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Q1) a) An operator T is a linear operator if

$T(au + bv) = aT(u) + bT(v)$

$\forall\ a, b \in \mathbb{F}$ & $u, v \in V$ ($\mathbb{R}^n$ for us)

~~Since~~

Let two vector $x, y \in V$ & $a, b \in \mathbb{R}$

$\Rightarrow T(ax + by) = \langle ax + by \mid v \rangle u$

since $\langle \cdot \mid \cdot \rangle$ is a standardised product

$$\Rightarrow \langle ax + by \mid v \rangle u = \langle ax \mid v \rangle u + \langle by \mid v \rangle u$$

$$= a \langle x \mid v \rangle u + b \langle y \mid v \rangle u$$

$$= a\,T(x) + b\,T(y)$$

Hence proved that T is a linear operator on $\mathbb{R}^n$

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Q2) $T^2 - 3T + 2 = 0$ — (1) (given eigen value 1 and 2)

To proove that $\Rightarrow$ $T^n = 2^n(T-I) - (T-2T)$ $\forall\ n \in N$

- For $n = 1$

$$T = 2T - 2I - T + 2I$$

$$= T$$

(hence its true for $n = 1$)

- Let us assume that it is true for some $n = k$

$$\therefore T^k = 2^k(T-I) - (T-2I)$$

- for $n = k+1$

$$T^{k+1} = 2^{k+1}(T-I) - (T-2I)$$

$$= 2^{k+1}T - 2^{k+1}I - (T-2I)$$

$$= 2^k(2^k(T-I)) - (T-2I)$$

$$\cancel{= 2^k(T-I) + T^k}$$

$$T^{k+1} - 2T^k = 2^k \ldots (T-2I)$$

$$T^{k+2} - 2T^{k+1} = \boxed{T^2 - 2T = T - 2I} \rightarrow \text{(From (1))}$$

Hence Proved by Mathematical Induction That $T^n = 2^n(T-I) - (T-2I)$

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For n = 2

$$T^2 = 2^2(T-I) - (T-2I)$$
$$= 3T - 2I \quad \text{— from (1)}$$

multiplying both sides by T

$$T^3 = 2^2(T^2 - T) - (T^2 - 2T) \quad \text{— (A)}$$
$$T^2 - T = 2(T-I) \quad \text{— (2)}$$
$$T^2 - 2T = T - 2I \quad \text{— (3)}$$

Putting (2) and (3) in (A) we will get

$$T^3 = 2^3(T-I) - (T-2I)$$

Suppose $T^n = 2^n(T-I) - (T-2I)$ holds

multiplying both sides with T and using (2) and (3) we will get

$$T^{n+1} = 2^{n+1}(T-I) - (T-2I)$$

this will also hold

We have already proved this for n = (2) and (3)

Hence by induction this will hold for all $n \in N$

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Q3) Given. $V = M_{n\times n}(\mathbb{C})$ $0 \neq A \in V$

Since A is not a null matrix, It has non zero eigen values, say λ

Then $\det(A - \lambda I) = 0$

expansion of above eq^n is:

$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$

as we know that matrix A satisfies its eigen eq^n (cayley hamilton theorem)

Thus,

$a_n A^n + a_{n-1} + A^{n-1} + \dots + a_1 A + A_0 = 0$

As $a_n \neq 0 \ \forall n$

Then $S = \{1, A, A^2 \dots A^{n-1}, A^n\}$ must be linearly dependent over C.

Thus the set S^n can't be part of any basis of V, as for being the part of basis it should be linearly independant

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Q4) b) $x(t) \leq L \int_{t_0}^{t} x(s)\, ds$ $x(t) \geq 0$

differentiating both sides

$x'(t) \leq L\, x(t)$

case 1 $x(t) > 0$

$\Rightarrow \frac{x'(t)}{x(t)} - L \leq 0$ since

$\ln(x(t)) - Lt \leq 0$

$x(t) \leq e^{Lt}$

4(b) Let $x \geq 0$ $L > 0$

$x(t) \leq L \int_{t_0}^{t} x(s)\, ds$

$0 \leq x(t_0) \leq L \times 0 = 0 \Rightarrow x(t_0) = 0$

$F(t) = \int_{t_0}^{t} x(s)\, ds$

$F'(t) = x(t)$

$F'(t) \leq L F(t)$

$\Rightarrow F'(t) - L F(t) \leq 0$

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$$\frac{d}{dt}\left(e^{-Lt} f(t)\right) \leq 0$$

$$\Rightarrow \quad e^{-Lt} f(t) \leq e^{-Lt_0} f(t_0) = 0$$

$$\Rightarrow \quad f(t) \leq 0$$

but as $x(t) \geq 0 \Rightarrow \int_{t_0}^{t} x(s)\,ds \geq 0$

$$f(t) \geq 0$$

$$\therefore \quad f(t) = 0$$

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Q 5) Case - 1) when $f_1 = 0$, $f_2 \neq 0$

Then $W(f_1, f_2)(t) = f_1(t) f_2'(t) - f_1'(t) f_2(t) = 0$

Also, f_1 & f_2 are linearly dependant on I as every zero function is linearly dependant to every function

Case - 2 when $f_1 \neq 0$, $f_2 \neq 0$

given $W(f_1, f_2)(t) = 0$

$\Rightarrow f_1(t) f_2'(t) - f_1'(t) f_2(t) = 0$

$\Rightarrow \dfrac{df_2}{f_2} = \dfrac{df_1}{f_1}$

$\Rightarrow \log f_2 = \log f_1 + \log C$

$\Rightarrow \phi_2 = C \phi_1$ (C is const of integration)

Hence ϕ_1 & ϕ_2 are ~~linearly~~ dependant on I ~~Hence are scalar multiple of each other~~

Case 3 $\quad f_1 = 0 \;\&\; f_2 = 0 \quad \forall f \in I$

$$f_1 = \lambda f_2$$

$$0 = \lambda(0) \qquad \lambda = \text{const}$$

$\therefore$ ϕ_1 and ϕ_2 are linearily dependant on I. both are scalar multiple of each other

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Q6) given, $\alpha \in \mathbb{R}$

$$L(x(t)) = t^3 x''' + 3(1-\alpha) t^2 x'' + (3\alpha^2 - 3\alpha + 1) t x' - \alpha^3 x \quad , t > 0$$

a) we substitute $t = e^{s}$

$$\frac{dx}{ds} = \frac{dx}{dt} \times \frac{dt}{ds} = t x'$$

$$\frac{d^2x}{ds^2} = \frac{d}{ds}(t x') = x' \frac{dt}{ds} + t \frac{dx'}{ds}$$

$$= x' t + t^2 x''$$

$$\frac{d^3x}{ds^3} = \frac{d}{dx}(x' t + t^2 x'') = x' t + t^2 x'' + 2t^2 x'' + t^3 x'''$$

$$= x' t + 3t^2 x'' + t^3 x'''$$

We rewrite $L(x(t)) = 0$ as

$$(t^3 + 3t^2 x' + t x) - 3\alpha(t^2 x'' + x' t) + 3\alpha^2 t x' - \alpha^3 x = 0$$

$$\frac{d^3 x}{ds^3} - 3\alpha \frac{d^2 x}{ds^2} + 3\alpha^2 \frac{dx}{ds} - \alpha^3 x = 0$$

$\therefore$ This is a differential equation with const coefficients

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b) Characteristic eq^n is

$$m^3 - 3\alpha m^2 + 3\alpha^2 m - \alpha^3 = 0$$

$\Rightarrow$ $m = \alpha, \alpha, \alpha$ (Repeated Roots)

In this case,
general solution is given by

$$x(t) = (c_1 + c_2 s + c_3 s^2) e^{\alpha s}$$

$$\Rightarrow x(t) = (c_1 + c_2 \ln t + c_3 (\ln t)^2) t^{\alpha}$$

c) $\alpha = 1$, $L(x(t)) = 1$

$$y_h(t) = (c_1 + c_2 \ln t + c_3 (\ln t)^2) t^{\alpha}$$

Homo part of the eq^n = 0
Non homo part = 1

sol^n of $M(x(t)) = 1$
is given by

$$x(s) = x_h(s) + x_p(s)$$

where $x_p(s)$ sol^n
Since particular sol^n for n^{th} order linear non homo ODE is given by

$$x_p(s) = \sum_{k=1}^{n} (-1)^{n+k} x_k(s) \int \frac{w_k(s) g(s)\, ds}{W(s)}$$

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where $g(s)$ = non homo part = 1 (here)

& $W_k(s)$ = wronskian of gen sol^n (fundamental set)

here $W_k(s) = W(x_1, x_2, x_3)$

& $W_k(s)$ = Wronskian submatrix det obt by deleting K^{th} col & last row

$$\therefore x_p(s) = -\left\{ \sum_{k=1}^{3} (-1)^k x_k(s) \int \frac{W_k(s) g(s) ds}{W(s)} \right\}$$

Now wronskian of $x_1(s)$, $x_2(s)$, $x_3(s)$ is given by

$$W(e^s, se^s, s^2e^s) = \begin{vmatrix} e^s & se^s & s^2e^s \\ (e^s)' & (se^s)' & (s^2e^s)' \\ (e^s)'' & (se^s)'' & (s^2e^s)'' \end{vmatrix}$$

On solving we get

$$W(x_1, x_2, x_3) = 2e^{3s}$$

$$\therefore W_1(s) = s^2 e^{2s}$$

$$W_2(s) = 2se^{2s}$$

$$W_3(s) = e^{2s}$$

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$\therefore u_1(s) = e^s \int \frac{s^2 e^{2s}}{2e^{3s}} ds$

$= \frac{e^s}{2} \int s^2 e^{-s} ds$

$u_1(s) = -\frac{1}{2}(s^2 + 2s + 2)$

Similarly $u_2(s) = s e^s \int s e^{-s} ds$

$= s(s+1)$

$u_3 = s^2 e^s \int \frac{e^{2s}}{2e^{3s}} ds$

$= -\frac{1}{2} s^2$

$x_p(s) = u_1 - u_2 + u_3$

$= -(2s^2 + 2s + 1)$

general sol^n of $M[x(s)] = 1$ is

$x = C_1 e^s + C_2 s e^s + C_3 s^2 e^s - (2s^2 + 2s + 1)$

$s = \ln t$

$x = C_1 t + C_2 \ln t + C_3 (\ln t)^2 t$

$- (2(\ln t)^2 + 2(\ln t) + 1)$

where C_1, C_2, C_3 are arbitrary constants.

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Q7) $X' = AX$

$$A = \begin{bmatrix} 8 & 12 & -2 \\ -3 & -4 & 1 \\ -1 & -2 & 2 \end{bmatrix}$$

eigen values

$$\begin{vmatrix} 8-\lambda & +12 & -2 \\ -3 & -\lambda-4 & 1 \\ -1 & -2 & 2-\lambda \end{vmatrix} = 0$$

we get

$$(8-\lambda)\left[(-\lambda-4)(2-\lambda)+2\right] - 12\left(-3(2-\lambda)+1\right]$$
$$-2(6-(\lambda+4)) = 0$$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$\Rightarrow \quad (\lambda-2)^3 = 0$

$\lambda = 0$

For $\lambda = 2$, eigen space :-

$$\begin{bmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$c = 0$

$a = -2b$

so $\begin{bmatrix} -2b \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

So one soln $\Rightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} e^{2t} = x_1(t)$

$x_2(t) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} te^{2t} + ue^{2t}$

$(A - 2I)u = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

one set $\Rightarrow \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

So $x_2(t) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} e^{2t}$

$x_3(t) \Rightarrow \left[\frac{t^2}{2} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} e^{2t} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} e^{2t} + ve^{2t} \right]$

$(A - 2I)v = u$

one possible set $\Rightarrow \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

$$X_3(t) \Rightarrow \frac{t^2}{2}\begin{bmatrix}-2\\1\\0\end{bmatrix}e^{2t} + \begin{bmatrix}-2\\1\\1\end{bmatrix}te^{2t} + \begin{bmatrix}-2\\1\\2\end{bmatrix}e^{2t}$$

As x_1, X_2, X_3 are linearly independent

so $X(t) = C_1X_1 + C_2X_2 + C_3X_3$

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8) $x'' + tx' + x = 0$

since $a_1(t) = t$ and $a_2(t) = 1$ are real analytic at $t=0$

$\therefore\ x(t) = \sum_{n=0}^{\infty} \frac{x^{(n)}(0)}{n!} t^n$ represents a solution to the given DE

$\therefore$ Let $x(t) = \sum_{k=0}^{\infty} c_k t^k$

$\therefore\ x'(t) = \sum_{k=1}^{\infty} k c_k t^{k-1} = \sum_{k=0}^{\infty} k c_k t^{k-1}$

and $x''(t) = \sum_{k=2}^{\infty} k(k-1) c_k t^{k-2}$

$= \sum_{k=0}^{\infty} (k+1)(k+2) c_k t^k$

Now, $x'' + tx' + x = 0$

$\therefore \sum_{k=0}^{\infty} \left((k+1)(k+2) c_{k+2} + k c_k + c_k \right) t^k = 0$

This implies that

$(k+1)(k+2) c_{k+2} = -(k+1) c_k$

$\therefore \boxed{c_{k+2} = \frac{-c_k}{k+2}} \quad (\because k \neq -1)$

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By induction we then have

$$c_{2n} = \frac{(-1)^n}{2 \cdot 4 \cdot \ldots (2n)} c_0$$

$$c_{2n+1} = \frac{(-1)^{n+1}}{1 \cdot 3 \cdot \ldots (2n-1)} c_1$$

c_0 and c_1 are arbitrary

$$\therefore y = c_0 \left(\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2 \cdot 4 \ldots (2n)} \right) + c_1 \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{1 \cdot 3 \cdots (2n-1)} \right)$$