Lecture 3 (Infinite series)

1) $\sum_{n\geq 2} \frac{1}{n\log n}$ does not converge.

2) $\sum_{n\geq 3} \frac{1}{n\log n\log\log n}$ does not converge.

We know, if $x > 1$, then $\log x > 0$.

So if $x > y > 0$, then $\log x > \log y$.

Let $n \geq 3$.

$$n+1 > n$$

$$\Rightarrow \log(n+1) > \log n$$

$$\Rightarrow \log\log(n+1) > \log\log n.$$

$$(n+1)\log(n+1)\log\log(n+1)$$

$$> n\log n\log\log n.$$

$\therefore \left(\frac{1}{n\log n\log\log n}\right)_{n\geq 3}$ is a decreasing sequence.

By Cauchy condensation test,

$$\sum_{n\geq 3} \frac{1}{n\log n \log\log n} \text{ is convergent}$$

$$\Leftrightarrow \underbrace{\sum_{n\geq 2} \frac{2^n}{2^n \log 2^n \log(\log 2^n)}}_{} \text{ is convergent.}$$

$$\sum_{n\geq 2} \frac{1}{n\log 2 \log(n\log 2)}$$

$$= \sum_{n\geq 2} \frac{1}{n\log 2(\log n + \log\log 2)}$$

$$\Leftarrow \quad \underbrace{\frac{1}{n\log 2(\log n + \log\log 2)} > \frac{1}{(\log 2)\cdot n\cdot(\log n)}}$$

as $\log\log 2 < 0$

Denote $\displaystyle S_k := \sum_{n=2}^{k} \frac{1}{\log 2^n \log\log 2^n}$

$\displaystyle t_k := \frac{1}{\log 2} \sum_{n=2}^{k} \frac{1}{n\log n}$

where $k \geqslant 2$.

We get $S_k > t_k$.

We know, $\frac{1}{\log 2}\sum_{n\geq 2}\frac{1}{n\log n}$ does not converge.

As it is an infinite series of +ve terms, its sequence of partial sums is increasing.

We can conclude, $t_k \to \infty$ as $k \to \infty$.

$\therefore S_k \to \infty$ as $k \to \infty$.

$\therefore \sum_{n\geq 2}\frac{1}{2^n \log\log 2^n}$ does not converge.

$\sum_{n\geq 3}\frac{1}{n\log n \log\log n}$ does not converge.

Comparison test

Let $(a_n)_{n\geqslant 1}$, $(b_n)_{n\geqslant 1}$ be two Sequences of non-negative real numbers such that $a_n \leqslant b_n \ \forall\ n \geqslant 1$. Then

a) $\sum_{n\geqslant 1} b_n$ is convergent $\Rightarrow \sum_{n\geqslant 1} a_n$ is convergent.

b) $\sum_{n\geqslant 1} a_n$ is not convergent $\Rightarrow \sum_{n\geqslant 1} b_n$ is not convergent,

Proof Denote,

$S_n :=$ Seq. of partial sums of $\sum_{n\geq 1} a_n$

$t_n :=$ Seq. of partial sums of $\sum_{n\geq 1} b_n$

As $0 \leq a_n \leq b_n \ \forall\ n \geq 1$,

then $0 \leq S_n \leq t_n \ \forall\ n \geq 1$

Now, (S_n), (t_n) both are non-decreasing sequences as a_n, b_n are non-negative.

$\therefore$ the convergence of (S_n) (respectively (t_n)) is equivalent to the statement that (S_n) (respectively (t_n)) is bounded.

Now, $\sum_{n\geq 1} b_n$ is convergent

$\Rightarrow (t_n)_{n\geq 1}$ is convergent

$\Rightarrow (t_n)_{n\geq 1}$ is bounded.

$\Rightarrow (S_n)_{n\geq 1}$ is bounded.

$\Rightarrow \sum_{n\geq 1} a_n$ is convergent.

Next if $\sum_{n\geq 1} a_n$ is not convergent,

then $(S_n)_{n\geq 1}$ is not convergent.

$\therefore$ $(S_n)_{n\geq 1}$ is not bounded.

In particular, $(S_n)_{n\geq 1}$ is not bounded above as $(S_n)_{n\geq 1}$ is bounded below by 0.

$\therefore$ $(t_n)_{n\geq 1}$ is also not bounded above as $S_n \leq t_n \ \forall\ n\geq 1$.

$\therefore$ $(t_n)_{n\geq 1}$ does not converge. i.e. $\sum_{n\geq 1} b_n$ does not converge.

Remark If $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ are such that $0 \leq a_n \leq b_n \;\forall\; n \geq n_0$ for some $n_0 \in \mathbb{N}$, then still the conclusions of the comparison test hold.

If a series converges then all its k-tails converge. On the other hand if one of its k-tail converges then the series itself converges. In this case, we can work with n_0-tail of $\sum_{n\geq 1} a_n$, $\sum_{n\geq 1} b_n$.

Applications of comparison test

1) $\sum_{n\geq 1} \frac{1}{n^p}$, $0<p<1$, is not convergent.

As $n^p < n \ \forall \ n \geq 1$

we get $\frac{1}{n} < \frac{1}{n^p} \ \forall \ n \geq 1$.

As $\sum_{n\geq 1} \frac{1}{n}$ does not converge,

$\sum_{n\geq 1} \frac{1}{n^p}$ does not converge where $0<p<1$.

2) $\sum_{n\geq 0} \frac{1}{n!}$ is convergent.

Note that for $n \geq 2$,

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}}.$$

We know, $\sum_{n\geq 1} \frac{1}{2^n}$ is convergent.

$\therefore \sum_{n\geq 2} \frac{1}{2^{n-1}}$ is convergent.

So, $\sum_{n\geq 2} \frac{1}{n!}$ is convergent.

$\therefore \sum_{n\geq 0} \frac{1}{n!}$ is convergent.

3) Let $a_n \geqslant 0 \ \forall \ n \geqslant 1$ and $\sum_{n\geqslant 1} a_n$ is convergent. Then $\sum_{n\geqslant 1} a_n^2$ is convergent.

Since, $\sum_{n\geqslant 1} a_n$ is convergent, by n-th term test we have,

$$a_n \to 0 \text{ as } n \to \infty.$$

$\therefore \ \exists \ n_0 \in \mathbb{N}$ such that $0 \leqslant a_n \leqslant 1$
$\forall \ n \geq n_0$.

$\therefore \ 0 \leqslant a_n^2 \leqslant a_n \ \forall \ n \geqslant n_0$.

By comparison test, $\sum_{n\geqslant 1} a_n^2$ is convergent.

Remark If $a_n \geq 0$ and $\sum_{n\geq 1} a_n^2$ is convergent, that need not imply $\sum_{n\geq 1} a_n$ is also convergent.

For example $a_n = \frac{1}{n}$.

$\sum_{n\geq 1} \frac{1}{n^2}$ is convergent but $\sum_{n\geq 1} \frac{1}{n}$ does not converge.

4) Let $a_n \geq 0$ $\forall$ $n \geq 1$ and $\sum_{n\geq 1} a_n$ is convergent. Then $\sum_{n\geq 1} \sqrt{a_n a_{n+1}}$ is convergent.

By AM-GM inequality,

$$\sqrt{a_n a_{n+1}} \leq \frac{(a_n + a_{n+1})}{2}.$$

Now as $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} a_{n+1}$ both are convergent, we get $\sum_{n\geq 1} \frac{(a_n + a_{n+1})}{2}$ is convergent.

$\therefore \sum_{n\geq 1} \sqrt{a_n a_{n+1}}$ is convergent.

Theorem (Limit comparison test)

Suppose $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of +ve real numbers such that $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ exists. If $r = \lim\limits_{n\to\infty} \frac{a_n}{b_n}$, then

i) if $r \neq 0$, then $\sum\limits_{n\geq 1} a_n$ converges $\iff \sum\limits_{n\geq 1} b_n$ converges.

ii) if $r = 0$, then $\sum\limits_{n\geq 1} b_n$ converges $\Rightarrow \sum\limits_{n\geq 1} a_n$ converges.

Consider, $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\sum_{n\geq 1} \frac{1}{n^2}$ is convergent but

$\sum_{n\geq 1} \frac{1}{n}$ does not converge.

So in the case $\rho = 0$, $\sum_{n\geq 1} a_n$ is convergent need not imply $\sum_{n\geq 1} b_n$ is convergent.

Proof Since $\lim_{n\to\infty} \frac{a_n}{b_n} = r$, we know

$\exists$ $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2} r < \frac{a_n}{b_n} < 2r \quad \forall \ n \geq n_0.$$

i.e. $\frac{1}{2} r b_n < a_n < 2 r b_n \quad \forall \ n \geq n_0.$

Since $\frac{a_n}{b_n} > 0$, we have $r \geq 0$.

Case-I $r \neq 0$.

$\therefore \ r > 0.$

$\therefore \; 0 < \frac{1}{2}\ell b_n < a_n < 2\ell b_n.$

$\therefore$ By comparison test, if $\sum_{n\geq 1} a_n$ converges then $\sum_{n\geq 1} \frac{1}{2}\ell b_n$ converges.

i.e. if $\sum_{n\geq 1} a_n$ converges then $\sum_{n\geq 1} b_n$ converges.

Next if $\sum_{n\geq 1} b_n$ converges, then

$\sum_{n\geq 1} 2r b_n$ converges.

$\therefore$ By comparison test we get

$\sum_{n\geq 1} a_n$ converges.

Case-II $r = 0$.

we can find $n_0 \in \mathbb{N}$ so that

$0 < \frac{a_n}{b_n} < 1 \quad \forall \ n \geq n_0.$

$\therefore \ 0 < a_n < b_n \ \forall \ n \geq n_0.$

By comparison test, if $\sum_{n\geq 1} b_n$ converges then $\sum_{n \geq 1} a_n$ converges.

An application of the limit comparison test

Let $a_n = \dfrac{1}{n\sqrt[n]{n}}$ and $b_n = \dfrac{1}{n}$.

Note that $a_n \leq b_n \ \forall \ n \geq 1$.

But since $\sum_{n \geq 1} \frac{1}{n}$ does not converge,

we cannot conclude anything about

the convergence of $\sum_{n\geq 1} a_n$ by comparing it with $\sum_{n\geq 1} b_n$.

If we use limit comparison test, we can conclude that $\sum_{n\geq 1} \frac{1}{n\sqrt[n]{n}}$ does not converge.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \text{ as } \lim_{n\to\infty} \sqrt[n]{n} = 1.$$

$\therefore$ By limit comparison test we conclude $\sum_{n\geq 1} \frac{1}{n\sqrt[n]{n}}$ does not converge as $\sum_{n\geq 1} \frac{1}{n}$ does not converge.

Absolute convergence of an infinite series of real numbers

Let $\sum_{n\geq 1} a_n$ be an infinite series of real numbers. We say $\sum_{n\geq 1} a_n$ is absolutely convergent if $\sum_{n\geq 1} |a_n|$ is convergent.

For example, any convergent series of non-negative real numbers is absolutely convergent.

$\sum_{n\geq 1} \frac{\cos n}{n^2}$ is absolutely convergent.

$|\cos n| \leq 1$

$\therefore \left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$.

By comparison test as $\sum_{n\geq 1} \frac{1}{n^2}$ is convergent, we conclude that $\sum_{n\geq 1} \left|\frac{\cos n}{n^2}\right|$ is convergent.

i.e. $\sum_{n\geq 1} \frac{\cos n}{n^2}$ is absolutely convergent.

Theorem Any absolutely convergent series is convergent.

Proof Let $\sum_{n\geq 1} a_n$ be absolutely convergent.

We have, $\sum_{n\geq 1} |a_n|$ is convergent.

$S_n :=$ the seq. of partial sums of $\sum_{n\geq 1} a_n$

$t_n :=$ the seq. of partial sums of $\sum_{n\geq 1} |a_n|$.

We show that $(S_n)_{n\geq 1}$ is Cauchy and therefore it is convergent.

Let $\epsilon > 0$. As $(t_n)_{n\geq 1}$ is convergent, it is Cauchy.

$\therefore \exists\, n_0 \in \mathbb{N}$ so that $\forall\, n \geq m \geq n_0$, we have, $|t_n - t_m| < \epsilon$.

Note, $0 \leq |S_n - S_m| = \left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k\right|$

$$= \left|\sum_{k=m+1}^{n} a_k\right|$$

$$|s_n - s_m| \leq \sum_{k=m+1}^{n} |a_k| = t_n - t_m$$
$$= |t_n - t_m| < \epsilon$$

$\therefore$ $(s_n)_{n \geq 1}$ is cauchy.

This proves, $\sum_{n \geq 1} a_n$ is convergent.

Remark The converse of the above theorem is not true.

For example, $\sum_{n \geq 1} \frac{(-1)^{n+1}}{n}$ is convergent, but not absolutely convergent.

Defn

An infinite series $\sum_{n\geq 1} a_n$ is said to be conditionally convergent if $\sum_{n\geq 1} a_n$ converges but $\sum_{n\geq 1} |a_n|$ does not converge.

For example, $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.