Marginal PDF & PMF

(X, Y) be a Random vector with joint PMF $\{p_{ij}\}_{i=1, j=1}^{\infty \ \infty}$ ✓

what is PMF of X & Y?

$$p_{i\cdot} = \sum_{j=1}^{\infty} p_{ij} = P\{X = x_i\}$$

$\{p_{i\cdot}\}_{i=1}^{\infty}$ is PMF of X.

Similarly, PMF of Y is given by

$$\underline{P\{Y = y_j\} = p_{\cdot j}} = \sum_{i=1}^{\infty} p_{ij}$$

$\{p_{\cdot j}\}_{j=1}^{\infty}$ is PMF of Y.
↳ (Marginal PMF of Y)

Marginal CDF of X and Y

can be obtained from marginal PMF of X & Y, respectively.

$$F_X(x) = \sum_{x_i \le x} p_{i\cdot}$$

$$F_Y(y) = \sum_{y_j \le y} p_{\cdot j}$$

<u>Example</u>: A fair coin is tossed three times.

X = number of heads in three tossings

Y = Absolute difference between number of heads and number of tails

$X \in \{0, 1, 2, 3\}$

$Y \in \{1, 3\}$

What is joint PMF of (X, Y)?

What is j...

$Y \backslash X$	0	1	2	3	$P\{Y=y\}$
✓ 1	0	3/8	3/8	0	6/8
✓ 3	1/8	0	0	1/8	2/8
$P\{X=x\}$	1/8	3/8	3/8	1/8	1

$\underline{P\{X=1, Y=1\}}$

$= P\{HTT, THT, TTH\}$

$= 3/8$

$P\{X=2, Y=1\}$

$= P\{HHT, HTH, THH\}$

$= 3/8$

\# Let (X, Y) be continuous RV with joint PDF $f(x,y)$.

What are PDFs of X & Y?

Take a Borel set B

$$\underline{\{X \in B\}} = \{X \in B, Y \in \mathbb{R}\}$$

$$\underline{P\{X \in B\}} = P\{\underline{X \in B, Y \in \mathbb{R}}\}$$

$$= \underline{\int_B \int_{-\infty}^{\infty} f(x,y)\, dy\, dx} \checkmark$$

$$\boxed{P\{X \in B\} = \int_B f_X(x)\, dx}, \quad P \ldots$$

$$\boxed{\underbrace{P\{X \in B\}}_{= P_X(B)} = \int_B f_X(x)\,dx}, \qquad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_X(x) \geq 0, \qquad \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$\Rightarrow$ $f_X(x)$ is PDF of X

$\downarrow$

marginal PDF of X

Similarly marginal PDF of Y is given by

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx}$$

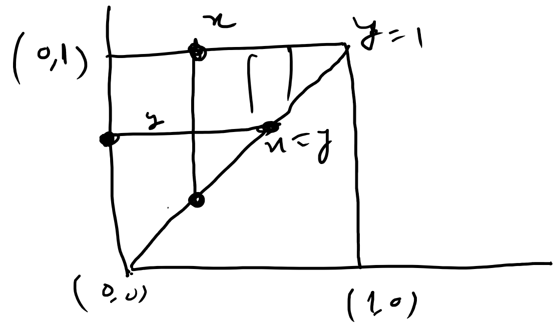
\# Marginal CDFs can be obtained from marginal PDFs.

Example:- Let (X, Y) be jointly distributed with joint PDF

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \end{cases}$$

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$



$$= \int_x^1 2\,dy$$

$$f_X(x) = 2 - 2x, \quad 0 < x < 1$$

$$= 0 \quad \text{o.w.}$$

$$f_Y(y) = \int_0^y 2\,dx = 2y, \quad 0 < y < 1$$

$$f_Y(y) = \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Conditional distributions:

Let (X,Y) be a discrete RV

with PMF

$$p_{ij} = P\{X = x_i, Y = y_j\}$$

If $P\{Y = y_j\} > 0$. Then

$$P\{X = x_i \mid Y = y_j\} = \frac{P\{X = x_i, Y = y_j\}}{P\{Y = y_j\}}$$

$$= \frac{p_{ij}}{p_{\cdot j}}$$

Conditional PMF of X given $\{Y = y_j\}$

Similarly, Conditional PMF of Y given $\{X = x_i\}$, provided $P\{X = x_i\} > 0$ is given by

$$P\{Y = y_j \mid X = x_i\} = \frac{p_{ij}}{p_{i\cdot}}$$

<u>Exercise</u>: Consider the example of 3 times coin tossing and calculate

of 3 times coin tossing and calculate

$P\{X = i \mid Y = 1\}$ for all $i = 0, 1, 2, 3$.

Suppose that (X, Y) is a continuous RV with joint PDF $f(x, y)$.

$$P\{X = x\} = 0 \quad , \quad P\{Y = y\} = 0$$

How do we define ?

$$P\{X \le x \mid \underline{Y = y}\} \quad \text{or} \quad P\{\underline{Y \le y} \mid X = x\}$$

suppose there exists $\varepsilon > 0$ such that

$$P\{y - \varepsilon < Y \le y + \varepsilon\} > 0$$

$$P\{X \le x \mid y - \varepsilon < Y \le y + \varepsilon\} = \frac{P\{X \le x, y - \varepsilon < Y \le y + \varepsilon\}}{P\{y - \varepsilon < Y \le y + \varepsilon\}}$$

Conditional CDF $\underline{F_{X|Y}(x|y)}$ is defined as

$$\lim_{\varepsilon \to 0} P\{ X \le x \mid y-\varepsilon < Y \le y+\varepsilon \}$$

Provided limit exists.

Suppose $F_{X|Y}(x|y)$ exists then

Conditional PDF $f_{X|Y}(x|y)$ is a non-negative function satisfying.

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(t|y)\, dt.$$

At every point (x, y) where f is continuous function and marginal PDF $f_Y(y) > 0$ and continuous, we have

$$F_{X|Y}(x|y) = \lim_{\varepsilon \to 0} \frac{P\{ X \le x,\ Y \in (y-\varepsilon, y+\varepsilon] \}}{P\{ Y \in (y-\varepsilon, y+\varepsilon) \}}$$

$$= \int_{-\infty}^{x} \left[\lim_{\varepsilon \to 0} \int_{y+\varepsilon}^{y+\varepsilon} \frac{f(u, v)}{2\varepsilon}\, dv \right] du$$

$$\int_{-\infty} \left[\lim_{\varepsilon \to 0} \int_{y-\varepsilon}^{y+\varepsilon} \frac{\quad}{2\varepsilon} \right]$$

$$= \lim_{\varepsilon \to 0} \int_{y-\varepsilon}^{y+\varepsilon} \frac{f_Y(v)\,dv}{2\varepsilon}$$

$$= \frac{\int_{-\infty}^{x} f(u,y)\,du}{f_Y(y)}$$

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} \left(\frac{f(u,y)}{f_Y(y)} \right) du$$

(c) Conditional PDF of X given $Y=y$ is given by.

$$\boxed{f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}} \quad , \quad f_Y(y) > 0$$