Scalar product, Dual vectors & Cauchy-Schwarz inequality

Scalar product: A scalar product on a vector space V over field F is a map

$$\langle * | * \rangle : V \times V \longrightarrow F$$

which takes two vectors as input and gives a scalar as output & satisfies the following properties

1. $\langle \alpha | \beta \rangle = \overline{\langle \beta | \alpha \rangle}$ $\quad \{ \bar{a} \rightarrow$ complex conjugate of a

2. $\langle \alpha | a_1 \beta_1 + a_2 \beta_2 \rangle = a_1 \langle \alpha | \beta_1 \rangle + a_2 \langle \alpha | \beta_2 \rangle$

   $\langle a_1 \alpha_1 + a_2 \alpha_2 | \beta \rangle = \bar{a}_1 \langle \alpha_1 | \beta \rangle + \bar{a}_2 \langle \alpha_2 | \beta \rangle$

3. $\langle \alpha | \alpha \rangle \geqslant 0 \quad \forall \ |\alpha\rangle \in V$ $\quad \{ \langle \alpha | \alpha \rangle$ is real $\}$

   & $\langle \alpha | \alpha \rangle = 0 \Rightarrow |\alpha\rangle = |0\rangle$

- $\|\alpha\| = +\sqrt{\langle \alpha | \alpha \rangle}$ is usually referred to as the <u>length</u> or <u>Norm of vector $|\alpha\rangle$</u>.

- If $F = \mathbb{C}$, then $\langle \alpha | \beta \rangle \neq \langle \beta | \alpha \rangle$ generally.

- If $\langle \alpha | \beta \rangle = 0 = \langle \beta | \alpha \rangle$, vectors $|\alpha\rangle$ & $|\beta\rangle$ are referred to as <u>orthogonal</u> to each other.

Cauchy-Schwarz inequality:

Given any two arbitrary vectors $|\alpha\rangle, |\beta\rangle \in V$

$$\sqrt{\langle \alpha | \alpha \rangle . \langle \beta | \beta \rangle} \geqslant |\langle \alpha | \beta \rangle|$$

Consider $\quad |\gamma\rangle = |\alpha\rangle - x \langle \beta | \alpha \rangle |\beta\rangle \qquad x \in F$

$$\langle \gamma | \gamma \rangle \geqslant 0 \quad \forall \quad x \ \& \ |\alpha\rangle, |\beta\rangle$$

$$\langle \gamma | \gamma \rangle = \langle \alpha | \alpha \rangle + x^2 |\langle \beta | \alpha \rangle|^2 \langle \beta | \beta \rangle - 2x |\langle \beta | \alpha \rangle|^2 \geqslant 0$$

$$0 \geqslant D = 4 |\langle \beta | \alpha \rangle|^4 - 4 \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle |\langle \beta | \alpha \rangle|^2$$

$$\Rightarrow \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geqslant |\langle \alpha | \beta \rangle|^2$$

Gram-Schmidt orthogonalization procedure:

Given any Basis $B = \{|\alpha_i\rangle\}_{i=1,2,..n}$ of a vector space V_F on which a scalar product $\langle * | * \rangle$ is defined, it is always possible to construct an Orthonormal Basis $\{|e_i\rangle\}_{i=1,2,..n}$ satisfying

$$\langle e_i | e_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Let us explicitly construct the $|e_i\rangle$s

- $$|e_1\rangle = \frac{|\alpha_1\rangle}{\sqrt{\langle \alpha_1 | \alpha_1 \rangle}} \Rightarrow \langle e_1 | e_1 \rangle = 1$$
- $$|e_2\rangle = \frac{|\alpha_2\rangle - |e_1\rangle\langle e_1 | \alpha_2\rangle}{N_2} \Rightarrow \begin{cases} \langle e_2 | e_2 \rangle = 1 \\ \langle e_1 | e_2 \rangle = 0 \end{cases}$$

  where $N_2 = \| (|\alpha_2\rangle - \langle e_1 | \alpha_2 \rangle |e_1\rangle) \|$

- $$|e_3\rangle = \frac{|\alpha_3\rangle - |e_1\rangle\langle e_1 | \alpha_3\rangle - |e_2\rangle\langle e_2 | \alpha_3\rangle}{N_3}$$

  with $N_3 = \| (|\alpha_3\rangle - |e_1\rangle\langle e_1 | \alpha_3\rangle - |e_2\rangle\langle e_2 | \alpha_3\rangle) \|$

  $\Rightarrow \langle e_3 | e_3 \rangle = 1$; $\langle e_1 | e_3 \rangle = 0$, $\langle e_2 | e_3 \rangle = 0$

Continue the process...

In general: $$|e_i\rangle = \frac{1}{N_i}\left(|\alpha_i\rangle - \sum_{j=1}^{i-1} |e_j\rangle\langle e_j | \alpha_i \rangle \right)$$

with $N_i = \left\| \left(|\alpha_i\rangle - \sum_{j=1}^{i-1} |e_j\rangle\langle e_j | \alpha_i \rangle \right) \right\|$

Side comment:

$P_i = |e_i\rangle\langle e_i|$ is referred to as the "projection operator" in in $|e_i\rangle$ & satisfies $P_i^2 = P_i$. We will come back to these operators later again.