Lecture 11 Fri 18 Aug 2023

Fundamental theorem of LP.

(LP) $\max\ z = c^T x$

subject to $x \in S$.

$$S = \left\{ x \in \mathbb{R}^n : \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\}$$

If $S \neq \phi$, then the system describing S must have a BFS.

Algorithm:

(i) $S = \phi \Rightarrow$ (LP) is infeasible $\Rightarrow$ exit algorithm
(ii) $S \neq \phi \Rightarrow S$ must have a BFS

Start your algo with this BFS.

Proof: Since $S \neq \phi \Rightarrow \exists$ some $x \in S$.

$$x = (x_1, x_2, \text{—}, x_n)^T$$

$Ax = b$, $x_i \geq 0$, $\forall i$

Let k components of x are +ve where, $1 \leq k \leq n$

$$x = (\underbrace{x_1, \text{—}, x_k}_{(k)}, \underbrace{0, \text{—}, 0}_{(n-k)})^T$$

Let $A = [a_1, \text{—}, a_n]_{m \times n}$

Look for those columns ($a_i : m \times 1$) of A which corresponds to $x_i > 0$.

Let $\{a_1, \ldots, a_k\}_{m\times 1}$ be such column vectors

$$\sum_{i=1}^{k} a_i x_i = b \quad , \quad x_i > 0$$

Case 1: Let $\{a_1, \ldots, a_k\}$ be LI. in $\mathbb{R}^m$.

$\Rightarrow k \leq m$

$k = m$

$B = [a_1, \ldots, a_k]$ is a basis matrix invertible

$\Rightarrow x$ is BFS.

$k < m$

$\Downarrow$

$B = [a_1, \ldots, a_k, \ldots]$

columns from A s.t. B^{-1} exist. This we can do as $\text{rank}(A) = m$.

$\Rightarrow x$ is BFS.

Case 2: Let $\{a_1, a_2, \ldots, a_k\}$ be LD in $\mathbb{R}^m$

$\Rightarrow \exists$ scalar $\lambda_1, \ldots, \lambda_k$, not all zeros s.t.

$$\sum_{i=1}^{k} \lambda_i a_i = 0$$

Let $\lambda_r > 0$ for some r $1 \leq r \leq k$.

$$a_r = -\sum_{\substack{i=1 \\ i \neq r}}^{k} \frac{\lambda_i}{\lambda_r} a_i$$

Put this a_r in eqn (1).

$$a_1x_1 + a_2x_2 + \cdots + \left(-\frac{\lambda_1}{\lambda_r}a_1 - \frac{\lambda_2}{\lambda_r}a_2 - \frac{\lambda_{r-1}}{\lambda_{r-1}}a_{r-1} - \frac{\lambda_{r+1}}{\lambda_{r+1}}a_{r+1}\right.$$

$$\left. - \frac{\lambda_k}{\lambda_r}a_k\right)x_r + \cdots$$

$$+ a_k x_k = b.$$

Reduce $\to$ $k-1 \Rightarrow k-2 \Rightarrow k-3$
in the similar way as explained above

$$\Rightarrow \sum_{\substack{i=1 \\ i\neq r}}^{k} \left(x_i - \frac{\lambda_i}{\lambda_r}x_i\right) a_i = b$$

$$\Rightarrow \sum_{\substack{i=1 \\ i\neq r}}^{k} \hat{x}_i a_i = b, \quad \hat{x}_i = x_i - \frac{\lambda_i}{\lambda_r}x_r.$$

Define a vector,

$$\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{r-1}, 0, \hat{x}_{r+1}, \cdots, \hat{x}_k, 0, \cdots 0)^T \in \mathbb{R}^m$$

then, $A\hat{x} = b.$

Note: $\lambda_r > 0$, $x_i > 0$, $x_r > 0$

If $\lambda_i \leq 0 \Rightarrow \hat{x}_i > 0$

If $\lambda_i > 0$ then $\hat{x}_i > 0$

$$\Leftrightarrow x_i > \frac{\lambda_i}{\lambda_r}x_r \Leftrightarrow \frac{x_i}{\lambda_i} = \frac{x_r}{\lambda_r}$$

$\forall\ 1 \leq i \leq r$ with $\lambda_i > 0.$

We choose index s.t.

$$\Rightarrow \frac{x_r}{\lambda_r} = \min\left\{ \frac{x_i}{\lambda_i} : \lambda_i > 0 \right\}$$

with the choice of index r, we will make sure that $\hat{x}_i \geq 0$.

$\Rightarrow \hat{x}$ is feasible for (LP).

continue above arguments with x replaced with $\hat{x}$.

So, either $\hat{x}$ is BFS or not
(in this case we can find $\hat{x}$ as in case 2)

- Continue the process, we land up at BFS.