Lecture 2 - Riemann Integration

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$R[a,b]$ = Set of all functions defined on $[a,b]$ which are R-integrable.

Q: How large $R[a,b]$ is ?

Theorem: A bdd fn $f \in R[a,b]$ iff for every $\epsilon > 0$, $\exists$ a partition P_ϵ such that

$$U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$$

Proof: Let $f \in R[a,b]$.

Then $\underline{\int_a^b} f = \overline{\int_a^b} f$

Let us choose $\epsilon > 0$.

Since $\underline{\int_a^b} f = \sup_P \{L(P,f)\}$

So then $\exists\, P_1$ of $[a,b]$ s.t

$$\boxed{\underline{\int_a^b} f - \frac{\epsilon}{2} < L(P_1, f) \leq \underline{\int_a^b} f} \quad - (1)$$

Similarly $\exists\, P_2$ of $[a,b]$ s.t

$$\boxed{\overline{\int_a^b} f \leq U(P_2, f) \leq \overline{\int_a^b} f + \frac{\epsilon}{2}} \quad - (2)$$

Consider,

$$\boxed{P_\epsilon = P_1 \cup P_2 \text{ (Refinement)}}$$

Hence,

$$U(P_\epsilon, f) - L(P_\epsilon, f)$$

$$\leq U(P_1, f) - L(P_2, f)$$

$$\leq \overline{\int_a^b} f + \frac{\epsilon}{2} - \underline{\int_a^b} f + \frac{\epsilon}{2}$$

$$= \epsilon$$

Converse

For any 'P' of $[a, b]$

$$L(P, f) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f$$

$$\leq U(P, f)$$

So then,

$$\boxed{\overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P, f) - L(P, f)}$$

Let $\epsilon > 0$. So by the given condition $\exists P_\epsilon$ s.t

$$U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$$

This implies

$$\boxed{\overline{\int_a^b} f - \underline{\int_a^b} f < \epsilon}$$

We know,

$$\boxed{\overline{\int_a^b} f \geq \underline{\int_a^b} f}$$ (always)

So then,

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < \epsilon.$$

$$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f$$

$$\Rightarrow f \in R[a,b]$$

[Proved]

Example

$$f(x) = x^2, \quad \text{on } [0, 1].$$

$$P_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots \frac{n}{n} \right\}$$

We calculated

$$U(P_n, f) = \frac{n(n+1)(2n+1)}{6n^3}$$

&

$$L(P_n, f) = \frac{n(n-1)(2n-1)}{6n^3}$$

Now,

$$U(P_n, f) - L(P_n, f)$$

$$= \frac{1}{2n^2}\Big(n(n+1) - n(n-1) \Big)$$

$$= \frac{1}{n} \quad (< \epsilon)$$

So for any $\epsilon > 0$ we can find n (large) & P_n s.t

$$U(P_n, f) - L(P_n, f) < \epsilon.$$

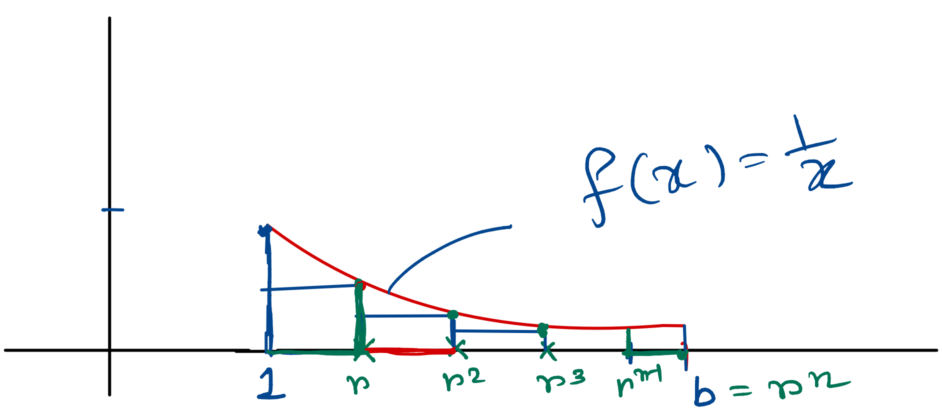
Theo: $f : [a, b] \to \mathbb{R}$ is integrable if and only if $\exists$ a seq $\{P_n\}$ of $[a, b]$ s.t.,

$$\lim_{n \to \infty} U(P_n, f) - L(P_n, f) = 0$$

Example.

Consider.

$f(x) = \frac{1}{x}$ on $[1, b]$.



Soln. Consider

$$P_n = \{1, r, r^2, \ldots \ldots r^{n-1}, r^n = b\}$$

where $r = b^{1/n}$.

$$U(P_n, f) = f(1)(r-1) + f(r)(r^2 - r) + \cdots + f(r^{n-1})(r^n - r^{n-1})$$

$$= 1(r-1) + \frac{1}{\cancel{r}}\cancel{r}(r-1) + \cdots + \frac{1}{\cancel{r^{n-1}}} \cdot \cancel{r^{n-1}}(r-1)$$

$$= \underbrace{(r-1) + \cdots\cdots + (r-1)}_{n \text{ times}}$$

$$= n(r-1) = n(b^{1/n} - 1)$$

$$L(P_n, f) = f(r)(r-1) + \cdots$$

$$+ f(r^n)(r^n - r^{n-1})$$

$$= \frac{n}{r}\left(b^{1/n} - 1\right)$$

$$= n\left(1 - \frac{1}{b^{1/n}}\right)$$

Then

$$U(P_n, f) - L(P_n, f)$$

$$= n\, b^{1/n} - n - n + \frac{n}{b^{1/n}}.$$

$$= n\left(b^{1/n} - 2 + \frac{1}{b^{1/n}}\right)$$

$$= \frac{b^{1/n} - 2 + \frac{1}{b^{1/n}}}{1/n} \quad \left(\frac{0}{0}\right) \quad b > 1$$

So

$\frac{1}{b} < 1.$

(Using the fact that

$$\lim_{n \to \infty} b^{1/n} = 1 \quad b>0)$$

Claim is,

$$U(P_n, f) - L(P_n, f) \to 0$$

as $n \to \infty$. (Check).

So from the previous theo.

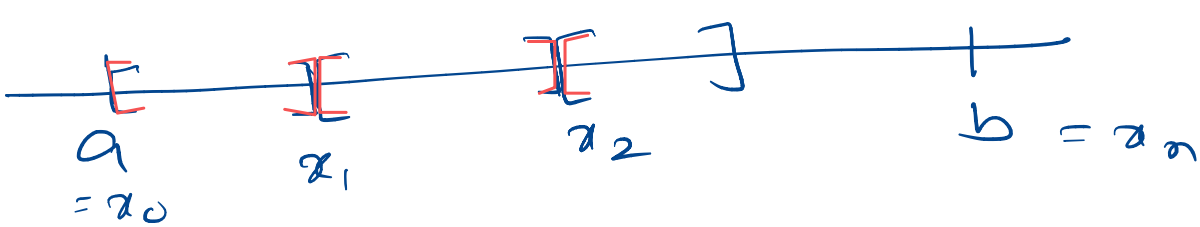
we say $f \in \mathcal{R}[1, b]$.

$\boxed{\text{Condition Another of integrability}}$

Let P be such a partition.

given by

$$P = \{a = x_0, x_1, x_2, \cdots x_n = b\}$$



$[a, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$

Norm of the partition

$$\|P\| = \max\{(x_1 - x_0), (x_2 - x_1), \ldots, (x_n - x_{n-1})\}$$

= $\boxed{\text{maximum length}}$ of the sub intervals of $[a, b]$.

Q be a refinement of P.

Then $\|Q\| \leq \|P\|$.

Converse - ?

Converse is not true.

<u>Example</u>: $[0, 1]$

$$P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

$$Q = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$$

Here

$\|P\| = \frac{1}{4}$ &

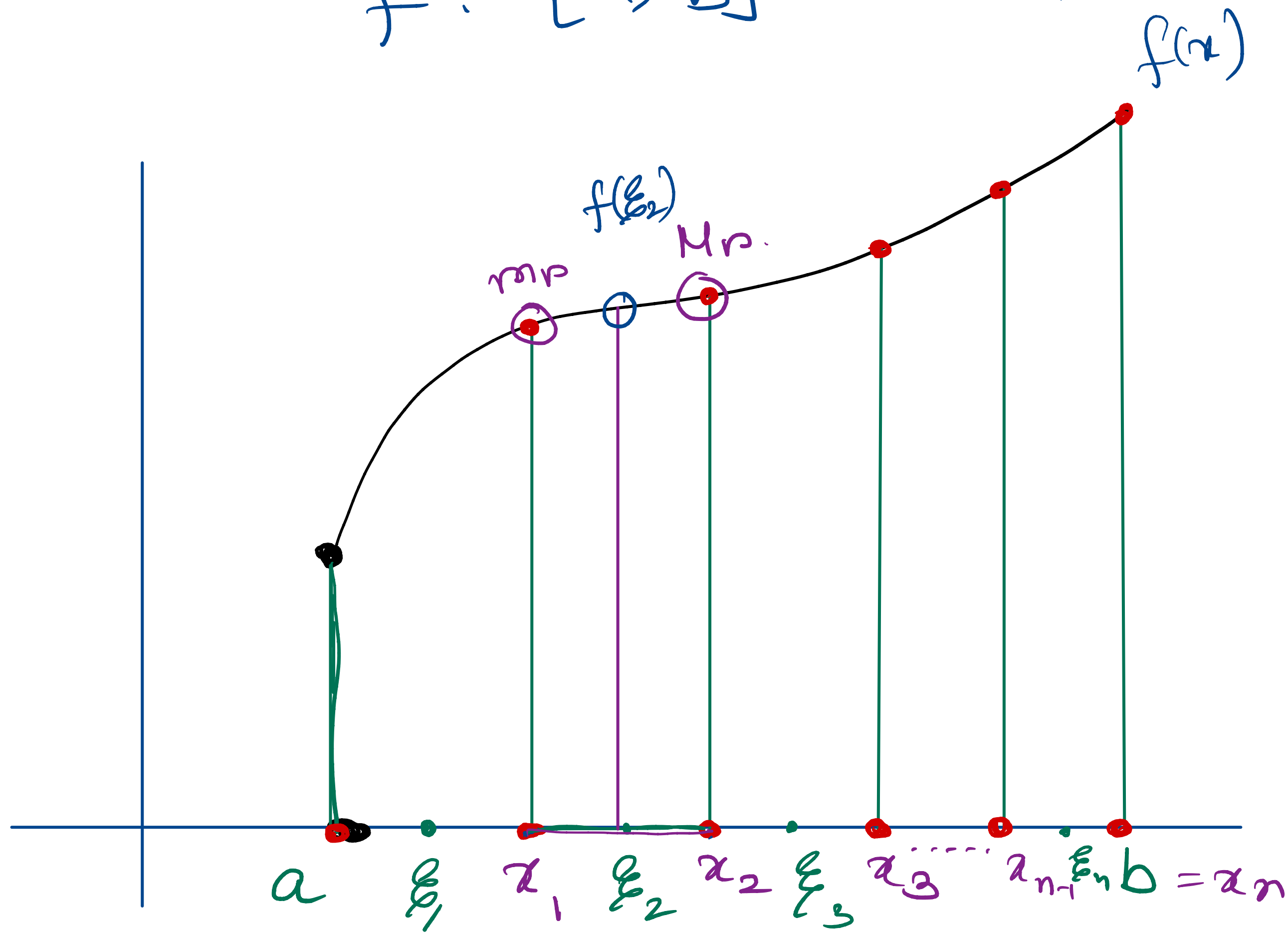
$\|Q\| = \frac{1}{6}$.

But Q is not a refinement.

(Why?)

Now let

$f : [a, b] \to \mathbb{R}$.



$P = \{x_0, x_1, \ldots \ldots x_n = b\}$

Pick up some arbitrary points

$\xi_1, \xi_2, \ldots \xi_n$ s.t

$$x_{r-1} \le \xi_r \le x_r$$

&

We Consider the sum.

$$f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(x_n - x_{n-1})$$

$=$ $\boxed{\text{Riemann sum}}$ corresponding to the partition P.

& the choosen intermediate points.

We denote it by,

$$S(P, f, \xi), \quad \xi = \{\xi_1, \xi_2 \ldots \xi_n\}$$

It is easy to check.

$$\boxed{L(P,f) \leq S(P,f) \leq U(P,f)}$$

Why the above inequality is true?

This is because.

$$m_r \leq f(\xi_r) \leq M_r.$$

$$\Rightarrow \quad \sum m_r(x_r - x_{r-1}) \leq S(P,f,\xi)$$

$$\leq \sum M_r(x_r - x_{r-1})$$

$$\Rightarrow \quad L(P,f) \leq S(P,f,\xi) \leq U(P,f)$$

$$L(P,f) \leq S(P,f) \leq U(P,f)$$

Also if

$$m = \inf_{x \in [a,b]} f(x)$$

$$M = \sup_{x \in [a,b]} f(x)$$

then,

$$\boxed{m(b-a) \leq L(P,f) \leq S(P,f) \leq U(P,f) \leq M(b-a)}$$

Note: If m_r & M_r are attained by f at some points in $[x_{r-1}, x_r]$ then for a particular choice of the intermediate points, ξ_r,

$S(P,f) = L(P,f)$

or

$S(P,f) = U(P,f)$.

However in general

$U(P,f)$ or $L(P,f)$ are

not the Riemann sums.

Theorem

Let $f : [a,b] \to \mathbb{R}$ be

Riemann integrable. Then

for each $\epsilon > 0$ $\exists\, \delta > 0$ s.t

$$\left| S(P,f) - \int_a^b f \right| < \epsilon \quad \forall\, P$$

s.t $\|P\| < \delta$.

[If $f \in R[a,b]$ then $\lim_{\|P\| \to 0} S(P,f) = \int_a^b f$]

Suppose

$\{P_n\}$ s.t $\|P_n\| \to 0$ as $n \to \infty$.

Let $f \in \mathcal{R}[a,b]$. then for any $\epsilon > 0$ $\exists\ \delta > 0$ s.t

$$\left| S(P,f) - \int_a^b f \right| < \epsilon,$$

$$\|P\| < \delta. \quad \text{——— } (1)$$

For $\delta > 0$ $\exists\ k \in \mathbb{N}$ s.t

$$\|P_n\| < \delta, \quad \forall\ n \geq k \quad \text{——— } (2)$$

We can conclude that

$$\left| S(P_n, f) - \int_a^b f \right| < \epsilon, \quad \forall\, n \geq k.$$

$$\Rightarrow \boxed{\lim_{n\to\infty} S(P_n, f) = \int_a^b f \qquad \|P_n\| \to 0 \text{ as } n \to \infty.}$$

Ex. 1 If $f \in \mathcal{R}[a,b]$ then for any $\epsilon > 0$ $\exists\, \delta > 0$ s.t

$$\boxed{\begin{aligned} U(P,f) &< \int_a^b f + \epsilon \\ L(P,f) &> \int_a^b f - \epsilon \end{aligned} \quad \forall\, \|P\| < \delta.}$$

• $L(P,f) \le S(P,f) \le U(P,f)$

(Recall)

Now then from Ex 1,

$$U(P_n,f) < \int_a^b f + \epsilon \quad \forall n \ge k.$$

$$L(P_n,f) > \int_a^b f - \epsilon$$

Using,

$$\int_a^b f - \epsilon < L(P_n,f) \le \int_a^b f$$

$$\le U(P_n,f) < \int_a^b f + \epsilon$$

$$, \forall n \ge k.$$

Then, $$\lim_{n\to\infty} U(P_n,f) = \int_a^b f$$

$$= \lim_{n\to\infty} L(P_n, f).$$

Cor. If $f \in R[a,b]$, then for any sequence of partitions $\{P_n\}$ with $\|P_n\| \to 0$ we have

$$L(P_n, f) \to \int_a^b f \quad \&$$

$$U(P_n, f) \to \int_a^b f \quad \text{as}$$

$$n \to \infty.$$

Remark:

If $\exists$ a seq $\{P_n\}$ s.t $\|P_n\| \to 0$

& $U(P_n, f) - L(P_n, f) \not\to 0$ as $n \to \infty$.

then f is not R-integrable.

Example $f: [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1+x, & x \in \mathbb{Q} \\ 1-x, & x \in \mathbb{Q}^c \end{cases}$$

is not R-integrable.

Soln. We consider

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots \frac{n}{n}\right\}$$

Observe $P_1 = \{0, 1\}$

$P_2 = \{0, \frac{1}{2}, 1\}$

$P_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and so on.

Then

$$\|P_n\| = \max\left\{\left(\frac{1}{n}-0\right), \left(\frac{2}{n}-\frac{1}{n}\right), \ldots \left(\frac{n}{n}-\frac{n-1}{n}\right)\right\}$$

$$= \frac{1}{n} \longrightarrow 0 \quad \text{as } n \to \infty$$

Compute

$$U(P_n, f) = \sum_i M_i \Delta x_i,$$

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = \left(1 + \frac{i}{n}\right)$$

(why?)

$$= f\left(\frac{i}{n}\right)$$

Rewrite the $U(P_n, f)$,

$$U(P_n, f) = f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \cdots + f\left(\frac{n}{n}\right)\frac{1}{n}$$

$$= \left(1+\frac{1}{n}\right)\frac{1}{n} + \left(1+\frac{2}{n}\right)\frac{1}{n} + \cdots + \left(1+\frac{n}{n}\right)\frac{1}{n}$$

$$= \frac{3}{2} + \frac{1}{2n}$$

as $n \to \infty$ we get

$$U(P_n, f) \to \frac{3}{2}.$$

We will not compute $L(P_n, f)$.

Now, $L(P_n, f) = \sum_i m_i \Delta x_i$

Claim $\quad m_i = \left(1 - \frac{i}{n}\right), \ \forall i = 1, \ldots, n$

(Exercise)

Then,

$$L(P_n, f) = 1 - \frac{1}{2} - \frac{1}{2n}$$

$$\longrightarrow \frac{1}{2} \quad \text{as } n \to \infty$$

Then,

$$U(P_n, f) - L(P_n, f) \not\to 0,$$

as $n \to \infty$.

So f is $\boxed{\underline{\text{not}}}$ Riemann integrable.