LIMIT & CONTINUITY

Theorem :- Let f and g be two real valued functions such that f and g both are continuous at 'c'. Then we prove that

(i) $f \pm g$ is continuous at c.

(ii) $(fg)(x) = f(x)g(x)$ is continuous at c.

(iii) If $g(c) \neq 0$ then $(f/g)(x) = f(x)/g(x)$ is continuous at c.

(iv) $|f|$ is also continuous at 'c'

(v) $\max(f,g)$ & $\min(f,g)$ both are continuous at c.

Proof (i) Let (x_n) be an arbitrary sequence in the domain such that $x_n \to c$ as $n \to \infty$.

Since f & g both are continuous at c, $f(x_n)$ & $g(x_n)$ converge to $f(c)$ and $g(c)$ respectively.

$\Rightarrow f(x_n) \pm g(x_n)$ converges to $f(c) \pm g(c)$.

Thus, $f \pm g$ is continuous at 'c'.

(ii) Similar proof.

(iii) Since $g(c) \neq 0$, $\exists\ \delta > 0$ s.t. $g(x) \neq 0\ \forall x \in (c-\delta, c+\delta)$.

Therefore, $f(x)/g(x)$ is well defined on $(c-\delta, c+\delta)$.

Let (x_n) be an arbitrary sequence in the domain s.t. $x_n \to c$ as $n \to \infty$.

Since f & g are continuous at c, $f(x_n)$ & $g(x_n)$ converges to $f(c)$ & $g(c)$ respectively.

Let $\exists\ N$ s.t. $x_n \in (c-\delta, c+\delta)\ \forall\ n \geq N$

then $g(x_n) \neq 0\ \forall\ n \geq N$

Then $f(x_n)/g(x_n) \longrightarrow f(c)/g(c)$ as $n \to \infty$

$\therefore (f/g)(x)$ is continuous at 'c'

(iv) Similar.

(v) Note that, $\max(f, g) = \frac{1}{2}\left[(f+g) + |f-g|\right]$

$$\min(f, g) = \frac{1}{2}\left[(f+g) - |f-g|\right]$$

Since f and g are continuous at 'c', $f+g$ & $f-g$ are also continuous at 'c'.

Then $|f-g|$ is also continuous at 'c'.

Thus, $(f+g) + |f-g|$ & $(f+g) - |f-g|$ are also continuous at 'c'.

Therefore, $\max(f, g)$ & $\min(f, g)$ are continuous at 'c'.

Theorem :- Composition of two continuous functions is also continuous. In other words, if f is continuous at c and g is continuous at $f(c)$ and $(g \circ f)(x) = g(f(x))$ is defined. Then $g \circ f$ is also continuous at 'c'.

Proof :- Discussed.

Types of discontinuity

Removable discontinuity :- Let $f : D \to \mathbb{R}$ be a function and $a \in D$. Then we say f has removable discontinuity at 'a' if $\lim_{x \to a} f(x)$ exists but not equal to $f(a)$.

If f is not defined at 'a' but $\lim_{x \to a} f(x)$ exists then also we say that f has removable discontinuity at 'a'.

Example

$f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Here $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{\sin x}{x} = 1 \neq f(0)$

Here we have removable discontinuity at $x = 0$.

Jump discontinuity (Discontinuity of first kind) :-

Let $f: D \to \mathbb{R}$ be a function and $a \in D$. Then we say f has Jump discontinuity at 'a' if $\lim_{x \to a-} f(x)$ & $\lim_{x \to a+} f(x)$ both exist but not equal.

Eg :- $f : \mathbb{R} \to \mathbb{R}$ $f(x) = 1, \; x \le 0$

$= 2, \; x > 0$

Here $\lim_{x \to 0-} f(x) = 1$

$\lim_{x \to 0+} f(x) = 2$

$\therefore$ Thus f has jump discontinuity at '0'.

Essential discontinuity (Discontinuity of second kind)

Let $f : D \to \mathbb{R}$ be a function and $a \in D$. If $\lim_{x \to a-} f(x)$ or $\lim_{x \to a+} f(x)$ does not exist (or infinite) then we say f has essential discontinuity at 'a'.

Infinite discontinuity:- Let $f: D \to \mathbb{R}$ be a function and $a \in D$. If $\lim\limits_{x \to a-} f(x)$ or $\lim\limits_{x \to a+} f(x)$ is $+\infty$ (or $-\infty$) then we say f has infinite discontinuity at 'a'.

Ex:- $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that f has essential discontinuity at each point $c \in \mathbb{R}$.

Sol^n:- Check that for $r \in \mathbb{Q}$ or $c \in \mathbb{R} \setminus \mathbb{Q}$

$\lim\limits_{x \to r+} f(x)$, $\lim\limits_{x \to r-} f(x)$, $\lim\limits_{x \to c+} f(x)$ & $\lim\limits_{x \to c-} f(x)$

all do not exist.

$\therefore$ Let $r \in \mathbb{Q}$

$x_n = r + \frac{1}{n}$ & $y_n = r + \frac{\sqrt{2}}{n}$

Then $x_n \to r$ & $y_n \to r$, $x_n, y_n > r$

but $f(x_n) = 0 \ \forall n$ & $f(y_n) = 1 \ \forall n$

$\Rightarrow f(x_n) \to 0$ & $f(y_n) \to 1$ as $n \to \infty$

Thus $\lim\limits_{x \to r+} f(x)$ does not exist.

For $c = \pm c_0 . c_1 c_2 c_3 c_4 \ldots$

$x_n = c + \frac{1}{n}$ & $r_n = \pm c_0 . c_1 c_2 \ldots c_n$

Eg:- $f: \mathbb{R} \to \mathbb{R}$ $f(x) = 0, \ x \le 1$

$= \frac{1}{x-1} \quad x > 1$ Thus f has infinite

$\therefore \lim\limits_{x \to 1-} f(x) = 0$ and $\lim\limits_{x \to 1+} f(x) = \infty$. discontinuity at 1.

f also has essential discontinuity at 1 as well.

Theorem:- Let $f: A \to \mathbb{R}$ be a continuous function and A is a closed and bounded set. Then f is bounded.

Bolzano Weierstrass Theorem:- Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof:- If possible, let f be not bounded. Then for each $n \in \mathbb{N}$, $\exists\, x_n \in A$ s.t. $|f(x_n)| > n$.

Thus, we get a sequence (x_n) in A s.t. $|f(x_n)| > n$.

Since A is bounded, the sequence (x_n) is a bounded sequence. By Bolzano-Weierstrass theorem, $\exists$ a subsequence (x_{r_n}) of (x_n) such that $x_{r_n} \to c$ as $n \to \infty$.

Since A is closed & (x_{r_n}) is a convergent sequence in A & $x_{r_n} \to c$, $c \in A$

Since f is continuous at $c \in A$ & $x_{r_n} \to c$,

$$f(x_{r_n}) \longrightarrow f(c) \text{ as } n \to \infty.$$

$$\Rightarrow |f(x_{r_n})| \longrightarrow |f(c)| \text{ as } n \to \infty$$

But, by our construction $|f(x_{r_n})| > r_n \geq n$

$$\Rightarrow |f(x_{r_n})| \longrightarrow \infty \text{ as } n \to \infty.$$

This is a contradiction.

$\therefore$ The function f must be bounded.

<u>Theorem</u> :- Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then the supremum and infimum of the function f are achieved in $[a,b]$.

<u>Proof</u> :- By the previous theorem, f is bounded, i.e, $f([a,b])$ is a bounded set in $\mathbb{R}$.

Thus, by the Completeness Property of $\mathbb{R}$, we have $f([a,b])$ has a supremum & an infimum in $\mathbb{R}$.

Let $M = \sup f([a,b]) = \sup\{f(x) : x \in [a,b]\}$

$m = \inf f([a,b]) = \inf\{f(x) : x \in [a,b]\}$

Thus, for each $n \in \mathbb{N}$, $\exists\ x_n \in [a,b]$ such that

$$M - \frac{1}{n} < f(x_n) \le M.$$

Thus (x_n) is a bounded sequence in $[a,b]$ such that $f(x_n) \longrightarrow M$ as $n \to \infty$.

Similarly, for each $n \in \mathbb{N}$, $\exists\ y_n \in [a,b]$ such that

$$m \le f(y_n) < m + \frac{1}{n}$$

Thus (y_n) is a bounded sequence in $[a,b]$ s.t. $f(y_n) \to m$ as $n \to \infty$

By Bolzano-Weierstrass theorem, $\exists$ convergent subsequences (x_{r_n}) and (y_{r_n}) of (x_n) and (y_n) such that $x_{r_n} \to c$ and $y_{r_n} \to d$ respectively.

Since $a \le x_{r_n} \le b$ & $a \le y_{r_n} \le b$,

$a \le c \le b$, $a \le d \le b$, i.e., $c, d \in [a, b]$

Since f is continuous at c & d,

$f(x_{r_n}) \to f(c)$ & $f(y_{r_n}) \to f(d)$ as $n \to \infty$.

On the other hand, $f(x_n) \to M$ & $f(y_n) \to m$.

Thus, $f(x_{r_n}) \to M$ & $f(y_{r_n}) \to m$ as $n \to \infty$

$\therefore$ $f(c) = M$ & $f(d) = m$.

This completes the proof.

Example :- (i) $f(x) = \frac{1}{x}$ on $(0,1)$

Then f is continuous on $(0,1)$ but f is not bounded.

(ii) $f(x) = \sin\left(\frac{1}{x}\right)$ on $(0,1)$

Then f is continuous & bounded on $(0,1)$.

(iii) $f(x) = x$ on $\mathbb{R}$

f is continuous but not bounded.

(iv) $f(x) = c$ on $\mathbb{R}$

f is continuous and bounded.

Theorem :- Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function.

If $f(a)f(b) < 0$ then $\exists\ c \in (a,b)$ s.t. $f(c) = 0$.

Proof:- Without loss, assume $f(a) < 0$ and $f(b) > 0$.

Since $f(a) < 0$, $\exists\, \delta_1 > 0$ s.t. $f(x) < 0 \ \forall x \in [a, a+\delta_1)$

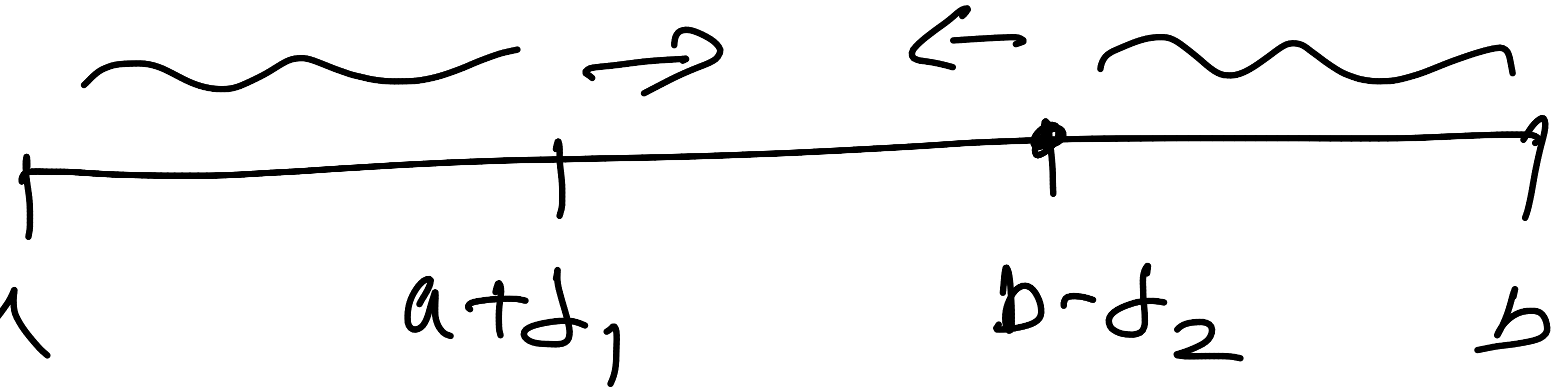
Since $f(b) > 0$, $\exists\, \delta_2 > 0$ s.t. $f(x) > 0 \ \forall x \in (b-\delta_2, b] \subseteq [a,b]$.

Let $S = \{ x \in [a,b] : f(x) < 0 \}$

Then $[a, a+\delta_1) \subseteq S$ & $S \cap (b-\delta_2, b] = \phi$

Since $S \subseteq [a,b]$, S is bound above, $b-\delta_2$ is an upper bound for S.

Let $c = \sup(S)$



Here $c \in [a+\delta_1, b-\delta_2]$, $(c-\delta_1, c+\delta_2) \subseteq [a,b]$.

Let $x_n = c + \frac{1}{n}$. Since $c = \sup(S)$, therefore

$x_n \notin S \quad \forall n$.

Here $x_n \to c$ as $n \to \infty$. Thus, $\exists\, N \in \mathbb{N}$ s.t.

$x_n \in (c-\delta_1, c+\delta_2) \subseteq [a,b] \quad \forall\, n \geq N$

$\Rightarrow f(x_n) > 0 \quad \forall\, n \geq N$.

Since f is continuous at 'c' & $x_n \to c$, $f(x_n) \to f(c)$ as $n \to \infty$

Since $f(x_n) > 0 \;\; \forall\, n \geq N$, $f(c) \geq 0$ — ①

Since $\sup(S) = c$. Thus for each n, $\exists\, y_n \in S$

s.t. $c - \frac{1}{n} < y_n \leq c$.

Thus $y_n \to c$ as $n \to \infty$. Since f is continuous

at 'c', $f(y_n) \to f(c)$.

Since $y_n \in S \ \forall\, n \in \mathbb{N}$, $f(y_n) < 0$

$\therefore \ f(c) \le 0$ — ②

Thus by ① & ②, $f(c) = 0$

Since $c \in [a+\delta_1, b-\delta_2] \subset (a,b)$, we are done.