Discrete-Time Signals and Systems

Lecture 27

Extra class

- March 19, 2 pm onwards

Examples of FT pairs

- $\mathrm{F}\{\delta[n]\} = 1$
- $\mathrm{F}\{a^n u[n]\} = \frac{1}{1-ae^{-j\Omega}}$
- $x[n] = \begin{cases} 1, |n| \leq N_1, \\ 0, |n| > N_1 \end{cases}$ $\quad \mathrm{F}\{x[n]\} = \frac{\left(\sin\left(\Omega\left(N_1+\frac{1}{2}\right)\right)\right)}{\sin(\Omega/2)}$
- $\mathrm{F}\{1\} = 2\pi \sum_m \delta(\Omega - m2\pi)$

Examples of FT pairs

- $X(e^{j\Omega}) = 2\pi \sum_m \delta(\Omega - \Omega_0 - m2\pi)$
- $x[n] = ?$

Examples of FT pairs

- $X(e^{j\Omega}) = 2\pi \sum_m \delta(\Omega - \Omega_0 - m2\pi)$
- $x[n] = e^{j\Omega_0 n}$

$$x[n] = \int_0^{2\pi} \delta(\Omega - \Omega_0) e^{jn\Omega} d\Omega = e^{j\Omega_0 n}$$

Properties of DTFT

Properties	time	frequency
Linearity	$Ax[n] + By[n]$	$AX\left(e^{j\Omega}\right) + BY\left(e^{j\Omega}\right)$
Time-shift	$x[n - n_o]$	$e^{-j\Omega n_o}X\left(e^{j\Omega}\right)$
Modulation	$x[n]e^{j\Omega_o n}$	$X\left(e^{j(\Omega-\Omega_o)}\right)$
Conjugation	$\overline{x[n]}$	$\overline{X(e^{-j\Omega})}$
Differencing	$x[n] - x[n-1]$	$X\left(e^{j\Omega}\right)(1 - e^{-j\Omega})$

Caution in Differencing/Differentiation

$$y(t) = \frac{dx(t)}{dt}$$

$$x(t) = \int_{-\infty}^{t} y(t)dt$$

$$Y(\omega) = j\omega X(\omega)$$

$$X(\omega) = \frac{Y(\omega)}{j\omega} + \pi Y(0)\delta(\omega)$$

$$X(\omega) = \frac{Y(\omega)}{j\omega}$$

Caution in Differencing/Differentiation

$$u[n] - u[n-1] = \delta[n]$$

$$U\left(e^{j\Omega}\right)\left(1 - e^{-j\Omega}\right) = 1$$

$$U\left(e^{j\Omega}\right) = \frac{1}{1 - e^{-j\Omega}}$$

Examples of FT pairs

- $\mathrm{F}\{\mathrm{u}[n]\} = ?$

Example

- F{u$[n]$} $=?$
- $u[n] = u_e[n] + u_o[n]$

Example

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    - $u[n] = u_e[n] + u_o[n]$
    - $u_e[n] = \frac{1}{2}\delta[n] + \frac{1}{2}$

Example

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  - $u[n] = u_e[n] + u_o[n]$
  - $u_e[n] = \frac{1}{2}\delta[n] + \frac{1}{2}$
    - $F\{u_e[n]\} = \frac{1}{2} + \frac{1}{2}\sum_{p=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi p)$

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  - $u_o[n] - u_o[n-1] = \frac{1}{2}(\delta[n] + \delta[n-1])$

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  - $u_o[n] - u_o[n-1] = \frac{1}{2}(\delta[n] + \delta[n-1])$
    - $F\{u_o[n]\} = \frac{1+e^{-j\Omega}}{2(1-e^{-j\Omega})}$

Example

- F{u[n]} $=$ *?*
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  - $u_e[n] = \frac{1}{2}\delta[n] + \frac{1}{2}$
    - $F\{u_e[n]\} = \frac{1}{2} + \frac{1}{2}\sum_{p=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi p)$
  - $u_o[n] - u_o[n-1] = \frac{1}{2}(\delta[n] + \delta[n-1])$
    - $F\{u_o[n]\} = \frac{1+e^{-j\Omega}}{2(1-e^{-j\Omega})}$
  - F{u[n]} $= F\{u_e[n]\} + F\{u_o[n]\} = \frac{1}{(1-e^{-j\Omega})} + \sum_{p=-\infty}^{\infty} \pi\delta(\Omega - 2\pi p)$

Example

- $\Omega_o = \frac{2\pi}{N}$
- $F\left\{\sum_{k=0}^{N-1} a_k\, e^{jk\Omega_o n}\right\} =?$
- $F\left\{\sum_{k=0}^{N-1} a_k\, e^{jk\Omega_o n}\right\} = \sum_{k=0}^{N-1} a_k \sum_{p\in I} 2\pi\delta(\Omega - k\Omega_o - 2\pi p)$

$$=\sum_{k=0}^{N-1} 2\pi a_k \sum_{p\in I} \delta(\Omega - k\Omega_o - 2\pi p)$$

Example

- $F\left\{\sum_{k=0}^{N-1} a_k \, e^{jk\Omega_o n}\right\} = \sum_{k=0}^{N-1} 2\pi a_k \sum_{p \in I} \delta(\Omega - k\Omega_o - 2\pi p)$

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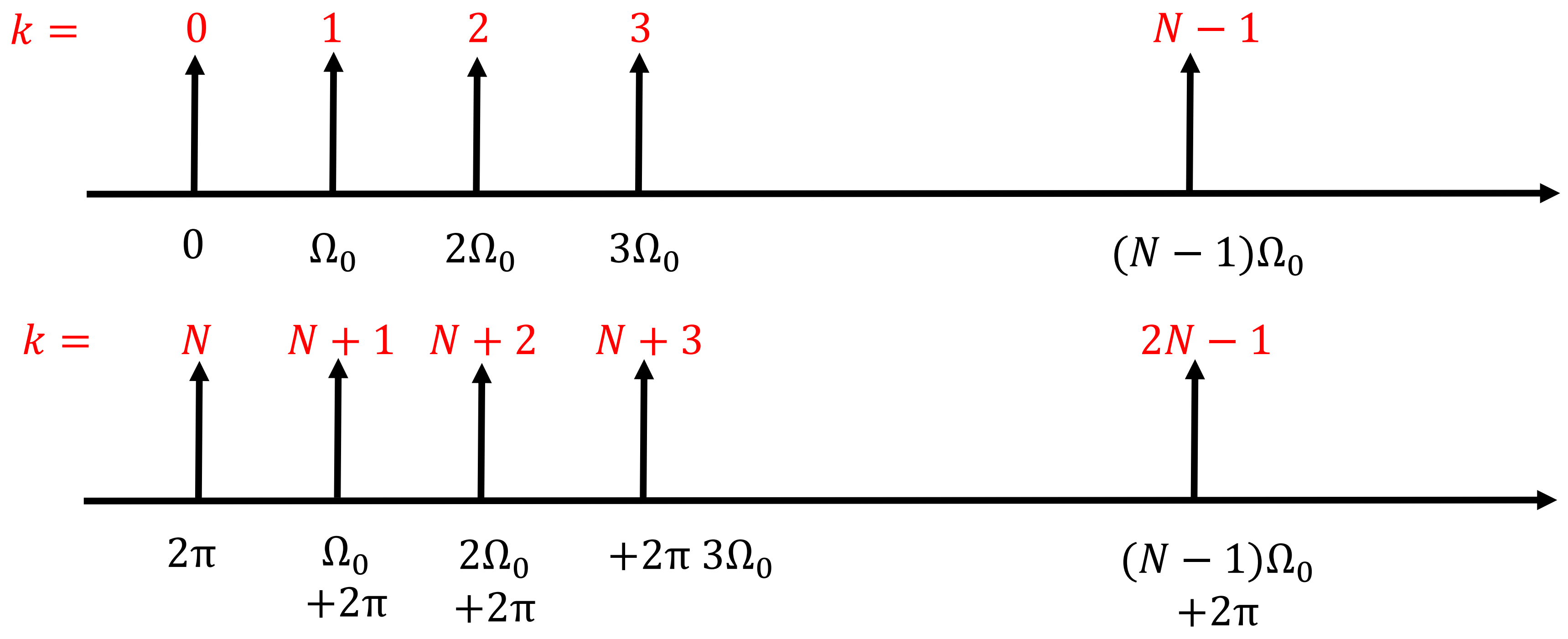

Example

- $F\left\{\sum_{k=0}^{N-1} a_k\, e^{jk\Omega_o n}\right\}=\sum_{k=0}^{N-1} 2\pi a_k \sum_{p\in I} \delta(\Omega - k\Omega_o - 2\pi p) =$

$$\sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\Omega - k\Omega_o)$$

Properties of DTFT

- Convolution $x[n] * h[n] = X(e^{j\Omega})\,Y(e^{j\Omega})$

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- $y[n] = \sum_{m=-\infty}^{\infty} x[m]u[n-m] = (x * u)[n]$

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Proof:

- $y[n] = \sum_{m=-\infty}^{\infty} x[m]u[n-m] = (x * u)[n]$
- $Y(e^{j\Omega}) = X(e^{j\Omega}) \times \{\frac{1}{(1-e^{-j\Omega})} + \sum_{p=-\infty}^{\infty} \pi\delta(\Omega - 2\pi p)\}$
- $Y(e^{j\Omega}) = \frac{X(e^{j\Omega})}{(1-e^{-j\Omega})} + \sum_{p=-\infty}^{\infty} \pi X(e^{j0})\delta(\Omega - 2\pi p)$

Properties of DTFT

- Differentiation in frequency

$$nx[n] \rightarrow j\frac{dX(e^{j\Omega})}{d\Omega}$$

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- Differentiation in frequency

$$nx[n] \to j\frac{dX(e^{j\Omega})}{d\Omega}$$

Proof:

$$X(e^{j\Omega}) = \sum_n x[n]e^{-j\Omega n}$$

$$\frac{dX(e^{j\Omega})}{d\Omega} = \sum_n -jnx[n]e^{-j\Omega n}$$

Properties of DTFT

- Multiplication

$$x[n]y[n] \rightarrow \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\Omega-\theta)}\right)d\theta$$

Properties of DTFT

$$\hat{X}(e^{j\Omega}) = \begin{cases} X(e^{j\Omega})\ for\ -\pi < \Omega \leq \pi \\ 0 \quad otherwise \end{cases}$$

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\,Y(e^{j(\Omega-\theta)})d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}(e^{j\Omega})\,Y(e^{j(\Omega-\theta)})d\theta$$

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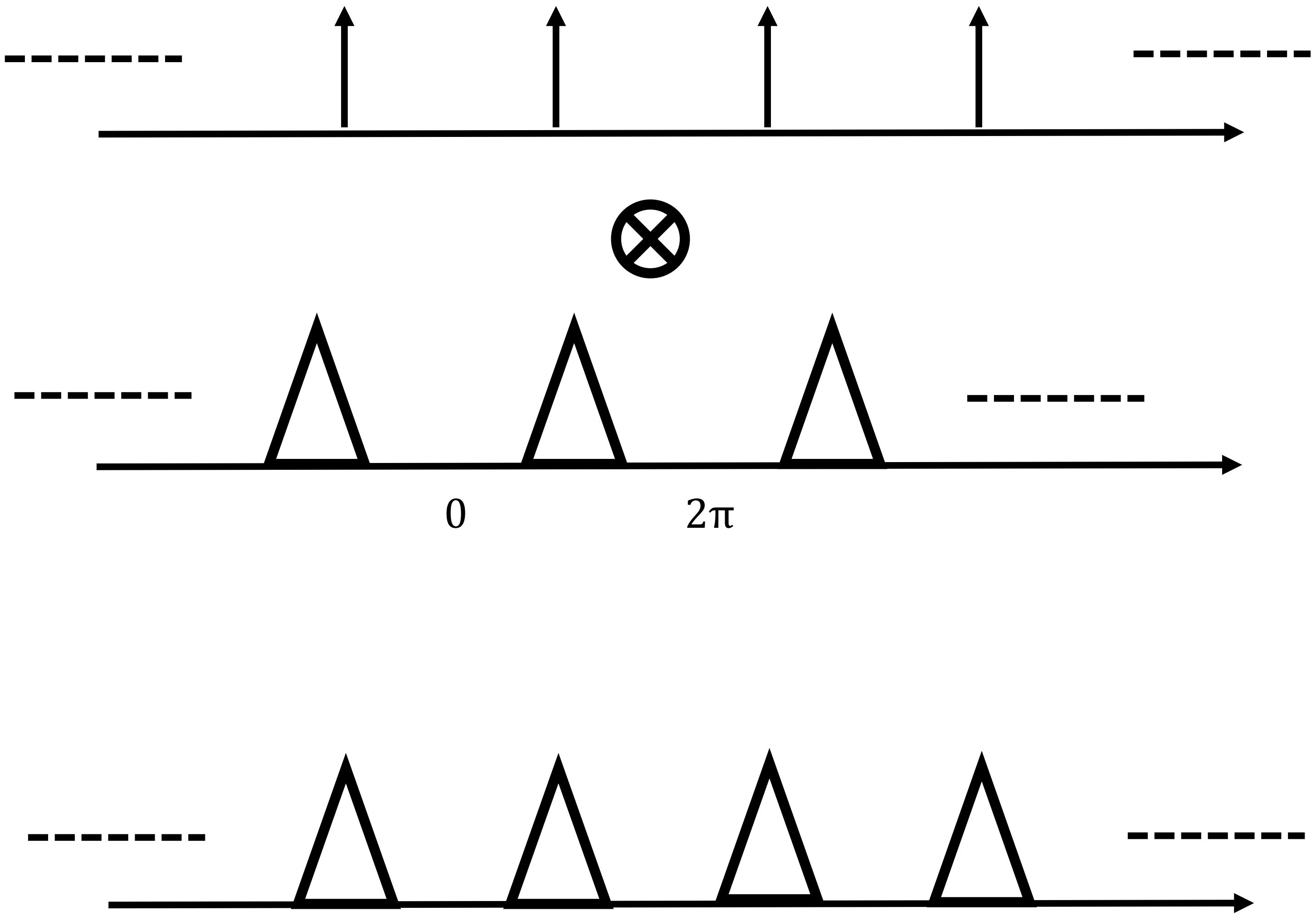
Mechanics: Aperiodic convolution with one signal as aperiodic

Properties of DTFT



Properties of DTFT

- Parsevals

$$\sum_{n} x[n]y[n]e^{-j\Omega n} = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\Omega-\theta)}\right)d\theta$$

$$\sum_{n} x[n]\overline{y[n]}e^{-j\Omega n} = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\theta}\right)\overline{Y\left(e^{j(\theta-\Omega)}\right)}\, d\theta$$

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Duality between CTFS and DTFT

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$$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\frac{2\pi}{T}t}dt$$

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$$a_k = \frac{1}{2\pi}\int_{2\pi} x(t)e^{-jkt}dt$$

$t \rightarrow \Omega$

$$a_k = \frac{1}{2\pi}\int_{2\pi} x(\Omega)e^{-jk\Omega}d\Omega$$

Properties of DTFT

Duality between CTFS and DTFT

$$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\frac{2\pi}{T}t}dt$$

$$x(t) = \sum_{k} a_k e^{jk}\frac{2\pi}{T}t$$

$$a_k = \frac{1}{2\pi}\int_{2\pi} x(t)e^{-jk\frac{2\pi}{2\pi}t}dt$$

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$$a_k = \frac{1}{2\pi}\int_{2\pi} x(t)e^{-jkt}dt$$

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$t \rightarrow \Omega$

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$$x(\Omega) = \sum_{k} a_k e^{jk\Omega}$$