COL 202 HOMEWORK 2

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Problem 1.

We will prove the given claim by considering two cases.
Consider the graph F_1. Let there be k connected components in $F_1, k \geq 0$.

Note that every connected component of F_1 has to be acyclic since the overall graph is acyclic.

Consider any edge $e \in E_2 \backslash E_1$. We have the following two cases for e

Case 1 : The two endpoints of e , say {u,v}, both belong to different connected components.
In this case, it is easy to see that graph $F(V, E_1 \cup \{e\})$ is also an acyclic graph. This is because the edge e is a bridge edge and cannot form a cycle between two connected components. (Atleast 2 edges are needed to form a cycle between two connected components).

Case 2 : The two endpoints of e, namely, {u,v}, both belong to the same connected component.
This is true for all $e \in E_2 \backslash E_1$. (Else we can find such e not following this condition and consider case 1).

We will use contradiction here.

In this case, since all e belong to some of the connected components, the total number of edges in E_2 is less than or equal to the sum of number of edges in all connected components.

Consider a connected component with n vertices. For this to be acyclic and connected, the max number of edges can be n-1 (as proven in class).

This implies that $|E_2|$ is less than or equal to sum of (number of vertices in each connected component - 1).

Also, for any acyclic graph, number of vertices in a graph is less than equal to the number of edges.

All this combined, gives $|E_2| \leq |E_1|$. This is in contradiction with the fact that $|E_1| < |E_2|$.

Thus, there exists an edge in $E_2 \backslash E_1$ such that F = (V, $E_1 \cup \{e\}$) is an acyclic graph.

Problem 2.

Proof by Induction on the number of vertices.

Base Case: When the number of vertices is 2, say v_1, v_2. There can be only 1 edge. So, moving from v_1 to v_2 and back forms a closed graph with all edges traversed exactly twice.

Induction Hypothesis: Let the claim be true for $|V| = n - 1$.

Induction step: Now lets add another vertex say, v_n to the graph, and lets say it is connected to certain other vertices. Since by the induction hypothesis our claim is true for the graph with $n - 1$ vertices, we just have to show that even if we add another vertex, we can still form a closed path traversing all edges twice.

Lets say that v_n is attached to k other vertices, where $k < n$, let those vertices be $\{v_1, v_2, ..., v_k\}$. Now, there already exists a closed path which traverses the $n - 1$ vertices, and a closed path can begin and end at any vertex visited in it. So lets say it ends on v_1. Now, from v_1 we move to v_n first, then from there we move to and fro to all other vertices in the set $\{v_2, ..., v_k\}$ back to v_n and then finally back to v_1.

We can see that this again forms a closed path which traverses all the new edges twice too, and the older edges were already traversed by our Induction Hypothesis.

Therefore, our claim is true by P.M.I. $\forall\ |V| \geq 2$

Hence, Proved

Now for the second claim that every connected graph $G = (V, E)$ has a closed walk of length $2|V| - 2$ which visits every vertex in V at least once.

We will prove this by induction.

Base Case: When the number of vertices is 2, say v_1, v_2. There can be only 1 edge. So, moving from v_1 to v_2 and back forms a closed graph which visits every vertex of length 2 which is $2|V| - 2$, $since\ |V| = 2$.

Induction Hypothesis: Let the claim be true for $|V| = n - 1$.

Induction Step: Now lets add another vertex say, v_n to the graph, and lets say it is connected to certain other vertices. Since by the induction hypothesis our claim is true for the graph with $n - 1$ vertices, we just have to show that even if we add another vertex, we can still form a closed path of length $2|V| - 2$, which visits every vertex.

Lets say that v_n is connected to v_i where $i < n$. Now we know that v_i is already a part of a closed path that satisfies our claim by Induction Hypothesis. We also know that we can assume the closed path to end at v_i. Now we traverse from v_i to v_n and then back to v_i after traversing the older closed path containing $n - 1$ vertices.

We can see that this again forms a closed path which satisfies our claim since earlier the length of path was $2(n - 1) - 2$, and we add 2 more steps to it, so the length becomes, $2(n) - 2$, which satisfies the claim that $length\ = 2|V| - 2$.

Therefore, our claim is true by P.M.I. $\forall\ |V| \geq 2$

Hence, Proved

Problem 3.

Lemma: A regular bipartite graph always has a perfect matching.

Let us take 2 sets say $X\, and\, Y$ be the parts of the bipartite graph.

Now lets take a subset S of the set X, and the set of the points $P(S)$ to which the points in S are connected.

We know that $|S| \leq |P(S)|$ because every matching with a vertex in S has a vertex in $P(S)$ but not vice-versa. Thus we can say that if d is the degree of each vertex, then cardinality of the edge set of S, $d|S|$ would definitely be less than or equal to the cardinality of the edge set of $P(S)$.

Now, using $Hall's\, Theorem$, we can say that there exists a matching, say, M, which matches every vertex in X. Because the graph is $regular$ $|X| = |Y|$. Therefore, the matching M would be $perfect$.

Now let's prove that the edge set of every bipartite regular graph can be partitioned into $perfect$ matchings by induction on the degrees of the vertices, d.

Base Case: When $d = 0$, Edge set has no elements, so there are 0 perfect matchings. When $d = 1$, the edge set is perfectly matched because all vertices will be the end point of exactly one edge.

Induction Hypothesis: Assume that our claim holds for a graph where $d \leq n - 1$.

Induction Step: Let's consider a graph $G = (V, E)$, with degree $d = n$. Since G is bipartite and regular, it will have a $perfect$ matching, say M.
Consider another graph $G' = (V, E \backslash M)$
G' is also a regular bipartite graph, but with degree $d < n$ (As every vertex will have the same number of edges(≥ 1) in M).
As G' satiesfies our induction hypothesis, $E \backslash M$ can be partitioned into perfect matchings. Therefore G too can be partitioned into perfect matchings, since E is has all elements perfectly matched ($\{E \backslash M \cup M\}$ all elements in this set is perfectly matched)

So by P.M.I, our claim is true $\forall\, d \geq 0$.
Hence Proved

Problem 4.

Claim : The number of perfect matchings in a complete graph on k vertices is given by :

$$N(k) = \begin{cases} \frac{k!}{(\frac{k}{2})! \times 2^{\frac{k}{2}}} & \text{if k is even} \\ 0 & \text{otherwise (i.e. if k is odd)} \end{cases} \tag{1}$$

Proof : We will prove the claim by construction.
Consider a graph having n vertices. Clearly, if n is odd, then number of perfect matchings is zero. (This is straightforward from the definition of perfect matchings, since every vertex needs to be matched to exactly one vertex, the total number of vertices has to be even).
So, consider the case when n is even. Let n = 2m, for some natural number m.
Now, our goal is to divide these n vertices into m pairs, such that every vertex belongs to exactly one pair. Call this matching M.
Claim : M is a perfect matching.
Proof : The proof is easy to see. Since every vertex of the graph is included in a pair, and every vertex is included exactly once, by the definition of perfect matching, M is a perfect matching.
So, the number of ways to divide 2m objects into m groups such that each group has exactly 2 elements is given by

$$\frac{(2m)!}{m! \times 2^m},$$

which proves the claim.

Problem 5.

Claim : The number of ways in which n passengers can occupy m seats following social distancing norms is given by :

$$N(n,m) = \frac{(n-2m+2)!}{(n-3m+2)!}$$

Proof : Seat m passengers on random m seats out of the given n seats.
Let $x_1, x_2, x_3, ..., x_{m-1}$ be the number of vacant seats between passenger 1 and 2, passenger 2 and 3, passenger 3 and 4 , ... , passenger m -1 and m respectively.
According to the conditions given in the question, we need $x_i \geq 2 \;\; \forall \;\; i \;\; \in [1, m-1]$.
Also, let y_1, y_2 be the number of vacant seats to the left of passenger 1 and to the right of passenger m respectively.
Clearly, $y_1 \geq 0$ and $y_2 \geq 0$.
Now, since m seats are occupied by m passengers, the number of vacant seats is n - m.
So,

$$x_1 + x_2 + ... + x_{m-1} + y_1 + y_2 = n - m, \;\; \text{where} \;\; x_i \geq 2, y_i \geq 0$$

Substitute $z_i = x_i - 2$, or in other words, $x_i = z_i + 2$. So, the equation becomes,

$$z_1 + z_2 + ... + z_{m-1} + 2(m-1) + y_1 + y_2 = n - m$$

$$z_1 + z_2 + z_3 + ... + z_{m-1} + y_1 + y_2 = n - 3m + 2, \;\; \text{where} \;\; z_i \geq 0, y_i \geq 0$$

The problem has now reduced to a similar problem as discussed in class. We have to arrange n - 3m $+$ 2 identical balls in m bins. The number of solutions for this, using formula discussed in class is :

$$\binom{n-3m+2+m-1+1}{m-1}$$

$=$

$$\binom{n-2m+2}{m-1}$$