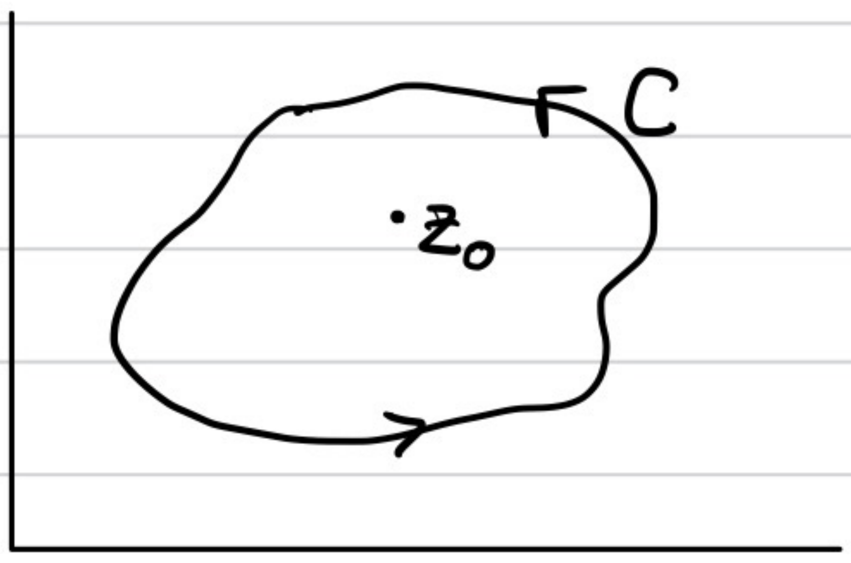
Lecture – 20

Cauchy's Integral formula :

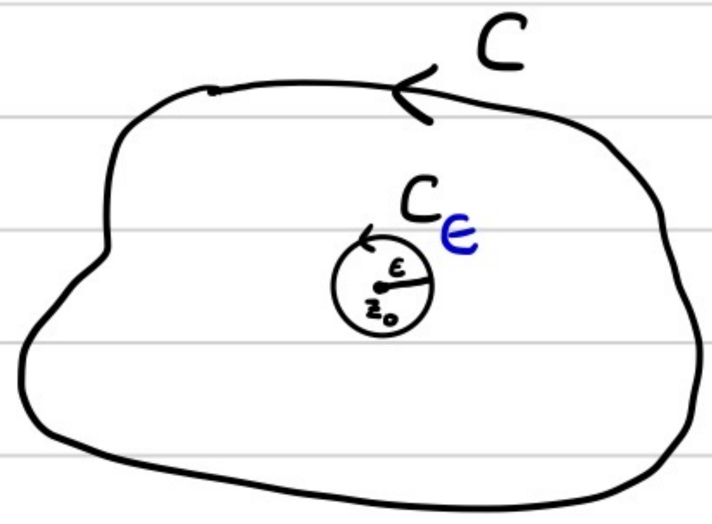
If $f(z)$ is analytic on and inside a simple closed contour C (running counterclockwise)

$$\oint_C dz \, \frac{f(z)}{z - z_0} = 2\pi i \, f(z_0)$$



where z_0 is any point inside the contour C.

Since $\frac{f(z)}{z - z_0}$ is analytic everywhere inside C except at $z = z_0$, Using the Cauchy-Gursat theorem we can rewrite



$$I = \oint_C dz \, \frac{f(z)}{z - z_0} = \oint_{C_\epsilon} dz \, \frac{f(z)}{z - z_0}$$

where C_ϵ is circle contour of radius ϵ & centre z_0.

On C_ϵ : $\quad z = z_0 + \epsilon e^{i\theta} \Rightarrow dz = i \epsilon e^{i\theta} d\theta$

$$I_\epsilon = \int_0^{2\pi} d\theta \left(i \cancel{\epsilon e^{i\theta}} \right) . \frac{f(z_0 + \epsilon e^{i\theta})}{\cancel{\epsilon e^{i\theta}}}$$

$$= i \int_0^{2\pi} d\theta \, f(z_0 + \epsilon e^{i\theta})$$

CG theorem further tells us that I_ϵ is independent of ϵ, Since the integrand is analytic $\forall \, \epsilon > 0$. Thus

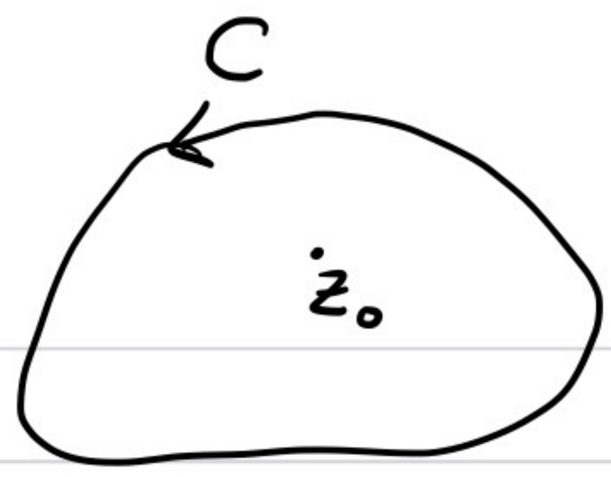
$$I = \lim_{\epsilon \to 0} I_\epsilon = \lim_{\epsilon \to 0} i \int_0^{2\pi} d\theta \, f(z_0 + \epsilon e^{i\theta})$$

$$= i \int_0^{2\pi} d\theta \, f(z_0) = 2\pi i \, f(z_0)$$

Thus

$$\boxed{\oint_C dz \, \frac{f(z)}{z - z_0} = 2\pi i \, f(z_0)}$$

Generalization of Cauchy's Integral formula:



$$f(z_0) = \frac{1}{2\pi i}\oint dz\, \frac{f(z)}{z-z_0}$$

Differentiate both sides w.r.t. z_0 successively

$$f'(z_0) = \frac{1}{2\pi i}\oint dz\, \frac{f(z)}{(z-z_0)^2}$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint dz\, \frac{f(z)}{(z-z_0)^3}$$

$$\vdots$$

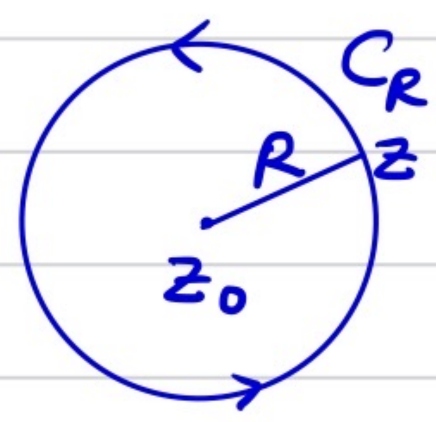
$$\boxed{f^{n}(z_0) = \frac{n!}{2\pi i}\oint dz\, \frac{f(z)}{(z-z_0)^{n+1}}}$$

We will later justify this generalization better after studying Laurant series expansion of complex analytic function

* Note that the above generalization of Cauchy integral formula implies that all order derivatives of an analytic function exist & are analytic themselves!

Ex: Show that if $f(z)$ is analytic then so is $f'(z)$ (CR eqns).

- The above expression for $f^{(n)}(z_0)$ implies a bound on the magnitude of $f^{(n)}(z_0)$



$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{C_R} dz\, \frac{f(z)}{(z-z_0)^{n+1}}$$

$$= \frac{n!}{2\pi}\int_0^{2\pi} d\theta\,.\,Re^{i\theta}.\ \frac{f(z_0+Re^{i\theta})}{R^{n+1}.\,e^{i\theta(n+1)}}$$

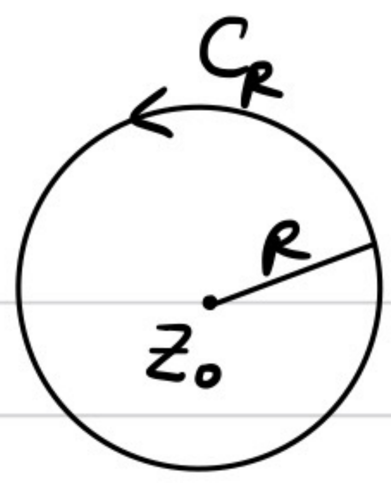
$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\int_0^{2\pi} d\theta\, \frac{|f(z_0+Re^{i\theta})|}{R^n}$$

$$\leq \frac{n!}{R^n}M_R$$

where $M_R \geq 0$ is the maximum value of $|f(z)|$ on C_R.

Examples:

1. $$I_n = \oint_{C_R} \frac{dz}{(z-z_0)^{n+1}} \qquad \forall \; n \in \text{Integers}$$

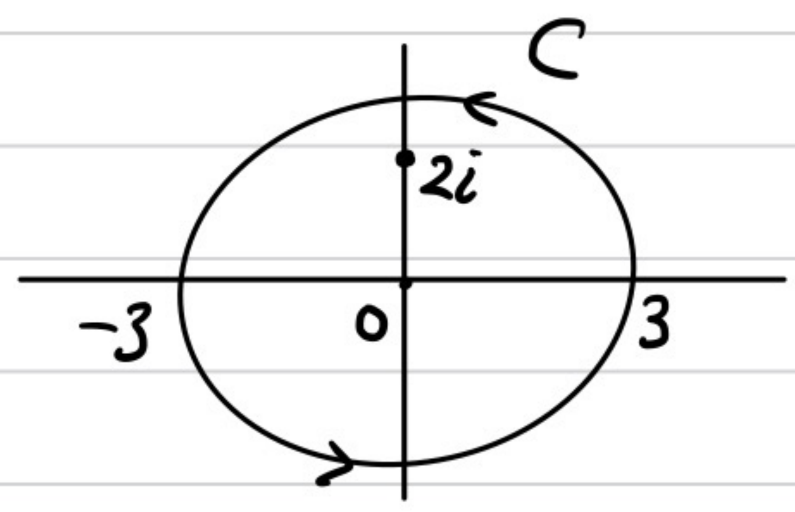


$$z = z_0 + Re^{i\theta} \qquad \theta \in (0, 2\pi)$$
$$dz = iRe^{i\theta} d\theta$$

$$I_n = \int_0^{2\pi} \frac{iRe^{i\theta} d\theta}{R^{n+1} e^{i\theta(n+1)}} = \frac{i}{R^n} \int_0^{2\pi} e^{-in\theta} d\theta = \frac{i}{R^n} 2\pi \delta_{n,0} = \begin{cases} 2\pi i \, ; & n=0 \\ 0 & \text{else} \end{cases}$$

The above also follows from Cauchy's integral formula & its generalization.

2. $$I = \oint_C \frac{2z^2 - z - 2}{z - 2i} dz$$



$$= \frac{1}{2\pi i} \oint_C \frac{2\pi i (2z^2 - z - 2)}{z - 2i} dz$$

use cauchy's integral formula.

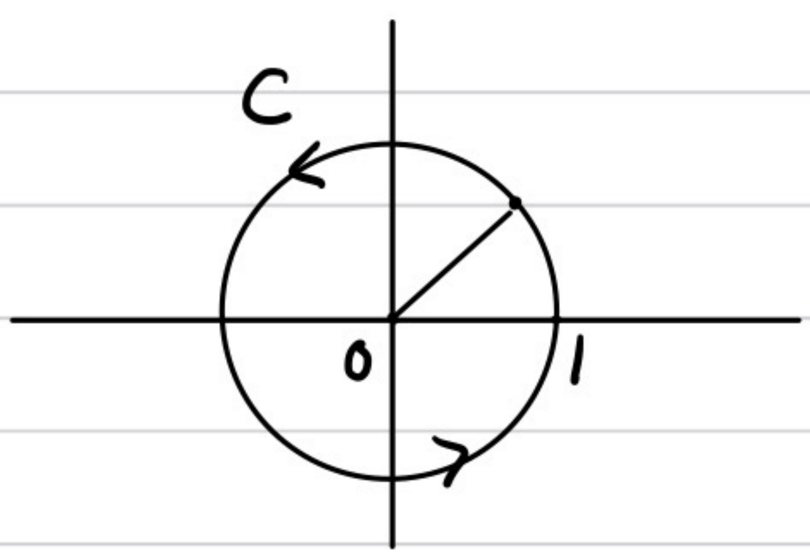
$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{z - z_0}$$

$$\therefore \; I = 2\pi i \, (2z^2 - z - 2)\Big|_{z=2i} = 2\pi i \left(2(2i)^2 - 2i - 2\right)$$

$$= 2\pi i \left[-8 - 2i - 2\right] = -4\pi i (5 + 2i)$$

$$= 8\pi - i 20\pi$$

- $$I = \oint_C dz \cdot \frac{e^{az}}{z}$$



$$= 2\pi i \; e^{az}\Big|_{z=0} = 2\pi i$$

$$I = i \int_{-\pi}^{\pi} \cancel{e^{i\theta}} d\theta \cdot \frac{e^{ae^{i\theta}}}{\cancel{e^{i\theta}}} = i \int_{-\pi}^{\pi} d\theta \cdot e^{a\cos\theta + i a \sin\theta}$$