Differentiation

we denote I is an open interval in $\mathbb{R}$.
Let $f: I \to \mathbb{R}$ be a function.

Defn

We say that f is differentiable at a given point 'a' in I if the limit
$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ exists and finite.}$$

i.e. $\forall (x_n)$, $x_n \neq a$, $x_n \to a$, we get the sequence $\left(\frac{f(x_n)-f(a)}{x_n-a}\right)_{n\geq 1}$ convergers to a unique real number.

The value of the limit is called the derivative of f at 'a' and it's denoted by $f'(a)$.

Examples:

1. $f(x) = 1$, $x \in \mathbb{R}$.
Let $a \in \mathbb{R}$. Consider $\frac{f(x)-f(a)}{x-a} = \frac{1-1}{x-a}$, $x\in\mathbb{R}, x\neq a$
$$= 0$$
$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = 0$$
i.e. $f'(a) = 0$

2. $f(x) = x^2$, $x \in \mathbb{R}$.
At $a = 2$, Consider $\frac{f(x)-f(2)}{x-2} = \frac{x^2-2^2}{x-2}$, $x\neq 2$
$$= x+2$$
$$\to 4 \text{ as } x\to 2$$
$$\therefore f'(2) = 4$$
$\parallel^{ly}$ $f'(x) = 2x$, $x\in\mathbb{R}$.

Remark f' (itself) is a function from $\{x \mid f \text{ is differentiable at } 'x'\}$ to $\mathbb{R}$.
$$\therefore (f')'(x) = 2, \quad x\in\mathbb{R}$$
$$f''(x) := (f')'(x)$$
$$(f'')' = 0$$

Exc: Consider $f(x) = x^n$, $n \geq 1$.
Show that $f'(x) = n x^{n-1}$ and deduce that the nth derivative of f is $n!$.

(3) $f(x) = |x|$, $x\in\mathbb{R}$.
At $a = 0$, consider
$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x} = \begin{cases} 1, & x>0 \\ -1, & x<0\end{cases}$$
$*$ $\lim_{x\to 0}\frac{|x|}{x}$ does not exist (why?).
Because choose $x_n = \frac{1}{n} \to 0$ & $y_n = \frac{-1}{n} \to 0$
but $\frac{|x_n|}{x_n} = 1$ and $\frac{|y_n|}{y_n} = -1$
$\therefore$ $|x|$ is not diffenticable at '0'.

(4) $f(x) = \sqrt{x}$, $x>0$.
Let $a>0$. consider
$$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$$
$$= \frac{1}{\sqrt{x}+\sqrt{a}}$$
WKT, $\sqrt{x}\to\sqrt{a}$ as $x\to a$ and $\because a>0$ & using the limit properties
$$\frac{1}{\sqrt{x}+\sqrt{a}} \to \frac{1}{2\sqrt{a}} \text{ as } x\to a.$$
$$\therefore f'(x) = \frac{1}{2\sqrt{x}}, \quad x>0.$$

Exc. Discuss the diff at '0' for the following functions

1. $f(x) = \begin{cases} x\sin(1/x), & 0\neq x\in\mathbb{R} \\ 0, & x=0\end{cases}$

2. $g(x) = \begin{cases} x^2\sin(1/x), & 0\neq x\in\mathbb{R} \\ 0, & x=0\end{cases}$

Thm Suppose f is diff. at 'a'. Then f is continuous at 'a'.

Pf: WST $\lim_{x\to a} f(x) = f(a)$
$$\iff \lim_{x\to a}[f(x)-f(a)] = 0$$
Consider $f(x)-f(a) = \frac{f(x)-f(a)}{x-a}\cdot(x-a)$, $x\neq a$
As $x\to a$, $\frac{f(x)-f(a)}{x-a}\to f'(a)$ & $x-a\to 0$
we get $\lim_{x\to a}(f(x)-f(a)) = 0$.

Rmk: If f is not cts at 'a' then f is not diff. at 'a'.

Properties
Suppose f & g are diff. at 'a'. Then
(i) $f+g$ is diff at 'a' and
$$(f+g)'(a) = f'(a)+g'(a).$$
Pf:
$$\frac{f+g(x) - f+g(a)}{x-a} = \frac{f(x)+g(x)-[f(a)+g(a)]}{x-a}$$
$$= \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}$$
$$\to f'(a)+g'(a) \text{ as } x\to a.$$

(2) $(cf)'(a) = c f'(a)$. $\boxed{\text{Exc.}}$
where c is a constant.

(3) The product function $(fg)(x) := f(x)g(x)$ is diff. at 'a' and
$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$
[product rule]

Proof: Consider
$$\frac{fg(x)-fg(a)}{x-a} = \frac{f(x)g(x) - f(a)g(a)}{x-a}$$
$$= \frac{[f(x)-f(a)]}{x-a}g(x) + f(a)\frac{[g(x)-g(a)]}{x-a}$$
$$\to f'(a)g(a) + f(a)g'(a) \text{ as } x\to a.$$

(4) $\frac{f}{g}$ is diff is at 'a' if $g(a)\neq 0$.
And $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{[g(a)]^2}$ (quotient rule)

proof:
Since g is cts at 'a' and $g(a)\neq 0$, there exists a $\delta>0$ s.t.
$$g(x)\neq 0 \quad \forall\ |x-a|<\delta$$
$$\frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(a)}{x-a} = \frac{1}{x-a}\left[\frac{f(x)g(a) - g(x)f(a)}{g(x)g(a)}\right]$$
$$= \frac{1}{g(x)g(a)}\left[\frac{[f(x)-f(a)]}{x-a}\cdot g(a) - \frac{g(x)-g(a)}{x-a}\cdot f(a)\right]$$
$$\to \frac{1}{[g(a)]^2}\left[f'(a)g(a) - g'(a)f(a)\right].$$

Exc. (1). Consider $f(x) = x^{-m}$, $x\neq 0$, $m\geq 1$.
Find $f'(x)$?
(2) Consider $g(x) = x|x|$. At $a=0$, can we use product rule?
Is g diff at '0'?

Chain Rule:

If f is diff at 'a' and g is diff at '$f(a)$' then the composite function gof is diff at 'a' and
$$(gof)'(a) = g'(f(a))\,f'(a).$$

Proof

Idea: with flaw.
$$\frac{gof(x) - gof(a)}{x-a} = \cancel{\frac{g(f(x)) - g(f(a))}{f(x)-f(a)}}\ \frac{f(x)-f(a)}{x-a}$$
$\to \circledast$

The $\circledast$ is valid, if $f(x)\neq f(a)$ for x around 'a' ($x\neq a$).
Example, $f(x) = \begin{cases} x^2\sin 1/x, & 0\neq x\in\mathbb{R} \\ 0, & x=0\end{cases}$
$f(x) = 0$, $x = \frac{1}{\pi n}$, $n\in\mathbb{N}$.
and choose $g(x) = x$.

Proof begin: Define $h(y) = \begin{cases}\frac{g(y)-g(f(a))}{y-a}, & y\neq f(a) \\ g'(f(a)) & y = f(a)\end{cases}$

Then h is cts at $f(a)$.
Becaus, $y_n\to f(a)$, $\frac{g(y_n)-g(f(a))}{y_n-a}\to g'(f(a)) = h(f(a))$

Consider
$$\frac{gof(x)-gof(a)}{x-a} = h(f(x))\cdot\frac{f(x)-f(a)}{x-a} \text{ (verify).}$$
$x\neq a$.

For any seq. (x_n), $x_n\neq a$, $x_n\to a$,
we get $f(x_n)\to f(a)$
$$\Rightarrow h(f(x_n)) \to h(f(a)) = g'(f(a))$$
$$\Rightarrow \frac{gof(x_n) - gof(a)}{x_n-a} \to g'(f(a))\,f'(a)$$
as $x\to a$.
$$\therefore (gof)'(a) = g'(f(a))\,f'(a).$$

eg: $f(x) = (1+x^2)^{1/2}$, $x\in\mathbb{R}$.
$g(x) = 1+x^2$, $x\in\mathbb{R}$ $\quad h(x) = \sqrt{x}$, $x>0$
$f = hog$. $\quad f'(x) = h'(g(x))\,g'(x)$.
$$= \frac{1}{2\sqrt{g(x)}}\cdot 2x$$

Exc: Consider $f(x) = \left(\frac{1-x}{x}\right)^2$, $0<x<1$.
Find f'?

Relation between local maxima/minima and derivative:

Local maximum: we say that f has a maximum at the point 'a' if $\exists\ \delta>0$ s.t.
$$f(a)\geq f(x), \quad \forall x\in(a-\delta, a+\delta)\subset I.$$

Local minimum: we say that f has a minimum at the point 'a' if $\exists\ \delta>0$ such that
$$f(a)\leq f(x), \quad \forall x\in(a-\delta, a+\delta)\subset I.$$

eg: $f(x) = x^2-x^4$, $x\in\mathbb{R}$
$$= x^2(1-x^2).$$
At $a=0$, $f(0) = 0 \leq f(x)$, $\forall |x|<1$.
i.e. choose $\delta=1$, observe that f has a local mini at '0'.

Qn: Is f as a local maximum?

Interior extremum theorem.

Suppose $f: I\to\mathbb{R}$ has a local max/min. at a point 'a' in I and f is diff at 'a'. Then $f'(a) = 0$.

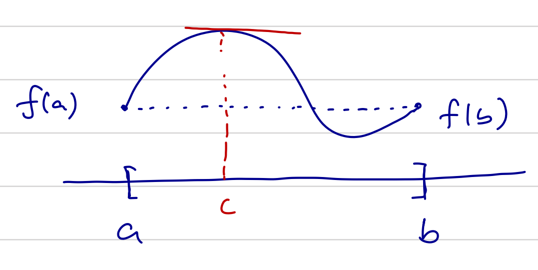
Pf: Assume that f has a local max at 'a'.
i.e. $\exists\ \delta>0$ s.t.
$$f(a)\geq f(x), \quad \forall x\in(a-\delta,a+\delta)\subset I.$$
$$f(x)-f(a)\leq 0 \quad ''$$
For $x\in(a, a+\delta)$, consider
$$\frac{f(x)-f(a)}{x-a}\leq 0$$
$$\Rightarrow \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}\leq 0$$
i.e. $f'(a)\leq 0$.

$\parallel^{ly}$ for $x\in(a-\delta, a)$, consider
$$\frac{f(x)-f(a)}{x-a}\geq 0$$
As $x\to a^-$, we get $f'(a)\geq 0$
$$\therefore f'(a) = 0.$$

eg: $f(x) = x^3$, $x\in\mathbb{R}$.
$f'(0) = 0$. Qn: Is the point '0' a local max/min of f?

Rolle's Theorem

If f is a continuous function on $[a,b]$,
f is diff. on (a,b) and
$f(a) = f(b)$, then
there exists $c \in (a,b)$ s.t. $f'(c) = 0$.



Proof: Idea: S.T. f has a local extremum value inside the interval (a,b).

Recall, every cts function on $[a,b]$ attains its maximum & minimum in $[a,b]$.

Say, $x_0, t_0 \in [a,b]$ s.t.

$$f(x_0) \le f(x) \le f(t_0) \quad \forall x \in [a,b].$$

Case-1 f is constant. Choose any $c \in (a,b)$, then $f'(c) = 0$.

Case-2 f is not a constant.

Then $f(x_0) \neq f(t_0)$.

$\Rightarrow$ either $x_0 \in (a,b)$ or $t_0 \in (a,b)$

$\Rightarrow$ either $f'(x_0) = 0$ or $f'(t_0) = 0$

$\therefore \exists\ c \in (a,b)$ s.t. $f'(c) = 0$.

Exc: Suppose f is diff. on (a,b) and $f'(x) \neq 0$ $\forall x \in (a,b)$. Then show that f is 1-1 function.