Laplace Transforms

Lecture 33

Connection between Laplace and Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$s = \sigma + j\omega \qquad X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(j\omega) = X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\} = X(\omega)$$

$$\boxed{X(j\omega) = \mathcal{F}\{x(t)\}}$$

New notation

Connection between Laplace and Fourier Transform

$$\boxed{X(s)\big|_{s=j\omega} = X(j\omega)}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

$$\boxed{X(s) = \mathcal{F}\{x(t)e^{-\sigma t}\}}$$

$\mathcal{L}$ may converge when $\mathcal{F}$ does not

Difference between Laplace and Fourier Transform

- $X(s) = \mathbb{F}\{x(t)e^{-\sigma t}\}$
  - In Laplace transforms, we play with σ to achieve convergence
  - In Fourier transforms, we allowed impulses.

Inverse Laplace Transform

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(s)\}$$

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{j\omega t}\,d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{(\sigma + j\omega)t}\,d\omega$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds$$

Inverse Laplace Transform

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$$x(t) = \cancel{\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}} X(s)e^{st}\,ds$$

Example 9.1

$$x(t) = e^{-at}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt$$

$$X(s) = \int_{0}^{\infty} e^{-at}e^{-st}dt$$

$$X(s) = \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$X(s) = \frac{-1}{s+a}[e^{-(s+a)t}]_0^{\infty}$$

$$= \frac{-1}{s+a}[0-1] = \frac{1}{s+a}$$

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$$X(s) = \int_{0}^{\infty} e^{-(\sigma+j\omega+a)t}dt$$

$$= \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt$$

$$\mathrm{Re}\{s\} > -a$$

Example 9.1

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

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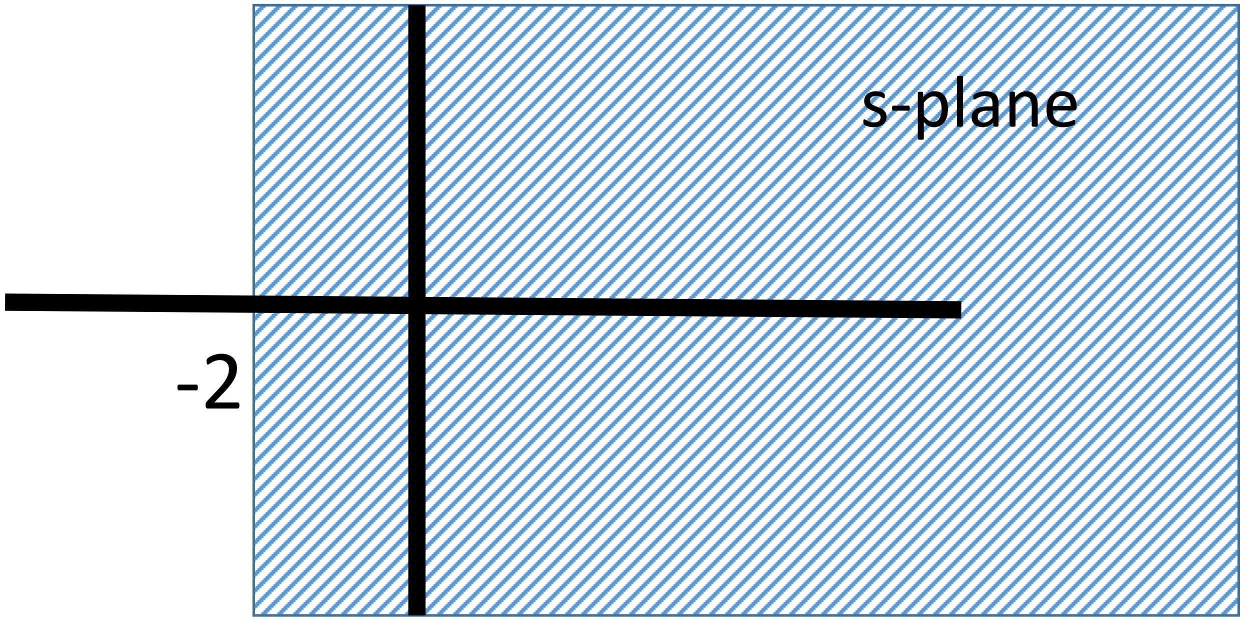
$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

If a = 2



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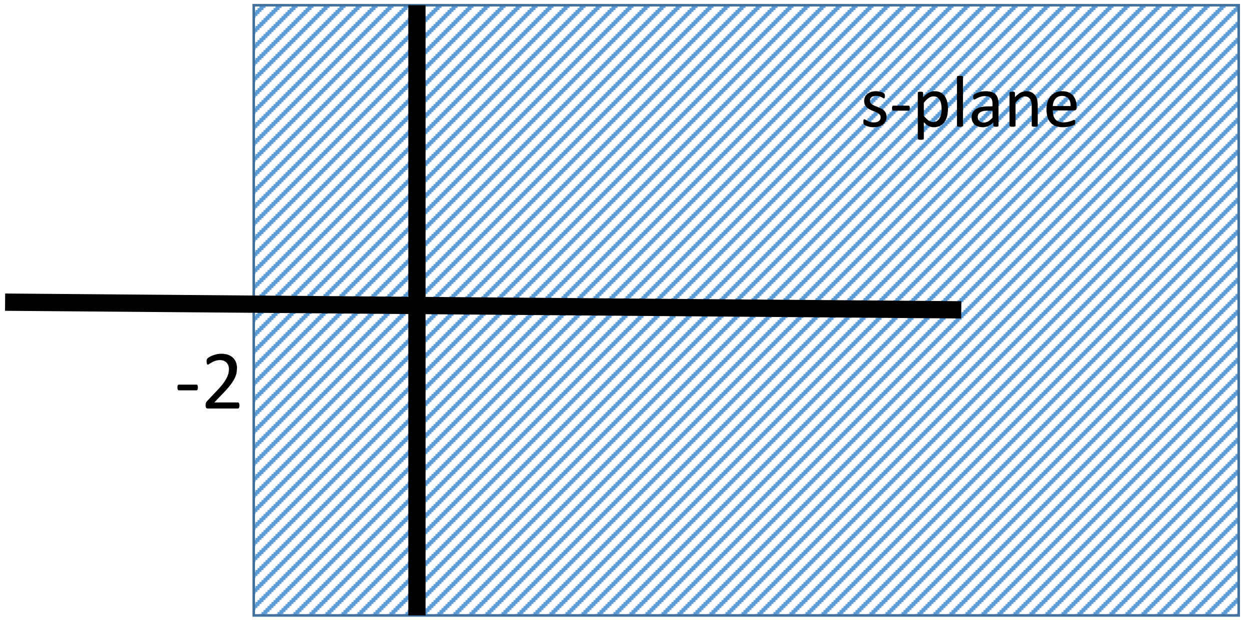


ROC (Region of Convergence)

Example 9.2

$$x(t) = -e^{-at}u(-t)$$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt$$

$$X(s) = \int_{0}^{-\infty} e^{-at}e^{-st}dt$$

$$X(s) = \int_{0}^{-\infty} e^{-(s+a)t}dt$$

$$X(s) = \frac{-1}{s+a}[e^{-(s+a)t}]_0^{-\infty}$$

$$= \frac{-1}{s+a}[0-1] = \frac{1}{s+a}$$

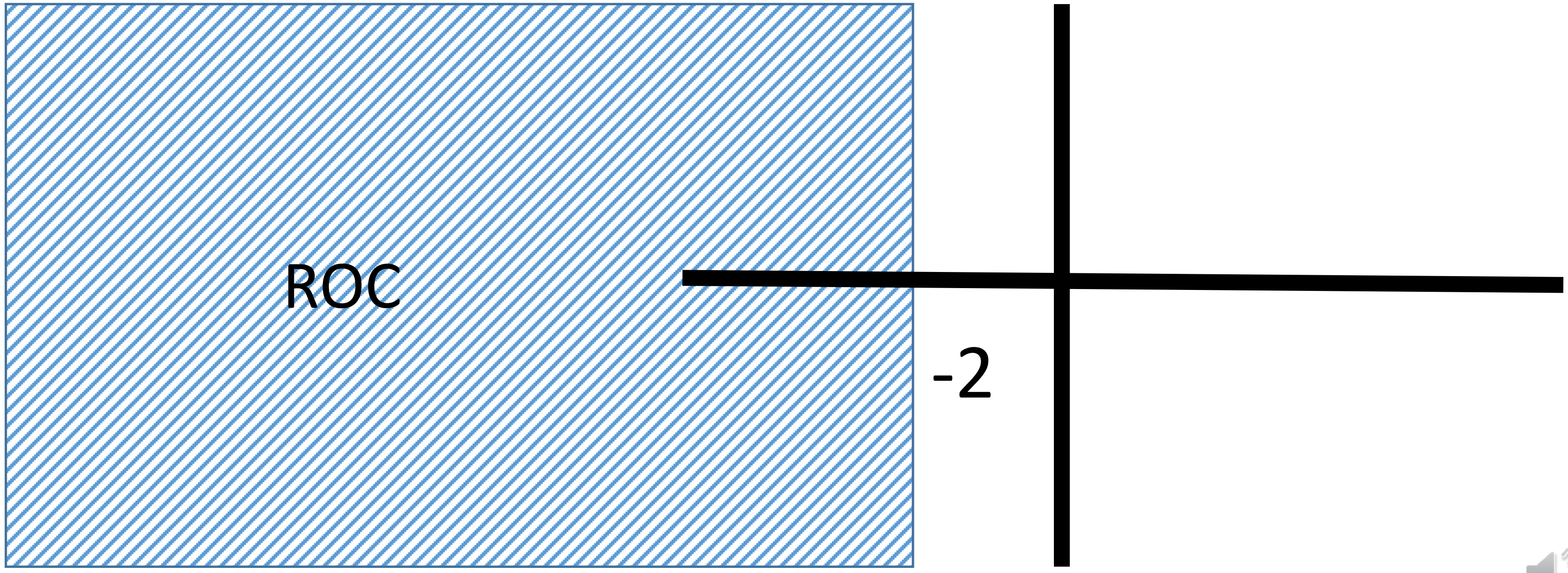
Example 9.2

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$$

$$a+\sigma<0$$

$$\sigma<-a$$

$$\mathrm{Re}\{s\}<-a$$



Importance of ROC

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \mathrm{Re}\{s\} < -a$$

Importance of ROC

$$\underbrace{e^{-at}u(t)}_{\text{right}} \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \underbrace{\mathrm{Re}\{s\} > -a}_{\text{right}}$$

$$\underbrace{-e^{-at}u(-t)}_{\text{left}} \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \underbrace{\mathrm{Re}\{s\} < -a}_{\text{left}}$$

Example 9.3

$$e^{-t}u(t)+e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}$$

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$$\mathrm{Re}\{s\} > -1$$

Example 9.3

$$e^{-t}u(t)+e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}+\frac{1}{s+2}$$

$$\mathrm{Re}\{s\}>-1 \quad \& \quad \mathrm{Re}\{s\}>-2$$

Example 9.3

$$e^{-t}u(t)+e^{-2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{2s+3}{(s+1)(s+2)}$$

$$\mathrm{Re}\{s\}>-1$$

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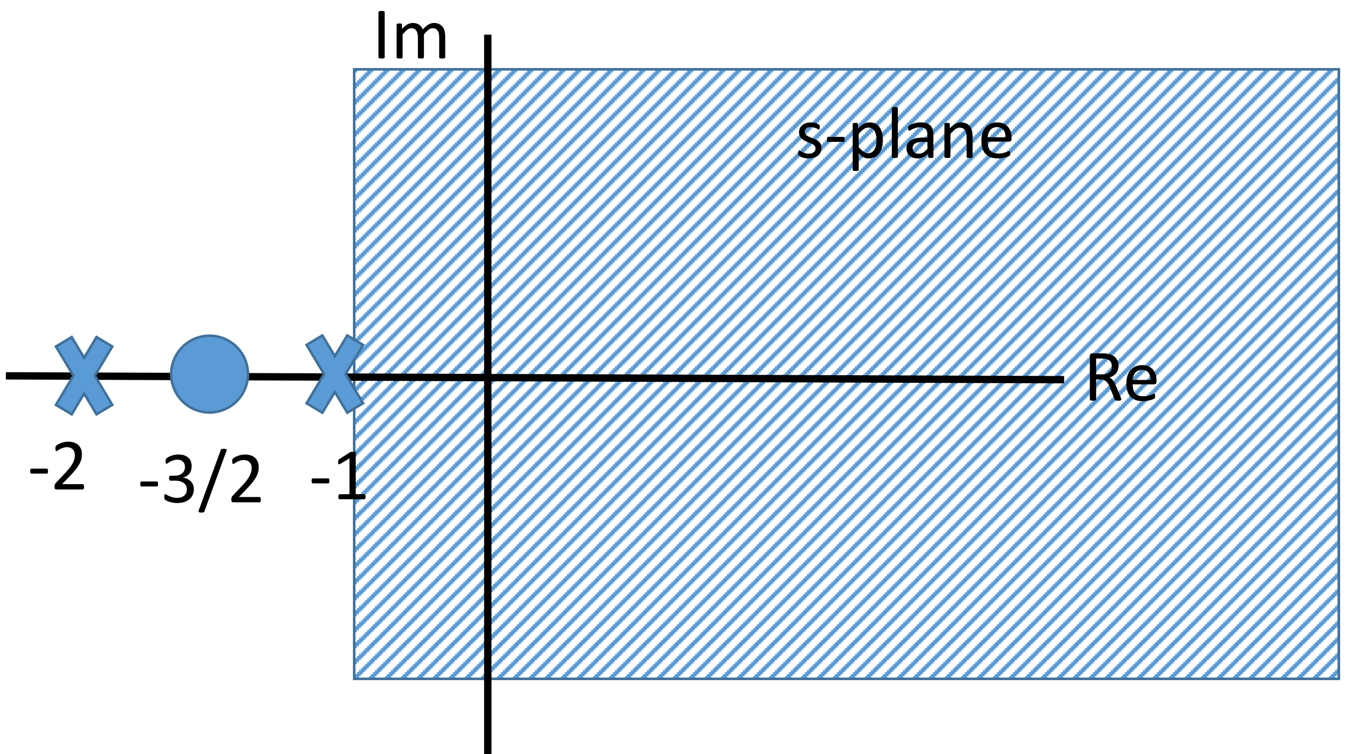

Laplace transform as a rational function

$$X(s) = \frac{N(s)}{D(s)}$$

Describe linear-constant coefficient differential equation

$$N(s) = 0$$ Zeros of X(s)

$$D(s) = 0$$ Poles of X(s)

ROC does not contain poles

Properties of the Region of Convergence

- The ROC contains no poles

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Poles of X(s) are where D(s) = 0

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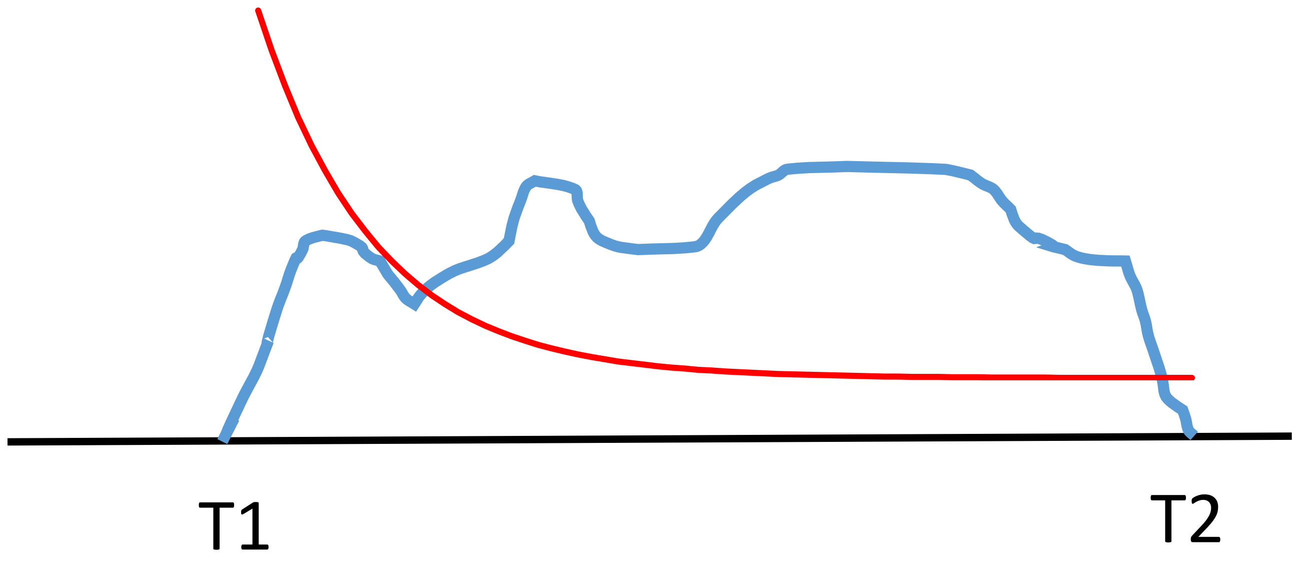
Poles of X(s) are where D(s) = 0

- The ROC of X(s) consists of a strip parallel to the jω-axis in the s-plane

- $\mathcal{F}$\{x(t)\} converges implies ROC includes the jω-axis in the s-plane

Properties of the Region of Convergence

- If x(t) is of finite duration
  - -ROC is entire s-plane



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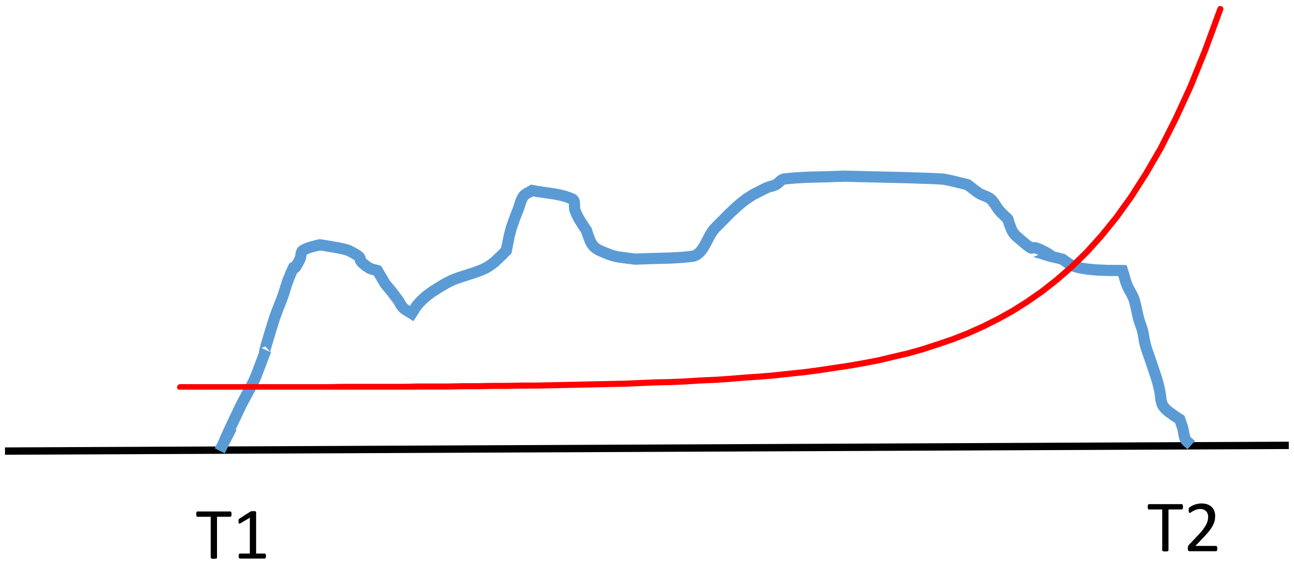
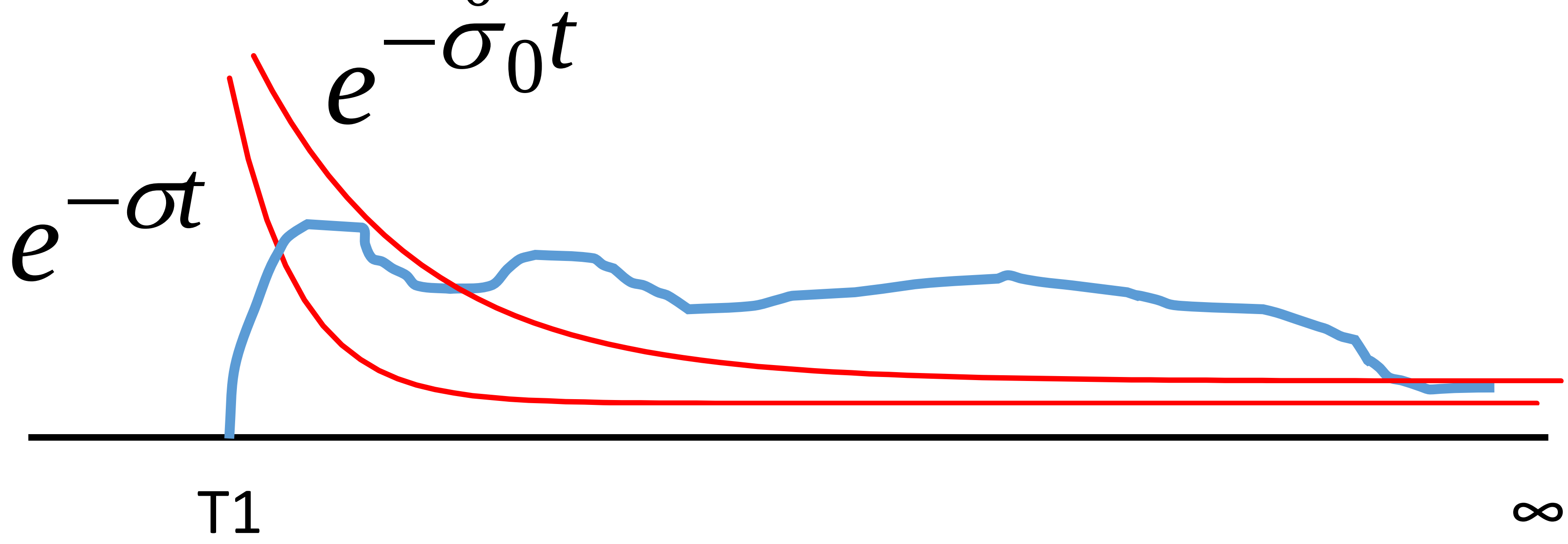


Properties of the Region of Convergence

- If x(t) is right-sided
  - -If σ_0 is in ROC, then $\sigma > \sigma_0$ is also in ROC



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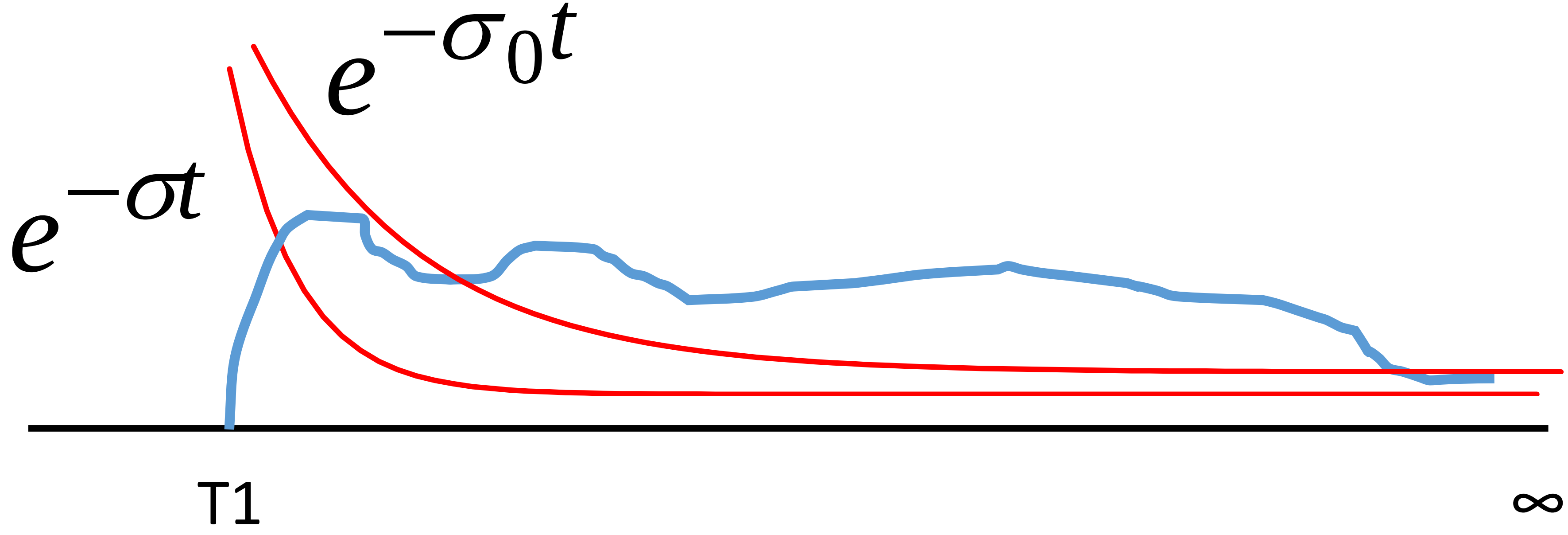

- If x(t) is right-sided and X(s) is rational
  - -ROC lies to the right of the rightmost pole

Properties of the Region of Convergence

- If x(t) is left-sided and Re{s} = σ_0 is in ROC
  - -all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC

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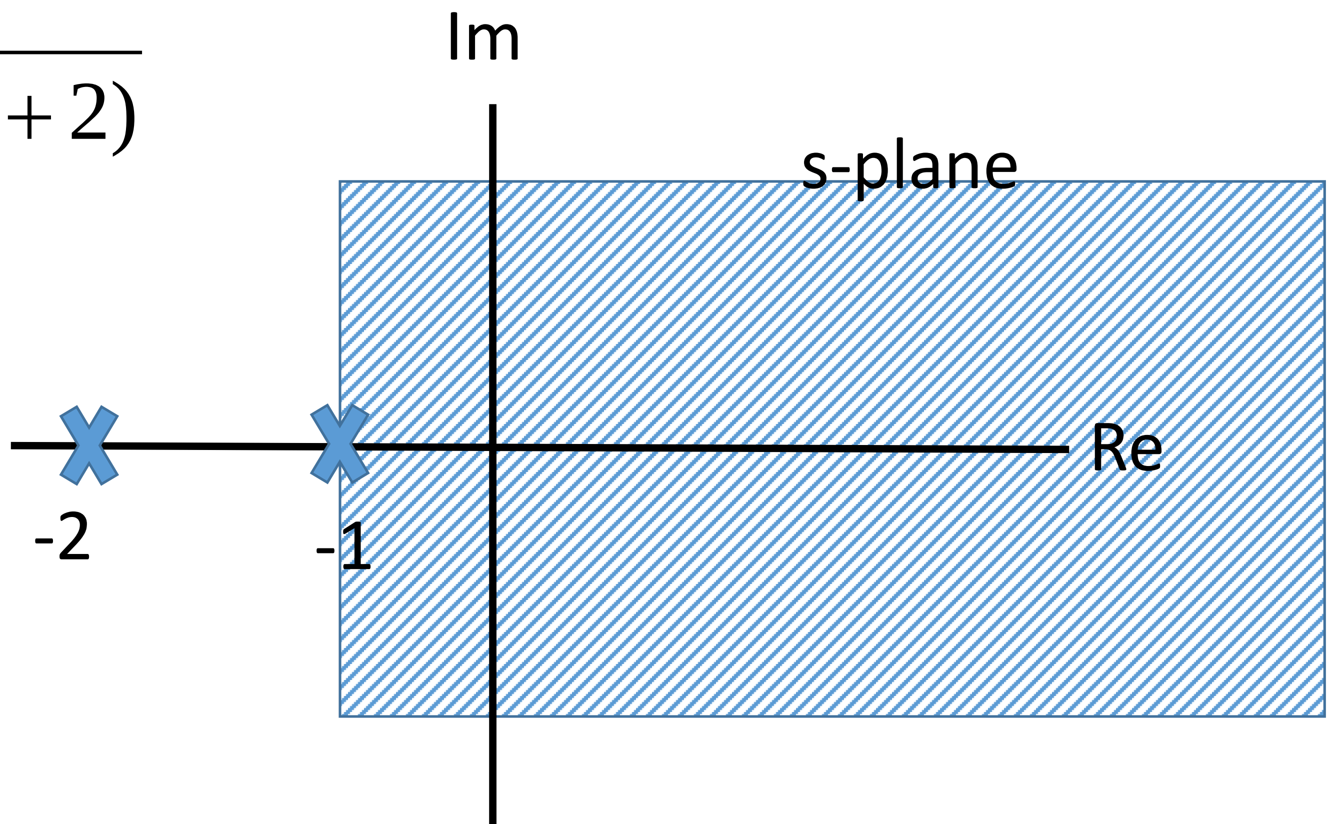
  - -all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC
- If x(t) is left-sided and X(s) is rational
  - -ROC lies to the left of the leftmost pole
- If x(t) is two-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC
  - -ROC is a strip in the s-plane

Properties of the Region of Convergence

- The ROC is a connected region
  - It cannot have multiple strips

$$X(s) = \frac{1}{(s+1)(s+2)}$$



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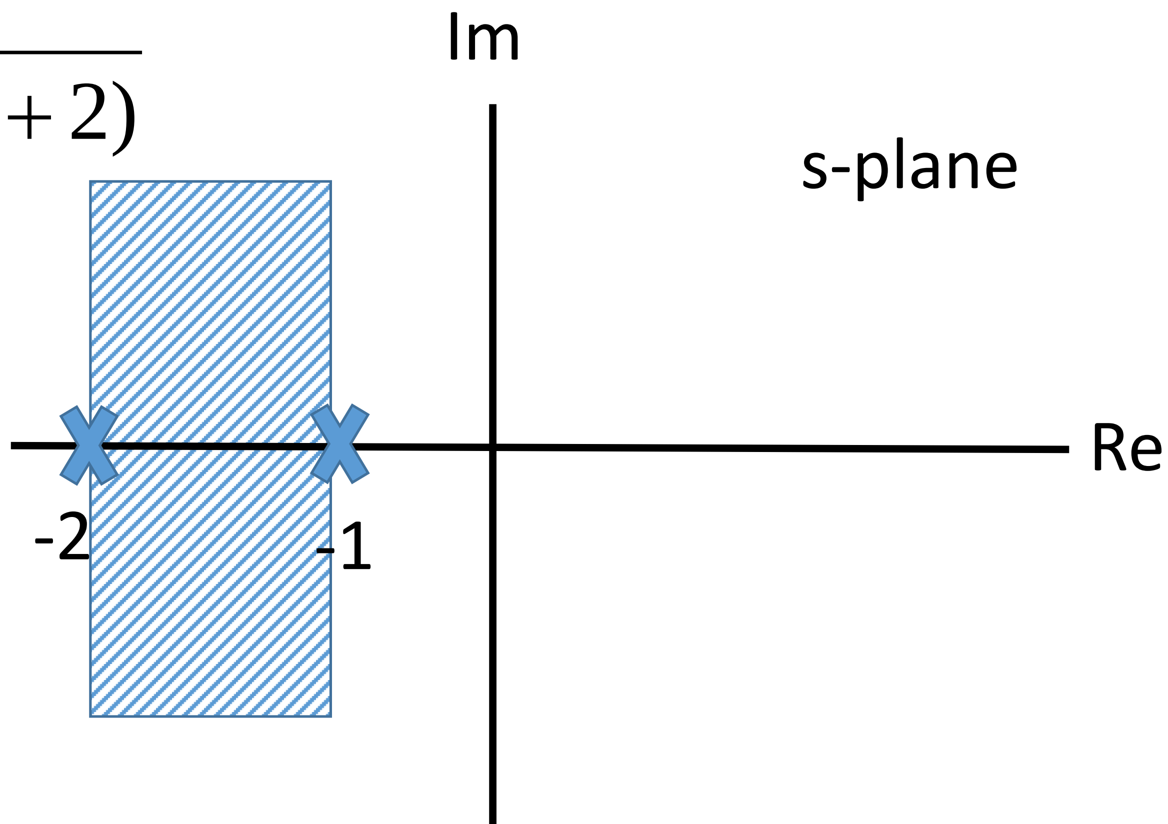

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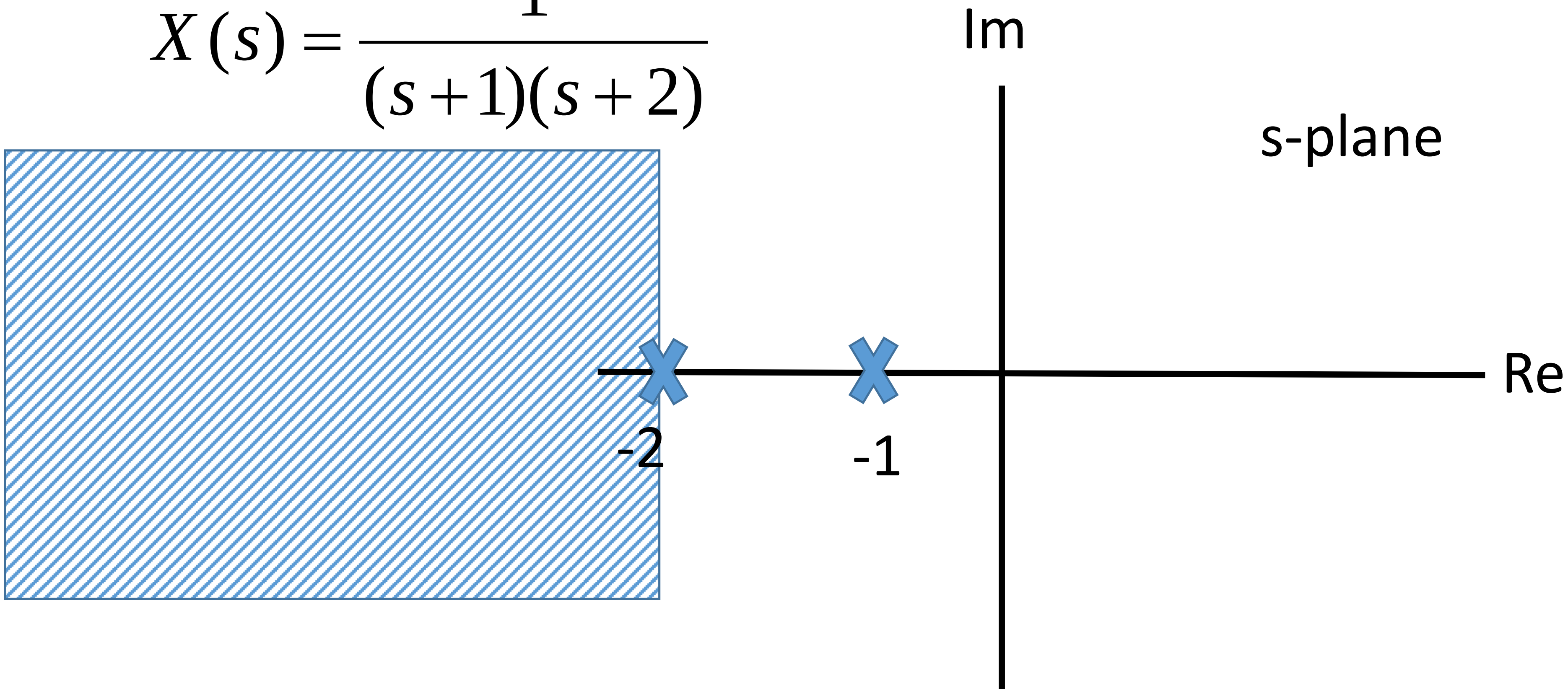

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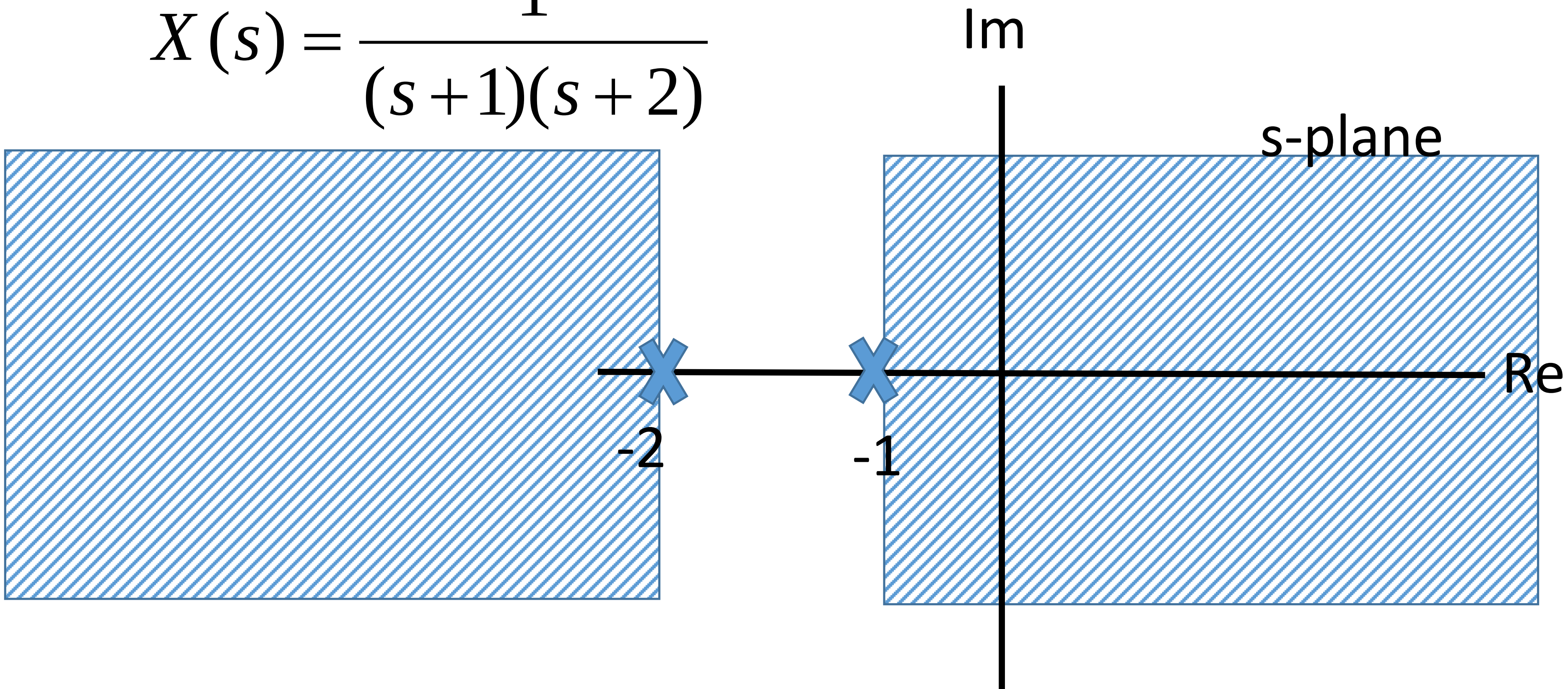

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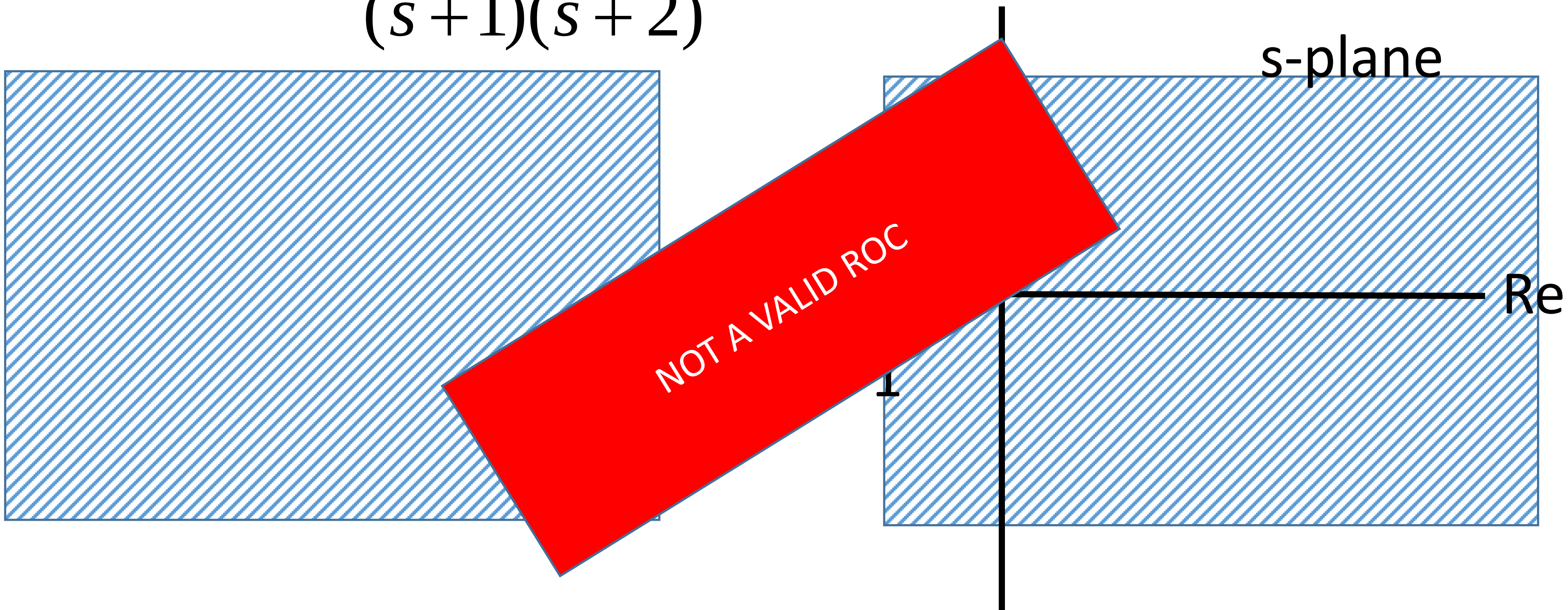

Inverse Laplace Transform

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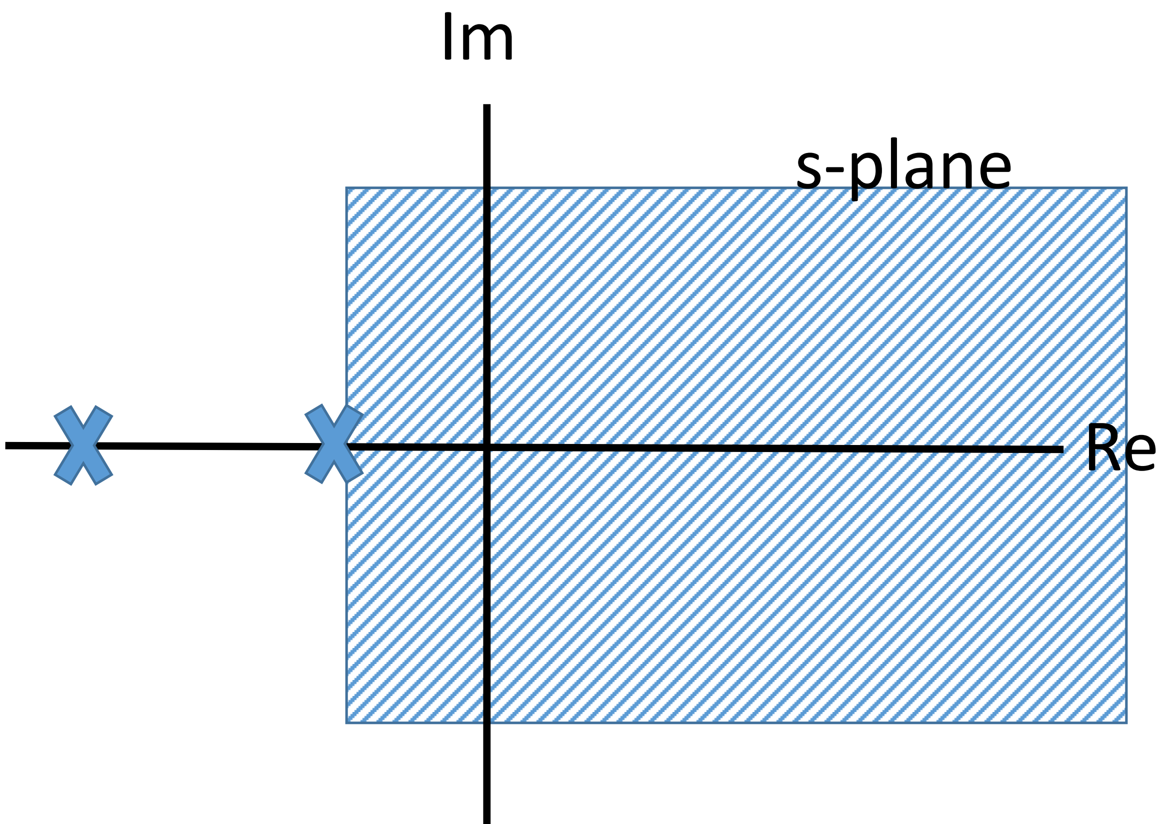

I	$x(t) = e^{-t}u(t) - e^{-2t}u(t)$
II	$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$
III	$x(t) = e^{-t}u(t) - e^{-2t}u(-t)$
IV	$x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)$

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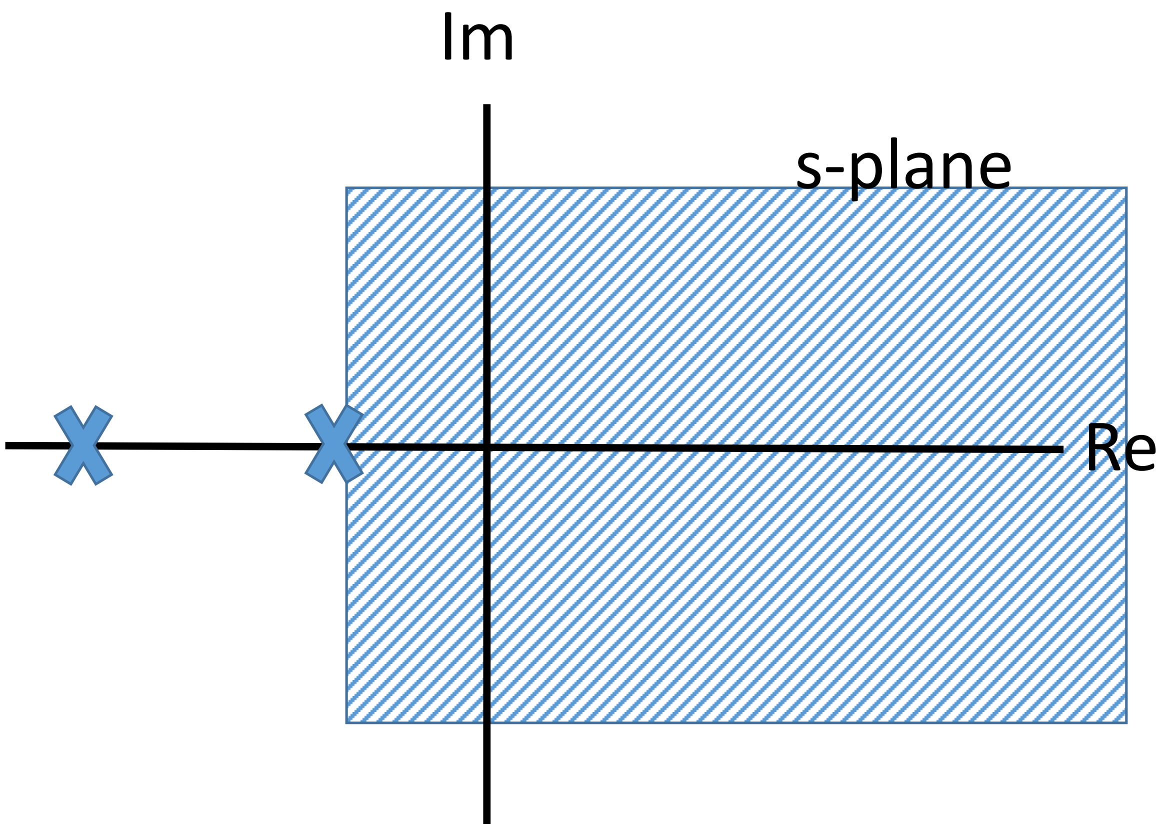

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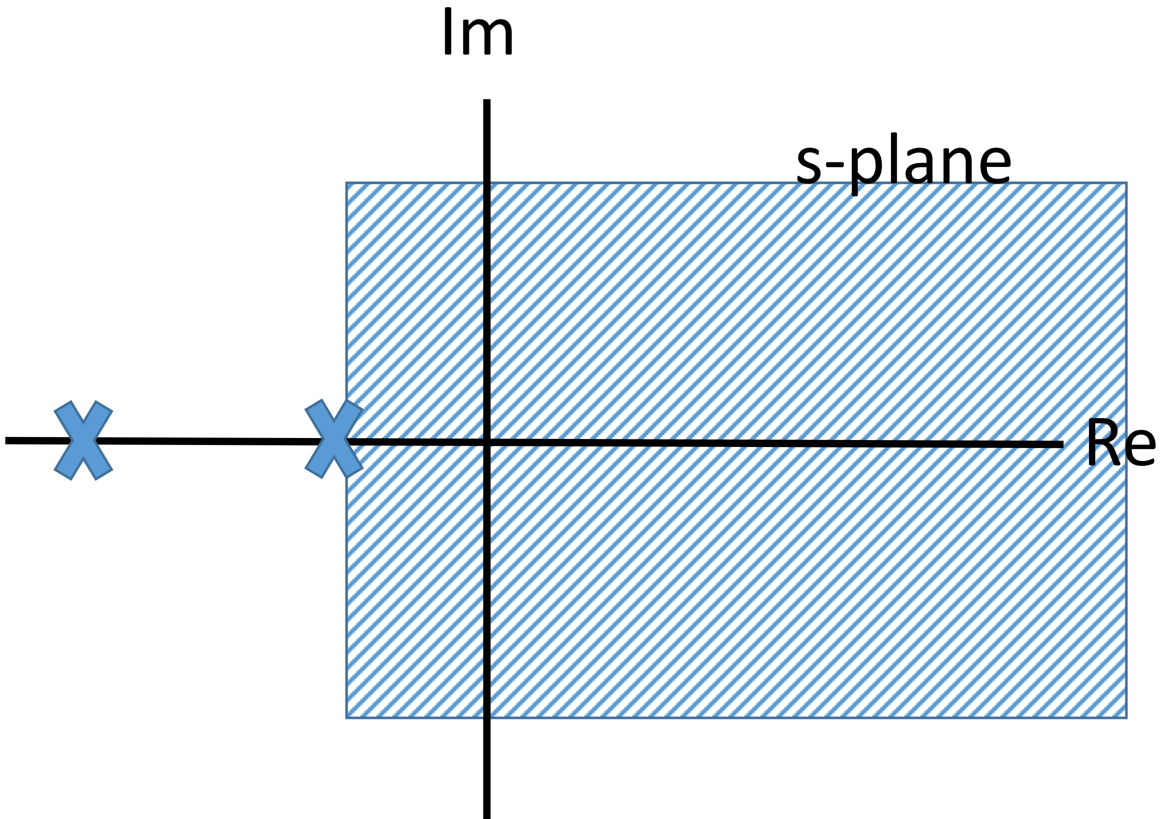


$$X(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$(s+2)A + (s+1)B = 1$$

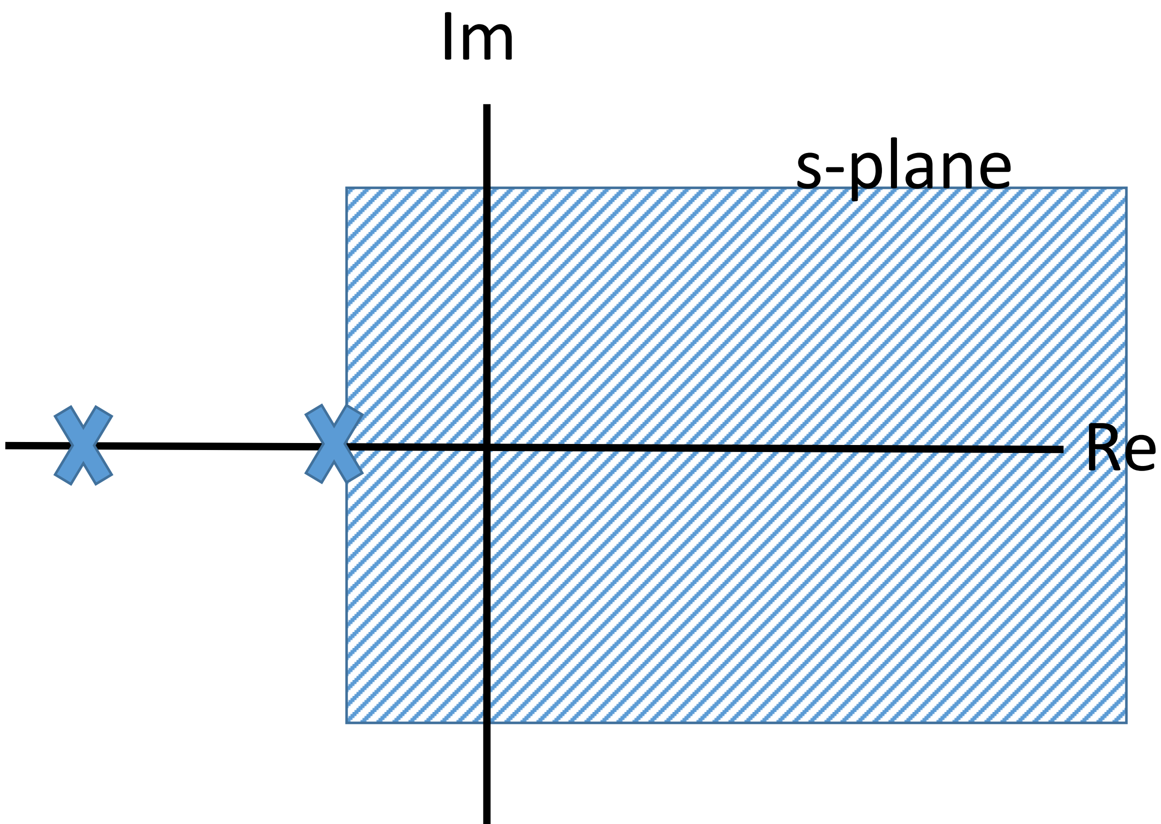
$$s(A+B) + 2A + B = 1$$

$$A + B = 0 \qquad 2A + B = 1$$

$$A = 1 \qquad B = -1$$

Inverse Laplace Transform

$$X(s) = \frac{1}{(s+1)(s+2)}$$



$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$x(t) = e^{-t}u(t) - e^{-2t}u(t)$$

Inverse Laplace Transform

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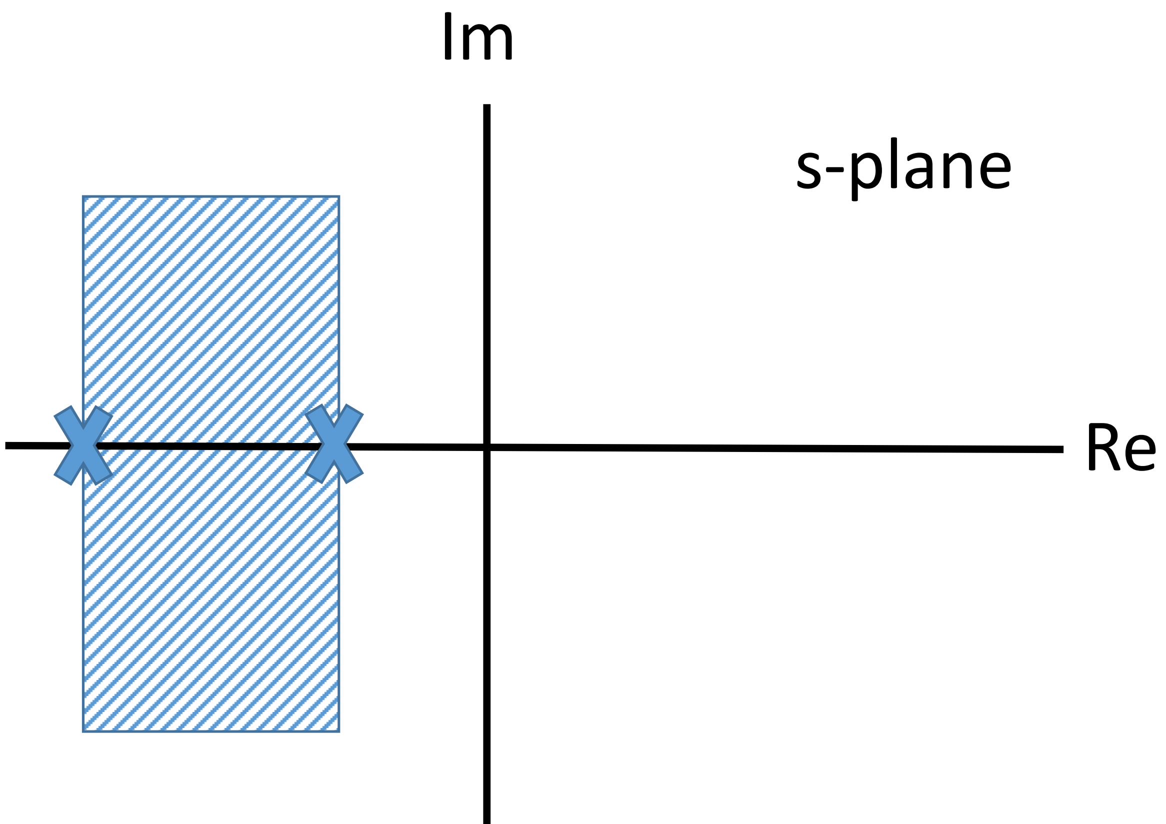

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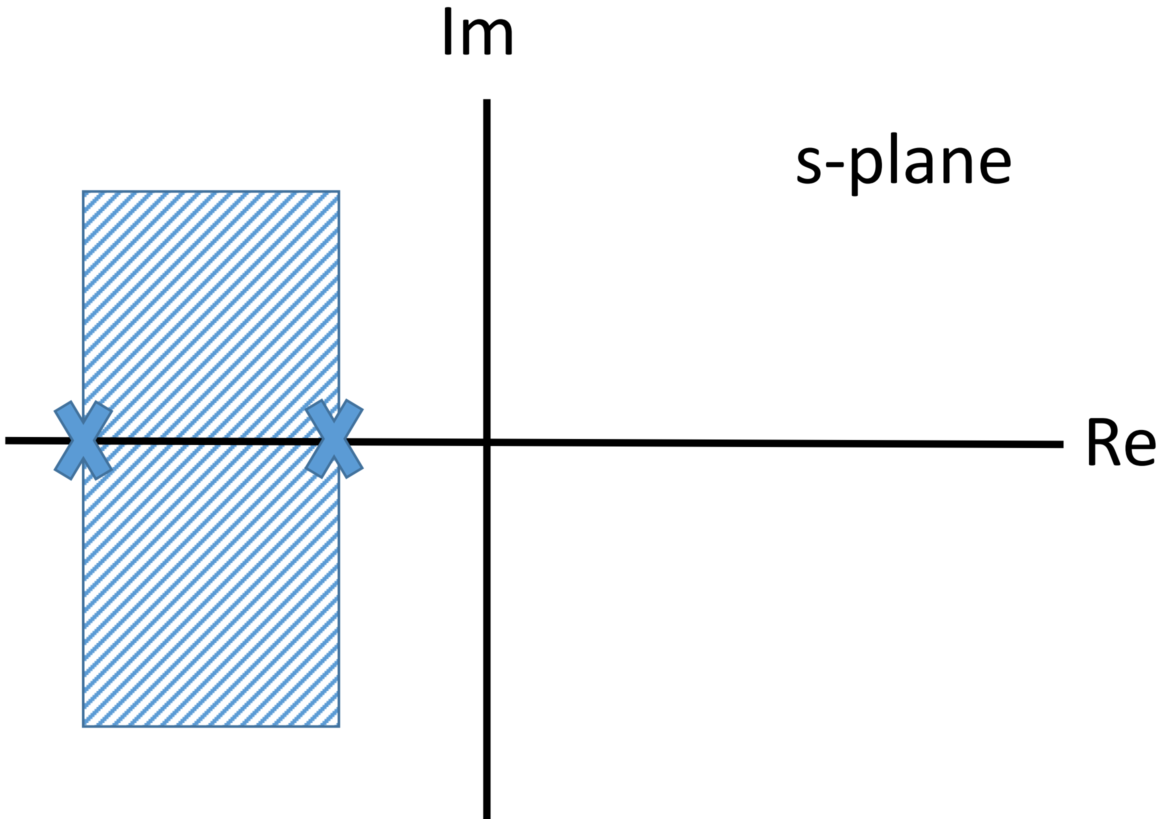

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Properties of Laplace Transforms

- **Linearity** $ax(t) + by(t) \leftrightarrow aX(s) + bY(s)$ $\supset (R_1 \cap R_2)$
- **Time shifting** $x(t - T) \leftrightarrow e^{-sT} X(s)$ R
- **Time scaling** $x(\mathrm{a}\,t) \leftrightarrow \dfrac{1}{|a|} X\left(\dfrac{s}{a}\right)$ aR