COL 351:
Analysis and Design of Algorithms

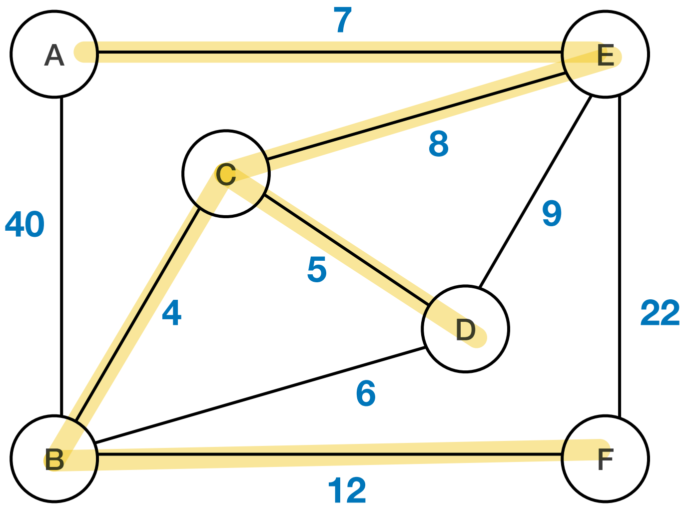
Lecture 4

Metro Layout

Given: There are n locations in a city connected by roads.

Question: Compute a "metro-network" on top of road map having minimum cost such that each pair of location is connected by metro.

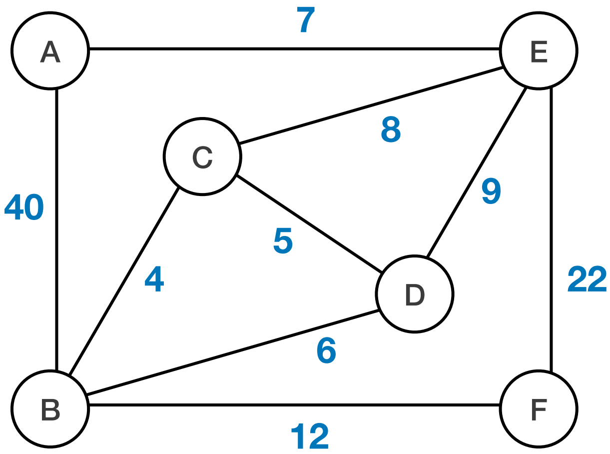
Example:



Minimum Spanning Tree

Given: A connected weighted graph $G = (V, E, \text{wt})$ with n vertices. $wt : E \to \mathbb{R}^+$

Find: A spanning tree $T = (V, E_T \subseteq E)$ of graph G such that $\underbrace{\sum_{e \in E_T} wt(e)}_{wt(T)}$ is minimized.

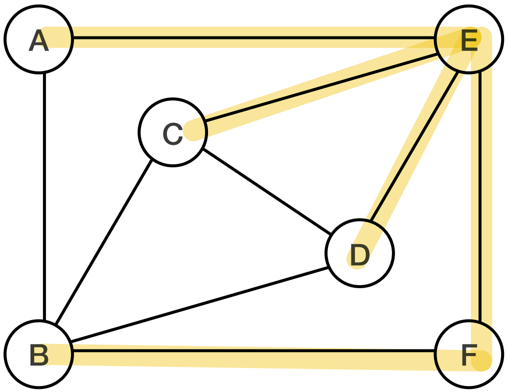


Can we find optimal solution efficiently?

What is a Spanning Tree?

Equivalent def of Trees

* A connected graph with n vertices and (n-1) edges
* An acyclic graph with n vertices and (n-1) edges

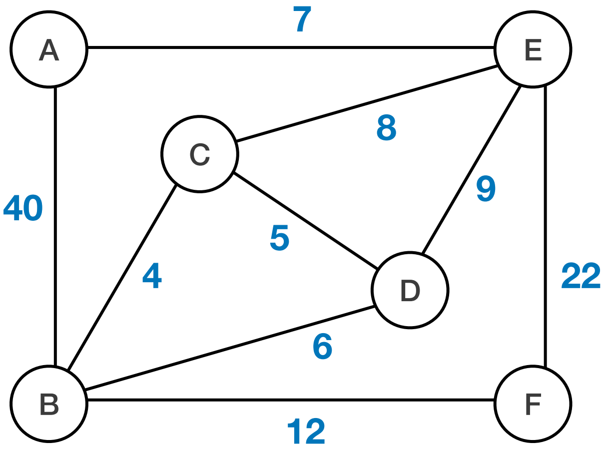


Tree:
An undirected connected acyclic graph.

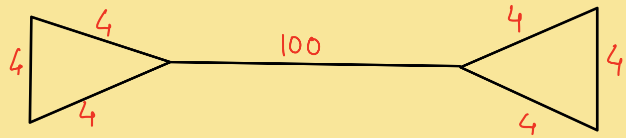
Spanning Tree:
A subgraph of G that is a tree and includes all of the vertices of G

BFS tree / DFS tree are examples of spanning tree.

Greedy Approaches for MST?



Approach I : Skip edge of largest weight (if possible).



Remark : Here we can't skip edge of weight 100.

Approach II : Include in MST edge(s?) of smallest weight.

Remark : In above example we can't include ALL edges of weight 4.

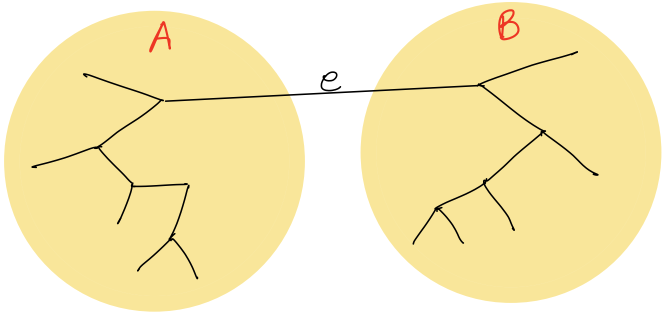
Greedy Approach I

Lemma: Let $e = (x, y)$ be edge of **largest** weight in G that satisfies $(G - e)$ is connected. Then,

there exists an MST of G that doesn't contain e.

Proof:

$T = MST(G)$



Let A, B be two subtrees of $T \setminus e$.
There must exists an edge connecting A and B other than e (say e').

Define $T' = (T \setminus e) \cup \{e'\}$

Claim 1 : T' is spanning tree.
Claim 2 : $wt(e') \leq wt(e)$.

These two together imply T' is MST not containing e.

H.W.

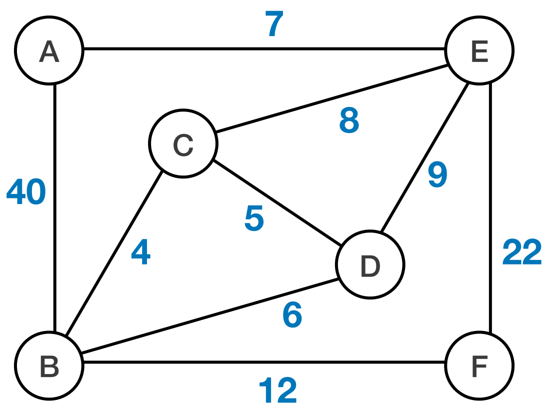
Greedy Algorithm (Deletion based)

While (G is not acyclic):

— Let e be edge of **largest** weight in G for which $G - e$ is connected

— Remove e from G.

Return G.



Time complexity (naive)

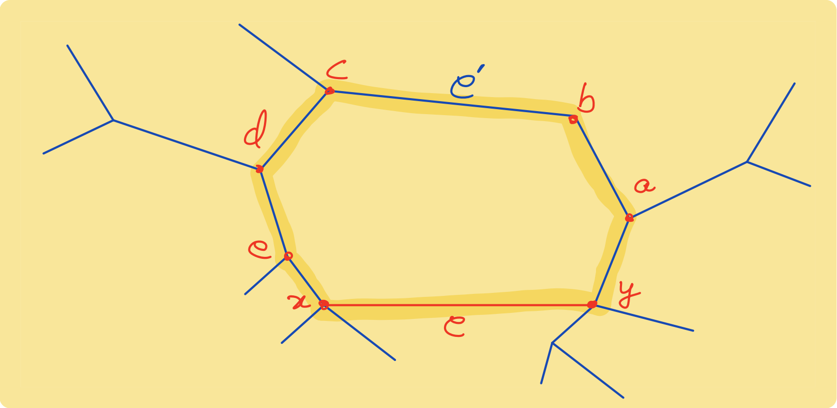
$= O(m \log m + m^2)$

$= O(m^2)$

Greedy Approach II

H.W. Prove all edges in C have same weight.

Lemma: Let $e = (x, y)$ be edge of **smallest** weight in G. Then there exists an MST of G containing e.



Proof:

Consider the case where $T = MST(G)$ doesn't contain e.

Let "C" be unique cycle in $T+(x,y)$, and let e' be be any other edge in C.

Define $T' = (T \setminus e') \cup \{e\}$

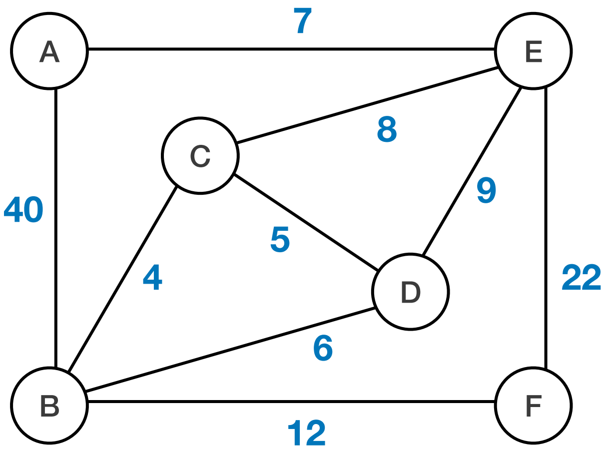
Claim 1: T' is a spanning tree (WHY??)

claim 2: $wt(T) = wt(T')$.

Claim 1 & 2 together prove that there is an MST containing e.

Greedy Approach II

Lemma: Let $e = (x, y)$ be edge of **smallest** weight in G. Then there exists an MST of G containing e.



We must prevent cycles!

How to iteratively use this lemma?

Greedy Approach II

Lemma: Let H be a partial solution to MST of G. Let $e = (x, y)$ be edge of **smallest** weight in G connecting two different components in H. Then $(H + e)$ is also partial solution.

Proof:

Partial sol^n $\equiv$ $\left\{\begin{array}{l} \text{A forest } H \text{ for which} \\ \exists \text{ MST of } G \text{ containing } H. \end{array}\right.$

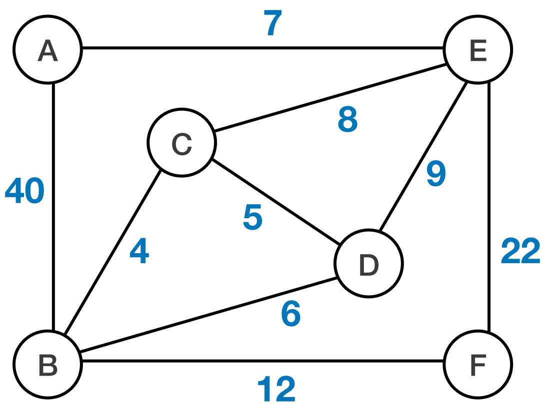
Greedy Algorithm (Incremental)

Set $H = (V, \varnothing)$.

While $|E(H)| \neq (n-1)$:

— Let $e = (x, y)$ be edge of **smallest** weight in G connecting two different components in H.

— Add $e = (x, y)$ to H.



Time complexity ?

(homework)

Challenge Problem: Unique MST

Given: A connected edge-weighted undirected graph $G = (V, E)$.

Problem 1: If all edge weights of $G = (V, E)$ are distinct then has a unique MST.

Problem 2: Provide sufficient and necessary condition for $G = (V, E)$ to have a unique MST.

Questions for Next class

- Can you provide an $O(mn)$ or implementation of the incremental MST algorithm? Is this optimal?

- Here, $n = |V|$ and $m = |E|$