Matrix Rep. in orthonormal basis

: $E \equiv \{|e_i\rangle\}_{i=1,2,\dots n}$ w/ $\langle e_i|e_j\rangle = \delta_{ij}$

$$\hat{A}|e_i\rangle = A_{ki}|e_k\rangle$$

$$\Rightarrow \quad \langle e_j|A|e_i\rangle = A_{ki}\langle e_j|e_k\rangle = A_{ki}\delta_{jk} = A_{ji}$$

$$\boxed{A_{ji} = \langle e_j|A|e_i\rangle}$$

Using the notion of dual vectors we can represent the operator $\hat{A}$ as follows

$$\hat{A} = \underbrace{A_{ij}}_{\substack{\text{Matrix}\\ \text{element}\\ \text{in } \{|e_i\rangle\}\\ \text{basis.}}} \underbrace{|e_i\rangle}_{\substack{\text{basis}\\ \text{vector}}} \underbrace{\langle e_j|}_{\substack{\text{dual of}\\ \text{the basis}\\ \text{vector}}}$$

Given the orthonormality of $\{|e_i\rangle\}$, the above expression immediatly reproduces the correct matrix elements

$$\langle e_i|\hat{A}|e_j\rangle = \langle e_i\left(A_{mn}|e_m\rangle\langle e_n|\right)|e_j\rangle$$
$$= A_{mn}\langle e_i|e_m\rangle\langle e_n|e_j\rangle$$
$$= A_{mn}\delta_{im}\delta_{nj} = A_{ij} \quad \star$$

- More generally, for any two vectors $|\alpha\rangle, |\beta\rangle \in V_F$

$$\hat{P}_{\alpha\beta} = |\alpha\rangle\langle\beta|$$

acts as a linear operator on V_F with the action

$$\hat{P}_{\alpha\beta}|\gamma\rangle = |\alpha\rangle\langle\beta|\left(|\gamma\rangle\right) \equiv \underbrace{\left(\langle\beta|\gamma\rangle\right)}_{\text{number}}\underbrace{|\alpha\rangle}_{\text{Vector.}}$$

- For any basis $B \equiv \{|\beta_i\rangle\}$ of V_F, the N^2 operators $P_{ij} \equiv |\beta_i\rangle\langle\beta_j|$ <u>form Basis of the vector space of operators on V_F.</u>

i.e. for any linear operator $\hat{A}$ on V_F

$$\hat{A} = a_{ij}|\beta_i\rangle\langle\beta_j|$$

For general Basis its complicated to write a_{ij} coefficients in terms of $\langle \beta_i | \hat{A} | \beta_j \rangle$ & $\langle \beta_i | \beta_j \rangle$ but for orthonormal basis these coefficients are simply the elements of Matrix rep. of $\hat{A}$ in the orthonormal basis.

$A_{ij} = \langle e_i | \hat{A} | e_j \rangle$ as discussed above.

Projection operators: Any linear operator P satisfying

(1) $P^2 = P$, (2) $P^\dagger = P$ is called a projection operator.

- Note that (1) $\Rightarrow$ $P^n = P$ $\forall \; n \in \mathbb{N}$
  (2) $\Rightarrow$ $(P^n)^\dagger = P^n = P$
- Every projection operator is associated with a particular subspace of the full vector space V.
- The projection operator P_S associated with subspace S of V has the important property that it projects any arbitrary vector in V to its components along S.

Projection operators in Orthormal basis $E \equiv \{ |e_i\rangle ; \langle e_i | e_j \rangle = \delta_{ij} \}$

Consider the following $n (= \dim(V))$ Projection operators

$$P_i = |e_i\rangle\langle e_i| \quad \{\text{no summation here}\}^*$$

$$\Rightarrow \quad P_i \cdot P_j = \left(|e_i\rangle\langle e_i|\right)\left(|e_j\rangle\langle e_j|\right)$$

$$= |e_i\rangle\langle e_i | e_j\rangle\langle e_j|$$

$$= \delta_{ij} \, |e_i\rangle\langle e_j| = \delta_{ij} P_i = \begin{cases} P_i & i = j \\ 0 & i \neq j \end{cases}$$

i.e. $\{P_i\}_{i=1,2,\dots,n}$ form a set of "orthogonal" projection operators.

- P_i projects any arbitrary vector along $|e_i\rangle$

$$P_i|\alpha\rangle = \left(|e_i\rangle\langle e_i|\right)\left(\sum_j \alpha_j|e_j\rangle\right)$$
$$= \sum_j \alpha_j |e_i\rangle \underbrace{\langle e_i|e_j\rangle}_{\to\,\delta_{ij}}$$
$$= \sum_j \alpha_j|e_i\rangle\delta_{ij} = \alpha_i|e_i\rangle$$

- Consider the subspace W of V spanned by the basis vector $\{|e_1\rangle, |e_2\rangle, \dots |e_m\rangle\}$ $m < \dim(V)$. then

$P = P_1 + P_2 + \cdots P_m$ is the projection operator onto the subspace W

$$P^2 = P.P = (P_1 + P_2 + \cdots P_m).(P_1 + P_2 + \cdots P_m)$$
$$= P_1^2 + P_2^2 + \cdots P_m^2 \qquad \{P_iP_j = \delta_{ij}P_i$$
$$= P_1 + P_2 + \cdots P_m = P$$

$$P|\alpha\rangle = (P_1 + P_2 + \cdots P_m)\left(\alpha_1|e_1\rangle + \alpha_2|e_2\rangle + \cdots \alpha_n|e_n\rangle\right)$$
$$= \alpha_1|e_1\rangle + \alpha_2|e_2\rangle + \cdots \alpha_m|e_m\rangle$$

- Given an arbitrary vector $|\alpha\rangle$

$$P_\alpha = \frac{|\alpha\rangle\langle\alpha|}{\langle\alpha|\alpha\rangle}$$

is a projection operator which projects any vector to the 1-dimensional subspace spanned by $|\alpha\rangle$.

Ex: Let $|\alpha\rangle$ & $|\beta\rangle$ be two linearly independent vectors in V. Does $P = P_\alpha + P_\beta$ work as a projection operator onto the subspace spanned by $|\alpha\rangle$ & $|\beta\rangle$?

If not, Can you construct such a projection operator?

- Note that the Identity operator can be written as

$$I = P_1 + P_2 + P_3 + \cdots + P_n \quad ; \quad n = \dim(V).$$
$$= \sum_{i=1}^{n} |e_i\rangle\langle e_i|$$

This is often referred to as the decomposition of Identity (operator) in the orthonormal basis $\{|e_i\rangle\}$.

Ex. Verify the decomposition of Identity in $V = \mathbb{R}^2$ for the following two orthonormal basis.

1. $\{|e_1\rangle, |e_2\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$
2. $\{|\tilde{e}_1\rangle, |\tilde{e}_2\rangle\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$