Lecture - 25

Multivariable Calculus

Maxima and Minima

Let $f: D (\subseteq \mathbb{R}^2) \to \mathbb{R}$, D is an open set in $\mathbb{R}^2$ and $(a,b) \in D$.

Then the point (a,b) is a point of local maxima if there exists a $\delta > 0$ such that

$$f(a+h, b+k) - f(a,b) \leq 0, \text{ whenever } \sqrt{h^2+k^2} < \delta.$$

Minima : $$f(a+h, b+k) - f(a,b) \geq 0, \text{ whenever } \sqrt{h^2+k^2} < \delta.$$

Defn: We say (a,b) is a point of local Extremum if it is either a maxima or minima.

Necessary Condition: Suppose (a,b) is a point of local Extremum for f & f is diffble. Then

$$\boxed{\frac{\partial f}{\partial x}(a,b) = 0 = \frac{\partial f}{\partial y}(a,b)}$$

Pf:

Let us Consider a fn Ψ, defined by

$$\Psi(x) = f(x, b)$$

This is a fn of one variable. Then clearly Ψ has local extremum at a. Hence,

$$\Psi'(x)\Big|_{x=a} = 0$$

$$\Rightarrow \quad \frac{\partial f(a,b)}{\partial x} = 0.$$

Similarly, $\dfrac{\partial f(a,b)}{\partial y} = 0.$

Higher Order Derivative Test. (Assume (a,b) is a pt. of Extremum)

Let ~~f(x,y)~~ $f(x,y)$ satisfies the assumptions of Taylor's Theorem with $n \geq 3$. ~~(Atleast)~~ Then using Taylor's Thm we write: (upto 3rd Order)

$$f(a+\Delta x, b+\Delta y) - f(a,b) = \frac{1}{2!}\Big((\Delta x)^2 f_{xx}(a,b) + 2 f_{xy}(a,b)\,\Delta x\,\Delta y + (\Delta y)^2 f_{yy}(a,b)\Big) + \frac{1}{3!}\left(\Delta x\, f_x + \Delta y\, f_y\right)^3 f\Big|_{(a+\theta h,\, b+\theta k)},$$

$$0 < \theta < 1.$$

So, one can write:

$$\Delta f := f(a+\Delta x, b+\Delta y) - f(a,b) = \frac{1}{2!}\Big((\Delta x)^2 f_{xx}(a,b) + 2 f_{xy}(a,b)\,\Delta x\,\Delta y$$

$$+ (\Delta y)^2 f_{xy}(a,b)\Big)$$

$$\underbrace{+\frac{1}{3!}\left(\Delta x\, f_x + \Delta y\, f_y\right)^3 f \Big|_{(a+\theta \overset{\Delta x}{\cancel{h}},\; b+\theta \overset{\Delta y}{\cancel{k}})}}_{\parallel}$$

$$= \alpha\,(\Delta \rho)^3,$$ for some $\alpha \in \mathbb{R}$

&

$$\Delta \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Hence, we can write:

$$\Delta f = \frac{1}{2!}\Big((\Delta x)^2 f_{xx}(a,b) + 2 f_{xy}(a,b)\,\Delta x\,\Delta y$$

$$+ (\Delta y)^2 f_{yy}(a,b)\Big)$$

$$+ \alpha\,(\Delta \rho)^3.$$

for some (~~This~~ $\alpha \in \mathbb{R}$
(involves derivative of f at $a+\theta \overset{\Delta x}{\cancel{h}}, b+\theta \overset{\Delta y}{\cancel{k}}$).

Put: $\Delta x = \Delta\rho \cos\phi$, $\Delta y = \Delta\rho \sin\phi$, $A = f_{xx}(a,b)$

$B = f_{xy}(a,b)$

$C = f_{yy}(a,b)$.

$$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left[A\cos^2\phi + 2B\cos\phi\sin\phi + C\sin^2\phi + 2\alpha\,\Delta\rho \right].$$

Suppose, $A \neq 0$, then

$$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left[\frac{(A\cos\phi + B\sin\phi)^2 + (AC - B^2)\sin^2\phi}{A} + 2\alpha\Delta\rho \right]$$

Case-I: $AC - B^2 > 0$, $A < 0$. Then $(A\cos\phi + B\sin\phi)^2 \geq 0$, $\sin^2\phi > 0$

$$\Rightarrow \quad \Delta f = \frac{1}{2}(\Delta s)^2 \left(-m^2 + \underline{2\alpha\Delta s} \right)$$

m is indep. of Δs, also, $\alpha\Delta\rho \to 0$ Hence $\Delta\rho \approx 0$,

$\underline{\Delta f \leq 0}$ Hence, (a,b) is a point local maximum.

Case-II: Let $AC - B^2 > 0$, $A > 0$

$$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left(m^2 + 2\alpha\Delta\rho \right)$$

$\Rightarrow$ $\Delta f \geq 0$ Hence, (a,b) is a point of local minima.

Case-III:

$AC - B^2 < 0$, $A > 0$. We move along $\phi = 0$, we have:

$$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left(A + 2\alpha\Delta\rho \right) > 0$$

move: Along $\tan\phi_0 = -A/B$, then.

$$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left(\frac{AC - B^2}{A}\sin^2\phi_0 + 2\alpha\underline{\Delta\rho} \right) \leq 0$$

This is called Saddle point

For $\Delta\rho \approx 0$. We don't have constant sign along all directions. Hence (a,b) is neither a point of maximum nor a point of minimum.

- $AC - B^2 < 0$, $A = 0$

$\Rightarrow B \neq 0$ & $\Delta f = \frac{1}{2}(\Delta\rho)^2 \left(2B\cos\phi\sin\phi + C\sin^2\phi + 2\alpha\Delta\rho\right)$

$= \frac{1}{2}(\Delta\rho)^2\left[\sin\phi\left(2B\cos\phi + C\sin\phi\right) + 2\alpha\Delta\rho\right]$.

For $\phi \approx 0$, $2B\cos\phi + C\sin\phi \approx 2B$, but $\sin\phi$ changes sign for $\phi\ (<>0)$. Hence (a,b) is a saddle point.

Case-IV: $AC - B^2 = 0$

$$\Delta f = \frac{1}{2}(\Delta\rho)^2\left(\frac{(A\cos\phi + B\sin\phi)^2}{A} + 2\alpha\Delta\rho\right)$$

$$= \frac{1}{2}(\Delta\rho)^2\left[\frac{\cos^2\phi\,(A + B\tan\phi)^2}{A} + 2\alpha\Delta\rho\right]$$

Choose: $\tan\phi = -A/B$, then

$$= \frac{1}{2}(\Delta\rho)^2\left[2\alpha\Delta\rho\right]$$

$$= \alpha(\Delta\rho)^3.$$

So, the sign depends on the sign of α. So, it is inconclusive.

Example: $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$f_x = y - 2x - 2 = 0$, $f_y = x - 2y - 2 = 0$.

$\Rightarrow (-2,-2)$ is the only critical point.

~~Therefore~~ $\underset{A}{f_{xx}} = -2$, $\underset{C}{f_{yy}} = -2$, $\underset{B}{f_{xy}} = 1$.

$AC - B^2 = -2\cdot -2 - 1 = 3 > 0$, $A = -2 < 0$.

$\Rightarrow (-2,-2)$ is a pt of local maxima.

Example Consider the function $f(x,y) = (x-y)^2$

$f_x = 2(x-y)$, $f_y = -2(x-y)$

$\Rightarrow f_x = 0 = f_y \Rightarrow x = y.$

$f_{xx} = 2$, $f_{yy} = 2$ $\qquad f_{xx} f_{yy} - f_{xy}^2 \quad (AC - B^2)$

$f_{yx} = -2.$ $\qquad = 4 - 4 = 0$

- Further all partial derivatives are zero

No further informations can be derived.

Example $f(x,y) = x^3 + 3xy + y^3$

$f_x = 3x^2 + 3y = 0$, $f_y = 3x + 3y^2 = 0.$

$\Rightarrow x = y = 0$ & $x = -1, y = -1$

$f_{xx} = 6x$, $f_{xy} = 6y$, $f_{yy} = 3.$

At $(0,0)$: $AC - B^2 < 0 \Rightarrow (0,0)$ is saddle.

At $(-1,1)$: $AC - B^2 > 0$, $A < 0$. $\Rightarrow (-1,-1)$ is a point of local minima.