Course : MTL-100 (Calculus)

Chapter 1 : Sequences of Real Numbers

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Reference Book : Elementary Analysis : The Theory of Calculus by Kenneth A. Ross (Chapters 1 and 2)

Lecture 1 : A brief introduction to real numbers

- We will assume that the students are familiar with natural numbers, integers and rational numbers to some extent.
- Notations:

$\mathbb{N}$ = the set of all natural numbers
$= \{1, 2, 3, 4, \cdots\}$

$\mathbb{Z}$ = the set of all integers
$= \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$

$\mathbb{Q}$ = the set of all rational numbers
$= \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0, p \,\&\, q \text{ have no common factors} \right\}$

$\mathbb{R}$ = the set of all real numbers.

Natural numbers ($\mathbb{N}$) :

- $\mathbb{N}$ is an infinite set.
- 1 is the least element of $\mathbb{N}$.
- $\mathbb{N}$ has no maximum element.

On each of the above sets, we have two binary operations : addition ($+$) and multiplication ($\cdot$) defined.

Also, there is an "ordering" given two numbers x & y, we have
$x < y$ or $x = y$ or $y < x$.

- **Well-ordering property of $\mathbb{N}$:**
  Every nonempty subset S of $\mathbb{N}$ has a least element.
- **Remark:** "Well-ordering property of $\mathbb{N}$" is equivalent to the "principle of mathematical induction".
- **Integers ($\mathbb{Z}$):** $\mathbb{Z} \supseteq \mathbb{N}$
  Extra properties:
  - It has an additive identity 0. $a+0=a \ \forall a \in \mathbb{Z}$
  - Every element in $\mathbb{Z}$ has an additive inverse, i.e. for any $x \in \mathbb{Z}$, $\exists -x \in \mathbb{Z}$ s.t. $\boxed{x+(-x)=0}$.

- Rational Numbers ($\mathbb{Q}$):

Every nonzero rational number has a multiplicative inverse.

$\mathbb{Q}$ with $+$ and $\cdot$ forms what is known as a "field".

Properties of a field:

- $+$ & $\cdot$ are "commutative" and "associative".
- Existence of additive identity 0 and multiplicative identity 1.
- Every element has an additive inverse.
- Every nonzero element has a mult. inverse.
- (Distributive Law): $a\cdot(b+c) = a\cdot b + a\cdot c$

$\mathbb{Q}$ also has an order structure $(<)$

- $\frac{a}{b} < \frac{c}{d} \iff ad < bc$
- This order $<$ on $\mathbb{Q}$ satisfies the usual properties.

- $\mathbb{Q}$ is an "ordered field" with respect to the addition, multiplication and ordering defined.

- Shortcomings:

  $\mathbb{Q}$ has "gaps" in some sense.

  For example, there is no $r \in \mathbb{Q}$ such that $r^2 = 2$

- **Rational root theorem:**

Consider the polynomial equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where $n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, $a_0 \neq 0, a_n \neq 0$.

Suppose that a rational number $\frac{p}{q}$ satisfies this equation where p & q have no common factors. Then p must divide a_0 and q must divide a_n.

Proof: We have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$\Rightarrow \quad a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0$$

$\Rightarrow a_n p^n = -q\,(a_{n-1}p^{n-1} + \cdots + a_0 q^{n-1})$

$\Rightarrow q$ divides $a_n p^n \Rightarrow q$ divides a_n ($\because p$ & q have no common factors)

Also, $a_0 q^n = -p\,(a_n p^{n-1} + \cdots + a_1 q^{n-1})$

$\Rightarrow p$ divides $a_0 q^n \Rightarrow p$ divides a_0.

<u>Corollary</u>: There is no $x \in \mathbb{Q}$ such that $x^2 = 2$.

<u>Proof</u>: If $x = \frac{p}{q}$ satisfies $x^2 - 2 = 0$, then by the prev. thm, p must divide -2 and q must divide 1.

So, the only possibilities for x are ± 1 and ± 2.

But clearly none of these satisfies $x^2 = 2$.

Real Numbers ($\mathbb{R}$)

Addition & multiplication.

$\mathbb{R} \supseteq \mathbb{Q}$.

$(\mathbb{R}, +, \cdot)$ is also an "ordered field".

- Extra property : "completeness axiom"!
- We will not describe any explicit construction of $\mathbb{R}$.
- Let A be a nonempty subset of $\mathbb{R}$.
- We say M is the maximum element of A if $M \in A$ and $x \le M \ \forall x \in A$.

- m is the minimum element of A is $m \in A$ and $m \leq x \quad \forall x \in A$.

Upper bounds and lower bounds:

- An element $u \in \mathbb{R}$ is said to be an upper bound of A if $\boxed{x \leq u \quad \forall x \in A}$.
- Similarly, l is a lower bound of A if $\boxed{x \geq l \quad \forall x \in A}$.
- A is said to be bounded above if it has an upper bound.
- A is said to be bounded if it has a lower bound as well as an upper bound.

Supremum and Infimum

Let A be a nonempty subset of $\mathbb{R}$.

- $\alpha \in \mathbb{R}$ is said to be supremum or least upper bound of A if
  - (i) α is an upper bound of A
  - (ii) if u is any upper bound of A, then $\alpha \leq u$.

- Similarly, $\beta \in \mathbb{R}$ is said to be the infimum or the greatest lower bound of A if
  - (i) β is a lower bound of A and
  - (ii) if ℓ is any lower bound of A, then $\beta \geq \ell$.

Completeness Axiom (or the least upper bound property)

Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound (or supremum).

Equivalently, every nonempty subset of $\mathbb{R}$ that is bounded below has an infimum.

Exercise: $\sup(-A) = -\inf(A)$ for any nonempty subset A of $\mathbb{R}$.

Remark: The completeness axiom does not hold for $\mathbb{Q}$. e.g. $A = \{x \in \mathbb{Q} : x^2 < 2\}$. Show that A is bounded above in $\mathbb{Q}$ but it has no least upper bound in $\mathbb{Q}$.

Archimedean property: For any $x \in \mathbb{R}$, there exists a natural number N (depending on x) s.t. $x < N$.

Theorem: Completeness axiom $\Rightarrow$ Archimedean property.

Proof: Suppose the Archimedean property does not hold.

Then $\exists\, x \in \mathbb{R}$ s.t. $n \le x \quad \forall n \in \mathbb{N}$.

$\Rightarrow$ x is an upper bound for the set $\mathbb{N} \subseteq \mathbb{R}$

Completeness axiom $\Rightarrow$ $\mathbb{N}$ has a l.u.b. say $\alpha \in \mathbb{R}$.

Then $\alpha - 1$ is not an upper bound for $\mathbb{N}$.

$\Rightarrow \exists\, n \in \mathbb{N}$ s.t. $\alpha - 1 < n$.

$\Rightarrow \quad \alpha < \underbrace{n+1}_{\in \mathbb{N}}$

$\Rightarrow$ α is not an upper bound for $\mathbb{N}$,

which is a contradiction.

· **Remark:** The converse of the previous theorem is not true.

$\mathbb{Q}$ satisfies the Archimedean property (why?) but not the completeness axiom.

Consequences of the Archimedean property:

① Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $\inf(S) = 0$.

Proof: Since $\frac{1}{n} > 0$ $\forall n \in \mathbb{N}$, S is bounded below.

Let $\alpha = \inf(S)$. Then $\alpha \geq 0$ ($\because$ 0 is a lower bdd. for S)

If $\alpha > 0$, then by the Archimedean property, $\exists\, n \in \mathbb{N}$ s.t. $n > \frac{1}{\alpha} \Rightarrow \alpha > \frac{1}{n} \in S$

$\Rightarrow$ α is not a lower bound for S, which is a contradiction.

Hence, $\alpha = 0$.

② **The Density Theorem:**

Let $x, y \in \mathbb{R}$ such that $x < y$.
Then $\exists$ a rational number r s.t. $x < r < y$.
(ie. between any ~~two~~ real numbers there is a rational number).

Proof: First let's assume $x \geq 0$.

$x < y \Rightarrow y - x > 0$

By the Archimedean property, $\exists\, n \in \mathbb{N}$ s.t.

$n > \frac{1}{y-x}$ ie. $\frac{1}{n} < y - x$.

Now, let $S = \{ m \in \mathbb{N} : \frac{m}{n} > x \}$.
Then $S \subseteq \mathbb{N}$. Also, by the Arch. property,

$S \neq \phi$.

By the well-ordering of $\mathbb{N}$, S has a least element, say, m_0.

Then, $\frac{m_0}{n} > x$ ($\because m_0 \in S$)

and $\frac{m_0-1}{n} \leq x$ (If $m_0 \geq 2$, then $m_0-1 \notin S$, $m_0-1 \in \mathbb{N}$ so $\frac{m_0-1}{n} \leq x$.
If $m_0=1$, then $x \geq 0 = \frac{m_0-1}{n}$)

$\frac{m_0-1}{n} \leq x \Rightarrow \frac{m_0}{n} \leq x+\frac{1}{n} < x+(y-x) = y$

$\therefore \quad x < \frac{m_0}{n} < y$. Hence done.

If $x<0$, by the Arch. prop., $\exists$ $n \in \mathbb{N}$ s.t. $n > -x$

$\Rightarrow \quad x+n > 0$.

Also, $\quad x < y \quad \Rightarrow 0 < x+n < y+n$.

$\therefore$ By the above proof, $\exists$ a rational no. r
s.t. $\quad x+n < r < y+n$

$\Rightarrow \quad x < r-n < y$

Also, $r-n \in \mathbb{Q}$. Hence we are done.

<u>Exercise</u>: Let $x, y \in \mathbb{R}$ s.t. $x < y$.
Show that $\exists$ an irrational number z
s.t. $x < z < y$.

<u>Proof</u>: Since $x < y$,

$$x - \sqrt{2} < y - \sqrt{2}$$

By the density theorem, $\exists\, r \in \mathbb{Q}$ s.t.

$$x - \sqrt{2} < r < y - \sqrt{2}$$

$$\Rightarrow \quad x < r + \sqrt{2} < y$$

Let $z = r + \sqrt{2}$. Then $x < z < y$
Also, z is an irrational number.