Commutator : The multiplication of linear operators defined above is not commutative in general

i.e. $A \cdot B \neq B \cdot A$

$$[A, B] = A \cdot B - B \cdot A$$

is referred to as the commutator of A & B.

* An operator equation $A = B$ mean $A|\alpha\rangle = B|\alpha\rangle \quad \forall\, |\alpha\rangle \in V$

E.g. Let V = space of polynomial (differentiable) functions on $\mathbb{R}$. Consider the following two linear operators on V

$$\hat{x}[f(x)] = x f(x)$$

$$\hat{p}[f(x)] = \frac{d}{dx} f(x)$$

Ex: Verify that these are linear

$$\Rightarrow \quad \hat{x}\cdot\hat{p}[f(x)] = \hat{x}\left[\frac{d}{dx} f(x)\right] = x\frac{d}{dx} f(x)$$

$$\hat{p}\cdot\hat{x}[f(x)] = \hat{p}[x f(x)] = \frac{d}{dx}[x f(x)] = f(x) + x\frac{d f(x)}{dx}$$

$$\therefore \quad (\hat{x}\cdot\hat{p} - \hat{p}\cdot\hat{x})[f(x)] = -f(x) \equiv -\mathbb{1}[f(x)] \quad \forall\, f(x) \in V$$

$$* \; [\hat{x}, \hat{p}] = \boxed{\hat{x}\cdot\hat{p} - \hat{p}\cdot\hat{x} = -\mathbb{1}}$$

$$\boxed{\mathbb{1}[f(x)] = f(x)}$$

Functions of linear operators :

Using the elementary notions of addition & multiplication discussed above we can define various functions of linear operators in anology with the usual functions of real/complex numbers

Polynomials : $a_0\mathbb{1} + a_1 A + a_2 A^2 + a_3 A^3 + \cdots + a_n A^n$

Exponential & Trignometric functions : defined by their Taylor series

- $$e^A = \mathbb{1} + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

- $$\sin(A) = \frac{e^{iA} - e^{-iA}}{2i} \; ; \quad \cos(A) = \frac{e^{iA} + e^{-iA}}{2}$$

& so on.

Inverse of a Linear operator :

Let $A : V \rightarrow V$ be a linear operator on V_F. An operator A_I satisfying

$$A_l^{-1} \cdot A |\alpha\rangle = |\alpha\rangle \quad \forall \; |\alpha\rangle \in V_F$$

is called the left inverse of A. $\boxed{A_l^{-1} \cdot A = \mathbb{1}}$; $\boxed{A \cdot A_r^{-1} = \mathbb{1}}$

* Inverse of an operator does not always exist !

Consider an operator which maps a non zero vector in V_F to $|0\rangle$

i.e. $A|\alpha_0\rangle = |0\rangle$

If A_l^{-1} exists then we would have

$$A_l^{-1}|0\rangle = A_l^{-1} A |\alpha_0\rangle = |\alpha_0\rangle$$

Since no linear operator can map $|0\rangle$ to a non zero vector, we are lead to a contradiction i.e. A_l^{-1} doesn't exist if A has a non trivial Kernal.

Such operators are often referred to as Singular operators & the set of vectors which are mapped to $|0\rangle$ by A is called the Kernal or Null space of A.

- The inverse of an operator exists only when it is a one-to-one & onto map on V i.e.

$$\forall \; |\alpha\rangle \neq |\beta\rangle \in V$$

$$A|\alpha\rangle \neq A|\beta\rangle \Rightarrow A(|\alpha\rangle - |\beta\rangle) \neq |0\rangle$$

i.e. the operator A is non singular.

- For non singular / Invertible operators left & right inverse are the same $\boxed{A^{-1} \cdot A = \mathbb{1} = A \cdot A^{-1}}$

$A^{-1} \cdot A = \mathbb{1}$
multiplying both sides with A on left
$[A \cdot A^{-1}] \cdot A = A \quad \Rightarrow \quad A \cdot A^{-1} = \mathbb{1}$

Ex.
If $A \cdot B = A$ (or $B \cdot A = A$) for non singular A then $B = \mathbb{1}$

- For non-singular A & B : $\boxed{(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}}$

$$(A \cdot B)^{-1} \cdot (A \cdot B) = B^{-1} A^{-1} \cdot A \cdot B = B^{-1} \mathbb{1} \cdot B = \mathbb{1}$$

$$(A \cdot B) \cdot (A \cdot B)^{-1} = (A \cdot B) \cdot B^{-1} A^{-1} = A \cdot \mathbb{1} \cdot A^{-1} = \mathbb{1}$$

(Scalar/inner)

Hermitian Conjugate (Adjoint) of an Operator : Let V_F be a complex vector space equiped with an Scalar product $\langle * | * \rangle$. The Hermitian conjugate of an operator A is then defined as

$$A^{\dagger} \quad \text{s.t.} \quad \langle \beta | (A^{\dagger} | \alpha \rangle) = \left(\langle \alpha | (A | \beta \rangle) \right)^{*} \qquad \forall \; |\alpha\rangle, |\beta\rangle \in V$$

- Note that the notion of Hermitian conjugate requires & depends upon the scalar product. Different scalar products on same vector space lead to different Hermitian conjugates of same operator.

- $$\langle \beta | (A^{\dagger})^{\dagger} | \alpha \rangle = \left(\langle \alpha | A^{\dagger} | \beta \rangle \right)^{*} = \left(\left(\langle \beta | A | \alpha \rangle \right)^{*} \right)^{*} = \langle \beta | A | \alpha \rangle$$
  $$\Rightarrow \boxed{(A^{\dagger})^{\dagger} = A} \qquad \forall \; |\alpha\rangle, |\beta\rangle \in V$$

- $|\gamma\rangle = A|\beta\rangle$ then the dual vector $\langle \gamma | = \langle \beta | A^{\dagger}$
  $$\langle \gamma | \alpha \rangle = \langle \alpha | \gamma \rangle^{*} = \left(\langle \alpha | A | \beta \rangle \right)^{*} = \langle \beta | A^{\dagger} | \alpha \rangle \qquad \forall \; |\alpha\rangle \in V$$
  $$\Rightarrow \boxed{\langle \gamma | = \langle \beta | A^{\dagger}}$$

Ex: Show 1. $(aA + bB)^{\dagger} = a^{*} A^{\dagger} + b^{*} B^{\dagger}$

2. $(A \cdot B)^{\dagger} = B^{\dagger} \cdot A^{\dagger}$

Hermitian operators : An operator A satisfying : $A^{\dagger} = A$.

{ Hermition operators play a central role in Q.M. where they represent the operators correspondings to observable properties of the system like Energy, momentum, position of particles e.t.c. }

Unitary operators : An operator U satisfying : $U^{\dagger} U = \mathbb{1}$

Equivalently $U^{\dagger} = U^{-1}$

{ The evolution of states & the action of symmetry transformations in Q.M. are governed by <u>Unitary operators</u> Since they <u>preserve the "norm of states"</u>

We will later see a connection between Unitary & Hermitian operators. }