**Shivam Jain, Tanish Tuteja, Viraj Agashe** **Homework - 1**
COL202: Discrete Mathematical Structures September 2021

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**Problem 1**
An IPL tournament is played between $n$ cricket teams, where each team plays exactly one match with every other team. How many matches are played? (This is easy.) Assume that no match ends in a tie. We say that a subset $S$ of teams is *consistent* if it is possible to order teams in $S$ as $T_1, \ldots, T_{|S|}$ (think of this as the strongest to weakest ordering) such that for every $i, j$ with $1 \leq i < j \leq |S|$, $T_i$ beats $T_j$. Prove that irrespective of the outcomes of the matches, there always exists a consistent subset $S$ with $|S| \geq \log_2 n$.

*Solution:* The number of matches played is given by $\binom{n}{2} = \frac{n(n-1)}{2}$. For the second claim, we proceed using the Strong Induction. Consider the claim:

$$p(k) : \text{In a tournament of } k \text{ teams, always exists a consistent subset } S \text{ such that } |S| \geq \log_2(n+1)$$

Base Case: For $n = 2$, there are only two teams $T_1$ and $T_2$. Say $T_1$ wins against $T_2$. Then $T_1, T_2$ form the required consistent subset. $|S| = 2 \geq \log_2 3$. So $P(2)$ is true.

Induction Hypothesis: Let $P(k)$ be true $\forall\ k < n$.

Induction Step: Consider a tournament of $n$ teams. We will construct a consistent subset for this tournament. Choose any team $T_i$. We will consider two cases:

**Case 1:** $n = 2m$. Note that the $n - 1 = 2m - 1$ matches $T_i$ plays can end in a win or a loss. By Pigeon-Hole principle, either $T_i$ wins against at least $m$ teams or loses at least $m$ teams. In the case $T_i$ wins against at least $m$ teams, notice that by the induction hypothesis ($m < n$), exists a consistent subset $S_1$ within the $m$ teams of size $|S_1| \geq \log_2(m+1)$. Since $T_i$ wins against all $T_j \in S_1$, $S_2 = S_1 \cup T_i$ forms a consistent subset. Now we have, $|S_2| \geq \log_2(m+1) + 1$, or

$$|S_2| \geq \log_2(2m+2) \geq \log_2(n+1)$$

As required. In case $T_i$ loses at least $m$ matches, we note that for the consistent subset $S_1$ among the $m$ teams, $T_i$ loses to all $T_j \in S_1$ and proceed as earlier.

**Case 2:** $n = 2m + 1$. Again by PHP, either $T_i$ wins against at least $m$ teams or loses at least $m$ teams. By the earlier argument, we can construct a consistent subset $S_2$ such that,

$$|S_2| \geq \log_2(2m+2) = \log_2(n+1)$$

As required. Thus in either case, exists a consistent subset $S_2$ such that, $|S_2| \geq \log_2(n+1)$ and thus $|S_2| \geq \log_2(n)$ ■

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**Problem 2**
Call a non-empty subset $S$ of integers *nice* if, for every $x, y \in S$ and every two integers $a, b$, we have $ax + by \in S$. Observe that the set of multiples of any integer is a nice set. Prove that, in fact, these are the only nice sets. In other words, prove that for every nice set $S$ there exists an integer $x$ such that $S = \{ax \mid a \in \mathbb{Z}\}$.

*Solution:* We will prove by construction, i.e. finding an $x$ such that $S = \{ax \mid a \in \mathbb{Z}\}$. Recall the result from problem 8 of the tutorial:

$$\text{For every } x, y \in \mathbb{Z}\ \exists\ a, b \in \mathbb{Z} \text{ such that } \gcd(x, y) = ax + by$$

By construct of $S$, $ax + by \in S \; \forall \; x, y \in \mathbb{S}$ and $a, b \in \mathbb{Z}$. Thus we have, $\gcd(x, y) \in S \; \forall \; x, y \in S$. Further, consider the set $T = \{|s| \mid s \in S\}$. Note that $T \subseteq S$. Also since $gcd(x, y) \geq 0 \; \forall \; x, y$, we must have $gcd(x, y) \in T \; \forall \; x, y \in S$.

Now, $T \subset \mathbb{N}$, thus by Well-Ordering Principle, exists a minimum element (say $t_0$) in $T$. Consider $gcd(t_0, s)$ for any $s \in S$. We know that

$$gcd(t_0, s) \leq t_0$$

But since $gcd(t_0, s) \in T$ by minimality of $t_0$ we must have $gcd(t_0, s) = t_0 \; \forall \; s \in S$. Thus, for all $s \in S$ we may write $s = at_0$ for some $a \in \mathbb{Z}$, or in other words, $S = \{at_0 \mid a \in \mathbb{Z}\}$ ■

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**Problem 3**

Prove "Claim 2" from the proof of Schröder-Bernstein Theorem discussed in Lecture 5. Here is the statement of the claim. Let $A$ and $B$ be infinite sets, $f$ be an injection from $A$ to $B$, and $g$ be an injection from $B$ to $A$. Let $B' = \{b \in B \mid \exists b^* \in B \setminus \mathrm{Im}(f) \; \exists k \in \mathbb{N} \cup \{0\} : \; b = (f \circ g)^k(b^*)\}$, and $A' = \{g(b) \mid b \in B'\}$. Then for every $b \in B$, the following statements are equivalent.

1. $b \in B'$.
2. If $f^{-1}(b)$ exists, then it is in $A'$.
3. $g(b) \in A'$.

*Solution:* ($1 \implies 2$) Consider any $b \in B'$. By definition, suppose $a = f^{-1}(b)$, or $f(a) = b$. Now by the definition of $B'$, we have that $b = (f \circ g)^k(b^*)$ for some $k \in \mathbb{N} \cup \{0\}$ and $b^* \in B \setminus \mathrm{Im}(f)$. Thus we may write,

$$f(a) = (f \circ g)^k(b^*)$$

By definition of $B'$, $b \neq b^*$ since $b \in Im(f)$ and so $k \geq 1$. Also, by injectivity of $f$ we must have,

$$a = g\left((f \circ g)^{k-1}(b^*)\right) = g(\bar{b}), \; \bar{b} \in B'$$

Hence, $a \in A'$.

($2 \implies 3$) If $f^{-1}(b)$ exists and in $A'$, then we have $a = g(\bar{b})$ for some $b \in B'$. Write $\bar{b} = (f \circ g)^k(b^*)$ for some $k \in \mathbb{N} \cup \{0\}$ and $b^* \in B \setminus \mathrm{Im}(f)$, and so we have,

$$b = f(a) = f \circ g\left((f \circ g)^k(b^*)\right) = (f \circ g)^{k+1}(b^*)$$

Hence, $b \in B'$ and $g(b) \in A'$.

($3 \implies 1$) Since $g(b) \in A'$, by definition of $A'$ it follows that $b \in B'$ ■

**Problem 4**

Given a set $A$, the set of finite length strings over $A$ is denoted by $A^*$. Prove that if $A$ is a finite set, then $A^*$ is necessarily countable. What can you say about the cardinality of $A^*$ if $A$ is countably infinite instead?

*Solution:* Let $s_P$ denoted a string over a set $P$ and $l_s$ denoted the length of a string $s$. Define the set $A_i$ as follows:

$$A_i = \{s_A \mid l_s = i\}$$

Note that,

$$A^* = \bigcup_{i=1}^{\infty} A_i$$

Given that $A$ is a finite set, let us suppose it's cardinality is $n$. Then, cardinality of each $A_i$ is, $|A_i| = n^i$. Thus, each $A_i$ is finite.

We recall that if we have a countably infinite collection of sets, each of which is countable, then $\bigcup_{i=1}^{\infty} A_i$ is countable. Therefore, $A^*$ is countable. ■

**Problem 5**
Prove by mathematical induction that every graph has at least two vertices having equal degree.

*Solution:*