MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 8 (DTMC)

1. The owner of a local one-chair barber shop is thinking of expanding because there seem to be too many people waiting. Observations indicate that in the time required to cut one person's hair there may be 0, 1 and 2 arrivals with probability 0.3, 0.4 and 0.3 respectively. The shop has a fixed capacity of six people whose hair is being cut. Let $X(t)$ be the number of people in the shop at any time t and $X_n = X(t_n^+)$ be the number of people in the shop after the time instant of completion of the nth person's hair cut. Prove that $\{X_n, n = 1, 2, \ldots\}$ is a Markov chain assuming i.i.d arrivals. Find its one step transition probability matrix.

2. Let X_0 be an integer-valued random variable, $P(X_0 = 0) = 1$, that is independent of the i.i.d. sequence $Z_1, Z_2, \ldots$, where $P(Z_n = 1) = p$, $P(Z_n = -1) = q$, and $P(Z_n = 0) = 1 - (p + q)$. Let $X_n = \max(0, X_{n-1} + Z_n), n = 1, 2, \ldots$. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain.

3. Suppose that a machine can be in two states: 0 = working and 1 = out of order on a day. The probability that a machine is working on a particular day depends on the state of the machine during two previous days. Specifically assume that $P(X(n+1) = 0/X(n-1) = j, X(n) = k) = q_{jk} \quad j, k = 0, 1$ where $X(n)$ is the state of the machine on day n.

   (a) Show that $\{X(n), n = 1, 2, \ldots\}$ is not a discrete time Markov chain.

   (b) Define a new state space for the problem by taking the pairs (j, k) where j and k are 0 or 1. We say that machine is in state (j, k) on day n if the machine is in state j on day $(n - 1)$ and in state k on day n. Show that with this changed state space the system is a discrete time Markov chain.

   (c) Suppose the machine was working on Monday and Tuesday. What is the probability that it will be working on Thursday?

4. The transition probability matrix of a discrete time Markov chain $\{X_n, n = 0, 1, \ldots\}$ having three states 1, 2 and 3 is $P = \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.6 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.1 \end{pmatrix}$ and the initial distribution is $\pi = (0.7, 0.2, 0.1)$

   (a) Compute $P(X_2 = 3)$. (b) Compute $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$.

5. Consider a time-homogeneous discrete time Markov chain $\{X_n, n = 0, 1, \ldots\}$ with state space $S = \{0, 1, 2, 3, 4\}$ and one-step transition probability matrix $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

   (a) Classify the states of the chain as transient, +ve recurrent or null recurrent.

   (b) When $P(X_0 = 2) = 1$, find the expected number of times the Markov chain visit state 1 before being absorbed.

   (c) When $P(X_0 = 1) = 1$, find the probability that the Markov chain absorbs in state 0.

6. Consider a DTMC with transition probability matrix $\begin{pmatrix} 0.4 & 0.6 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.4 & 0.6 \end{pmatrix}$. Find the stationary distribution for this Markov chain.

7. Two gamblers, A and B, bet on successive independent tosses of an unbiased coin that lands heads up with probability p. If the coin turns up heads, gambler A wins a rupee from gambler B, and if the coin turns up tails, gambler B wins a rupee from gambler A. Thus the total number of rupees among the two gamblers stays fixed, say N. The game stops as soon as either gambler is ruined; i.e., is left with no money! Assume the initial fortune of gambler A is i. Let X_n be the amount of money gambler A has after the nth toss. If

$X_n = 0$, then gambler A is ruined and the game stops. If $X_n = N$, then gambler B is ruined and the game stops. Otherwise the game continues. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain.

8. One way of spreading information on a network uses a rumor-spreading paradigm. Suppose that there are 5 hosts currently on the network. Initially, one host begins with a message. In every round, each host that has the message contacts another host chosen independently and uniformly at random from the other 4 hosts, and sends the message to the host. The process stops when all hosts has the message. Model this process as discrete time Markov chains with
(a) X_n be state of host $(i = 1, 2, \ldots, 5)$ who received the message at the end of nth round.
(b) Y_n be number of hosts having the message at the end of nth round.
Find one step transition probability matrix for the above discrete time Markov chains. Classify the states of the chains as transient, positive recurrent or null recurrent.

9. For $j = 0, 1, \ldots$, let $P_{jj+2} = v_j$ and $P_{j0} = 1 - v_j$, define the transition probability matrix of Markov chain. Discuss the character of the states of this chain.

10. For a Markov chain $\{X_n, n = 1, 2, \ldots\}$ with state space $E = \{0, 1, 2, 3, 4\}$ and transition probability matrix P given below, classify the states of the chain. Also determine the closed communicating classes.

$$(a)\ P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad (b)\ P = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

11. Show that if a Markov chain is irreducible and $P_{ii} > 0$ for some state i then the chain is aperiodic.

12. Let 0 be an absorbing state and for $j > 0$, $P_{jj} = p$, $P_{jj-1} = q$ where $p + q = 1$. Find $f_{j0}^{(n)}$, the probability that absorption takes place exactly at n^{th} step given initial state is j. Also find the expectation of this distribution.

13. Determine the equilibrium probability distribution of a three state Markov chain whose transition probability matrix is $\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.

14. Suppose there are three white and three black balls in two urns (labeled 1, 2) distributed so that each urn contains three balls. We say that the system is in state $i \in \{0, 1, 2, 3\}$, if there are i white balls in Urn 1. At each stage one ball is drawn at random from each urn and interchanged. Let X_n denote the state of the system after the nth draw. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Write the one-step probability transition matrix or draw the state transition diagram for this Markov chain.

15. Consider a branching process, denoted by Galton-Watson process, that model a population in which each individual in generation n produces some random number of individuals in generation $n+1$, according, in the simplest case, to a fixed probability distribution that does not vary from individual to individual. That is, the first generation of individuals is the collection of off-springs of a given individual. The next generation is formed by the off-springs of these individuals. Let X_n denote the number of individuals of the nth generation, starting with $X_0 = 1$ individual (the size of the zeroth generation). Let Y_i (or $Y_{i,n}$) be the number of offspring of the ith individual of the nth generation. Suppose that, $\{Y_i, i = 1, 2, \ldots\}$ are non-negative integer valued i.i.d. random variables with probability mass function $p_j = P(Y_i = j), j = 0, 1, \ldots$ and independent of the size of the generation. Then

$$X_n = \sum_{i=1}^{X_{n-1}} Y_i, \quad n = 1, 2, \ldots$$

and $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Classify the states of the chain.

16. Consider a DTMC on the non negative integers such that, starting from i, the chain goes to state $i+1$ with probability $p, 0 < p < 1$ and goes to state 0 with probability $1-p$. Show that this DTMC has a unique steady state distribution π and then find π.

17. Consider an aperiodic irreducible finite state space DTMC $\{X_n, n = 0, 1, \ldots\}$ with one step transition probability matrix $P = [P_{ij}], i, j \in S$ satisfying $\sum_j P_{ij} = \sum_i P_{ij} = 1$. Find the steady state distribution for this DTMC, if it exist.

18. Consider the simple random walk on a circle. Assume that K odd number of points labeled $0, 1, \ldots, K-1$ are arranged on a circle clockwise. From i, the walker moves to $i+1$ (with K identified with 0) with probability p $(0 < p < 1)$ and to $i-1$ (with -1 identified with $K-1$) with probability $1-p$. Find the steady state distribution for this random walk, if it exist.

19. A mathematics professor has 2 umbrellas. He keeps some of them at home and some in the office. Every morning, when he leaves home, he checks the weather and takes an umbrella with him if it rains. In case all the umbrellas are in the office, he gets wet. The same procedure is repeated in the afternoon when he leaves the office to go home. The professor lives in a tropical region, so the chance of rain in the afternoon is higher than in the morning; it is 1/5 in the afternoon and 1/20 in the morning. Whether it rains or not is independent of whether it rained the last time he checked. On day 0, there is an umbrella at home, and 1 in the office. Note that, there are two trips each day. What is the expected number of days that will pass before the professor gets wet? What is the probability that the first time he gets wet it is on his way home from the office?

20. Consider a DTMC model arises in an insurance problem. To compute insurance or pension premiums for professional diseases such as silicosis, we need to compute the average degree of disability at pre-assigned time periods. Suppose that, we retain m degrees of disability $S_1, S_2, \ldots, S_m$. Assume that an insurance policy holder can go from degree S_i to degree S_j with a probability P_{ij}. This strong assumption leads to the construction of the DTMC model in which $P = [P_{ij}]$ is the one step transition probability matrix related to the degree of disability. Using real observations recorded in India, we considered the following transition matrix P:

$$P = \begin{pmatrix} 0.90 & 0.10 & 0 & 0 & 0 \\ 0 & 0.95 & 0.05 & 0 & 0 \\ 0 & 0 & 0.90 & 0.05 & 0.05 \\ 0 & 0 & 0 & 0.90 & 0.10 \\ 0 & 0 & 0.05 & 0.05 & 0.90 \end{pmatrix}$$

   (a) Classify the states of the chain as transient, +ve recurrent or null recurrent along with period.
   (b) Find the limiting distribution for the degree of disability.

21. A particle starts at the origin and moves to and fro on a straight line. At any move it jumps either 1 unit to the right or 1 unit to the left each with probability 1/3 or 1/3 respectively. With probability 1/3, the particle may stay at the same position in any move. Model this process as a stochastic process. Write the stochastic process with state space and parameter space. Verify whether this process satisfies Markov property.

22. Consider a DTMC $\{X_n, n = 0, 1, \ldots\}$ with $S = \{1, 2, 3, 4\}$ and its one-step transition probability matrix

$$P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \end{pmatrix}$$

(a) Classify the states as transient, + recurrent or null recurrent.

(b) Find the stationary distribution, $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$, if it exists.

23. Suppose graduate students exhibit 4 states of mind: 1 (suicidal); 2 (severe depression); 3 (mild depression); 4 (seeking for professional psychiatric help). Admit changes in state of mind can be modeled as a DTMC $\{X_n, n = 0, 1, \ldots\}$ with one-step transition probability matrix given by $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0 & 0.25 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(a) Draw the state transition diagram for this DTMC model.

(b) Find the expected number of changes of state of mind until a student seeks for professional psychiatric help, considering the initial state $X_0 = 2$.

(c) Compute the probability the student will eventually be suicidal starting from state $X_0 = 2$?