Multiple Integral — lecture 3

let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain in $\mathbb{R}^3$
and f be a cont^n function on Ω.

$$f : \Omega \longrightarrow \mathbb{R}$$

split Ω into small subdomains Ω_k and
define $\quad S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, |\Omega_k|$ (volume of Ω_k)

$$(x_k, y_k, z_k) \in \Omega_k$$

$$\iiint_{\Omega} f \, dv = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\sum_k f(x_k, y_k, z_k) |\Omega_k| \right)$$

We start with the simplest region in $\mathbb{R}^3$ "a box", $\Omega = [a,b] \times [c,d] \times [r,s]$. Then

$$\iiint_{\Omega} f \, dv \; (dx\,dy\,dz) = \int_a^b \left(\int_c^d \left(\int_r^s f \, dz \right) dy \right) dx$$

We have six different order to integrate.

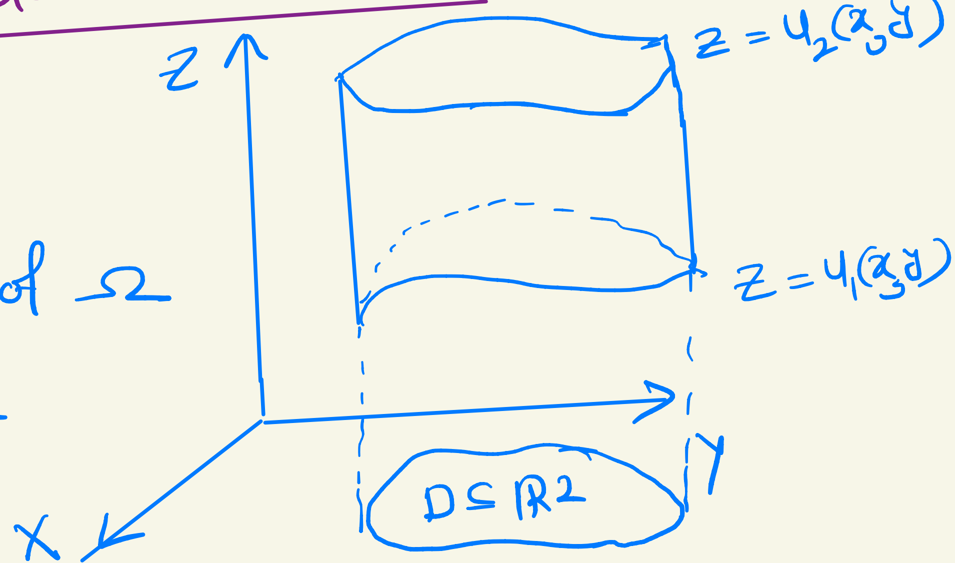
Remark: If $f(x,y,z) = 1$. Then

$$\boxed{\iiint_\Omega f\,dv = \iiint_\Omega dv = \text{Volume}(\Omega).}$$

Possibilities of domain in $\mathbb{R}^3$.

I^{st} possibility:

$D \rightarrow$ projection of Ω in XY-plane



$$\Omega = \{(x, y, z) \mid u_1(x, y) \le z \le u_2(x, y) \text{ \& } (x, y) \in D\}$$

In this is we have the following Fubini's theorem to compute the triple integral

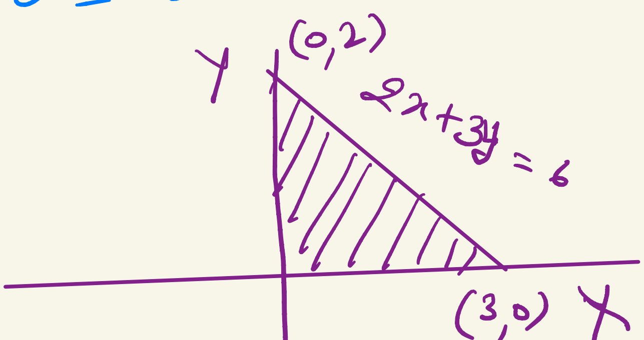
$$\boxed{\iiint_{\Omega} f \, dv = \iint_{D} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \, dz \right) dA}$$

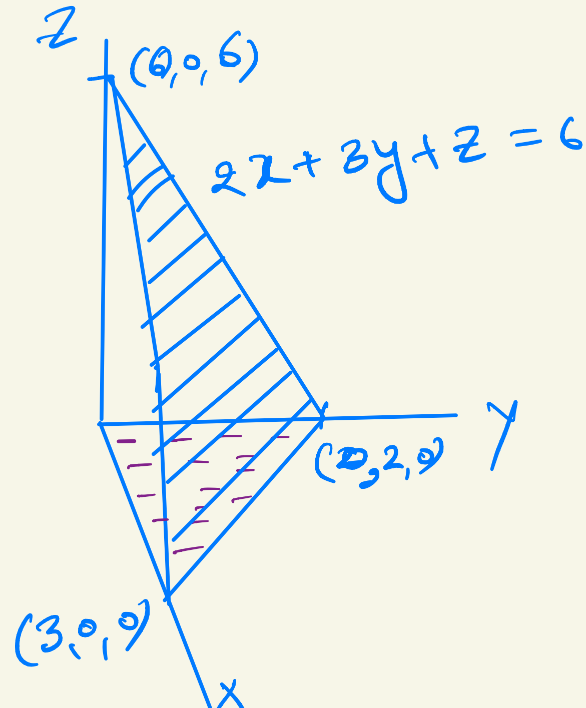
— $\circledast$.

Example ① Evaluate $\iiint 2x\,dv$, where Ω is the region under Ω the plane $2x+3y+z=6$ lies in the I^{st} octant.

Sol^n. clearly

$$0 \le z \le 6-2x-3y$$





$$0 \le x \le 3$$
$$0 \le y \le -\tfrac{2}{3}x+2$$

$$\Omega = \{ (x,y,z) \mid 0 \le z \le 6-2x-3y, \ (x,y) \in D \}$$

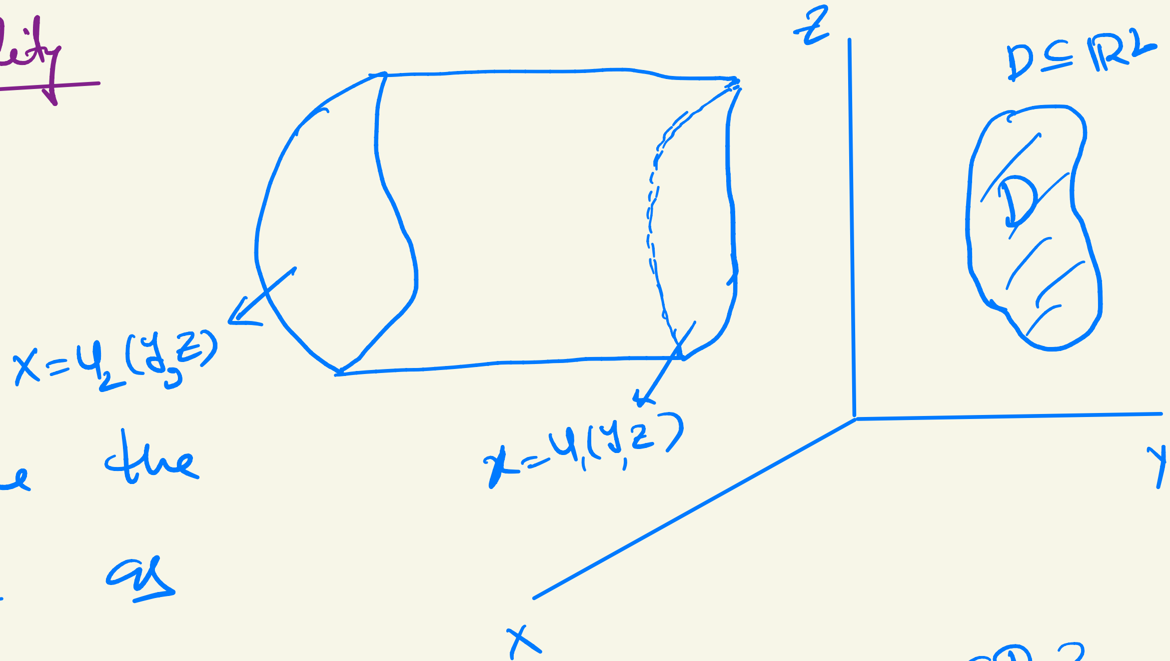
and $D = \{ (x,y) \mid 0 \le x \le 3 \ \& \ 0 \le y \le -\frac{2}{3}x + 2 \}$

$$\iiint_{\Omega} f\,dv = \iint_{D} \left(\int_0^{6-2x-3y} 2x \; dz \right) dA$$

$$= \int_0^3 \left(\int_0^{-\frac{2}{3}x+2} \left(\int_0^{6-2x-3y} 2x\,dz \right) dy \right) dx$$

$=$ compute.

IInd possibility



We can define the domain Ω as

$$\Omega = \{(x,y,z) \mid \psi_1(y,z) \leq x \leq \psi_2(y,z) \;\&\; (y,z) \in D\}$$

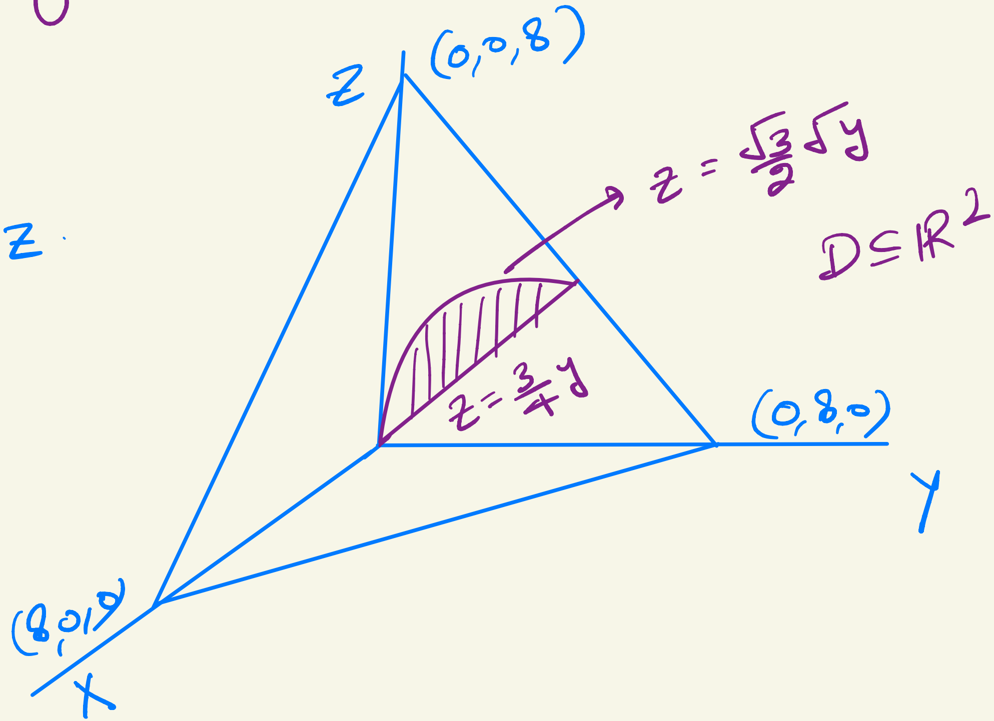
$$\boxed{\iiint_\Omega f\,dV = \iint_D \left(\int_{\psi_1}^{\psi_2} f(x,y,z)\,dx \right) dA}$$

Example ② Compute volume of solid below plane $z = 8 - x - y$ and above the region in ZY-plane bounded by $z = \frac{\sqrt{3}}{2}\sqrt{y}$ and $z = 3/4\, y$.

Sol^n.

$x = 8 - y - z$.

(surface)



$$\text{Volume} = \iint_D \left(\int_0^{8-y-z} dx \right) dA$$

Easy to see $0 \le y \le 4$ & $\frac{3}{4} y \le z \le \frac{\sqrt{3}}{2} \sqrt{y}$

$$\text{Volume} = \int_0^4 \left(\int_{3/4 y}^{\frac{\sqrt{3}}{2}\sqrt{y}} \left(\int_0^{8-y-z} dx \right) dz \right) dy.$$

$$= \text{compute.}$$

$\underline{\text{III}^{rd} \text{ possibility}}$:→ If the domain Ω is given by

$$\Omega = \{ (x, y, z) \mid u_1(x, z) \le y \le u_2(x, z) \ \& \ (x, z) \in D \}$$

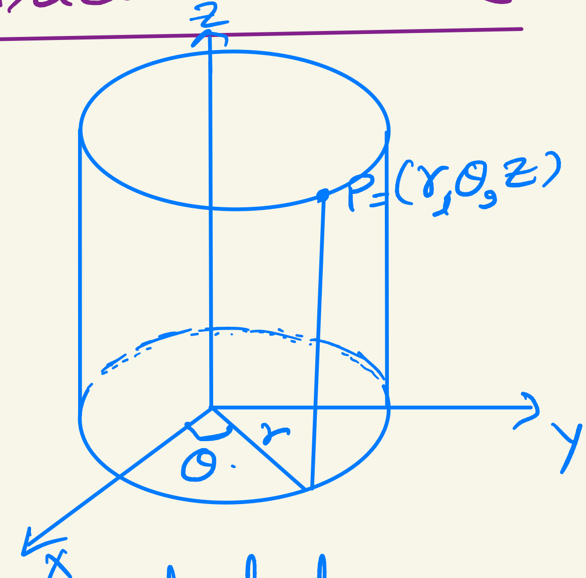
In this case we have

$$\iiint_{\Omega} f \, dr = \iint_{D} \left(\int_{u_1}^{u_2} f(x, y, z) \, dy \right) dA$$

Example ③ Find volume of the region bounded by $x + z = 1$ and $y + 2z = 2$ in I^{st} octant.

Sol^n :- Exercise.

Triple Integral in Cylindrical co-ordinates



Recall:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Then triple integral

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\Omega} f(r,\theta,z)\, r\,dr\,d\theta\,dz$$

$$dv = dx\,dy\,dz = (dx\,dy)\,dz = (r\,dr\,d\theta)\,dz$$

Example (4). Evaluate $\iiint_{\Omega} y\,dv$, where Ω is the region lies below $z = x+2$ and above XY-plane b/w the cylinders $x^2+y^2 = 1$ and $x^2+y^2 = 4$.

Sol^n :→

$$\iiint_{\Omega} y\,dv = \iint_{\Omega} \left(\int_0^{x+2} y\,dz \right) dA$$

$$D = \{ (r,\theta) \mid 1 \le r \le 2 \ \& \ 0 \le \theta \le 2\pi \}$$

$$\iiint_{\Omega} y\,dv = \int_0^{2\pi} \int_1^{2} \int_0^{r\cos\theta+2} (r\sin\theta)\,dz\, r\,dr\,d\theta$$

Example ⑤ Evaluate $I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dv.$

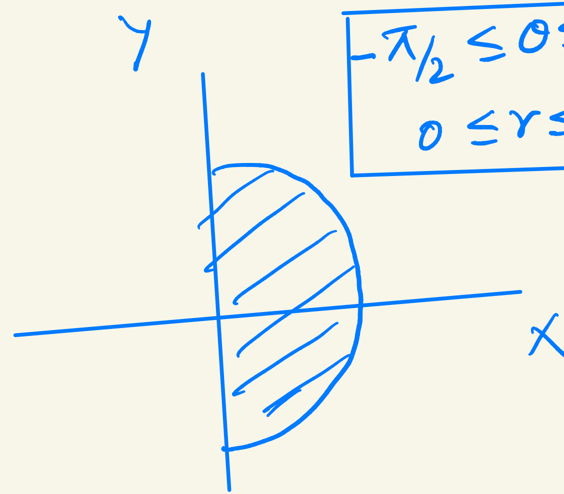
Sol^n :→

$$-1 \le y \le 1$$

$$0 \le x \le \sqrt{1-y^2}$$

$$x^2+y^2 \le z \le \sqrt{x^2+y^2}$$

$$r^2 \le z \le r$$



$$\boxed{\begin{array}{c} -\pi/2 \le \theta \le \pi/2 \\ 0 \le r \le 1 \end{array}}$$

$$x^2+y^2 = r^2$$

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}}$$

$$I = \int_{-\pi/2}^{\pi/2} \int_{0}^{1} \int_{r^2}^{r} (r\cos\theta \, r\sin\theta \, z)(r\,dr\,d\theta\,dz).$$

Triple Integral in Spherical Co-ordinates

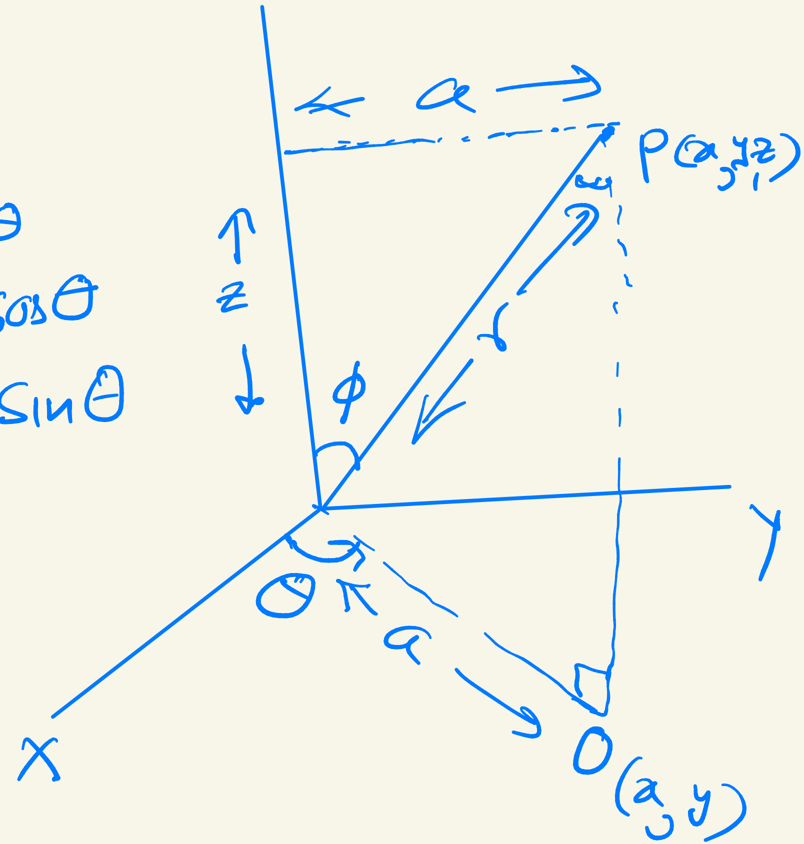
Spherical Co-ordinates

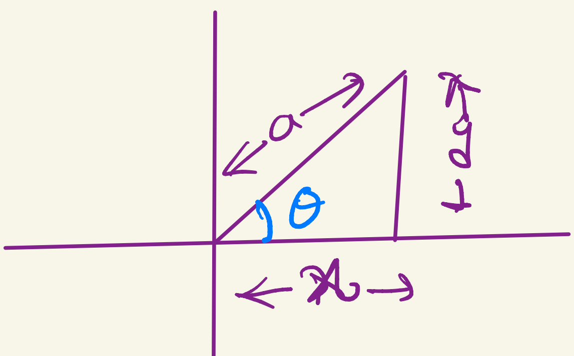
$$z = r\cos\phi$$

$$a = r\sin\phi$$

$$x = a\cos\theta$$
$$= r\sin\phi\cos\theta$$

$$y = r\sin\phi\sin\theta$$



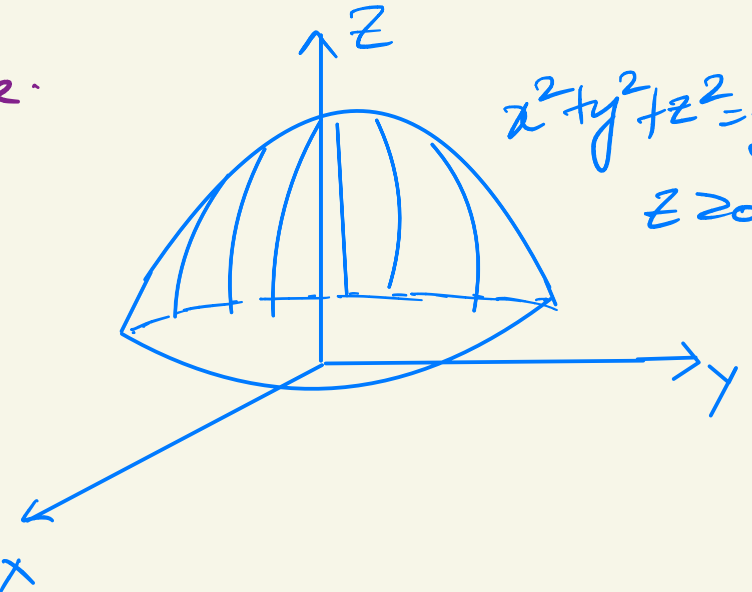


$$\iiint_\Omega f\,dv = \iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \iiint_\Omega f(r,\theta,\phi)\,(dv)\,?$$

$$\boxed{dv = dx\,dy\,dz = (r^2\sin\phi)\,dr\,d\theta\,d\phi}$$

$$\iiint_\Omega f\,dv = \iiint_\Omega f(r,\theta,\phi)\,(r^2\sin\phi)\,dr\,d\theta\,d\phi.$$

Example ⑥ Evaluate $\iiint_{\Omega} z\,dv$ where Ω upper half of the unit sphere.



Sol^n:-

$$x^2+y^2+z^2=1$$

$$z \geq 0$$

$$\left.\begin{aligned} 0 &\leq \phi \leq \pi/2 \\ 0 &\leq \theta \leq 2\pi \\ 0 &\leq r \leq 1 \end{aligned}\right\} \circledast$$

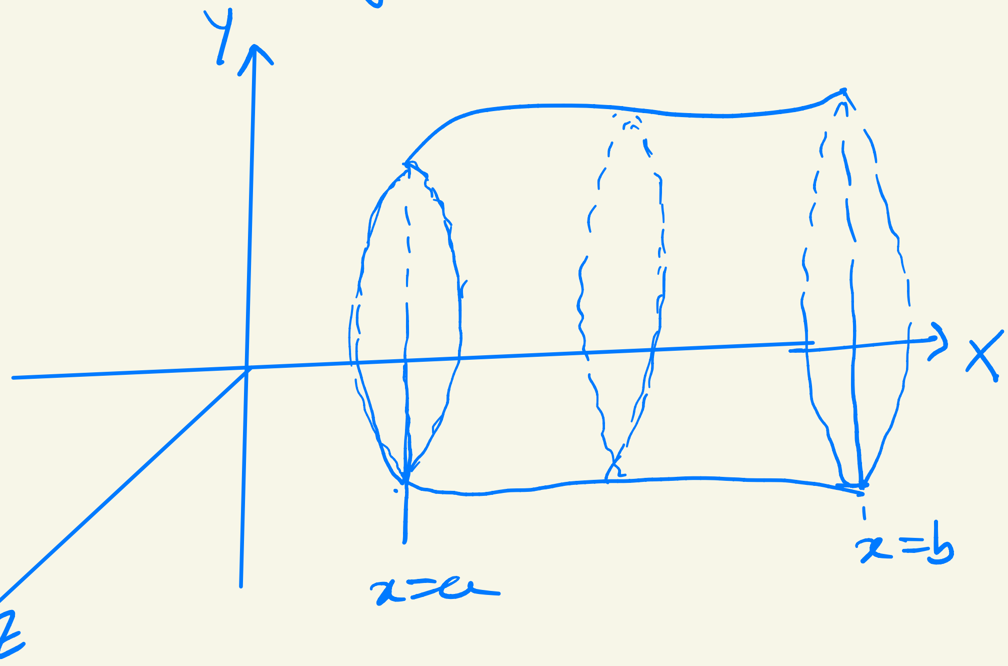
$$\iiint_{\Omega} z\,dv = \int_0^{\pi/2}\int_0^{2\pi}\int_0^{1} (r\cos\phi) \times (r^2 \sin\phi \, dr\, d\theta\, d\phi)$$

Example 7 Volume of the solid of revolution

We need to find volume of the solid obtained by revolving the curve $y = f(x)$ along the x-axis

$a \le x \le b$

$y^2 + z^2 = (f(x))^2$



Volume $V = \iiint_{\Omega} dv$ and Ω is given by

$$\Omega = \{(x, y, z) \mid a \le x \le b \ \& \ y^2 + z^2 = (f(x))^2\}$$

$$= \{(x, r, \theta) \mid a \le x \le b \ \& \ 0 \le r \le f(x), \ 0 \le \theta \le 2\pi\}$$

where $y = r\cos\theta$
$z = r\sin\theta$

$$\iiint dv = \int_a^b \left(\int_0^{f(x)} \int_0^{2\pi} r \, dr \, d\theta \right)$$

$$\text{Volume } V = \int_a^b 2\pi \left[\frac{r^2}{2}\right]_0^{f(x)} dx$$

$$= \int_a^b 2\pi \frac{(f(x))^2}{2} dx$$

$$\boxed{\text{Volume} = \pi \int_a^b (f(x))^2 dx.}$$

If the curve is $x = g(y)$ b/w $y = c$ to $y = d$

$$\boxed{\text{Volume} = \pi \int_c^d (g(y))^2 dy}$$

Exercise 1 $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dV$

Evaluate by changing to spherical co-ordinates.

Exercise 2. Find the volume of the region above the surface $\phi = \pi/3$ and below the surface $r = 4\cos\phi$.