2]$$

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Euler's second axiom takes the form $\dot{\vec{H}}_{A|I} = \vec{M}_A - \vec{r}_{CA} \times m\vec{a}_{A|I}$ for an arbitrary point A (Fig. 2.30) Using eq. (2.30): $\vec{H}_{A|I} = \vec{H}_{C|I} + \vec{r}_{CA} \times m\vec{v}_{CA|I}$,

$$\vec{H}_{O|I} = \vec{H}_{C|I} + \vec{r}_{CO} \times m\vec{v}_{CO|I} \quad \Rightarrow \quad \vec{H}_{C|I} = \vec{H}_{O|I} - \vec{r}_C \times m\vec{v}_{C|I}$$

$$\begin{aligned}
\Rightarrow \quad \vec{H}_{A|I} &= \vec{H}_{O|I} - \vec{r}_C \times m\vec{v}_{C|I} + \vec{r}_{CA} \times m\vec{v}_{CA|I} \\
\dot{\vec{H}}_{A|I} &= \dot{\vec{H}}_{O|I} - \vec{v}_{C|I} \times m\vec{v}_{C|I} - \vec{r}_C \times m\vec{a}_{C|I} \\
&\quad + \vec{v}_{CA|I} \times m\vec{v}_{CA|I} + \vec{r}_{CA} \times m\vec{a}_{CA|I} \\
&= \vec{M}_O - \vec{r}_C \times \vec{F} + \vec{r}_{CA} \times m(\vec{a}_{C|I} - \vec{a}_{A|I}) \\
&= \vec{M}_O - \vec{r}_C \times \vec{F} + \vec{r}_{CA} \times \vec{F} - \vec{r}_{CA} \times m\vec{a}_{A|I} \\
&= \vec{M}_O - \vec{r}_A \times \vec{F} - \vec{r}_{CA} \times m\vec{a}_{A|I} \quad \Rightarrow
\end{aligned}$$

$$\dot{\vec{H}}_{A|I} = \vec{M}_A - \vec{r}_{CA} \times m\vec{a}_{A|I} \tag{2.35}$$

since $\dot{\vec{H}}_{O|I} = \vec{M}_O$, $m\vec{a}_{C|I} = \vec{F}$, $\vec{M}_O - \vec{r}_A \times \vec{F} = \vec{M}_A$ (Fig. 2.30a). Hence,

$$\dot{\vec{H}}_{A|I} = \vec{M}_A \tag{2.36}$$

provided point A satisfies at least one of the following three conditions:

1. $\vec{a}_{A|I} = \vec{0}$, i.e., A has zero acceleration in I,
2. $\vec{a}_{A|I} \parallel \vec{r}_{CA}$, i.e., acceleration of A is along AC,
3. $\vec{r}_{CA} = \vec{0}$, i.e., A coincides with C. It yields the most useful form

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If $\vec{M}_A(t) \equiv \vec{0}$, then eq. (2.36) yields $\vec{H}_A(t) = \vec{H}_A(0)$.

Equation 2.36)/(2.37) is derived using $\vec{F} = m\vec{a}_{C|I}$, $\dot{\vec{H}}_{O|I} = \vec{M}_O$, and so is not independent of these. Hence, once $\vec{F} = m\vec{a}_{C|I}$ and a moment equation have been written for a system, writing another moment equation does not yield an independent equation.

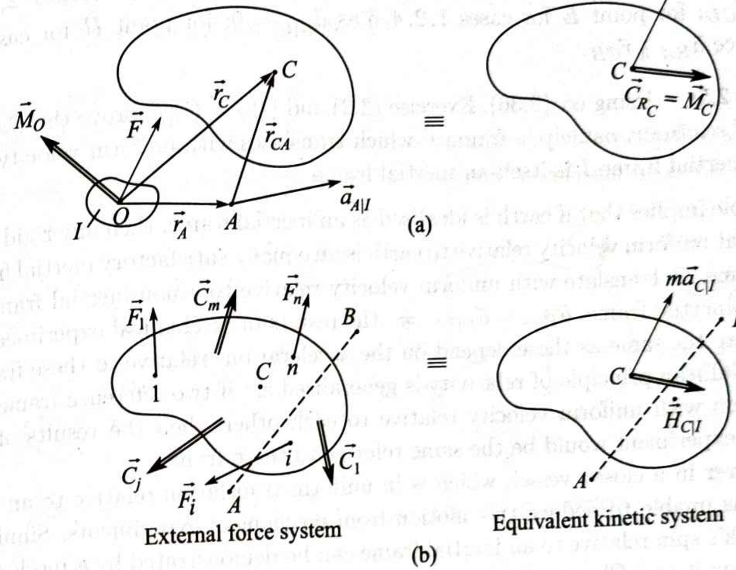


Fig. 2.30a,b

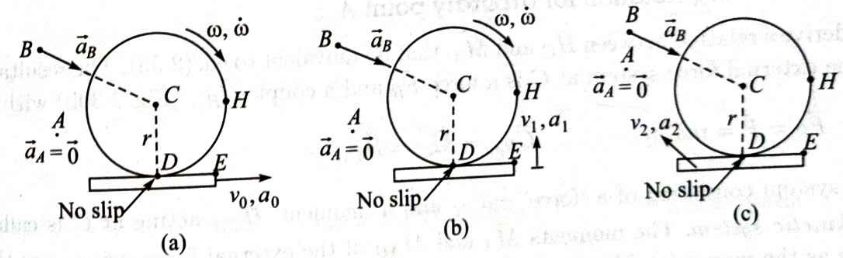


Fig. 2.31

Consider 7 types of motion of a uniform disc rolling without slip on

(a) a conveyor (Fig. 2.31a) with

1. $v_0 = 0$, $a_0 = 0$; 2. $v_0 \neq 0$, $a_0 = 0$; 3. $v_0 \neq 0$, $a_0 \neq 0$;

(b) an elevator (Fig. 2.31b) with

4. $v_1 \neq 0$, $a_1 = 0$; 5. $v_1 \neq 0$, $a_1 \neq 0$;

(c) the step of an escalator (Fig. 2.31c) with

6. $v_2 \neq 0,\ a_2 = 0,$ and   7. $v_2 \neq 0,\ a_2 \neq 0.$

Euler's second axiom takes the form of eq. (2.36) for points A, B, C for all 7 cases since $\vec{a}_{A|I} = \vec{0}$, $\vec{a}_{B|I} \parallel \vec{r}_{CB}$ and C is the centre of mass; for point D for cases 1, 2, 4, 5, 6 as $\vec{a}_{D|I} \parallel \vec{r}_{CD}$; for point E for cases 1, 2, 4, 6 as $\vec{a}_{E|I} = \vec{0}$; for point H for case 5 if $a_1 = \dot{\omega} r$ since $\vec{a}_{H|I} \parallel \vec{r}_{CH}$.

§ **Exercise 2.5:** Using eq. (2.36), Exercise (2.2) and $(\dot{\ })_{|I} = (\dot{\ })_{|T}$, prove the *Galilean principle of relativity*, namely 'a frame T which translates with uniform velocity relative to an inertial frame I is itself an inertial frame'.

This principle implies that if earth is idealised as an inertial frame, then any rigid body translating at uniform velocity relative to earth is an equally satisfactory inertial frame.

Let a frame T^* translate with uniform velocity relative to a non-inertial frame m. T^* is a non-inertial frame. $\vec{a}_{P|m} = \vec{a}_{P|T^*} \Rightarrow$ the results of mechanical experiments in m and T^* are the same as these depend on the accelerations relative to these frames. Thus, the Galilean principle of relativity is generalised as: if two reference frames are in translation with uniform velocity relative to each other, then the results of any mechanical experiment would be the same relative to these frames.

An observer in a closed vessel, which is in uniform translation relative to an inertial frame, is unable to deduce this motion from mechanical experiments. Similarly, though earth's spin relative to an inertial frame can be demonstrated by a mechanical experiment on it (see Chapter 3 for gyrocompass), its translation is undetermined.

2.8.1 $\dot{\vec{H}}_C - \vec{M}_A$ Relation for arbitrary point A

We derive a relation between $\dot{\vec{H}}_C$ and $\vec{M}_A$, that is equivalent to eq. (2.35). The resultant of the external force system at C is a force $\vec{F}_R$ and a couple $\vec{C}_{R_C}$ (Fig. 2.30b) with:

$$\vec{F}_R = \vec{F} = m\vec{a}_{C|I}, \qquad \vec{C}_{R_C} = \vec{M}_C = \dot{\vec{H}}_{C|I}$$

The system consisting of a 'force' $m\vec{a}_{C|I}$ and a 'moment' $\dot{\vec{H}}_{C|I}$ acting at C is called the *kinetic system*. The moments $\vec{M}_A$ and M_{AB} of the external force system are the same as the moments of its equivalent kinetic system at C:

$$\vec{M}_A = \vec{M}_C + \vec{r}_{CA} \times \vec{F}_R = \dot{\vec{H}}_{C|I} + \vec{r}_{CA} \times m\vec{a}_{C|I} \tag{2.38}$$

$$\Rightarrow \quad M_{AB} = [\dot{\vec{H}}_{C|I} + \vec{r}_{CA} \times m\vec{a}_{C|I}] \cdot \vec{e} \tag{2.39}$$

These are more useful than eq. (2.36), as A can be chosen where many unknown forces act and AB as that line through which many unknown forces act, to eliminate them.

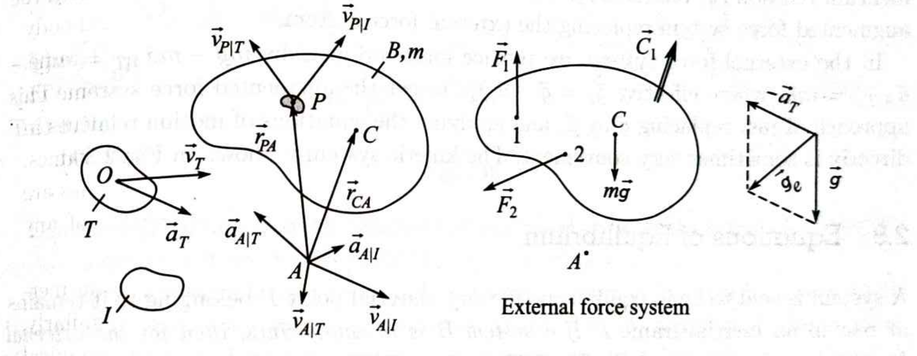
2.8.2 $\dot{\vec{H}}_A - \vec{M}_A$ Relation for any point A for a translating frame T

Consider a frame T translating relative to inertial frame I at $\vec{v}_T(t), \vec{a}_T(t)$ and body B of mass m in motion relative to T (Fig. 2.30c). $\vec{v}_{P|I} = \vec{v}_{P|T} + \vec{v}_T$, $\vec{a}_{P|I} = \vec{a}_{P|T} + \vec{a}_T$, $(\dot{\ })_{|I} = (\dot{\ })_{|T} \Rightarrow \vec{v}_{PA|I} = \vec{v}_{PA|T}$. For body B:

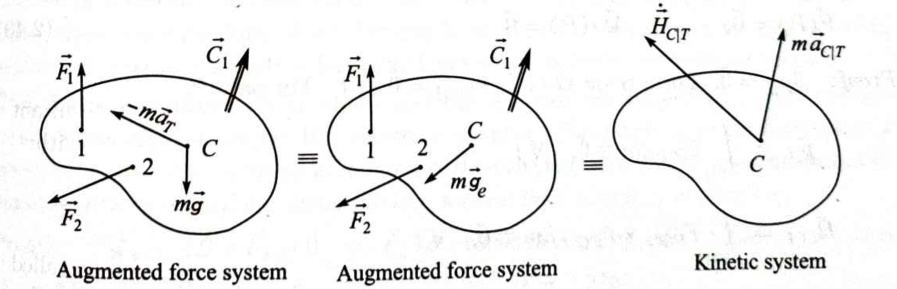
$$\vec{F} = m\vec{a}_{C|I} = m\vec{a}_{C|T} + m\vec{a}_T$$

$$\Rightarrow \quad m\vec{a}_{C|T} = \vec{F}' \tag{1}$$

where $\vec{F}' = \vec{F} + (-m\vec{a}_T)$ is the sum of forces of an *augmented force system* consisting of external loads on body augmented by an inertia 'force' $-m\vec{a}_T$ acting at C (Fig. 2.30d).



(c)



(d)

Fig. 2.30c,d

Let $\vec{M}'_A = \vec{M}_A + \vec{r}_{CA} \times (-m\vec{a}_T)$ be the moment of the augmented force system about A. Note that $\vec{M}'_C = \vec{M}_C$. For body B:

$$\vec{H}_{A|I} = \int_m \vec{r}_{PA} \times \vec{v}_{PA|I}\, dm = \int_m \vec{r}_{PA} \times \vec{v}_{PA|T}\, dm = \vec{H}_{A|T}$$

$$\vec{H}_{A|T} = \vec{H}_{A|I} = \vec{H}_{C|I} + \vec{r}_{CA} \times m\vec{v}_{CA|I} = \vec{H}_{C|I} + \vec{r}_{CA} \times m\vec{v}_{CA|T}$$

$$\dot{\vec{H}}_{A|T} = \dot{\vec{H}}_{C|I} + \vec{r}_{CA} \times m\vec{a}_{CA|T} = \vec{M}_C + \vec{r}_{CA} \times m(\vec{a}_{C|T} - \vec{a}_{A|T})$$

$$= \vec{M}'_C + \vec{r}_{CA} \times (\vec{F}' - m\vec{a}_{A|T}) = \vec{M}'_A - \vec{r}_{CA} \times m\vec{a}_{A|T}$$

$$\Rightarrow \quad \dot{\vec{H}}_{A|T} = \vec{M}'_A \tag{2}$$

if point A is coincident with $C \,/\, \vec{a}_{A|T} = \vec{0} \,/\, \vec{a}_{A|T}$ is along AC.

We conclude that the force–momentum relation (1) and moment–moment of momentum relation (2) relative to T have the same form as those relative to I, with the augmented force system replacing the external force system.

In the external force system, we replace force $m\vec{g}$ at C by $m\vec{g} - m\vec{a}_{A|T} = m(\vec{g} - \vec{a}_{A|T}) = m\vec{g}_e$ where effective $\vec{g}_e = \vec{g} - \vec{a}_{A|T}$, to get the augmented force system. This approach of just replacing $\vec{g}$ by $\vec{g}_e$ and applying the equations of motion relative to T directly is sometimes very convenient. The kinetic system is shown in Fig. 2.30d.

2.9 Equations of Equilibrium

A system is said to be in equilibrium if every material point P belonging to it remains at rest in an inertial frame I. *If a system B is in equilibrium, then for the external load system on every part P_i* (Fig. 2.32):

$$\vec{F}(P_i) = \vec{0}, \qquad \vec{M}_A(P_i) = \vec{0} \tag{2.40}$$

Proof: $\vec{v}_{P|I} = \vec{0}$. For a point O of I, $\vec{v}_{PO|I} = \vec{0}$, $\forall\, t$. For part P_i

$$\vec{p}_{|I} = \int_{m_i} \vec{v}_{P|I}\, dm \equiv \vec{0}, \quad \forall\, t$$

$$\vec{H}_{O|I} = \int_{m_i} \vec{r}_{PO|I} \times \vec{v}_{PO|I}\, dm \equiv \vec{0}, \quad \forall\, t$$

Euler's axioms yield $\quad \vec{F}(P_i) = \dot{\vec{p}}_{|I} = \vec{0}, \qquad \vec{M}_O(P_i) = \dot{\vec{H}}_{O|I} = \vec{0} \Rightarrow$

$$\vec{M}_A(P_i) = \vec{M}_O(P_i) + \overrightarrow{AO} \times \vec{F}(P_i) = \vec{0}$$

A could be any point of the system or outside and may be in arbitrary motion. In particular, eq. (2.40) holds for the complete system B in equilibrium, i.e.,

$$\vec{F}(B) = \vec{0}, \qquad \vec{M}_A(B) = \vec{0} \tag{2.41}$$

Equations (2.41) are necessary conditions of equilibrium, but not sufficient conditions even for a single rigid body. For example, a uniform rigid circular cylinder rolls without slip with uniform angular velocity on a horizontal plane, and free vibrations occur in an unconstrained deformable body, even though $\vec{F}(B) = \vec{0}$, $\vec{M}_A(B) = \vec{0}$.

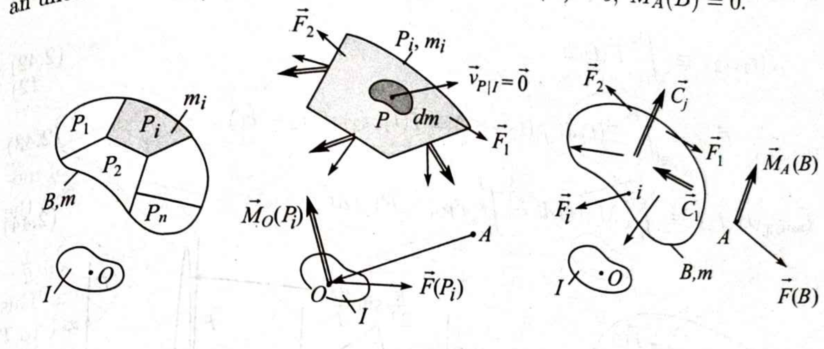


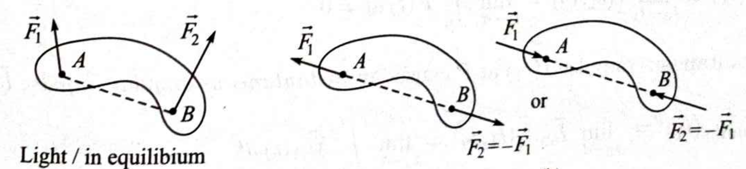
Fig. 2.32

Equations (2.40) are also the *equations of motion of an idealised inertia-less ('light') system*, since $m = 0 \Rightarrow$ for all parts P_i : $\vec{p}_{|I} = \vec{0}$, $\vec{H}_{O|I} = \vec{0}$, $\forall\, t$.

In statics, a system is said to be *light* if the gravitational force on it is very small (modelled as zero) compared to other forces acting on it. In dynamics, if the product of mass of a body and its acceleration is small compared to individual forces, over the interval of interest, then the mass is taken as zero. Such an analysis is called *quasi-static* since the equations of equilibrium hold. However, we should be careful while neglecting inertia of a system for impact problems, as large accelerations occur.

Two-force member: A two-force member is a member subjected to only two external forces and no couple. If a two-force member (Fig. 2.33a) is *in equilibrium* or a two-force *inertia-less* member is *in motion*, then the two forces have *equal magnitudes, opposite directions and act along the line joining their points of application.*

Proof: $\quad \vec{M}_A = \overrightarrow{AB} \times \vec{F}_2 = \vec{0} \;\Rightarrow\; \vec{F}_2$ acts through A along AB. Similarly, $\vec{F}_1$ acts along AB. $\;\vec{F} = \vec{F}_1 + \vec{F}_2 = \vec{0} \;\Rightarrow\; \vec{F}_2 = -\vec{F}_1$ (Fig. 2.33b).



(a) (b)

Fig. 2.33

2.10 Impulse and Angular Impulse

The *impulse* $\vec{I}(t_1, t_2)$ of a force $\vec{F}(t)$ acting at P (Fig. 2.34), the time average of $\vec{F}(t)$, and the *angular impulse* $\vec{I}_{\text{ang}_A}(t_1, t_2)$ of $\vec{F}$ about A for interval t_1 to t_2 are defined by

$$\vec{I}(t_1, t_2) \equiv \int_{t_1}^{t_2} \vec{F}(t)\, dt \tag{2.42}$$

$$\vec{F}_{\text{av}} = \int_{t_1}^{t_2} \vec{F}(t)\, dt \,/\, (t_2 - t_1) = \vec{I}(t_1, t_2) \,/\, (t_2 - t_1) \tag{2.43}$$

$$\vec{I}_{\text{ang}_A}(t_1, t_2) \equiv \int_{t_1}^{t_2} \vec{M}_A(t)\, dt = \int_{t_1}^{t_2} \vec{r}_{PA} \times \vec{F}(t)\, dt \tag{2.44}$$

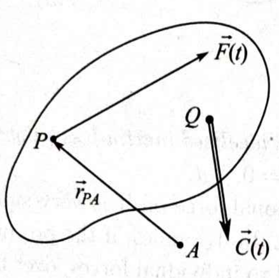


Fig. 2.34

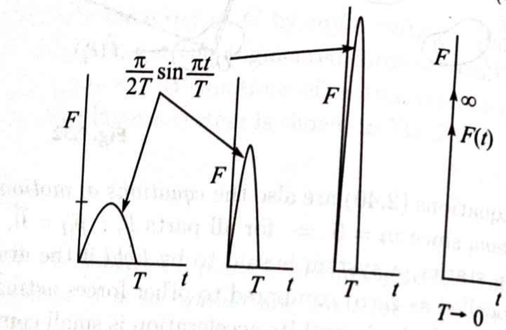


Fig. 2.35

The impulse $\vec{I}(t_1, t_2)$ of a moment (or couple–moment) $\vec{C}(t)$ (Fig. 2.34) acting at Q for interval t_1 to t_2 is zero and its angular impulse $\vec{I}_{\text{ang}_A}(t_1, t_2)$ about A for this interval is

$$\vec{I}_{\text{ang}_A}(t_1, t_2) = \int_{t_1}^{t_2} \vec{M}_A(t)\, dt = \int_{t_1}^{t_2} \vec{C}(t)\, dt \tag{2.45}$$

An *instantaneous impulse* $\vec{I}(t_1)$ (also called an *impulsive 'force'* with dimensions of impulse) is said to act at point P at time t_1 if the force $\vec{F}(t)$ acting at P is such that

$$\vec{I}(t_1) \equiv \lim_{t_2 \to t_1} \vec{I}(t_1, t_2) = \lim_{t_2 \to t_1} \int_{t_1}^{t_2} \vec{F}(t)\, dt \neq \vec{0} \tag{2.46}$$

An instantaneous impulse $\vec{I}(t_1)$ at P causes an *instantaneous angular impulse* $\vec{I}_{\text{ang}_A}(t_1)$

$$\vec{I}_{\text{ang}_A}(t_1) \equiv \lim_{t_2 \to t_1} \vec{I}_{\text{ang}_A}(t_1, t_2) = \lim_{t_2 \to t_1} \int_{t_1}^{t_2} \vec{M}_A(t)\, dt$$

$$= \lim_{t_2 \to t_1} \int_{t_1}^{t_2} \vec{r}_{PA}(t) \times \vec{F}(t)\, dt = \vec{r}_{PA}(t_1) \times \vec{I}(t_1)$$

An *instantaneous angular impulse* (or *impulsive 'moment'* with dimensions of angular impulse) $\vec{I}_{\text{ang}_Q}(t_1)$ is said to act at Q at t_1 if the primitive of moment $\vec{C}(t)$ is such that

$$\vec{I}_{\text{ang}_Q}(t_1) = \lim_{t_2 \to t_1} \vec{I}_{\text{ang}_Q}(t_1, t_2) = \lim_{t_2 \to t_1} \int_{t_1}^{t_2} \vec{C}(t)\, dt \neq \vec{0} \tag{2.47}$$

Its instantaneous angular impulse about any point A is the same: $\vec{I}_{\text{ang}_A}(t_1) = \vec{I}_{\text{ang}_Q}(t_1)$.

Since $\vec{I}(t_1, t_2) = \vec{F}_{\text{av}}(t_2 - t_1)$, a finite force does not cause an instantaneous impulse. A step force, i,e., the sudden application or removal of a force, does not cause an instantaneous impulse. A force causes an instantaneous impulse only if it becomes infinite for an infinitesimal duration. Consider the force of Fig. 2.35:

$$\vec{F}(t) = \begin{cases} (\pi/2T)\sin(\pi t/T)\,\vec{e} & \text{for } 0 < t < T \\ \vec{0} & \text{for } t \notin (0, T) \end{cases} \tag{1}$$

$$\Rightarrow \quad \vec{I}(0, T) = \int_0^T \vec{F}(t)\, dt = \frac{\pi}{2T}\int_0^T \sin\frac{\pi t}{T}\, dt\, \vec{e} = 1\,\vec{e} \quad \forall\, T$$

In the limit as $T \to 0$, $|\vec{F}_{\text{av}}| \to \infty$, but $\vec{I}(0, t)$ has still the same value $1\,\vec{e}$. The force defined by eq. (1) for $T \to 0$ causes an instantaneous impulse $\vec{I}(0) = 1\,\vec{e}$ at $t = 0$.

Very large forces acting for a very short duration, such as those present during impact of bodies, are mathematically modelled as impulsive forces.

2.11 Impulse–Momentum Relations

All entities in this section are with respect to an inertial frame. The solution of $\vec{F} = m\vec{a}_C$ requires two integrations with respect to t. If $\vec{F}(t)$ is known and $\vec{v}_C(t)$ is required, then the first integral of $\vec{F} = m\vec{a}_C$, i.e., the impulse–momentum relation, is useful.

$$\vec{I}(t_1, t_2) = \int_{t_1}^{t_2} \vec{F}\, dt = \int_{t_1}^{t_2} \dot{\vec{p}}\, dt = \int_{\vec{p}(t_1)}^{\vec{p}(t_2)} d\vec{p}$$

$$\vec{I}(t_1, t_2) = \vec{F}_{\text{av}}(t_2 - t_1) = \vec{p}(t_2) - \vec{p}(t_1)$$

$$= \Delta\vec{p} = m\Delta\vec{v}_C = \sum m_i \Delta\vec{v}_{C_i} \tag{2.48}$$

i.e., the *impulse of the* **external** *force $\vec{F}$ equals the change in momentum.* For a pile-driver hammer, the impact interval $(t_2 - t_1)$ is kept small to have large $\vec{F}_{\text{av}}$. In contrast, devices meant to prevent injury in accidents, such as a helmet and a seat belt, are designed to increase the time $(t_2 - t_1)$ of absorption of the momentum to decrease $\vec{F}_{\text{av}}$.

Equation (2.48) holds for every Cartesian component, as $a_x = \dot{v}_x$, $a_y = \dot{v}_y$, $a_z = \dot{v}_z$, e.g.,

$$I_x(t_1, t_2) = \int_{t_1}^{t_2} F_x\, dt = F_{x\text{av}}(t_2 - t_1) = \Delta p_x = m\Delta v_{C_x} = \sum m_i \Delta v_{C_{ix}}$$

Such relations are not valid for $\vec{e}_r$, $\vec{e}_\phi$, $\vec{e}_n$ components, since $a_r \neq \dot{v}_r$, $a_\phi \neq \dot{v}_\phi$, $a_n \neq \dot{v}_n$.

The first integral of the equation of motion, $F_\phi = ma_{C_\phi} = m[\frac{1}{r_C}\frac{d}{dt}(r_C^2\dot{\phi}_C)] \Rightarrow$

$$m\Delta(r_C^2\dot{\phi}_C) = m\Delta(r_C v_{C_\phi}) = \int_{t_1}^{t_2} r_C F_\phi dt \tag{2.49}$$

In particular, $F_\phi \equiv 0$[5] $\Rightarrow$

$$r_C^2\dot{\phi}_C = r_C v_{C_\phi} = \text{constant} \tag{2.50}$$

For an *instantaneous impulse* at t_1, eq. (2.48) $\Rightarrow$

$$\begin{aligned} \vec{I}(t_1) &= \Delta\vec{p} = \vec{p}(t_1^+) - \vec{p}(t_1^-) = m[\vec{v}_C(t_1^+) - \vec{v}_C(t_1^-)] \\ &= \sum m_i[\vec{v}_{C_i}(t_1^+) - \vec{v}_{C_i}(t_1^-)] \end{aligned} \tag{2.51}$$

$\vec{p}$, $\vec{v}_C$, and $\vec{v}_{C_i}$ change instantaneously *without change in position* since $\vec{v}_{C_{av}}$ is finite:

$$\vec{r}_C(t_1^+) - \vec{r}_C(t_1^-) = \lim_{t_2 \to t_1} \int_{t_1}^{t_2} \vec{v}_C(t)\, dt = \lim_{t_2 \to t_1} [\vec{v}_{C_{av}}(t_2 - t_1)] = \vec{0}$$

If $\vec{I}(t_1, t_2) = \vec{0}$, then eq. (2.48) implies *conservation of momentum*:

$$\vec{p}(t_2) = \vec{p}(t_1), \qquad \vec{v}_C(t_2) = \vec{v}_C(t_1)$$

$$\sum m_i \vec{v}_{C_i}(t_2) = \sum m_i \vec{v}_{C_i}(t_1) \tag{2.52}$$

Internal forces alone cannot change the velocity of C, though the velocities $\vec{v}_{C_i}$ of different parts can be changed. If a particular component of impulse in some fixed direction is zero, then the corresponding component of momentum is conserved, for example if only $I_y(t_1, t_2) = 0$, then the component $p_y = mv_{C_y} = \sum m_i v_{C_{iy}}$ is conserved.

Example 2.17: Discuss the impact of a rigid block of mass m attached to a spring of stiffness k, moving with velocity v_0 on a smooth horizontal plane, with a smooth rigid vertical wall (Fig. E2.17).

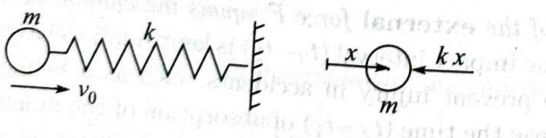


Fig. E2.17

[5]Note that for a mass-point m, $H_{O_z} = [(r\vec{e}_r + z\vec{e}_z) \times m(\dot{r}\vec{e}_r + r\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z)] \cdot \vec{e}_z = mr^2\dot{\phi} = mrv_\phi$.

Solution: Let x be the compression of the spring (Fig. E2.17). The initial conditions are: $x(0) = 0$, $\dot{x}(0) = v_0$. The equation of motion for $x \geq 0$ is

$$m\ddot{x} = -kx \quad \Rightarrow \quad x(t) = \frac{v_0}{\omega}\sin\omega t, \quad \text{with } \omega^2 = \frac{k}{m}, \quad x_{max} = \frac{v_0}{\omega} = v_0\sqrt{m/k}$$

The impact ends at t_1 when the spring force becomes zero, i.e., when

$$x(t_1) = \frac{v_0}{\omega}\sin\omega t_1 = 0 \quad \text{at } t_1 = \pi/\omega = \pi\sqrt{m/k}, \quad \text{and} \quad \dot{x}(t_1) = -v_0$$

$$\vec{I}(0, t_1) = \Delta\vec{p} = m(-v_0\,\vec{i} - v_0\,\vec{i}) = -2mv_0\,\vec{i}$$

This impulse on the block is independent of k. In the limit as $k \to \infty$ (stiff spring), $t_1 \to 0$, $x_{max} \to 0 \Rightarrow$ there is an instantaneous change of $-2v_0\,\vec{i}$ in velocity without change in position and the force from the wall, kx, tends to infinity in such a way that it exerts an instantaneous impulse $\vec{I}(0) = -2mv_0\,\vec{i}$.

2.12 Angular Impulse–Moment of Momentum Relations

All entities in this section are with respect to an inertial frame. If A is a point such that $\vec{a}_A(t) \equiv \vec{0}$, or $A \equiv C$, or $\vec{a}_A(t)$ is along AC, $\forall\, t \in (t_1, t_2)$, then $\vec{M}_A = \dot{\vec{H}}_A$ yields the angular impulse–moment of momentum relation:

$$\begin{aligned} \vec{I}_{\text{ang}_A}(t_1, t_2) &= \int_{t_1}^{t_2} \vec{M}_A\, dt = \int_{\vec{H}_A(t_1)}^{\vec{H}_A(t_2)} d\vec{H}_A \\ \vec{I}_{\text{ang}_A}(t_1, t_2) &= \vec{H}_A(t_2) - \vec{H}_A(t_1) = \Delta\vec{H}_A \end{aligned} \tag{2.53}$$

$\Rightarrow$ **1.** *The external angular impulse about A equals the change in the moment of momentum $\vec{H}_A$.*

2. If $\vec{I}_{\text{ang}_A}(t_1, t_2) = \vec{0}$, then eq. (2.53) implies that $\vec{H}_A$ is conserved.

Since the results 1 and 2 are valid for a point A fixed in I, if a component of $\vec{I}_{\text{ang}_A}(t_1, t_2)$ along a fixed direction in I is zero, then the component of $\vec{H}_A$ in that direction is conserved. Similarly, if a component of $\vec{I}_{\text{ang}_C}$ along a fixed direction of I is zero, then the component of $\vec{H}_C$ in that direction is conserved.

If an instantaneous angular impulse acts at t_1, then without change in position there is an instantaneous change in moment of momentum about the centre of mass C, about a point O fixed in I and *about any point A in arbitrary motion with respect to I whose velocity does not change instantaneously at t_1 (A may be a material point of the system or a point outside it)*, with

$$\vec{I}_{\text{ang}_C}(t_1) = \Delta\vec{H}_C, \qquad \vec{I}_{\text{ang}_O}(t_1) = \Delta\vec{H}_O, \qquad \vec{I}_{\text{ang}_A}(t_1) = \Delta\vec{H}_A \tag{2.54}$$

The last equation is valid (Fig. 2.36) since the positions of C, A and $\vec{v}_A$ remain the same, (i.e., $\Delta\vec{v}_A = \vec{0}$) and

$$\Delta\vec{H}_A = \Delta[\vec{H}_C + \vec{r}_{CA} \times m\vec{v}_{CA}]$$
$$= \Delta\vec{H}_C + \vec{r}_{CA} \times m\Delta\vec{v}_C$$
$$= \vec{I}_{\text{ang}_C}(t_1) + \vec{r}_{CA} \times \vec{I}(t_1) = \vec{I}_{\text{ang}_A}(t_1)$$

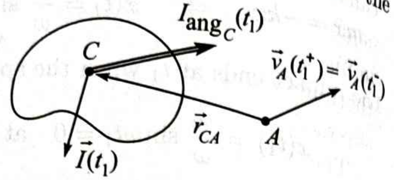


Fig. 2.36

But $\vec{I}_{\text{ang}_B}(t_1)$ is not equal to $\Delta\vec{H}_B$ for any other point B of the system or otherwise.

2.13 Work–Energy Relation for Centre of Mass C of any System

We define $\dot{W}^*$ and T_C by

$$\dot{W}^* \equiv \vec{F} \cdot \vec{v}_C = \text{power of all the } \textit{external} \text{ forces } \textit{as if} \text{ acting at } C \tag{2.55}$$

$$T_C \equiv \tfrac{1}{2}mv_{C|I}^2 = \text{kinetic energy } \textit{as if} \text{ all mass is concentrated at } C \tag{2.56}$$

In general, $\dot{W}^* \neq \dot{W}$, $T_C \neq T$.

$$\vec{F} = m\vec{a}_{C|I} \;\Rightarrow\; \vec{F}\cdot\vec{v}_{C|I} = m\vec{a}_{C|I}\cdot\vec{v}_{C|I} = \frac{d}{dt}\left[\tfrac{1}{2}m\vec{v}_{C|I}\cdot\vec{v}_{C|I}\right]_{|I} = \frac{d}{dt}\left[\tfrac{1}{2}mv_{C|I}^2\right]_{|I}$$

$$\Rightarrow \dot{T}_C = \dot{W}^* \quad\Rightarrow\quad T_{C_2} - T_{C_1} = W_{1-2}^* \tag{2.57}$$

T_{C_1}, T_{C_2} are the values of T_C in configurations 1 and 2 of the system and W_{1-2}^* is the work done from configuration 1 to 2 by the external forces as if they were acting at C. The work–energy relation is *not applicable when instantaneous impulsive loads act*, since at that instant $|\vec{a}_C| = \infty$ and the proof of eq. (2.57) requires $\vec{a}_C$ to remain finite.

2.14 Friction

The mechanical interaction at the surfaces of contact of two bodies is modelled by a distributed surface force whose components normal and tangent to the contact surface are called normal reaction and force of friction. Friction obstructs relative sliding between the bodies. Its direction on body 1 is opposite to the possible or existing relative velocity of point P_1 of body 1 relative to the point P_2 of body 2 with which it is in instantaneous contact (Fig. 2.37a). Since the bodies press against each other, the normal reaction on a body always points into its interior.

For the case of slip, friction causes energy dissipation, partially as thermal energy. This energy 'loss', can be high (20% for automobiles), but it is the resulting wear causing rough operation and premature failure of components, which is a principal engineering worry. Friction is undesirable in bearings, power screws, gears, fluid flow in pipes, aircraft propulsion, etc. Sometimes, the conversion of mechanical energy into thermal energy due to friction is beneficial, e.g., in brakes. Friction is utilised in its no slip mode in land transport, belt and clutch drives, holding and fastening devices, etc.

The friction between dry rubbing solid bodies is called *dry friction*. Fluid friction will not be covered in this text. Rolling friction occurs between bodies in rolling contact. The major cause of dry friction is bonding between the elements of the bodies over the microscopic roughness of the surfaces in contact. Interlocking microscopic protuberances and troughs of the contacting surfaces oppose relative motion giving rise to frictional resistance force. When sliding is present between such surfaces, some of these protuberances are sheared off or melt due to high local temperature, causing wear. Experiments broadly reveal that the frictional force at the instant of impending slip is proportional to the normal reaction and is independent of the surface area of contact. The friction force during continued slip, called kinetic friction, is less than that at impending slip because some of the interlocking protuberances have sheared off and there is no time to form new bonds between the elements of the bodies in contact. Over a limited range of speed, friction force is independent of the relative velocity between two points in contact. Friction force depends on local temperatures generated, adhesion at contact points, relative hardness of mating surfaces and presence of surface films of oxide, oil, dirt or other substances.

2.14.1 Axiom of dry friction or Coulomb friction

Consider two bodies with a surface contact (Fig. 2.37a). At a point P_1 of body 1, which at time t is in contact with point P_2 of body 2, the normal (*pressure*) and tangential (*frictional*) components, $\vec{p}$ and $\vec{\tau}$ respectively, of the surface force density (Fig. 2.37b) with magnitudes $p = |\vec{p}|$, and $\tau = |\vec{\tau}|$, are related as follows.

1. For no slip between P_1 and P_2:

$$\tau < \mu_s p \tag{2.58}$$

$\vec{\tau}$ is found from the equation of motion.

2. For impending slip between P_1 and P_2:

$$\tau = \mu_s p \tag{2.59}$$

and $\vec{\tau}$ is directed opposite to the direction of impending slip of P_1 relative to P_2, i.e., $\vec{\tau}$ is opposite to the component of $\vec{a}_{P_1P_2}(t^+)$ in the tangent plane.

3. For slip between P_1 and P_2:

$$\tau = \mu_k p \tag{2.60}$$

and $\vec{\tau}$ is directed opposite to $\vec{v}_{P_1P_2}(t)$. $\mu_s(>0)$, $\mu_k(>0)$ are called the static and kinetic (dynamic) coefficients of friction with $\mu_k \le \mu_s$. These are independent of p, $\vec{v}_{P_1P_2}$ and the area of contact. If a tentative assumption of no slip is made, then the direction and magnitude of τ can be shown arbitrarily in an FBD. However, it must be verified that the magnitude of τ, determined from the equation of motion, lies within the bound of $\mu_s p$. If the check holds, the assumption of no slip is correct, else there is slip. Often the direction of friction forces for the latter case are the same as those obtained with the assumption of no slip.

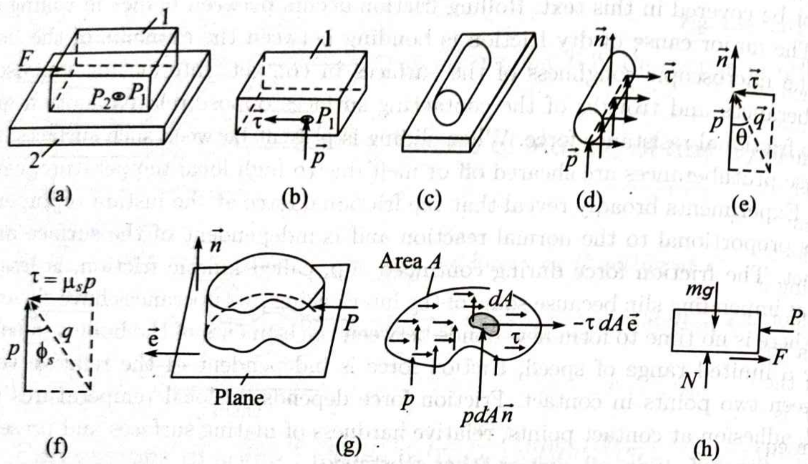


Fig. 2.37

For two bodies with idealised line contact (Fig. 2.37c), eqs (2.58) to (2.60) hold with p and τ being the normal and tangential (*frictional*) components of the line force density respectively (Fig. 2.37d).

In reality, μ is not exactly independent of p or the area of contact. The axiom is satisfactory for most metals in air. For diamond, rubber and plastics, μ decreases with increase in p. One should avoid using μ from tables. A variation of 25–100% can occur depending on the cleanliness, surface finish, pressure, lubrication and velocity. It is preferable to find the actual μ for the bodies to be used. The typical values are $\mu_s \simeq$: 1 for copper, cast iron, glass rubber tyres on road; 0.7 for mild steel, hemp rope on wood; 0.5 for leather on metal, wood on wood; 0.4 for steel on babbitt, brake lining on cast iron, rubber on metal / wood; 0.3 for hemp rope on metal, 0.2 for wire rope on steel, 0.1 for rubber on ice; 0.04 for teflon on steel/teflon, metal on ice. These data are for commercially available materials having some impurities. For pure metals with fine surface finish, the surfaces could bond and μ_s could become very large.

The resultant $\vec{q}$ of $\vec{p}$ and $\vec{\tau}$ makes angle $\theta = \tan^{-1}(\tau/p)$ with the normal (Fig. 2.37e).

The limiting value of θ for no slip (Fig. 2.37f) is called the *angle of static friction* ϕ_s:

$$\tan\phi_s = \mu_s, \qquad \tau/p \le \mu_s \;\Rightarrow\; \tan\theta \le \tan\phi_s \;\Rightarrow\; \theta \le \phi_s \tag{2.61}$$

Most often, we work with $\vec{p}$ and $\vec{\tau}$ and not with their resultant.

If the frictional force $\tau = 0$ for no slip, impending slip and actual slip, i.e., $\mu_s = \mu_k = 0$, then the contact is defined to be a *perfectly smooth contact*. If the frictional force τ can be arbitrarily high, i.e., $\mu_s = \infty$ and slip between points in contact is not possible, then the contact is defined to be a *perfectly rough contact*. However, it should be noted that these definitions in our mathematical model are not based on the physical waviness of surfaces in contact. The coefficient of friction for a highly ground and polished surface (i.e., a physically smooth surface) is more than for an unground surface with more waviness (i.e., a physically rough surface). This is due to greater adhesion for the former case as the surface elements are much closer over most of the surface.

Like any other force system, the simplest resultants of the distributed contact force system and the distributed frictional force system are, in general, wrenches. For example, the simplest resultants of the contact force and the frictional force on a screw moving in a fixed nut (see Section 2.20) are wrenches (Fig. 2.61b). The simplest resultants of the contact force and the frictional force on an axisymmetric thrust bearing are a wrench and a single couple, respectively (Fig. 2.60).

If the contact surface of area A is plane (Fig. 2.37g,h) and the direction and sense of $\vec{\tau}$ is the same at all points (for example, when 1 translates with respect to 2), then eqs (2.58) to (2.60) imply that the resultant normal force $N = \int_A p\,dA$, and the resultant frictional force $F = \int_A \tau\,dA$, are related by

$$\text{for no slip: } F < \mu_s N, \qquad \text{impending slip: } F = \mu_s N, \qquad \text{for slip: } F = \mu_k N \tag{2.62}$$

Most often, $F \ne 0$ for no slip, and its direction is a priori arbitrary and is obtained from equations of motion. *In general, N does not act at the centroid of the contact area.*

Equations (2.62) are valid (i) for idealised straight line contact if the direction and sense of $\vec{\tau}$ is the same at all points, (ii) for idealised point contact (Fig. 2.38a,b) with the normal reaction N being in the direction of the normal to one (if the radius of curvature of one of the surfaces is zero) or both surfaces of contact. $N \ge 0$ for a one-sided constraint, but the sign of N is arbitrary for a two-sided constraint. For the one-sided constraint, check whether the computed value of N is non-negative. If this check fails, then rework the problem with $N = 0$.

We consider two cases of point contact.

1. A body makes contact at a point on a curved thin body (Fig. 2.38c) idealised as a curve C (for example, a small bead on a wire or a slider in a slot). Then,

$$\vec{F} = -F\vec{e}_t, \qquad \vec{N} = N_1\vec{e}_n + N_2\vec{e}_b,$$

$$N = (N_1^2 + N_2^2)^{1/2} \quad \text{and for no slip:} \quad |F| < \mu_s (N_1^2 + N_2^2)^{1/2}$$

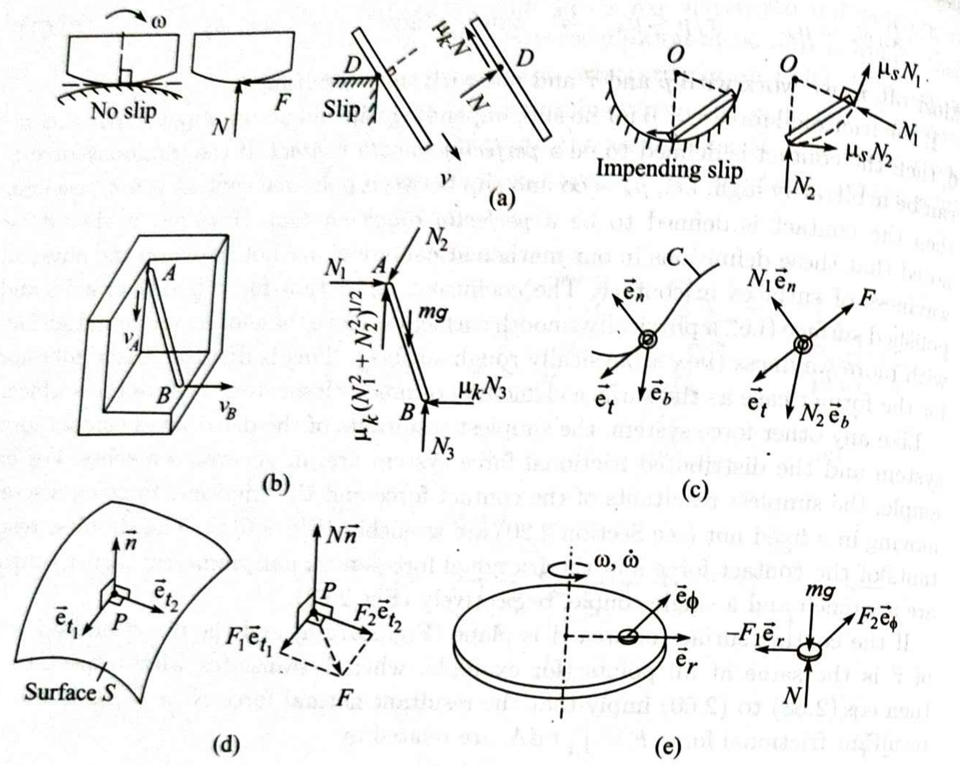


Fig. 2.38

2. A body makes point contact on a surface S (Fig. 2.38d) of another body (for example, a small slider on a cylinder),

$$\vec{F} = F_1\,\vec{e}_{t_1} + F_2\,\vec{e}_{t_2}, \qquad F = (F_1^2 + F_2^2)^{1/2}, \qquad \vec{N} = N\vec{n}$$

and for no slip: $\qquad (F_1^2 + F_2^2)^{1/2} < \mu_s |N|$

where $\vec{e}_{t_1}$, $\vec{e}_{t_2}$ are orthogonal unit vectors in the plane tangent to the surface S and $\vec{n}$ is its unit normal vector. For example, for impending slip of a small coin on a disc (Fig. 2.38e) rotating about a fixed vertical axis with angular acceleration, there are frictional force components in the radially inward and the circumferential directions, with $|\vec{F}| = \sqrt{F_1^2 + F_2^2} = \mu_s N$.

Example 2.18: Draw FBD of body 1 of mass m for each case shown in Fig. E2.18a. In case 3, there is no slip at A and B and body 3 translates at v. Given μ_s and μ_k.

Solution: Find $\vec{v}_{AB}$ for points A and B in contact. $\vec{F}$ at A for the 3 cases is:

1. If $\vec{v}_{AB} \neq \vec{0}$, then there is actual slip $\Rightarrow \vec{F}$ has magnitude $F = \mu_k N$ and is directed opposite to $\vec{v}_{AB}$, where N is the normal reaction.
2. If $\vec{v}_{AB} = \vec{0}$ and $\vec{a}_{AB} \cdot \vec{e}_t = 0$, where $\vec{e}_t$ is any vector in the tangent plane of contact, then there is no slip $\Rightarrow \vec{F}$ has arbitrary magnitude (bounded by $\mu_s N$) and arbitrary direction, which are obtained by solving the equations of motion.
3. If $\vec{v}_{AB} = \vec{0}$ and $\vec{a}_{AB} \cdot \vec{i} \neq 0$, where $\vec{i}$ is in the tangent plane of contact, then there is impending slip $\Rightarrow F = \mu_s N$ and is directed opposite to $(\vec{a}_{AB} \cdot \vec{i})\,\vec{i}$. $F = \mu_s N$ exists momentarily and becomes $\mu_k N$ immediately afterwards.

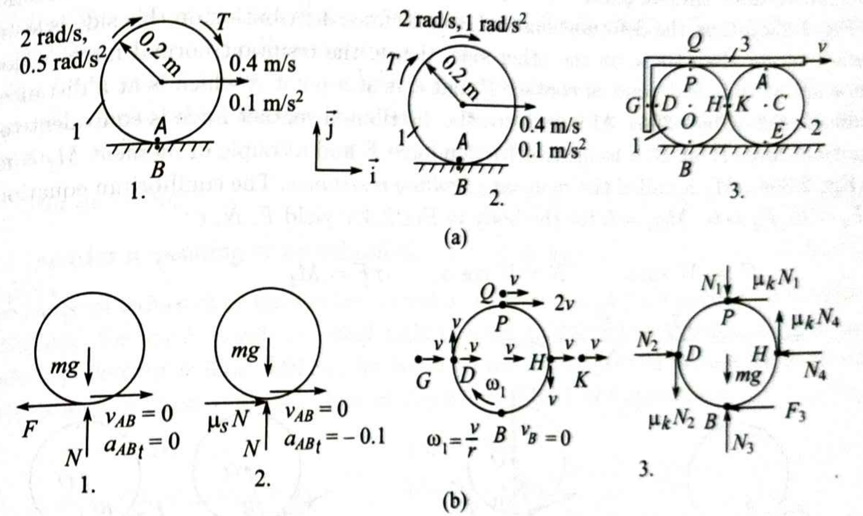


Fig. E2.18

Case 1: $v_{AB} = 0.4 - 2(0.2) = 0$, $a_{AB_t} = 0.1 - 0.5(0.2) = 0 \;\Rightarrow\;$ no slip. $|F| < \mu_s N$.

Case 1: $v_{AB} = 0.4 - 2(0.2) = 0$, $a_{AB_t} = 0.1 - 1(0.2) = -0.1 \Rightarrow$ impending slip to left.

Case 3: $\vec{v}_A = v\,\vec{i}$, $\quad \vec{v}_C = v\,\vec{i} \;\Rightarrow\;$ body 2 translates $\Rightarrow\; \vec{v}_K = v\,\vec{i}$.

$\vec{v}_B = \vec{0}$, $\quad \vec{v}_O = v\,\vec{i} \;\Rightarrow\;$ body 1 rotates at $\omega_1 = v/r$. $\;\vec{v}_Q = \vec{v}_G = v\,\vec{i}$.

$\vec{v}_P = \omega_1(2r)\,\vec{i} = 2v\,\vec{i}$, $\qquad \vec{v}_{PQ} = v\,\vec{i} \;\Rightarrow\; P$ slips in direction $\vec{i}$.

$\vec{v}_D = \vec{v}_O + \omega_1 r\,\vec{j} = v\,\vec{i} + v\,\vec{j}$, $\qquad \vec{v}_{DG} = v\,\vec{j} \;\Rightarrow\; D$ slips in direction $\vec{j}$.

$\vec{v}_H = \vec{v}_O + \omega_1 r(-\vec{j}) = v\,\vec{i} - v\,\vec{j}$, $\qquad \vec{v}_{HK} = -v\,\vec{j} \;\Rightarrow\; H$ slips in direction $-\vec{j}$.

No slip at $B \;\Rightarrow\; |F_3| < \mu_s N_3$.

2.14.2 Rolling friction

Consider a wheel resting in equilibrium on an inclined plane of small inclination $\alpha \neq 0$ (Fig. 2.39a). Its FBD, on the assumption of line contact, is shown in Fig. 2.39b. There do not exist any values of F and N which can satisfy $F_x = 0,\ F_y = 0,\ M_{O_z} = 0 \Rightarrow$

$$F = W \sin\alpha, \qquad N = W\cos\alpha, \qquad rF = 0.$$

$\Rightarrow$ No matter how small α ($\neq 0$) is, and how large μ is, the wheel should roll. This contradicts our experience. Hence, our modelling of the contact surface as a line contact is not adequate. The bodies are not exactly rigid but deform enough near the contact region to have a small surface of contact. If the body tends to roll in the x-direction (Fig. 2.39c), then the deformations and contact force distribution on this side is somewhat larger than those on the other side. Hence, the resultant normal force N does not act at the ideal point of contact B, but acts at a point A which is at a distance e ahead of B (Fig. 2.39d). Alternatively, the distributed contact force is equivalent to a normal force N at B, a tangential friction force F and a couple of moment $M_f = eN$ (Fig. 2.39e). M_f is called the *moment of rolling resistance*. The equilibrium equations: $F_x = 0,\ F_y = 0,\ M_{O_z} = 0$ for the body in Fig. 2.39e yield $F,\ N,\ e$:

$$F = W\sin\alpha, \qquad N = W\cos\alpha, \qquad rF = M_f$$

$$\Rightarrow \quad e = \frac{M_f}{N} = \frac{rF}{N} = r\tan\alpha$$

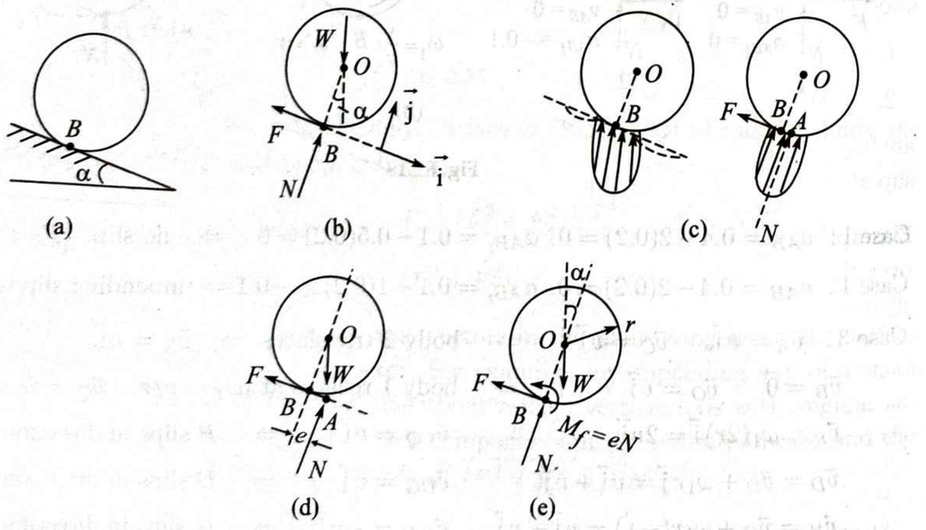


Fig. 2.39

Experiments show that the wheel remains at rest only for small values of angle α, i.e., the eccentricity e of N has an upper limit e_1, i.e., $|e| \le e_1$ and $M_f \le e_1 N$. It is called the *inequality of rolling resistance* and e_1 is called the *length of rolling resistance* (also called the 'coefficient' of rolling friction). It has the dimensions of length and it may depend on the curvature of the contacting curves, speed of travel, elastic and plastic properties of the contacting materials and roughness of surfaces. Tests indicate little variation of e_1 with the wheel radius. So, e_1 (also denoted by a) is taken as independent of the radii of curvatures of the curves of the bodies rolling on each other at the point of contact. In case of an impending and actual rolling motion, the inequality of rolling resistance is replaced by an *equality of rolling resistance* with usually the same length of rolling resistance e_1 as in the inequality. We can thus state the following *axiom of rolling resistance*.

If a body tends to roll over another body, then the resultant normal force N acts through a point at a distance e ahead of the ideal line or point of contact in the direction in which impending or actual subsequent contact occurs, such that

$$\text{for no rolling}: \qquad |e| < e_1 \tag{2.63}$$

$$\text{and for impending or actual rolling}: \qquad |e| = e_1 \tag{2.64}$$

The range of values of e_1 for hardened steel on hardened steel (ball bearing) is 0.0005–0.0013 cm, for steel wheels on steel rails is 0.006–0.010 cm, for pneumatic tyre on smooth pavement is 0.05–0.07 cm, for wood on wood is 0.05–0.08 cm and for steel on wood is 0.15–0.25 cm. For the wheel of Fig. 2.39a, if both the conditions

1. $F/N = \tan\alpha < \mu_s$ (for no slip)
2. $e = M_f/N = r\tan\alpha < e_1$ (for no rolling)

are fulfilled, then it remains at rest. If α is gradually increased, then the wheel starts to slip at $\alpha = \tan^{-1}\mu_s$ if $\mu_s < e_1/r$, and starts to roll at $\alpha = \tan^{-1}(e_1/r)$ if $e_1/r < \mu_s$.

Example 2.19: A cylindrical roller of mass m is at rest. Find M to initiate rolling over the step (Fig. E2.19a). The length of rolling resistance is e. Find the minimum μ_s to ensure no slip.

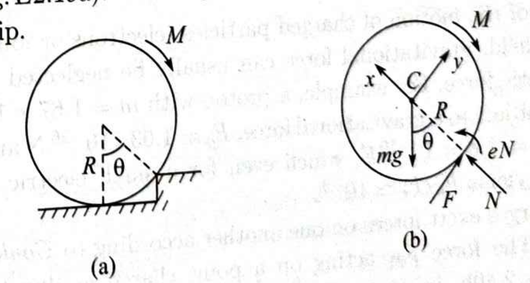


Fig. E2.19

Solution: N acts at a distance e ahead of the ideal contact point on the side which is about to make contact subsequently. Hence, in the FBD for impending rolling, the direction of moment eN is as shown (Fig. E2.19b). M, N, F are obtained from:

$$F_x = N - mg\cos\theta = 0 \quad \Rightarrow \quad N = mg\cos\theta$$

$$F_y = F - mg\sin\theta = 0 \quad \Rightarrow \quad F = mg\sin\theta$$

$$M_{C_z} = M - eN - FR = 0 \quad \Rightarrow \quad M = mg(e\cos\theta + R\sin\theta)$$

$$\mu_{s_{\min}} = F/N = \tan\theta$$

2.15 Axiom of Electromagnetic Force

Force $\vec{F}$ on a mass-point with charge q moving with velocity $\vec{v}$ in a static electromagnetic field with electric field vector $\vec{E}$ and a magnetic field vector $\vec{B}$ (Fig. 2.40a) is given by the Lorentz law as

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}. \tag{2.65}$$

$\vec{F}$ is in newtons if q is in coulombs (C), $\vec{E}$ is in N.C^{-1}, $\vec{B}$ is in N.C^{-1}.m^{-1}.s and $\vec{v}$ is in m.s^{-1} (1 C = 1 ampere second= 1 A.s).

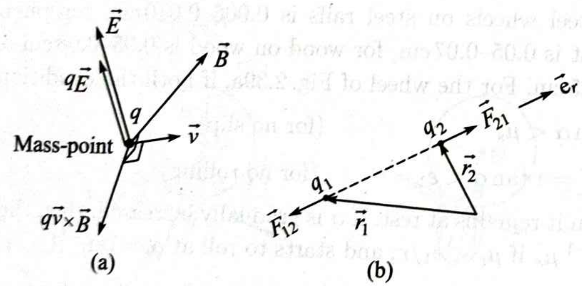


Fig. 2.40

For the study of the motion of charged particles (electrons or ionised atoms) in an electromagnetic field, gravitational force can usually be neglected in comparison to the electromagnetic force. For example, a proton with $m = 1.67\times10^{-27}$ kg, and $q = 1.6\times10^{-19}$ C is subject to a gravitational force, $F_g = 1.63\times10^{-26}$ N and an electrostatic force, $F_e = qE = 1.6\times10^{-19}E$, which even for a small electric field of strength, $E = 10^{-4}$N.C^{-1}, yields $F_g/F_e \simeq 10^{-3}$.

Stationary charges exert forces on one another according to *Coulomb's law of electrostatic forces.* The force F_{21} acting on a point charge q_2 due to a point charge q_1, shown in Fig. 2.40b, in an intervening medium of permittivity ϵ (for vacuum, $\epsilon = 8.854\times10^{-12}$ C^2.N^{-1}.m^{-2}) is given by

$$\vec{F}_{21} = -\vec{F}_{12} = \frac{q_1q_2}{4\pi\epsilon r^2}\vec{e}_r = \frac{q_1q_2}{4\pi\epsilon}\frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3} \tag{2.66}$$

2.16 Axiom of Gravitational Force

Newton's law of gravitation for mass-points (particles) is extended to finite bodies in the following axiom of gravitation: 'The total gravitational force $\vec{F}_{21}$ on body 2 of mass m_2 due to body 1 of mass m_1 (Fig. 2.41) is given by

$$\vec{F}_{21} = -\int_{m_2}\left[\int_{m_1} G\frac{\vec{r}}{r^3}\,dm_1\right]dm_2 \tag{2.67}$$

where $G = (6.672\pm0.004)\times10^{-11}$ m^3.kg^{-1}.s^{-2} is the universal gravitational constant.' The total force $\vec{F}_{21}$ is called the *weight* of body 2 due to body 1. The *simplest resultant* of the spatially distributed force $d\vec{F}_{21}$ is, in general, a *wrench.* If the simplest resultant is a single force, then the *centre of gravity* G of body 1 due to body 2 is defined as the unique point where the whole mass m_1 can be concentrated so that the force on it due to body 2 equals that on the actual distributed mass m_1. For a uniform sphere, G coincides with C and remains unchanged as the body moves. In general, the position of G relative to body 1 depends on its orientation and its distance from body 2 and it is distinct from C. In such cases, it is more useful to show an equivalent force–couple system at C.

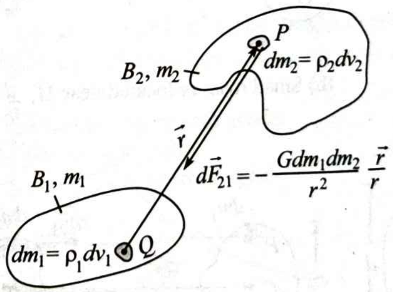


Fig. 2.41

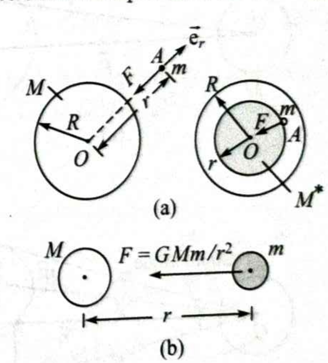


Fig. 2.42

The following results can be proved from eq. (2.67).

1. For a *radially symmetric* [$\Rightarrow$ $\rho(r)$] sphere of mass M, radius R and a *mass-point* of mass m (Fig. 2.42a) at distance r from its centre:

   for m outside M, i.e., $r \geq R$: $\quad F = \dfrac{GMm}{r^2} \quad (2.68)$

for m inside M, i.e., $r < R$: $\qquad F = \dfrac{GM^*m}{r^2}$ (2.69)

where M^* is the mass inside a sphere of radius r. For a uniform sphere:

$$\frac{M^*}{M} = \frac{r^3}{R^3} \Rightarrow \text{ for a uniform sphere: } \quad F = \frac{GMmr}{R^3} \quad \text{for } r < R \tag{2.70}$$

2. The resultant force between *two separate radially symmetric spheres* of mass M and m (Fig. 2.42b), with centres at a distance r apart, is $F = GMm/r^2$.
3. The *simplest resultant* of the concurrent gravitational force between *a radially symmetric sphere of mass M and an arbitrary body of mass m* (Fig. 2.43a) is a single force $\vec{F} = -GM\int_m(\vec{r}_P/r_P^3)\,dm$ through O. The *centre of gravity G* of m due to M exists and is given by

$$-GMm\frac{\vec{r}_G}{r_G^3} = -GM\int_m \frac{\vec{r}_P}{r_P^3}\,dm \tag{2.71}$$

In general, the location of G differs from C. If the *size of the body is small* compared to its distance from the sphere (Fig. 2.43b,c), then $\vec{r}_P = \vec{r}_{PC} + \vec{r}_C \simeq \vec{r}_C$ and eq. (2.71) $\Rightarrow \vec{r}_G \simeq \vec{r}_C$, i.e., model G to be at C and $\vec{F} = -GMm\vec{r}_C/r_C^3$.

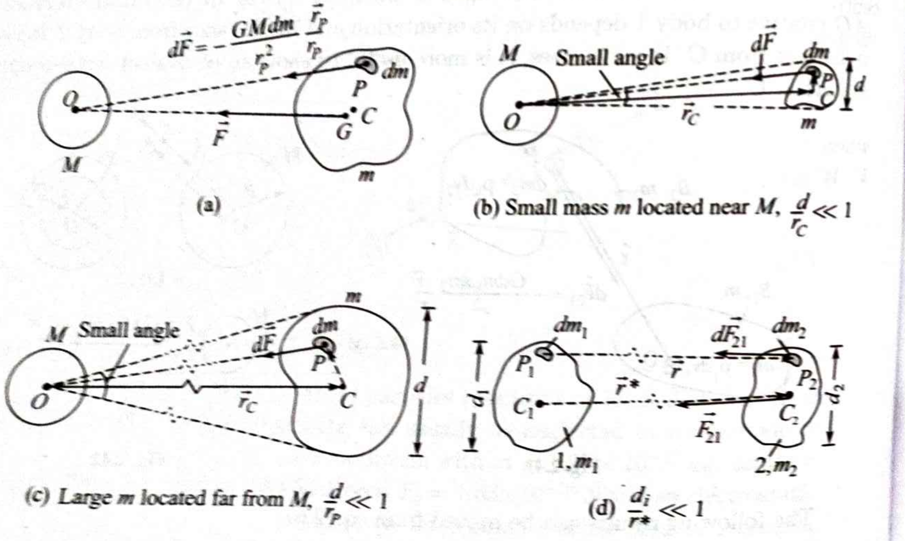


Fig. 2.43

Note: The distinction between C and G is rarely of importance in engineering applications on earth, but it is of interest in celestial mechanics and dynamics of

space vehicles. In Skylab, 3 control moment gyros were installed to cancel $\vec{H}_C$ caused by moment $\vec{M}_C$ of a gravitational force of only a few N.m. Since the earth is not an exact sphere, the resultant of the gravitational force of the sun and the moon does not always pass through the same point. It exerts a variable moment about the centre of mass of the earth causing a gradual change in the direction of the earth's axis of rotation, which is called precession of the equinoxes.

4. For *two bodies of arbitrary shapes whose sizes d_1, d_2 are small* compared to the distance r^* separating them (Fig. 2.43d), since $\vec{r} = \overrightarrow{P_1P_2} = \overrightarrow{P_1C_1} + \overrightarrow{C_1C_2} + \overrightarrow{C_2P_2} \simeq \overrightarrow{C_1C_2} = \vec{r}^*$, eq. (2.67) yields $\vec{F}_{21} \simeq -Gm_1m_2\vec{r}^*/r^{*3}$.

2.16.1 Model of earth's gravitational field

For most practical problems, the variation of the earth's gravitational force due to its non-spherical shape, radially un-symmetric mass density and its motion can be neglected. Earth is modelled as a radially symmetric sphere of radius $R = 6395$ km with mass M equal to that of the physical earth. Its gravitational force F on a mass-point m at a distance r from its centre is given by eqs (2.68) and (2.69), and whose variation with r is shown in Fig. 2.44. The variation in the interior is not linear in r (as in a uniform sphere) since the density increases as its centre is approached. Consider a body of mass m whose dimensions are much smaller than its distance r from earth's centre. Its weight W and acceleration g of its centre of mass C in 'free fall' (i.e., in vacuum) under the influence of the earth's gravitational field (Fig. 2.45) are given by

$$W = GMm/r^2 = mg = \rho g V = \gamma V \quad \Rightarrow \quad g = GM/r^2 \tag{2.72}$$

where $\gamma = \rho g = \rho GM/r^2$ is the weight density of the body having density ρ and volume V. W and g at the surface of the earth ($r = R$) are: $W_0 = GMm/R^2$, $g_0 = GM/R^2$.

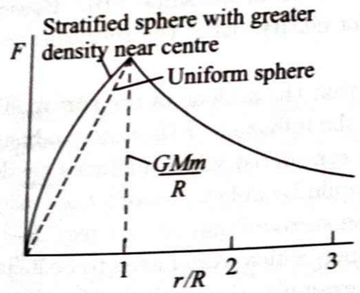


Fig. 2.44

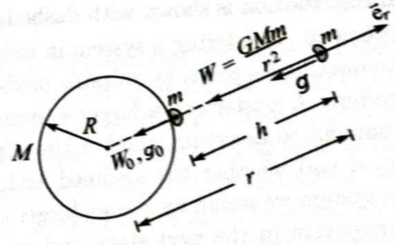


Fig. 2.45

Equation (2.72) $\Rightarrow W \simeq W_0$, $g \simeq g_0$ if $h/R \ll 1$, with only 1% error for $h < 30$ km. For 1° variation in the direction of g_0 on the earth's surface, the corresponding arc-length $s = R(\pi/180) = 111.7\,\text{km}$ $\Rightarrow$ the direction of g_0 varies by less than 0.01° for $s = 1\,\text{km}$. Thus, for a body of dimensions very much less than R, moving through small distances ($s \ll R$, $h \ll R$) near the surface of the earth, the distributed gravitational force is modelled as a parallel uniform field with force $d\vec{F} = dm\, g\vec{e}$ on mass element dm, where $\vec{e}$ is the local vertically downward direction. Their resultant is a constant force, $\vec{F} = mg\vec{e}$ acting at the centre of mass C.

Two uniform spheres of mass 5 kg each, with centres 10 cm apart, attract with a force of $16.68{\times}10^{-8}$ N, which is extremely small compared to their weight of 49.05 N at the earth's surface. Hence, the mutual gravitational forces between objects on the earth's surface are negligible compared to their weights unless the masses involved are great.

2.17 Free Body Diagram and Reactions of Supports

To **apply Euler's axioms to a system** consisting of n connected/unconnected bodies or mass-points, or a body or a finite/an infinitesimal part of a body,

1. draw a sketch of the *well-identified* system *in isolation* from its surroundings, and
2. show the *external forces and moments* exerted by the surroundings on the system.

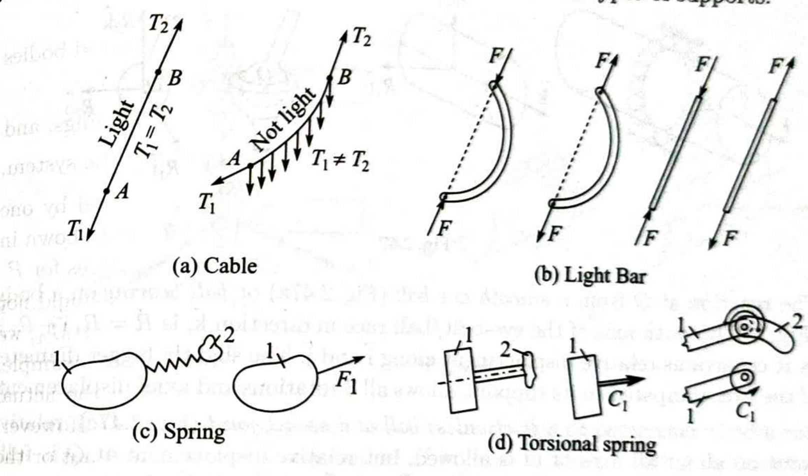
Such a diagram is called a **Free body diagram** (FBD). The forces exerted by one part of the system on another part P are called *internal forces* and are not shown in an FBD. Such forces appear in the FBD of part P as these are external forces for P.

Inertial forces like centrifugal force, Coriolis force, $-m\vec{a}_A$, $-m\vec{a}_C$, etc., should not appear in an FBD. Since the purpose of drawing an FBD is to obtain $\vec{F}$ and $\vec{M}_A$, we often replace some external forces by their simpler equivalent systems, for example, uniform gravitational force is often shown as mg at C. In general, both the actual force distribution and its equivalent should not be shown in the same FBD. However, sometimes for convenience, both are shown and for clarity either the resultant or the actual distribution is shown with dashed arrows.

Implicit in considering a system in isolation is that the actions of the surroundings can be obtained a priori by suitable modelling of the influence of the surroundings on the system. A posteriori, a larger system can be considered which includes the local environment to ascertain whether the supports would be able to absorb the reactions obtained and whether the assumed actions of the surroundings are correct. For this larger system we would have even larger surroundings which would need to be included in our system in the next stage and so on. For example, the analysis of a dam is a complex problem of structure, foundation, water and soil interaction including the effect of waves due to earthquakes whose intensity and frequency could be affected by the huge water reservoir and the dam. We need to consider the tectonic plates involved to address the problem of earthquakes. For the analysis of the dam, the actions of its

surroundings are suitably modelled. The accuracy of a model can only be 'checked' a posteriori by actually measuring the response of the surrounding systems.

The loads exerted by the supports on the system are called *reactions*. The nature of the load system that body B_2 can exert on body B_1 through some connector or support depends on the type of constraint on their relative motion provided by it. It exerts a force component for each of the relative displacement components constrained by it, and a moment component for each of the relative rotation components constrained by it. To find out which relative displacement or rotational components are constrained, we mentally try to relatively translate or rotate the bodies in three convenient orthogonal directions. While drawing an FBD, *no attempt should be made to decide the direction of a constraint force/moment component, except for the friction force for impending slip/actual slip.* We consider several types of supports.



(a) Cable (b) Light Bar (c) Spring (d) Torsional spring

Fig. 2.46

1. A *flexible cable* (belt, chain, rope, string) is idealised as a perfectly flexible one-dimensional body with the internal force resultant being a tensile axial force if it is taut and a zero force if it is slack. At a point of fastening to a body, it provides a one-sided displacement constraint as it can only apply a pull in its axial direction and not a push (Fig. 2.46a). A *'light' cable* subjected to only two forces (pulls) at its ends is a two-force member. Hence, the end forces must be along the line joining the ends, the cable remaining straight. The cable is often modelled as *inextensible*. The forces in inextensible cables can change instantaneously.
2. A *light bar* subjected to only two forces at its points of connection is a two-force member and provides a two-sided displacement constraint at the points of connection along the line joining them, since it can apply a force F (as a pull or

as a push) in this direction (Fig. 2.46b).

3. For a body connected to an *ideal (inertia-less) spring* under extension e, a force $F_1(e)$ is exerted on the body along the axis of the spring (Fig. 2.46c).
4. For a body connected to an *ideal (inertia-less) torsional spring* under twist θ, a moment $C_1(\theta)$ is exerted on the body along the axis of the spring (Fig. 2.46d).

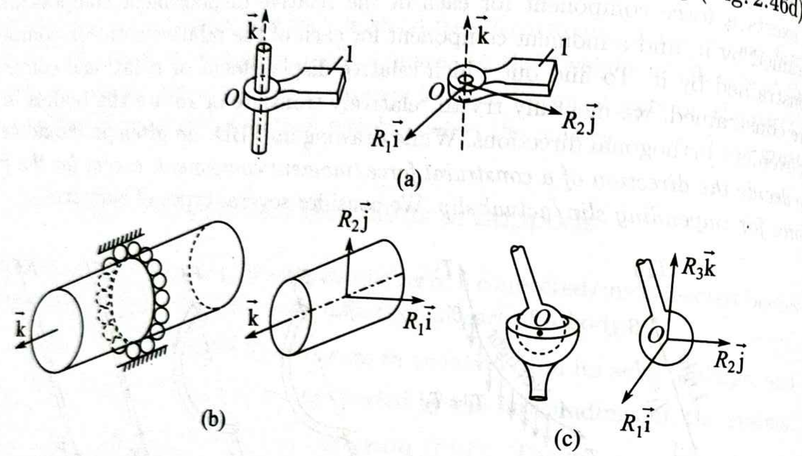


Fig. 2.47

5. The reaction at O from a *smooth eye-bolt* (Fig. 2.47a) or *ball bearing* on a body (Fig. 2.47b), with axis of the eye-bolt/ball race in direction $\vec{k}$, is $\vec{R} = R_1\vec{i} + R_2\vec{j}$, as it constrains relative displacement along $\vec{i}$ and $\vec{j}$. The slightly bigger diameter of the eye, compared to its support, allows all 3 rotations and axial displacement.
6. For a body connected to a *frictionless ball and socket joint* (Fig. 2.47c), relative rotation about all axes at O is allowed, but relative displacement at O is fully prevented. Hence, the reaction at O is $\vec{R} = R_1\vec{i} + R_2\vec{j} + R_3\vec{k}$.
7. In a *frictionless clevis joint* (Fig. 2.48a), the relative displacement along $\vec{k}$ and relative rotation about $\vec{j}$ and $\vec{k}$ are allowed. The reaction is $\vec{R} = R_1\vec{i} + R_2\vec{j}$ and $\vec{C} = C_1\vec{i}$. A similar reaction is exerted by the *rail on a flanged wheel* (Fig. 2.48b).
8. The reaction at O (Fig. 2.48c) from a *smooth rod on a collar*, with the axis of the rod in direction $\vec{k}$, is $\vec{R} = R_1\vec{i} + R_2\vec{j}$ and $\vec{C} = C_1\vec{i} + C_2\vec{j}$, since relative displacement along the rod axis $\vec{k}$ and relative rotation about it are allowed. A similar reaction is exerted by a *smooth roller bearing on a shaft* (Fig. 2.48d).
9. In a *frictionless pin and bracket* or a *frictionless hinge joint* or a *frictionless thrust bearing* with axis $\vec{k}$, relative displacement is constrained but relative rotation about axis $\vec{k}$ is allowed (Fig. 2.49a). The reaction at O is $\vec{R} = R_1\vec{i} + R_2\vec{j} + R_3\vec{k}$ and $\vec{C} = C_1\vec{i} + C_2\vec{j}$.

If the relative axial displacement of the pin is not constrained, then $R_3 = 0$, $\vec{R} = R_1\vec{i} + R_2\vec{j}$, $\vec{C} = C_1\vec{i} + C_2\vec{j}$ (Fig. 2.49b). This joint is same as a collar joint.

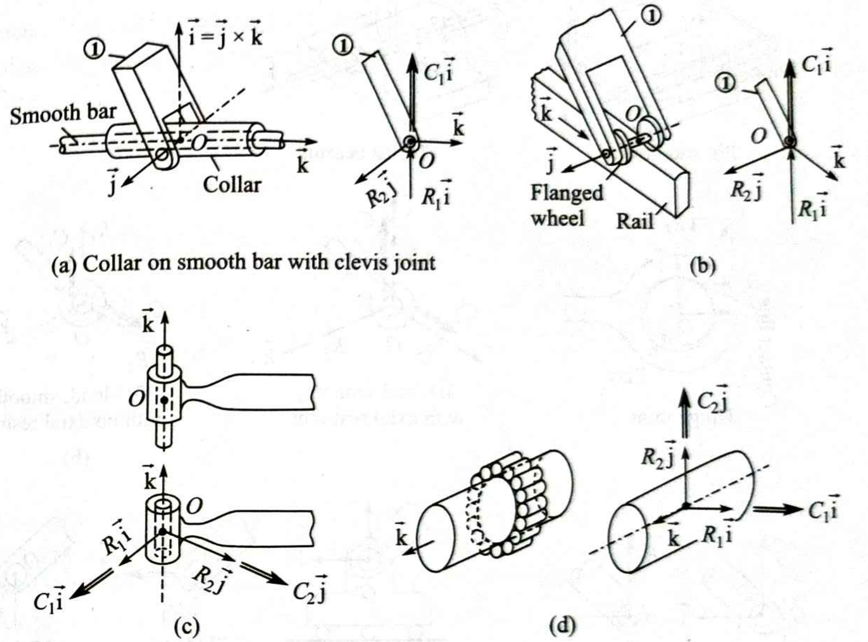


Fig. 2.48

If there are only coplanar loads in the mid-plane (xy-plane) of the joint, then $R_3\vec{k}$ and $C_1\vec{i} + C_2\vec{j}$ may not be of interest in the governing equations. Hence, for 2D loads, the relevant reaction is $\vec{R} = R_1\vec{i} + R_2\vec{j}$ (Fig. 2.49c).

For frictionless pin joints with a short pin, C_1, C_2, being of the order of force reaction times length of the pin, are small. Hence, only force reactions are included in the analysis unless it is found that C_1, C_2 are required to satisfy the equations of motion. For example, *the couple reactions are neglected for a shaft with two hinge joints, but are included for a shaft with only one hinge support.*

10. Consider a *smooth pin joint which itself is supported on smooth light spherical balls ('rollers')* on a plane (Fig. 2.49d) with normal $\vec{j}$. The pin support is free to move in directions $\vec{i}$ and $\vec{k}$, while the pin itself is free to rotate about the z axis relative to its support. Each ball is subjected to only normal reaction from the plane, as it is a two-force light member. The simplest resultant of these normal reactions is $N\vec{j}$ which has a moment $C_1\vec{i} + C_3\vec{k}$ about O. Reaction at O

is $\vec{F} = N\vec{j}$ and $\vec{C} = C_1\vec{i}$ since $C_3 = 0$ as pin allows free rotation about z axis.

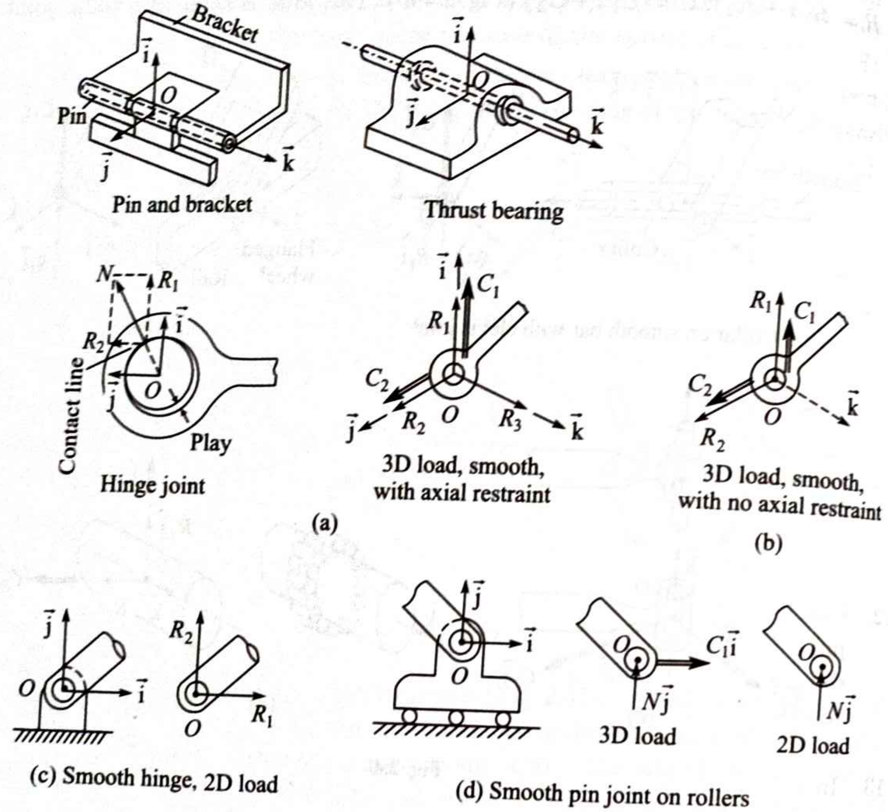


Fig. 2.49

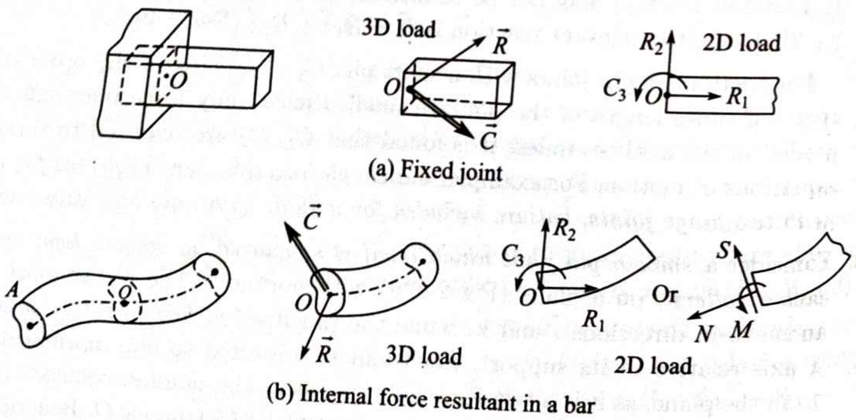


Fig. 2.50

11. In a *welded, 'fixed', 'built-in'* or *'encastre'* joint or a transverse *internal section of a thin bar*, all the relative displacements and rotations are constrained (Fig. 2.50a,b). Hence, the reaction is a force $\vec{R}$ and a moment $\vec{C}$ at O having arbitrary directions. For a coplanar force system acting on the body, only R_1, R_2, C_3, may be relevant, which for a straight/curved bar are usually shown as a normal force N, a shear force S normal to the axis and a bending moment M.

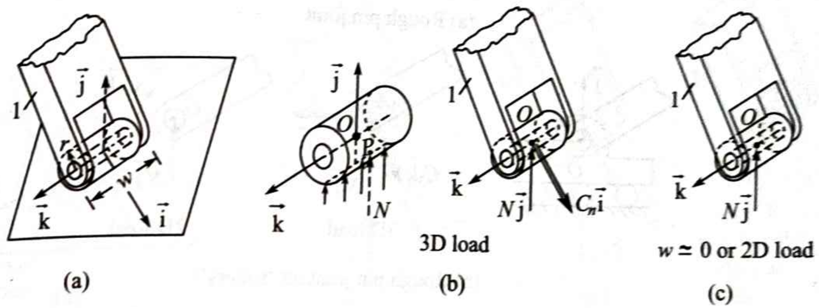


Fig. 2.51

12. Consider a *light circular cylindrical roller on a smooth surface* with a smooth pin (axis $\vec{k}$) connection to a body (Fig. 2.51a). It makes a line contact in the $\vec{k}$ direction with a plane normal to $\vec{j}$. Normal reaction on the pin is $N\vec{j}$ at $P \Rightarrow$ reaction at O is a force $N\vec{j}$ and a couple $C_n\vec{i}$ (Fig. 2.51b). If w is small, then $C_n \simeq 0$ and the reaction is $N\vec{j}$ (Fig. 2.51c). For 2D loads also, the reaction is $N\vec{j}$.
13. In a *rough pin joint*, the friction force $F\vec{e}_\phi$ at the line of contact causes a moment $C_3\vec{k}$ about O (Fig. 2.52a). Hence, the reaction at O is $\vec{R} = R_1\vec{i} + R_2\vec{j} + R_3\vec{k}$ and $\vec{C} = C_1\vec{i} + C_2\vec{j} + C_3\vec{k}$ having arbitrary directions. If the pin is not axially constrained, then R_3 is due to the axial friction force. For a 2D load system, the relevant reaction is $\vec{R} = R_1\vec{i} + R_2\vec{j}$ and $\vec{C} = C_3\vec{k}$.

    Consider a rough hinge joint supported on light spherical balls ('rollers') (Fig. 2.52b). Each ball is subjected to only normal reaction from the plane, since it is a two-force light member. The simplest resultant of these normal reactions on the balls is $N\vec{j}$ which has a moment $C_1\vec{i} + C_3\vec{k}$ about O. Hence, the reaction at O is $\vec{R} = N\vec{j}$ and $\vec{C} = C_1\vec{i} + C_3\vec{k}$. For 2D loads, it reduces to $N\vec{j}$ and $C_3\vec{k}$.
14. For a *rough ball and socket joint* (Fig. 2.52c), the friction force $F_\theta\vec{e}_\theta + F_\phi\vec{e}_\phi$ causes moment about all three axes. The reaction at O is $\vec{R} = R_1\vec{i} + R_2\vec{j} + R_3\vec{k}$ and $\vec{C} = C_1\vec{i} + C_2\vec{j} + C_3\vec{k}$. For 2D loads, it reduces to $\vec{R} = R_1\vec{i} + R_2\vec{j}$, $\vec{C} = C_3\vec{k}$.
15. A *universal joint* consists of a cross 3 whose two arms are pinned to shafts 1 and 2 (Fig. 2.53). Cross 3 can rotate relative to 1 about the $\vec{j}$ axis alone and 2 can rotate relative to 3 about the $\vec{i}$ axis alone. The point O of the cross is on the

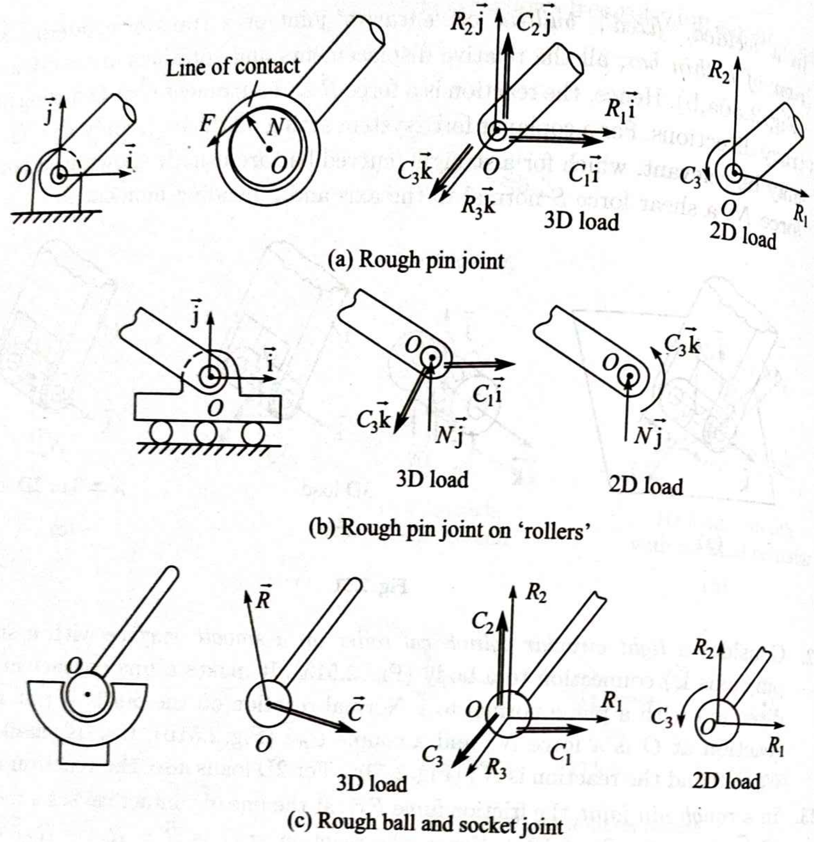


Fig. 2.52

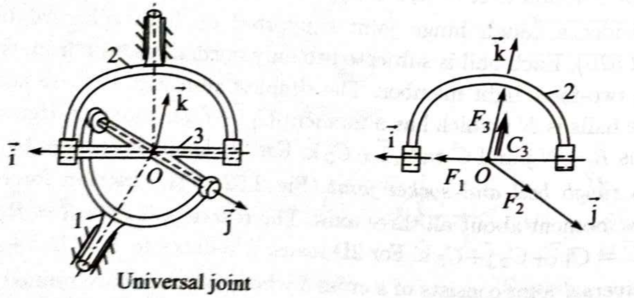


Fig. 2.53

rigid extensions of 1 and 2. It constrains translation of 2 relative to 1, but allows rotation of 2 relative to 1 along the axes $\vec{i}$ and $\vec{j}$ of the cross. For a *frictionless universal joint*, the reaction on 2 at O is $\vec{R} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ and $\vec{C} = C_3\vec{k}$.

16. Consider the reaction on a *light circular cylindrical roller on a rough surface* with a *smooth pin* connection to a body with axis $\vec{k}$ (Fig. 2.54a). It makes a line contact in direction $\vec{k}$ with a plane normal to $\vec{j}$.

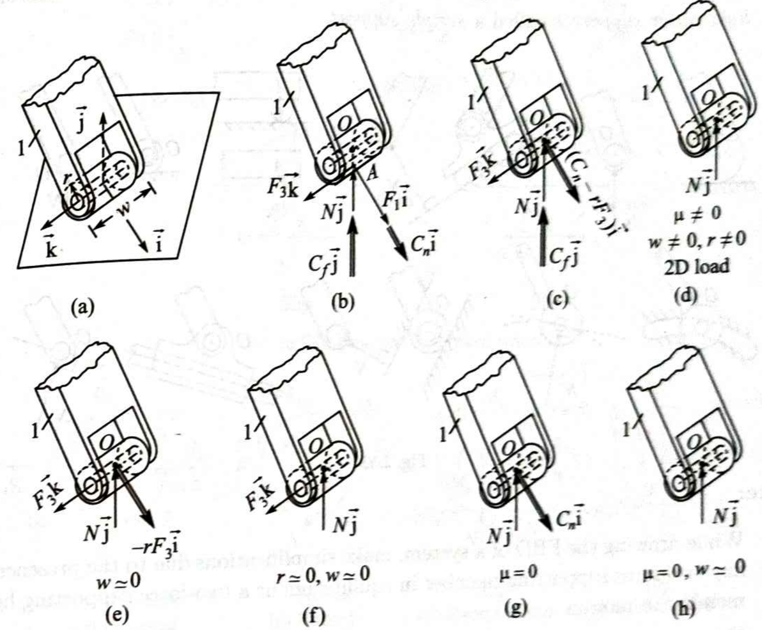


Fig. 2.54

The resultant (Fig. 2.54b) of the coplanar friction forces (in $\vec{i}$-$\vec{k}$ plane) at A is a force $F_1\vec{i} + F_3\vec{k}$ and a couple $C_f\vec{j}$. The resultant of the coplanar normal force (in $\vec{j}$-$\vec{k}$ plane) at A is a force $N\vec{j}$ and a couple $C_n\vec{i}$. The force–couple reaction at O is $\vec{R}_O = F_1\vec{i} + N\vec{j} + F_3\vec{k}$ and $\vec{C}_O = C_n\vec{i} + C_f\vec{j} - r\vec{j} \times (F_1\vec{i} + N\vec{j} + F_3\vec{k}) = (C_n - rF_3)\vec{i} + C_f\vec{j} + rF_1\vec{k} \Rightarrow rF_1 = 0$, since the roller can rotate freely about the pin axis $\vec{k}$. Hence, the force–couple reaction of the roller support on the body (Fig. 2.54c) at O is a force $\vec{R}_O = N\vec{j} + F_3\vec{k}$ and a couple $\vec{C}_O = (C_n - rF_3)\vec{i} + C_f\vec{j}$.

For 2D load system in xy plane, the relevant reaction at O is a force $\vec{R}_O = N\vec{j}$ alone (Fig. 2.54d), irrespective of the size (r, w) of the roller and its coefficient of friction μ. If the width w of the roller is small, then $C_f \simeq 0$, $C_n \simeq 0$ and the reaction at O

is $\vec{R}_O = N\vec{j} + F_3\vec{k}$ and $\vec{C}_O = -rF_3\vec{i}$ (Fig. 2.54e). If the radius r of the roller is also small, it reduces to only a force reaction $\vec{R}_O = N\vec{j} + F_3\vec{k}$ at O (Fig. 2.54f).

If the roller is smooth, then the reaction at O is $N\vec{j}$ and $C_n\vec{i}$ (Fig. 2.54g). If the width of the smooth roller is small, then $C_n \simeq 0$ and the reaction is $N\vec{j}$ (Fig. 2.54h).

For coplanar loads, for all the joints shown in Fig. 2.55, the relevant reaction is just a normal force N with no couple, irrespective of the size of the smooth or rough roller. A *light roller support* is called a *simple support*.

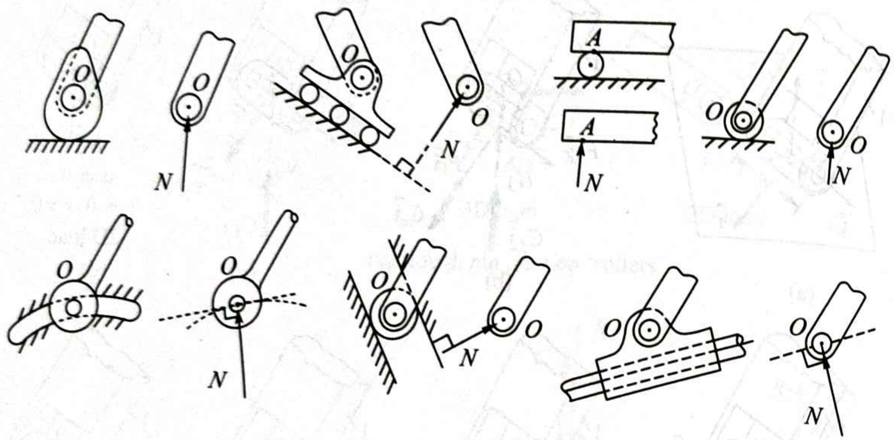


Fig. 2.55

Note:

1. While drawing the FBD of a system, make simplifications due to the presence of any two-force supporting member in equilibrium or a two-force supporting light member in motion.
2. Once support reactions on body B_1 due to body B_2 have been shown in the FBD of B_1, then in the FBD of B_2 show the same support reactions but with opposite sense. If these reactions are $\vec{R}$, $\vec{C}$ on B_1 then show $-\vec{R}$, $-\vec{C}$ on B_2.
3. Most often, the FBD of the pulley is not useful. It is more useful to draw the FBD of the system of the pulley and a part of the belt which overlaps it, since the internal contact force between the pulley and the belt does not appear in it.
4. If more than two members meet at a hinge joint, then either the FBD of the pin is drawn separately, or preferably, the FBDs of the members are drawn with the pin forming part of one of the members along with the load, if any, acting on it.
5. Consider FBDs of wheels of a vehicle, in which their inertia is included. One equation of motion is $M_{C_z} = I^C_{zz}\dot{\omega}$. Hence, the FBDs for the un-powered and un-braked wheels, shown in Fig. 2.56a, include a friction force to ensure angular

acceleration. For the case of no slip, the direction of friction force F follows from the equation $M_{C_z} = I^C_{zz}\dot{\omega}$, though it could be drawn in either direction.

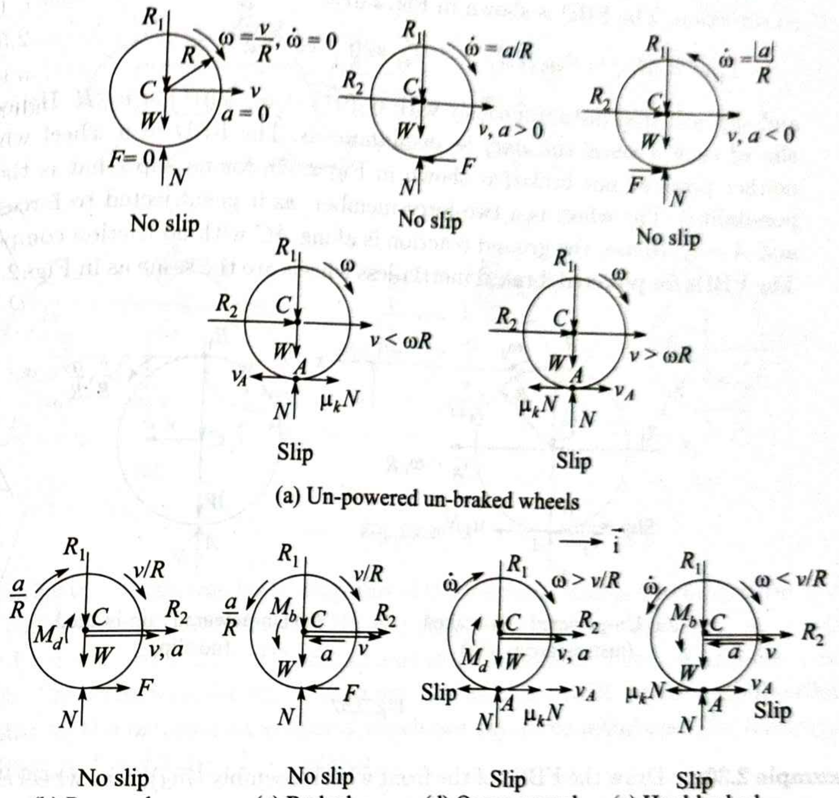


Fig. 2.56

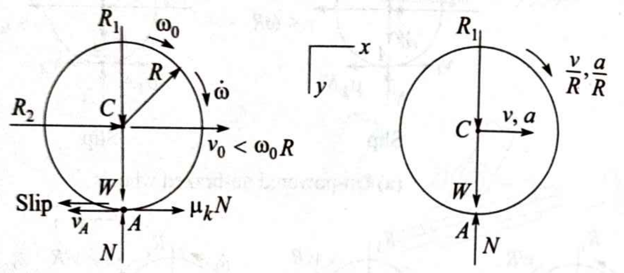
The FBDs for the case of powered and braked wheels rolling without slip are shown in Figs 2.56b,c. F could be drawn in either direction since there is no slip. The corresponding FBDs for the case of over-powered and hard-braked wheels for which slip occurs are shown in Figs 2.56d,e. For the over-powered case, greater driving moment $\Rightarrow \dot{\omega} > a/R \Rightarrow \omega > v/R$, $v_A = v - \omega R < 0$, i.e., A slips in $-\vec{i}$ direction and $F = \mu_k N$ is in $\vec{i}$ direction. For the hard-braked case, since a and $\dot{\omega}$ are negative, and greater braking torque implies that $|\dot{\omega}| > |a/R| \Rightarrow \omega < v/R$, $v_A = v - \omega R > 0$, i.e., A slips in $\vec{i}$ direction and $F = \mu_k N$ is in $-\vec{i}$ direction.

6. Consider FBDs of wheels of a vehicle in which their inertia is neglected. First

consider the case of an un-powered, un-braked wheel (Fig. 2.57a) with $v_C(0) = v_0$, $\omega(0) = \omega_0$, $v_0 < \omega_0 R$. $\vec{v}_A(0) = -(\omega_0 R - v_0)\vec{i} \Rightarrow$ initial slip of A is in the $-\vec{i}$ direction. The FBD is shown in Fig. 2.57a.

$$I_{zz}^C \dot{\omega} = M_{C_z} = -\mu_k N R, \qquad I_{zz}^C \approx 0 \quad \Rightarrow \quad \dot{\omega} = \infty$$

and slip will stop instantaneously with $v_C(0^+) = v_0$, $\omega(0^+) = v_0/R$. Hence, the *slip of such a wheel can only be instantaneous.* The FBD of a wheel which is neither powered nor braked is shown in Fig. 2.57b for no slip (that is the only possibility). The wheel is a two-force member, as it is subjected to forces at C and A only. Hence, the ground reaction is along AC with no friction component. The FBDs for powered/braked inertia-less wheels are the same as in Figs 2.56b-e.



(a) Un-powered / un-braked (instantaneous slip)

(b) Un-powered / un-braked (no slip)

Fig. 2.57

Example 2.20: Draw the FBDs of the front wheel assembly (m_3), rear wheel assembly (m_1), chassis (m_2), and the complete road roller accelerating under driving torque M_0 on the rear wheels (Fig. E2.20a). Assume no slip.

Solution: The friction force F_2 on the front wheels (Fig. E2.20b) is in the backward direction to provide a moment about C_3 in the direction of its angular acceleration a/r. The reaction from the axle on it has a component R_3 in the forward direction to provide forward acceleration a (since F_2 is backward) and a component R_4 downward due to the weight of the chassis. In the FBD of the complete vehicle (Fig. E2.20c), the friction force F_1 on the rear wheels is in the forward direction to provide forward acceleration to the vehicle (since F_2 is backward). The driving torque, being internal to the system, does not appear in this FBD. In the FBD of the chassis (Fig. E2.20d), the reaction R_1 from the axle is in the forward direction to provide it a forward acceleration (since R_3 is in the backward direction) and the reaction R_2 is upwards to support its weight. The FBD of the rear wheel assembly (Fig. E2.20e) is now completed with the

driving moment M_0 on the rear axle shown in the direction of its angular acceleration, and an opposite moment is shown as acting on the chassis (Fig. E2.20d).

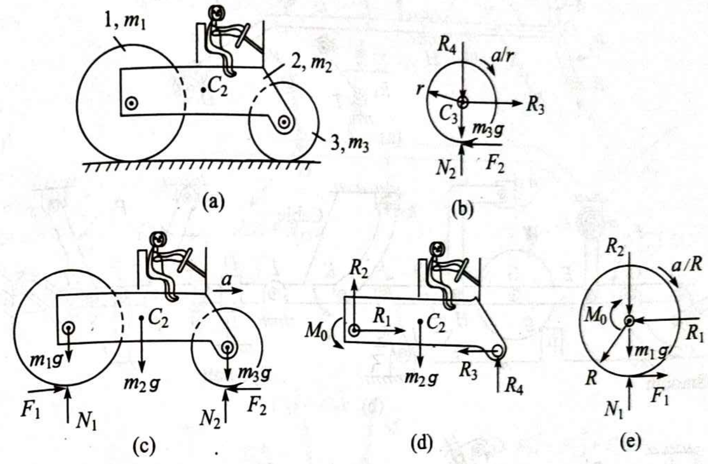


Fig. E2.20

Example 2.21: Draw free body diagrams of the following bodies for the systems given in Fig. E2.21a–d: (a) $ABOD$, $ABOE$, (b) AB, AE, (c) 1, 2, 3, 4+5, 4, 5, AB, 1+2+3+4+5, (d) 1, 2, 3. All contact surfaces are smooth. The belt and the cables are light. The members, for which no mass is shown, are light. Make simplifications, if any, due to the presence of supports which are two-force members. The loading on the systems in Fig. E2.21b–d is coplanar.

Solution: (a) The FBDs are shown in Fig. E2.21e. All relative rotations are allowed at an eye-bolt, but the relative displacements normal to the axis of the bolt are prevented. Hence, the eye-bolt exerts two components of force in the two directions normal to the axis of its eye. The supporting members PQ and RS, being two-force members, exert forces along PQ and RS only. However, the member TU exerts 3 force components, since it is not a two-force member. At the support B in the first FBD and at the internal section at E in the second FBD, three force components and three couple components are shown, because there is complete constraint on the relative rotation and relative displacement and the applied force system is three-dimensional. The reaction at the ball and socket joint D has three force components as complete constraint on translation occurs, but there is no rotational constraint. The hinge at H with axis y provides a force reaction $R_{14}\vec{i} + R_{15}\vec{k}$ and a moment reaction $C_4\vec{i} + C_5\vec{k}$, since it permits axial displacement and prevents rotation about $\vec{k}$ and $\vec{i}$ only.

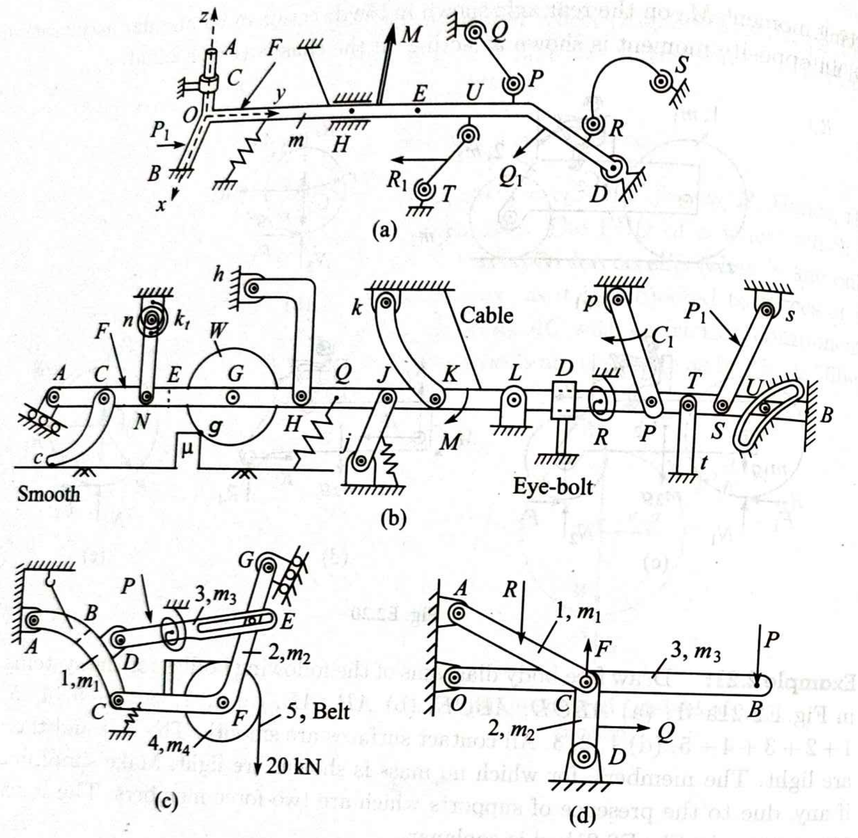


Fig. E2.21a-d

(b) The FBDs are depicted in Fig. E2.21f. The supporting members Kk and Hh, being two-force members, exert forces along Kk and Hh only. Member Cc is a two-force member, and the support reaction $N\vec{e}_n$ must be zero as it is not in the direction Cc. Thus, Cc does not apply any reaction at C on member AB. The member Gg is subjected to a force at G (pin reactions F_1, F_2 and weight W) and a force at g (normal and friction forces). Hence, it is a two-force member and reaction R_3 at g is along Gg only $\Rightarrow$ $\vec{W} + \vec{R}_3 - F_1\vec{i} - F_2\vec{j} = \vec{0} \Rightarrow F_1\vec{i} + F_2\vec{j} = \vec{W} + \vec{R}_3$. The members Ss, Tt, Pp, Jj, Nn are not two-force members since an additional force/couple acts on them in between or a couple also acts at one end. Each of them provides a force reaction on AB with two non-zero components in general, since the loading is coplanar. At the support B in the first FBD and at the internal section at E in the second FBD, two in-plane force components and one out-of-plane couple component are shown, because

there is complete constraint on the relative rotation and relative displacement, and the applied force system is coplanar.

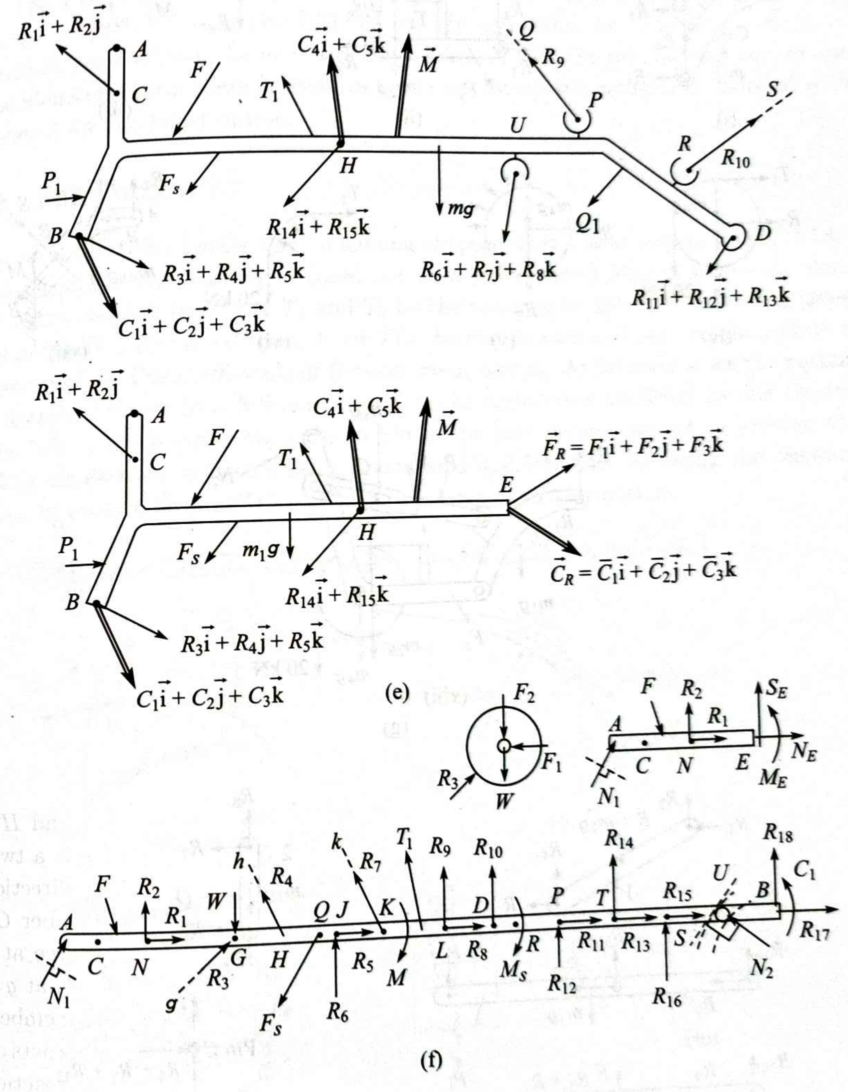


Fig. E2.21e,f

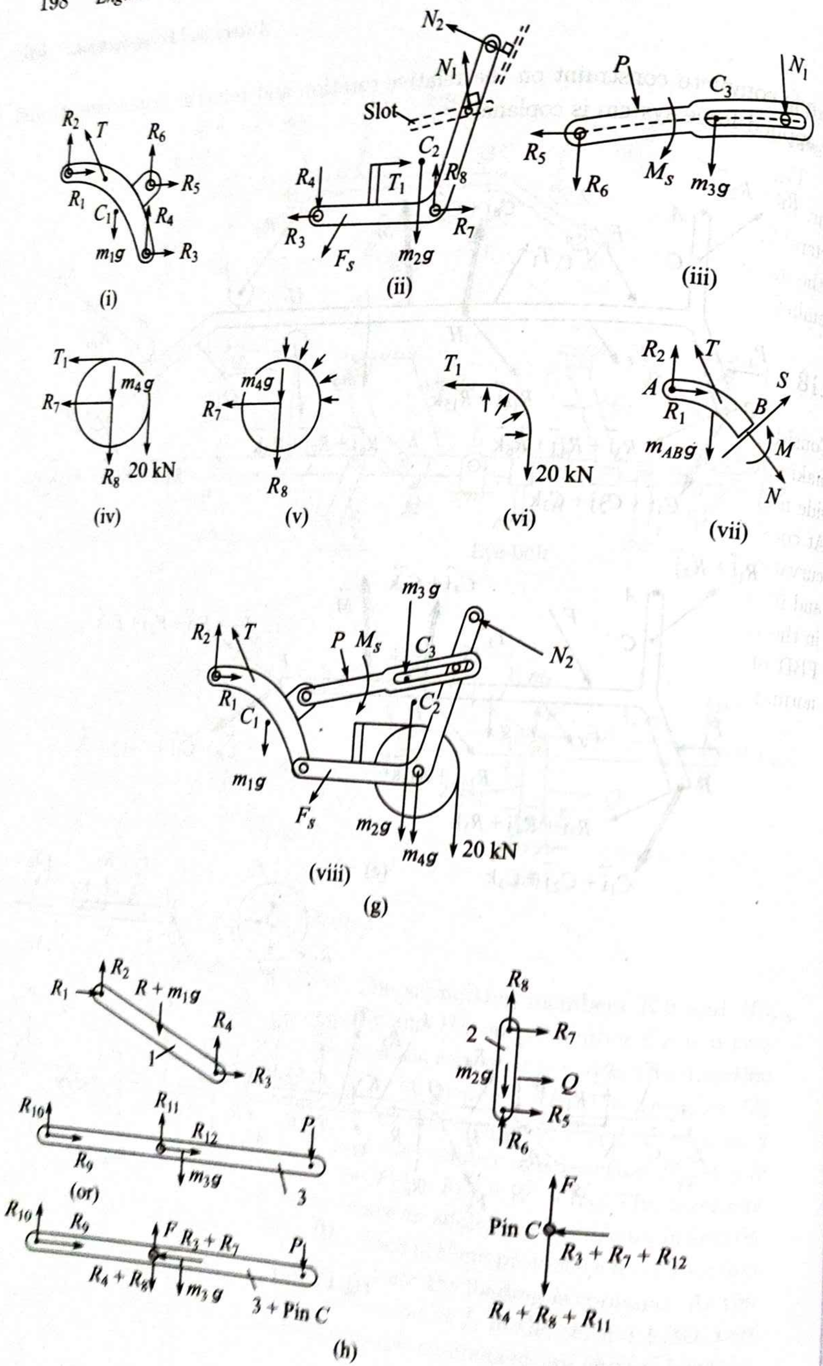


Fig. E2.21g,h

(c) The FBDs are shown in Fig. E2.21g. The FBD of the pulley has been drawn for the system consisting of the pulley and the part of the belt which overlaps it.

(d) The FBDs are shown in Fig. E2.21h. If more than 2 members meet at a hinge joint, as at joint C, then the FBD of the pin can either be drawn separately, or preferably, the FBDs of the members can be drawn, with the pin forming part of one of the members. Both these procedures have been illustrated with pin C included with member 3 for the latter option.

2.18 Belt Friction

Consider a flat belt of mass λ kg/m moving at speed v on a *fixed surface* S (Fig. 2.58a) making contact over a curve C (need not be a plane curve) from A where the slack side makes contact to B. Let T_1 and T_2 be the tensions on the tight and slack sides. At coordinate s measured from A, let $T(s)$ be the tension and $\rho(s)$ be the radius of curvature of C. The coefficients of friction are μ_k and μ_s. At location s, let the normal and frictional forces be p N/m and $\mu_k p$ N/m. In engineering applications, the tension in the belt is much larger than the weight of the belt. Hence, neglecting gravity, the FBD of an element of length Δs is shown in Fig. 2.58b with $\vec{e}_N$ being the outward normal to surface S at s. The equation of motion of this element is:

$$(T\vec{e}_t)|_{s+\Delta s} - (T\vec{e}_t)|_s + (p\vec{e}_N)_{av}\Delta s - (\mu_k p\vec{e}_t)_{av}\Delta s = \lambda\Delta s\left(\frac{v^2}{\rho}\vec{e}_n\right)_{av}$$

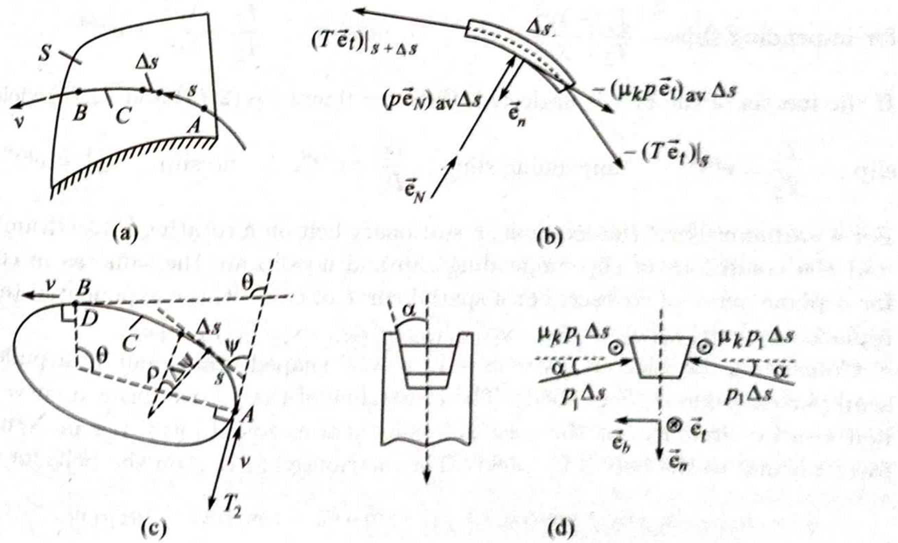


Fig. 2.58

Divide by Δs, take limit as $\Delta s \to 0$ and use $d\vec{e}_t/ds = \vec{e}_n/\rho$, $\vec{e}_t \perp \vec{e}_N$:

$$\frac{d}{ds}(T\vec{e}_t) + p\vec{e}_N - \mu_k p\vec{e}_t = \frac{\lambda v^2}{\rho}\vec{e}_n \;\Rightarrow$$

$$\frac{dT}{ds}\vec{e}_t + \frac{T}{\rho}\vec{e}_n + p\vec{e}_N - \mu_k p\vec{e}_t = \frac{\lambda v^2}{\rho}\vec{e}_n$$

$$\vec{e}_n: \; p = \frac{T-\lambda v^2}{\rho} \quad \text{and} \quad \vec{e}_N = -\vec{e}_n$$

$$\vec{e}_t: \; \frac{dT}{ds} = \mu_k p = \mu_k\frac{T-\lambda v^2}{\rho} \quad \Rightarrow \quad \int_{T_2}^{T}\frac{1}{T-\lambda v^2}\,dT = \int_0^s \frac{\mu_k}{\rho}\,ds$$

$$\frac{T-\lambda v^2}{T_2-\lambda v^2} = \exp\left[\int_0^s \frac{\mu_k}{\rho}\,ds\right], \quad p = \frac{T_2-\lambda v^2}{\rho}\exp\left[\int_0^s \frac{\mu_k}{\rho}\,ds\right] \tag{2.73}$$

For a plane curve C shown in Fig. 2.58c, the contact is over an arc AB, the belt changes its direction by angle θ, $ds/\rho = d\psi$ and eq. (2.73) reduces to

$$\frac{T-\lambda v^2}{T_2-\lambda v^2} = e^{\mu_k\psi}, \quad p = \frac{T_2-\lambda v^2}{\rho}e^{\mu_k\psi}, \quad \frac{T_1-\lambda v^2}{T_2-\lambda v^2} = e^{\mu_k\theta} \tag{2.74}$$

where ψ and θ are in radians. $T_{max}(=T_1)$ occurs at B, where the tension is in the direction of $\vec{v}_{BD}$. If the belt is on a *rotating pulley*, then eq. (2.74) holds for slip and

$$\text{for impending slip}: \; \frac{T_1-\lambda v^2}{T_2-\lambda v^2} = e^{\mu_s\theta}, \qquad \text{no slip}: \; \frac{T_1-\lambda v^2}{T_2-\lambda v^2} < e^{\mu_s\theta} \tag{2.75}$$

If the inertia of the belt is neglected, then $\lambda = 0$ and eqs (2.74) and (2.75) yield for

$$\text{slip}: \; \frac{T_1}{T_2} = e^{\mu_k\theta}, \qquad \text{impending slip}: \; \frac{T_1}{T_2} = e^{\mu_s\theta}, \qquad \text{no slip}: \; \frac{T_1}{T_2} < e^{\mu_s\theta} \tag{2.76}$$

For a *stationary* belt (for example, a stationary belt on a rotating brake drum), $v = 0$ and the conditions of slip, impending slip and no slip are the same as in eq. (2.76) for a plane curve of contact. For a spatial curve of contact, the exponential terms are replaced by $\exp[\int_0^s(\mu_k/\rho)\,ds]$, $\exp[\int_0^s(\mu_s/\rho)\,ds]$, $\exp[\int_0^s(\mu_s/\rho)\,ds]$.

Consider a flexible rope/'vee' belt in a 'vee' shaped groove cut in a pulley, with semi-vertex angle α (Fig. 2.58d). The centre line of the groove forms a curve C. The belt can be circular, but the wear is less for a trapezoidal form. Let p_1 N/m be the force normal to the belt (Fig. 2.58d). The distributed force $\vec{q}$ on the belt, for slip is

$$\begin{aligned}\vec{q} &= p_1(-\sin\alpha\,\vec{e}_n + \cos\alpha\,\vec{e}_b) + p_1(-\sin\alpha\,\vec{e}_n - \cos\alpha\,\vec{e}_b) - 2\mu_k p_1\vec{e}_t \\ &= -2\mu_k p_1\vec{e}_t - 2p_1\sin\alpha\,\vec{e}_n\end{aligned}$$

The forces in $-\vec{e}_n$ and $-\vec{e}_t$ directions are $p = 2p_1\sin\alpha$ and $2\mu_k p_1 = (\mu_k/\sin\alpha)p$ instead of $\mu_k p$ for a flat belt. Hence, eqs (2.73)–(2.76) are valid with an effective coefficient of friction $\mu_{eff} = \mu/\sin\alpha\ (>\mu)$ replacing μ. For a 'vee' belt, eq. (2.76) becomes for

$$\text{slip}: \; \frac{T_1}{T_2} = e^{\mu_k\theta/\sin\alpha}, \qquad \text{impending slip}: \; \frac{T_1}{T_2} = e^{\mu_s\theta/\sin\alpha}, \qquad \text{no slip}: \; \frac{T_1}{T_2} < e^{\mu_s\theta/\sin\alpha}$$

A 'vee' belt is better for power transmission than a flat belt since $\mu_{eff} = \mu_k/\sin\alpha > \mu_k$. For example, for $\alpha = 18°$, let $(T_1/T_2)_f, (T_1/T_2)_v$ be the limiting tension ratios for a flat and a 'vee' belt. Then $\mu_{eff} = \mu_s/\sin 18° = 3.24\mu_s \Rightarrow (T_1/T_2)_v = e^{\mu_s\theta/\sin\alpha} = [e^{\mu_s\theta}]^{1/\sin\alpha} = (T_1/T_2)_f^{3.24} \gg (T_1/T_2)_f$.

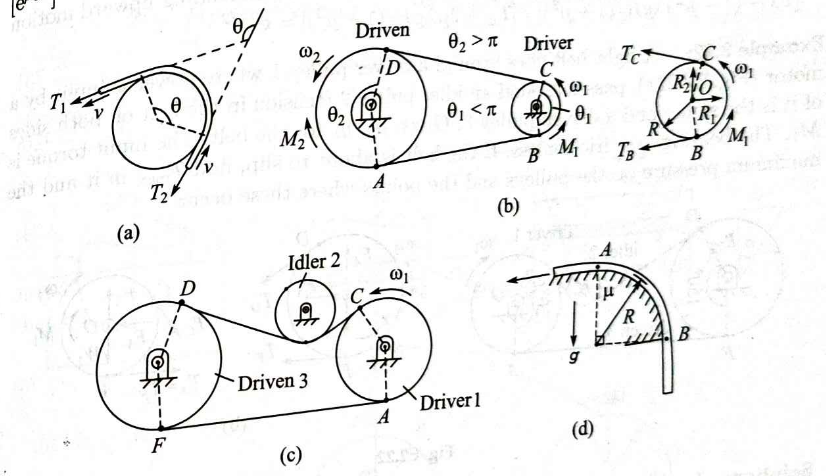


Fig. 2.59

For a circular pulley, angle θ is the angle of lap of the belt on the pulley (Fig. 2.59a).

For power transmission, if μ_s are the same for both pulleys (Fig. 2.59b), then the smaller angle of lap θ_1 is used in the above expressions. Idler pulleys are used to increase this value (Fig. 2.59c). On a driving pulley, the input torque M_1 is in the same direction as ω_1, whereas on the driven pulley, the load torque M_2 is in a direction opposite to ω_2 (Fig. 2.59b). For uniform rotation, from the FBD of the driver, $M_{Oz} = M_1 + (T_C - T_B)R = 0 \Rightarrow T_B > T_C$. In Fig. 2.59b, the belt tension is maximum in the segment AB and minimum in the segment CD, and the maximum pressure on the pulleys is at B, where T is maximum and ρ is smaller. Let T_0 be the allowable tension in the belt. Substituting T_2 from eq. (2.75)$_1$, the power P transmitted is

$$P = T_1 v - T_2 v = (T_1 - \lambda v^2)v(1 - e^{-\mu_s\theta})$$

For maximum P, $\dfrac{dP}{dv} = 0 \;\Rightarrow\; T_1 = 3\lambda v^2 = T_0$.

Hence, the maximum power is transmitted when $v^2 = T_0/3\lambda$.

§ **Exercise 2.6:** The speed v of the belt on a circular fixed pulley of radius R is increasing at the rate $\dot{v}$. Include acceleration a_t in the equation of motion of Δs to prove that eq. $(2.74)_3$ is replaced by $[T_1 - \lambda(v^2 - R\dot{v}/\mu_k)]/[T_2 - \lambda(v^2 - R\dot{v}/\mu_k)] = e^{\mu_k\theta}$.

§ **Exercise 2.7:** A heavy chain of mass λ kg/m is at rest (Fig. 2.59d) around a corner of radius R and coefficient of friction μ. The weight has to be included in the FBD of an infinitesimal element of the chain. Show that for impending upward motion $[T_A - (1-\mu^2)\lambda gR/(1+\mu^2)]\,/\,[T_B + 2\mu\lambda gR/(1+\mu^2)] = e^{\mu\pi/2}$.

Example 2.22: A light belt goes around a driver pulley 1 where power is input by a motor (Fig. E2.22a), passes around an idler pulley 2 (tension in the belt on both sides of it is the same) and a driven pulley 3. Given μ_s, μ_k for the belt. The input torque is M_1. The bearings are frictionless. If the belt is about to slip, find T_{max} in it and the maximum pressure on the pulleys and the points where these occur.

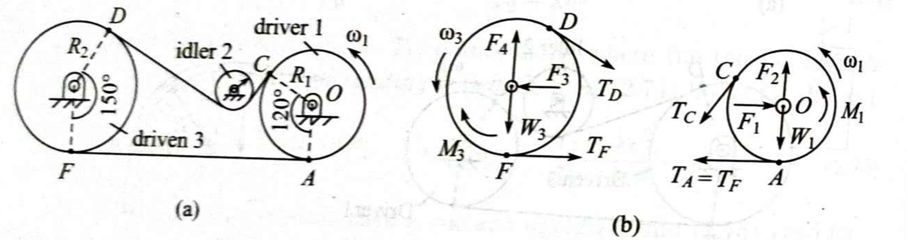


Fig. E2.22

Solution: In the FBDs (Fig. E2.22b), the driving torque is shown in the direction of ω_1 and the load torque is shown in the direction opposite to ω_3. For pulley 1:

$$M_{O_z} = M_1 + T_C R_1 - T_A R_1 = 0 \;\Rightarrow\; T_A = T_C + M_1/R_1 > T_C \qquad (1)$$

Hence, part AF carries $T_{max} = T_A$ and $p_{max} = T_{max}/R_1$ is at A where the radius of the pulley is smaller than at F. The angle of lap of 210° ($= 7\pi/6$ rad) on pulley 3 is less than that of 240° on pulley 1. Thus, the belt has impending slip on pulley 3:

$$\frac{T_A}{T_C} = \frac{T_F}{T_D} = \frac{T_{max}}{T_{min}} = e^{\mu_s\theta} = e^{7\pi\mu_s/6} \qquad (2)$$

(1), (2) $\Rightarrow$ $T_{max} = T_A = \dfrac{M_1}{R_1\left(1 - e^{-7\pi\mu_s/6}\right)}$, $\quad p_{max} = \dfrac{T_A}{R_1} = \dfrac{M_1}{R_1^2\left(1 - e^{-7\pi\mu_s/6}\right)}$

2.19 Frictional Torque at a Thrust Bearing and Frictional Clutches

Consider an axisymmetric body 1 (Fig. 2.60a) in contact with another over an axisymmetric surface (conical, flat, spherical, etc.). The axial torque M needed for impending axial rotation of body 1 under an axial thrust P is to be determined. The FBD of body 1 is shown in Fig. 2.60b. Consider an elementary ring of the contact surface of width ds and radius r with its normal making an angle $\pi/2 - \alpha$ with the axis. Let the normal force be p N/m². The frictional force is $\mu_s p$ N/m² in the circumferential direction at a radial distance r from the axis. Equations of equilibrium for axial force and axial moment yield

$$P = \int_s (p\sin\alpha)(2\pi r)\,ds, \qquad M = \int_s (r\mu_s p)(2\pi r)\,ds$$

$$\therefore\;\Rightarrow\; \frac{M}{P} = \frac{\int_s \mu_s r^2 p\,ds}{\int_s rp\sin\alpha\,ds} \qquad (1)$$

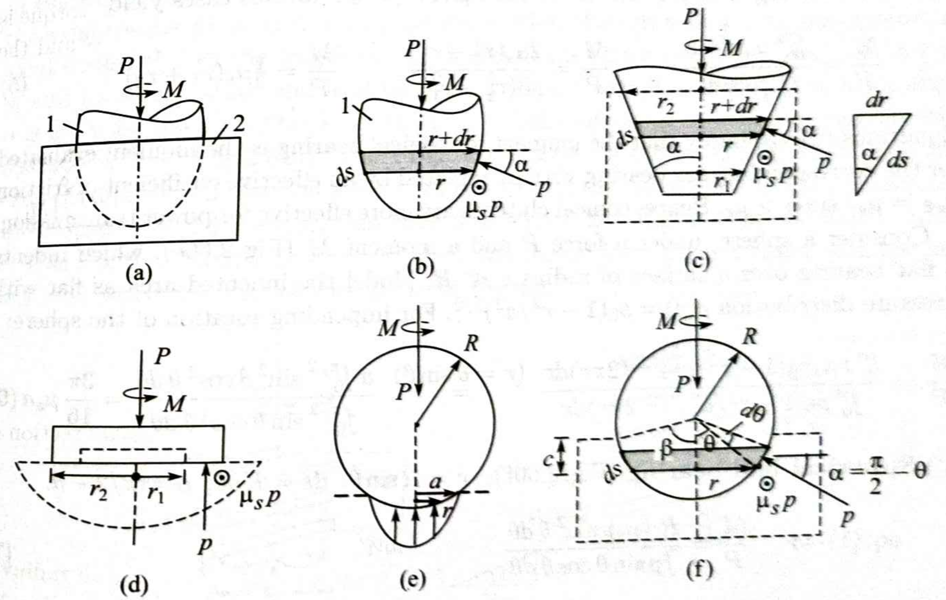


Fig. 2.60

(a) Conical bearing (Fig. 2.60c): α = constant, $ds = dr/\sin\alpha$, eq. (1)$\Rightarrow$

$$\frac{M}{P} = \frac{\int_{r_1}^{r_2} \mu_s r^2 p\,dr/\sin\alpha}{\int_{r_1}^{r_2} rp\,dr} \qquad (2)$$

1. For uniform p: $\quad M/P = (2\mu_s/3\sin\alpha)(r_2^3 - r_1^3)/(r_2^2 - r_1^2) \quad (3)$

2. For uniform wear: $p = C/r, \quad M/P = (\mu_s/\sin\alpha)(r_1 + r_2)/2 \quad (4)$

This approximation of uniform wear is good if the bearing has been worn out, since the surfaces retain their relative shape and hence further wear is constant over the surface. The amount of wear at radius r, in n revolutions, is proportional to (i) the distance $2\pi rn$ through which slip takes place, and (ii) the intensity of the frictional force $\mu_s p$. Hence, for uniform wear, $2\pi rn\mu_s p$ =constant, i.e., rp =constant= C (say). Let the moment for the cases 1 and 2 be denoted by M_1 and M_2. Then,

$$M_1 - M_2 = \frac{\mu_s P(r_1 - r_2)^2}{6(r_1 + r_2)} \sin\alpha > 0 \quad \Rightarrow \quad M_1 > M_2$$

i.e., the resisting moment due to friction decreases when the bearing is worn out. The decrease is 25% for $r_1 = 0$.

(b) **Flat bearing** (Fig. 2.60d): $\alpha = \pi/2$, eqs (2)–(4) for various cases yield

$$\frac{M}{P} = \frac{\int_{r_1}^{r_2} \mu_s r^2 p\, dr}{\int_{r_1}^{r_2} rp\, dr}, \qquad \frac{M}{P} = \frac{2\mu_s(r_2^3 - r_1^3)}{3(r_2^2 - r_1^2)}, \qquad \frac{M}{P} = \tfrac{1}{2}\mu_s(r_1 + r_2) \quad (5)$$

Equations (2)–(5) imply that the moment for conical bearing is the moment evaluated for the corresponding flat bearing with μ_s replaced by an effective coefficient of friction $\mu_{\text{eff}} = \mu_s/\sin\alpha > \mu_s$. Hence, conical clutches are more effective for power transmission.

Consider a sphere, under a force P and a moment M (Fig. 2.60e), which indents a flat bearing over a surface of radius $a \ll R$. Model the indented area as flat with pressure distribution $p(r) = p_0(1 - r^2/a^2)^{1/2}$. For impending rotation of the sphere:

$$\frac{M}{P} = \frac{\int_0^a r\mu_s p_0(1 - r^2/a^2)^{1/2}(2\pi r)dr}{\int_0^a p_0(1 - r^2/a^2)^{1/2}(2\pi r)dr} \underset{=}{(r = a\sin\theta)} \frac{a\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta}{\int_0^{\pi/2}\sin\theta\cos^2\theta\, d\theta} = \frac{3\pi}{16}\mu_s a \quad (6)$$

(c) **Spherical ball bearing** (Fig. 2.60f): $r = R\sin\theta$, $ds = R\,d\theta$, $\alpha = \pi/2 - \theta$,

$$\text{eq. (1)} \Rightarrow \quad \frac{M}{P} = \frac{R\int \mu_s p\sin^2\theta\, d\theta}{\int p\sin\theta\cos\theta\, d\theta} \quad (7)$$

For a ball bearing in a race, p is assumed to be either p = constant or $p = C/r$. For a sphere pressed into an elastic plate, we assume that p is proportional to normal indentation $c = R(\cos\theta - \cos\beta)$. For these three cases, M/P is obtained from eq. (7) as follows:

1. for p = constant: $\quad \dfrac{M}{P} = \dfrac{R\mu_s\int_0^\beta \sin^2\theta\, d\theta}{\int_0^\beta \sin\theta\cos\theta\, d\theta} = \dfrac{\mu_s R(2\beta - \sin 2\beta)}{1 - \cos 2\beta}$

2. for $p = \dfrac{C}{r} = \dfrac{C}{R\sin\theta}$: $\quad \dfrac{M}{P} = \dfrac{R\mu_s\int_0^\beta \sin\theta\, d\theta}{\int_0^\beta \cos\theta\, d\theta} = \mu_s R\tan\tfrac{1}{2}\beta$

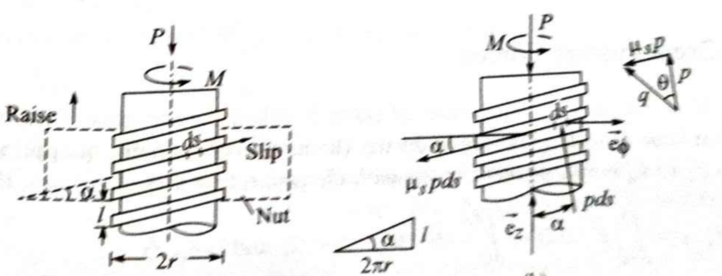
3. for $p \propto c$, $p = AR(\cos\theta - \cos\beta)$:

$$\frac{M}{P} = \frac{R\mu_s\int_0^\beta \sin^2\theta(\cos\theta - \cos\beta)\, d\theta}{\int_0^\beta \sin\theta(\cos^2\theta - \cos\theta\cos\beta)\, d\theta} = \frac{\mu_s R(3\sin\beta - 3\beta\cos\beta - \sin^3\beta)}{\cos^3\beta - 3\cos\beta + 2}$$

For actual slip, μ_k should replace μ_s.

2.20 Frictional Torque for a Square-Threaded Screw

Square-threaded screws are commonly used in screw-jacks and power-transmission (as in lead-screw of a lathe machine) and in testing machines and presses. We compute the axial torque M needed to impend the motion of a screw in a fixed nut against an axial thrust P. Let the length of the thread of the screw in contact with the nut be s_1 and its mean radius and lead be r and l respectively (Fig. 2.61a). The helix angle α is given by $\tan\alpha = l/2\pi r$ since the helix axially advances by l for a circumferential distance of $2\pi r$ (Fig. 2.61a). The FBD of the screw is shown in Fig. 2.61b. The nut exerts a distributed reaction force from one side of the thread of the screw. Since the width of the thread is small, this force is modelled as a line load at the mean radius r. This reaction extends from $s = 0$ where the thread first makes contact with the nut to $s = s_1$ where the thread emerges from the nut.



(a) (b)

Fig. 2.61

Let the normal force (per unit length) on the thread be p. This makes an angle α with the axis as shown. The frictional force (per unit length) is $\mu_s p$ along the helix in the direction opposite to the direction of impending slip. Hence, the force $d\vec{F}$ on an element of length ds at a distance r from the axis is

$$d\vec{F} = p\,ds(-\sin\alpha\,\vec{e}_\phi + \cos\alpha\,\vec{e}_z) + \mu_s p\,ds(-\cos\alpha\,\vec{e}_\phi - \sin\alpha\,\vec{e}_z)$$

Equations of equilibrium for axial force and axial moment yield

$$P = \int_0^{s_1} (p\cos\alpha - \mu_s p\sin\alpha)ds, \qquad M = \int_0^{s_1} r(p\sin\alpha + \mu_s p\cos\alpha)ds$$

$$\Rightarrow \quad \frac{M}{P} = \frac{r(\sin\alpha + \mu_s\cos\alpha)}{\cos\alpha - \mu_s\sin\alpha} = \frac{r(\tan\alpha + \mu_s)}{1 - \mu_s\tan\alpha} = r\tan(\alpha+\theta) \tag{1}$$

$$q_{av} = \left[\int_0^{s_1} \frac{p}{\cos\theta} ds\right] / s_1 = \frac{P\sec\theta}{s_1(\cos\alpha - \mu_s\sin\alpha)} = P\sec(\alpha+\theta)/s_1$$

where q_{av} is the average force per unit length on the thread and $\tan\theta = \mu_s$. The threads will be sheared off if they are not strong enough to sustain q_{av}. q_{av} can be reduced by increasing s_1, i.e., by increasing the number of threads in contact with the nut. The opposite torque $-M_1\,\vec{e}_z$ needed for impending motion in the direction of the axial thrust P can be obtained from eq. (1) by replacing M with $-M_1$ and μ_s with $-\mu_s$ since the directions of the torque and the friction force in this case are opposite to those of the previous case:

$$\frac{M_1}{P} = \frac{r(\mu_s - \tan\alpha)}{1 + \mu_s\tan\alpha} = r\tan(\theta - \alpha) \tag{2}$$

The screw is *self-locking* if it does not advance under P in the absence of M_1. Hence, it is self-locking if $M_1 > 0$, i.e., if $\tan\alpha < \mu_s$.

2.21 Conservative Forces

A force $\vec{F}$ acting at the same material point is called *conservative* if the work done by it from time t_1 to t_2 (Fig. 2.62a) during the motion of its point of application from location $\vec{r}_1$ to $\vec{r}_2$ is *independent of the path C_1 connecting these locations.* Hence,

$$W_{1-2} = \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = \int_{C_1:\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}, \qquad \forall\, C_1 \text{ and } \forall\, \vec{r}_1, \vec{r}_2 \tag{2.77}$$

in the region R in which $\vec{F}$ is defined. R may be the whole of space or a part of it. Hence, a conservative force *acts on the same material point of the body and does not depend explicitly on its velocity and time*, i.e., $\vec{F} = \vec{F}(\vec{r})$. The force due to friction is non-conservative as its direction is velocity dependent in the case of dry friction and its magnitude and direction are velocity dependent if a lubricating film is present.

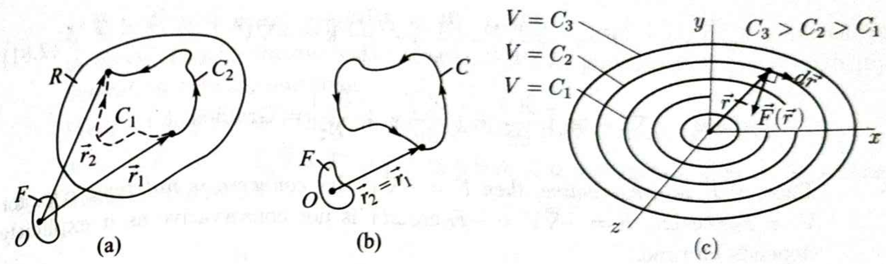


Fig. 2.62

The following inferences can be drawn for conservative forces.

1. Equation (2.77) implies that the integrand is a perfect differential of a position dependent single-valued function $V(\vec{r})$, called the *potential energy*, such that

$$dW = \vec{F}\cdot d\vec{r} = -dV \tag{2.78}$$

$$\Rightarrow \quad W_{1-2} = -\int_{V(\vec{r}_1)}^{V(\vec{r}_2)} dV = -[V(\vec{r}_2) - V(\vec{r}_1)] \tag{2.79}$$

   i.e., *the work done from $\vec{r}_1$ to $\vec{r}_2$ equals negative of the change in V from $\vec{r}_1$ to $\vec{r}_2$.* The customary negative sign in the definition of V ensures that in work–energy relations [e.g., eq. (2.57)], when the work done by a conservative force is transferred to the left hand side, then we get the sum of kinetic and potential energies, rather than their difference.

2. Hence, the *work done by a conservative force in any closed path C* ($\vec{r}_2 = \vec{r}_1$) *is zero* (Fig. 2.62b).

3. Let $\vec{r}_O$ be the *datum* for V, i.e., $V(\vec{r}_O) = 0$. Equation (2.78) $\Rightarrow$

$$V(\vec{r}) = -\int_{\vec{r}_O}^{\vec{r}} \vec{F}\cdot d\vec{r} \tag{2.80}$$

   i.e., potential energy at a given position $\vec{r}$ equals the negative of the work done from the datum to this position and also equals the *potential* (actual) work done from the given position $\vec{r}$ to the datum. For variable forces, the datum is usually chosen at the position where the force is zero.

4. Writing $\vec{F}\cdot d\vec{r} = -dV$ in terms of Cartesian components of $\vec{F}$ and $d\vec{r}$, we have:

$$F_x\,dx + F_y\,dy + F_z\,dz = -\frac{\partial V}{\partial x}dx - \frac{\partial V}{\partial y}dy - \frac{\partial V}{\partial z}dz \quad \forall\, dx, dy, dz.$$

   $\Rightarrow$ The coefficients of dx, dy, dz on the two sides must be the same, i.e.,

$$F_x = -\frac{\partial V}{\partial x}, \qquad F_y = -\frac{\partial V}{\partial y}, \qquad F_z = -\frac{\partial V}{\partial z}$$

$$\Rightarrow \qquad \vec{F} = -\frac{\partial V}{\partial x}\,\vec{i} - \frac{\partial V}{\partial y}\,\vec{j} - \frac{\partial V}{\partial z}\,\vec{k} = -\vec{\nabla} V \tag{2.81}$$

$$\text{where} \qquad \vec{\nabla}(\,) \equiv \vec{i}\,\frac{\partial(\,)}{\partial x} + \vec{j}\,\frac{\partial(\,)}{\partial y} + \vec{k}\,\frac{\partial(\,)}{\partial z} = \text{Gradient }(\,)$$

Thus, *if $\vec{F}$ is conservative, then* $\vec{F} = -\vec{\nabla}V$. Its *converse is not true*, e.g., for $V = F_0 x \cos\omega t$, $\vec{F} = -\vec{\nabla}V = -F_0 \cos\omega t\,\vec{i}$ is not conservative as it explicitly depends on time.

For any $d\vec{r}$ along a contour surface $V(x, y, z) = \text{constant}$ (Fig. 2.62c):

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \vec{\nabla}V \cdot d\vec{r} = 0 \quad \Rightarrow \quad \vec{\nabla}V \perp d\vec{r} \;\; \forall\, d\vec{r} \Rightarrow$$

$\vec{F} = -\vec{\nabla}V$ is along normal to contour surface of V in the direction of decreasing V

Using the expressions for dW in cylindrical, spherical and path coordinates [eq. (2.28)], we have:

$$F_r = -\frac{\partial V}{\partial r}, \qquad F_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi}, \qquad F_z = -\frac{\partial V}{\partial z}$$

$$F_r = -\frac{\partial V}{\partial r}, \qquad F_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}, \qquad F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \tag{2.82}$$

$$F_t = -\frac{\partial V}{\partial s}$$

5. The force component F_e in direction $\vec{e}$ is obtained as

$$F_e = \vec{F}\cdot\vec{e} = -\vec{\nabla}V\cdot\vec{e} = -(\text{directional derivative of } V \text{ along } \vec{e}).$$

6. *If $\vec{F}$ is conservative, then $\vec{\nabla}\times\vec{F} = \vec{0}$:*

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right]\vec{i} + \ldots = \left[-\frac{\partial^2 V}{\partial y \partial z} + \frac{\partial^2 V}{\partial z \partial y}\right]\vec{i} + \ldots = \vec{0}$$

In cylindrical coordinates:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \dfrac{1}{r}\vec{e}_r & \vec{e}_\phi & \dfrac{1}{r}\vec{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_r & rF_\phi & F_z \end{vmatrix} \tag{2.83}$$

7. The following theorem provides a simple condition to check whether a given force is conservative.

If $\vec{\nabla}\times\vec{F} = \vec{0}$ in the region R of the definition of $\vec{F}$, then $\vec{F}$ is conservative in every **simply-connected** *sub-region R_1 of R, provided $\vec{F}$ does not explicitly depend on velocity and time.*

Proof: Using Stoke's theorem, for any closed path C in R_1 (Fig. 2.63), $W = \oint_C \vec{F}\cdot d\vec{r} = \int_A \vec{\nabla}\times\vec{F}\cdot d\vec{A} = 0$, where A is any surface area in R_1 whose bounding curve is C.

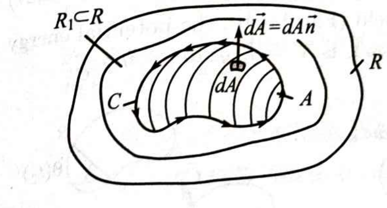


Fig. 2.63

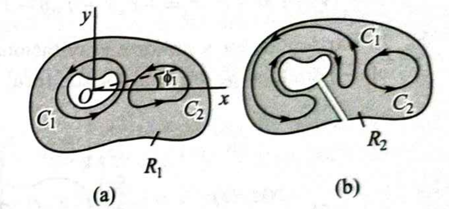


Fig. 2.64

Consider a planar force field $\vec{F} = (c/r)\vec{e}_\phi$ which is not defined for $r = 0$. $\vec{\nabla}\times\vec{F} = (0/r)(\vec{e}_r + \vec{e}_z) = \vec{0}$ for $r \neq 0$ in the doubly-connected region R_1 in the xy-plane (Fig. 2.64a). $\oint_{C_1} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} c\,d\phi = 2\pi c \neq 0$ for curves like C_1 which enclose the hole, and hence the force is non-conservative in R_1, with a potential $V = -c\phi$ that is multi-valued in R_1: $V(r, \phi, z) = -c\phi, -c(\phi + 2\pi), \ldots$. For any closed curve C_2 not enclosing the hole: $W = \int_{\phi_1}^{\phi_1} c\,d\phi = 0$. If the region is modified to a simply-connected region R_2 by making a slit in R_1 (Fig. 2.64b), then $\vec{F}$ is conservative in R_2.

2.21.1 Potential energies of some conservative forces

$V(\vec{r}) = -\int_{\vec{r}_O}^{\vec{r}} \vec{F}\cdot d\vec{r}$ is computed by integrating along a convenient path.

1. Integrating along the piecewise linear curves $C_1(OABD)$ and $C_2(OBD)$, shown in Fig. 2.65a, yields in general

$$V(\vec{r}) - V(\vec{0}) = -\int_{C_1:\,\vec{0}}^{\vec{r}} \vec{F}\cdot d\vec{r}$$

$$= -\int_0^x F_x(x, 0, 0)\,dx - \int_0^y F_y(x, y, 0)\,dy - \int_0^z F_z(x, y, z)\,dz \tag{2.84}$$

$$V(\vec{r}) - V(\vec{0}) = -\int_{C_2:\,\vec{0}}^{\vec{r}} \vec{F}\cdot d\vec{r}$$

$$= -\int_0^r F_r(r, \phi, 0)\,dr - \int_0^z F_z(r, \phi, z)\,dz \tag{2.85}$$

2. For $\vec{F} = F_x(x)\,\vec{i} + F_y(y)\,\vec{j} + F_z(z)\,\vec{k}$, with its components dependent only on the corresponding coordinate, eq. (2.84) yields

$$V(\vec{r}) - V(\vec{0}) = -\int_0^x F_x(x)\,dx - \int_0^y F_y(y)\,dy - \int_0^z F_z(z)\,dz \tag{2.86}$$

3. For $\vec{F} = F_x(x)\,\vec{i}$, eq. (2.86) $\Rightarrow$ $V(\vec{r}) - V(\vec{0}) = -\int_0^x F_x(x)\,dx$.
4. For a constant force $\vec{F}$, with datum at $\vec{0}$, eq. (2.80) $\Rightarrow$

$$V(\vec{r}) = -\vec{F}\cdot\vec{r} = -F_x x - F_y y - F_z z \tag{2.87}$$

5. In particular, for a uniform gravitational field (Fig. 2.65b), the potential energy of $-g\,dm\,\vec{k}$ is $gz\,dm$ and for the total $-mg\,\vec{k}$ is $V = \int_m gz\,dm = mgz_C$.

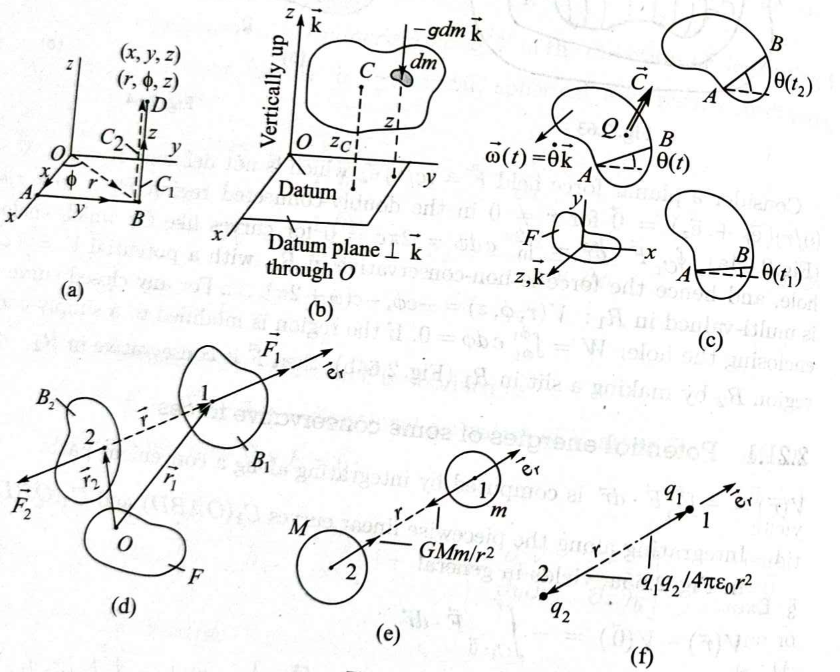


Fig. 2.65

6. The work done by a constant moment (or couple–moment) $\vec{C}$ acting at point Q of a rigid body (Fig. 2.65c) moving with $\vec{\omega}(t) = \omega(t)\,\vec{k} = \dot{\theta}(t)\,\vec{k}$ from orientation $\theta(t_1)$ at time t_1 to orientation $\theta(t_2)$ at time t_2 is given by

$$W = \int_{t_1}^{t_2} \vec{C}\cdot\vec{\omega}(t)\,dt = \vec{C}\cdot[\theta(t_2) - \theta(t_1)]\,\vec{k} = C_z[\theta(t_2) - \theta(t_1)] \tag{1}$$

which is independent of the path followed by the body from its initial configuration to its final configuration. Hence, a constant moment acting on a rigid body, moving with angular velocity having only z-component, is conservative with potential energy $V(\theta) = -C_z\theta$ with datum at $\theta = 0$. However, a constant moment acting on a rigid body having arbitrary $\vec{\omega}$ is *non-conservative.*

§ **Exercise 2.8:** Prove the above result by considering two motions between the same initial and final configurations: (1) $\vec{\omega} = \omega\,\vec{i}$ for time 0 to π/ω, followed by $\vec{\omega} = \omega\,\vec{j}$ for time π/ω to $2\pi/\omega$, and (2) $\vec{\omega} = \frac{1}{2}\omega\,\vec{k}$ for time 0 to $2\pi/\omega$.

7. *For a pair of mutual central forces* $\vec{F}_1$ *and* $\vec{F}_2$ (Fig. 2.65d), *with* $\vec{F}_1 = -\vec{F}_2 = F_1\,\vec{e}_r = F(r,\theta,\phi)\,\vec{e}_r$:

$$\begin{aligned} dW &= \vec{F}_1\cdot d\vec{r}_1 + \vec{F}_2\cdot d\vec{r}_2 = \vec{F}_1\cdot d(\vec{r}_1 - \vec{r}_2) = F_1\,\vec{e}_r\cdot d\vec{r} \\ &= F_1\,\vec{e}_r\cdot(dr\,\vec{e}_r + r\,d\theta\,\vec{e}_\theta + r\sin\theta\,d\phi\,\vec{e}_\phi) = F(r,\theta,\phi)\,dr \end{aligned} \tag{1}$$

Equation (1) $\Rightarrow$ this pair of forces is conservative if $F(r,\theta,\phi) = F(r)$, i.e., if the forces depend only on the separation distance between points 1 and 2. For $\vec{F}_1 = F(r)\,\vec{e}_r$, the potential energy of the pair of forces with datum at $r = r_0$ is:

$$V(r) = -\int_{r_0}^{r} F(r)\,dr \tag{2.88}$$

If point 2 remains fixed, then the work done by $\vec{F}_2$ is zero and eq. (2.88) gives the potential energy of the single central force $\vec{F}_1 = F(r)\,\vec{e}_r$ alone (with centre of force at point 2), which by itself is conservative.

We derive V of gravitational, electrostatic, spring forces and torsional spring moment.

(a) For a pair of mutual gravitation forces $F(r) = -GMm/r^2$ (Fig. 2.65e), eq. (2.88) yields $V = -GMm/r$ with datum at $r_0 = \infty$. This is also potential energy of gravitational force $\vec{F} = (-GMm/r^2)\,\vec{e}_r$ on m due to a large mass M which remains fixed.

§ **Exercise 2.9:** Consider the gravitational force on a mass-point of mass m (inside or outside the shell) due to a fixed uniform thin spherical shell of radius a and mass M. Show that, with datum at $r_0 = \infty$, its potential energy $V = -GMm/r$ if $r > a$ and $V = -GMm/a$ if $r < a$.

(b) For a pair of mutual electrostatic forces $F(r) = q_1q_2/4\pi\epsilon_0 r^2$ on two point charges q_1 and q_2 (Fig. 2.65f) with datum at $r_0 = \infty$, eq. (2.88) yields $V = q_1q_2/4\pi\epsilon_0 r$. This is also the potential energy of a central electrostatic force $\vec{F} = (q_1q_2/4\pi\epsilon_0 r^2)\,\vec{e}_r$ on q_1 due to a point charge q_2 which remains fixed.

(c) Consider a pair of forces due to an *ideal elastic* spring (Fig. 2.66a,b) such as a helical spring. Ideal means that its inertia is neglected. The spring is an inertia-less two-force member and the two forces at its ends must be equal in magnitude, opposite in direction and act along the line joining the end-points. Elastic means that the spring recovers its un-deformed length on removal of the end-loads and the load-extension relation is the same during loading and unloading. Let L_0 be its un-stretched length.

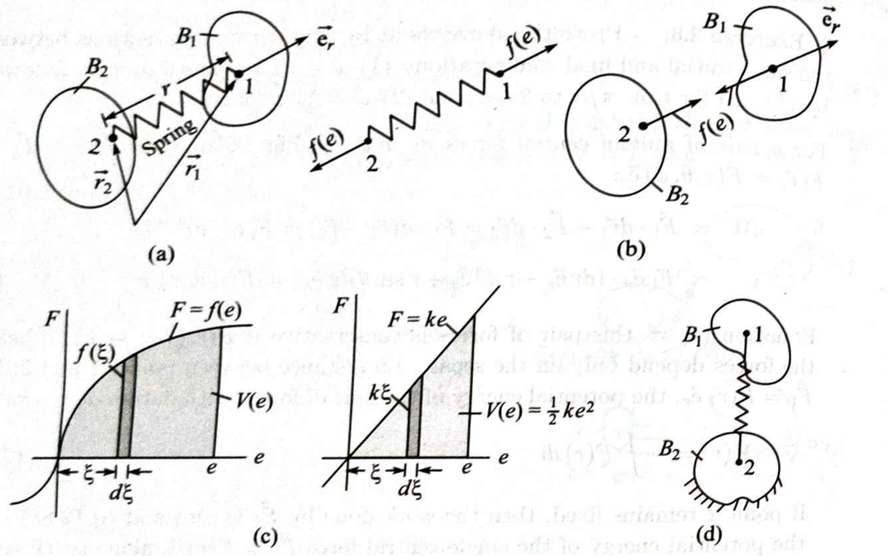


Fig. 2.66

Let the pull on the spring be $f(e)$ for an extension $e = r - L_0$. Hence, $F(r) = -f(e)$, $de = dr$ and eq. (2.88) yields potential energy of pair of spring forces

$$V(e) = \int_0^e f(e)\, de \tag{2.89}$$

with datum at un-stretched (un-deformed) configuration $r_0 = L_0$. The integral in eq. (2.89) represents the area under the force-extension curve (Fig. 2.66c). This is also the potential energy of a central spring force $\vec{F} = -f(e)\,\vec{e}_r$ acting on body B_1 with the other end of the spring fixed (Fig. 2.66d). For example, for

$$f(e) = k_1 e + k_2 e^3, \qquad V(e) = \tfrac{1}{2}k_1 e^2 + \tfrac{1}{4}k_2 e^4 \qquad \text{(a)}$$

For a *linear spring* of *stiffness* k: $\quad f(e) = ke, \qquad V = \tfrac{1}{2}ke^2 = \tfrac{1}{2}k\delta^2 \qquad$ (b)

with δ $(= e)$ being the extension. The expressions (2.89), (a), (b) are valid for compression where e is negative since both $f(e)$ and de change sign.

(d) Consider a pair of moments due to an *ideal elastic torsional spring*, such as a torque bar and a spiral spring (Fig. 2.67a). Ideal means that its inertia is neglected. Hence, the end-moments must be $\pm M_t\,\vec{e}$ along the axis $\vec{e}$ of the spring. Elastic means that the spring recovers its un-deformed shape and size on removal of the end-moments and the moment–twist relation is the same during loading and unloading.

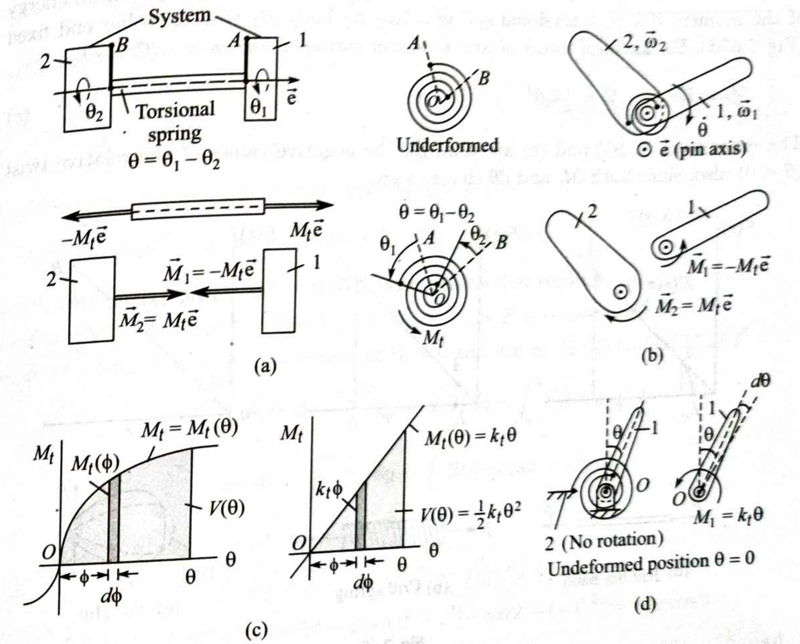


Fig. 2.67

Let the axial torsional (twisting) moment on the spring at the ends be $M_t(\theta)$ for axial twist (relative axial rotation of the two ends of the spring) θ, where for the un-deformed spring, $\theta = 0$ and $M_t(0) = 0$. Using $\vec{\omega}_1 = \vec{\omega}_2 + \dot{\theta}\,\vec{e}$ and $\vec{M}_1 = -M_t(\theta)\,\vec{e} = -\vec{M}_2$, the rate of work done by the pair of moments $\vec{M}_1$ and $\vec{M}_2$, acting on the rigid bodies 1 and 2 due to a torsional spring (Fig. 2.67b), is given by

$$\dot{W} = \vec{M}_1 \cdot \vec{\omega}_1 + \vec{M}_2 \cdot \vec{\omega}_2 = \vec{M}_1 \cdot (\vec{\omega}_1 - \vec{\omega}_2)$$

$$= -M_t(\theta)\,\vec{e} \cdot (\dot{\theta}\,\vec{e}) = -M_t(\theta)\,\dot{\theta} \quad \Rightarrow \quad dW = -M_t(\theta)\, d\theta$$

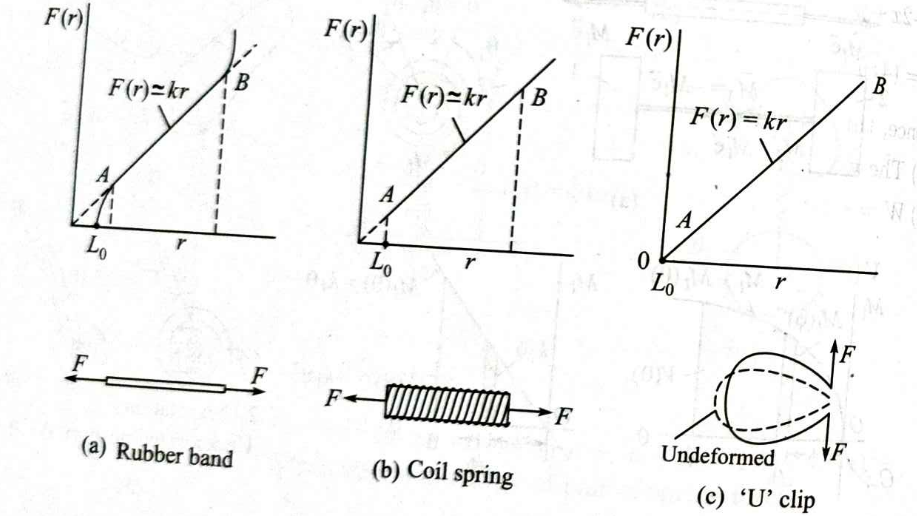
Hence, the pair of moments together are conservative with potential energy

$$V(\theta) = \int_0^\theta M_t(\theta)\, d\theta \tag{2.90}$$

with the untwisted configuration $\theta = 0$ as the datum. The integral in eq. (2.90) is the area under the moment–twist curve (Fig. 2.67c). This is also the potential energy of the moment due to a torsional spring acting on body B_1 with its other end fixed (Fig. 2.67d). For an *ideal linear elastic torsional spring* of *torsional stiffness* k_t:

$$M_t = k_t\theta, \qquad V = \tfrac{1}{2}k_t\theta^2 \tag{c}$$

The expressions (2.90) and (c) are valid for the negative values of the relative twist ($\theta < 0$) also, since both M_t and $d\theta$ change sign.



(a) Rubber band (b) Coil spring (c) 'U' clip

Fig. 2.68

(e) **'Zero-length' linear spring:** In some types of springs and their specific form of attachment, the relation of force F to extension e can be approximated as:

$$F(r) = f(e) = f(r - L_0) \simeq kr \qquad \Rightarrow \qquad V(r) = \tfrac{1}{2}kr^2$$

i.e., effectively it is a *'zero-length' linear spring*. For example, in range A to B of force-extension relation (Fig. 2.68): a rubber band, a pre-stressed spring with coils touching each other, a pinched 'U' clip, and springs with $r \gg L_0$ behave as zero-length springs.

Example 2.23: (a) Check whether the forces: (1) $\vec{F} = (-2x + y^2 + z^2\sin x)\,\vec{\imath} + 2(x + z^2)y\,\vec{\jmath} + 2z(-\cos x + y^2)\,\vec{k}$, (2) $\vec{F} = (2x - y + yz^2)\,\vec{\imath} + (y^2x - x + xz^2)\,\vec{\jmath} + 2xyz\,\vec{k}$, are conservative force fields. (b) Find the work done by these forces in a closed path shown in Fig. E2.23. (c) Find the work done by these forces along the curve $C: \vec{r}(\tau) = 2\tau\,\vec{\imath} - \tau^3\,\vec{\jmath} + \tau^2\,\vec{k}$ from $\tau = 1$ to $\tau = 2$.

Solution: (1) $\vec{F}$ is independent of velocity and time. $F_x = -2x + y^2 + z^2\sin x$, $F_y = 2(x + z^2)y$, $F_z = 2z(-\cos x + y^2)$

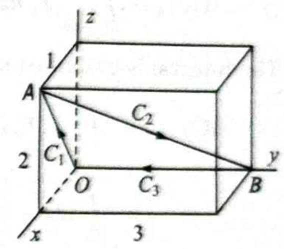


Fig. E2.23

(a) $\vec{\nabla} \times \vec{F} =$

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (-2x + y^2 + z^2\sin x) & 2(x + z^2)y & 2z(-\cos x + y^2) \end{vmatrix}$$

$$= (4zy - 4zy)\,\vec{\imath} - (2z\sin x - 2z\sin x)\,\vec{\jmath} + (2y - 2y)\,\vec{k} = \vec{0}$$

Hence, this force is conservative for all simply-connected regions.

(b) The work done over the closed path is zero since $\vec{F}$ is conservative.

(c) $W = -\Delta V$. We choose the datum at $\vec{r}_0 = \vec{0}$ and use eq. (2.84) to find V of $\vec{F}$.

$$\begin{aligned} V(\vec{r}) &= -\int_0^x F_x(x,0,0)\,dx - \int_0^y F_y(x,y,0)\,dy - \int_0^z F_z(x,y,z)\,dz \\ &= -\int_0^x (-2x)\,dx - \int_0^y 2xy\,dy - \int_0^z 2z(-\cos x + y^2)\,dz \\ &= x^2 - xy^2 - z^2(-\cos x + y^2) \end{aligned}$$

$C: \quad \vec{r}_1 = \vec{r}(1) = 2\vec{\imath} - \vec{\jmath} + \vec{k}, \quad \vec{r}_2 = \vec{r}(2) = 4\vec{\imath} - 8\vec{\jmath} + 4\vec{k},$

$$\begin{aligned} V(\vec{r}_1) &= V(2,-1,1) = 2^2 - 2(-1)^2 - 1^2[-\cos 2 + (-1)^2] = 1 + \cos 2 \\ V(\vec{r}_2) &= V(4,-8,4) = 4^2 - 4(-8)^2 - 4^2[-\cos 4 + (-8)^2] = -1264 + 16\cos 4 \\ W(\vec{r}_1 \to \vec{r}_2) &= -[V(\vec{r}_2) - V(\vec{r}_1)] = 1265 + \cos 2 - 16\cos 4 = 1275 \text{ units} \end{aligned}$$

(2) $\vec{F}$ is not explicitly dependent on $\vec{v}$ and t.

$$F_x = 2x - y + yz^2, \quad F_y = y^2x - x + xz^2, \quad F_z = 2xyz$$

(a) $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2x - y + yz^2) & (y^2x - x + xz^2) & 2xyz \end{vmatrix} = y^2\,\vec{k} \neq \vec{0}$$

Hence, this force is non-conservative.

(b) C_1: $O(0,0,0) \to A(1,0,2)$, $y=0$, $(x-0)/(1-0)=(z-0)/(2-0)$

$$\Rightarrow \quad z = 2x, \quad dz = 2\,dx, \quad dy = 0$$

$$W_{O\text{-}A} = \int_{C_1:O}^{A} (F_x\,dx + F_y\,dy + F_z\,dz)$$

The integral is evaluated after replacing y, z in terms of x for points on C_1:

$$W_{O\text{-}A} = \int_{C_1:O}^{A} (F_x\,dx + F_y\,dy + F_z\,dz)$$

$$= \int_0^1 [2x - 0 + (0)(2x)^2]\,dx + \int_0^1 [2x(0)(2x)]2\,dx = 1$$

C_2: $A(1,0,2) \to B(0,3,0)$, $\dfrac{x-1}{0-1} = \dfrac{y-0}{3-0} = \dfrac{z-2}{0-2} \Rightarrow$

$$y = -3x + 3, \quad z = 2x, \quad dy = -3\,dx, \quad dz = 2\,dx.$$

$$\int_{C_2:A}^{B} F_x\,dx = \int_{C_2:A}^{B} (2x - y + yz^2)\,dx$$

$$= \int_1^0 [2x - (-3x+3) + (-3x+3)(2x)^2]\,dx = -0.5$$

$$\int_{C_2:A}^{B} F_y\,dy = \int_{C_2:A}^{B} (y^2x - x + xz^2)\,dy$$

$$= \int_1^0 [(-3x+3)^2x - x + x(2x)^2](-3)\,dx = 3.75$$

$$\int_{C_2:A}^{B} F_z\,dz = \int_{C_2:A}^{B} (2xyz)\,dz$$

$$= \int_1^0 [2x(-3x+3)(2x)]2\,dx = \int_1^0 (24x^2 - 24x^3)\,dx = -2$$

$$W_{A\text{-}B} = \int_{C_2:A}^{B} (F_x\,dx + F_y\,dy + F_z\,dz) = -0.5 + 3.75 - 2 = 1.25$$

C_3: $B(0,3,0) \to O(0,0,0)$, $x=0$, $z=0 \Rightarrow dx = 0$, $dz = 0$,

$$W_{B\text{-}O} = \int_{C_3:B}^{O} (F_x\,dx + F_y\,dy + F_z\,dz) = \int_3^0 F_y\,dy$$

$$= \int_3^0 \left[y^2(0) - 0 + 0(0)^2\right] dy = 0$$

$$W_{O\text{-}A\text{-}B\text{-}O} = W_{O\text{-}A} + W_{A\text{-}B} + W_{B\text{-}O} = 1 + 1.25 + 0 = 2.25 \text{ units}$$

The work done for this closed path is non-zero. The work done by a non-conservative force in a closed path is, in general, non-zero.

(c) C: $\vec{r}(\tau) = 2\tau\,\vec{i} - \tau^3\,\vec{j} + \tau^2\,\vec{k} \Rightarrow x = 2\tau$, $y = -\tau^3$, $z = \tau^2$,

$$\Rightarrow \quad dx = 2\,d\tau, \quad dy = -3\tau^2\,d\tau, \quad dz = 2\tau\,d\tau,$$

$$W = \int_{C:\vec{r}(1)}^{\vec{r}(2)} (F_x\,dx + F_y\,dy + F_z\,dz) = \int_1^2 (2F_x - 3\tau^2F_y + 2\tau F_z)\,d\tau$$

$$= \int_1^2 \Big[2\left\{2(2\tau) - (-\tau^3) - \tau^3(\tau^2)^2\right\} - 3\tau^2\left\{(-\tau^3)^2(2\tau) - 2\tau + 2\tau(\tau^2)^2\right\}$$
$$+ 2\tau\left\{2(2\tau)(-\tau^3)(\tau^2)\right\}\Big]\,d\tau$$

$$= \int_1^2 \left[8\tau + 8\tau^3 - 16\tau^7 - 6\tau^9\right] d\tau = -1081.8 \text{ units}$$

Example 2.24: (a) Is the force $\vec{F} = (C/r^3)\,\vec{e}_r + b\,\vec{e}_\phi$ conservative? If so find its potential energy V. (b) Find the work done by $\vec{F}$ over the closed path $ABCD$ (Fig. E2.24a) and the quarter circular path from F to G. C and b are constants. (c) Is the force $\vec{F} = (C/r^3)\,\vec{e}_r + (b/r)\,\vec{e}_\phi$ conservative? If so find its potential energy V.

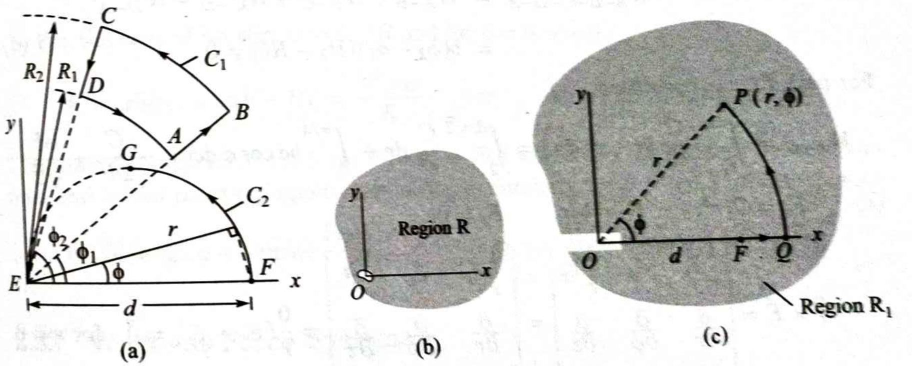


Fig. E2.24

Solution: (a)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \frac{1}{r}\vec{e}_r & \vec{e}_\phi & \frac{1}{r}\vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_r & rF_\phi & F_z \end{vmatrix} = \begin{vmatrix} \frac{1}{r}\vec{e}_r & \vec{e}_\phi & \frac{1}{r}\vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ C/r^3 & rb & 0 \end{vmatrix} = \frac{b}{r}\vec{e}_z \neq \vec{0}$$

$\Rightarrow \vec{F}$ is non-conservative and V does not exist. $F_r = C/r^3$, $F_\phi = b$, $F_z = 0$.

Alternative method: Assume V exists and try to obtain V by indefinite integration.

$$F_r = -\frac{\partial V}{\partial r} \Rightarrow \frac{\partial V}{\partial r} = -\frac{C}{r^3} \Rightarrow V = \frac{C}{2r^2} + g(\phi, z)$$

$$F_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi} \Rightarrow \frac{\partial g}{\partial \phi} = -br \Rightarrow g = -br\phi + f(z)$$

This is an inconsistency as $g(\phi, z)$ is showing dependence on $r \Rightarrow V$ does not exist.

(b) $W = \int \vec{F}\cdot d\vec{r} = \int (F_r\,dr + F_\phi r\,d\phi + F_z\,dz) = \int [(C/r^3)dr + rb\,d\phi]$ (1)

$$AB: \quad \phi = \phi_1, \quad d\phi = 0, \quad W_{A\text{-}B} \overset{(1)}{=} \int_{R_1}^{R_2} \frac{C}{r^3}\,dr = -\tfrac{1}{2}C(R_2^{-2} - R_1^{-2})$$

$$BC: \quad r = R_2, \quad dr = 0, \quad W_{B\text{-}C} \overset{(1)}{=} \int_{\phi_1}^{\phi_2} R_2 b\,d\phi = R_2 b(\phi_2 - \phi_1)$$

Similarly,

$$W_{C\text{-}D} \overset{(1)}{=} \int_{R_2}^{R_1} \frac{C}{r^3}\,dr = -\tfrac{1}{2}C(R_1^{-2} - R_2^{-2})$$

$$W_{D\text{-}A} \overset{(1)}{=} \int_{\phi_2}^{\phi_1} R_1 b\,d\phi = R_1 b(\phi_1 - \phi_2)$$

$$\begin{aligned} W_{A\text{-}B\text{-}C\text{-}D\text{-}A} &= W_{A\text{-}B} + W_{B\text{-}C} + W_{C\text{-}D} + W_{D\text{-}A} \\ &= b(\phi_2 - \phi_1)(R_2 - R_1) \neq 0 \end{aligned}$$

For path FG: $\quad r = d\cos\phi$,

$$W_{F\text{-}G} \overset{(1)}{=} \int_{r_F}^{r_G} \left(\frac{C}{r^3}\,dr + rb\,d\phi\right) = \int_d^{d/\sqrt{2}} \frac{C}{r^3}\,dr + \int_0^{\pi/4} bd\cos\phi\,d\phi = -\frac{C}{2d^2} + \frac{bd}{\sqrt{2}}$$

(c) $\quad F_r = C/r^3, \quad F_\phi = b/r, \quad F_z = 0,$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \frac{1}{r}\vec{e}_r & \vec{e}_\phi & \frac{1}{r}\vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_r & rF_\phi & F_z \end{vmatrix} = \begin{vmatrix} \frac{1}{r}\vec{e}_r & \vec{e}_\phi & \frac{1}{r}\vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ C/r^3 & b & 0 \end{vmatrix} = \frac{0}{r}(\vec{e}_r + \vec{e}_z) = \vec{0} \quad \text{for } r \neq 0$$

$\vec{F}$ is non-conservative in the doubly-connected region R, but is conservative in the simply-connected region R_1 (Fig. E2.24b,c). O cannot be datum as $O \notin R_1$. A simple path from the datum at F to $P(r, \phi)$ is radial from F to Q and circular from Q to P:

$$V_P = -\int_d^r (F_r|_{\phi=0})\,dr - \int_0^\phi rF_\phi\,d\phi = \frac{C}{2r^2} - \frac{C}{2d^2} - b\phi$$

Example 2.25: (a) Find $\dot{W}$ for the force $F = F_0 e^{-\alpha t}$ at time t when angular velocity of the rigid wheel is ω and velocity of O is v_O (Fig. E2.25). Angle ϕ and the distances h and b are constant. (b) Find the work done by $F = F_0$ ($\alpha = 0$), during the time O covers distance d. Assume no slip. Consider cases: $\phi = 0$, $\phi \neq 0$.

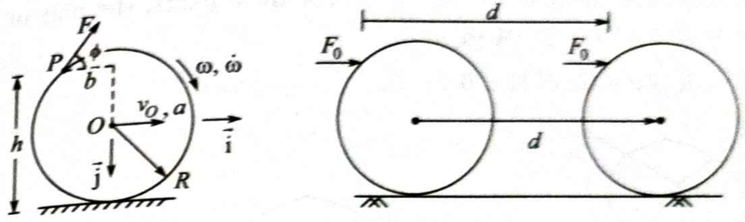


Fig. E2.25

Solution: (a) The force $\vec{F} = F_0 e^{-\alpha t}(\cos\phi\,\vec{\imath} - \sin\phi\,\vec{\jmath})$ acts on different material points P at different instants. Hence, work is obtained by integrating $\dot{W}$ with respect to time.

$$\begin{aligned} \vec{v}_P &= \vec{v}_O + \vec{\omega}\times\overrightarrow{OP} = v_O\,\vec{\imath} + \omega\,\vec{k}\times[-b\,\vec{\imath} - (h-R)\,\vec{\jmath}\,] \\ &= [v_O + \omega(h-R)]\,\vec{\imath} - \omega b\,\vec{\jmath} \\ \dot{W} &= \vec{F}\cdot\vec{v}_P = F_0 e^{-\alpha t}[\{v_O + \omega(h-R)\}\cos\phi + \omega b\sin\phi] \end{aligned} \quad (1)$$

(b) For the case of no slip, $\omega = v_O/R$ and for $\phi = 0, \alpha = 0$:

$$\dot{W} \overset{(1)}{=} F_0[v_O + \omega(h-R)] = \frac{hF_0 v_O}{R} \quad \Rightarrow \quad W = \frac{hF_0}{R}\int_0^t v_O\,dt = \frac{hF_0 d}{R}$$

The work done by constant force F_0: $W \neq F_0 d$ where d is the distance between the final and initial points of application of force. Similarly, for $\omega = v_O/R$, $\alpha = 0$, $\phi \neq 0$:

$$\dot{W} \overset{(1)}{=} (h\cos\phi + b\sin\phi)\frac{F_0 v_O}{R} \quad \Rightarrow \quad W = (h\cos\phi + b\sin\phi)\frac{F_0 d}{R}$$

2.22 Workless Force Systems

A force system which does no work on a body in any motion is called a workless system. Their identification is useful in the application of work–energy relations. The rate of work done by a pair of force reactions $\vec{R}, -\vec{R}$ and moment reactions $\vec{C}, -\vec{C}$ on rigid bodies 1 and 2 (Fig. 2.69a) at a joint/contact at A and B, or by $\vec{R}$ and $\vec{C}$ alone if body 2 is fixed, is given by

$$\dot{W} = \vec{R}\cdot(\vec{v}_A - \vec{v}_B) + \vec{C}\cdot(\vec{\omega}_1 - \vec{\omega}_2) \quad (a)$$

We first prove three specific force systems to be workless, which are applied later.

(i) Consider frictionless joints (Fig. 2.69b) with A coinciding with B, and with constraint on relative displacement and relative rotation such that $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = v\vec{e}$, $\vec{\omega}_{1|2} = \vec{\omega}_1 - \vec{\omega}_2 = \Omega\vec{e}^*$. Hence, $\vec{R}$ has no component along $\vec{e}$ ($\vec{R}\cdot\vec{e} = 0$) and $\vec{C}$ has no component along $\vec{e}^*$ ($\vec{C}\cdot\vec{e}^* = 0$). For these joints, the pair of reactions are together workless since by eq. (a):

$$\dot{W} = \vec{R}\cdot\vec{e}v + \vec{C}\cdot\vec{e}^*\Omega = 0 \qquad (b)$$

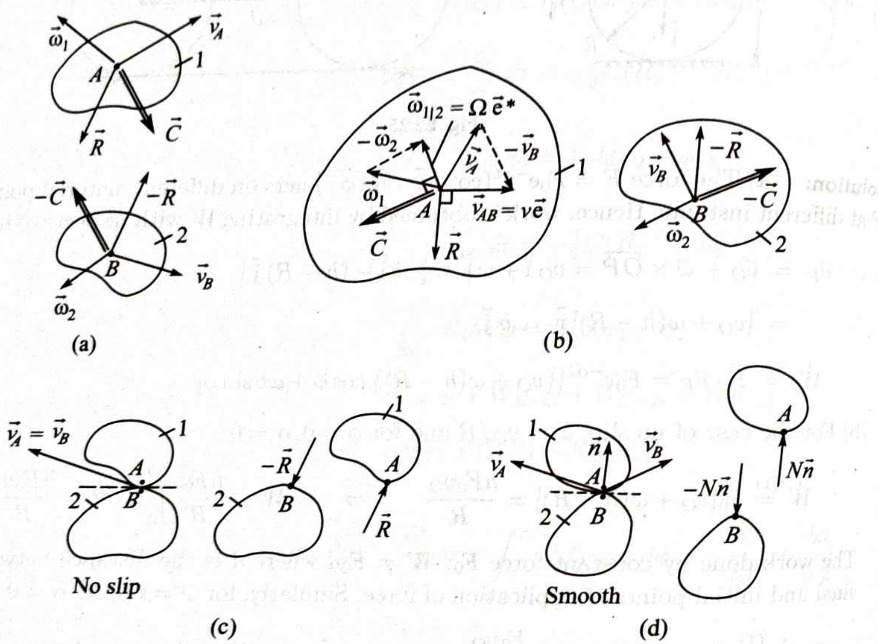


Fig. 2.69

(ii) For point contact (Fig. 2.69c) with no slip, $\vec{v}_A = \vec{v}_B$ and $\vec{C} = \vec{0}$, eq. (a) $\Rightarrow \dot{W} = 0$.

(iii) For smooth point contact (Fig. 2.69d) with or without slip, $\vec{C} = \vec{0}$, $\vec{R} = N\vec{n}$, where $\vec{n}$ is normal to the tangent plane at the point of contact, and eq. (a) yields

$$\dot{W} = \vec{R}\cdot(\vec{v}_A - \vec{v}_B) = N\vec{n}\cdot(\vec{v}_A - \vec{v}_B) = N(v_{An} - v_{Bn}) = 0 \qquad (c)$$

as $N = 0$ for impending separation, or $v_{An} = v_{Bn}$ for no separation.

Using the expression (a) for $\dot{W}$, some examples of pairs of reactions/reactions which are not workless occur at: sliding contact with friction, joint with driving force, joint with driving torque, joint with frictional torque; and connections involving extensible cable, spring, damper, rotational spring or rotational damper.

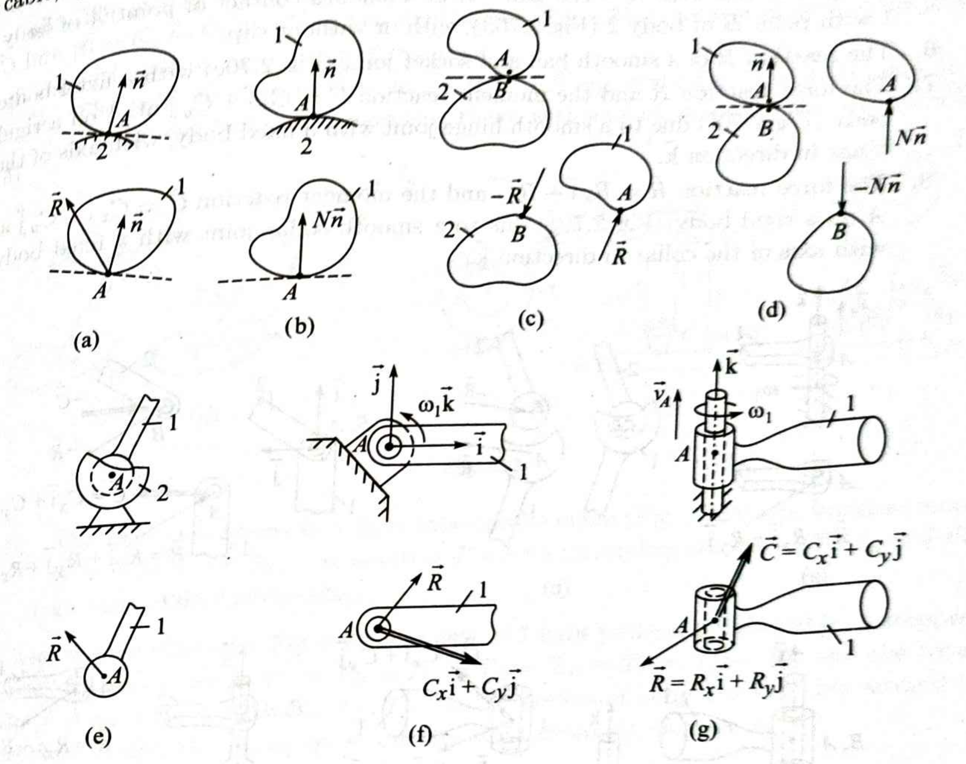


Fig. 2.70

Based on eqs (b) and (c), we list below in items 3 to 13 some force systems which are workless while acting on a body and some pairs of force systems acting on two bodies which together are workless, though individually each may perform work.

1. *Gyroscopic force* is a velocity dependent force $\vec{F}(\vec{r}, \vec{v})$, which always acts at right angles to the velocity $\vec{v}$ of its point of application. It is workless since $\dot{W} = \vec{F}\cdot\vec{v} = 0$. For example, the force $\vec{F} = q\vec{v}\times\vec{B}$ on a point charge q moving with velocity $\vec{v}$ in a magneto-static field $\vec{B}$ is workless.
2. The reaction $\vec{R}$ at contact A with a fixed body with no slip (Fig. 2.70a), is workless since $\dot{W} = \vec{R}\cdot\vec{v}_A = 0$ as $\vec{v}_A = \vec{0}$.
3. The reaction $\vec{R} = N\vec{n}$ at A with a smooth fixed body (Fig. 2.70b), with or without slip.

4. The pair of reactions $\vec{R}$ and $-\vec{R}$ at the contact of point A of body 1 with point B of body 2 (Fig. 2.70c) with no slip.
5. The pair of reactions $\vec{R} = N\vec{n}$ and $-\vec{R}$ at a smooth contact of point A of body 1 with point B of body 2 (Fig. 2.70d), with or without slip.
6. The reaction $\vec{R}$ of a smooth ball and socket joint (Fig. 2.70e) with a fixed body.
7. The force reaction $\vec{R}$ and the moment reaction $\vec{C} = C_x\vec{i} + C_y\vec{j}$ at A on a rigid body (Fig. 2.70f) due to a smooth hinge joint with a fixed body, with axis of the hinge in direction $\vec{k}$.
8. The force reaction $\vec{R} = R_x\vec{i} + R_y\vec{j}$ and the moment reaction $\vec{C} = C_x\vec{i} + C_y\vec{j}$ at A on a rigid body (Fig. 2.70g) due to a smooth collar joint with a fixed body, with axis of the collar in direction $\vec{k}$.

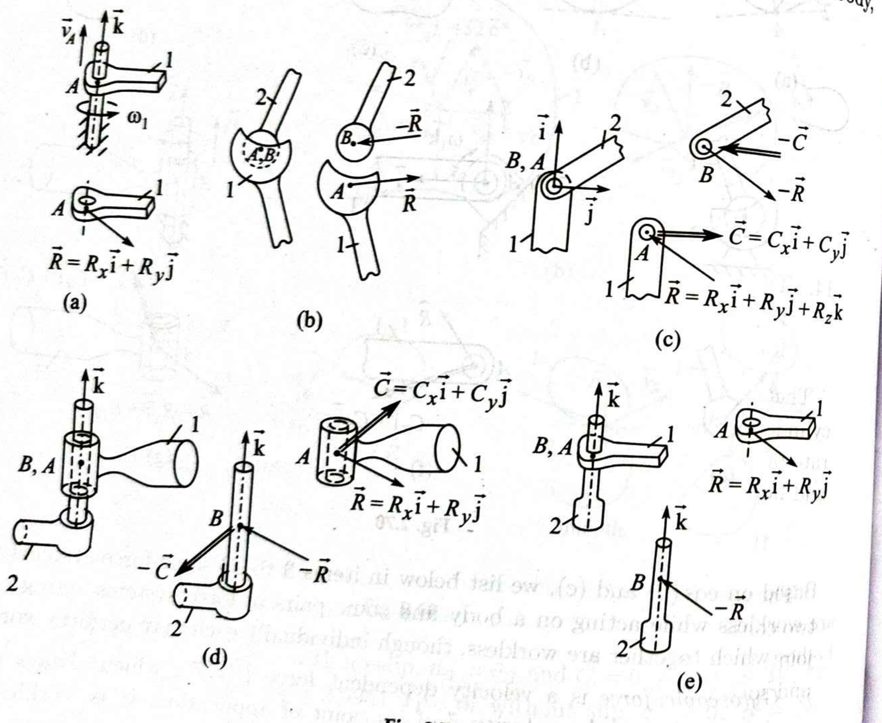


Fig. 2.71

9. The force reaction $\vec{R} = R_x\vec{i} + R_y\vec{j}$ at A on a rigid body (Fig. 2.71a), due to a smooth eye-bolt joint with a fixed body, with the bolt axis in direction $\vec{k}$.
10. The pair of reactions $\vec{R}$ and $-\vec{R}$ on two bodies (Fig. 2.71b) due to a smooth ball and socket joint connecting them.
11. The pair of force reactions $\vec{R}$, $-\vec{R}$ and the pair of moment reactions $\vec{C} = C_x\vec{i} + C_y\vec{j}$, $-\vec{C}$ on two rigid bodies (Fig. 2.71c) due to a smooth hinge joint connecting them, with axis of hinge in direction $\vec{k}$.
12. The pair of force reactions $\vec{R} = R_x\vec{i} + R_y\vec{j}$, $-\vec{R}$ and the pair of moment reactions $\vec{C} = C_x\vec{i} + C_y\vec{j}$, $-\vec{C}$ on two rigid bodies (Fig. 2.71d) due to a smooth collar joint, with axis of the collar in direction $\vec{k}$.
13. The pair of reactions $\vec{R} = R_x\vec{i} + R_y\vec{j}$, $-\vec{R}$ on two bodies (Fig. 2.71e) due to a smooth eye-bolt joint connecting them, with axis of the eye-bolt in direction $\vec{k}$.

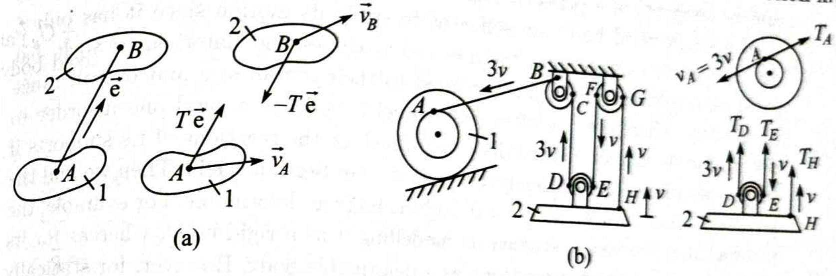


Fig. 2.72

14. The pair of tensions in a light inextensible cable (Fig. 2.72a) are workless since $\dot{W} = T(\vec{e}\cdot\vec{v}_A - \vec{e}\cdot\vec{v}_B) = 0$, as either $T = 0$ for impending slackness or $\vec{e}\cdot\vec{v}_A = \vec{e}\cdot\vec{v}_B$ for taut inextensible cable.

The system shown in Fig. 2.72b consists of 3 light pulleys, a light cable, a stepped cylinder 1 and body 2. $T_A = T_B = T_C = T_D = T_E = T_F = T_G = T_H$ and the total rate of work done by T_A, T_D, T_E, T_H, on the system of body 1 with cable around it and body 2 with the pulley on it and the cable around it, is given by

$$\dot{W} = T_A v_A + T_D v_D + T_E v_E + T_H v_H = T_A(-3v + 3v - v + v) = 0$$

The reactions on a rigid body connected to a fixed body, and the pair of reactions on two connected rigid bodies, are workless if the connection is by (i) a smooth universal joint, (ii) a clevis-type joint, (iii) a light rod connected by two hinge joints, two ball and socket joints or by a hinge joint and a ball and socket joint to the two bodies.

2.23 Deformable Body, Rigid Body and Mass-point Models

The 3 models for a physical system are: deformable body, rigid body and mass-point.

1. **Deformable body model:** The dof of a deformable body are infinite since the position of every material point is to be specified. Hence, Euler's axioms are applied to an infinitesimal volume which in the limit yield partial differential

equations of motion with the number of unknowns in them being more than the number of equations. The constitutive equations of the material of the body, relate the local internal force distributions (stresses) point-wise and the deformation measures (strains, strain-rates). The strain–displacement or strain-rate–velocity relations complete the set of governing equations.

2. **Rigid body model:** A rigid body has invariant distances between its material points for all time, for all motions and loadings. This is just an idealization and no example of a perfectly rigid body can be given. Euler's axioms applied to the whole rigid body are sufficient to study its motion since it has only 6 dof. For many engineering structures and machines, the materials are such that local deformations are small compared to their overall size and overall range of motion. Therefore, we often first model a body as a rigid one in order to determine its motion if partially constrained, or the reactions of its supports if fully constrained and statically determinate (see Section 4.1.1). Then we find the stresses and deformations by idealising the body as deformable. For example, the motion of a flywheel is studied by modelling it as a rigid body, whereas for its design for strength it is modelled as a deformable body. However, for statically indeterminate problems (see Section 4.1.1), it is not possible to separate the two analyses *even when the deformations are small* and the body has to be modelled as deformable.

3. **Mass-point model ('particle' or point-body):** $\vec{F} = m\vec{a}_{C|I}$ implies that the motion of the centre of mass C of a system is the same as that of a point-body of mass m under the action of a force $\vec{F}$. Thus, a *mass-point/point-mass/particle/point-body* is a model for a body in which the mass and the forces are lumped at its centre of mass C. It has a position and mass m but no size. Irrespective of the size of the system, the particle model is good if we only seek the motion of C, and the total external force on the system is not significantly affected by the motion of the system relative to C. For example, the motion of the centre of mass of a small lead pellet falling in still air can be determined by modelling it as a mass-point, but the motion of the centre of mass of a leaf falling in still air cannot be determined by modelling it as a mass-point since the total external force on the leaf depends on the aerodynamic forces on it, which are sensitive to the rotation of the leaf about its centre of mass. The leaf has to be modelled as a deformable body since it is also quite flexible. A missile falling in still air can be modelled as a rigid body since its deformations are small, but the aerodynamic forces on it also depend on its angular velocity. Hence, it cannot be modelled as a mass-point.

Consider a uniform rigid cylinder of mass m rolling without slip down an inclined plane at an angle α with the horizontal. It can be shown that irrespective of its size, i.e., its radius R, its centre has acceleration $2g\sin\alpha/3$ and the friction force is $mg\sin\alpha/3$. Slip occurs if the coefficient of friction $\mu < \tan\alpha/3$. These results differ from those of a cuboid which slips down the plane with acceleration $g(\sin\alpha - \mu\cos\alpha)$, and the friction force is $\mu mg\cos\alpha$ if $\mu < \tan\alpha$. Thus, a mass-point model of the cylinder to determine the motion of its centre C is wrong, even when R is very small. This is because the friction-force part of the total external force $\vec{F}$ depends on the rotational motion of the body.

The mass-point model may be convenient for a physical problem in which a body is very small in comparison with the distances involved. If we observe a system (a galaxy, a star or a spinning sphere) from a very great distance, it appears as a moving point, and the detailed local motions of rotation and vibration are not observable. Our only interest is about the *observable* motion of a body's centre of mass. This motion is independent of the way the external forces are distributed and the way its parts interact and move relative to each other, *provided the total external force $\vec{F}$ is the same.* Such a system can be modelled as a mass-point.

From tiny protein molecules, particles of pollutants, satellites, planets, stars, up to large galaxies, encompassing a wide range—10^{40} in length size and 10^{80} in mass—many physical systems are modelled as mass-points for the study of the motion of their centre of mass.

Review 2 A

1. The moment of $\vec{F}$ about a line parallel to it, and about a line intersecting it is 0.
2. The moment of a couple about all points is the same.
3. $\vec{p} = m\vec{v}_C = \sum m_i\vec{v}_{C_i}$, $\quad \vec{H}_A = \vec{H}_C + \vec{r}_{CA} \times m\vec{v}_{CA}$, $\quad T = \frac{1}{2}mv_C^2 + \frac{1}{2}\int_m v_{PC}^2\, dm$.
4. $\vec{F} = \dot{\vec{p}} = m\vec{a}_C = \sum m_i\vec{a}_{C_i}$, $\quad \dot{\vec{H}}_O = \vec{M}_O$ for O fixed in I.
5. Two equivalent force systems acting separately on the same material system with the same initial conditions cause the same $\vec{a}_{C|I}$. For *a rigid body,* they cause the same motion and $\dot{W}$, but may cause *different internal force distributions.*
6. The resultant/equivalent system at A is a force $\vec{F}_R = \vec{F}$ and a couple $\vec{C}_{R_A} = \vec{M}_A$.
7. The simplest resultant is a wrench. It simplifies to a force/couple/null system if $\vec{F}_R \cdot \vec{C}_{R_A} = 0$, which holds for concurrent, coplanar, and parallel force systems.
8. The centre R of parallel forces is located at $\vec{r}_R = \sum \vec{r}_i F_i / \sum F_i$.
9. For a distributed parallel force $\perp$ to a plane curve, $F_R =$ algebraic area of loading (if $\neq 0$) and acts through its C^*. For a distributed force $\perp$ to a plane surface (if $\neq 0$) and acts through its C^*. area, $F_R =$ algebraic volume of pressure space (if $\neq 0$) and acts through its C^*.
10. $W = \int \vec{F}(t) \cdot \vec{v}(t)\, dt$. For $\vec{F}$ acting at same material point, $W = \int_{C:\vec{r}(t_1)}^{\vec{r}(t_2)} \vec{F}(t) \cdot d\vec{r}$.
11. A is a point of a rigid body, $\dot{W} = \vec{F} \cdot \vec{v}_A + \vec{M}_A \cdot \vec{\omega}$. For a couple $\vec{C}$: $\dot{W} = \vec{C} \cdot \vec{\omega}$.

12. $\dot{\vec{H}}_{A|I} = \vec{M}_A$ if $\vec{a}_{A|I} = \vec{0} \,/\, A \equiv C \,/\, \vec{a}_{A|I} \parallel \vec{r}_{CA}$.

13. For plane motion of a rigid body: $\vec{M}_A = \dot{\vec{H}}_A$ is *not always* valid for A on instantaneous axis of rotation $(\vec{e})$ in a plane through C normal to $\vec{e}$.

14. For any system and any point A: $\vec{M}_A = \dot{\vec{H}}_C + \vec{r}_{CA} \times m\vec{a}_C$.

15. For T translating at $\vec{a}_T$ relative to I, the equations of motion relative to T are identical to those relative to I if we replace $m\vec{g}$ by $m\vec{g}_e$ where $\vec{g}_e = \vec{g} - \vec{a}_T$.

16. For part P of a system in equilibrium/light system in motion: $\vec{F}(P) = \vec{0}$, $\vec{M}_{A}(P) = \vec{0}$

17. For a member in equilibrium/light member in motion: (i) If subjected only to $\vec{F}_1$ at A and $\vec{F}_2$ at B, then $\vec{F}_1 = -\vec{F}_2$ and both act along AB. (ii) If subjected to 3 forces, then these are coplanar, and concurrent/parallel.

18. $\vec{I} = \Delta\vec{p} = m\Delta\vec{v}_C = \sum m_i \Delta\vec{v}_{C_i}$.

19. $\vec{I}_{\text{ang}_A} = \Delta\vec{H}_A$ if $\vec{a}_{A|I}(t) \equiv \vec{0} \,/\, A \equiv C \,/\, \vec{a}_{A|I}(t) \parallel \vec{r}_{CA}(t)$.

20. For impulsive loads at t_1: $\vec{I}_{\text{ang}_C}(t_1) = \Delta\vec{H}_C$. $\vec{I}_{\text{ang}_A}(t_1) = \Delta\vec{H}_A$ if $\Delta\vec{v}_A = \vec{0}$.

21. $\dot{W}^* = \vec{F} \cdot \vec{v}_C$, $T_C = \frac{1}{2}mv_C^2$, $\dot{T}_C = \dot{W}^*$, $T_{C_2} - T_{C_1} = W^*_{1-2}$.

22. For no slip, $\vec{F}$ is arbitrary in magnitude and direction but bounded by $|\vec{F}| < \mu_s N$.

23. Displacement constraint $\Rightarrow$ force reaction. Rotation constraint $\Rightarrow$ moment reaction. Smooth hinge with axis z, reactions are $R_1\vec{\imath} + R_2\vec{\jmath} + R_3\vec{k}$ and $C_1\vec{\imath} + C_2\vec{\jmath}$.

24. Impending slip of belt: $\dfrac{T_1 - \lambda v^2}{T_2 - \lambda v^2} = e^{\mu_s\theta/\sin\alpha}$, $p = \dfrac{T - \lambda v^2}{\rho}$. T_1 along slip.

25. Thrust bearing: $M/P = [\int_s \mu_s r^2 p\, ds] \,/\, [\int_s rp\sin\alpha\, ds]$, p at r at $\pi/2 - \alpha$ to axis.

26. Screw-jack: $M/P = r\tan(\alpha+\theta)$, where $\theta = \tan^{-1}\mu_s$. Self-locking if $\tan\alpha < \mu_s$.

27. Conservative $\vec{F} \Rightarrow \vec{F}(\vec{r})$: work path-independent, work in closed path is 0, $\vec{\nabla} \times \vec{F} = \vec{0}$, $dW = -dV$, $F_y = -\partial V/\partial y$, $F_r = -\partial V/\partial r$, $F_\phi = -\partial V/r\partial\phi$.

28. $V(\vec{r}) - V(\vec{0}) = -\int_0^x F_x(x,0,0)dx - \int_0^y F_y(x,y,0)dy - \int_0^z F_z(x,y,z)dz$
$= -\int_0^r F_r(r,\phi,0)dr - \int_0^z F_z(r,\phi,z)dz$.

29. For constant $\vec{F}$: $V = -\vec{F} \cdot \vec{r} = -F_x x - F_y y - F_z z$.

30. For constant couple $\vec{C}$ on a rigid body with $\vec{\omega}(t) = \dot{\theta}(t)\vec{k}$: $V = -C_z\theta$.

31. For $\vec{F} = F_r(r)\vec{e}_r$: $V(r) = -\int_{r_0}^r F_r(r)\, dr$.

32. Spring, extension e, $F(e)$, $V(e) = \int_0^e F(e)\, de$. Linear $k \Rightarrow V = \frac{1}{2}ke^2$.

33. Torsional spring, twist θ, $M(\theta)$, $V(\theta) = \int_0^\theta M(\theta)\, d\theta$. Linear $k_t \Rightarrow V = \frac{1}{2}k_t\theta^2$.

34. For pair of $\vec{R}, -\vec{R}$ and $\vec{C}, -\vec{C}$ on bodies 1, 2 at smooth joint/contact or at rough contact with no slip: $\dot{W} = 0$. Not so for internal driving moment/driving force.

2.24 Concept Review Questions 2A

1. Reference frame 1 is in translation with respect to frame 2. Which of the measurements of $\vec{p}$, $\dot{\vec{p}}$, $\vec{H}_A$, $\dot{\vec{H}}_A$, T, $\dot{T}$, W, $\dot{W}$ for a system made by two observers in these frames will be the same? [Only $\vec{H}_A$, $\dot{\vec{H}}_A$]

2. (a) The resultant of a force system at A is a force $\vec{F}_{R_A}$ and a couple $\vec{C}_{R_A}$. Find its resultant at B. (b) $\vec{F}_{R_A} = 4\vec{\imath} - 3\vec{\jmath} + 5\vec{k}$ kN, $\vec{C}_{R_A} = -2\vec{\imath} - 4\vec{\jmath} + \vec{k}$ kN.m. Find the couple of minimum magnitude to balance this force system in addition to an appropriate force. [(a) $\vec{F}_{R_A}$, $\vec{BA} \times \vec{F}_{R_A} + \vec{C}_{R_A}$ (b) $-0.72\vec{\imath} + 0.54\vec{\jmath} - 0.9\vec{k}$ kN.m]

3. Find the simplest resultant of the force systems shown in Fig. Q2.3a,b.
[(a) $-10\vec{k}$ kN at $12.8\vec{\imath} - 11.6\vec{\jmath}$ m (b) $-80\vec{\imath} + 210\vec{\jmath}$ kN through $(0, 11.5)$ m]

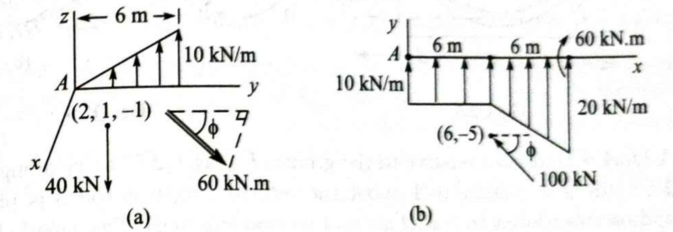


Fig. Q2.3

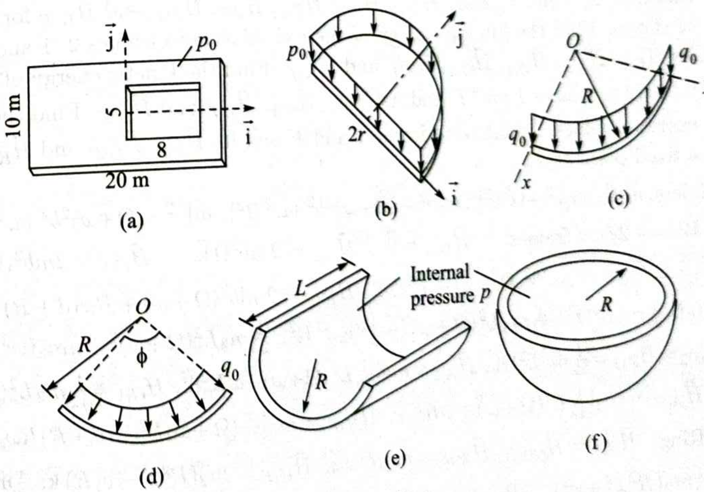


Fig. Q2.4

4. Find the simplest resultant of the uniform normal force distributions on plates (Fig. Q2.4a,b), rods (Fig. Q2.4c,d) and in shells (Fig. Q2.4e,f).
[(a) $160p_0$ at $-\vec{\imath}$ m (b) $-\frac{1}{2}p_0\pi r^2\,\vec{k}$ at $(4r/3\pi)\,\vec{\jmath}$ (c) $-(q_0\pi R/2)\,\vec{k}$ at $(2R/\pi)(\vec{\imath}+\vec{\jmath})$ (d) $2q_0R\sin\frac{\phi}{2}\,\vec{e}_r$ thro' O bisecting ϕ (e) $2RLp$ at centre (f) $\pi R^2 p$ at centre]

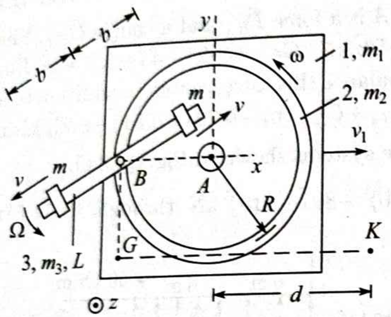


Fig. Q2.5

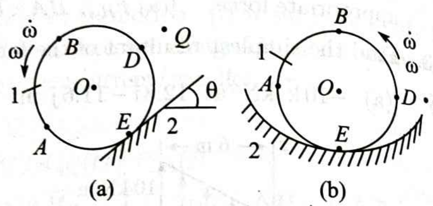


Fig. Q2.6

5. A block 1 translates relative to the ground F (Fig. Q2.5) at v_1 along x. A ring 2 rotates at ω relative to 1 about the vertical z axis. A rod 3 is pinned to 2 and rotates relative to 2 at Ω about a vertical axis at B. Two small sliders slide at v relative to 3 and are subjected to friction force f each. Find the kinetic energy of the pair of sliders, $\dot W$ by the friction forces on the pair relative to frames 3, 2, 1 and F. Find $\vec{H}_{B|3}$, $\vec{H}_{A|2}$, $\vec{H}_{G|1}$, $\vec{H}_{A|1}$, $\vec{H}_{A|F}$ and $\vec{H}_{K|F}$ for the pair of sliders. Find the kinetic energy of the rod relative to frames 2, 1 and F, and its $\vec{H}_{B|2}$, $\vec{H}_{A|2}$, $\vec{H}_{A|1}$, $\vec{H}_{G|1}$, $\vec{H}_{A|F}$ and $\vec{H}_{K|F}$. Find the kinetic energy of the ring relative to frames 1 and F and its $\vec{H}_{A|1}$, $\vec{H}_{G|1}$, $\vec{H}_{A|F}$ and $\vec{H}_{K|F}$. Find the kinetic energy of block 1 relative to frame 1 and F and its $\vec{H}_{A|1}$, $\vec{H}_{G|F}$ and $\vec{H}_{K|F}$. K is a fixed point in F.

[sliders: mv^2, $m[v^2+\Omega^2b^2]$, $m[v^2+(\Omega+\omega)^2b^2+\omega^2R^2]$, $m[v^2+(\Omega+\omega)^2b^2+\omega^2R^2+v_1^2]$
$\dot W=-2fv\ \forall$ frames, $\vec{H}_{B|3}=\vec{0}$, $\vec{H}_{A|2}=2mb^2\Omega\,\vec{k}$, $\vec{H}_{G|1}=2mb^2(\Omega+\omega)\,\vec{k}$,
$\vec{H}_{A|1}=\vec{H}_{A|F}=2m[b^2(\Omega+\omega)+R^2\omega]\vec{k}$, $\vec{H}_{K|F}=2m[b^2(\Omega+\omega)+R\omega(d+R)-v_1R]\vec{k}$
Rod: $\frac{1}{24}m_3L^2\Omega^2$, $\frac{1}{24}m_3L^2(\Omega+\omega)^2+\frac{1}{2}m_3\omega^2R^2$, $\frac{1}{24}m_3L^2(\Omega+\omega)^2+\frac{1}{2}m_3(\omega^2R^2+v_1^2)$
$\vec{H}_{B|2}=\vec{H}_{A|2}=\frac{1}{12}m_3L^2\Omega\vec{k}$, $\vec{H}_{A|1}=m_3[\frac{1}{12}L^2(\Omega+\omega)+\omega R^2]\vec{k}$, $\vec{H}_{G|1}=\frac{1}{12}m_3L^2(\Omega+\omega)\vec{k}$,
$\vec{H}_{A|F}=m_3[\frac{1}{12}L^2(\Omega+\omega)+\omega R^2]\,\vec{k}$, $\vec{H}_{K|F}=m_3[\frac{1}{12}(\Omega+\omega)L^2+(d+R)R\omega-v_1R]\,\vec{k}$
Ring: $\vec{H}_{A|1}=\vec{H}_{G|1}=\vec{H}_{A|F}=m_2R^2\omega\,\vec{k}$, $\vec{H}_{K|F}=m_2(R^2\omega-v_1R)\,\vec{k}$, $\frac{1}{2}m_2R^2\omega^2$,
$\frac{1}{2}m_2(R^2\omega^2+v_1^2)$, Block 1: 0, $\frac{1}{2}m_1v_1^2$, $\vec{H}_{A|1}=\vec{H}_{G|F}=\vec{0}$, $\vec{H}_{K|F}=-m_1v_1R\,\vec{k}$]

6. For which of the points A, B, D, E, O, Q is $\vec{M}_P=\dot{\vec{H}}_P$ valid for each of the uniform cylinders in Fig. Q2.6 rolling without slip with $\omega\neq 0$, $\dot\omega\neq 0$? Work out by considering i surface 2 is (1) fixed; (ii) surface 2 is fixed to a conveyor translating to the right with (2) uniform velocity, (3) uniform acceleration; (iii) surface 2 is fixed to an elevator translating up with (4) uniform velocity, (5) uniform acceleration; (iv) surface 2 is fixed to an escalator translating up at angle θ to the horizontal (angle θ same as in Fig. Q2.6a) with: (6) uniform velocity, (7) uniform acceleration. How will the answers change if $\omega=0$?

[No for A, B, D; yes for O, Q; yes for E in cases (1), (2), (4), (6); no for E in cases (3), (7). For E in case (5): no for (a) & yes for (b). No change]

7. Draw FBD of member $ABCDEFJ$, making simplifications for supports which are two-force members (Fig. Q2.7). All contacts are smooth and all members are light. (a) The loading is coplanar. (b) The loading is three dimensional.

[(a) C_0, Q, triangle load, load left of J; R_1, R_2, C_1 at A; N, S, M at J; cable T, normal reactions at slot and roller, 1 reaction component from a straight and a curved 2-force member, 2 reaction components at remaining supports and D.
(b) At hinge of 2-force members of part (a) also couple $c_1\vec{\imath}+c_2\vec{\jmath}$ and force $F_3\vec{k}$. Other hinges: 3 reaction force components and couples $C_{i1}\vec{\imath}+C_{i2}\vec{\jmath}$, 3 reaction force components at E, 3 force reaction components and 3 couple components at A and J, cable tension, reactions at slot and roller support consist of a normal force N and a couple $\perp N$ in xy-plane and applied loads]

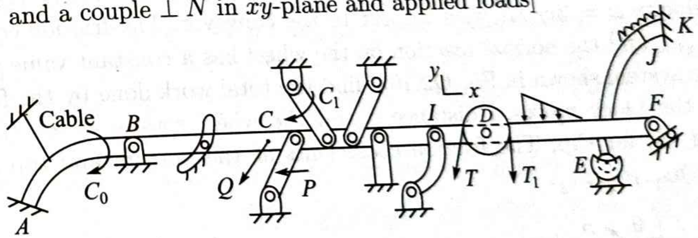


Fig. Q2.7

8[d]. (a) Find the condition to be satisfied by μ_1, μ_2 so that the light wedges gripping a specimen ensure no slip (Fig. Q2.8). [Hint: As the specimen deforms under gripping force, the wedges will slip a little on their supports.] (b) A door stopper is a light link hinged to the door. It operates at an angle θ to the vertical. The coefficient of friction with the floor may vary from 0.1 to 0.8. Design the maximum value of θ for the stopper. [(a) $\mu_1+\tan\alpha<\mu_2(1-\mu_1\tan\alpha)$ (b) 5.71°]

9[d]. (a) Find h if the sluice gate (Fig. Q2.9a)is designed to open when the water depth equals d. The hydrostatic gauge pressure at a depth y is ρgy. (b) Compare the efficacy of the two designs for supporting two loads P (Fig. Q2.9b). The load is transferred to the tie rods via nuts and washers. The criteria for design are the loads carried by nut connections and the tie rods. [(a) $\frac{1}{3}d$ (b) 1 is better]

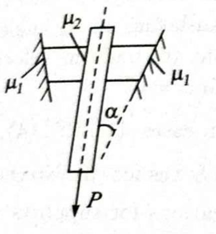


Fig. Q2.8

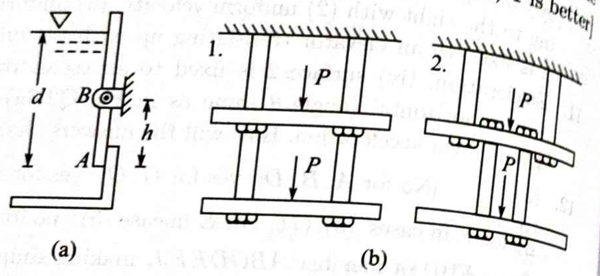


Fig. Q2.9

10. (a) A rigid wheel rolls without slip on a conveyor T (Fig. Q2.10a) translating relative to the ground G. Constant forces acting are: F_1 at C, Q at the end of a string wrapped around the axle, P at an angle θ with the vertical at a point at distance d above C and the friction force F in the direction of motion of the conveyor. A moment M acts as shown. Find the work done by P, Q, F_1, F, M relative to frames T and G in the period in which C advances a distance b relative to the conveyor and the conveyor moves a distance s in the same direction. (b) Rework part (a) if the wheel rolls with slip with velocity v_C and angular velocity $\omega = 3v_C/R$ with respect to the conveyor. The friction coefficients are μ_s, μ_k and the normal reaction on the wheel has a constant value of N. (c) For the system shown in Fig. Q2.10b, find the total work done by the friction forces as the block moves a distance d and the wheel rotates by θ: (i) for no slip and (ii) for slip. The friction coefficients at the bottom and top contacts are $\mu_{s_1}, \mu_{k_1}, \mu_{s_2}, \mu_{k_2}$.

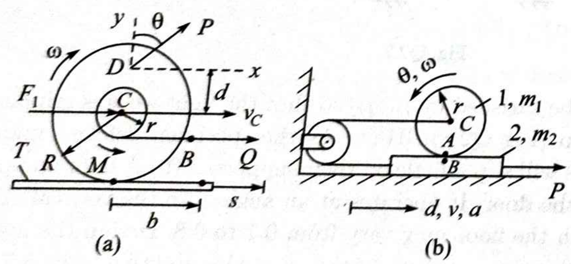


Fig. Q2.10

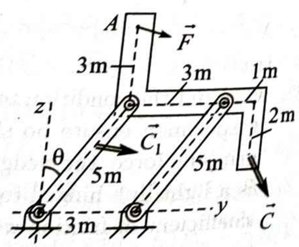


Fig. Q2.11

[(a) $W_{|T} = [Pb\sin\theta(d+R)/R,\ Q(R-r)b/R,\ F_1 b,\ 0,\ -Mb/R]$

(b) $W_{|T} = [Pb\sin\theta(1+3d/R),\ Q(1-3r/R)b,\ F_1 b,\ -2\mu_k Nb,\ -3Mb/R]$

For both (a) & (b): $W_{|G} = W_{|T} + [P\sin\theta s,\ Qs,\ F_1 s,\ Fs,\ 0]$

(c) (i) $-\mu_{k_1}(m_1+m_2)gd$ (ii) $-\mu_{k_1}(m_1+m_2)gd - \mu_{k_2}m_1 g|2d - r\theta|$]

11. Find the work done by $\vec{F} = 20\vec{i} + 5\vec{j} - 4\vec{k}$ kN (Fig. Q2.11) and couples $\vec{C} = -3\vec{i}+2\vec{j}-3\vec{k}$, $\vec{C_1} = -4\vec{i}-2\vec{j}-4\vec{k}$ kN.m when the system moves from position 1 at $\theta = \cos^{-1}0.8$ to position 2 at $\theta = \cos^{-1}0.6$. [10.14 kN.m]

12. (a) Find the work done by a force P acting in the circumferential direction at the tip of a telescopic link which is rotated at uniform ω during the interval in which its length L increases from L_1 to L_2 at: (i) uniform rate $\dot{L} = v$, and (ii) uniform $\ddot{L} = a$ with initial $\dot{L} = 0$. (b) Rework part (a) if the link has uniform $\dot{\omega}$ with initial $\omega = 0$. [(a) (i) $P\omega(L_2^2 - L_1^2)/2v$ (ii) $P\omega\sqrt{2(L_2-L_1)/a}(2L_1+L_2)/3$

(b) (i) $P\dot{\omega}(L_2-L_1)^2(L_1+2L_2)/6v^2$ (ii) $P\dot{\omega}(L_2^2-L_1^2)/2a$]

13. Find the work done by: (a) $\vec{F} = \phi(1/r + 1/r^3)\vec{e}_r + [(1+\cos\phi)/r^2]\vec{e}_\phi$ along the straight path from $A(r_1, \phi_1)$ to $B(r_2, \phi_1)$ followed by the circular path from $B(r_2, \phi_1)$ to $C(r_2, \phi_2)$, (b) $\vec{F} = (c_1\cos\phi/r)\vec{e}_r + (c_2/r)\vec{e}_\phi$ along the path $r = 1/(a+b\cos\phi)$ from $(1/(a+b), 0)$ to $(1/a, \pi/2)$, (c) $\vec{F} = x\vec{i} + yx^2\vec{j} + 2z\vec{k}$ along the circular path of radius 1 m centred at $(2,0,0)$ from A $(1,0,0)$ to B $(2,1,0)$.

[(a) $\phi_1[\ln(r_2/r_1) - \frac{1}{2}(r_2^{-2} - r_1^{-2})] + [\phi_2 - \phi_1 + \sin\phi_2 - \sin\phi_1]/r_2$

(b) $\frac{1}{2}\pi c_2 + c_1[a\ln\{a/(a+b)\} + b]/b$ (c) 29/12]

14. Find the potential energy of the following forces, if conservative. (a) $4\vec{i}-3\vec{j}+5\vec{k}$, (b) $f(x)\vec{i}+g(y)\vec{j}+h(z)\vec{k}$, (c) $f[(x^2+y^2)^{1/2}](x\vec{i}+y\vec{j})$, (d) $(x\vec{i}+y\vec{j}+z\vec{k})g(r)/r$ with $r^2 = x^2+y^2+z^2$, (e) $c\vec{r}/r^5$, (f) $3\dot{x}\vec{i} - \dot{z}\vec{j}$, (g) $(\alpha/r)[\cos 2\phi(\vec{e}_r + \cos\phi\,\vec{e}_\phi) + \sin\phi\,\vec{e}_\phi]$, (h) $\vec{A}\sin\omega t$.

[(a) $-4x+3y-5z$ (b) $-\int_0^x f(x)\,dx - \int_0^y g(y)\,dy - \int_0^z h(z)\,dz$ (c) $-\int_0^r rf(r)\,dr$

(d) $-\int_0^r g(r)\,dr$ (e) $c/3r^3$, (f) no (g) no (h) no]

15[d]. A flat belt of width b, thickness t and density $\rho = 1000\,\text{kg/m}^3$ is designed to transmit power P with the driver pulley of radius $r = 0.1$ m, rotating at n rpm, with angle of lap $\theta = 150°$ and $\mu = 0.25$. For belt: $T_{max} = T_1$ and $T_{min} = T_2$. It was designed on the basis of $(T_1 - T_2)r\omega = P$, $T_1/T_2 = e^{\mu\theta}$, $T_1 = \sigma bt$, where σ is the allowable stress. The belt designed for $n = 150$ rpm, $\sigma = 2$ MPa, $P = 200$ W, $b = 10t$ performed well. However, a belt designed similarly for $n = 1500$ rpm broke down. Find the dimensions of the two belts, analyse the reason for their contrasting behaviour, and hence re-design the belt for $n = 1500$ rpm.

[150 rpm: 3.641 mm × 36.41 mm. 1500 rpm: 1.151 mm × 11.51 mm, fails due to neglect of λv^2 in tension ratio in design, re-design: 1.23 mm × 12.3 mm]

2.25 Practice Problems 2A

1. Find $\vec{M}_O$, M_{OB} and the equivalent (resultant) force system at O (Fig. P2.1) for (a) tension 700 N in cable ED and couple $\vec{C} = 200\,\vec{i} + 300\,\vec{j} - 400\,\vec{k}$ N.m acting on a hinged plate, (b) force applied to a crank, and (c) weight mg acting at C, where AC has rotated by angle θ about OB from its equilibrium position in the vertical yz plane.

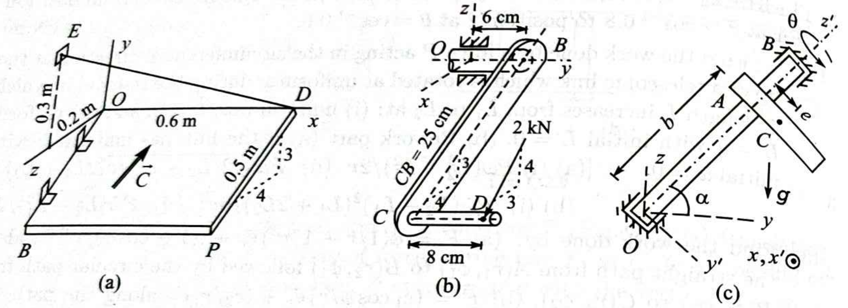


Fig. P2.1

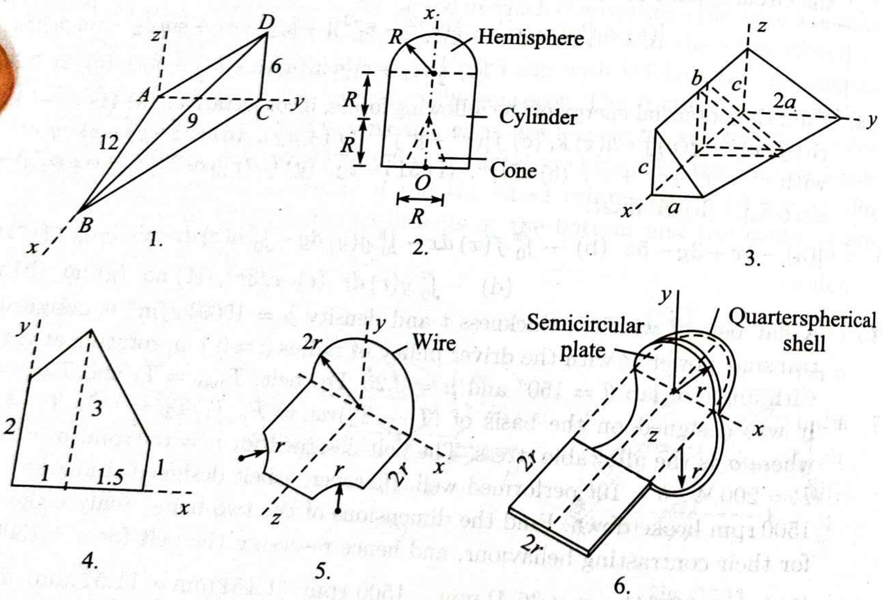


Fig. P2.2

[(a) $\vec{C}_{RO} = \vec{M}_O = 200\vec{i} + 180\vec{j} - 220\vec{k}$, $M_{OB} = -284$ N.m, $\vec{F}_R = -600\vec{i} + 300\vec{j} + 200\vec{k}$ N

(b) $\vec{C}_{RO} = \vec{M}_O = -0.224\vec{i} + 0.14\vec{j} - 0.168\vec{k}$, $M_{OB} = 0.14$ kN.m, $\vec{F}_R = 1.2\vec{i} - 1.6\vec{k}$ kN

(c) $\vec{C}_{RO} = \vec{M}_O = -(e\cos\theta\sin\alpha + b\cos\alpha)mg\,\vec{i}' - e\sin\theta\sin\alpha\, mg\,\vec{j}' - emg\sin\theta\cos\alpha\,\vec{k}'$,

$M_{OB} = -emg\sin\theta\cos\alpha$, $\vec{F}_R = -mg\vec{k}$]

2. Find the centre of mass of the uniform solids 1, 2, 3, plate 4, wire 5 and shell 6 (Fig. P2.2). For part 1, find the centre of mass for a thin box too.

[1. $3\vec{i} + 4.5\vec{j} + 1.5\vec{k}$, $3.434\vec{i} + 4.131\vec{j} + 1.434\vec{k}$ 2. $1.379R\vec{i}$ 3. $\frac{1}{27}(12b\vec{i} + 14a\vec{j} + 9c\vec{k})$ 4. $1.129\vec{i} + 1.167\vec{j}$ 5. $0.5186\,r\vec{j} + 1.130\,r\vec{k}$ 6. $-0.1175r\vec{j} + 1.114r\vec{k}$]

3. Find the centroid of a quarter turn of a helical curve of radius R and pitch p.

[$(2R/\pi)(\vec{i} + \vec{j}) + p\vec{k}/8$]

4. Find C^* of the areas shown in Fig. P2.4.

[(a) $0.4194\,r_0\vec{i} + 0.3500\,r_0\vec{j}$ (b) $2.133\vec{i} + 6.095\vec{j}$]

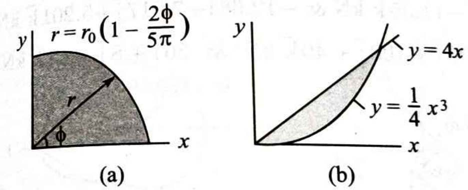


Fig. P2.4

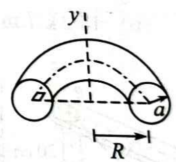


Fig. P2.5

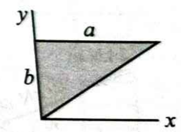


Fig. P2.6

5. Find the centroid of the volume enclosed and of the surface area of the solid half-torus shown in Fig. P2.5. [$(4R^2 + a^2)\vec{j}/2\pi R$, $(2R^2 + a^2)\vec{j}/(\pi R + a)$]

6. Find the centre of mass of a thin plate (Fig. P2.6) whose density σ varies as: $\sigma = \sigma_0[1 + (x/a)(1 + y^2/b^2)]$. [$\frac{55}{138}a\vec{i} + \frac{65}{92}b\vec{j}$]

7. Find the centre of mass of a hemisphere of radius R, shown in Fig. P2.7, if its density ρ varies as: (i) $\rho = \rho_0(1 + cr/R)$, (ii) $\rho = \rho_0(1 + cz/R)$ and (iii) $\rho = \rho_0(1 + c\xi/R)$.

[(i) $\frac{3R(1+0.5333c)}{8(1+0.5890c)}\vec{k}$ (ii) $\frac{3R(1+0.5333c)}{8(1+0.375c)}\vec{k}$ (iii) $\frac{3R(1+0.8c)}{8(1+0.75c)}\vec{k}$]

8. Find the resultant force on (a) the bracket and (b) the antenna (Fig. P2.8) due to the applied forces shown. [(a) $22\vec{i} - 1.68\vec{j}$ kN (b) $5.410\vec{i} + 4\vec{j} - 25.27\vec{k}$ kN]

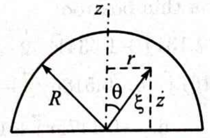


Fig. P2.7

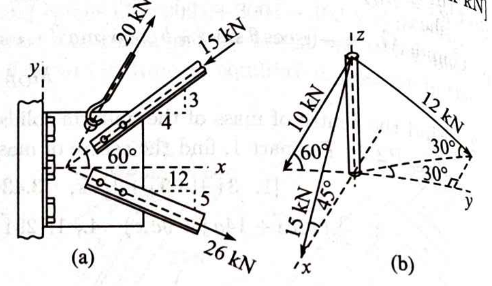


Fig. P2.8

9. A column of a machine and a concrete anchor block are subjected to the forces shown in Fig. P2.9. In Fig. P2.9a, $\alpha = 60°, \beta = 45°, \gamma = 120°$, $BC = CD = 40$ cm, $EF = 20$ cm. Find M_{AB} and resultant at A for each force system.

[(a) 5.201 kN.m, $.9797\vec{i}+13.08\vec{j}-11.35\vec{k}$ kN & $-12.68\vec{i}+7.017\vec{j}+5.201\vec{k}$ kN.m

(b) 43.2 kN.m, $30\vec{i}+100\vec{j}+40\vec{k}$ kN & $56\vec{i}+8\vec{j}-12\vec{k}$ kN.m]

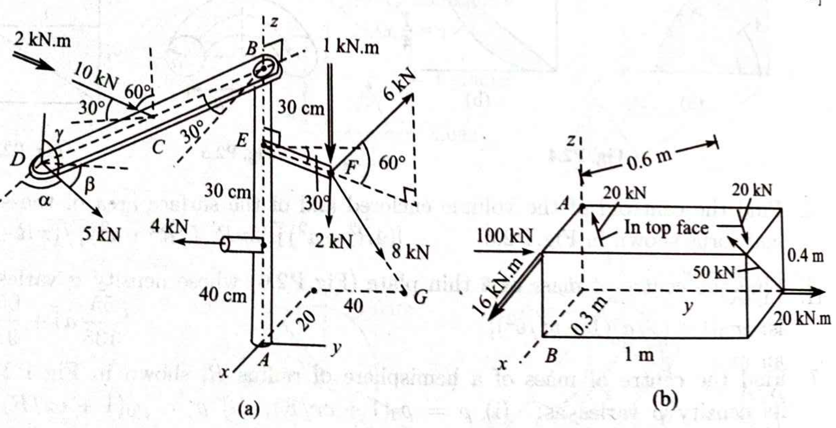


Fig. P2.9

10. (a) A hollow cylinder of radii R_1, R_2 is twisted by end loads distributed tangentially in the circumferential direction $\vec{e}_\phi$ as $\tau = cr$ N/m^2 (Fig. P2.10a). Prove that its simplest resultant is a couple $\vec{C}_R = \frac{1}{2}\pi c(R_2^4 - R_1^4)\,\vec{k}$. (b) The residual shear stress distribution on a circular section of a shaft after unloading is shown in Fig. P2.10b. Find its simplest resultant. (c) Find the simplest resultant of the shear stress distribution on a fully plastic rectangular cross-section of a shaft under torsion (Fig. P2.10c). [(b) null system (c) $C_R = (3b - a)a^2\tau_0/6$]

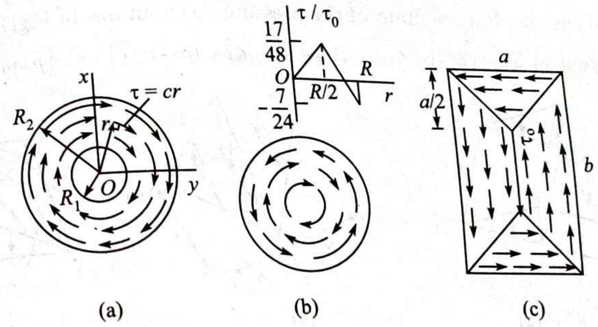


Fig. P2.10

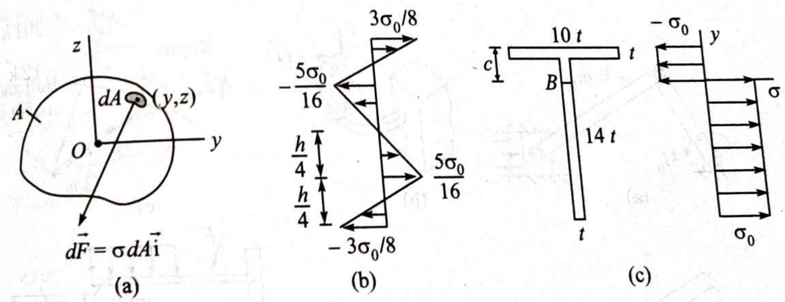


Fig. P2.11

11. (a) A normal stress distribution σ on the cross-section of a beam (Fig. P2.11a) is: $\sigma = a + by + cz$. Find its resultant at the centroid O. If this resultant is an axial force $N\vec{i}$ and a bending moment $M_y\vec{j} + M_z\vec{k}$, then show that $\sigma = N/A + [(I_{yz}M_y - I_{yy}M_z)y + (I_{zz}M_y - I_{yz}M_z)z] / (I_{yy}I_{zz} - I_{yz}^2)$, where $I_{yy} = \int_A z^2\,dA$, $I_{zz} = \int_A y^2\,dA$, $I_{yz} = -\int_A yz\,dA$ are the elements of the inertia matrix of the area of cross-section[6]. If the axes y, z are such that $I_{yz} = 0$, then $\sigma = N/A - M_z y / I_{zz} + M_y z / I_{yy}$ (according to engineering theory of bending of elastic beams.) (b) The residual normal stress, on a rectangular section of width b of a beam, across its depth h is shown in Fig. P2.11b. Find its simplest resultant.

[6] The inertia matrix of area is discussed in detail in Appendix C.

(c) The distribution of normal stress σ on a fully plastic T-section of a beam in pure bending is shown in Fig. P2.11c. Find c so that its simplest resultant is a couple and find its magnitude.

[(a) $\vec{F}_R = aA\vec{i}$ & $\vec{C}_{R_O} = (I_{yy}c - I_{yz}b)\vec{j} + (I_{yz}c - I_{zz}b)\vec{k}$ (b) null (c) $3t$, $99\sigma_0 t^2$]

12. Find the simplest resultant of the pressure distributions in Fig. P2.12.

[(a) $p_0 ab$ at $5a\vec{j}/12$ (b) $\frac{1}{6}(b+a)cp_0$ at $\frac{1}{4}[c\vec{i}+(b-a)\vec{j}]$ c) $\frac{1}{4}p_0 ab$ at $\frac{2}{3}(a\vec{i}+b\vec{j})$]

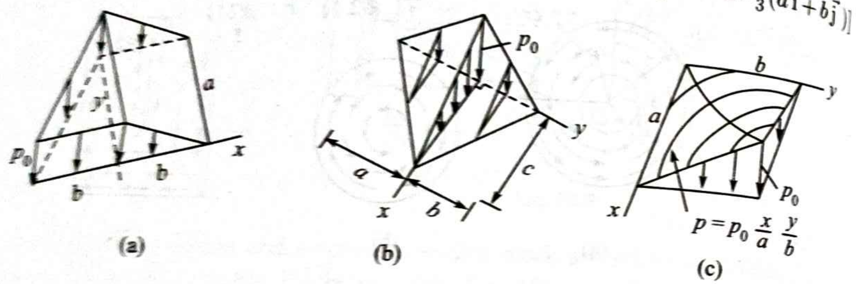


Fig. P2.12

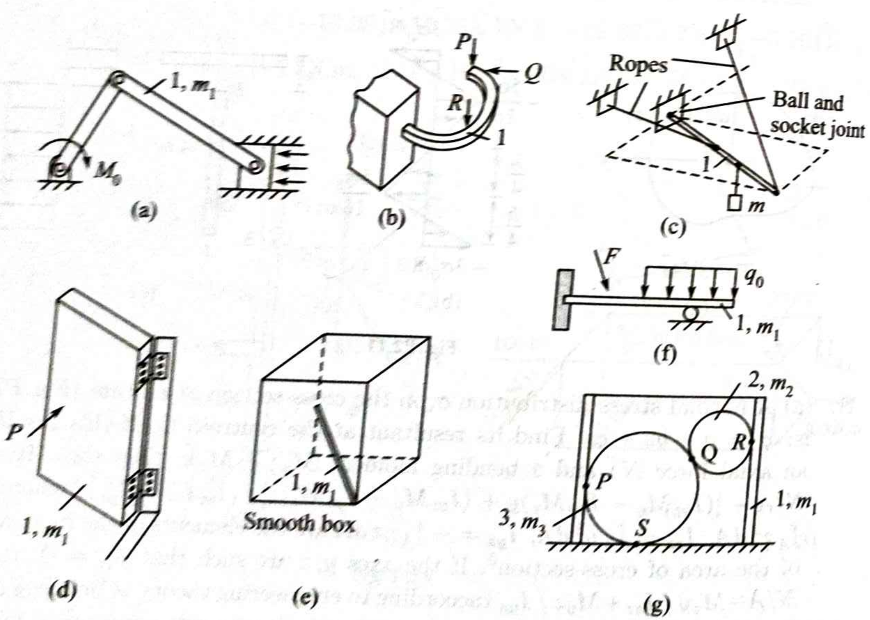


Fig. P2.13

13. Draw FBD of body 1 of mass m_1 in the systems in Fig. P2.13. The cables are light. Assume all hinge and ball and socket joints and surfaces to be smooth. Make simplifications due to the presence of two-force members, if any. Rework case (a) if member 1 is light.

[(a) R_1, R_2; R_3, R_4 at ends, $m_1 g$ at C_1. For light case: only collinear F_1 at ends
(b) $\vec{R}, \vec{C}$ at support, weight $m_1 g$ at C_1 and given loads P, Q, R
(c) Cable tensions T_1, T_2; weights $m_1 g, mg$; reactions R_1, R_2, R_3
(d) P, weight $m_1 g$ at C; $R_1, R_2, R_3, C_1\vec{i} + C_2\vec{j}$ at bottom & R_4, R_5, R_6, $C_3\vec{i} + C_4\vec{j}$ at top hinges where $\vec{k}$ along hinge axis. $C_1 = C_2 = C_3 = C_4 \simeq 0$
(e) weight $m_1 g$; R_1, R_2, R_3 at bottom, $R_4\vec{i} + R_5\vec{j}$ at top ($\vec{k}$ vertical)
(f) R_1, R_2, C_1 at left support; N at roller, $F, q_0, m_1 g$ (g) weight $m_1 g$, normal reactions N_2, N_3 on walls at P, R; N_1 on bottom at distance d from left corner]

14. Draw FBD of body 1 of mass m for each case in Fig. P2.14. The coefficients of friction are μ_s, μ_k. [Given loads, mg, normal reactions N's and (1) $|F| < \mu_s N$ (2) $\mu_s N$ to right (3) F at incline, $\mu_s N_1$ down (4) F at bottom, $\mu_k N_1$ down]

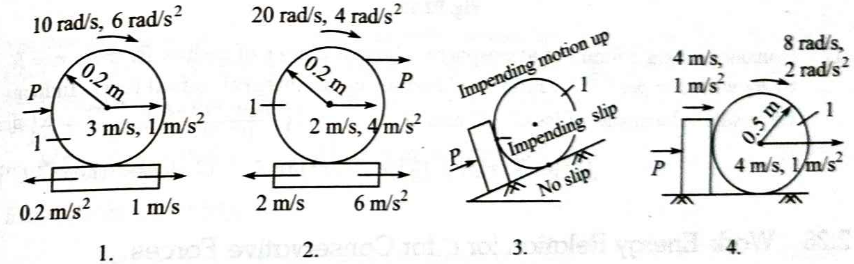


Fig. P2.14

15. (a) Find the work done by the force $\vec{F} = (x^2 + yt)\vec{i} + \cos t\vec{j} + t^2\vec{k}$ N acting on a material point which moves from $\vec{r}_1 = 2\vec{j} + 3\vec{k}$ m at $t = 1$ s to $\vec{r}_2 = 4\vec{i} - 5\vec{k}$ m at $t = 5$ s with uniform velocity. (b) Find the work done by the force $\vec{F} = (r + 8\phi + 2t)\vec{e}_\phi + 7t\vec{e}_z$ N acting on a material point which moves with uniform speed from $\vec{r}_1 = 0.1\vec{i}$ m at $t = 0$ to $\vec{r}_2 = 0.1\vec{j}$ m at $t = 5$ s along a circular arc of radius 0.1 m with centre at the origin. [(a) $-51.10\,J$ (b) $1.788\,J$]

16. (a) Is the force $\vec{F} = (-2xy + yz)\vec{i} + (-x^2 + xz - z)\vec{j} + (xy - y)\vec{k}$ conservative? Find its potential energy if conservative. Find the work done by it in a closed path C and in an open quarter circular path C_1 (Fig. P2.16a). (b) Is the force $\vec{F} = (5x^2 + xy)\vec{i} + (10xy + y^2)\vec{j}$ conservative? Find the work done by it in a closed path (Fig. P2.16b). (c) A plane force field has magnitude prx and is directed towards O. Is it conservative? Find the work done by it for the three paths ACB, ADB

and AB shown in Fig. P2.16c. In each case, the point of application of the force starts at point A and ends at point B. (d) Is the force $\vec{F} = c_1 r^2\,\vec{e}_r + c_2\,\vec{e}_\phi + c_3 z\,\vec{e}_z$ conservative? Find the work done by it when its material point of application moves over a conical helix: $\vec{r}(\phi) = (r_0 - \alpha\phi)\,\vec{e}_r + \beta\phi\,\vec{e}_z$ from $\phi = 0$ to $\phi = \pi$. (e) Rework part (d) for the force $\vec{F} = 2r\phi z^2\,\vec{e}_r + rz^2\,\vec{e}_\phi + 2r^2\phi z\,\vec{e}_z$.

[(a) Yes, $x^2y - (xy - y)z$, 0, -6 (b) No, 875 (c) No, $pa(a^2/3 - b^2/2)$, $p(a^3 - b^3)/3$, $pa(a^2/3 - b^2/6)$ (d) No, $\frac{1}{3}c_1[(r_0-\alpha\pi)^3 - r_0^3] + c_2[r_0\pi - \frac{1}{2}\alpha\pi^2] + \frac{1}{2}c_3\beta^2\pi^2$ (e) Yes, $\beta^2\pi^3(r_0-\alpha\pi)^2$]

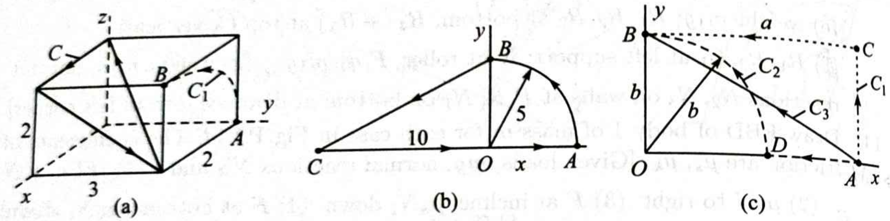


Fig. P2.16

17. Consider a long column of atmosphere, above a planet of radius R, from $r = R_1$ to R_2 with $\rho = \rho_0 e^{\alpha(R-r)}$. Find its C and set up an integral to find its G. Indicate the relative locations of its C^*, C and G. [$\left\{\frac{(R_2e^{-R_2\alpha} - R_1e^{-R_1\alpha})}{(e^{-R_2\alpha} - e^{-R_1\alpha})} + \frac{1}{\alpha}\right\}\vec{e}_r$, $\int_{R_1}^{R_2}(e^{-r\alpha}/r)dr \,/\, \int_{R_1}^{R_2}(e^{-r\alpha}/r^2)dr\;\vec{e}_r$, G closest then C, C^*]

2.26 Work–Energy Relation for C for Conservative Forces

In Sections 2.26 to 2.28, all statements are made with respect to an inertial frame I. For brevity this has not been mentioned explicitly in the equations used/derived.

Let $\dot{W}_c^*$ and $\dot{W}_{nc}^*$ be the rate of work done by the external conservative forces and the external non-conservative forces as if they were acting at the centre of mass C. Let V^* be the potential energy of the former with $\dot{W}_c^* = -\dot{V}^*$. Then, eq. (2.57): $\dot{T}_C = \dot{W}^* = \dot{W}_c^* + \dot{W}_{nc}^* = -\dot{V}^* + \dot{W}_{nc}$, yields

$$\dot{T}_C + \dot{V}^* = \dot{W}_{nc}^*, \qquad (T_{C_2} + V_2^*) - (T_{C_1} + V_1^*) = W_{nc1-2}^* \tag{2.91}$$

with obvious notation in use. If all the forces, as if acting at C, are conservative, then $\dot{W}_{nc}^* = 0$ and eq. (2.91) reduces to conservation of mechanical energy (= kinetic energy + potential energy) associated with C:

$$\dot{T}_C + \dot{V}^* = 0, \qquad T_{C_2} + V_2^* = T_{C_1} + V_1^* \tag{2.92}$$

2.27 Dynamics of a Translating System

A system is *defined to translate* relative to I *if the displacements of its material points are the same for all time.* Hence, $\vec{d}_P(t) = \vec{d}_Q(t) \Rightarrow \vec{v}_P(t) = \vec{v}_Q(t)$, $\vec{a}_P(t) = \vec{a}_Q(t) \Rightarrow \dot{\vec{r}}_{PQ} = \vec{v}_{PQ} = \vec{v}_P - \vec{v}_Q = \vec{0} \Rightarrow \vec{r}_{PQ}(t) =$ constant $\Rightarrow$ the distance between any two material points P and Q remains constant $\Rightarrow$ the system translates as a rigid system with no change in its orientation[7], irrespective of its constitution (rigid/deformable). $\vec{v}_{PA} = \vec{v}_P - \vec{v}_A = \vec{0} \Rightarrow \vec{v}_{PC} = \vec{0}$, $\vec{v}_{iC} = \vec{0}$. Its $\vec{p}$, $\vec{H}_A$ about its material point A, $\vec{H}_C$, $\vec{H}_B$ about any point B, T and $\dot{W}$ of external forces are given by

$$\vec{p} = m\vec{v}_C, \qquad \vec{H}_A(t) \equiv \vec{0} \;\Rightarrow$$
$$\vec{H}_C(t) \equiv \vec{0}, \qquad \vec{H}_B = \vec{r}_{CB} \times m\vec{v}_{CB}$$
$$T = \tfrac{1}{2}mv_C^2 = T_C, \qquad \dot{W} = \vec{F}\cdot\vec{v}_C = \dot{W}^*$$

$\Rightarrow \vec{M}_C = \dot{\vec{H}}_C = \vec{0}$. The equations of motion of a translating system are:

$$\vec{F} = m\vec{a}_C, \qquad \vec{M}_C = \vec{0} \tag{2.93}$$

Thus, the resultant at C of the external force system is a single force $\vec{F}_R = \vec{F} = m\vec{a}_C$ since $\vec{C}_R = \vec{M}_C = \vec{0}$ (Fig. 2.73b). For any arbitrary point B, and any arbitrary line BD in direction $\vec{e}$, $\vec{M}_B = \vec{M}_C + \vec{r}_{CB} \times \vec{F} = \vec{r}_{CB} \times m\vec{a}_C$, i.e.,

$$\vec{M}_B = \vec{r}_{CB} \times m\vec{a}_C, \qquad M_{BD} = [\vec{r}_{CB} \times m\vec{a}_C]\cdot\vec{e} \tag{2.94}$$

It follows from eq. $(2.94)_1$ that

$$\vec{M}_B = \vec{0}, \qquad \text{if } \overrightarrow{BC} \text{ is in the direction of } \vec{a}_C \tag{2.95}$$

i.e., moment is zero *only* about any point on the line of acceleration vector $\vec{a}_C$ through C. To avoid solving many simultaneous equations, one should apply eq. (2.94) at a point where most unknown reactions act, or about a line through which they act.

Note that for a translating system (Fig. 2.73c) the moment is *not* zero about *any arbitrary* point, e.g., $\vec{M}_C = \vec{M}_D = \vec{M}_E = \vec{M}_H = \vec{M}_G = \vec{M}_K = \vec{0}$, though $\vec{M}_K \neq \dot{\vec{H}}_K$; and $\vec{M}_a \neq \vec{0}$, $\vec{M}_b \neq \vec{0}$, $\vec{M}_d \neq \vec{0}$, $\vec{M}_e \neq \vec{0}$, though $\vec{M}_a = \dot{\vec{H}}_a$, $\vec{M}_b = \dot{\vec{H}}_b$ and $\vec{M}_d = \dot{\vec{H}}_d$.

For a translating system, the impulse–momentum relations are the same as given in Section 2.11. Further, since $\dot{W} = \dot{W}^*$, $V^* = V$ and $T = T_C$, the work–energy relations (2.57), (2.91), (2.92) become

[7] The converse is not true, e.g., for a thin rod with axial vibration superposed on its gross uniform axial motion, the orientation of all its material line elements remains the same (though their lengths may not), but it is not in translation as the displacements of all its material points are not the same.

$$\dot{T} = \dot{W}, \qquad T_2 - T_1 = W_{1-2}$$

$$\dot{T} + \dot{V} = \dot{W}_{nc}, \qquad (T_2 + V_2) - (T_1 + V_1) = W_{nc_{1-2}} \tag{2.96}$$

$$\text{conservative system: } \dot{T} + \dot{V} = 0, \qquad T_2 + V_2 = T_1 + V_1$$

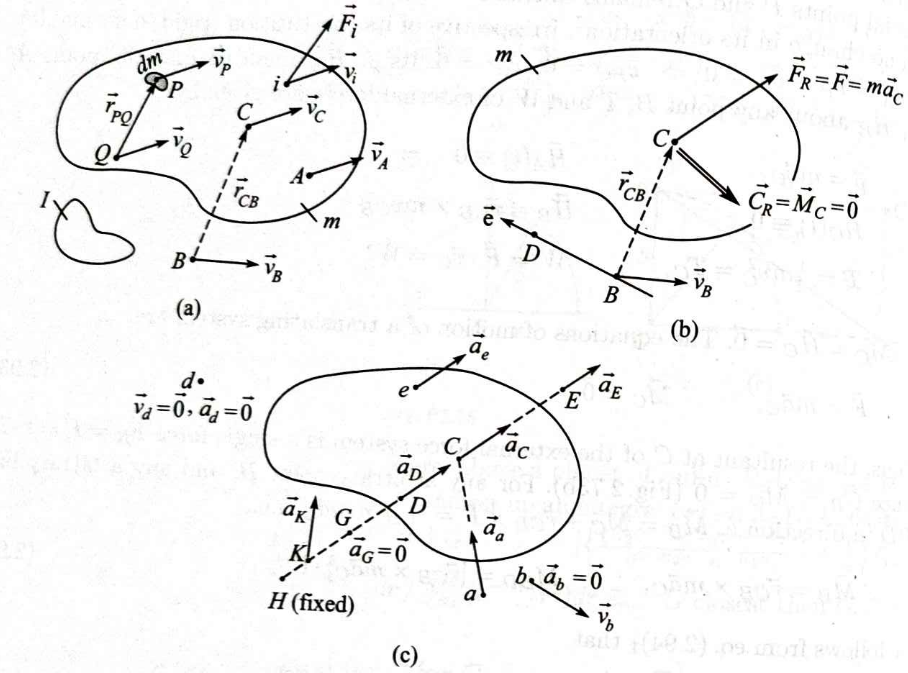


Fig. 2.73

For a system consisting of interconnected translating subsystems, eq. (2.96) is applied to each translating subsystem and added up to yield relations for the *whole system*. This may involve work done by the internal and external forces on the system since some of the forces which are external for an individual subsystem (Fig. 2.74) are in fact internal for the whole system. Denoting the contribution of internal and external forces by the subscript int + ext, the work–energy relations (2.96) become:

$$\dot{T} = \dot{W}_{\text{int+ext}}, \qquad T_2 - T_1 = W_{\text{int+ext}_{1-2}}$$

$$\dot{T} + \dot{V}_{\text{int+ext}} = \dot{W}^{nc}_{\text{int+ext}}$$

$$T_2 + V_{\text{int+ext}_2} - T_1 - V_{\text{int+ext}_1} = W^{nc}_{\text{int+ext}_{1-2}} \tag{2.97}$$

$$\text{conservative system: } \dot{T} + \dot{V}_{\text{int+ext}} = 0, \qquad T_2 + V_{\text{int+ext}_2} = T_1 + V_{\text{int+ext}_1}$$

If all internal forces of interaction are together workless, then eqs (2.97) reduce to eqs (2.96). The *work–energy relation is useful* if the interconnected system has *only one degree of freedom*. The *rate form is used to obtain acceleration*, which may then be integrated. If the work done can be computed independently of the actual motion, then the integrated form can be used directly to get the velocity in a given position.

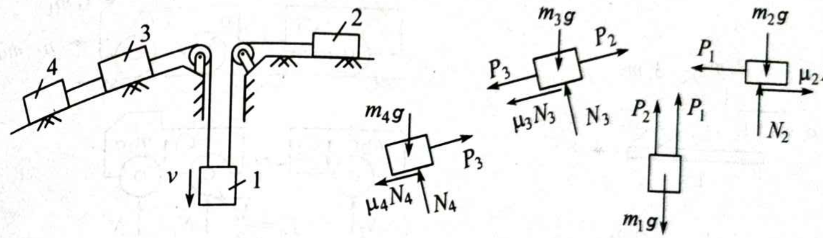


Fig. 2.74

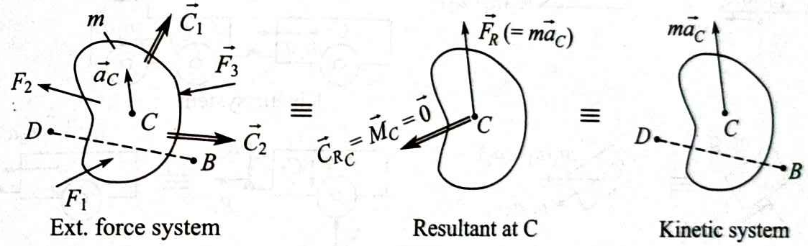


Fig. 2.75

The problem of several connected *translating bodies* can be solved by three methods.

1. Using $\vec{F} = m\vec{a}_C$, $\vec{M}_C = \vec{0}$ for individual bodies which involve internal and external forces on the system.
2. Using $\vec{F} = m\vec{a}_C$ for the whole system and appropriate $\dot{\vec{H}}_O = \vec{M}_O$ for convenient fixed points for the whole system and some individual bodies such that only the external forces on the system appear in the equations.
3. The resultant of the external force system at C is a single force $\vec{F}_R = \vec{F} = m\vec{a}_C$ (Fig. 2.75). In this method, the moment $\vec{M}_B$ or moment M_{BD} of the external force system about a convenient point B (where most unknown reactions act) or line BD (through which most reactions act) is equated to the corresponding moment of the equivalent kinetic system of $m\vec{a}_C$ acting at C. Method 2 would require proper computation of $\dot{H}_{O_z}$, though the resulting moment equation would be the same as in method 3. Methods 2 and 3 are illustrated in Ex. 2.47.

Note that using the equivalent kinetic systems $m_i\vec{a}_{C_i}$ at C_i for the individual translating systems and the fact that action and reaction are equal and opposite, the external force system on the whole system is equivalent to a kinetic system consisting of $m_i\vec{a}_{C_i}$ at C_i. Such equivalent systems for translating blocks on a conveyor and vehicles with light wheels, with driving moment M on rear wheels, are shown in Figs 2.76a,b.

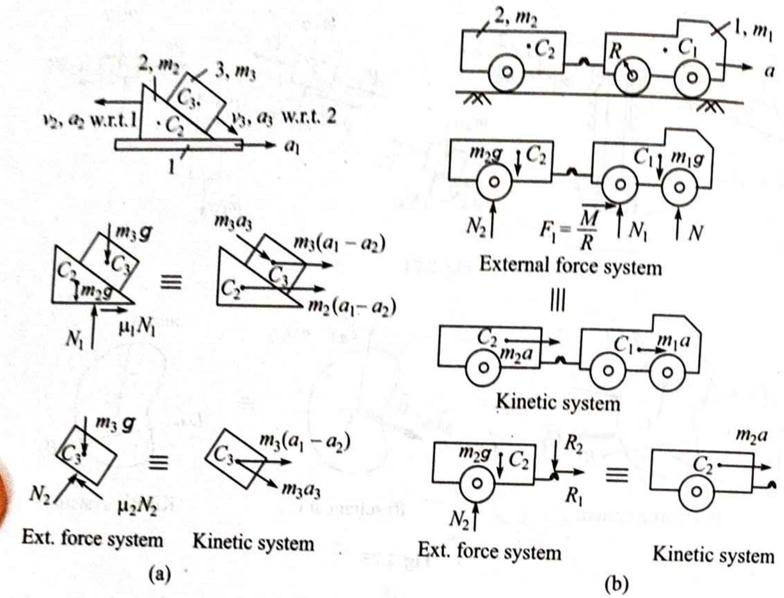


Fig. 2.76

2.27.1 Virtual work approach for equivalent force systems

The work done by equivalent force systems on a rigid body is the same. Therefore, for a translating rigid body, the virtual work, δW, done by the external force system on a rigid body in any infinitesimal virtual displacement to a neighbouring configuration in zero time, keeping the forces constant, equals the virtual work done by its equivalent system of $m\vec{a}_C$ acting at C. It is called virtual as this displacement is considered in zero time, whereas the actual displacement always occurs in a finite time interval. The virtual work in zero time is denoted by δW to distinguish it from the differential work dW in time dt. See Chapter 5 for a more detailed discussion of these concepts.

Hence, for a system of p connected rigid bodies with n dof q_i, $i = 1, \ldots, n$, with each one in translation, the virtual work of all the external and internal forces in any virtual displacement δq_i, $i = 1, \ldots, n$, equals the virtual work done by the equivalent system of $m_i\vec{a}_{C_i}$ acting at C_i. The internal forces are to be included since these are external forces for some individual bodies. The symbol δq_i for virtual displacement in zero time differs from the symbol of dq_i for differential displacement in time dt. Thus,

$$\sum_{j=1}^{p} m_j\vec{a}_{C_j} \cdot \delta\vec{r}_{C_j} = \delta W_{\text{ext+int}} = -\delta V_{\text{ext+int}} + \delta W^{\text{nc}}_{\text{ext+int}} \tag{2.98}$$

for all possible virtual displacements δq_i, where $\vec{r}_{C_j}(q_1, \ldots, q_n) \Rightarrow$

$$\delta\vec{r}_{C_j} = \sum_{i=1}^{n} \frac{d\vec{r}_{C_j}}{dq_i}\,\delta q_i$$

Unlike the usual work–energy relation which yields a single equation for the entire system, this procedure enables us to obtain n equations for an n dof system. This approach is explored further in Chapter 5.

2.28 Dynamics of a Rigid Body with Mass Concentrated at C

Sometimes, we model a stiff body with mass predominantly confined to a small volume as a rigid body with its mass m concentrated at its centre of mass C. For example, a light rigid rod with a heavy bob of mass m at one end (Fig. 2.77a) is modelled as a rigid body with mass m concentrated at C. However, the entire load on the body does not act at C. The FBD is shown in Fig. 2.77b. The spring is un-deformed for $\theta = 0$.

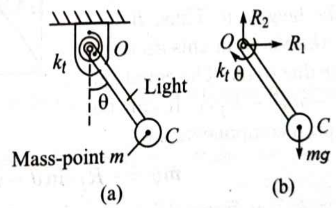


Fig. 2.77

For such a system: $\vec{H}_C(t) = \vec{r}_{CC} \times m\vec{v}_{CC}(t) \equiv \vec{0}$ and

$$\vec{M}_C = \dot{\vec{H}}_C = \vec{0}, \qquad \vec{F} = m\vec{a}_C \tag{a}$$

Hence, the resultant of the external force system at C is a single force $\vec{F}_R = \vec{F} = m\vec{a}_C$ and for an arbitrary point B: $\vec{M}_B = \vec{r}_{CB} \times \vec{F}_R \Rightarrow$

$$\vec{M}_B = \vec{r}_{CB} \times m\vec{a}_C \tag{b}$$

The kinetic energy T, $\dot{W}$ and work–energy relation for C yield:

$$T = \tfrac{1}{2}mv_C^2 = T_C, \qquad \dot{W} = \vec{F}\cdot\vec{v}_C + \vec{M}_C\cdot\vec{\omega} = \vec{F}\cdot\vec{v}_C = \dot{W}^*$$

$$\dot{T}_C = \dot{W}^* \quad \Rightarrow \quad \dot{T} = \dot{W} \tag{c}$$

(d)

Thus, the governing equations (a) to (d) of a 'rigid body mass-point' are exactly the same as those for a translating system. This is to be expected since, as for a translating system, for the present case also $\vec{H}_C \equiv \vec{0}$ and so rotation or $\vec{\omega}$ is irrelevant.

By eqs (3.2) and (3.37): $\vec{H}_C = I_{ij}^C\omega_j\,\vec{e}_i$, $T = T_C + \frac{1}{2}I_{ij}^C\omega_i\omega_j$, where I_{ij}^C are elements of inertia matrix at C. Consider the model of a **rigid body with negligible rotational inertia**, i.e., $I_{ij}^C \simeq 0\ \forall\, i,j$. Thus,

$$\vec{H}_C(t) = I_{ij}^C\omega_j(t)\,\vec{e}_i \equiv \vec{0}\ \ \forall\, t, \qquad T = T_C + \tfrac{1}{2}I_{ij}^C\omega_i\omega_j = T_C$$

as for the 'rigid body point-mass' model. Hence, the governing equations of a rigid body with negligible rotational inertia are the same as those for a translating system.

In particular, eqs (a) to (d) are valid for a 'mass-point' model of a physical system.

2.29 Projectiles with Resistance

The effect of air resistance, $-R_f\vec{e}_t$, is important in determining the motion of an artillery shell (Fig. 2.78) (mass m and centre of mass C). R_f depends on the speed of the projectile, v, and the density of air, ρ, which in turn depends on the height, y. Thus, $R_f = R_f(y, v)$. Note that here, as elsewhere in this section, we neglect the buoyancy force due to air. The equation of motion for C is, $m\vec{a}_C = -mg\vec{j} - R_f\vec{e}_t$. It can be expressed in Cartesian and path components as:

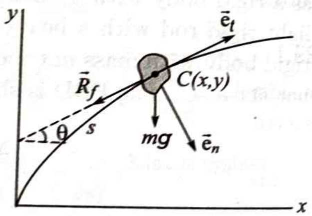


Fig. 2.78

$$m\ddot{x} = -R_f\cos\theta, \qquad m\ddot{y} = -R_f\sin\theta - mg,$$

$$m\dot{v} = mv\frac{dv}{ds} = -R_f - mg\sin\theta, \qquad \frac{mv^2}{\rho} = mg\cos\theta$$

with $v^2 = \dot{x}^2 + \dot{y}^2$, $\cos\theta = \dot{x}/v = \dot{x}/\sqrt{\dot{x}^2+\dot{y}^2}$, $\sin\theta = \dot{y}/\sqrt{\dot{x}^2+\dot{y}^2}$. These equations have to be solved numerically since the dependence of R_f on y and v is usually obtained from tables as analytical expressions are not always available.

To get a quantitative and qualitative idea of the effect of drag, we idealise the physical problem with $R_f(v) = B_1v + B_2v^2$, where B_1, B_2 are constants. For a sphere of radius r, $B_1 = c_1r$, $B_2 = c_2r^2$, where c_1, c_2 are constants and

$$R_f = c_1rv + c_2r^2v^2 = c_1rv(1 + v/v^*) = c_2r^2v^2(1 + v^*/v)$$

$$\Rightarrow\ R_f \simeq c_1rv \quad \text{if } v < 0.1v^*, \qquad R_f \simeq c_2r^2v^2 \quad \text{if } v > 10v^*$$

where $v^* = c_1/c_2r$ is the velocity for which the two terms in R_f are equal. For air, $c_1 \simeq 3.1\times10^{-4}\,\mathrm{kg.m^{-1}.s^{-1}}$, $c_2 \simeq 0.87\,\mathrm{kg.m^{-3}}$, and

$$v^* = \frac{c_1}{c_2r} = \frac{3.6\times10^{-4}}{r} \qquad \text{(SI units)} \tag{2.99}$$

For a macroscopic pebble-like object with $r \simeq 0.01\,\mathrm{m}$, $v^* = 0.036\,\mathrm{m/s}$. Starting from rest, in free fall under gravity, a speed of $10v^*$ occurs after a a fall of 0.8 cm in 0.04 s. Thus, for most practical problems with $r \sim 1\,\mathrm{cm}$, the motion occurs essentially under a resistive force $R_f = B_2v^2$. For a spherical rock of radius $r = 0.01\,\mathrm{m}$, density $\rho = 2500\,\mathrm{kg/m^3}$, mass $m \simeq 0.01\,\mathrm{kg}$, the terminal speed v_t occurs for $mg = B_2v_t^2$, i.e., $v_t = (mg/c_2)^{1/2}/r \simeq 30\,\mathrm{m/s}$. In free fall, the speed v_t occurs after a fall of 45 m in 3 s. The effect of R_f becomes significant at times and distances much less than these. Thus, some idealised problems of free fall under gravity, are not physically valid in air.

By contrast, for microscopic objects, such as particles of dust with $r \simeq 1\,\mu\mathrm{m} = 10^{-6}\,\mathrm{m}$, eq. (2.99) yields $v^* = 360\,\mathrm{m/s}$. Thus, for lower speeds ($0 < v < 0.1v^* = 36\,\mathrm{m/s}$), R_f may be approximated as B_1v, i.e., the resistance force is viscous.

Motion under force $R_f \propto$ (speed) and $R_f \propto$ (speed)^2 are considered next.

2.29.1 General motion under $\vec{R}_f = -mk\vec{v}$

Consider, motion of a body of mass m under linear resistance force $\vec{R}_f = -mk\vec{v}$, where k is a constant, with initial conditions: $\vec{v}(0) = \vec{v}_0$ and $\vec{r}(0) = \vec{0}$ (Fig. 2.79a). $\vec{F} = m\vec{a}_C \Rightarrow$

$$m\dot{\vec{v}} = m\vec{g} - mk\vec{v} \quad \Rightarrow \quad \dot{\vec{v}} + k\vec{v} = \vec{g} \tag{2.100}$$

Its complementary solution is $\vec{C}_1e^{-kt}$ and particular integral is $\vec{g}/k$:

$$\vec{v} = \vec{C}_1e^{-kt} + \frac{\vec{g}}{k} \quad \Rightarrow \quad \vec{v}(0) = \vec{C}_1 + \frac{\vec{g}}{k} = \vec{v}_0 \quad \Rightarrow \quad \vec{C}_1 = \vec{v}_0 - \frac{\vec{g}}{k} = \vec{v}_0 - \vec{v}_t$$

where, for $\dot{\vec{v}} = \vec{0}$, eq. (2.100) $\Rightarrow$ terminal velocity $\vec{v}_t = \vec{g}/k = -g\vec{j}/k = -v_t\vec{j}$, $v_t = g/k$.

$$\dot{\vec{r}} = \vec{v} = (\vec{v}_0 - \vec{v}_t)e^{-gt/v_t} + \vec{v}_t \tag{2.101}$$

$$\Rightarrow\ \vec{r}(t) = (\vec{v}_0 - \vec{v}_t)(1 - e^{-gt/v_t})v_t/g + \vec{v}_t t$$

The trajectory of C is in the plane of $\vec{v}_0$ and $\vec{v}_t = -g\vec{j}/k$, i.e., in the vertical plane through $\vec{v}_0$, say the xy-plane (Fig. 2.79a). The component form of eq. (2.101) is

$$\dot{x} = v_{x0}e^{-gt/v_t}, \qquad \dot{y} = (v_{y0} + v_t)e^{-gt/v_t} - v_t \tag{2.102}$$

$$x = (1 - e^{-gt/v_t})v_tv_{x0}/g, \qquad y = (v_{y0} + v_t)(1 - e^{-gt/v_t})v_t/g - v_tt$$

We non-dimensionalise eqs (2.102) with respect to the parameters associated with the maximum height in an un-resisted medium: $x_m = v_{x0}v_{y0}/g$, $y_m = v_{y0}^2/2g$, $t_m = v_{y0}/g$;

$$\xi = \frac{x}{x_m} = \frac{xg}{v_{x0}v_{y0}}, \qquad \eta = \frac{y}{2y_m} = \frac{yg}{v_{y0}^2}$$

$$\tau = \frac{t}{t_m} = \frac{gt}{v_{y0}}, \qquad \frac{v_{y0}}{v_t} = \frac{kv_{y0}}{g} = \alpha \tag{2.103}$$

Using $gt/v_t = v_{y0}\tau/v_t = \alpha\tau$, $d(\,)/d\tau = [d(\,)/dt][dt/d\tau] = (\dot{\,})v_{y0}/g$, eqs (2.102) reduce to

$$\frac{d\xi}{d\tau} = e^{-\alpha\tau}, \qquad \xi = (1 - e^{-\alpha\tau})/\alpha$$

$$\frac{d\eta}{d\tau} = [(1+\alpha)e^{-\alpha\tau} - 1]/\alpha, \qquad \eta = [(1+\alpha^{-1})(1 - e^{-\alpha\tau}) - \tau]/\alpha$$

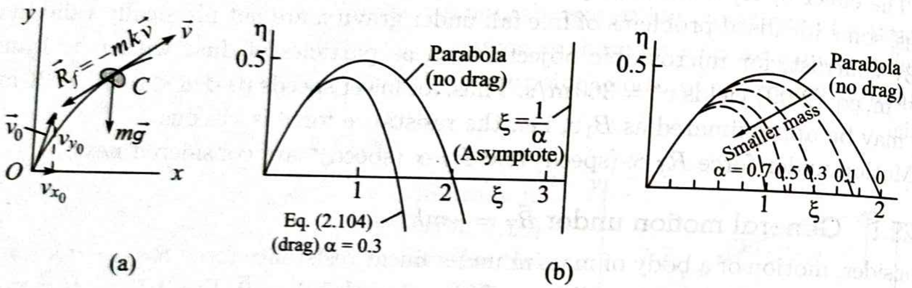


Fig. 2.79

Eliminating τ from the expressions of ξ, η and expanding the logarithmic term in Taylor series yields the equation of the trajectory:

$$\eta = (1+\alpha^{-1})\xi + \alpha^{-2}\ln(1 - \alpha\xi)$$

$$\eta = \xi - \tfrac{1}{2}\xi^2 - \sum_{n=3}^{\infty} \frac{1}{n}\alpha^{n-2}\xi^n \tag{2.104}$$

The first two terms outside the series in eq. (2.104) are for free fall ($\alpha = 0$). Unlike free fall under gravity, the trajectory for the resisted motion depends on the mass of the object as $\alpha = kv_{y0}/g = c_1 r v_{y0}/mg = 3c_1 v_{y0}/4\pi r^2 \rho g$ depends on m. Equation (2.104) implies that the trajectory is not significantly affected in the ascending part (small ξ), but is affected considerably in the descending part, which becomes steeper and the fall tends to be vertical (Fig. 2.79b) for a larger time. This effect increases with $\alpha \Rightarrow$

(1) for a given r, the effect is more for smaller ρ, i.e., for a body of smaller mass,

(2) for a given ρ, the effect is more for lesser r, i.e., for a smaller body of less mass.

The maximum height η_{max} is attained at $\tau = \tau_1$ when $\dfrac{d\eta}{d\tau} = [(1+\alpha)e^{-\alpha\tau_1} - 1]/\alpha = 0$,

$$\Rightarrow \quad \tau_1 = \alpha^{-1}\ln(1+\alpha), \qquad \eta_{max} = \eta(\tau_1) = -\alpha^{-2}\ln(1+\alpha) + \alpha^{-1}$$

Letting $\eta = 0$ in eq. (2.104) yields range $\xi_r = 2 - 2\sum_{n=3}^{\infty}\frac{1}{n}\alpha^{n-2}\xi_r^{n-1}$. Its solution ξ_r is less than the value of 2 for free fall. As $t \to \infty$, $\tau \to \infty$:

$$\frac{d\xi}{d\tau} \to 0, \qquad \frac{d\eta}{d\tau} \to -\frac{1}{\alpha}, \qquad \xi \to \frac{1}{\alpha}, \qquad \eta \to -\frac{\tau}{\alpha} + \frac{(1+\alpha)}{\alpha^2}$$

i.e., $\eta = -\tau/\alpha + (1+\alpha)/\alpha^2$ is an asymptote to the trajectory in the η-τ plane (not shown) and $\xi = 1/\alpha$ is an asymptote in the ξ-η plane (Fig. 2.79b).

2.29.2 Body falling vertically under $\vec{R}_f = -mk\vec{v}$

The settling of particles under gravity in a viscous fluid has applications to settlement of solid, liquid and air pollutants such as ash and smoke from thermal power plant and sewer sediments in a river. Consider the vertical settlement of solid particles with initial velocity v_0 (Fig. 2.80a) under $\vec{R}_f = -B_1\vec{v} = -mk\vec{v}$, $k = B_1/m = c_1 r/m$.

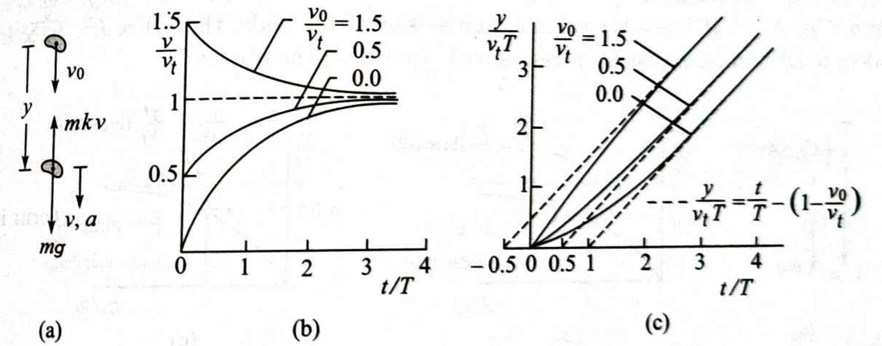


Fig. 2.80

The equation of motion: $ma = mg - mkv \Rightarrow$

$$a = \frac{dv}{dt} = g - kv = g\left(1 - \frac{v}{v_t}\right)$$

$$\Rightarrow \int_{v_0}^{v} \frac{1}{v_t - v}\,dv = \frac{g}{v_t}\int_0^t dt \quad \Rightarrow \quad -\ln\frac{v_t - v}{v_t - v_0} = \frac{gt}{v_t} \quad \Rightarrow$$

$$t = -T\ln\left[\frac{1 - v/v_t}{1 - v_0/v_t}\right], \qquad \dot{y} = v = v_t\left[1 - (1 - v_0/v_t)e^{-t/T}\right] \tag{2.105}$$

where $T = v_t/g$ is the time taken to attain v_t in free fall under gravity, starting from rest. Equation (2.105) implies that $v \to v_t$ as $t \to \infty$ and if $v_0 < v_t$, then $v(t) \le v_t$ and if $v_0 > v_t$, then $v(t) \ge v_t$. If L is a typical linear dimension of the body, then $m \propto L^3$, $k = B_1/m = c_1 r/m \propto 1/L^2$ and $v_t = g/k \propto L^2$. Hence, terminal velocity of larger-sized particles is more and they settle faster. Integration of eq. (2.105) yields:

$$y(t) = y(0) + \int_0^t v\,dt = v_t[t + T(1 - v_0/v_t)(e^{-t/T} - 1)] \qquad (2.106)$$

As $t \to \infty$: $y(t)/v_t T \to t/T - (1 - v_0/v_t)$, which is shown as dashed line in Fig. 2.80c. Relations (2.105) and (2.106) are illustrated in Fig. 2.80b and Fig. 2.80c.

Using $c_1 = 3.1 \times 10^{-4}\,\mathrm{kg.m^{-1}.s^{-1}}$, a particle of radius 10μm, with a density of $300\,\mathrm{kg/m^3}$ (soot), has $v_t \simeq 4\,$mm/s $\Rightarrow$ it will take 7 hours to settle to the ground from a height of 100 m in still air, while a particle of radius 2.5μm would take 4.5 days.

2.29.3 Body falling vertically under $R_f = \frac{1}{2}C_D\rho A v^2$

The resistance force R_f on a body, moving in a fluid of density ρ at velocity v with cross-sectional area A normal to it, is given by $R_f = \frac{1}{2}C_D\rho A v^2$ for turbulent flow, where C_D is the drag coefficient. Consider a pebble-sized body falling from rest (Fig. 2.81a). The terminal speed v_t ($a = 0$) is obtained from $\frac{1}{2}C_D\rho A v_t^2 = mg$, i.e., $v_t = (2mg/C_D\rho A)^{1/2}$. If L is a typical linear dimension of the body, then $A \propto L^2$, $m \propto L^3$ and $v_t \propto L^{1/2}$ implies that v_t increases with the size of the object.

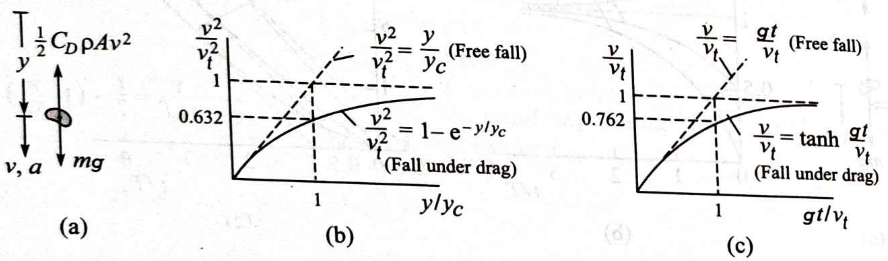


Fig. 2.81

The equation of motion

$$ma = mg - \tfrac{1}{2}C_D\rho A v^2 \quad \Rightarrow \quad a = \dot v = v\frac{dv}{dy} = \frac{g(v_t^2 - v^2)}{v_t^2}$$

$$\Rightarrow \quad \frac{v_t^2}{2g}\int_0^v \frac{2v}{v_t^2 - v^2}\,dv = \int_0^y dy \quad \Rightarrow \quad y = -y_c \ln(1 - v^2/v_t^2)$$

$$\text{and} \quad \int_0^v \frac{1}{v_t^2 - v^2}\,dv = \frac{g}{v_t^2}\int_0^t dt \quad \Rightarrow \quad \tanh^{-1}(v/v_t) = \frac{g}{v_t}t$$

$$v^2 = v_t^2(1 - e^{-y/y_c}), \qquad v = v_t\tanh\frac{gt}{v_t} = v_t\frac{1 - e^{-2gt/v_t}}{1 + e^{-2gt/v_t}} \qquad (2.107)$$

where $y_c = v_t^2/2g$ is the distance covered in free fall to attain a speed of v_t. The v^2–y and v–t relations are shown in Fig. 2.81b,c with $v \to v_t$ as $t \to \infty$. Equations (2.107) $\Rightarrow$ at $y = y_c$, $v = 0.795v_t$; at $t = v_t/g$, $v = 0.762v_t$; and $v = 0.95v_t$ at $y = 2.33y_c$, $t = 1.832v_t/g$; $v = 0.99v_t$ at $y = 3.92y_c$, $t = 2.647v_t/g$. The typical values of y_c (m) and v_t (m/s) for some objects are: (2.1, 6.5) for a rain drop of radius 0.1 cm, (21, 20) for hailstorm of radius 1 cm, (200, 63) for a human being, (149, 54) for a parachutist with closed canopy and (1.49, 5.4) for a parachutist with open canopy.

The parachutist falls for about 1500 m in 30 s with a closed canopy with most of this descent at nearly $v_t = 54\,$m/s. Then the canopy is opened (about 150 m above the ground to account for the time of opening of the canopy) and attain a new $v_t = 5.4\,$m/s in a very short descent of 20 m. This v_t equals the velocity after a free fall from 1.5 m height and is safe. If the canopy is opened earlier, the time of fall increases and the parachutist becomes an easy target. If it is not opened in time, the fall may be fatal.

2.29.4 Motion of a shell fired up under $R_f = \frac{1}{2}C_D\rho A v^2$

Consider, upward motion of a shell fired vertically up with velocity v_0 (Fig. 2.82). The equation of motion $ma = -mg - \frac{1}{2}C_D\rho A v^2 \Rightarrow$

$$a = \dot v = v\frac{dv}{dy} = -\frac{g(v_t^2 + v^2)}{v_t^2}$$

$$\Rightarrow \quad \frac{v_t^2}{2g}\int_{v_0}^v \frac{2v}{v_t^2 + v^2}\,dv = -\int_0^y dy, \qquad \frac{v_t}{g}\int_{v_0}^v \frac{v_t}{v_t^2 + v^2}\,dv = -\int_0^t dt$$

$$t = \frac{v_t}{g}\left[\tan^{-1}\frac{v_0}{v_t} - \tan^{-1}\frac{v}{v_t}\right]$$

$$y = y_c \ln\frac{1 + v_0^2/v_t^2}{1 + v^2/v_t^2}, \qquad v = v_t\tan\left[\tan^{-1}\frac{v_0}{v_t} - \frac{gt}{v_t}\right] \qquad (2.108)$$

The maximum height y_m is attained at time t_m for $v = 0$:

$$y_m = y_c \ln\left[1 + \frac{v_0^2}{v_t^2}\right], \qquad t_m = \frac{v_t}{g}\tan^{-1}\frac{v_0}{v_t} \le \tfrac{1}{2}\pi\left(\frac{v_t}{g}\right) \qquad (2.109)$$

For small resistance, v_0/v_t is small. Expanding the logarithmic term by Taylor series:

$$y_m = y_c\left[v_0^2/v_t^2 - v_0^4/2v_t^4 + \ldots\right] \simeq (v_0^2/2g)\left[1 - v_0^2/2v_t^2\right]$$

The first term is the height for no resistance. The relations (2.108) are plotted in

Fig. 2.82b,c. Substituting y_m from eq. (2.109)$_1$ for y in eq. (2.107)$_1$ for down motion yields the velocity with which the shell hits the ground as $v = v_0/[1 + v_0^2/v_t^2]^{1/2} < v_0$.

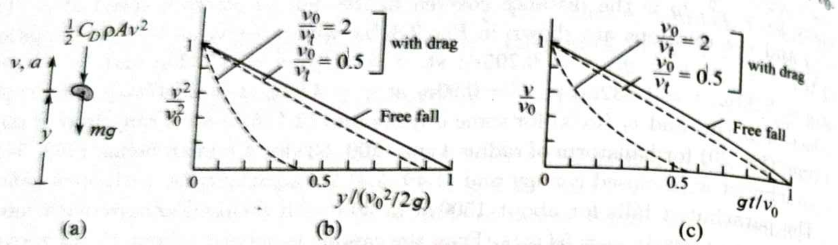


Fig. 2.82

2.30 Equation of Motion of C Relative to a Translating Frame

Consider a frame T translating with constant acceleration $\vec{a}_T$ relative to an inertial frame I. The equation of motion of the centre of mass C of any system, under external force sum $\vec{F} = \vec{F}_n + m\vec{g}$ where $\vec{F}_n$ is the sum of non-gravitational forces, is:

$$\vec{F} = \vec{F}_n + m\vec{g} = m\vec{a}_{C|I} = m(\vec{a}_{C|T} + \vec{a}_T)$$

$$\Rightarrow \quad \vec{F}_n + m\vec{g}_e = m\vec{a}_{C|T}, \qquad \text{where} \quad \vec{g}_e = \vec{g} - \vec{a}_T \tag{2.110}$$

$$\Rightarrow \quad (\vec{F}_n + m\vec{g}_e)\cdot\vec{v}_{C|T} = m\vec{a}_{C|T}\cdot\vec{v}_{C|T} = m\dot{\vec{v}}_{C|T}\cdot\vec{v}_{C|T} = \frac{d}{dt}\left(\tfrac{1}{2}mv_{C|T}^2\right)$$

$$\Rightarrow \quad \dot{T}_{C|T} = \dot{W}^*_{|T} \quad \Rightarrow \quad \dot{T}_{C|T} + \dot{V}^*_{|T} = \dot{W}^*_{nc|T} \tag{2.111}$$

where $\dot{W}^*_{|T}$ is the rate of work done relative to T by $\vec{F}_n + m\vec{g}_e$, as if acting at C; $\dot{W}^*_{nc|T}$ is the rate of work done by non-conservative part of forces, and $V^*_{|T}$ is the potential energy, relative to T, of the conservative part of forces. Thus, relative to T, the governing equation of motion of C and the work-energy relation are exactly of the same form as those relative to I, provided force $m\vec{g}$ and its potential energy $-m\vec{g}\cdot\vec{r}$ are replaced by force $m\vec{g}_e$ and its potential energy $-m\vec{g}_e\cdot\vec{r}$.

2.31 Examples on Motion of C and Translating Systems

In some cases, the motion of the centre of mass C can be obtained without studying the overall motion. For a translating system, this completes the study of dynamics. The following steps are useful for obtaining a solution.

1. List the given data and the entities to be computed.

2. Draw FBD of the system in general configuration. Select component directions to suit the description of kinematical quantities and the forces on the system.
3. Write the equations of motion and kinematic relations. List the unknowns. Go to step 5 if (number of unknowns) = (number of equations), else go to step 4.
4. List additional equations, if any, by using kinematic relationships for the constraints on the motion of various points, lines or bodies and using the conditions of friction for slip/no slip. If needed, draw FBDs of subsystems and repeat step 3.
5. Solve equations of motion and evaluate the constants of integration from initial conditions. Compute the required entities and analyse the results.

$\vec{F} = m\vec{a}_C$ or $\vec{F} = \sum m_i\vec{a}_{C_i}$ is used depending on whether the motion of C or C_is is reasonably simple and directly available. The force required for the given motion is obtained from $\vec{F} = m\vec{a}_C = m\ddot{\vec{r}}_C(t)$. Conversely, this equation is solved for $\vec{r}_C(t)$ for given $\vec{F}(t)$ and initial conditions $\vec{r}_C(0)$, $\dot{\vec{r}}_C(0)$. Closed form solution or a numerical solution is obtained by methods such as the fourth order Runge–Kutta method.

Generally, the energy relation is preferable if the force is a function of position and the system (even though interconnected) has a single dof, especially if the forces are conservative. The impulse–momentum relation is useful if the forces are known functions of time, or an impact occurs, or impulsive forces act. The moment of momentum equation is used if $F_\phi = 0$, or $M_{O_z} = 0$, or bodies rotate about a fixed axis, or the motion is under a central force $\vec{F} = F_r\vec{e}_r$.

§ Exercise 2.10: Prove that the tension on the two sides of a *mass-less belt* around a pulley is the same for the following three cases. (1) *The pulley is mass-less and has frictionless bearings*. This holds irrespective of the motion of the pulley and irrespective of whether the belt slips or not. (2) The pulley is heavy with its centre of mass on the axis of the frictionless bearings and is *in equilibrium*. (3) The pulley is *smooth*, i.e., $\mu = 0$. This holds irrespective of the mass, motion or the type of supports of the pulley.

Example 2.26: A system subjected to only 3 external forces and no couples is called a *three-force member*. Prove that these forces are necessarily coplanar and either parallel or concurrent for a mass-less 3-force member in motion and for a 3-force member with mass in equilibrium.

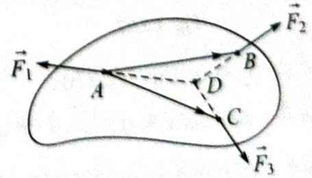


Fig. E2.26

Solution: Consider $\vec{F_1}$, $\vec{F_2}$, $\vec{F_3}$ acting at A, B, C (Fig. E2.26).

$$\vec{M}_A = \vec{AB}\times\vec{F_2}+\vec{AC}\times\vec{F_3}=\vec{0} \Rightarrow \quad \vec{AB}\times\vec{F_2} = -\vec{AC}\times\vec{F_3} = \vec{M} \quad \text{(say)}$$

$\Rightarrow$ $\vec{AB}$ and $\vec{F_2}$ lie in a plane $\perp \vec{M}$, also $\vec{AC}$ and $\vec{F_3}$ lie in a plane $\perp \vec{M}$. These planes having the same normal are identical planes since A is common to them. Hence, $\vec{F_2}$, $\vec{F_3}$ lie in the plane of A, B, C. Similarly, $\vec{M}_B = \vec{0} \Rightarrow \vec{F_1}$ also lies in the same plane. $\Rightarrow \vec{F_1}, \vec{F_2}, \vec{F_3}$ are coplanar. The coplanar $\vec{F_1}$, $\vec{F_2}$ can be parallel or intersect at D. In the former case, $\vec{F}=\vec{0} \Rightarrow \vec{F_3} = -\vec{F_1}-\vec{F_2} \Rightarrow \vec{F_3}$ is parallel to the parallel vectors $\vec{F_1}$ and $\vec{F_2}$. In the latter case, $\vec{M}_D = \vec{DC}\times\vec{F_3}=\vec{0} \Rightarrow \vec{F_3}$ passes through the point of intersection, D. Hence, the forces are coplanar and either parallel or concurrent.

Example 2.27: (a) A simple device for measuring reasonably uniform acceleration a is a pendulum bob (Fig. E2.27a) connected to a post by a light inextensible string. Find the $a-\theta$ relation. (b) If the pendulum is released from rest with respect to a vehicle at $\theta = 0$, find $\dot{\theta}$ and tension T for any position θ.

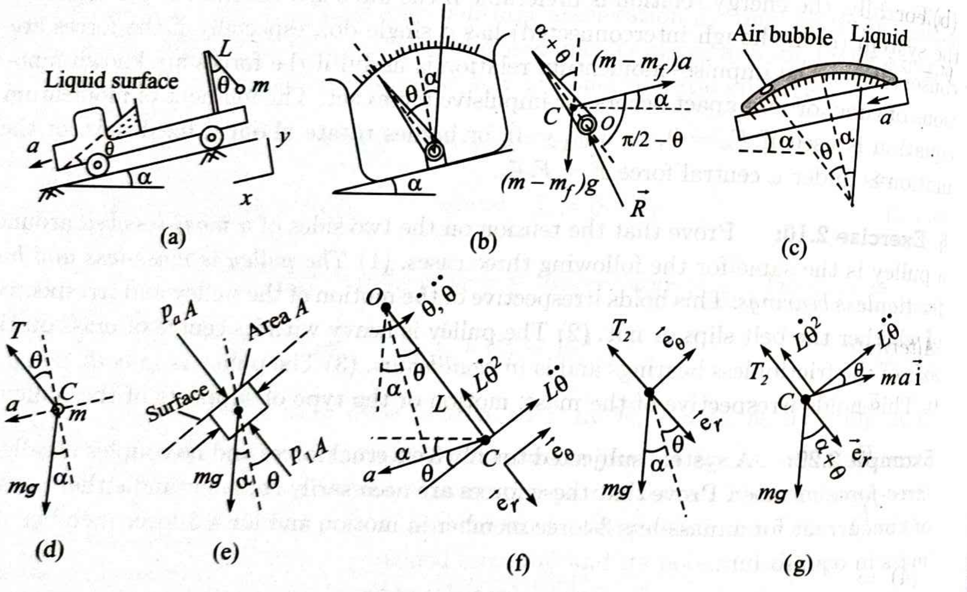


Fig. E2.27

Solution: From the FBD of the bob in Fig. E2.27d:

$$F_y = ma_{C_y}: \quad T\cos\theta - mg\cos\alpha = 0 \quad\Rightarrow\quad T = mg\cos\alpha/\cos\theta$$

$$F_x = ma_{C_x}: \quad T\sin\theta + mg\sin\alpha = ma \;\Rightarrow\; a = g(\tan\theta\cos\alpha+\sin\alpha) \quad (1)$$

$$\Rightarrow\ a = g\tan\theta \quad \text{for } \alpha = 0$$

The inclination θ of the free surface of a liquid in a tanker moving down a plane at acceleration a is also given by eq. (1) since the FBD of an element of the liquid at the surface (Fig. E2.27e) is similar to that for the pendulum with T replaced by $(p_1 - p_a)A$, which is the net force due to pressure. The $a-\theta$ relation (1) also holds for an accelerometer consisting of a uniform pivoted float immersed in a liquid container (Fig. E2.27b). The float is in equilibrium relative to the translating frame T_1 of the truck moving at acceleration $\vec{a} = a\,\vec{i}$. The augmented force system in terms of $\vec{g}_e = \vec{g} - \vec{a}$ consists of reaction $\vec{R}$ at O, $m\vec{g}_e$ and buoyancy force $-m_f\vec{g}_e$ at C. Let $OC = h$.

$$M_{O_z} = -(m-m_f)gh\sin(\alpha+\theta) + (m-mf)ah\sin(\pi/2-\theta) = 0 \quad\Rightarrow\quad \text{eq. (1)}$$

A sensitive accelerometer, with quick response without much overshoot or oscillation, is a circular tube filled with acetone with a small air bubble in it (Fig. E2.27c). The bubble acts as a bob of an inverted pendulum with the resultant fluid thrust on it replacing tension T. a is found from the position θ of the bubble, using eq. (1).

(b) For T_1, $\vec{a}_{C|T_1}$ and the augmented force system (add $-ma\,\vec{i}$) are shown in Fig. E2.27g.

$$V = -[m\vec{g} - m\vec{a}]\cdot\vec{r} = -m(\vec{g}-a\vec{i})\cdot L\vec{e}_r = -mL[g\cos(\alpha+\theta)+a\sin\theta], \quad T = \tfrac{1}{2}m(L\dot{\theta})^2$$

$$T + V = T_0 + V_0 = 0 - mL[g\cos\alpha]$$

$$\Rightarrow \quad \dot{\theta}^2 = 2[a\sin\theta + g\{\cos(\alpha+\theta)-\cos\alpha\}]/L \quad (2)$$

$$F_r = mg\cos(\alpha+\theta) + ma\sin\theta - T_2 = m(-L\dot{\theta}^2)$$

$$\Rightarrow \quad T_2 = m[3g\cos(\alpha+\theta) - 2g\cos\alpha + 3a\sin\theta] \quad (3)$$

Alternate Solution: Using equations of motion relative to ground frame (Fig. E2.77f).

$$\vec{a}_C = a(-\sin\theta\,\vec{e}_r - \cos\theta\,\vec{e}_\theta) - \dot{\theta}^2 L\vec{e}_r + \ddot{\theta}L\vec{e}_\theta$$

$$F_\theta = ma_\theta: \quad -mg\sin(\alpha+\theta) = m(L\ddot{\theta} - a\cos\theta) \quad (4)$$

$$F_r = ma_r: \quad mg\cos(\alpha+\theta) - T_2 = -m(L\dot{\theta}^2 + a\sin\theta) \quad (5)$$

$$(4) \Rightarrow \quad L\dot{\theta}\frac{d\dot{\theta}}{d\theta} = a\cos\theta - g\sin(\alpha+\theta) \quad\Rightarrow\quad \tfrac{1}{2}L\dot{\theta}^2 = [a\sin\theta + g\cos(\alpha+\theta)]\Big|_0^\theta$$

$$\dot{\theta}^2 = 2[a\sin\theta + g\{\cos(\alpha+\theta)-\cos\alpha\}]/L$$

$$(5) \Rightarrow \quad T_2 = m[3g\cos(\alpha+\theta) - 2g\cos\alpha + 3a\sin\theta]$$

Example 2.28: A fly-ball governor (Fig. E2.28a), used to regulate the speed of a device such as a steam engine/turbine, is rotated by the device. The bars are light. Neglect the rotational inertia of the balls and the collar at A. The configuration θ

depends on ω and force P on the collar. The motion of the collar due to change in ω is used to open or close a valve to regulate the speed. Find P and the maximum bending moment in the top and bottom rods when the governor rotates at $\omega, \dot{\omega}$ with constant θ.

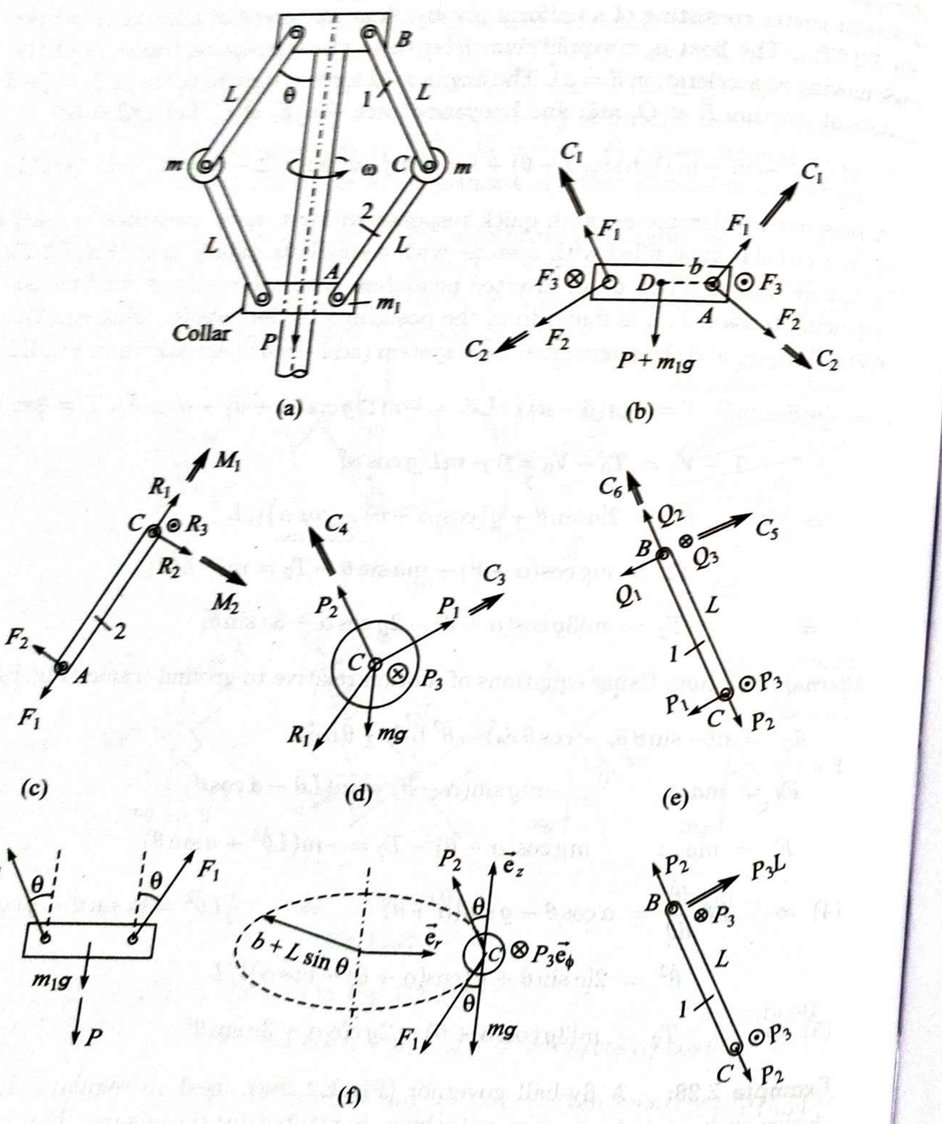


Fig. E2.28

Solution: For the transverse acceleration of the ball, the rod 2 must transmit a transverse force to C and hence there must be couple reactions on rod 1 and/or rod 2. The collar does not have any transverse acceleration and the pin reactions can provide the axial acceleration. Thus, for a determinate solution using rigid body models, we neglect the couple reactions C_1, C_2 from the pins on the collar (Fig. E2.28b). The reactions on the collar at the second pin are shown by rotating those on the first pin by 180° about the axis of rotation. Rotational inertia of the collar is neglected $\Rightarrow$ for collar:

$$\vec{M}_D = 2rF_3\vec{e}_z = \vec{0} \quad \Rightarrow \quad F_3 = 0$$

The FBD of rod 2 (light rod) is drawn in Fig. E2.28c. $C_1 = C_2 = 0$, $F_3 = 0$ and

$$\vec{M}_C = \vec{0} \quad \Rightarrow \quad F_2 = 0, \qquad M_1 = M_2 = 0$$

$$\vec{F} = \vec{0} \quad \Rightarrow \quad R_2 = R_3 = 0, \qquad R_1 = F_1$$

FBD (Fig. E2.28d) of the ball (no rotational inertia) using $M_1 = M_2 = 0$, $R_2 = R_3 = 0$,

$$\Rightarrow \quad \vec{M}_C = \vec{0} \quad \Rightarrow \quad C_3 = C_4 = 0$$

The FBD of rod 1 (light rod) is drawn in Fig. E2.28e using $C_3 = C_4 = 0$ and

$$\vec{M}_B = \vec{0} \quad \Rightarrow \quad P_1 = 0, \qquad C_6 = 0, \qquad C_5 = P_3L$$

$$\vec{F} = \vec{0} \quad \Rightarrow \quad Q_1 = 0, \qquad Q_2 = P_2, \qquad Q_3 = P_3$$

The FBDs of the collar, rod 1 and ball are presented in Fig. E2.28f showing only the non-zero actions. C moves in a circle of radius $r = b + L\sin\theta \Rightarrow$

$$\vec{a}_C = -\omega^2(b + L\sin\theta)\,\vec{e}_r + \dot{\omega}(b + L\sin\theta)\,\vec{e}_\phi$$

For the ball:

$$F_r = ma_{C_r}: \qquad -(P_2 + F_1)\sin\theta = -m\omega^2(b + L\sin\theta)$$

$$F_z = ma_{C_z}: \qquad (P_2 - F_1)\cos\theta - mg = 0$$

$$F_\phi = ma_{C_\phi}: \qquad P_3 = m\dot{\omega}(b + L\sin\theta)$$

$$\Rightarrow \qquad F_1 = \tfrac{1}{2}m[-g/\cos\theta + \omega^2(L + b/\sin\theta)]$$

Bending moment for bar 2 is zero and the maximum bending moment in bar 1 is:

$$M_{\max} = P_3L = m\dot{\omega}(b + L\sin\theta)L$$

For the collar: $\qquad F_z = m_1a_z: \qquad 2F_1\cos\theta - m_1g - P = 0 \quad \Rightarrow$

$$P = -m_1g + m[\omega^2(b\cot\theta + L\cos\theta) - g]$$

Example 2.29: Find the deceleration of the sled of a skier when its speed is v on a ground with elevation $y = h\sin(\pi x/L)$ (Fig. E2.29a). The coefficient of friction is μ.

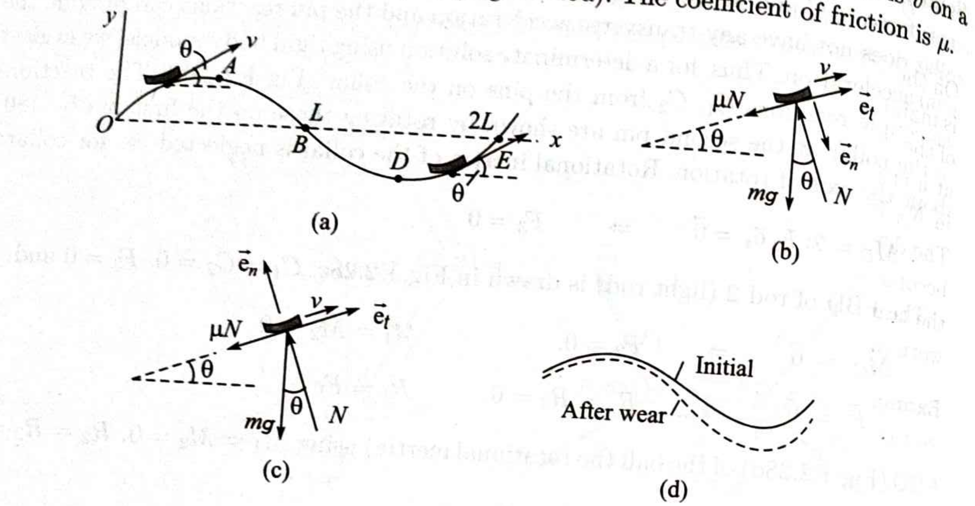


Fig. E2.29

Solution: We model the sled as a mass-point and describe its $\vec{a}$ in path coordinates:

$$y = h\sin\left(\frac{\pi x}{L}\right), \quad y' = \frac{h\pi}{L}\cos\left(\frac{\pi x}{L}\right) = \tan\theta, \quad y'' = -\frac{h\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right) \tag{1}$$

$$\rho = \frac{(1+y'^2)^{3/2}}{|y''|} = \frac{L^2[1+(h^2\pi^2/L^2)\cos^2(\pi x/L)]^{3/2}}{h\pi^2|\sin(\pi x/L)|} \tag{2}$$

In the FBD for $0 \le x \le L$, $\vec{e}_n$ is into the ground (Fig. E2.29b). Equations of motion:

$$F_n = ma_n: \quad mg\cos\theta - N = mv^2/\rho \quad \Rightarrow \quad N = m(g\cos\theta - v^2/\rho) \tag{3}$$

$$F_t = ma_t: \quad -\mu N - mg\sin\theta = m\dot{v} \quad \Rightarrow \quad \dot{v} = -\mu(g\cos\theta - v^2/\rho) - g\sin\theta \tag{4}$$

In the FBD for $L < x < 2L$, $\vec{e}_n$ is outwards (Fig. E2.29c). Equations of motion are:

$$-mg\cos\theta + N = mv^2/\rho, \qquad -\mu N - mg\sin\theta = m\dot{v}$$

$$\Rightarrow \quad N = m(g\cos\theta + v^2/\rho), \qquad \dot{v} = -\mu(g\cos\theta + v^2/\rho) - g\sin\theta \tag{5}$$

$$A: \; x = \frac{L}{2}, \quad \rho = \frac{L^2}{\pi^2 h}, \quad \theta = 0, \quad N = mg\left(1 - \frac{\pi^2 v^2 h}{L^2 g}\right), \quad \dot{v} = -\mu g\left(1 - \frac{\pi^2 v^2 h}{L^2 g}\right)$$

$$D: \; x = \frac{3L}{2}, \quad \rho = \frac{L^2}{\pi^2 h}, \quad \theta = 0, \quad N = mg\left(1 + \frac{\pi^2 v^2 h}{L^2 g}\right), \quad \dot{v} = -\mu g\left(1 + \frac{\pi^2 v^2 h}{L^2 g}\right)$$

$$B: \; x = L, \quad \rho = \infty, \quad \tan\theta = -\frac{\pi h}{L}, \quad [N, \dot{v}] = \frac{[mg, -g(\mu - \pi h/L)]}{(1+\pi^2h^2/L^2)^{1/2}}$$

On the valley at D, N and $\dot{v}$ are maximum and these decrease on a crest at A. Contact is maintained in part AB if $N = m(g\cos\theta - v^2/\rho) > 0$, i.e., if $v^2 < \rho g\cos\theta$. The wear of the ground, being proportional to the normal reaction, is more at the valley and less at a crest. As the depth h increases with wear: the increase in N_D and the decrease in N_A further accelerates the process of differential wear at D and A (Fig. E2.29d). The depressions in the road near turnings and at bus-stops, where brakes are applied, become pronounced over time. A level surface is therefore unstable. Similarly, where the train driver often applies brakes on seeing a distant red signal, the rails become worn out into short wavelength corrugations, producing a so-called 'roaring rail'.

Example 2.30: A cylindrical vessel containing water rotates at uniform ω about its vertical axis (Fig. E2.30a). Show that the water level becomes paraboloid: $z = \omega^2 r^2/2g$.

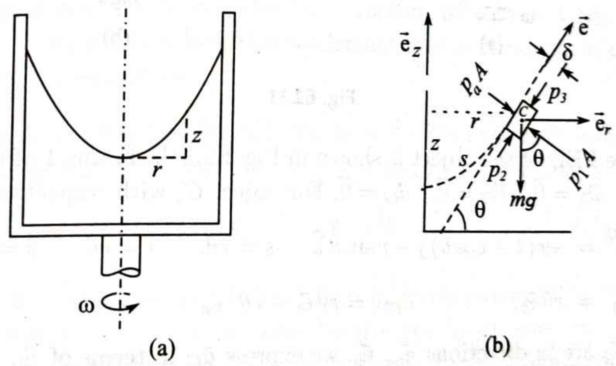


Fig. E2.30

Solution: At radius r, consider a volume element with outer face having small area $A = ds \times (r\,d\phi)$ and small thickness δ. Its mass $m = \rho A\delta$. In its FBD (Fig. E2.30b), $p_3 = p_2 + \dfrac{\partial p_2}{\partial s}ds$. In the limit as $\delta \to 0$, on the surface $p_2 = p_a = \text{constant} \Rightarrow \dfrac{\partial p_2}{ds} = 0$.

$$a_e = \vec{a}_C \cdot \vec{e} = (-\omega^2 r\,\vec{e}_r)\cdot\vec{e} = -\omega^2 r\cos\theta, \qquad F_e = ma_e$$

$$\Rightarrow \quad \left(p_2 - p_2 - \frac{\partial p_2}{\partial s}ds\right)(r\,d\phi)\delta - \rho(r\,d\phi)ds\,\delta g\sin\theta = -\rho(r\,d\phi)ds\,\delta\omega^2 r\cos\theta$$

$$\Rightarrow \quad \frac{\partial p_2}{\partial s} = -\rho(g\sin\theta - \omega^2 r\cos\theta) = 0 \qquad \Rightarrow \quad \tan\theta = \frac{\omega^2 r}{g}$$

$$\Rightarrow \quad \frac{dz}{dr} = \frac{\omega^2 r}{g}: \quad \int_0^z dz = \frac{\omega^2}{g}\int_0^r r\,dr \qquad \Rightarrow \quad z = \frac{\omega^2 r^2}{2g}$$

Example 2.31: In an automatic feeding system, small objects (modelled as mass-points) are released from rest at O at essentially zero radius and move out along a smooth circular path between radial vanes of the distributor which rotates at constant ω about a fixed vertical axis (Fig. E2.31a). Find the speed of the object and the reaction from the distributor on the object when it reaches the position θ.

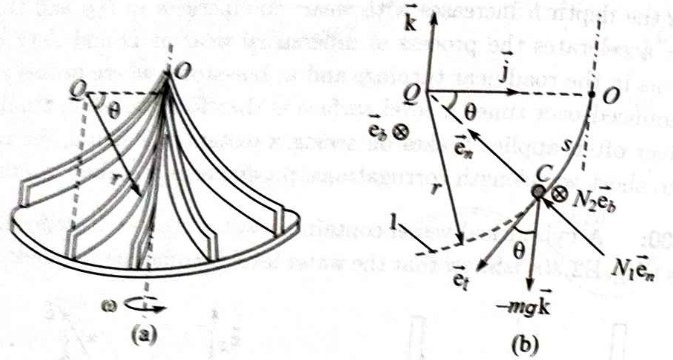


Fig. E2.31

Solution: The FBD of the object is shown in Fig. E2.31b. Frame 1 of the distributor has $\vec{\omega}_1 = \omega\vec{k}$, $\dot{\vec{\omega}}_1 = \vec{0}$, $\vec{v}_O = \vec{0}$, $\vec{a}_O = \vec{0}$. For object C, with respect to frame 1:

$$\overrightarrow{OC} = -r(1-\cos\theta)\vec{j} - r\sin\theta\,\vec{k}, \quad s = r\theta, \quad \dot{s} = r\dot{\theta}, \quad \ddot{s} = r\ddot{\theta}, \quad \rho = r$$

$$\Rightarrow \quad \vec{v}_{C|1} = r\dot{\theta}\vec{e}_t, \qquad \vec{a}_{C|1} = r\ddot{\theta}\vec{e}_t + r\dot{\theta}^2\vec{e}_n$$

Since N_1, N_2 are in directions $\vec{e}_n$, $\vec{e}_b$, we express $\vec{a}_C$ in terms of $\vec{e}_t$, $\vec{e}_n$, $\vec{e}_b$ using

$$\vec{i} = -\vec{e}_b, \qquad \vec{j} = -\sin\theta\,\vec{e}_t - \cos\theta\,\vec{e}_n, \qquad \vec{k} = -\cos\theta\,\vec{e}_t + \sin\theta\,\vec{e}_n$$

$$\vec{a}_C = \vec{a}_O + \dot{\vec{\omega}}_1 \times \overrightarrow{OC} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \overrightarrow{OC}) + 2\vec{\omega}_1 \times \vec{v}_{C|1} + \vec{a}_{C|1}$$

$$= \omega^2 r(1-\cos\theta)\vec{j} + 2\omega(-\cos\theta\,\vec{e}_t + \sin\theta\,\vec{e}_n) \times r\dot{\theta}\vec{e}_t + r\ddot{\theta}\vec{e}_t + r\dot{\theta}^2\vec{e}_n$$

$$= \omega^2 r(1-\cos\theta)(-\sin\theta\,\vec{e}_t - \cos\theta\,\vec{e}_n) - 2\omega r\dot{\theta}\sin\theta\,\vec{e}_b + r\ddot{\theta}\vec{e}_t + r\dot{\theta}^2\vec{e}_n$$

$$F_t = ma_{C_t}: \qquad m[-\omega^2 r\sin\theta(1-\cos\theta) + r\ddot{\theta}] = mg\cos\theta$$

$$\Rightarrow \quad \ddot{\theta} = g\cos\theta/r + \omega^2(\sin\theta - \tfrac{1}{2}\sin 2\theta) \tag{1}$$

Since $\ddot{\theta}$ is a function of θ, eq. (1) is integrated using $\ddot{\theta} = \dfrac{d\dot{\theta}}{d\theta}\dot{\theta}$ to get

$$\int_0^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = \int_0^{\theta} [g\cos\theta/r + \omega^2(\sin\theta - \tfrac{1}{2}\sin 2\theta)]\,d\theta$$

$$\Rightarrow \quad \dot{\theta}^2 = 2g\sin\theta/r + \omega^2(-2\cos\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{3}{2})$$

$$F_n = ma_{C_n}: \qquad m[-\omega^2 r\cos\theta(1-\cos\theta) + r\dot{\theta}^2] = N_1 - mg\sin\theta$$

$$F_b = ma_{C_b}: \qquad -2m\omega r\dot{\theta}\sin\theta = N_2$$

$$\Rightarrow \quad N_1 = m[3g\sin\theta + \omega^2 r(2 - 3\cos\theta + \cos 2\theta)]$$

$$N_2 = -2m\omega r\sin\theta[2g\sin\theta/r + \omega^2(\tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta)]^{1/2}$$

$$\vec{v}_C = \vec{v}_O + \vec{\omega}_1 \times \overrightarrow{OC} + \vec{v}_{C|1} = r\dot{\theta}\vec{e}_t + \omega r(1-\cos\theta)\,\vec{i} \qquad [\vec{i} = -\vec{e}_b]$$

$$v_C = r[\dot{\theta}^2 + \omega^2(1-\cos\theta)^2]^{1/2} = [2gr\sin\theta + \omega^2 r^2(3 - 4\cos\theta + \cos 2\theta)]^{1/2}$$

Example 2.32: (a) Find the steady settlement rate v of a small spherical body of radius R and density ρ in a liquid of density ρ_l assuming the fluid resistance force to be cRv. Find its settlement rate in water ($\rho_l = 1000\,\text{kg/m}^3$) if $\rho = 1.1\rho_l$, $R = 1\,\mu\text{m}$, $c = 0.02\,\text{kg.m}^{-1}.\text{s}^{-1}$. (b) Explain how this settlement rate is increased in the non-inertial rotating frame of a centrifuge.

Solution: (a) In the FBD of Fig. E2.32a, $m = \frac{4}{3}\pi R^3\rho$ and the buoyancy force $F_b = \frac{4}{3}\pi R^3\rho_l g$. For a steady settlement rate v, $a = 0$ and $F = ma$ yields

$$mg - F_b - cRv = \frac{4}{3}\pi R^3(\rho - \rho_l)g - cRv = 0 \quad \Rightarrow \quad v = \frac{4\pi}{3c}R^2 g(\rho - \rho_l) \tag{1}$$

$\Rightarrow v = 2.05 \times 10^{-7}\,\text{m/s}$ is so small that it is completely swamped by thermal motion.
(b) For given size, eq. (1) $\Rightarrow$ v can only be changed by an effective g_e using a non-inertial frame. A centrifuge 1 rotates at a very high ω (Fig. 2.32b). For the motion of a particle of mass m with $\vec{v}_{|1}$ and $\vec{a}_{|1}$ under $mg\vec{k}$ and other forces $\vec{P}$,

$$\vec{F} = m\vec{a}_C: \quad \vec{P} + mg\vec{k} = m[\vec{a}_A + \dot{\vec{\omega}} \times r\vec{e}_r + \vec{\omega} \times (\vec{\omega} \times r\vec{e}_r) + 2\vec{\omega} \times \vec{v}_{|1} + \vec{a}_{|1}]$$

$$\simeq m\vec{a}_{|1} - m\omega^2 r\vec{e}_r$$

where $\vec{a}_A = \vec{0}$, $\dot{\vec{\omega}} = \vec{0}$; the Coriolis term $2\omega v_{|1} \ll \omega^2 r$ since ω is very large, and is neglected. As above, $\vec{P}$ is the resultant of the viscous drag force and buoyancy force,

$$m\vec{a}_{|1} \simeq \vec{P} + m(g\vec{k} + \omega^2 r\vec{e}_r) = \vec{P} + m\vec{g}_e$$

$$\vec{g}_e = g\vec{k} + \omega^2 r\vec{e}_r \simeq \omega^2 r\vec{e}_r$$

i.e., effective $g_e \simeq \omega^2 r$. The outward radial velocity v of settlement is thus given by eq. (1) with g_e replacing g: $v = 4\pi R^2 g_e(\rho - \rho_l)/3c$. For a speed of 1500 rpm and

$r = 0.15\,\text{m}$, $g_e \simeq \omega^2 r = 3700\,\text{m/s}^2 \simeq 377g$ and $v \simeq 0.77 \times 10^{-4}\,\text{m/s} = 28\,\text{cm/h}$. A centrifuge makes possible effective separation of small bodies of different density, suspended in a fluid, in reasonable time. Ultra-centrifuges exist wherein $g_e = 50\,000g$.

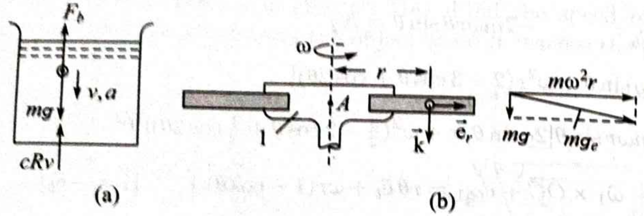


Fig. E2.32

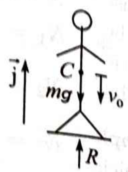


Fig. E2.33

Example 2.33: A person of mass m hits the ground with speed $v_0 = \sqrt{2gh}$ after a fall from a height h. He controls his impact by crouching allowing his knees and back to bend so that he comes to rest after the centre of mass C of his body descends a distance d. (a) Assuming constant deceleration of C, find the reaction from the ground during collision. (b) Find the average force across a horizontal section of the body during collision. Where is bone fracture most likely to occur? (c) Find the maximum height of fall without fracture if the compressive fracture stress of tibia is $\sigma_0 = 150$ MPa and $m = 60\,\text{kg}$, $d = 1\,\text{cm}$, and minimum area A of the tibia is $3\,\text{cm}^2$. (d) Discuss the forces acting during jumps into soft sand or water. (e) How does a parachutist manage to land safely, even though v_0 is very high?

Solution: (a) Let R be the ground reaction, T be the duration of collision and a be the acceleration of C. The FBD of the body is shown in Fig. E2.33. $ma = mg - R$,

$$0^2 - v_0^2 = 2ad = 2d \times \frac{mg - R}{m} \quad \Rightarrow \quad R = m(g + v_0^2/2d) = mg(1 + h/d) \qquad (1)$$

$$I_y = (mg - R)T = \Delta p_y = m(0 - v_0) \quad \Rightarrow \quad R - mg = mv_0/T$$

(b) The average force F_{av}, across a horizontal section of the body, is given by eq. (1) with m replaced by the mass m^* of the body above that cross-section. The average stress σ_{av} across any cross-section of area A^* is $\sigma_{av} = F_{av}/A^* = m^* g(1 + h/d)/A^*$. Failure occurs where $(\sigma_{av})_{max}$ occurs. The likely section is not at the neck, since m^* above the neck is small, but it is in the tibia just above the ankle where $m^* \simeq m$ and A^* has the least value equal to $2A$ (for 2 tibiae).

(c) For no fracture: $(\sigma_{av})_{max} \simeq \dfrac{mg(1 + h/d)}{2A} < \sigma_0 \Rightarrow h < (2\sigma_0 A/mg - 1)d = 1.52\,\text{m}$.

Thus, fractures can occur if the person falls from a height of 2 m. The safe height can be increased to 2.25 m by more crouching so that $d = 1.5\,\text{cm}$.

(d) $(R - mg) \propto 1/T \Rightarrow R$ can be decreased by increasing T. T for landing into loose earth, sand or water is hundreds of times larger than that of landing on hard ground [discussed in part (a)]. Hence, the safe jumping height for such cases is much more.
(e) Parachutists are trained to make their first contact with the ground on their toes and use the bending of the ankle joint to begin deceleration. The knees then bend and the body is turned sideways so as to fall first on the leg, then thigh and lastly the side of chest. Thereby, d is maximized to about 1 m, thus decreasing R_{av} and it is distributed over a large area of the body in contact with the ground to reduce stress.

Example 2.34: An arrangement shown in Fig. E2.34a is used for hoisting a block by a vehicle which goes up an icy incline, operating its winch b. The winch b draws in rope at 0.15 m/s when the vehicle is moving up at 0.3 m/s. The winch a is operated such that the tension T in cable AB is $(3600 - 200e^{-t})$ N. The coefficient of friction for snow is 0.1. Find the speed of the block at $t = 60$ s when winch b is drawing in rope at 0.3 m/s. Find the work done by T on the vehicle.

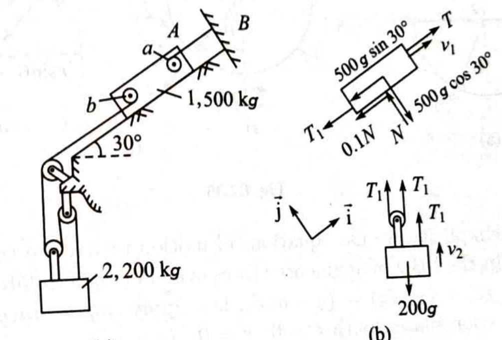


Fig. E2.34

Solution: The FBDs of the bodies are shown in Fig. E2.34b with $N = 500g\cos 30°$. Initially, $v_1(0) = 0.3\,\text{m/s}$, $v_2(0) = [v_1(0) + 0.15]/3 = 0.15\,\text{m/s}$ and finally let $v_1(60) = v\,\text{m/s}$. Then $v_2(60) = [v + 0.3]/3 = (0.1 + v/3)\,\text{m/s}$. We apply the impulse–momentum relation in the direction of the motion of the bodies and eliminate T_1:

$$500(v - 0.3) = \int_0^{60} (T - T_1 - 500g\sin 30° - 0.1N)\,dt \qquad (1)$$

$$200[(0.1 + v/3) - 0.15] = \int_0^{60} (3T_1 - 200g)\,dt \qquad (2)$$

$$3(1) + (2): \quad \frac{4700v}{3} - 460 = \int_0^{60} [3(T - 500g\sin 30° - 0.1N) - 200g]\,dt$$

$$= \int_0^{60} (206.14 - 600e^{-t})\, dt = 206.14 \times 60 + 600(e^{-60} - 1)$$

$$\Rightarrow \quad v = 7.81\,\text{m/s}, \qquad v_2(60) = 0.1 + v/3 = 2.70\,\text{m/s}$$

T does not do any work on the vehicle as $v_A = 0$.

Alternate Solution: Write $F_i = m_i a_i$ for the 2 bodies, eliminate T_1 and integrate with respect to t.

Example 2.35: A small slider of mass m moves in a smooth hemispherical bowl (Fig. E2.35a) fixed in an elevator moving up with uniform acceleration a. It is imparted a horizontal circumferential velocity v_1 in the position shown. Find v_ϕ, v_θ, $\dot{\phi}$, $\dot{\theta}$, $\ddot{\phi}$, $\ddot{\theta}$ for the general position θ. Set up an equation for the maximum angle θ_2 made by OC with the vertical.

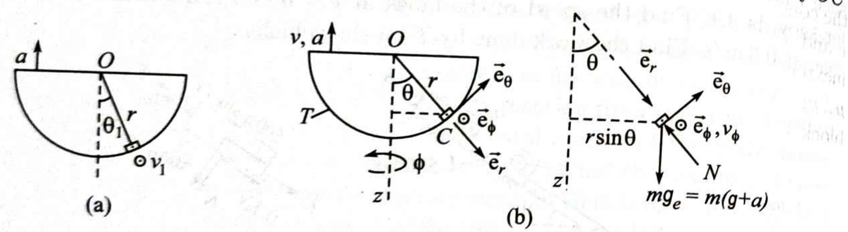


Fig. E2.35

Solution: For simplicity, we use equations of motion relative to translating frame T of the elevator. In the FBD of augmented force system (Fig. E2.35b), instead of $\vec{g}$, use $\vec{g}_e = \vec{g} - \vec{a}_T = g\,\vec{e}_z - (-a\,\vec{e}_z) = (g+a)\,\vec{e}_z$, i.e., apply $mg_e = m(g+a)$. Let r, θ, ϕ be the spherical coordinates, with $\dot{r} = 0$, $\ddot{r} = 0$. $\vec{v}_C = v_\theta\,\vec{e}_\theta + v_\phi\,\vec{e}_\phi$. From the FBD, $M_{O_z} = 0 \Rightarrow H_{O_z} = $ constant, since the z-axis through O is an axis fixed in T.

$$H_{O_z} = (mv_\phi) r \sin\theta = mv_1 r \sin\theta_1 \tag{1}$$

$$\Rightarrow \quad v_\phi = r\sin\theta\,\dot{\phi} = \frac{v_1 \sin\theta_1}{\sin\theta}, \qquad \dot{\phi} = \frac{v_1\sin\theta_1}{r\sin^2\theta} \tag{2}$$

$v_\theta = r\dot{\theta}$. Relative to translating frame T, N does not do any work as $\vec{v}_C \perp \vec{N}$ and:

$$T + V = \tfrac{1}{2}m(r^2\dot{\theta}^2 + v_\phi^2) - m(g+a)r\cos\theta = \tfrac{1}{2}mv_1^2 - m(g+a)r\cos\theta_1$$

Substitution of $\dot{\phi}$ from eq. (2), yields

$$\dot{\theta}^2 = \frac{2(g+a)}{r}(\cos\theta - \cos\theta_1) + \frac{v_1^2}{r^2}\left(1 - \frac{\sin^2\theta_1}{\sin^2\theta}\right) \tag{3}$$

$v_\theta = r\dot{\theta}$. $\ddot{\phi}$ is obtained using $\dot{H}_{O_z} = 0$, and $\ddot{\theta}$ is obtained by differentiating eq. (3):

$$\dot{H}_{O_z} = mr^2\frac{d}{dt}(\sin^2\theta\,\dot{\phi}) = mr^2(\sin 2\theta\,\dot{\theta}\,\dot{\phi} + \sin^2\theta\,\ddot{\phi}) = 0$$

$$2\dot{\theta}\ddot{\theta} = 2\left[-\frac{(g+a)}{r}\sin\theta + \frac{v_1^2\cos\theta\sin^2\theta_1}{r^2\sin^3\theta}\right]\dot{\theta}$$

For $\theta_{max} = \theta_2$, $\dot{\theta} = 0$ and eq. (3) yields

$$\frac{2(g+a)}{r}(\cos\theta_2 - \cos\theta_1) + \frac{v_1^2}{r^2}\left(1 - \frac{\sin^2\theta_1}{\sin^2\theta_2}\right) = 0$$

Example 2.36: (a) The system shown in Fig. E2.36a is released from rest. Specify the condition for downward displacement of block 2. Find its displacement s, velocity v and acceleration a at time t and $v(s)$. Assume light frictionless pulleys and a light inextensible string. The coefficients of friction for the inclined plane and block 1 are μ_s, μ_k. Assume $\theta < \tan^{-1}\mu_s$, so that all strings remain taut. (b) Rework part (a) if block 1 is subjected to additional resistance force $R = cv_1$, where v_1 is its speed.

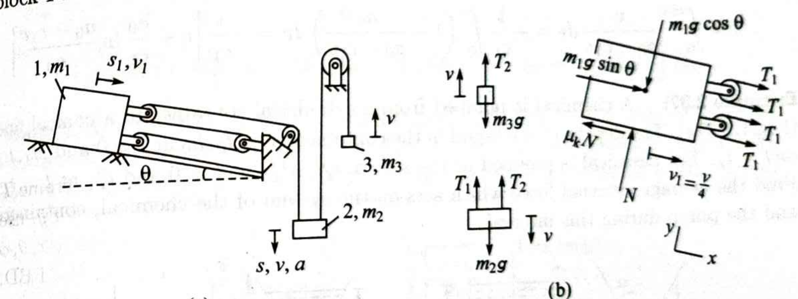


Fig. E2.36

Solution: (a) Inextensible string $\Rightarrow v_1 = v/4$. The FBDs are shown in Fig. E2.36b with $N = m_1 g\cos\theta$, since $a_y = 0$ for m_1. The work–energy relation in the rate form for the whole system is the best to get acceleration since it is a single dof system in which the internal tensions are together workless and other forces are known.

$$T = \tfrac{1}{2}m_2 v^2 + \tfrac{1}{2}m_1(v/4)^2 + \tfrac{1}{2}m_3 v^2 = \tfrac{1}{2}(m_1/16 + m_2 + m_3)v^2$$

$$\dot{T} = \tfrac{1}{2}(m_1/16 + m_2 + m_3)2v\dot{v} = \dot{W} \tag{1}$$

$$= m_2 g v - m_3 g v + (m_1 g\sin\theta - \mu_k N)v/4$$

$$\Rightarrow \quad a = \dot{v} = \frac{g[m_2 - m_3 + m_1(\sin\theta - \mu_k \cos\theta)/4]}{m_1/16 + m_2 + m_3} = \text{const.} = a_0 \quad (\text{say})$$

provided $m_2 + \frac{1}{4}m_1 \sin\theta > m_3 + \frac{1}{4}\mu_s m_1 \cos\theta$ to initiate motion. $v(0) = 0$, $s(0) = 0 \Rightarrow$

$$v(t) = v(0) + a_0 t = a_0 t, \qquad s(t) = s(0) + v(0)t + \tfrac{1}{2}a_0 t^2 = \tfrac{1}{2}a_0 t^2$$

$$v^2(s) - v^2(0) = 2a_0 s \quad \Rightarrow \quad v(s) = (2a_0 s)^{1/2}$$

(b) Add $-Rv_1 = -c(\frac{1}{4}v)(\frac{1}{4}v)$ in eq. (1) $\Rightarrow a = a_0 - c_1 v$ with $c_1 = \dfrac{c}{m_1 + 16(m_2 + m_3)}$.

$$\Rightarrow \quad \int_0^{v(t)} \frac{1}{a_0 - c_1 v}\, dv = \int_0^t dt \quad \Rightarrow \quad v = \frac{a_0}{c_1}(1 - e^{-c_1 t}) \tag{2}$$

$$\Rightarrow \quad s = \int_0^t \frac{a_0}{c_1}(1 - e^{-c_1 t})\, dt = \frac{a_0}{c_1}\left[t + \frac{1}{c_1}(e^{-c_1 t} - 1)\right] \tag{3}$$

Equation (2) implies that the terminal velocity is a_0/c_1 as $t \to \infty$. Separating the variables v and s in $\dfrac{dv}{dt} = v\dfrac{dv}{ds} = a_0 - c_1 v \Rightarrow$

$$s = \int_0^{v(s)} \frac{v}{a_0 - c_1 v}\, dv = -\frac{1}{c_1}\int_0^v \left(1 - \frac{a_0}{a_0 - c_1 v}\right) dv = -\frac{1}{c_1}\left[v + \frac{a_0}{c_1}\ln\frac{a_0 - c_1 v}{a_0}\right]$$

Example 2.37: A chemical is pumped from a cylindrical container to a conical one (Fig. E2.37a). The depths of the liquid in the containers are H_0, h_0 at $t = 0$ and H_1, h_1 at $t = t_1$. The chemical is pumped at the rate of $Q_0\,\text{m}^3/\text{s}$ at $t = 0$ and Q_1 at $t = t_1$. Find the average external force which acts on the system of the chemical, containers and the pump during this interval.

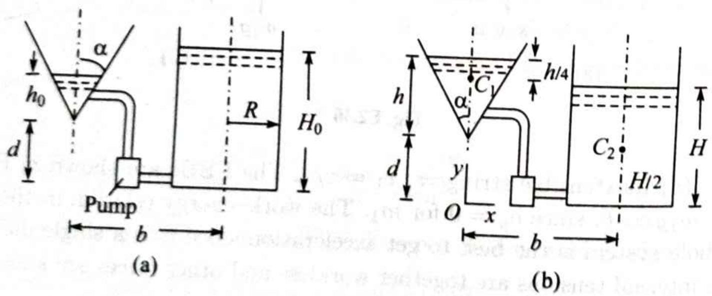


Fig. E2.37

Solution: The chemical has depths H and h in the two containers at time t and the flow rate is Q (Fig. E2.37b). The volumes V_1, V_2, mass m_1, m_2 and centres of mass C_1, C_2 of the chemical in the conical and cylindrical containers are

$$V_1 = \tfrac{1}{3}\pi(h\tan\alpha)^2 h, \qquad m_1 = \tfrac{1}{3}\pi\rho(\tan^2\alpha)h^3, \qquad \vec{r}_{C_1} = (d + \tfrac{3}{4}h)\,\vec{j}$$

$$V_2 = \pi R^2 H, \qquad m_2 = \pi\rho R^2 H, \qquad \vec{r}_{C_2} = b\vec{i} + \tfrac{1}{2}H\vec{j}$$

The mass m_3 of the rest of the system has centre of mass C_3 at a fixed location $\vec{r}_{C_3}$.

$$Q = \dot{V}_1 = \pi h^2 \dot{h}\tan^2\alpha \quad \Rightarrow \quad \dot{h} = Q/\pi h^2 \tan^2\alpha$$

$$Q = -\dot{V}_2 = -\pi R^2 \dot{H} \quad \Rightarrow \quad \dot{H} = -Q/\pi R^2$$

$\vec{F}_{av}$ on the system is obtained using the impulse–momentum relation:

$$m\vec{r}_C = \sum m_i \vec{r}_{C_i}$$

$$= \pi\rho\left[\tfrac{1}{3}\tan^2\alpha\left(dh^3 + \frac{3}{4}h^4\right)\vec{j} + R^2\left(bH\vec{i} + \tfrac{1}{2}H^2\vec{j}\right)\right] + m_3\vec{r}_{C_3}$$

$$m\vec{v}_C = \pi\rho\left[\tan^2\alpha\left(dh^2 + h^3\right)\dot{h}\vec{j} + R^2\left(b\vec{i} + H\vec{j}\right)\dot{H}\right] = \rho\left[(d + h - H)\,\vec{j} - b\vec{i}\right]Q$$

$$\vec{F}_{av} = \frac{\vec{I}}{t_1 - 0} = \frac{\Delta\vec{p}}{t_1} = \frac{m\vec{v}_C(t_1) - m\vec{v}_C(0)}{t_1}$$

$$= \rho\left[(bQ_0 - bQ_1)\,\vec{i} + \{(d + h_1 - H_1)Q_1 - (d + h_0 - H_0)Q_0\}\,\vec{j}\right]/t_1$$

Example 2.38: A body of mass m_1 falls through a height h on a body of mass m_2 supported on a spring of stiffness k (Fig. E2.38a) and does not separate from it after impact. Find the energy lost on impact and the maximum compression of the spring.

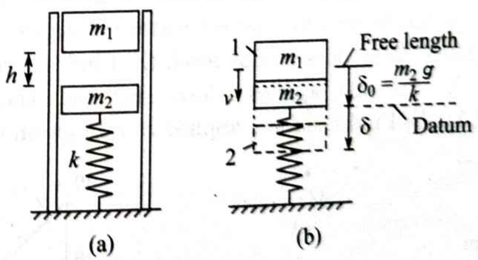


Fig. E2.38

Solution: The speed v_1 of m_1 just before impact is: $v_1 = (2gh)^{1/2}$. Let v be the common velocity of bodies just after impact. We assume that the duration of impact is very small compared to the period of the spring–mass system, i.e., the impact is modelled as instantaneous. The impulse of external forces on bodies 1 and 2 consisting of finite gravity and finite spring forces during the zero impact interval is zero. Hence, their momentum is conserved during impact:

$$(m_1+m_2)v = m_1 v_1 \quad \Rightarrow \quad v = (2gh)^{1/2} m_1/(m_1+m_2)$$

The relative kinetic energy lost on impact equals $[\frac{1}{2}m_1v_1^2 - \frac{1}{2}(m_1+m_2)v^2]/\frac{1}{2}m_1v_1^2 = \frac{1}{1+m_1/m_2}$, which is less if $m_2 \ll m_1$. The initial compression δ_0 of the spring under gravity force m_2g is $\delta_0 = m_2g/k$. Let the additional compression of the spring be δ when the velocities of the bodies are zero at their lowest position. After impact, the system is subjected to gravity and spring forces which are conservative (Fig. E2.38b):

$$T_1 + V_1 = \tfrac{1}{2}(m_1+m_2)v^2 + \tfrac{1}{2}k\delta_0^2 + 0 = T_2 + V_2 = \tfrac{1}{2}k(\delta_0+\delta)^2 - (m_1+m_2)g\delta$$

$$\Rightarrow \quad \delta^2 - \frac{2m_1g}{k}\delta - \frac{2m_1^2gh}{(m_1+m_2)k} = 0$$

The downward deflection corresponds to the positive root of this quadratic equation:

$$\delta = \left[1 + \left\{1 + \frac{2hk}{(m_1+m_2)g}\right\}^{1/2}\right]\frac{m_1g}{k} \tag{1}$$

The maximum spring compression is $\delta_0 + \delta$. Let $\delta_{st} = m_1g/k$ be the static deflection under gravity load m_1g. For $m_2 = 0$, eq.(1) yields $\delta/\delta_{st} = [1 + (1 + 2h/\delta_{st})^{1/2}] \gg 1$ if $h/\delta_{st} \gg 1$. If m_1 is released at the top of the spring, then $h = 0$ and $\delta/\delta_{st} = 2$.

Example 2.39: At the instant shown (Fig. E2.39a), a rope attached to O is thrown to the man in the rowboat at its centre of mass C. The man pulls on it to approach the deck. The mass of the person and the rowboat is 200 kg. Find v_r and v_ϕ of C after the rope has been pulled in by 2 m. Neglect water resistance. (a) The man applies a constant force of 240 N. Why does he not reach O? Find his minimum and maximum distance from O. (b) The man applies a force such that the rope is pulled in at a constant rate of 0.5 m/s. Find the force applied as a function of the radial distance.

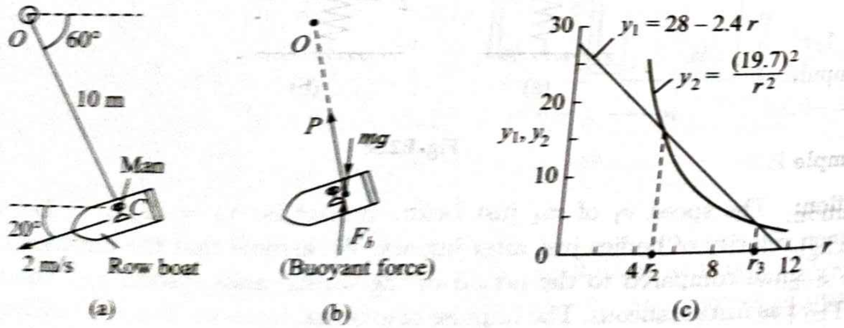


Fig. E2.39

Solution: The FBD of the man and the boat is shown in Fig. E2.39b. $F_b = mg \Rightarrow$ the resultant external force $\vec{F} = -P\vec{e}_r$ acts at C and passes through a fixed point O. Hence, the system moves under a central force and $F_\phi = 0 \Rightarrow$

$$F_\phi = ma_\phi = \frac{m}{r}\frac{d}{dt}(rv_\phi) = 0 \quad \Rightarrow \quad h_0 = rv_\phi = \text{constant}$$

Initial: $r_0 = 10\,\text{m}$, $v_0 = 2$, $v_{\phi_0} = v_0 \sin 80° = 1.970$, $v_{r_0} = -v_0 \cos 80° = -0.3473\,\text{m/s}$. Let v_1, v_{ϕ_1}, v_{r_1} be the values at $r_1 = 10 - 2 = 8\,\text{m}$.

$$h_0 = rv_\phi = r_0 v_{\phi_0} \quad \Rightarrow \quad v_\phi = \frac{r_0 v_{\phi_0}}{r} = \frac{19.7}{r} \quad \Rightarrow \quad v_{\phi_1} = \frac{19.7}{r_1} = 2.463\,\text{m/s}$$

(a) $F_r = -240\,\text{N}$. Work–energy relation for $m = 200\,\text{kg}$ for position r_0 and r_1:

$$T_1 - T_0 = W_{0-1}: \qquad \tfrac{1}{2}m(v_1^2 - v_0^2) = 240 \times 2 = 480$$

$$\Rightarrow \quad v_1 = 2.966\,\text{m/s}, \qquad v_r = -(v_1^2 - v_{\phi_1}^2)^{1/2} = -1.654\,\text{m/s}$$

The negative root is taken since r is decreasing.

The work–energy relation for positions r_0 and r yields

$$\tfrac{1}{2}(200)(v_\phi^2 + v_r^2 - v_0^2) = 240(10 - r) \quad \Rightarrow \quad v_r^2 = (28 - 2.4r) - 19.7^2/r^2$$

The terms $(28 - 2.4r)$ and $-19.7/r^2$ decrease to 28 and $-\infty$ as $r \to 0$. Hence, there exists $r_{min} = r_2$ where $v_r = 0$, then r increases to $r_{max} = r_3$. The boat does not reach O.

At the extremal values r_2 and r_3 of r: $v_r = \dot{r} = 0 \Rightarrow 28 - 2.4r = 19.7^2/r^2$. The smaller positive root of this cubic equation, obtained from the graph of Fig. 2.39c, is $r_2 = 4.882\,\text{m}$ and the larger positive root is $r_3 = 10.073\,\text{m}$.

(b) $v_r(t) = \dot{r} = -0.5\,\text{m/s} \Rightarrow \ddot{r} = 0$ and using $v_\phi = r_0 v_{\phi_0}/r$:

$$F_r = ma_r = m(\ddot{r} - r\dot{\phi}^2) = -mr\dot{\phi}^2 = -\frac{mv_\phi^2}{r} = -\frac{mr_0^2 v_{\phi_0}^2}{r^3} = -\frac{77620}{r^3}\,\text{N}$$

$F_r \propto 1/r^3$. It increases from 77.62 N at $r = 10\,\text{m}$ to 151.6 N at $r = 8\,\text{m}$. Moreover, an impulsive force $\hat{F}_r$ should be applied at $t = 0$ to change the radial speed from $v_{r_0} = -0.3473\,\text{m/s}$ to $v_r = -0.5\,\text{m/s}$, where $\hat{F}_r = m(v_r - v_{r_0}) = -30.54\,\text{N.s}$.

Example 2.40: Discuss the dynamics of a spherical pendulum.

Solution: This pendulum (Fig. E2.40a) consists of a small bob of mass m attached to a light rod with a smooth ball and socket joint at O. An equivalent problem is that of a small slider moving inside a smooth spherical bowl. Two dof of the bob are θ and ϕ. The FBD of the bob is shown in Fig. E2.40b. Using cylindrical coordinates (r, ϕ, z) for the centre of the bob:

$$r = L\sin\theta, \qquad \dot r = L\cos\theta\,\dot\theta, \qquad \ddot r = -L\sin\theta\,\dot\theta^2 + L\cos\theta\,\ddot\theta$$

$$z = L(1-\cos\theta), \qquad \dot z = L\sin\theta\,\dot\theta, \qquad \ddot z = L\cos\theta\,\dot\theta^2 + L\sin\theta\,\ddot\theta$$

$$\vec v = \dot r\,\vec e_r + r\dot\phi\,\vec e_\phi + \dot z\,\vec e_z = L\cos\theta\,\dot\theta\,\vec e_r + L\sin\theta\,\dot\phi\,\vec e_\phi + L\sin\theta\,\dot\theta\,\vec e_z$$

$$\Rightarrow \quad v^2 = L^2\dot\theta^2 + L^2\sin^2\theta\,\dot\phi^2 \qquad (1)$$

$$\vec a = (\ddot r - r\dot\phi^2)\,\vec e_r + (2\dot r\dot\phi + r\ddot\phi)\,\vec e_\phi + \ddot z\,\vec e_z$$

$$= [-\sin\theta\,(\dot\theta^2+\dot\phi^2) + \cos\theta\,\ddot\theta]L\,\vec e_r$$

$$+L(2\cos\theta\,\dot\theta\dot\phi + \sin\theta\,\ddot\phi)\,\vec e_\phi + L(\cos\theta\,\dot\theta^2 + \sin\theta\,\ddot\theta)\,\vec e_z \qquad (2)$$

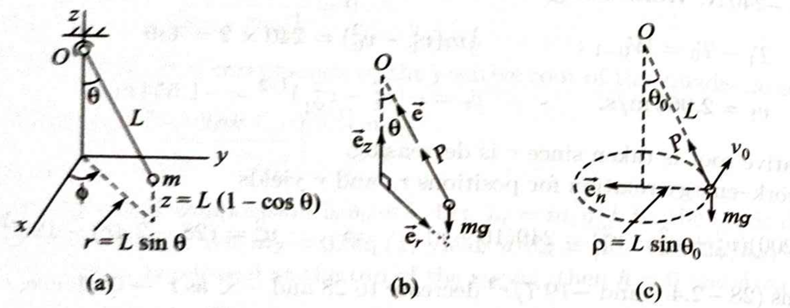


Fig. E2.40

For the bob, $-mg\vec e_z \parallel \vec e_z$ and $\vec P$ passes through $O \Rightarrow M_{Oz} = 0 \Rightarrow H_{Oz}$ = constant. The 2nd equation is $T_C + V^*$ = constant, as P is workless and mg is conservative.

$$H_{Oz} = mrv_\phi = mL^2\sin^2\theta\,\dot\phi \quad \Rightarrow \quad \sin^2\theta\,\dot\phi = c_1 \qquad (3)$$

$$T_C + V^* = \tfrac{1}{2}m(L^2\dot\theta^2 + L^2\sin^2\theta\,\dot\phi^2) + mgL(1-\cos\theta)$$

$$\Rightarrow \quad \dot\theta^2 + \sin^2\theta\,\dot\phi^2 - 2g\cos\theta/L = c_2 \qquad (4)$$

where c_1, c_2 are constants. Equations (3) and (4) are the first integrals of equations of motion and yield $\dot\phi$, $\dot\theta$ in any given position. Let the bob be projected circumferentially with speed v_0 at $\theta = \theta_0$. We get initial values $\dot\phi_0$, $\dot\theta_0$ of $\dot\phi$, $\dot\theta$ from eq. (1):

$$\vec v_0 = v_0\,\vec e_\phi = L\cos\theta_0\,\dot\theta_0\,\vec e_r + L\sin\theta_0\,\dot\phi_0\,\vec e_\phi + L\sin\theta_0\dot\theta_0\,\vec e_z$$

$$\Rightarrow \quad \dot\theta_0 = 0, \qquad \dot\phi_0 = v_0/L\sin\theta_0 \qquad (5)$$

$$(3)\ \&\ (5) \Rightarrow \quad \sin^2\theta\,\dot\phi = \sin^2\theta_0\,\dot\phi_0 \quad \Rightarrow \quad \dot\phi = v_0\sin\theta_0/L\sin^2\theta \qquad (6)$$

$$(4)-(6) \Rightarrow \quad \dot\theta^2 + \sin^2\theta\,\dot\phi^2 - 2g\cos\theta/L = \dot\theta_0^2 + \sin^2\theta_0\,\dot\phi_0^2 - 2g\cos\theta_0/L$$

$$\Rightarrow \quad \dot\theta^2 + v_0^2(\sin^2\theta_0 - \sin^2\theta)/L^2\sin^2\theta + 2g(\cos\theta_0 - \cos\theta)/L = 0 \qquad (7)$$

The maximum and minimum values of θ occur when $\dot\theta = 0$. At these positions, using $\sin^2\theta_0 - \sin^2\theta = \cos^2\theta - \cos^2\theta_0$, eq. (7) yields

$$(\cos\theta_0 - \cos\theta)[2g - (\cos\theta_0 + \cos\theta)v_0^2/L\sin^2\theta] = 0$$

$$\Rightarrow \quad \theta = \theta_0 \quad \text{or} \quad \sin^2\theta - (\cos\theta_0 + \cos\theta)v_0^2/2gL = 0$$

$$\Rightarrow \quad \cos^2\theta + 2\lambda^2\cos\theta - (1 - 2\lambda^2\cos\theta_0) = 0 \qquad (8)$$

with $\lambda^2 = v_0^2/4gL$. $0 < \theta < \pi \Rightarrow -1 < \cos\theta < 1 \Rightarrow$ acceptable root of eq. (8) is $\theta = \theta_1$, where $\cos\theta_1 = -\lambda^2 + (1 - 2\lambda^2\cos\theta_0 + \lambda^4)^{1/2}$. The motion is between θ_0 to θ_1.

$$T_C + V^* = \tfrac{1}{2}mv^2 + mgL(1-\cos\theta) = \tfrac{1}{2}mv_0^2 + mgL(1-\cos\theta_0) \qquad (9)$$

$\vec P = P\,\vec e$ is obtained from the equation of motion in direction $\vec e = -\sin\theta\,\vec e_r + \cos\theta\,\vec e_z$ and using $\vec a$ from eq. (2) and v^2 from eq. (9):

$$F_e = P - mg\cos\theta = ma_e = m\vec a\cdot\vec e = mL(\dot\theta^2 + \sin^2\theta\dot\phi^2) = mv^2/L$$

$$\Rightarrow \quad P = m(g\cos\theta + v^2/L) = m(3g\cos\theta - 2g\cos\theta_0 + v_0^2/L)$$

For a *conical pendulum* (Fig. E2.40c), $\theta(t) \equiv \theta_0 \Rightarrow \dot\theta = 0 \Rightarrow \theta_0$ is a root of eq. (8) $\Rightarrow$

$$\cos^2\theta_0 + 4\lambda^2\cos\theta_0 - 1 = 0$$

$$\Rightarrow \quad v_0^2 = gL\sin^2\theta_0/\cos\theta_0 \quad \& \quad \text{eq. (5)} \Rightarrow \quad \dot\phi = v_0/L\sin\theta_0$$

Direct proof for conical pendulum: $\vec a = (v_0^2/\rho)\,\vec e_n = (v_0^2/L\sin\theta_0)\,\vec e_n$,

$$F_z = P\cos\theta_0 - mg = 0 \quad \Rightarrow \quad P = mg/\cos\theta_0$$

$$F_n = P\sin\theta_0 = mv_0^2/L\sin\theta_0 \quad \Rightarrow \quad v_0^2 = gL\sin^2\theta_0/\cos\theta_0$$

Example 2.41: A small packet descends a helix whose axis is vertical (Fig. E2.41a). Find its speed after descending a height h if (a) the coefficient of friction for the floor is μ and for the side walls is μ_1, (b) the helix is rotating about its axis at uniform ω. (c) Find the speed of a bead on a helical wire after descending a height h. The coefficient of friction is μ.

Solution: (a) The FBD of the package is shown in Fig. E2.41b. The helix has pitch $p = 2\pi R\tan\alpha$. Cylindrical and path components are convenient for $\vec v$, $\vec a$:

$$r = R, \quad \dot r = 0, \quad \ddot r = 0, \quad z = p\phi/2\pi = R\phi\tan\alpha, \quad \dot z = R\dot\phi\tan\alpha$$

$$\vec v = \dot r\vec e_r + r\dot\phi\vec e_\phi + \dot z\vec e_z = R\dot\phi\vec e_\phi + R\dot\phi\tan\alpha\,\vec e_z = v\vec e_t$$

$$\Rightarrow \quad v = R\dot\phi(1+\tan^2\alpha)^{1/2}, \qquad \vec e_t = \cos\alpha\,\vec e_\phi + \sin\alpha\,\vec e_z$$

$$\dot\phi = v\cos\alpha/R, \qquad \ddot\phi = \dot v\cos\alpha/R$$

$$\Rightarrow \quad \dot z = R\dot\phi\tan\alpha = v\sin\alpha, \qquad \ddot z = \dot v\sin\alpha$$

$$\vec a = (\ddot r - r\dot\phi^2)\vec e_r + (2\dot r\dot\phi + r\ddot\phi)\vec e_\phi + \ddot z\vec e_z$$

$$= -(v^2\cos^2\alpha/R)\,\vec e_r + \dot v(\cos\alpha\,\vec e_\phi + \sin\alpha\,\vec e_z)$$

$$= \dot v\,\vec e_t - (v^2\cos^2\alpha/R)\,\vec e_r$$

$\Rightarrow$ $\vec e_n = -\vec e_r$, $\rho = R/\cos^2\alpha$. Equations of motion of the packet are:

$$F_n = ma_n: \qquad N_1 = mv^2\cos^2\alpha/R$$

$$F_b = 0: \qquad mg\cos\alpha - N = 0$$

$$F_t = ma_t: \qquad m\dot v = mg\sin\alpha - \mu N - \mu_1 N_1$$

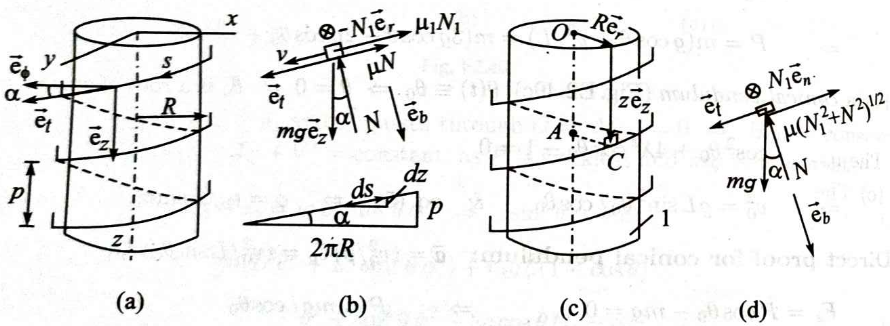


Fig. E2.41

Using expressions of N and N_1, the above equation yields:

$$\dot v = \frac{dv}{dz}\dot z = \frac{dv}{dz}v\sin\alpha = g(\sin\alpha - \mu\cos\alpha) - \mu_1 v^2\cos^2\alpha/R$$

$$\Rightarrow \quad \int_0^v \frac{v\sin\alpha}{g(\sin\alpha - \mu\cos\alpha) - \mu_1 v^2\cos^2\alpha/R}\,dv = \int_0^h dz$$

$$\Rightarrow \quad v^2 = gR(\sin\alpha - \mu\cos\alpha)[1 - \exp(-2\mu_1 h\cos\alpha\cot\alpha/R)]\,/\,\mu_1\cos^2\alpha$$

(b) $\vec\omega_1$ and $\dot{\vec\omega}_1$ of frame 1 of the helix are:

$$\vec\omega_1 = \omega\vec e_z = \omega(\sin\alpha\,\vec e_t + \cos\alpha\,\vec e_b), \qquad \dot{\vec\omega}_1 = \vec 0$$

$$\vec a_{C|1} = \vec a = \dot v\vec e_t + \frac{v^2\cos^2\alpha}{R}\vec e_n, \qquad \vec v_{C|1} = \vec v = v\vec e_t$$

In Fig. E2.41c, $\vec a_A = \vec 0$, $\overrightarrow{AC} = R\vec e_r$, and $\vec a_C$ with respect to the ground is

$$\vec a_C = \vec a_A + \dot{\vec\omega}_1\times\overrightarrow{AC} + \vec\omega_1\times(\vec\omega_1\times\overrightarrow{AC}) + 2\vec\omega_1\times\vec v_{C|1} + \vec a_{C|1}$$

$$= -\omega^2R\vec e_r + 2\omega(\sin\alpha\,\vec e_t + \cos\alpha\,\vec e_b)\times v\vec e_t + \vec a$$

$$= (2\omega v\cos\alpha + \omega^2R + v^2\cos^2\alpha/R)\,\vec e_n + \dot v\vec e_t$$

The equations of motion of the packet are:

$$F_n = ma_{C_n}: \qquad N_1 = m(2\omega v\cos\alpha + \omega^2R + v^2\cos^2\alpha/R)$$

$$F_b = 0: \qquad mg\cos\alpha - N = 0$$

$$F_t = ma_{C_t}: \qquad m\dot v = mg\sin\alpha - \mu N - \mu_1 N_1 \;\Rightarrow$$

$$\dot v = v\sin\alpha\frac{dv}{dz} = g(\sin\alpha - \mu\cos\alpha) - \mu_1(2\omega v\cos\alpha + \omega^2R + v^2\cos^2\alpha/R)$$

$$\int_0^v v[g(1-\mu\cot\alpha) - \mu_1\omega^2R\,\mathrm{cosec}\,\alpha - \mu_1\cot\alpha(2\omega v + v^2\cos\alpha/R)]^{-1}dv = h$$

The integral can be evaluated in closed form.

(c) The FBD of the bead is shown in Fig. 2.41d. Equations of motion:

$$F_n = ma_n: \qquad N_1 = mv^2\cos^2\alpha/R$$

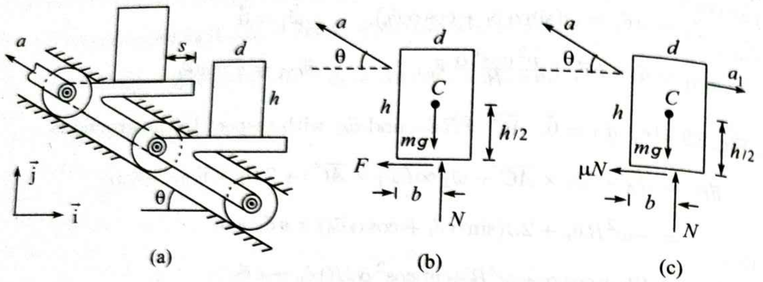
$$F_b = 0: \qquad mg\cos\alpha - N = 0$$

$$F_t = ma_t: \qquad m\dot v = mg\sin\alpha - \mu(N_1^2 + N^2)^{1/2}$$

$$\Rightarrow \quad \dot v = v\sin\alpha\frac{dv}{dz} = g\sin\alpha - \mu(v^4\cos^4\alpha/R^2 + g^2\cos^2\alpha)^{1/2}$$

which can be integrated numerically by the Runge–Kutta method.

Example 2.42: (a) Cylindrical cans are transported from one elevation to another by the translating horizontal arms (Fig. E2.42). The coefficient of friction is μ. Find the maximum upward acceleration a of the conveyor for which the can does not move relative to the arm. Rework for downward acceleration. (b) If relative motion starts with slip, find the relative acceleration and the time at which it starts to tilt.



(a) (b) (c)

Fig. E2.42

Solution: (a) The FBD of the can is shown in Fig. E2.42b. The four unknowns a, N, F, b are solved using $F_x = ma_{C_x}$, $F_y = ma_{C_y}$, $M_{C_z} = 0$ and the slip condition ($|F| = \mu N$) or tipping condition ($b = 0$ or $b = d$).

$$F_x = ma_{C_x}: \quad F = ma\cos\theta \qquad (1)$$

$$F_y = ma_{C_y}: \quad N - mg = ma\sin\theta \quad \Rightarrow \quad N = m(g + a\sin\theta) \qquad (2)$$

$$M_{C_z} = 0: \quad N(b - d/2) - Fh/2 = 0 \quad \Rightarrow \quad b = Fh/2N + d/2 \qquad (3)$$

For $a > 0$: eq. (2) $\Rightarrow$ $N > 0$ (no lift-off), eq. (1) $\Rightarrow$ F is positive and eq. (3) $\Rightarrow$ $b > d/2$. For no slip $F < \mu N$ and for no tipping $b < d$.

$$F < \mu N: \quad ma\cos\theta < \mu m(g + a\sin\theta)$$

$$\Rightarrow \quad a(\cos\theta - \mu\sin\theta) < \mu g$$

$$\Rightarrow \quad \text{no slip } \forall a \text{ if} \quad \cos\theta \le \mu\sin\theta \quad \Rightarrow \quad \text{if } \theta \ge \theta_s = \tan^{-1}(1/\mu),$$

$$\text{else no slip if} \quad a < \frac{\mu g}{\cos\theta - \mu\sin\theta}$$

$$b < d: \quad b = \frac{Fh}{2N} + \frac{d}{2} = \frac{ah\cos\theta}{2(g + a\sin\theta)} + \frac{d}{2} < d$$

$$\Rightarrow \quad a(\cos\theta - d\sin\theta/h) < dg/h$$

$$\Rightarrow \text{no tipping } \forall a \text{ if} \quad \cos\theta \le d\sin\theta/h \quad \Rightarrow \quad \text{if } \theta \ge \theta_t = \tan^{-1}(h/d),$$

$$\text{else no tipping if} \quad a < \frac{(d/h)g}{\cos\theta - (d/h)\sin\theta}$$

Hence, if $\theta \ge \max(\theta_s, \theta_t)$, then no relative motion of can $\forall a > 0$. Else, $a_{\max} = pg/(\cos\theta - p\sin\theta)$, with $p = \min(d/h, \mu)$ and the can tips before sliding if $d/h < \mu$.

For $a < 0$, no lift-off if: $N \overset{(2)}{=} m(g + a\sin\theta) \ge 0$, i.e., if $-a \le g\,\mathrm{cosec}\,\theta$. Equation (1) $\Rightarrow F$ is negative and eq. (3) $\Rightarrow b < d/2$. For no slip and no tipping:

$$-F < \mu N: \quad -ma\cos\theta < \mu m(g + a\sin\theta) \quad \Rightarrow \quad -a < \frac{\mu g}{\cos\theta + \mu\sin\theta}$$

$$b > 0: \quad b = \frac{Fh}{2N} + \frac{d}{2} = \frac{ah\cos\theta}{2(g + a\sin\theta)} + \frac{d}{2} > 0 \quad \Rightarrow \quad -a < \frac{(d/h)g}{\cos\theta + (d/h)\sin\theta}$$

Hence, $(-a)_{\max} = pg/(\cos\theta + p\sin\theta)$, and the can tips before sliding if $d/h < \mu$.

(b) Let $a_1\vec{\imath}$ be the acceleration with respect to the translating arm (Fig. E2.42c).

$$\vec{a}_C = a_1\vec{\imath} + a(-\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}) = (a_1 - a\cos\theta)\,\vec{\imath} + a\sin\theta\,\vec{\jmath}$$

$$F_y = ma_{C_y}: \quad N - mg = ma\sin\theta \quad \Rightarrow \quad N = m(g + a\sin\theta)$$

$$F_x = ma_{C_x}: \quad -\mu N = m(a_1 - a\cos\theta) \quad \Rightarrow \quad a_1 = a\cos\theta - \mu(g + a\sin\theta)$$

$$M_{C_z} = 0: \quad N(b - d/2) - \mu N h/2 = 0 \quad \Rightarrow \quad b = \tfrac{1}{2}d + \tfrac{1}{2}\mu h$$

It certainly tilts at time t when the can has moved a distance $s + d - b$ so that the line of action of N is just at the edge of the arm: $\frac{1}{2}a_1t^2 = s + d - b$.

Example 2.43: The mass of a fork-lift truck (Fig. E2.43a), excluding the fork, is m_1 with centre of mass at C_1. Neglect the inertia of the wheels. The mass of the fork and the crate is m_2 with centre of mass at C_2. The fork is supported by a smooth roller at A and a connection at B which supports both horizontal and vertical reactions. (a) Find the maximum deceleration that the truck can have with all wheels braked so that the crate neither slips nor tips and the wheels neither slip nor lift off. The mass of the crate is m and its centre of mass is at height h_3 above the centre of its base. The coefficient of friction for the wheels and the ground is μ_1 and for the crate and the fork is μ_2. (b) Find the ground reactions for a rear-wheel driven fork-lift truck at the instant when the fork has an upward acceleration a_1 relative to the truck and the driving torque is M_1. Assume that there is no slip and no lift-off at all contacts.

Solution: (a) Let the backward acceleration of the truck be a. Since the wheels are braked, the friction forces on them are in the backward direction. The external force systems acting on the crate and truck are shown in Fig. E2.43b,c along with the equivalent kinetic systems: $m_i\vec{a}_{C_i}$ at C_i. For the truck:

$$M_{E_z} = N_2(b_2 + b_3) - m_1gb_2 + m_2gb_1 = -m_1ah_1 - m_2ah_2$$

$$\Rightarrow \quad N_2 = [m_1 g b_2 - m_2 g b_1 - (m_1 a h_1 + m_2 a h_2)]/(b_2 + b_3)$$

$$F_y = N_1 + N_2 - (m_1 + m_2)g = 0 \quad \Rightarrow \quad N_1 + N_2 = (m_1 + m_2)g \tag{1}$$

$$F_x = F_1 + F_2 = (m_1 + m_2)a \quad \Rightarrow \quad a = (F_1 + F_2)/(m_1 + m_2) \tag{2}$$

(3)

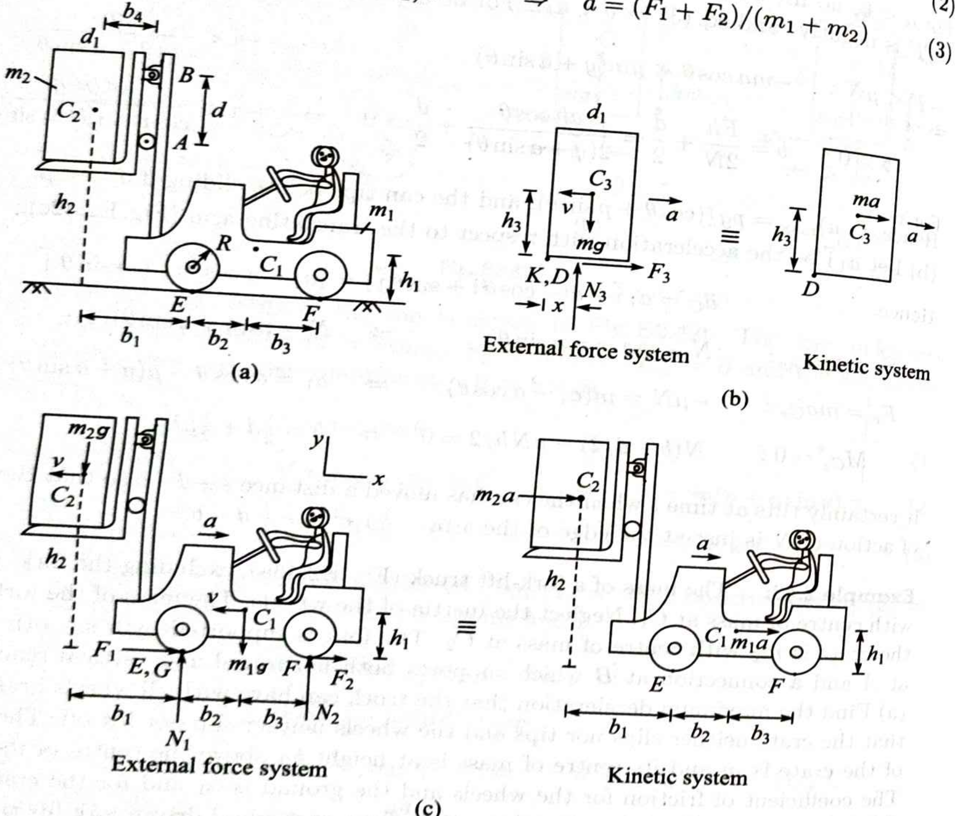


Fig. E2.43a–c

Equations (1) and (2) imply that N_2 decreases with increase in a and N_1 increases by the same amount. For no lift-off of rear wheels, using eq. (1): $N_2 \geq 0 \Rightarrow$

$$a \leq a_1 = (m_1 b_2 - m_2 b_1)g \,/\, (m_1 h_1 + m_2 h_2) \tag{4}$$

For no slip of wheels, using eqs (2), (3):

$$a = (F_1 + F_2)/(m_1 + m_2) \leq \mu_1 (N_1 + N_2) \,/\, (m_1 + m_2) = \mu_1 g \tag{5}$$

For the crate:

$$F_x = F_3 = ma \tag{6}$$

$$F_y = N_3 - mg = 0 \quad \Rightarrow \quad N_3 = mg \tag{7}$$

$$M_{Dz} = N_3 x - \tfrac{1}{2} m g d_1 = -m a h_3 \quad \Rightarrow \quad x = \tfrac{1}{2} d_1 - a h_3/g \tag{8}$$

$\Rightarrow x$ decreases with increase in a. For no slip of crate, using eqs (6), (7):

$$F_3 \leq \mu_2 N_3 \quad \Rightarrow \quad ma \leq \mu_2 mg \quad \Rightarrow \quad a \leq \mu_2 g \tag{9}$$

For no tipping of crate, using eq. (8):

$$x = \tfrac{1}{2} d_1 - a h_3/g \geq 0 \quad \Rightarrow \quad a \leq g d_1/2h_3 \tag{10}$$

Hence, the maximum retardation: $a = \min(a_1, \mu_1 g, \mu_2 g, g d_1/2h_3)$.

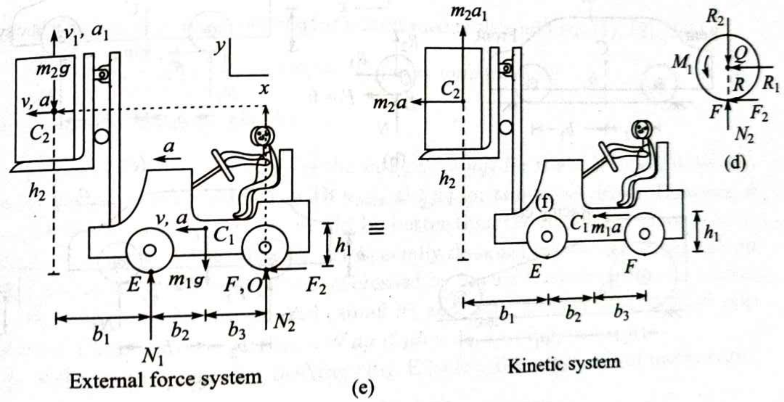


Fig. E2.43d,e

(b) The un-powered front wheels are two-force members with only normal ground reaction. For FBD of the driven rear wheels (Fig. E2.43d), $M_{Qz} = 0 \Rightarrow F_2 = M_1/R$. The external force system on the truck and its equivalent kinetic system of forces $m_i \vec{a}_{C_i}$ at C_i are shown in Fig. E2.43e:

$$F_x = -F_2 = -(m_1 + m_2)a \quad \Rightarrow \quad a = M_1/R(m_1 + m_2) \tag{11}$$

$$M_{Fz} = -N_1(b_2 + b_3) + m_1 g b_3 + m_2 g(b_1 + b_2 + b_3)$$

$$= -m_2 a_1 (b_1 + b_2 + b_3) + m_1 a h_1 + m_2 a h_2$$

$$\Rightarrow \quad N_1 = \frac{m_1 g b_3 + m_2(g+a_1)(b_1+b_2+b_3) - m_1 a h_1 - m_2 a h_2}{b_2+b_3} \tag{12}$$

$$F_y = N_1 + N_2 - m_1 g - m_2 g = m_2 a_1$$

$$\Rightarrow \quad N_2 = \frac{m_1 g b_2 - m_2(g+a_1)b_1 + m_1 a h_1 + m_2 a h_2}{b_2+b_3} \tag{13}$$

where a in eqs (12) and (13) is given by eq. (11).

Example 2.44: (a) Find the maximum acceleration of a self-propelled vehicle moving on a straight level road (Fig. E2.44a). Neglect the inertia of the wheels. Consider rear-wheel, front-wheel and four-wheel drives. The coefficient of friction is μ. (b) Find the maximum deceleration of the vehicle if the brakes are applied to the rear wheels only, to the front wheels only and to all wheels.

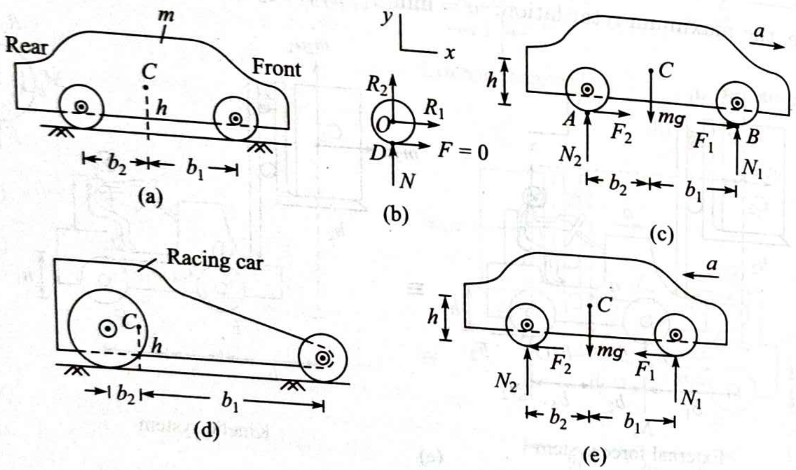


Fig. E2.44

Solution: For un-braked/un-powered light wheel (Fig. E2.44b), $M_{O_z} = 0 \Rightarrow F = 0$.
(a) The FBD of the vehicle is shown in Fig. E2.44c. The equations of motion of this translating body (since the inertia of rotating wheels is neglected) are:

$$F_x = ma_x : \quad F_1 + F_2 = ma \tag{1}$$

$$F_y = ma_y : \quad N_1 + N_2 - mg = 0 \tag{2}$$

$$M_{C_z} = 0 : \quad N_1 b_1 - N_2 b_2 + (F_1+F_2)h = 0 \tag{3}$$

$$\Rightarrow \quad N_1 = m(gb_2 - ah)/(b_1+b_2), \qquad N_2 = m(gb_1 + ah)/(b_1+b_2) \tag{4}$$

N_1 decreases with a and for no lift-off: $N_1 \geq 0 \Rightarrow$

$$m(gb_2 - ah)/(b_1+b_2) \geq 0 \quad \Rightarrow \quad a \leq gb_2/h \tag{5}$$

For **rear-wheel drive:** $F_1 = 0$ as there is no driving torque on the front wheels. Equations (1) and (4) imply that for no slip, $F_2 = ma \leq \mu N_2 \Rightarrow$

$$ma \leq \mu m(gb_1 + ah)/(b_1+b_2) \quad \Rightarrow \quad a \leq \mu g/(1 + b_2/b_1 - \mu h/b_1) \tag{6}$$

$$\Rightarrow \quad a_{max} = \min[\mu g/(1 + b_2/b_1 - \mu h/b_1),\ gb_2/h] \tag{7}$$

For most vehicles, $\mu g/(1+b_2/b_1-\mu h/b_1) < gb_2/h$ and so $a_{max} = \mu g/(1+b_2/b_1-\mu h/b_1)$.
For **front-wheel drive,** $F_2 = 0$ for un-powered rear wheels. For no slip, eqs (1), (4)

$$\Rightarrow \quad F_1 = ma \leq \mu N_1 = \mu m(gb_2 - ah)/(b_1+b_2)$$

$$\Rightarrow \quad a \leq a_1 = \mu g/(1 + b_1/b_2 + \mu h/b_2) \quad \Rightarrow \quad a_{max} = \min(a_1,\ gb_2/h) \tag{8}$$

For **four-wheel drive,** a_{max} occurs for simultaneous slip, and eqs (1), (2) yield

$$a \leq (F_1+F_2)/m = \mu(N_1+N_2)/m = \mu mg/m = \mu g \tag{9}$$

$$\Rightarrow \quad a_{max} = \min(\mu g,\ gb_2/h)$$

For $b_1 \simeq b_2$, eqs (7) to (9) $\Rightarrow$ a_{max} is the largest ($= \mu g$) for the four-wheel drive and smallest for the front-wheel drive, with $a_{max} \simeq \frac{1}{2}\mu g$ for two-wheel drives. However, if $b_1 < b_2$, then the front-wheel drive would be better than the rear-wheel drive.

Racing cars, shown in Fig. E2.44d, are specially designed rear-wheel driven cars in which the limit on a given by eq. (6) is increased by having C close to the rear wheels (small b_2/b_1) and close to the ground (small h) and using wide tyres with good grip on the ground (large $\mu \simeq 1$). Hence, $a = \mu g$ if $\mu h = b_2$, i.e., if $b_2 \simeq h$.

(b) F_1, F_2 and a are reversed for braking (Fig. E2.44e). The equations of motion are

$$F_1 + F_2 = ma, \qquad N_1 + N_2 - mg = 0$$

$$M_{C_z} = N_1 b_1 - N_2 b_2 - (F_1+F_2)h = 0$$

$$\Rightarrow \quad N_1 = m(gb_2 + ah)/(b_1+b_2), \qquad N_2 = m(gb_1 - ah)/(b_1+b_2) \tag{10}$$

N_2 decreases with a. For no lift-off: $N_2 \geq 0 \Rightarrow a \leq gb_1/h$ for all types of braking.
For only the rear wheels braked: $F_1 = 0$, $F_2 \leq \mu N_2 \Rightarrow a \leq \mu g/(1 + b_2/b_1 + \mu h/b_1)$.
For most vehicles, this provides a smaller bound than gb_1/h.
For only the front wheels braked: $F_2 = 0$, $F_1 \leq \mu N_1 \Rightarrow a \leq \mu g/(1 + b_1/b_2 - \mu h/b_2)$.
For all wheels braked, simultaneous slip $\Rightarrow a_{max} = \dfrac{F_1+F_2}{m} \leq \dfrac{\mu(N_1+N_2)}{m} = \mu g.$

Example 2.45: A platform 3, used for transport over small distances (Fig. E2.45a), is hinged to two light bars. Find $\dot{\omega}$ and the reactions at A, B, D, E in the given position.

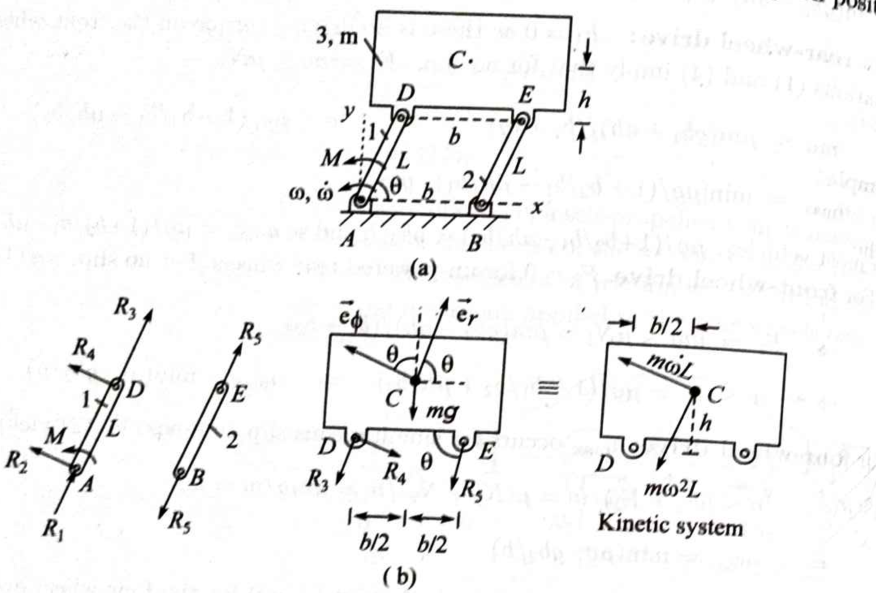


Fig. E2.45

Solution: Platform 3 translates. BE is a two-force light member. The reaction components are shown along $\vec{e}_r$, $\vec{e}_\phi$ in FBDs (Fig. 2.45b), since $\vec{a}_C = \vec{a}_D = -\omega^2 L\,\vec{e}_r + \dot{\omega} L\,\vec{e}_\phi$. The equivalent kinetic system for 3 is also shown. R_1 to R_5 and $\dot{\omega}$ are obtained from six equations of motion for 1 and 3.

$$1: \quad F_r = R_1 + R_3 = 0 \quad \Rightarrow \quad R_1 = -R_3 \tag{1}$$

$$F_\phi = R_2 + R_4 = 0$$

$$M_{A_z} = M + LR_4 = 0 \quad \Rightarrow \quad R_4 = -R_2 = -M/L \tag{2}$$

$$3: \quad F_\phi = ma_{C_\phi}: \qquad -R_4 - mg\cos\theta = m\dot{\omega}L$$

$$\Rightarrow \quad \dot{\omega} = (M - mgL\cos\theta)/mL^2 \tag{3}$$

$$F_r = ma_{C_r}: \qquad -R_3 - R_5 - mg\sin\theta = -m\omega^2 L \tag{4}$$

$$M_{D_z} = -\tfrac{1}{2}mgb - R_5\sin\theta\, b$$

$$= (m\dot{\omega}L\sin\theta + m\omega^2 L\cos\theta)h + (m\dot{\omega}L\cos\theta - m\omega^2 L\sin\theta)\tfrac{1}{2}b \tag{5}$$

R_5, R_3, R_1 are obtained in succession from eqs (5), (4), (1).

$\dot{\omega}$ could have been obtained directly from $\dot{T} + \dot{V} = \dot{W}_{nc}$:

$$\frac{d}{dt}\left(\tfrac{1}{2}mL^2\omega^2 + mgL\sin\theta\right) = M\omega$$

$$mL^2\omega\dot{\omega} + mgL\cos\theta\,\omega = M\omega \quad \Rightarrow \quad \dot{\omega} = (M - mgL\cos\theta)/mL^2$$

Example 2.46: The end plate of the truck (Fig. E2.46a) has mass m and is supported by a hinge A at the bottom edge and a chain BD at the top corner. Find the tension in the chain and the hinge reaction when the truck has acceleration a.

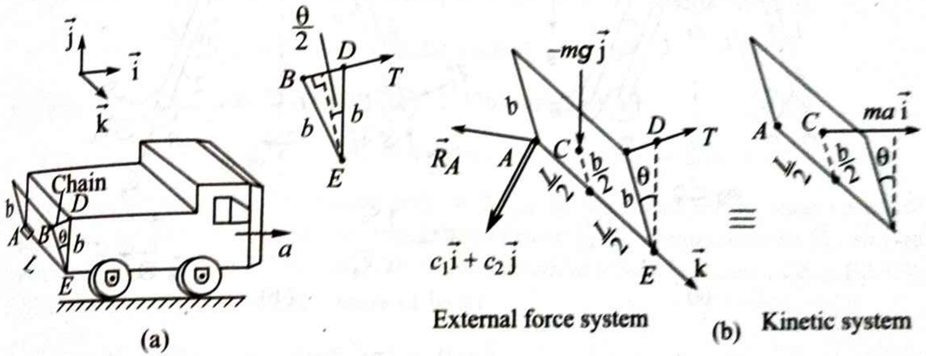


Fig. E2.46

Solution: The FBD of the end plate is shown in Fig. E2.46b, along with the equivalent kinetic system. $\vec{M}_A = \vec{AC} \times m\vec{a}_C$:

$$c_1\vec{i} + c_2\vec{j} + \left[\tfrac{1}{2}L\vec{k} + \tfrac{1}{2}b\left(\cos\theta\,\vec{j} - \sin\theta\,\vec{i}\right)\right] \times (-mg\vec{j}) + (L\vec{k} + b\vec{j})$$

$$\times T\left(\cos\tfrac{1}{2}\theta\,\vec{i} + \sin\tfrac{1}{2}\theta\,\vec{j}\right) = \left[\tfrac{1}{2}L\vec{k} + \tfrac{1}{2}b(\cos\theta\,\vec{j} - \sin\theta\,\vec{i})\right] \times ma\vec{i}$$

$$\vec{k}: \quad \tfrac{1}{2}mgb\sin\theta - Tb\cos\tfrac{1}{2}\theta = -\tfrac{1}{2}mab\cos\theta \quad \Rightarrow \quad T = \frac{m(a\cos\theta + g\sin\theta)}{2\cos\frac{1}{2}\theta}$$

$$\vec{i}: \quad c_1 + \tfrac{1}{2}mgL - TL\sin\tfrac{1}{2}\theta = 0 \quad \Rightarrow \quad c_1 = TL\sin\tfrac{1}{2}\theta - \tfrac{1}{2}mgL$$

$$\vec{j}: \quad c_2 + TL\cos\tfrac{1}{2}\theta = \tfrac{1}{2}maL \quad \Rightarrow \quad c_2 = \tfrac{1}{2}maL - TL\cos\tfrac{1}{2}\theta$$

$$\vec{F} = \vec{R}_A - mg\vec{j} + T\left(\cos\tfrac{1}{2}\theta\,\vec{i} + \sin\tfrac{1}{2}\theta\,\vec{j}\right) = ma\vec{i}$$

yields $\vec{R}_A$.

Example 2.47: The rotating system in an amusement park can be modelled as six light bars hinged at one end with a discrete mass m at the other end (Fig. E2.47a). The rotational springs are un-deformed for $\theta = 0$. The motor is shut-off when the system is in position θ_0 and has angular velocities ω_0 and $\dot\theta_0$. For the subsequent position θ, find ω, $\dot\theta$, $\dot\omega$, $\ddot\theta$ and the bearing reactions at A and O. Set up an equation for $\theta_{max} = \theta_1$ (say). Neglect the friction in the bearings.

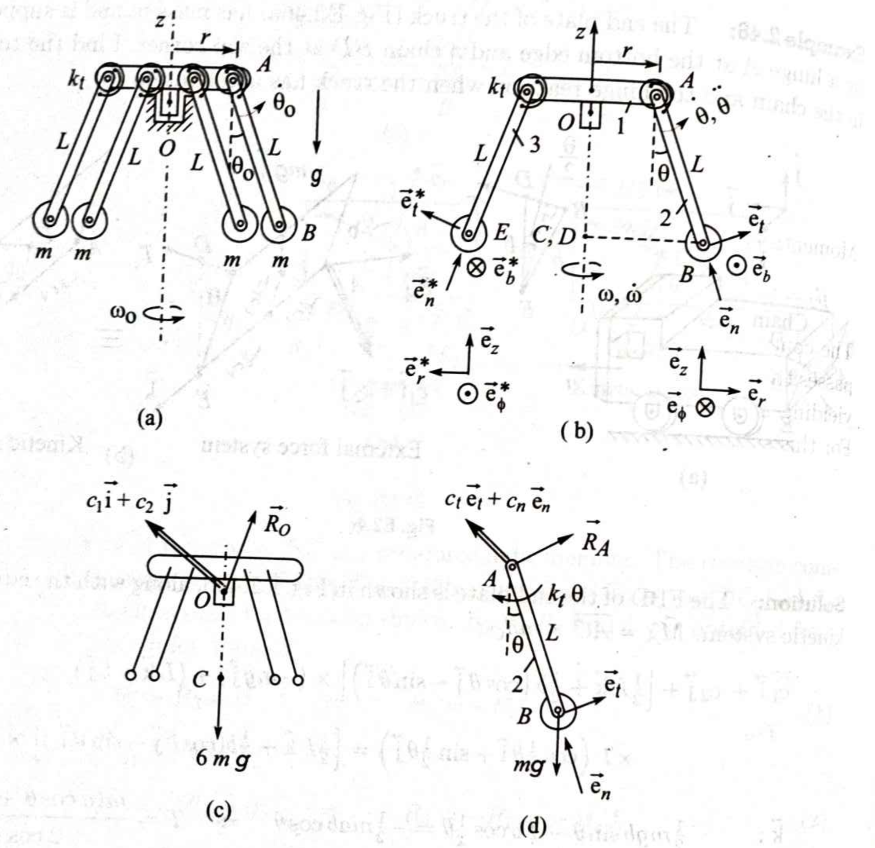


Fig. E2.47

Solution: From Fig. E2.47b, for point B: $\overrightarrow{DB} = (r + L\sin\theta)\,\vec{e}_r$,

$\vec{r}_C = -L\cos\theta\,\vec{k}$, $\quad \vec{v}_C = L\sin\theta\,\dot\theta\,\vec{k}$, $\quad \vec{a}_C = L(\cos\theta\,\dot\theta^2 + \sin\theta\,\ddot\theta)\,\vec{k}$.

For point D on rigid extension of 1: $\vec{v}_D = \vec{0}$, $\quad \vec{a}_D = \vec{0}$.

$$\vec{e}_r = \cos\theta\,\vec{e}_t - \sin\theta\,\vec{e}_n, \qquad \vec{e}_\phi = -\vec{e}_b, \qquad \vec{e}_z = \cos\theta\,\vec{e}_n + \sin\theta\,\vec{e}_t$$

$$\vec{v}_{B|1} = L\dot\theta\,\vec{e}_t, \qquad \vec{a}_{B|1} = L\ddot\theta\,\vec{e}_t + L\dot\theta^2\,\vec{e}_n$$

$$\vec{v}_B = \vec{v}_D + \omega\,\vec{e}_z \times \overrightarrow{DB} + \vec{v}_{B|1} = \omega(r + L\sin\theta)\,\vec{e}_\phi + L\dot\theta\,\vec{e}_t$$

$$\begin{aligned}
\vec{a}_B &= \vec{a}_D + \dot\omega\,\vec{e}_z \times \overrightarrow{DB} - \omega^2\overrightarrow{DB} + 2\omega\,\vec{e}_z \times \vec{v}_{B|1} + \vec{a}_{B|1} \\
&= \dot\omega(r + L\sin\theta)\,\vec{e}_\phi - \omega^2(r + L\sin\theta)\,\vec{e}_r \\
&\quad + 2\omega(\cos\theta\,\vec{e}_n + \sin\theta\,\vec{e}_t) \times L\dot\theta\,\vec{e}_t + L\ddot\theta\,\vec{e}_t + L\dot\theta^2\,\vec{e}_n \\
&= [\ddot\theta L - \omega^2(r + L\sin\theta)\cos\theta]\,\vec{e}_t + [L\dot\theta^2 + \omega^2(r + L\sin\theta)\sin\theta]\,\vec{e}_n \\
&\quad - [\dot\omega(r + L\sin\theta) + 2\omega L\dot\theta\cos\theta]\,\vec{e}_b
\end{aligned}$$

Momenta $\vec{p}_2$, $\vec{p}_3$ of bars 2 and 3 (Fig. E2.47b) are given by

$$\vec{p}_2 = m\vec{v}_B = m\omega(r + L\sin\theta)\,\vec{e}_\phi + mL\dot\theta\,\vec{e}_t, \qquad \vec{p}_3 = m\omega(r + L\sin\theta)\,\vec{e}_\phi^* + mL\dot\theta\,\vec{e}_t^*$$

The contributions of $\vec{e}_t$, $\vec{e}_t^*$ components of $\vec{p}_1$, $\vec{p}_2$ to $\vec{H}_O$ cancel out as their resultant passes through O, whereas the contributions of their $\vec{e}_\phi$, $\vec{e}_\phi^*$ components to $\vec{H}_O$ add up yielding $2m\omega(r + L\sin\theta)^2\,\vec{e}_z$. FBD of whole system of 6 bars is shown in Fig. E2.47c. For this system (considering 3 pairs of bars):

$$\vec{H}_O = 6m(r + L\sin\theta)^2\omega\,\vec{k} = H_{Oz}\,\vec{k}$$

$$\dot{\vec{H}}_O = \vec{M}_O: \qquad \dot{H}_{Oz}\,\vec{k} = c_1\,\vec{i} + c_2\,\vec{j} \;\Rightarrow\; c_1 = c_2 = \dot{H}_{Oz} = 0 \;\Rightarrow$$

$$H_{Oz} = 6m(r + L\sin\theta)^2\omega = 6m(r + L\sin\theta_0)^2\omega_0 \tag{1}$$

$$\Rightarrow \quad \omega = (r + L\sin\theta_0)^2\omega_0/(r + L\sin\theta)^2$$

$$\begin{aligned}
T + V &= \tfrac{1}{2}m\left[\omega^2(r + L\sin\theta)^2 + L^2\dot\theta^2\right] \times 6 - 6mgL\cos\theta + \tfrac{1}{2}k_t\theta^2 \times 6 = E_0 \\
&= 3m\left[\omega_0^2(r + L\sin\theta_0)^2 + L^2\dot\theta_0^2\right] - 6mgL\cos\theta_0 + 3k_t\theta_0^2
\end{aligned} \tag{2}$$

$$\Rightarrow \quad \dot\theta^2 = \dot\theta_0^2 + 2g(\cos\theta - \cos\theta_0)/L + k_t(\theta_0^2 - \theta^2)/mL^2$$
$$\qquad + \omega_0^2\,(r/L + \sin\theta_0)^2\left\{1 - \frac{(r + L\sin\theta_0)^2}{(r + L\sin\theta)^2}\right\}$$

$\dot\omega, \ddot\theta$ are obtained by differentiating eqs(1) and (2): $\dot{H}_{Oz} = 0$, $\dot{T} + \dot{V} = 0$,

$$(r + L\sin\theta)^2\dot\omega + 2(r + L\sin\theta)L\cos\theta\,\dot\theta\,\omega = 0 \tag{3}$$

$$m[\omega\dot\omega(r + L\sin\theta)^2 + \omega^2(r + L\sin\theta)L\cos\theta\,\dot\theta + L^2\dot\theta\,\ddot\theta] + mgL\sin\theta\,\dot\theta + k_t\theta\,\dot\theta = 0 \tag{4}$$

Substituting $\dot{\omega}$ from eq.(3) in eq.(4) yields

$$mL[L\ddot{\theta} - \omega^2(r + L\sin\theta)\cos\theta] + mgL\sin\theta + k_t\theta = 0 \tag{5}$$

This equation can be obtained directly from $\vec{M}_A \cdot \vec{e}_b = (\overrightarrow{AB} \times m\vec{a}_B) \cdot \vec{e}_b$ for bar 2. For the system, $\vec{F} = 6m\vec{a}_C$ yields

$$\vec{R}_O = [6mg + 6mL(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta})]\,\vec{k}$$

FBD of bar 2 is shown in Fig. E2.47d and $\vec{M}_A = \overrightarrow{AB} \times m\vec{a}_B \Rightarrow$

$$c_t\,\vec{e}_t + c_n\,\vec{e}_n - (mgL\sin\theta + k_t\theta)\,\vec{e}_b = -L\,\vec{e}_n \times m\vec{a}_B$$

$\vec{e}_n :\quad c_n = 0$

$\vec{e}_t :\quad c_t = -mL[\dot{\omega}(r + L\sin\theta) + 2\omega L\dot{\theta}\cos\theta]$

$\vec{e}_b :\quad -mgL\sin\theta - k_t\theta = mL[\ddot{\theta}L - \omega^2(r + L\sin\theta)\cos\theta]$

which is the same as eq. (5) derived earlier from $\dot{T} + \dot{V} = 0$, $\dot{H}_{O_z} = 0$. $\vec{R}_A$ is obtained using $\vec{F} = m\vec{a}_B$ for bar 2:

$$\vec{R}_A - mg(\cos\theta\,\vec{e}_n + \sin\theta\,\vec{e}_t) = m\vec{a}_B$$

At $\theta = \theta_1$: $\dot{\theta} = 0$, $\omega_1 = (r + L\sin\theta_0)^2\omega_0/(r + L\sin\theta_1)^2$, eq. (2) for $\theta = \theta_1$ yields:

$$\tfrac{1}{2}m\frac{(r + L\sin\theta_0)^4\omega_0^2}{(r + L\sin\theta_1)^2} \times 6 - 6mgL\cos\theta_1 + 3k_t\theta_1^2 = E_0$$

Example 2.48: Find the applied force P so that the system translates with the pendulum remaining vertical (Fig. E2.48a). The spring is un-deformed when the pendulum is at right angles to the inclined plane. Find the reactions at the pin and from the inclined plane. The centre of mass of the block is at O.

Solution: The FBD of the pendulum and its equivalent kinetic system (Fig. E2.48b)

$$\Rightarrow\quad M_{O_z} = k_t\theta = m_1 a \tfrac{1}{2}L\cos\theta \quad\Rightarrow\quad a = 2k_t\theta/m_1 L\cos\theta$$

$$F_x = R_1 - m_1 g\sin\theta = m_1 a \quad\Rightarrow\quad R_1 = m_1 g\sin\theta + 2k_t\theta/L\cos\theta$$

$$F_y = R_2 - m_1 g\cos\theta = 0 \quad\Rightarrow\quad R_2 = m_1 g\cos\theta$$

The FBD for the system and its equivalent kinetic system are shown in Fig. E2.48c.

$$F_y = N - (m + m_1)g\cos\theta = 0 \quad\Rightarrow\quad N = (m + m_1)g\cos\theta$$

$$F_x = P - (m + m_1)g\sin\theta - \mu N = (m + m_1)a$$

$$\Rightarrow\quad P = (m + m_1)[g\sin\theta + \mu g\cos\theta + 2k_t\theta/m_1 L\cos\theta]$$

$$M_{O_z} = (\eta - b)N - \mu N d - P(h - d) = m_1 a(\tfrac{1}{2}L\cos\theta) \qquad \text{yields } \eta.$$

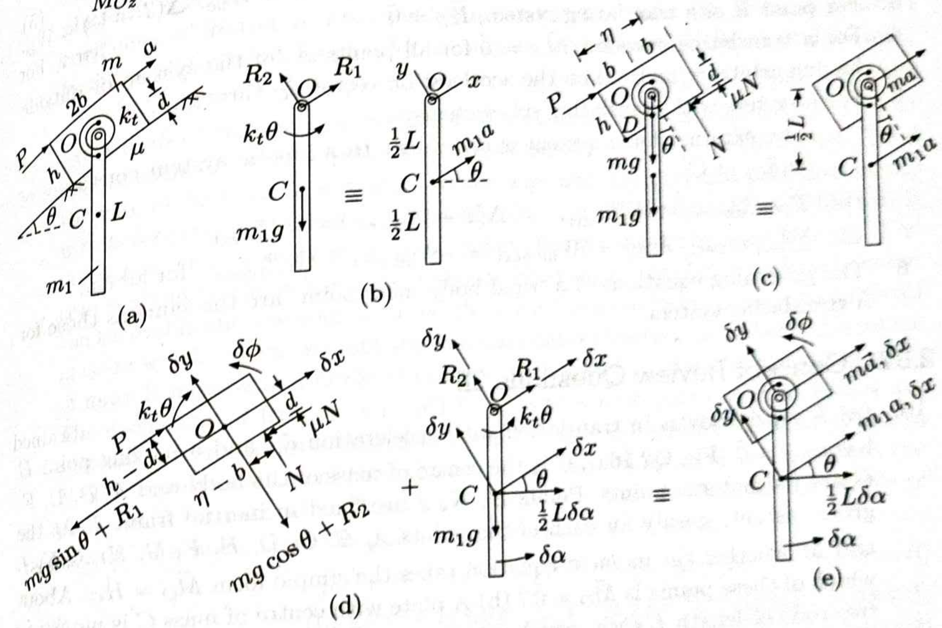


Fig. E2.48

Alternative solution using virtual work. Consider arbitrary virtual displacements $\delta x, \delta y$ of O, virtual rotation $\delta\phi$ of the block about O and virtual rotation $\delta\alpha$ of the pendulum about O. We equate the virtual work done by the internal and external loads mg, $m_1 g$, P, N, μN, $\pm R_1$, $\pm R_2$, $\pm k_t\theta$ (Fig. E2.48d) to that of the equivalent kinetic system (Fig. E2.48e). The virtual work of $\pm R_1$, $\pm R_2$ is zero.

$$\begin{aligned}\delta W &= P[\delta x - (h - d)\delta\phi] - mg\sin\theta\,\delta x - mg\cos\theta\,\delta y - \mu N(\delta x + d\,\delta\phi)\\ &\quad + N[\delta y + (\eta - b)\delta\phi] - k_t\theta\,\delta\phi - m_1 g(\delta x\sin\theta + \delta y\cos\theta) + k_t\theta\,\delta\alpha\\ &= ma\delta x + m_1 a(\delta x + \tfrac{1}{2}L\,\delta\alpha\cos\theta)\end{aligned}$$

The coefficients of δx, δy, $\delta\alpha$, $\delta\phi$ yield the governing equations obtained earlier.

Review 2B

1. $\dot{T}_C + \dot{V}^* = \dot{W}^*_{nc}$, $\Delta(T_C + V^*) = W^*_{nc}$
2. For a translating system: $\vec{F} = m\vec{a}_C$, $\vec{M}_C = \vec{0}$, $\dot{T} + \dot{V} = \dot{W}_{nc}$, $\Delta(T+V) = W_{nc}$
3. For point B of a translating system, $\vec{H}_B = \vec{0}$
4. For a translating system: $\vec{M}_A = \vec{0}$ for all points A (of the system or outside having arbitrary motion) on the acceleration vector $\vec{a}_C$ through C.
5. For a system with translating sub-systems:
   i. The external force system is equivalent to a kinetic system consisting of $m_i\vec{a}_{C_i}$ at C_i.
   ii. $\dot{T} + \dot{V}_{int+ext} = \dot{W}^{nc}_{int+ext}$, $\Delta(T + V_{int+ext}) = W^{nc}_{int+ext}$
   iii. $\sum_{j=1}^{p} m_j\vec{a}_{C_j} \cdot \delta\vec{r}_{C_j} = \delta W_{int+ext} = -\delta V_{int+ext} + \delta W^{nc}_{int+ext}$ for all δq_i.
6. The governing equations of a 'rigid body mass-point' are the same as those for a translating system.

2.32 Concept Review Questions 2B

16. (a) A rigid body is in translation with acceleration $\vec{a}_{|I}$ and a moving point D has $\vec{a}_{D|I} = \vec{0}$ (Fig. Q2.16a). C is the centre of mass of the body and A, B, E, F, G are its material points. Points H, N, J are fixed in inertial frame I. At the given instant, specify for each of the points $A, B, C, D, E, F, G, H, J, K, L$ and M whether the moment equation takes the simple form $\vec{M}_O = \dot{\vec{H}}_O$. About which of these points is $\vec{M}_O = \vec{0}$? (b) A plate with centre of mass C is pinned to two rods of length L each, which rotate in a vertical plane at constant angular velocity ω (Fig. Q2.16b) with $AB = b$. About which of the points A, C, O, P, Q, R, S or T is the moment zero for the external forces acting on the plate?
[(a) $C, D, E, F, H, J, L, M, N$; C, E, F, L, K, N (b) C, P, S, T

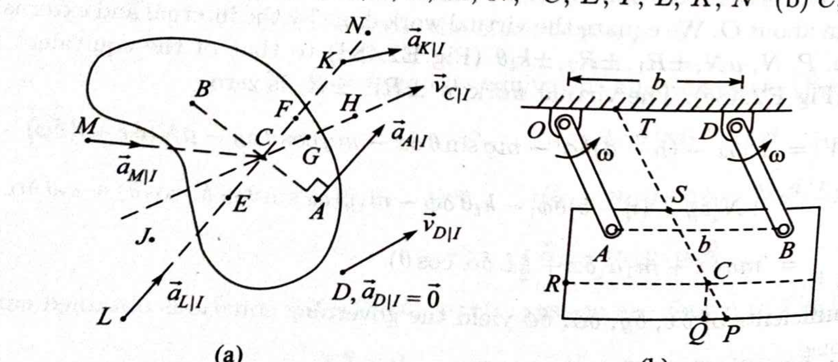


Fig. Q2.16

17. A mechanical shaker in a lift moving up with acceleration a_0 provides simple harmonic motion of amplitude A and circular frequency ω rad/s to a horizontal platform relative to the lift. A block of mass m rests on the platform with coefficient of friction μ. Find the minimum value of A for the block to separate from the platform or slip on it for (a) vertical shaking and (b) horizontal shaking.
[(a) $(g + a_0)/\omega^2$ (b) $(g + a_0)\mu/\omega^2$]

18. (a) A bead of mass m and coefficient of friction μ slides on a horizontal circular wire of radius R with speed v. The wire is translating up at acceleration a. Find the rate of decrease of v relative to the wire and the distance travelled on the wire before coming to rest. (b) A small coin is on a turntable (Fig. Q2.18) which is rotating at ω, $\dot{\omega}$ relative to an aeroplane translating with acceleration a. The coin is about to slip at the instant shown. Find the coefficient of friction between the turntable and the coin. (c) Find the total frictional force on an automobile of mass m which travels without slip in a horizontal plane around a bend. Its centre of mass is moving in a circle of radius r with speed v, which is increasing at $\dot{v}$. The coefficient of friction is μ. (d) In an amusement park, wooden horses are suspended from cables attached to periphery of a horizontal platform of radius R. The steady position of the cables is at an angle of θ to the vertical when the platform is rotating at ω about its vertical axis. The distance of the centre of mass C of a horse from the point of suspension is L. Find ω.
[(a) $\mu[(g+a)^2 + v^4/R^2]^{1/2}$, $(R/2\mu)\sinh^{-1}[v^2/(g+a)R]$
(b) $[\dot{\omega}^2r^2 + (g\sin\theta - \omega^2 r - a\cos\theta)^2]^{1/2}/(g\cos\theta + a\sin\theta)$
(c) $m[\dot{v}^2 + (v^4/r^2)]^{1/2}$ (d) $\{g\tan\theta/(R + L\sin\theta)\}^{1/2}$]

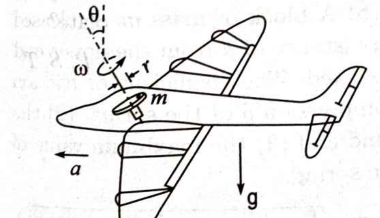


Fig. Q2.18

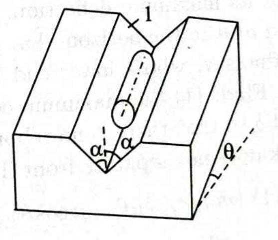


Fig. Q2.19

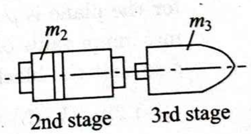


Fig. Q2.20

19. (a) Find the acceleration of a uniform cylinder which slides down a trough inclined at angle θ with the horizontal. The angle between the two flats of the trough is 2α (Fig. Q2.19) and these are symmetrical with respect to the vertical mid-plane. The coefficient of friction is μ. (b) Rework part (a) if face 1 is vertical.
[(a) $g\sin\theta - \mu g\cos\theta/\sin\alpha$ (b) $g\sin\theta - \mu g\cos\theta\cot\alpha$]

20. A rocket is moving at v_0 when the third stage engine is ignited by firing explosives, separating the 2nd and 3rd stages (Fig. Q2.20) in a time interval t_1. The stages separate at a velocity v_s. Assume that only η% of the chemical energy of explosives gets converted into mechanical energy. Neglect all forces other than that due to explosives. Find the final velocity of the 3rd stage, the average thrust on it during separation and the amount of chemical energy consumed.

$$[v_0 + m_2v_s/(m_2+m_3),\ m_2m_3v_s/(m_2+m_3)t_1,\ m_2m_3v_s^2/2(m_2+m_3)\eta]$$

21. A boy is riding a cycle at 10 m/s with the wheels rolling without slip when another boy running at 4 m/s in the same direction jumps on without a vertical component of velocity. The mass of each boy is 40 kg and that of the cycle is 20 kg. The radius of the wheel is 0.5 m. Find the speed of the cycle and angular speed of the wheels just after the boy gets on it. [7.6 m/s, 20 rad/s]

22. (a) A train has 3 cars, each of mass m and an engine of mass M. It is going up an incline, at elevation θ at speed v_0 when power is shut off and brakes are applied. The coefficient of friction is μ. The brakes are functioning only on the engine and the middle car and their wheels skid. Find the time t in which the train comes to rest. Assume that the couplings transmit only axial forces. (b) A driver moving at high speed v suddenly sees a blind T-crossing at distance d ahead. Which of the three options is the best, i.e., requires the least μ? (1) To try to stop by using brakes. (2) To try to turn in the maximum possible radius of d. (3) To try to stop for distance $\frac{1}{2}d$ and then take a turn of radius $\frac{1}{2}d$.

[(a) $(3m+M)v_0/g[(3m+M)\sin\theta + \mu(M+m)\cos\theta]$ (b) case (1) is the best]

23. (a) A block of mass m is released at the top of an un-deformed vertical spring of stiffness k. Find its maximum deflection. (b) A block of mass m is released on a plane inclined at θ to the horizontal at a distance of d from the upper end of a spring of stiffness k, whose lower end is fixed. The coefficient of friction for the plane is μ. Find: (1) the maximum compression δ of the spring, (2) the maximum value of δ so that there is no rebound and (3) the maximum value of δ so that the block does not separate from the spring.

[(a) $2mg/k$ (b) (1) $(mg/k)(\sin\theta - \mu\cos\theta)[1+\sqrt{1+2dk/mg(\sin\theta-\mu\cos\theta)}]$, (2) $mg(\sin\theta+\mu\cos\theta)/k$, (3) $2mg(\sin\theta+\mu\cos\theta)/k$]

24. A gunner can exert an average force of 175 N against the gun. Find the maximum number n of bullets of 50 g each at 1000 m/s that he can fire per minute. [210]

25. A man sits at the front right end of a stationary barge. He rises up and moves to the back left end of the barge and sits down again. The water resistance on the barge is cv, where v is the speed of the barge. Where will the barge come to rest? [Hint: $\Delta\vec{p} = \vec{0} = \int \vec{F}\,dt$] [At initial position]

26. Find the path of the block on a smooth inclined plane (Fig. Q2.26). Find d and when it hits the bottom. [$y = g\sin\theta(x^2/2v_0^2)$, $d = v_0\sqrt{2L/g\sin\theta}$, $\sqrt{2L/g\sin\theta}$]

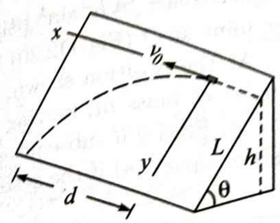


Fig. Q2.26

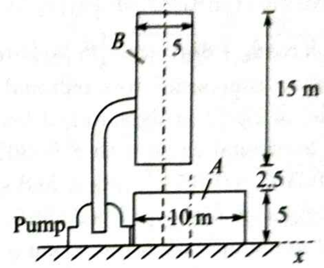


Fig. Q2.27

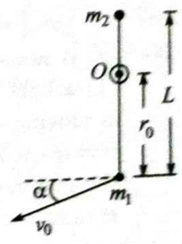


Fig. Q2.28

27. A cylindrical tank A is initially full of water and the cylindrical tank B is empty. Water is pumped from A to B (Fig. Q2.27). Initially the flow rate is 1 m^3/s, which increases at the rate of 0.1 m^3/s^2 for 30 s thereafter. What is the average force on the tanks and piping system from the water during this period aside from the weight of the water? [$-250\vec{i} + 886.7\vec{j}$ N]

28. Two mass-points with mass m_1, m_2 rest on a smooth horizontal plane and are connected by a string which passes through a small fixed smooth ring at O (Fig. Q2.28). A horizontal velocity as shown is given to m_1. In their subsequent motion, find v_r, v_ϕ of m_1 when it is at distance r from O.

$$[[(m_1 + m_2\sin^2\alpha - m_1r_0^2\cos^2\alpha/r^2)/(m_1+m_2)]^{1/2}v_0,\ r_0v_0\cos\alpha/r]$$

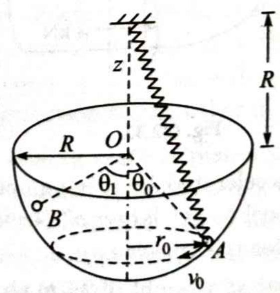


Fig. Q2.29

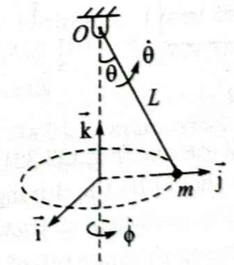


Fig. Q2.30

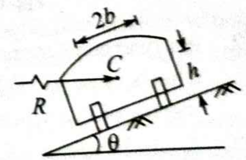


Fig. Q2.31

29. A mass-point is projected horizontally with speed v_0 at A on the inside of a smooth spherical surface of radius R (Fig. Q2.29). The spring stiffness is k and its free length is $2R$. (a) Find v_0 if it moves in a horizontal circle. (b) If it does

not move in a circle, set up two equations for finding the extreme angle θ_1 of its trajectory and the corresponding speed v_1. [with $P = 4kR\sin^2\frac{1}{4}\theta_0$

(a) $\left(\{(mg+P\cos\frac{1}{2}\theta_0)\tan\theta_0 - P\sin\frac{1}{2}\theta_0\}r_0/m\right)^{1/2}$ (b) $mR\sin\theta_1 v_1 = mr_0v_0$ & $\frac{1}{2}mv_1^2 - mgR\cos\theta_1 + 8kR^2\sin^4\frac{1}{4}\theta_1 = \frac{1}{2}mv_0^2 - mgR\cos\theta_0 + 8kR^2\sin^4\frac{1}{4}\theta_0]$

30. (a) A mass-point m is attached to a ball and socket joint at O (Fig. Q2.30) by (1) a light cable, or by (2) a light rod of length L. At the position shown, m is moving in a horizontal circle with $\theta = 30°$. A bullet of mass m, moving at $\vec{v} = v(-0.5\vec{i} + 0.707\vec{j} - 0.707\vec{k})$, hits it and sticks to it. Find v if subsequently the maximum height of A above O is $0.866L$. (b) Rework part (a) if the system is in an elevator going up with acceleration $a = \frac{1}{2}g$.

[For both (1) and (2): (a) $14.39\sqrt{gL}$ (b) $17.62\sqrt{gL}$]

31. A vehicle of mass m moves on a banked road (Fig. Q2.31), in a circle of radius R at speed v. The coefficient of friction for the ground is μ. Neglect the inertia of the wheels and the rotational inertia of the main body of the vehicle. (a) Find v for which the friction force is zero. (b) Find the range of v for which there is no slip. (c) Find the condition of tipping and show that tipping occurs before slip if $\mu > b/h$. [Let $s = \sin\theta$, $c = \cos\theta$ (a) $\sqrt{gR\tan\theta}$

(b) $\left[\dfrac{gR(s-\mu c)}{c+\mu s}\right]^{1/2} < v < \left[\dfrac{gR(\mu c+s)}{c-\mu s}\right]^{1/2}$ (c) $v^2 > gR(bc+sh)/(ch-sb)$]

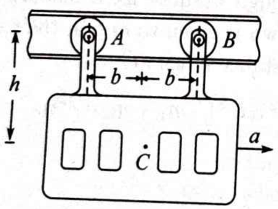


Fig. Q2.32

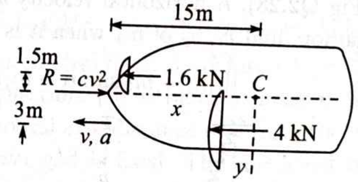


Fig. Q2.33

32. An overhead mono-rail system (Fig. Q2.32) is accelerated by driving one of its wheels A or B. Which should be the driving wheel to get larger a? Neglect the inertia of the wheels. The coefficient of friction for the wheels is μ. [B]

33. A wind force of 4 kN acts on the main sail of a boat at a height of 2.5 m above the level of centre of mass C of the boat (Fig. Q2.33). The wind force on the minor sail is 1.6 kN at a height of 2 m above the level of C. The water resistance $R = cv^2$ acts at 0.25 m above C. The weight of the boat is 800 kg and $a = 0.8\,m/s^2$. Find R, the buoyancy force B and the resultant of the lateral forces on the boat.

[$R = 4960\,N$, $B = 7848\,N$ at 1.524 m ahead of C, $\vec{F}_R = \vec{0}$ & $\vec{C}_{R_C} = 9600\vec{k}$ N.m]

34. A jet-propelled sled with light wheels moves over an inclined plane (Fig. Q2.34) Find the maximum value of the thrust T so that there is no lift-off. [$2.165mg$]

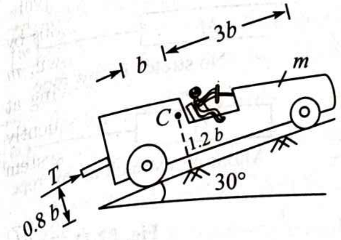


Fig. Q2.34

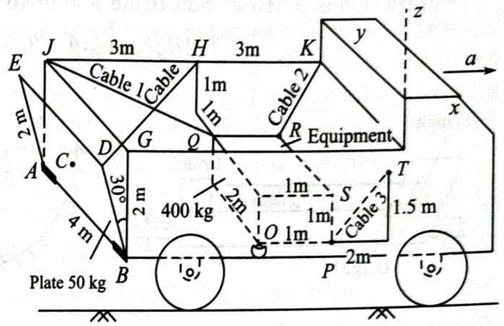


Fig. Q2.35

35. A truck is moving with acceleration $a = 0.6\,m/s^2$ (Fig. Q2.35). (a) A hinged plate of mass 50 kg at the back is restrained by cable DH. Find the cable tension and the hinge reactions. The hinge at A does not provide axial restraint. (b) An equipment of mass 400 kg carried by the truck is supported by a ball and socket joint at O and three cables (JQ, KR and PT). Find the reaction at O.

[(a) 213.4, $-22.69\vec{i}+89.80\vec{k}$, $-98.08\vec{i}-150.8\vec{j}+390.6\vec{k}$ (b) $-2059\vec{i}-3924\vec{j}-642\vec{k}$ N]

2.33 Practice Problems 2B

18. An athlete is running at speed v_1 when he throws the javelin at speed v_0 relative to himself at angle α with the horizontal. Prove that the range is maximum if $\cos\alpha = -v_1/4v_0 + [(v_1/4v_0)^2 + \frac{1}{2}]^{1/2}$. Neglect the height of the athlete in comparison to the other dimensions.

19. A 1000 kg crate is placed on a 4000 kg barge, which is equipped with a winch for pulling the crate along its smooth deck (Fig. P2.19). Initially the barge and the crate are at rest relative to the water. (a) Find their velocities when the winch is drawing the load towards it at 5 m/s. (b) Find the position of the barge when 10 m of the rope has been drawn in. (c) Find the velocities of the crate and the barge after the winch stops operating. (d) Rework parts (a), (b), (c) if the coefficient of friction for the crate and the deck is 0.4. Neglect water resistance.

[(a) & (c) $\vec{i}$ m/s, $-4\vec{i}$ m/s (b) $2\vec{i}$ m (d_a) $\vec{i}$ m/s, $-4\vec{i}$ m/s (d_b) $2\vec{i}$ m (d_c) $(1-0.1gt)\vec{i}$, $(0.4gt-4)\vec{i}$ for $t \le 1.019$ s, no motion after 1.019 s]

20*. A small packet is released from rest at position 1 and slides down a circular path of radius R (Fig. P2.20). Find its velocity v at position 2. The coefficient of friction is μ. [Hint: Substitute $u = v^2$ to solve the differential equation for v.]

$[\,[2Rg\{3\mu\cos\theta - 3\mu e^{-2\mu\theta} + (1-2\mu^2)\sin\theta\}/(1+4\mu^2)]^{1/2}]$

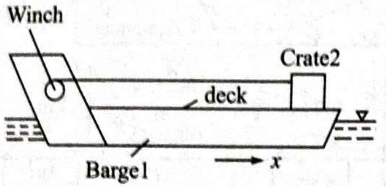


Fig. P2.19

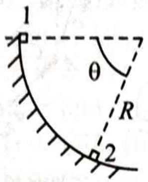


Fig. P2.20

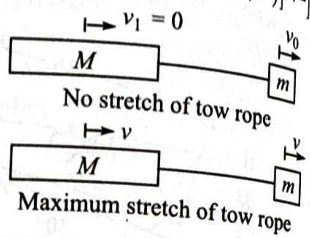


Fig. P2.21

21. A tugboat of mass m moving at v_0 starts towing a stationary ship of mass M (Fig. P2.21). Neglect the impulse of the propeller thrust of the tugboat and the resistance of water during the period the tow-rope is stretched to its maximum when the boat and the ship have common velocity. The tow-rope extends e m/m length for 1 kN load and its allowable load is P_0. Prove that the minimum required length L of the rope is: $L = 1000(mv_0^2/eP_0^2)M/(M+m)$.

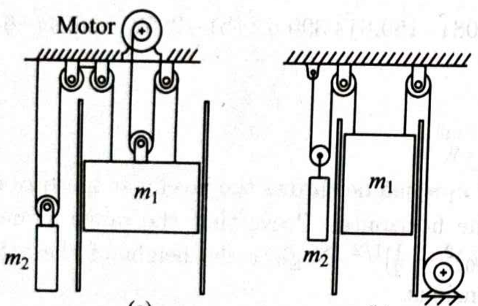


Fig. P2.22

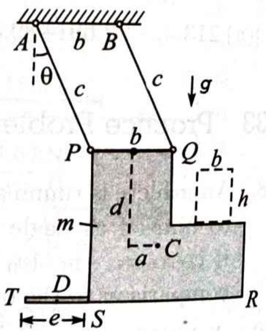


Fig. P2.23

22. Find the power that must be supplied to the motor of an elevator if its efficiency is η and the elevator is going up with velocity v and acceleration a. Neglect the inertia of the pulleys and friction at their axles. Solve for the two cases shown in Fig. P2.22.

[(a) & (b) $[(m_1 - \frac{1}{2}m_2)g + (m_1 + \frac{1}{4}m_2)a]v/\eta$]

23. (i) Model tests of a control panel $PQRS$ of mass m are conducted by using two light bars AP and BQ (Fig. P2.23). The support at A is frictionless and a frictional couple M_f acts at support at B. The system is released from rest at $\theta = \theta_0$. Find (a) the support reactions at release and (b) velocity, acceleration and the support reactions at position θ. If $\dot\theta = \omega$, $\ddot\theta = \dot\omega$ in position θ, find the normal force, shear force and bending moment at the middle D of the bar ST of mass m_1. (c) Find the minimum speed to be imparted to the panel at $\theta = 0$ so that it makes a complete circle. (ii) A cuboidal block of height h and base b rests on the panel. The bars are rotated at constant ω. Find the minimum value of ω for the impending motion of the block relative to the panel. The coefficient of static friction is μ_s. [Hint: Consider sliding, tipping and separation.]

[(i) (b) $v = [2cg(\cos\theta - \cos\theta_0) - \frac{2M_f}{m}(\theta_0 - \theta)]^{1/2}$, $\vec a = -\frac{v^2}{c}\vec e_r + (\frac{M_f}{mc} - g\sin\theta)\vec e_\theta$,
$\vec R_A = -\frac{1}{2}(P + mg\cos\theta + \frac{mv^2}{c})\vec e_r$, $\vec R_B = -\frac{1}{2}(mg\cos\theta + \frac{mv^2}{c} - P)\vec e_r + \frac{M_f}{c})\vec e_\theta$
$P = [2m(g\cos\theta + v^2/c)(d\sin\theta - a\cos\theta) + M_f\{(b-2a)\sin\theta - 2d\cos\theta\}/c]/b\cos\theta$,
$N = \frac{1}{2}m_1(\dot\omega c\cos\theta - \omega^2 c\sin\theta)$, $S = \frac{1}{2}m_1(g + \dot\omega c\sin\theta + \omega^2 c\cos\theta)$, $M = -.25Se$
(a) as in (b) with $v = 0$, $\theta = \theta_0$ (c) $4gc + 2\pi M_f/m$
(ii) $[\min\{1,\ \mu^2/(1+\mu^2),\ b^2/(b^2+h^2)\}]^{1/4}\sqrt{g/c}$]

24. (a) A body of mass m and area of cross-section A falls from a height h above an absorbing medium and comes to rest after sinking a distance δ. Find the average pressure p_{av} from the medium on the body. (b) A person of mass 60 kg jumps from the window of a burning building into a life net 10 m below. How much should the net sink to avoid injury which occurs for $p > 20\,\mathrm{N/cm^2}$? $A = 0.2\,\mathrm{m}^2$ and assume that $p_{max} \simeq 3p_{av}$. (c) An elastic resistance band of stiffness k is stretched horizontally with a large initial tension T_0 (Fig. P2.24). It catches a descending parcel of mass m. Find h_{max} from which the parcel can fall if the maximum force in the band: $T_{max} \le F_m$. [(a) $(1 + h/\delta)mg/A$ (b) > 0.4618 m
(c) $(F_m^2 - T_0^2)/2mgk - [\{(F_m - T_0)/2k + a\}^2 - a^2]^{1/2}$]

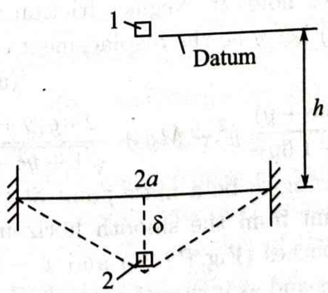


Fig. P2.24

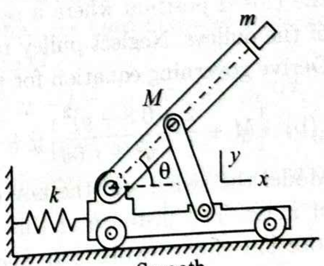


Fig. P2.25

25. A gun of mass M with a recoil spring of stiffness k fires a shell of mass m with velocity v_0 with respect to the barrel (Fig. P2.25). Assume that the recoil velocity

of the gun is instantaneously achieved and the ground is smooth. Find (a) the maximum compression of the spring, (b) the impulsive reaction from the ground and (c) the ratio of the kinetic energy T_s of the shell to the total kinetic energy T just after firing. Show that $T_s/T \simeq 1$ if $m/M \ll 1$. [(a) $mv_0 \cos\theta\sqrt{M/k}/(M+m)$ (b) $mv_0 \sin\theta$ (c) $1/[1+m/M\{1+\tan^2\theta(1+m/M)^2\}]$]

26. A compressed air gun consists of a reservoir of volume V_0 containing compressed air at pressure p_0 (Fig. P2.26). When fired, the air expands according to $pV^n =$ constant, to propel a striker of mass m in a barrel of length L and cross-sectional area A. The coefficient of friction for the barrel is μ. The atmospheric pressure is p_a. Find the speed of the striker at exit.

$$[\,[2\{p_0V_0\{(1+AL/V_0)^{1-n}-1\}/(1-n)-p_aAL-\mu mgL\}/m]^{1/2}]$$

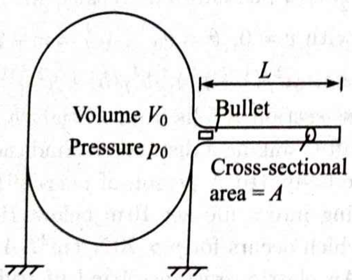


Fig. P2.26

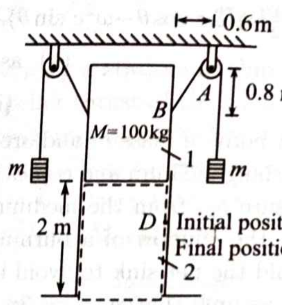


Fig. P2.27

27. (a) A 100 kg overhead door (Fig. P2.27) is held in open position by a catch. Design counterweights so that when the catch is released, the door gently reaches the closed position where a second catch holds it. Neglect friction and inertia of the pulleys. Neglect pulley radius. (b) Let y be the displacement of the door. Derive governing equation for y. [(a) 53.65 kg

(b) $\left[M+\dfrac{2m(0.8+y)^2}{1+y^2+1.6y}\right]\ddot{y}+\dfrac{0.72m(0.8+y)}{(1+y^2+1.6y)^2}\dot{y}^2-Mg+\dfrac{2mg(y+0.8)}{\sqrt{1+y^2+1.6y}}=0$]

28. Model the swing and the boy on the swing to be a mass-point at their centre of mass. The distance of this mass-point from the smooth horizontal axis of rotation of the swing is L when he crouches (Fig. P2.28) and $L-h$ when he stands up. As the swing falls, he crouches and as it rises, he stands. Assume that the changeover to standing position is instantaneous at the vertical position of the swing and changeover to crouching position at the extreme positions of the swing is also instantaneous. The maximum displacement at the start of a cycle is θ_1. Find position θ_3 at the end of one cycle. [$2\sin^{-1}\{\sin(\theta_1/2)/(1-h/L)^3\}$]

29. The system shown in Fig. P2.29 is rotating about a fixed vertical axis at ω_0. A catch keeps the collar of mass m_2 from moving relative to the shaft. The catch is removed. When the collars of mass m_1 have moved out to radius r, find their velocity v and acceleration a relative to the shaft and ω, $\dot{\omega}$ of the shaft. The contacts are smooth. Neglect inertia of the pulleys and the rods, and the rotational kinetic energy of m_2.

$[v^2 = 2[(r-r_0)(2m_1\cos\theta-m_2)g\operatorname{cosec}\theta+m_1\omega_0^2r_0^2(1-r_0^2/r^2)]/(m_2+2m_1),$
$[(2m_1\cos\theta-m_2)g\operatorname{cosec}\theta+2m_1\omega_0^2r_0^4/r^3]\sin\theta/(m_2+2m_1),\ \frac{\omega_0r_0^2}{r^2},\ -\frac{2\omega_0r_0^2v\sin\theta}{r^3}]$

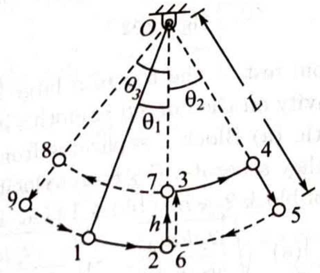


Fig. P2.28

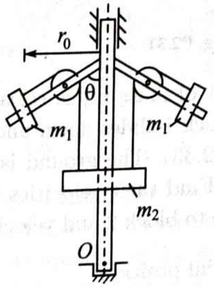


Fig. P2.29

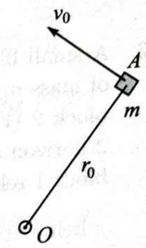


Fig. P2.30

30. A block of mass m rests at A on a smooth horizontal table and is tied to a fixed pin O by an elastic cord of stiffness k (Fig. P2.30). The free length of the cord is $L(< r_0)$. The block is imparted a horizontal speed v_0. (a) Find the radius of curvature of the trajectory of the block at location A (b) Find the speed and transverse component of velocity of the block when it is at a distance $r\,(> L)$ from O. (c) Find $(v_0)_{\min}$ so that the elastic cord remains taut at all times. (d) If $v_0 < (v_0)_{\min}$, find the speed of the block when the cord becomes slack and the subsequent closest distance of the block from O.

[(a) $mv_0^2/k(r_0-L)$ (b) $[v_0^2+k\{(r_0-L)^2-(r-L)^2\}/m]^{1/2}$, v_0r_0/r (c) $[(r_0-L)kL^2/(r_0+L)m]^{1/2}$ (d) $v=[v_0^2+k(r_0-L)^2/m]^{1/2}$, v_0r_0/v]

31. Small sliders, each of mass m, move along smooth slots in a disc which rotates about the vertical axis at constant ω (Fig. P2.31). Each slider starts from rest at $\theta = 0$. Find the reaction of the slot on the slider at θ and the speed of the slider relative to the disc at exit. [$mg\vec{k}+2m\omega^2r\sin\frac{\theta}{2}[3\sin\frac{\theta}{2}-2]\,\vec{e}_n$, $\sqrt{2}\omega r$]

32. Two blocks are connected by a cord. A spring between them, not connected to either, has compression x_0 (Fig. P2.32). The system moves on a smooth horizontal plane with speed v_0. Find the final velocities of the blocks if the cord is severed. [Hint: Choose the frame translating with C to simplify calculations]

$[\vec{v}_1 = v_0\vec{e} - [km_2/m_1(m_1+m_2)]^{1/2}x_0\vec{i},\ \vec{v}_2 = v_0\vec{e} + [km_1/m_2(m_1+m_2)]^{1/2}x_0\vec{i}]$

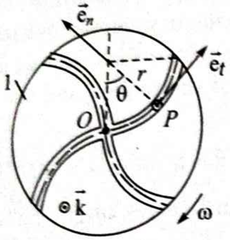


Fig. P2.31

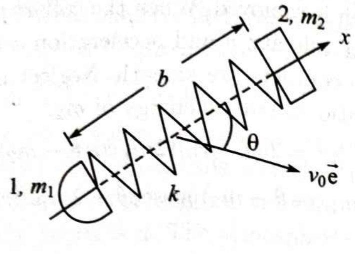


Fig. P2.32

33. A small block 1 of mass m_1 is released from rest at the top of a large block 2 of mass m_2. Block 1 slides down under gravity on the curved smooth surface of block 2 (Fig. P2.33). The ground is smooth. (a) Block 1 separates from block 2 horizontally. Find their velocities when they separate. (b) Find velocity v_1 of block 1 relative to block 2 and velocity v_2 of block 2, when block 1 is at distance z below its initial position. [(a) $\sqrt{\frac{2ghm_2}{m_1+m_2}},\ -\sqrt{\frac{2ghm_1^2}{m_2(m_1+m_2)}}$

(b) $v_1 = \left[\frac{2gz(m_1+m_2)}{m_2+m_1\sin^2\alpha}\right]^{1/2},\quad v_2 = -m_1\cos\alpha\left[\frac{2gz}{(m_1+m_2)(m_2+m_1\sin^2\alpha)}\right]^{1/2}$]

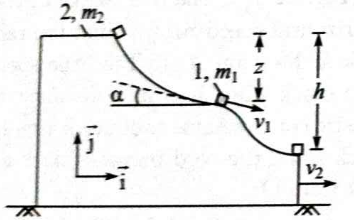


Fig. P2.33

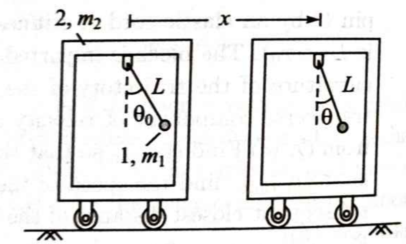


Fig. P2.34

34. A simple pendulum 1 (Fig. P2.34) is suspended in a trolley 2 which has light wheels. It is released from rest at $\theta = \theta_0$. Find the displacement, velocity and acceleration of the trolley for position θ of the pendulum.

$$[\ x = \frac{m_1L(\sin\theta_0 - \sin\theta)}{m_1+m_2},\quad \dot{x} = -\text{sign}(\dot{\theta})\left[\frac{2m_1gL(\cos\theta-\cos\theta_0)}{(m_1+m_2)(\tan^2\theta + m_2\sec^2\theta/m_1)}\right]^{1/2},$$

$$\ddot{x} = \frac{m_1(g\cos\theta + L\dot{\theta}^2)\sin\theta}{m_1\sin^2\theta + m_2}\quad \text{with}\quad \dot{\theta}^2 = \frac{2(m_1+m_2)g(\cos\theta - \cos\theta_0)}{(m_1\sin^2\theta + m_2)L}]$$

35[d]. A ramp is to be designed for a shipping warehouse so that the boxes come to rest within distance b after landing (Fig. P2.35). Assume a rounded transition at corner A. Find the optimum value of b for given $\mu_k = \mu$. However, for a lesser coefficient of friction, they will not come to rest at B. Design an appropriate value of stiffness k so that the boxes, having mass m each, do not rebound and the maximum compression is δ for $\mu_k = 0.5\mu,\ \mu_s = 0.6\mu$. Neglect m_1.

$[b = h(1-\mu\cot\theta)/\mu,\quad mg(h-\mu\delta)/\delta^2\quad \text{with}\quad h/\mu > \delta > h/1.6\mu]$

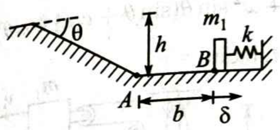


Fig. P2.35

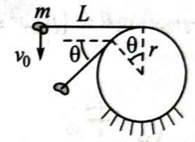


Fig. P2.36

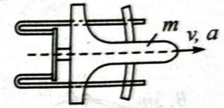


Fig. P2.37

36. (a) A taut light cable (Fig. P2.36) has a small bead of mass m tied to it at one end. Its other end is tied to a fixed disc with horizontal axis. The bead is imparted an initial velocity v_0. Find its speed and the cable tension after the cable has turned through angle θ. Find the force per unit length, p, on the disc from the cable if it is smooth. (b) Rework if the axis of the disc is vertical and the bead moves on a horizontal surface with: (1) Coulomb friction coefficient μ and (2) viscous drag cv. The cable is inextensible.

[(a) $v^2 = v_0^2 + 2g[r(1-\cos\theta) + (L-r\theta)\sin\theta],\quad T = m[g\sin\theta + v^2/(L-r\theta)],\quad T/r$

(b) (1) $v = [v_0^2 - 2\mu g(L\theta - \frac{1}{2}r\theta^2)]^{1/2},\quad T = mv^2/(L-r\theta),\quad T/r$

(2) $v = v_0 - c(L\theta - \frac{1}{2}r\theta^2)/m,\quad T = mv^2/(L-r\theta),\quad T/r$]

37. An aeroplane lands with speed v_0 on a runway of limited length. Brakes are applied and it gets hooked to two chains (mass λ kg/m) lying on the ground (Fig. P2.37). The coefficient of friction for the light wheels is μ and for the chain is μ_1. Find v, a for displacement x. $[a = -(\mu mg + 2\mu_1\lambda xg + 2\lambda v^2)/(m+2\lambda x)$

$$v = [m^2v_0^2 - 2g\{\mu m^2x + \tfrac{4}{3}\mu_1\lambda^2x^3 + m\lambda(\mu+\mu_1)x^2\}]^{1/2}/(m+2\lambda x)]$$

38. (a) Two beads, connected by a light inextensible string, move over a smooth vertical ring (Fig. P2.38). These are released in the given position. Find their initial acceleration. Find their speed v and $\dot{v}$ when the bead A is in position A'. Does it matter whether the string is in contact with the ring? (b) A and B are connected by a light rigid rod. Find their maximum speed. If, instead, the bar AB is at rest in its lowest position, find the minimum velocity that has to be imparted to B so that it makes a complete revolution. [(b) $1.936\sqrt{gR},\ 1.936\sqrt{gR}$

(a) $[.5gR(1.5\sin\theta - 3.598\cos\theta + 3.598)]^{1/2},\ .25g(1.5\cos\theta + 3.598\sin\theta),\ .375g$, no]

39. A square tube attached to a vertical shaft is rotating at constant angular velocity ω (Fig. P2.39). A slider of mass m has coefficient of friction μ_1 for its slanting faces and μ_2 for its vertical faces. (a) Find the range of s: $s_1 \le s \le s_2$, for which the slider does not move with respect to the tube if $\mu_1 < \tan\theta < 1/\mu_1$. (b) Find the solution $s(t)$ if (1) $s(0) = s_1$, $\dot{s}(0) = v_0 < 0$ and (2) $s(0) = s_2$, $\dot{s}(0) = v_0 > 0$.

[(a) $s_1,\ s_2 = g(\cot\theta \mp \mu_1)/\omega^2(\sin\theta \pm \mu_1\cos\theta)$

(b) (1): $i = 1$, $c_1 = 1$; (2): $i = 2$, $c_2 = -1$; $\quad s = s_i + \frac{v_0}{\lambda_1-\lambda_2}(e^{\lambda_1 t} - e^{\lambda_2 t})$,

$\lambda_1,\ \lambda_2 = -\mu_2\omega\sin\theta \pm \{\mu_2^2\omega^2\sin^2\theta + \omega^2\sin\theta(\sin\theta + c_i\mu_1\cos\theta)\}^{1/2}$]

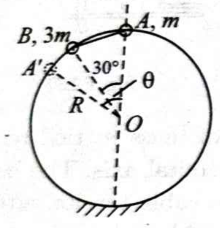


Fig. P2.38

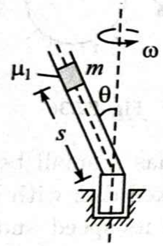


Fig. P2.39

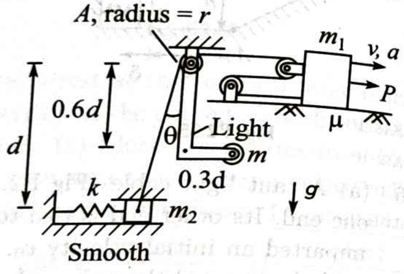


Fig. P2.40

40*. For the system shown in Fig. P2.40, the light bar rigidly attached to pulley, A, has a point-mass m at its end. Find the acceleration a in this position. The spring is undeformed for $\theta = 30°$ and the light cable does not slip on pulley A.

$$\left[\frac{P - \mu m_1 g + 0.9mgd/r + 3kd(\tan\theta - \tan 30°)/\sin\theta - 27v^2 m_2\cot^3\theta/d\sin\theta}{m_1 + 4.05md^2/r^2 + 9m_2/\sin^2\theta}\right]$$

41. (a) Two blocks, of mass m each, are connected by an inextensible cable and rest on a smooth table (Fig. P2.41a). An arrow moving at v_0 hits the cable at its centre. Find the speed of the arrow when the cable segments make angle $2\theta = 60°$ between them. (b) Two blocks are at rest in the position shown in Fig. P2.41b. They are pulled by a force F such that there is no lift-off. Find the initial acceleration of the blocks. Find their velocity when θ has increased to α and also the velocity with which they make an impact. Find the maximum value of F for no lift-off. The coefficient of friction for the floor is μ.

[(a) $0.8088v_0$ (b) $[F\cot\theta - \mu(2mg - F)]/2m$, For impact: $v(\frac{1}{2}\pi)$.

$v(\alpha) = [\{F(\sin\alpha - \sin\theta) - \mu(2mg - F)(\cos\theta - \cos\alpha)\}b/m]^{1/2}$, $F_{max} = 2mg$]

42. A jet plane of mass m kg (Fig. P2.42) reduces its speed on landing from $v_1 = 180$ km/h to $v_2 = 54$ km/h by applying negative thrust from its jet thrust reversers in distance $d = 500$ m with constant deceleration. This operation occurs before application of mechanical brakes. Find the reaction under the nose wheel and the reverse thrust R for $q = 14$ m, $p = 2.5$ m, $b = 2$ m, $h = 3$ m. Neglect aerodynamic forces and inertia of the wheels. [$1.914m$ N, $2.275m$ N]

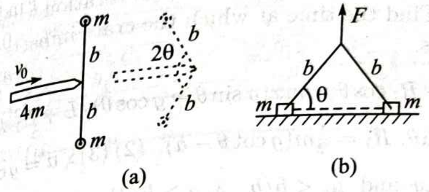


Fig. P2.41

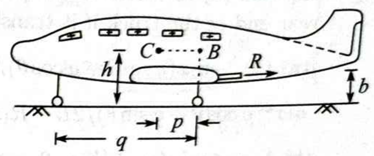


Fig. P2.42

43d. An accelerometer is installed in a vehicle moving at constant acceleration a at an angle θ to the horizontal (Fig. P2.43). Its mass is m and C is its centre of mass. The moment from the torsional spring is M_0 to ensure larger contact force. Find the value of a at which the electric contact will open.

[$(M_0 + mgb)/(p\cos\theta + b\sin\theta)m$]

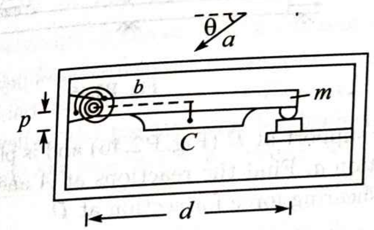


Fig. P2.43

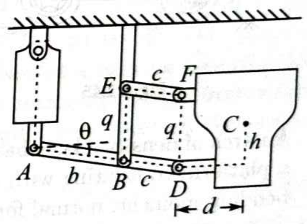


Fig. P2.44

44. Find the initial acceleration of a container of mass m with centre of mass at C, which is lifted from rest at the given position (Fig. P2.44) by the application of force P at A by the piston of the hydraulic cylinder. Neglect inertia of the bars. Find also the supporting forces at pins B and E.

[$(Pb/cm - g)\cos\theta$, $R_E = [mgd + (Pb/c - mg)\cos\theta(d\cos\theta - h\sin\theta)]/q\cos\theta$, $(Pb/c - mg)\cos\theta$ along BD & $P\cos\theta(1 + b/c) \perp BD$]

$\vec{R}_B$: $R_E - (P + mg)\sin\theta$ along BD & $P\cos\theta(1 + b/c) \perp BD$]

45. (a) A truck with rear-wheel drive has uniform acceleration a. A bar of mass m and length L is hinged to it at A and is supported on a smooth surface at B (Fig. P2.45). (1) Find the normal force, shear force and bending moment in the bar at a distance x from B. (2) What is the minimum value of a for which the

contact is lost at B? (3) Find the minimum value of a for which the bar topples over. (b) A crate at the back is at distance d from the rear end. The coefficients of friction are μ_s, μ_k. Find the value(s) of a for which the initial motion of the crate relative to the truck is: (1) no motion, (2) translation, (3) rotation without slip and (4) rotation with slip. (c) Find the time at which the crate impacts the rear end of the truck if it translates.

[(a) (1) $-mx(g\sin\theta + a\cos\theta)/L - R_1\cos\theta$, $mx(a\sin\theta - g\cos\theta)/L + R_1\sin\theta$, $mx^2(g\cos\theta - a\sin\theta)/2L - R_1 x\sin\theta$, $R_1 = \frac{1}{2}m(g\cot\theta - a)$, (2) (3): $a = g\cot\theta$

(b) 1. $a < \min(\mu_s, b/h)g$ 2. $a > \mu_s g$ and $\mu_s < b/h$ 3. $a > bg/h$ and $\mu_s > b/h$

4. $a > \max(\mu_s, b/h)g$ (c) $[2d/(a-\mu_k g)]^{1/2}$]

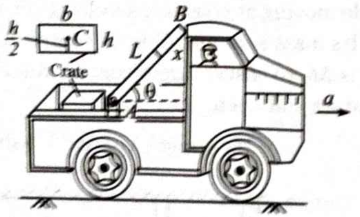


Fig. P2.45

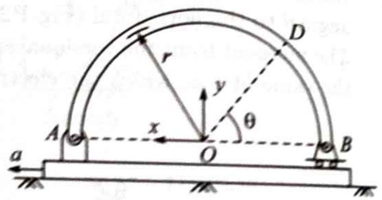


Fig. P2.46

46. An arch of density λ kg/s has a roller support at B (Fig. P2.46) and is pinned to a platform translating with acceleration a. Find the reactions at A and B and bending moment, normal force and shearing force for section at D.

$$[r\lambda[\pi a\vec{i} + \tfrac{1}{2}(\pi g - 2a)\vec{j}], \quad \tfrac{1}{2}\lambda r(\pi g + 2a)\vec{j};$$

$$N = \lambda r\theta(g\cos\theta + a\sin\theta) - \tfrac{1}{2}\lambda r(\pi g + 2a)\cos\theta,$$

$$S = \lambda r\theta(g\sin\theta - a\cos\theta) - \tfrac{1}{2}\lambda r(\pi g + 2a)\sin\theta,$$

$$M = \lambda r^2[g\{\sin\theta - \theta\cos\theta - \pi\sin^2(\theta/2)\} - a\theta\sin\theta]]$$

47. A rear-wheel driven pick-up truck has driving moment M and is connected to a trolley of mass m by a hitch (Fig. P2.47). Find its acceleration a. Find the reactions at A, B, D, E. Neglect inertia of the wheels.

[$a = M/r(m+m_1)$, $\vec{R}_A = m[gb - (p-h)a]\vec{j}/d$, $\vec{R}_B = P_1\vec{i} + P_2\vec{j}$ with $P_1 = ma$, $P_2 = m[(p-h)a + g(d-b)]/d$, $N_E = [m_1 g b_2 - P_2 b_3 - P_1(p-h_1)]/(b_1+b_2)$, $F_D = M/r$, $N_D = m_1 g + P_2 - N_E$]

48. Find the force P for which the system (Fig. P2.48) translates. Find the corresponding reactions at A and B. [$0.7826mg$, $1.630mg$, $1.370mg$]

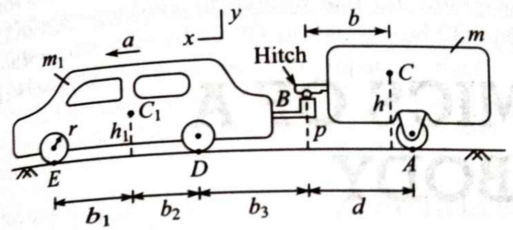


Fig. P2.47

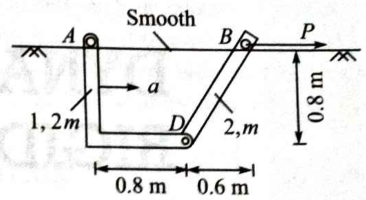


Fig. P2.48

49 Consider a suspension cable supporting a suspension bridge (Fig. 4.25), under uniform load q_0 per unit projected horizontal length. From the FBD of part CD, show that $T\cos\theta = H =$ constant, $T\sin\theta = q_0 x \Rightarrow y' = \tan\theta = q_0 x/H$. Hence, show that the shape of the cable is a parabola: $y = q_0 x^2/2H$. (b) Consider transmission line cable subjected to uniform self-weight of q per unit length. As in part (a), show that $T\cos\theta = H =$ constant, $T\sin\theta = qs \Rightarrow y' = \tan\theta = qs/H \Rightarrow ds/dx = (1+y'^2)^{1/2} = (1+q^2s^2/H^2)^{1/2} \Rightarrow s = \frac{H}{q}\sinh\frac{qx}{H} = \frac{H}{q}y' \Rightarrow$ the shape of the cable is a common catenary: $y = \frac{H}{q}(\cosh\frac{qx}{H} - 1)$.

50. The force-deflection relation of the flexible base of a packing case of a delicate instrument of mass m is $P = cx^2$. Find the maximum height from which the base of the package can be dropped if the maximum force on the instrument from the base is not to exceed P_1.

[$\sqrt{mg/c} - \sqrt{P_1/c} + c[(P_1/c)^{3/2} - (mg/c)^{3/2}]/3mg$]

51. A bomb is dropped from an airplane flying horizontally at height h with speed v_0. Assume that a *small* resistance $R = mkv$ acts on it. Show that the time of its fall is approximately $(2h/g)^{1/2}[1 + (2h/g)^{1/2}k/6]$ and the horizontal distance covered in its fall is approximately $v_0(2h/g)^{1/2}[1 - (2h/g)^{1/2}k/3]$.

52. A small object is projected from origin O at speed v_0 at angle θ_0 to the horizontal direction $\vec{i}$ in a wind having horizontal velocity $-u_1\vec{i}$. The resistance force on the object is $-mk \times$ (velocity relative to wind). Find its position at any time. Show that (a) it returns to its point of projection if $\tan\theta_0 = g/ku_1$ and (b) the highest point of its trajectory is located above its point of projection if $(1 + g/kv_0\sin\theta_0)\ln(1 + kv_0\sin\theta_0/g) = 1 + v_0\cos\theta_0/u_1$.

[$x = (v_0\cos\theta_0 + u_1)(1 - e^{-kt})/k - u_1 t$, $\quad y = [v_0\sin\theta_0 + g/k](1 - e^{-kt})/k - gt/k$]