Eigenvectors & Eigenvalues of Operators :

An Eigenvector of a Linear operator A on vector space V_F is a vector $|\alpha\rangle \in V$ Satisfying

$$A|\alpha\rangle = \lambda_\alpha |\alpha\rangle \qquad \text{where } \lambda_\alpha \in F$$

The scalar λ_α is called an Eigenvalue of A.

Comments

- Clearly, if $|\alpha\rangle$ is an eigenvector of A then $a|\alpha\rangle$ is also an Eigenvector with the same Eigenvalue.
- An operator can have many different Eigenvectors & Eigenvalues.
- Distinct eigenvectors of an operator A can have same eigenvalues.

Some examples :

- Consider the simplest example, the Identity operator

$$I|\alpha\rangle = 1.|\alpha\rangle$$

  - Every vector is an eigenvector of I with eigenvalue 1.
  - Thus identity operator has $n = \dim(V)$ Linearly independent Eigenvectors.

- The projection operator $P_\alpha = \dfrac{|\alpha\rangle\langle\alpha|}{\langle\alpha|\alpha\rangle}$ has

  ① $P_\alpha|\alpha\rangle = |\alpha\rangle$

  ② $\forall\ |\beta\rangle$ s.t. $\langle\alpha|\beta\rangle = 0$

$$P_\alpha|\beta\rangle = 0$$

$$\boxed{\begin{aligned} P &= P_1 + P_2 \\ &= |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| \\ \& \ |\alpha\rangle &= \alpha_1|e_1\rangle + \alpha_2|e_2\rangle \\ \Rightarrow P|\alpha\rangle &= |\alpha\rangle \end{aligned}}$$

  - Thus P_α has 1 Eigenvector with Eigenvalue 1 and $(n-1)$ linearly independent (orthogonal to $|\alpha\rangle$) eigenvectors with eigenvalue 0.

Ex. Consider $P = a_1|\alpha_1\rangle\langle\alpha_1| + a_2|\alpha_2\rangle\langle\alpha_2|$ with $\langle\alpha_1|\alpha_2\rangle = 0$. Can you find all the Eigenvectors & Eigenvalues of P ?

- <u>Eigenvalues & Eigenvectors of Hermitian operators</u> :

As mentioned earlier, Hermitian operators play a central role in many physical application, in particular in Quantum mechanics where they represent operators corresponding to observable quantities.

We will now discuss two important results regarding Eigenvalues & Eigenvectors of Hermition operators.

1. All eigenvalues of a hermitian operator are real.

$$H|\alpha\rangle = \lambda_\alpha|\alpha\rangle$$

$$\Rightarrow \lambda_\alpha = \frac{\langle\alpha|H|\alpha\rangle}{\langle\alpha|\alpha\rangle}$$

$$\lambda_\alpha^* = \frac{\langle\alpha|H|\alpha\rangle^*}{\langle\alpha|\alpha\rangle^*} = \frac{\langle\alpha|H^\dagger|\alpha\rangle}{\langle\alpha|\alpha\rangle} = \frac{\langle\alpha|H|\alpha\rangle}{\langle\alpha|\alpha\rangle} = \lambda_\alpha$$

2. Eigenvectors of Hermitian operators with different Eigenvalues are orthogonal to each other.

Let
$$H|\alpha\rangle = \lambda_\alpha|\alpha\rangle$$
$$H|\beta\rangle = \lambda_\beta|\beta\rangle$$

$$\langle\beta|H|\alpha\rangle = \lambda_\alpha\langle\beta|\alpha\rangle$$
$$\langle\alpha|H|\beta\rangle = \lambda_\beta\langle\alpha|\beta\rangle$$

$$\underset{\lambda_\alpha\langle\beta|\alpha\rangle^* \;=\; \lambda_\alpha\langle\alpha|\beta\rangle}{\langle\beta|H|\alpha\rangle^*} = \langle\alpha|H^\dagger|\beta\rangle = \underset{\lambda_\beta\langle\alpha|\beta\rangle}{\langle\alpha|H|\beta\rangle}$$

$$\therefore \quad \boxed{\lambda_\alpha \langle \alpha | \beta \rangle = \lambda_\beta \langle \alpha | \beta \rangle}$$

$\Rightarrow$ If $\lambda_\alpha \neq \lambda_\beta$ then $\langle \alpha | \beta \rangle = 0$

- If we have two linearly independent Eigenvectors of Hermition operator H with same Eigenvalue then they may not necessarily be orthogonal!

In fact, if $|\alpha_1\rangle, |\alpha_2\rangle$ are s.t. $H|\alpha_1\rangle = \lambda|\alpha_1\rangle$
$H|\alpha_2\rangle = \lambda|\alpha_2\rangle$, then

$\forall \ |\alpha\rangle = a_1|\alpha_1\rangle + a_2|\alpha_2\rangle$, we have

$$H|\alpha\rangle = \lambda|\alpha\rangle$$

☆ This implies that within the subspace spanned by eigenvectors (of Hermition operator) with same eigenvalue, we can simply use the Gram-Schmidt orthogonalization to construct eigenvectors which are orthogonal.