Lecture 22
Signals and Systems (ELL205)

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Lecture 22: Introduction to Fourier Transforms

Problems

Fourier Transform of $u(t)$?

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Fourier Transform of $\boldsymbol{u(t)}$?

Method 1: $U(\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-j\omega t}dt = \frac{-1}{j\omega}\left(e^{-j\omega t}\right)_0^{\infty}$ (Unsolvable)

Problems

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Method 2:

$$U(\omega) = F\{\lim_{a\to 0}(e^{-at}u(t))\} = \lim_{a\to 0}(F\{e^{-at}u(t)\}) = \lim_{a\to 0}\left(\frac{1}{a+j\omega}\right)$$

Problems

Fourier Transform of $\boldsymbol{u(t)}$?

Method 1: $U(\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-j\omega t}dt = \frac{-1}{j\omega}\left(e^{-j\omega t}\right)_0^{\infty}$ (Unsolvable)

Method 2:

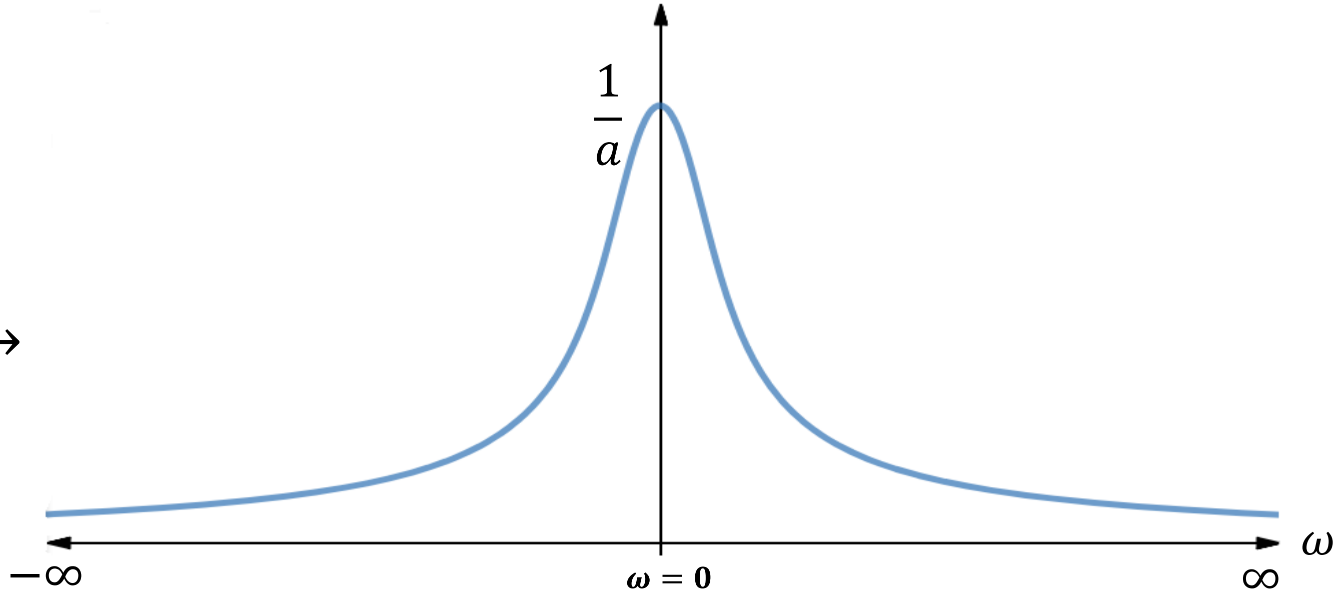
$$U(\omega) = F\{\lim_{a\to 0}(e^{-at}u(t))\} = \lim_{a\to 0}(F\{e^{-at}u(t)\}) = \lim_{a\to 0}\left(\frac{1}{a+j\omega}\right)$$

Multiplying both numerator and denominator by $a - j\omega$

$$\lim_{a\to 0}\left(\frac{1}{a+j\omega}\right) = \lim_{a\to 0}\left(\frac{a-j\omega}{a^2+\omega^2}\right) = \lim_{a\to 0}\left(\frac{a}{a^2+\omega^2}\right) - \lim_{a\to 0}\left(\frac{j\omega}{a^2+\omega^2}\right)$$

Problems

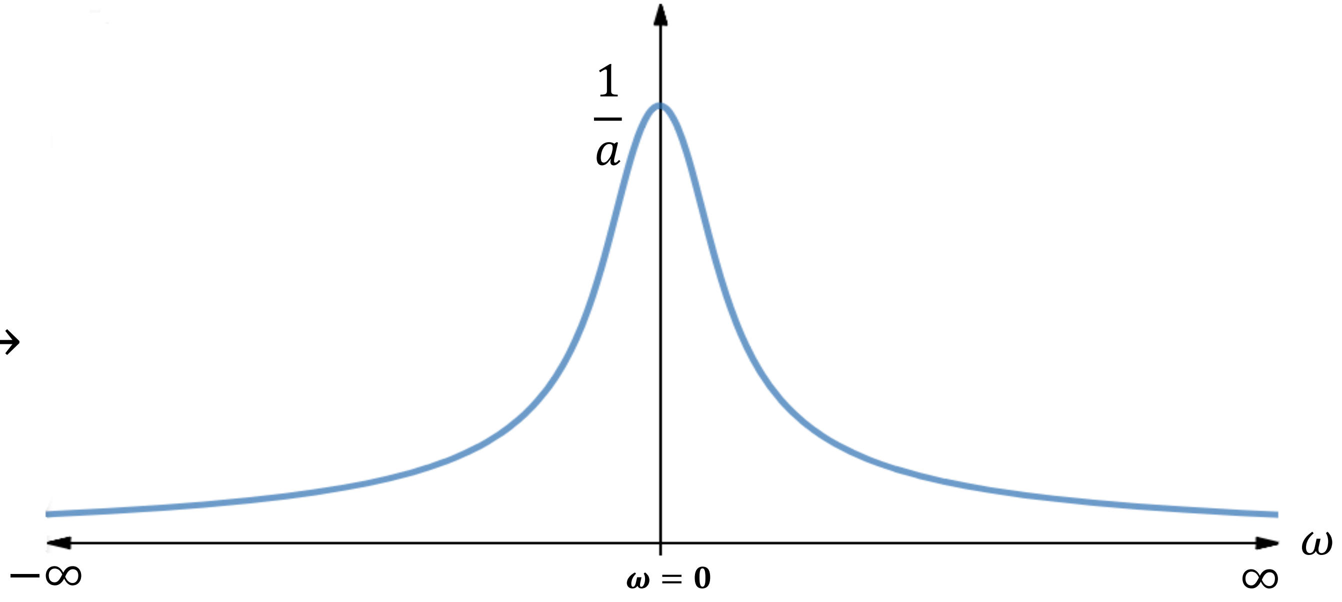
$$\frac{a}{a^2+\omega^2} \quad \rightarrow$$



Area under the curve = $\int_{-\infty}^{\infty} \frac{a}{a^2+\omega^2} d\omega = \pi$ (not a function of a)

Problems

$$\frac{a}{a^2+\omega^2} \quad \rightarrow$$



Area under the curve = $\int_{-\infty}^{\infty} \frac{a}{a^2+\omega^2} d\omega = \pi$ (not a function of a)

As $a \rightarrow 0$, the curve of $\frac{a}{a^2+\omega^2}$ becomes narrower and narrower so that it results into a impulse at $\omega = 0$.

Therefore $$\lim_{a \to 0}\left(\frac{a}{a^2+\omega^2}\right) = \pi\delta(\omega)$$

Problems

$$\lim_{a\to 0}\left(\frac{a}{a^2+\omega^2}\right)=\pi\delta(\omega)$$

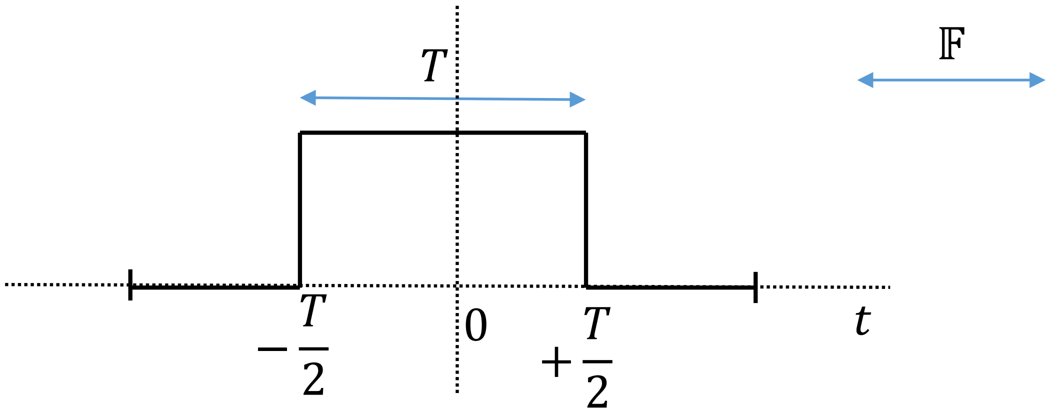
$$\lim_{a\to 0}\left(\frac{j\omega}{a^2+\omega^2}\right)=\begin{cases}0 & when\ \omega=0\\ \dfrac{j}{\omega} & when\ \omega\neq 0\end{cases}$$

$$\lim_{a\to 0}\left(\frac{1}{a+j\omega}\right)=\lim_{a\to 0}\left(\frac{a}{a^2+\omega^2}\right)\text{-}\lim_{a\to 0}\left(\frac{j\omega}{a^2+\omega^2}\right)=\begin{cases}\pi\delta(\omega) & when\ \omega=0\\ \dfrac{1}{j\omega} & when\ \omega\neq 0\end{cases}$$

$$u(t)\overset{F.T.}{\longleftrightarrow}\frac{1}{j\omega}+\pi\delta(\omega)$$

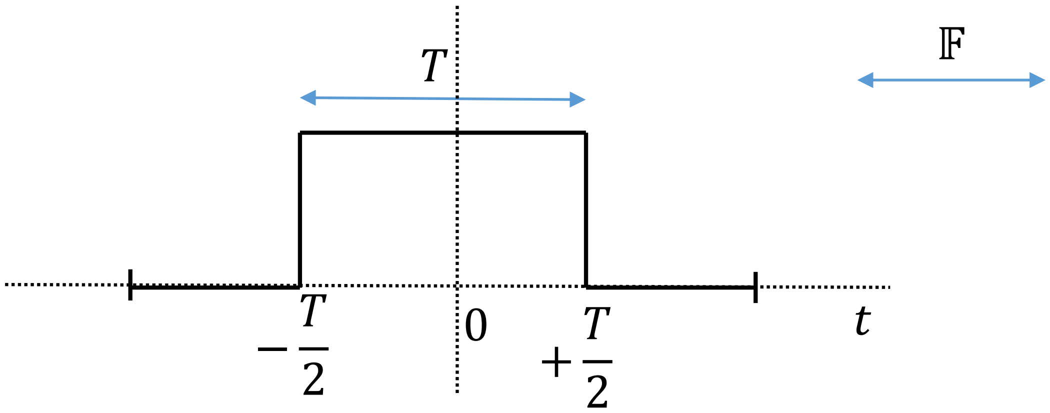
Problems

FT of $x(t) = rect\left(\frac{t}{T}\right)$?



Problems

FT of $x(t) = rect\left(\frac{t}{T}\right)$?



$$X(\omega) = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(\omega) = \frac{-1}{j\omega}\left[e^{-j\omega T/2} - e^{j\omega T/2}\right]$$

$$X(\omega) = \frac{2}{\omega 2j}\left[e^{j\omega T/2} - e^{-j\omega T/2}\right]$$

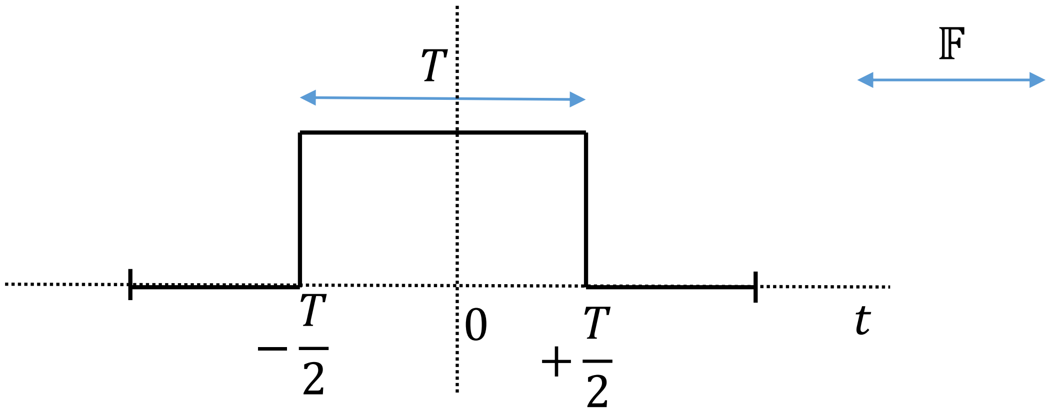
$$X(\omega) = \frac{2\sin(\omega T/2)}{\omega}$$

$$X(\omega) = \frac{T\sin(\omega T/2)}{\omega T/2}$$

$$X(\omega) = T sinc(\omega T/2)$$

Problems

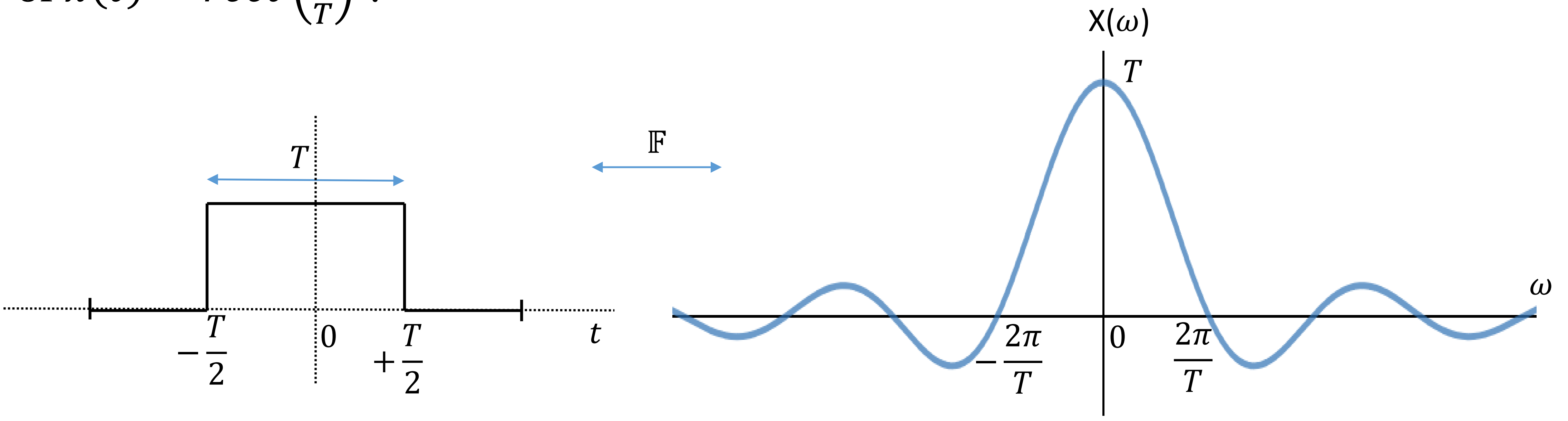
FT of $x(t) = rect\left(\frac{t}{T}\right)$?



Proof: $X(\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega t} dt = Tsinc\left(\frac{\omega T}{2}\right)$

Problems

FT of $x(t) = rect\left(\frac{t}{T}\right)$?



Proof: $X(\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega t} dt = T sinc\left(\frac{\omega T}{2}\right)$

Problems

What is the area of $T sinc\left(\frac{\omega T}{2}\right)$?

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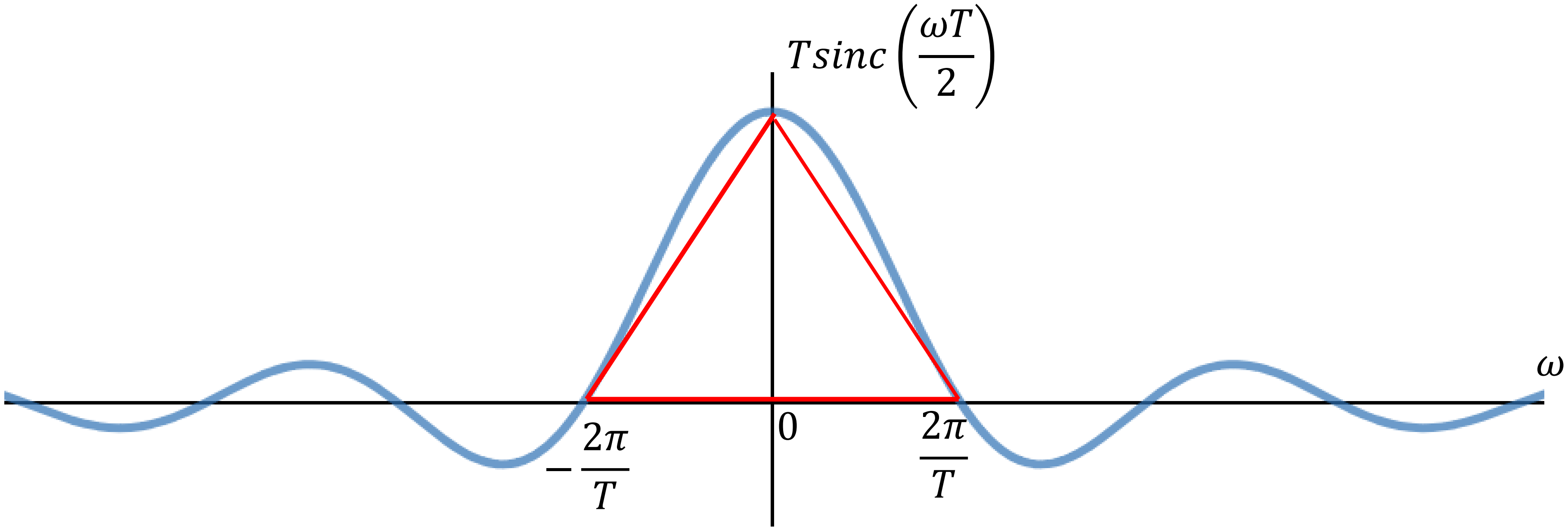
What is the area of $Tsinc\left(\frac{\omega T}{2}\right)$?



Area=$\frac{1}{2} \times \frac{4\pi}{T} \times T = 2\pi$

Moments theorem

Area of spectrum = 2π × $x(0)$

Synthesis: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

t= 0

$$2\pi x(0) = \int_{-\infty}^{\infty} X(\omega)\, d\omega$$

Area of time domain signal = X(0)

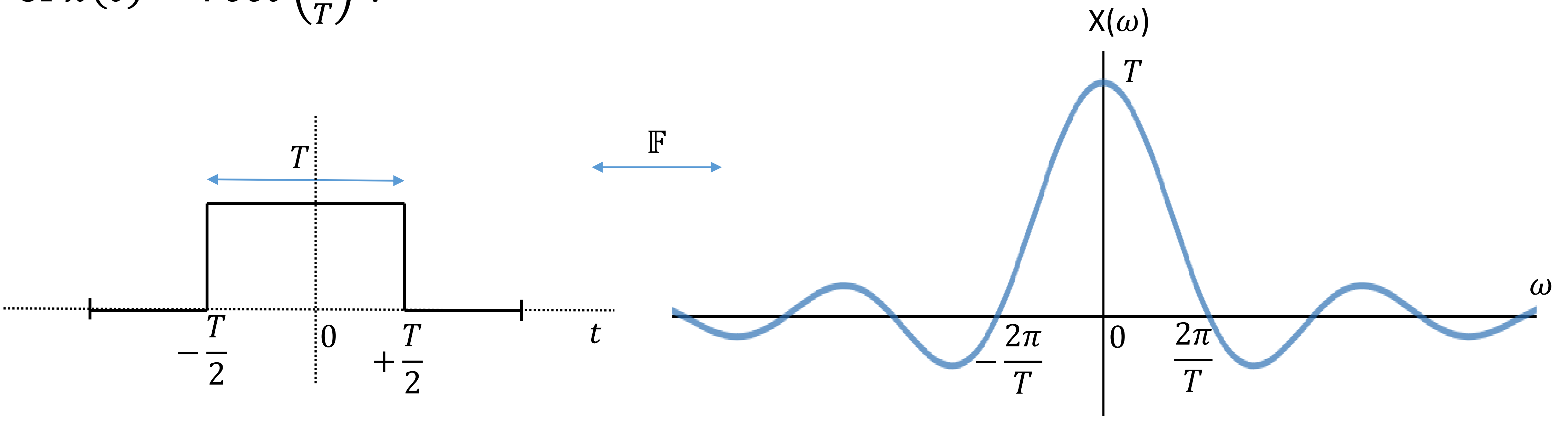
Analysis: $\mathrm{X}(\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$

ω= 0

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

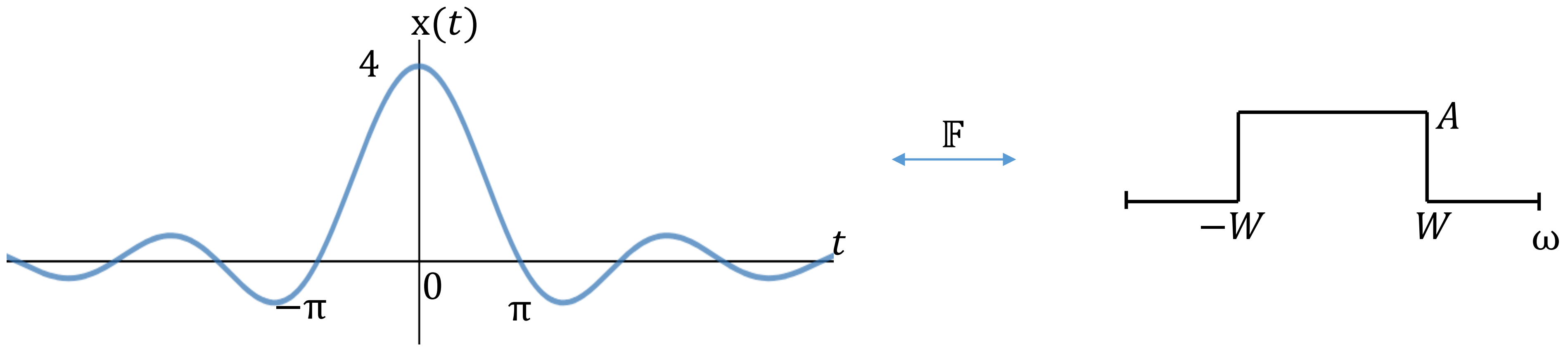
Problems

FT of $x(t) = rect\left(\frac{t}{T}\right)$?



Proof: $X(\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega t} dt = T sinc\left(\frac{\omega T}{2}\right)$

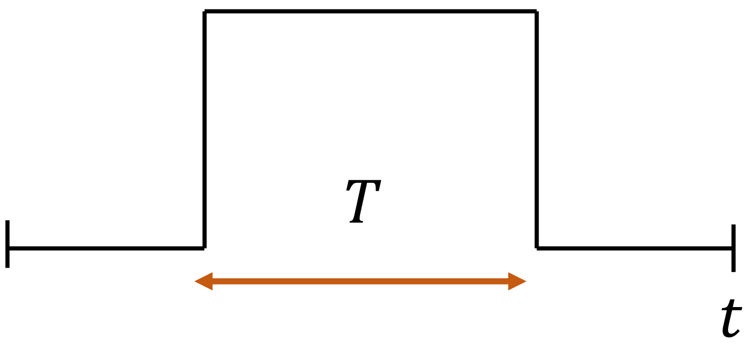
Try it yourself

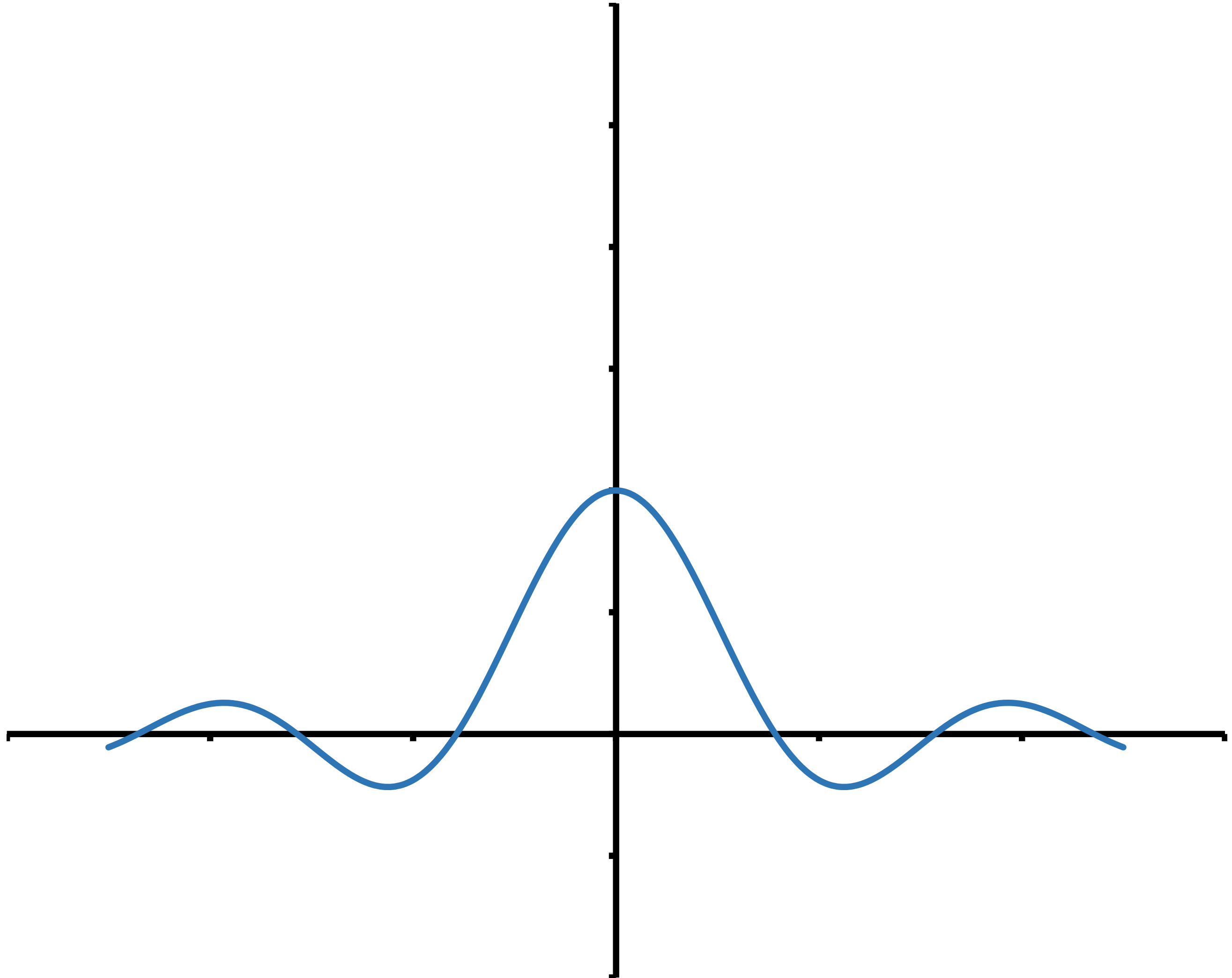


What are W and A?

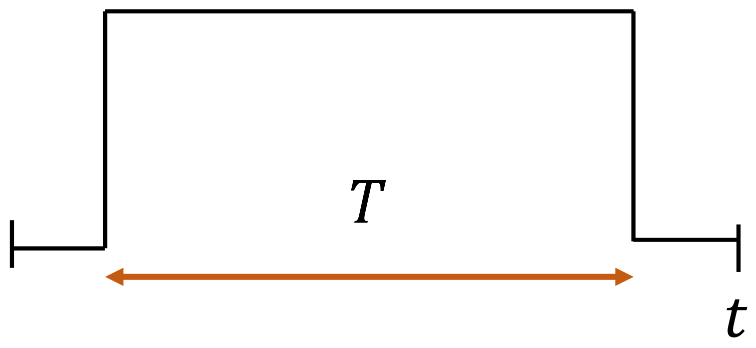
1)	$A = 4\pi \;\&\; W = 1$	**2)**	$A = 4\pi \;\&\; W = 2$
3)	$A = 2\pi \;\&\; W = 2$	**4)**	$A = 2\pi \;\&\; W = 1$

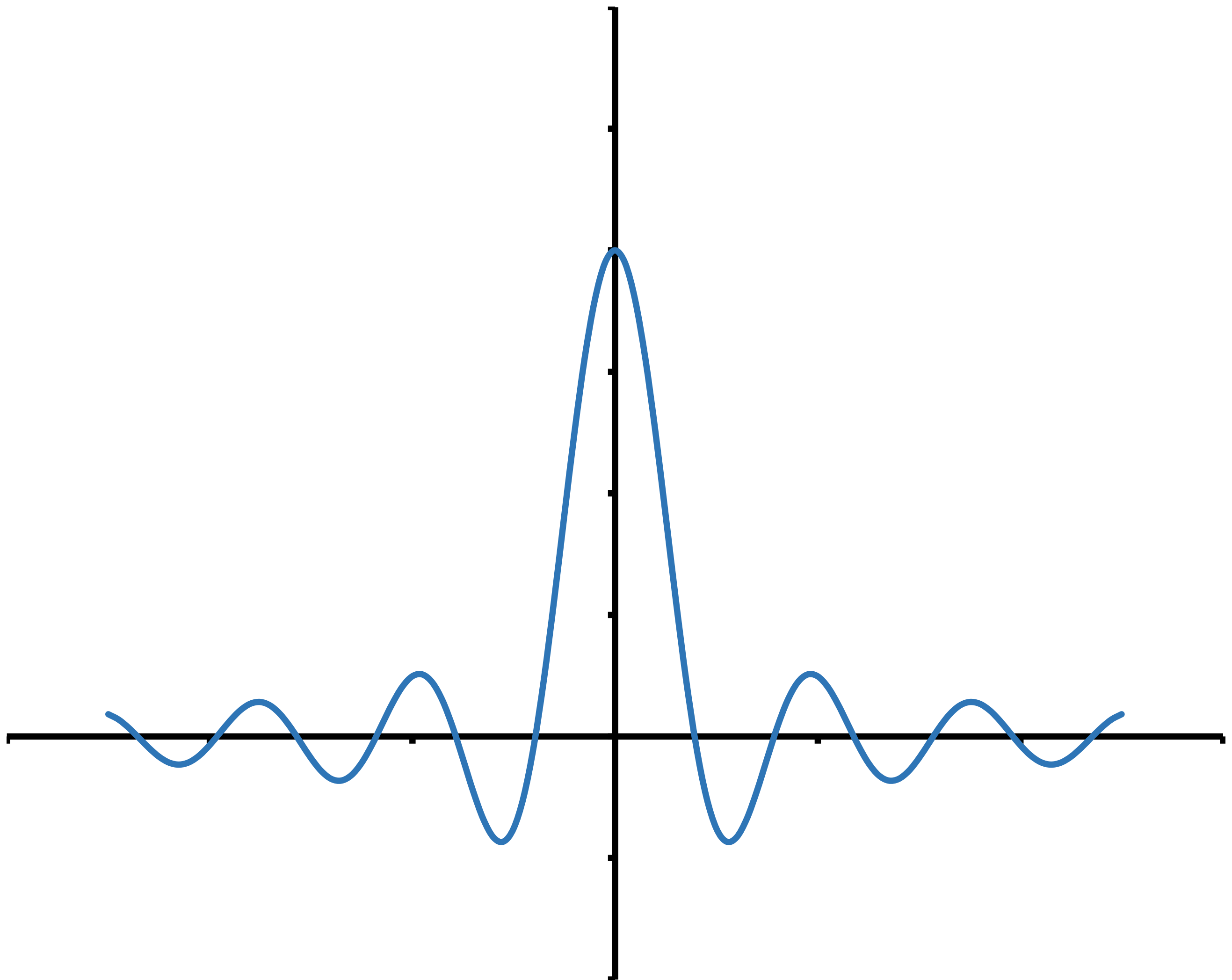
Rect & Sinc Pair



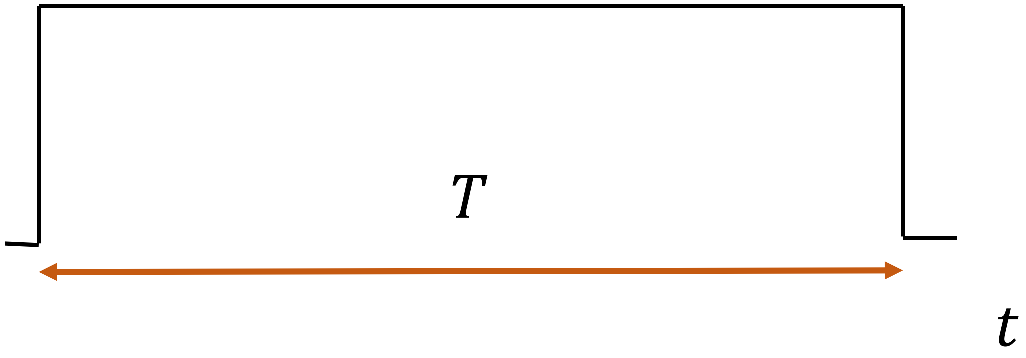


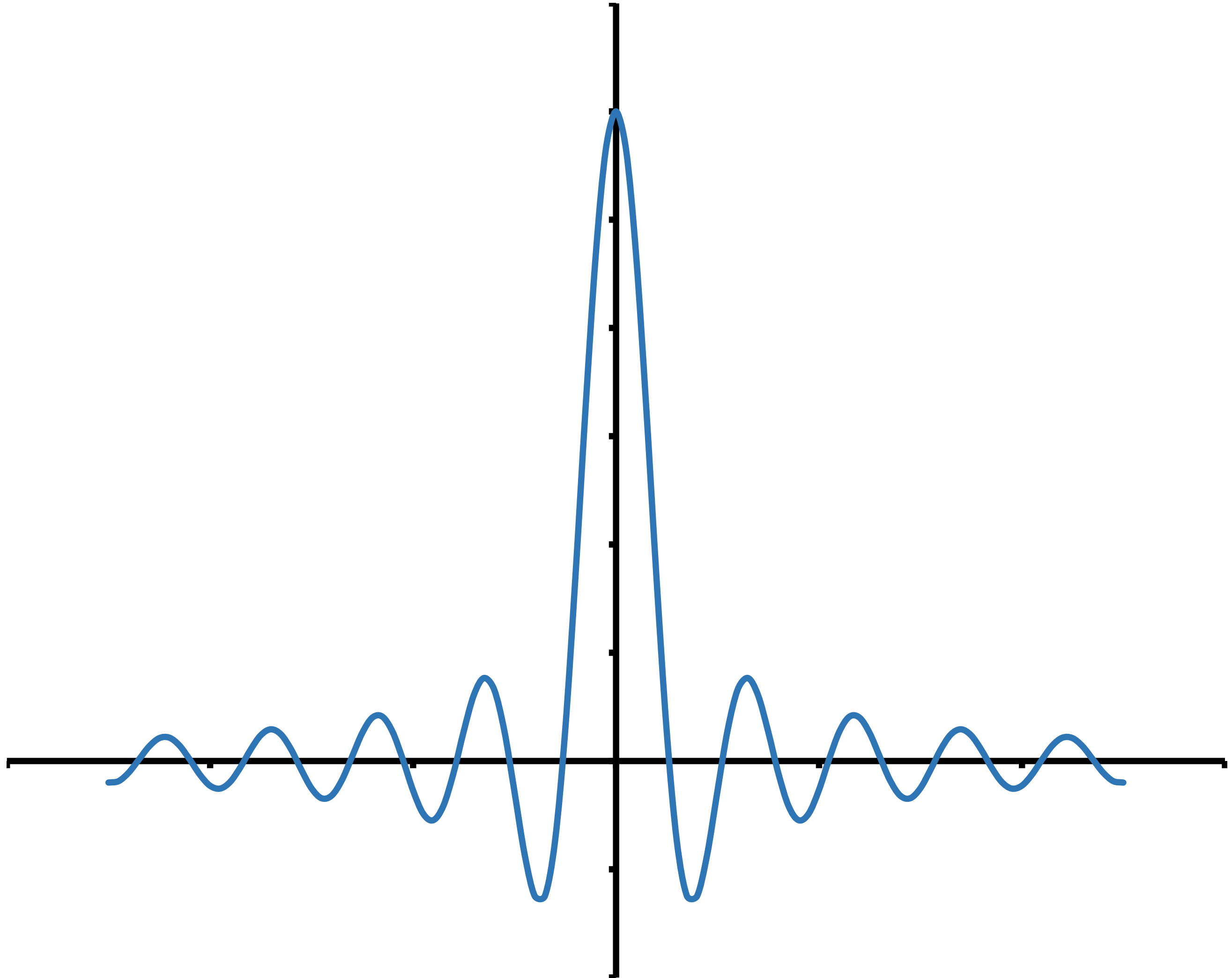
Rect & Sinc Pair





Rect & Sinc Pair





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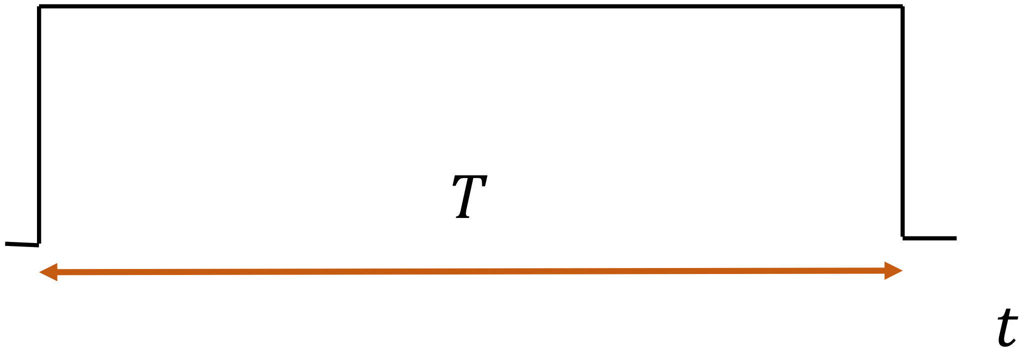
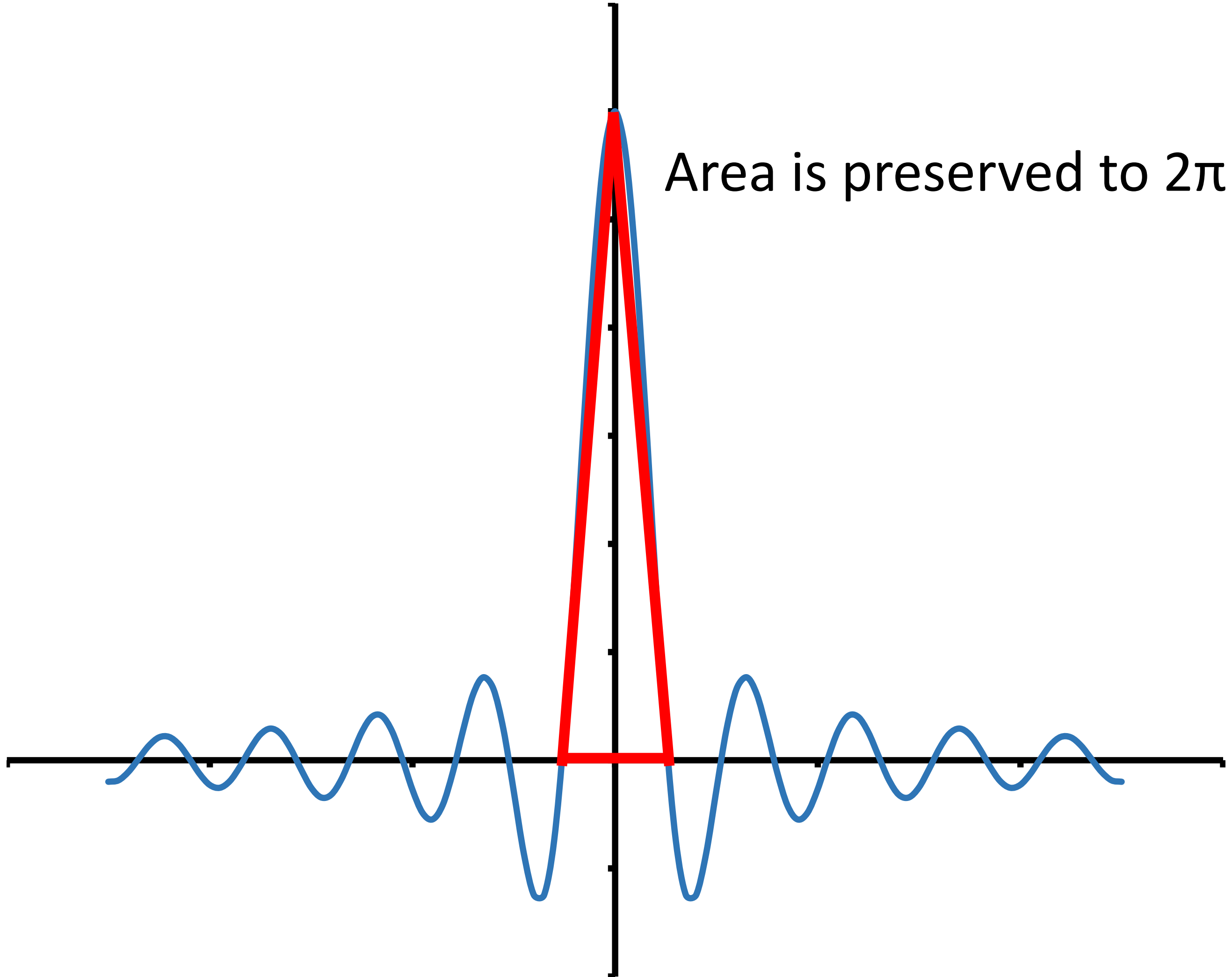
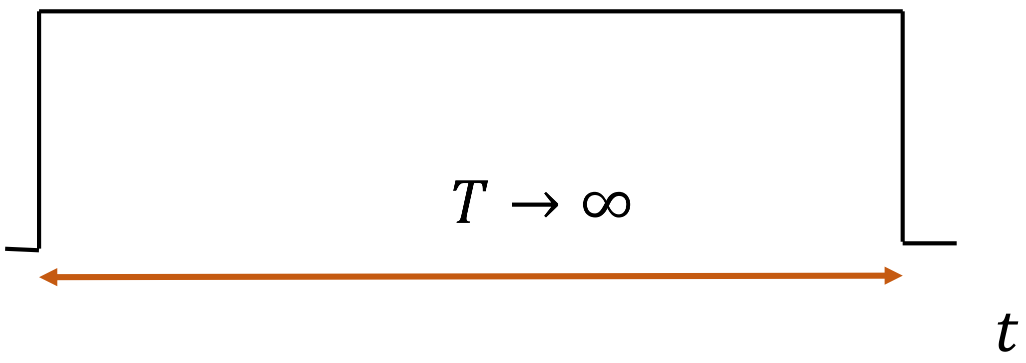
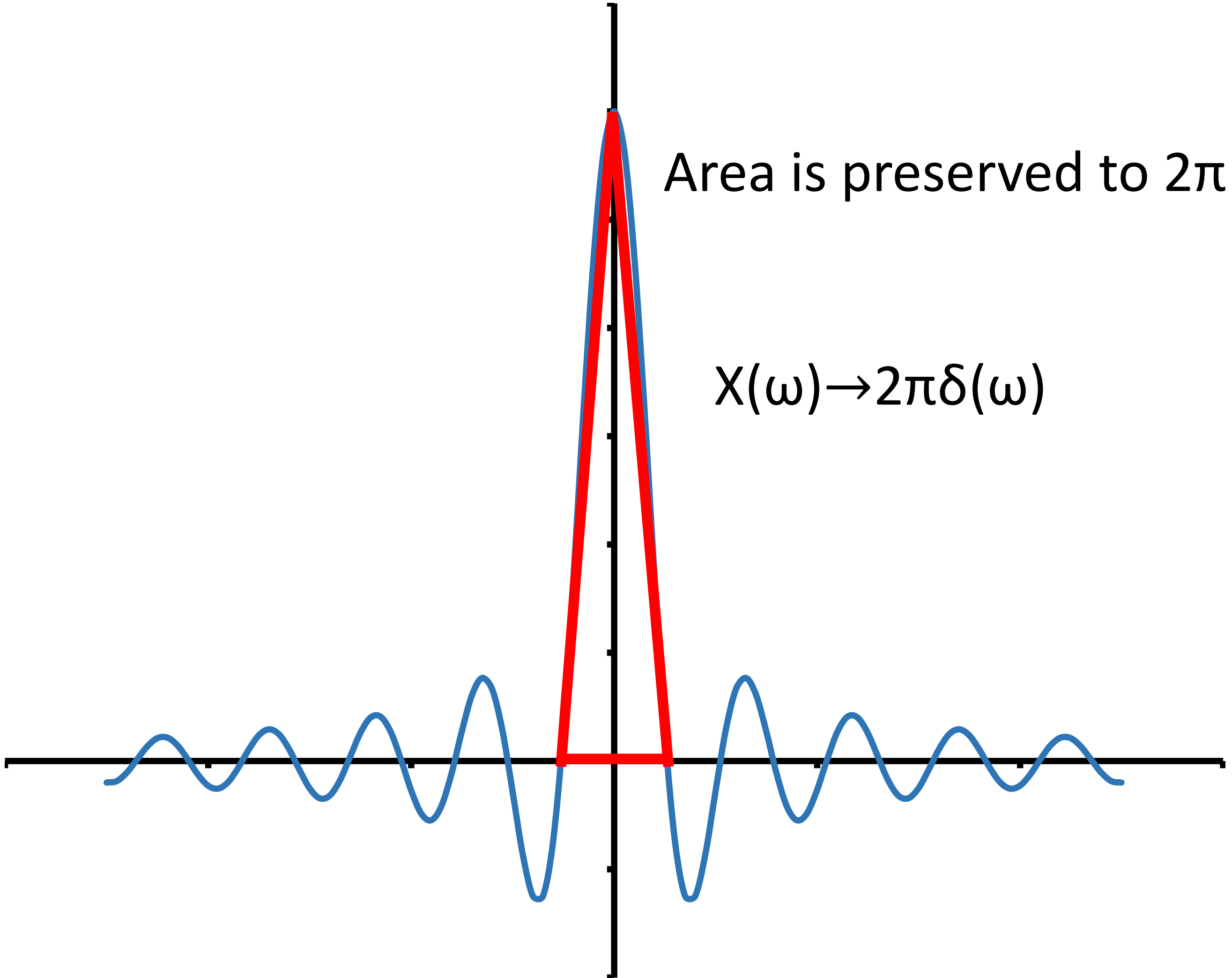




Rect & Sinc Pair





Fourier Transforms

Fourier Transform of 1 is $X(\omega)$

How many statements are correct?

1) $X(\omega) = 2\pi\delta(\omega)$	2) $X(\omega) = 2\delta(\omega)$
3) $X(\omega) = \pi\delta(\omega)$	4) $X(\omega) = \delta(\omega)$

Fourier Transforms

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$$2\pi\delta(\omega - \omega_0) = \int_{-\infty}^{\infty} e^{-j(\omega - \omega_0)t} dt$$

Fourier Transforms

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$$2\pi\delta(\omega - \omega_0) = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt$$

$$2\pi\delta(\omega - \omega_0) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt$$

Fourier Transforms

Fourier Transform of 1 is $2\pi\delta(\omega)$

$$2\pi\delta(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} dt$$

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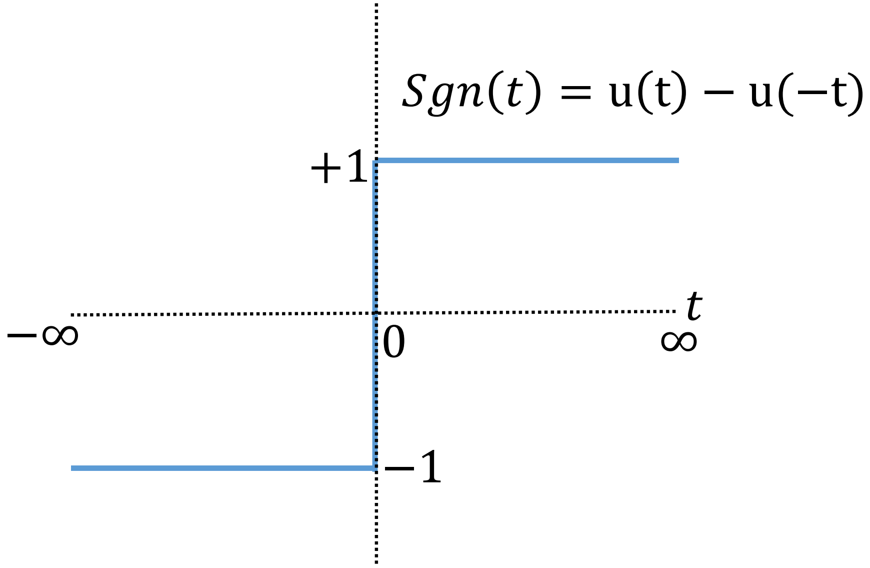
$$2\pi\delta(\omega - \omega_0) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt$$

$$2\pi\delta(\omega - \omega_0) = F[e^{j\omega_0 t}]$$

Problems

Fourier Transform of sgn$(\boldsymbol{t})$?

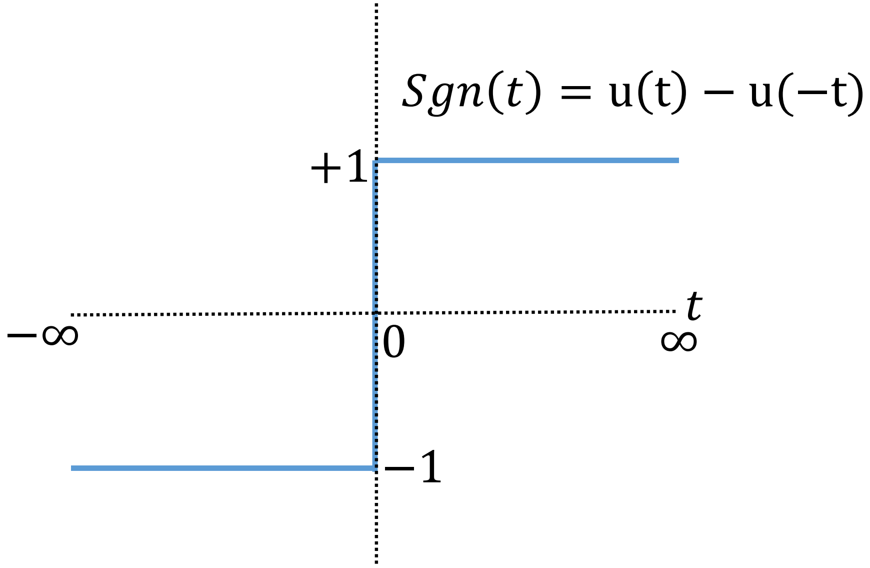
where $sgn(t) = \begin{cases} 1 & t > 0 \\ -1 & t \leq 0 \end{cases}$



Problems

Fourier Transform of sgn(t) ?

where $sgn(t) = \begin{cases} 1 & t > 0 \\ -1 & t \leq 0 \end{cases}$



Step 1: $sgn(t) = 2u(t) - 1$

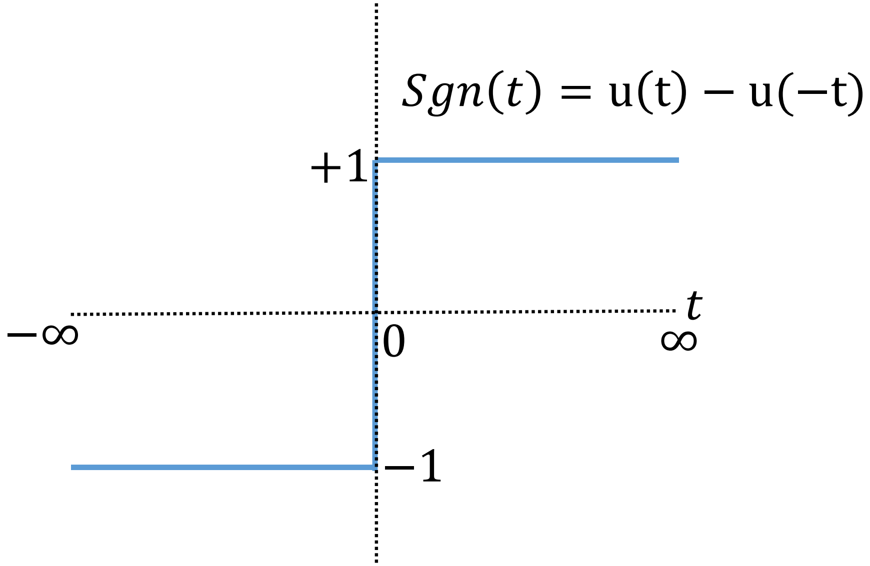
Step 2: F{sgn(t)}= F{$2u(t) - 1$} =$2(\pi\delta(\omega) \ + \frac{1}{j\omega}) - 2\pi\delta(\omega) = \frac{2}{j\omega}$

Note: Precisely, F{sgn(t)}=$\begin{cases} 0 \ \ when\ \omega = 0 \\ \frac{2}{j\omega} \ \ when\ \omega \neq 0 \end{cases}$ However, this expression is equivalent in energy sense to above expression and thus both expressions are used.

Problems

Fourier Transform of sgn(t) ?

where $sgn(t) = \begin{cases} 1 & t > 0 \\ -1 & t \leq 0 \end{cases}$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t)dt$$

$\mathrm{F\{sgn(t)\}} = \frac{2}{j\omega}$ $\mathrm{F\{sgn(t)\}} = \begin{cases} 0 & when\ \omega = 0 \\ \frac{2}{j\omega} & when\ \omega \neq 0 \end{cases}$

Fourier Transforms

Fourier Transform of $\delta(t)$ is $X(\omega)$

How many statements are correct?

1) $X(\omega) = 1$	2) $\|X(\omega)\|$ is an even function
3) $\angle X(\omega)$ is an odd function	4) $X(\omega)$ is a low pass filter

Fourier Transforms

Fourier Transform of $\delta(t)$ is $X(\omega)$

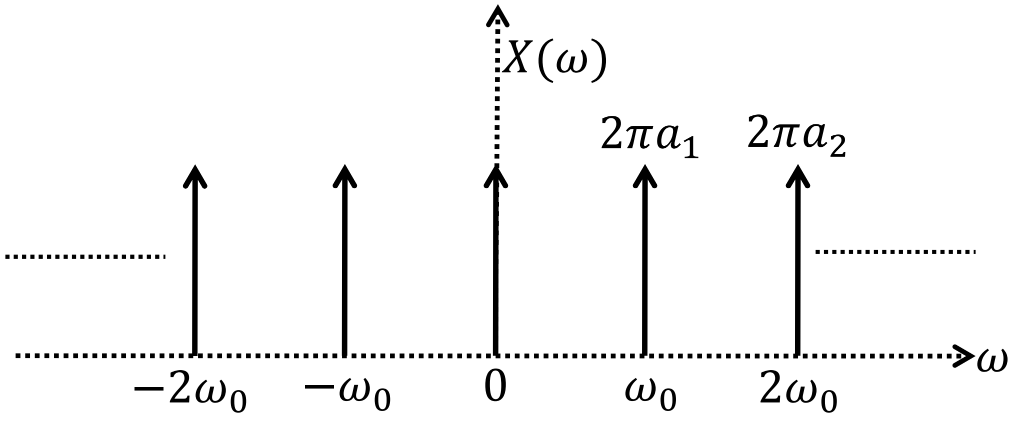
How many statements are correct?

1) $X(\omega) = 1$	2) $\lvert X(\omega) \rvert$ is an even function
3) $\angle X(\omega)$ is an odd function	4) $X(\omega)$ is a low pass filter

F. T. of Periodic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

F. T. $\quad X(\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$ (F.T. of a periodic signal)



Spectrum is composed of impulses (Discrete spectrum).

In F. S. we plot a_k by k.

In F. T. we plot $X(\omega)$ by ω.

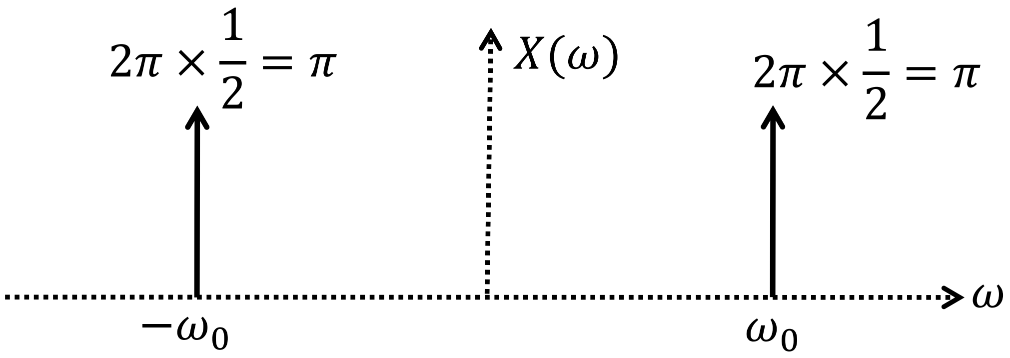
Problem

Draw the spectrum of x(t) $= \cos(\omega_0 t)$.

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$X(\omega) = \frac{1}{2} \times 2\pi\delta(\omega - \omega_0) + \frac{1}{2} \times 2\pi\delta(\omega + \omega_0)$$

$$X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



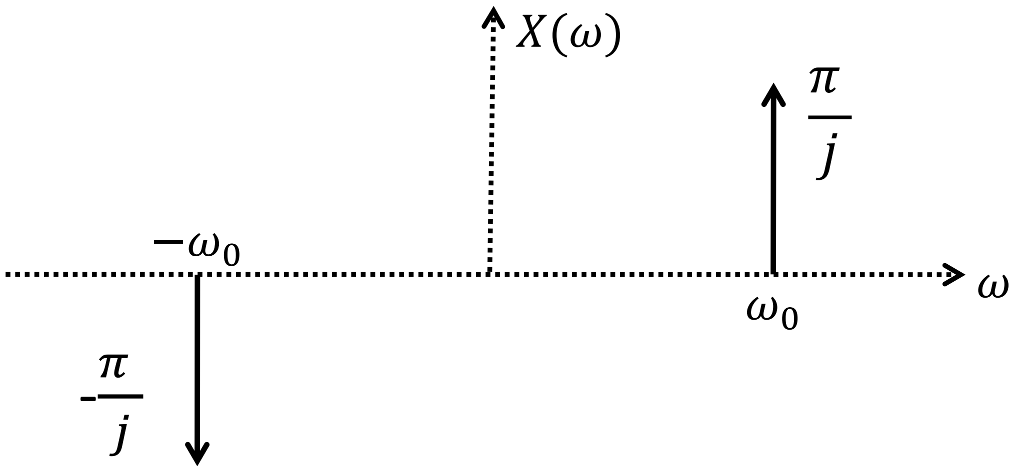
Problem

Draw the spectrum of x(t) = $\sin(\omega_0 t)$.

$$\sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$X(\omega) = \frac{1}{2j} \times 2\pi\delta(\omega - \omega_0) - \frac{1}{2j} \times 2\pi\delta(\omega + \omega_0)$$

$$X(\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$



Fourier Transforms pairs

$\boldsymbol{x(t)}$	$\mathbf{X}(\omega)$
$e^{-at}u(t)$, where $a > 0$	$\dfrac{1}{a + j\omega}$
$\boldsymbol{e^{-a\|t\|}}$, where $a > 0$	$\dfrac{2a}{a^2 + \omega^2}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$rect\left(\dfrac{t}{T}\right)$	$Tsinc\left(\dfrac{\omega T}{2}\right)$
$\dfrac{W}{2\pi}sinc\left(\dfrac{Wt}{2}\right)$	$rect\left(\dfrac{\omega}{W}\right)$

Fourier Transforms pairs

$\boldsymbol{x(t)}$	$\mathbf{X}(\omega)$
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$sgn(t)$	$\frac{2}{j\omega}$
$\delta(t - t_o)$	$e^{-j\omega t_o}$
$e^{-j\omega_0 t}$	$2\pi\delta(\omega + \omega_0)$
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$