Lecture - 3

Riemann Integration

Theo: If f is a continuous fn on the closed interval $[a, b]$ then f is Riemann Integrable.

Proof:

Claim: Given any $\epsilon > 0$ $\exists$ a partition P s.t

$$U(P,f) - L(P,f) < \epsilon$$

Let $\epsilon > 0$,

We know, f is uniformly continuous on $[a, b]$.

So $\exists \delta > 0$, s.t for all $x \in [a, b]$

$$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon.$$

Now let P be a partition of $[a,b]$, given by, $P=\{\underset{a}{\overset{\shortparallel}{x_0}}, x_1, \ldots, \underset{b}{\overset{\shortparallel}{x_n}}\}$

with the following property that $\Delta x_i = x_i - x_{i-1} < \delta, \forall i$

= length of the subinterval.

Then consider

$$U(P,f) - L(P,f)$$

$$= \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$

We know,

$$M_i = \sup_{[x_{i-1}, x_i]} f(x)$$

&

$$m_i = \inf_{[x_{i-1}, x_i]} f(x)$$

We also know that,

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x - y| < \delta$$

($\because$ f is uniformly continuous).

But by the choice of partition we have,

$$\Delta x_i < \delta, \forall i.$$

Also note that the sup & inf of f ie,

M_i & m_i will be attained in the $[x_{i-1}, x_i]$, since f is continuous.

So, $$|M_i - m_i| = |f(\alpha_i) - f(\beta_i)|,$$
$$\alpha_i, \beta_i \in [x_{i-1}, x_i]$$

Hence we have

$$|M_i - m_i| < \epsilon.$$

This means

$$|U(P,f) - L(P,f)|$$
$$< \epsilon \sum_{i=1}^{n} \Delta x_i$$
$$= \epsilon(b-a)$$

Given $\epsilon > 0$ we just start with $\epsilon_1 = \frac{\epsilon}{b-a}$,

So then,

$$U(P, f) - L(P, f) < \epsilon_1 (b-a)$$
$$= \epsilon.$$

$$\Rightarrow \quad f \in \mathcal{R}[a, b].$$

[Proved].

$\boxed{\text{Continuity is not necessary for R- integration.}}$

Theo If $f : [a, b] \to \mathbb{R}$ is a monotone fn the f is again Riemann integrable

Proof: We will prove this assuming f is m.i. (m.d is analogous).

We will check that for any $\epsilon > 0$ $\exists P_\epsilon$ s.t

$$U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$$

Assume f is m.i.

Choose $\epsilon > 0$.

Let 'n' be s.t

$$\frac{b-a}{n} < \epsilon.$$

Choose a partition P s.t

$$\Delta x_i < \frac{b-a}{n}, \ \forall i$$

Since f is monotone increasing,

$$\left.\begin{aligned} M_i &= f(x_i) \\ \& \quad m_i &= f(x_{i-1}) \end{aligned}\right\} \text{(Why?)}$$

To see this try to check the following:

Ex: 1) Let f be monotone increasing on (a, b). Then $f(x+)$ & $f(x-)$ exist at every point of x of (a, b). More precisely,

$$\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t)$$

Furthermore, if

$a < x < y < b$ then

$$f(x+) \le f(y-)$$

2) Monotone $\underline{fns}$ have no discontinuities of 2nd kind.

Then we have

$$U(P, f) - L(P, f)$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$

$$= \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) \Delta x_i$$

$$\leq \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) \left(\frac{b-a}{n}\right)$$

$$< \left(f(b) - f(a)\right) \epsilon$$

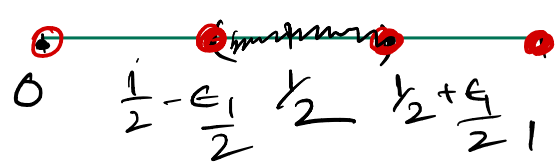
This implies that 'f' is Riemann integrable.

[Proved]

Example

$$f : [0,1] \to \mathbb{R},$$

$$f(x) = \begin{cases} 1, & x \neq \frac{1}{2} \\ 0, & x = \frac{1}{2} \end{cases}$$



Let P be a partition.

Choose $\epsilon > 0$ &

$\epsilon_1 = \frac{\epsilon}{4}$ (say)

So for $\epsilon > 0$ we choose the following partition,

$$P_\epsilon = \left\{ 0, \frac{1}{2} - \frac{\epsilon_1}{2}, \frac{1}{2} + \frac{\epsilon_1}{2}, 1 \right\}$$

Compute.

$$L(P_\epsilon, f) = 1\left(\frac{1}{2} - \frac{\epsilon_1}{2}\right) + 0\left(\frac{1}{2} + \frac{\epsilon_1}{2} - \frac{1}{2} + \frac{\epsilon_1}{2}\right) + 1\left(1 - \frac{1}{2} - \frac{\epsilon_1}{2}\right)$$

$$= 1 - \epsilon_1.$$

ie

$$\boxed{L(P_\epsilon, f) = 1 - \epsilon_1}$$

Similarly

$$U(P_\epsilon, f) = 1\left(\frac{1}{2} - \frac{\epsilon_1}{2}\right) + 1\left(\frac{1}{2} + \frac{\epsilon_1}{2} - \frac{1}{2} + \frac{\epsilon_1}{2}\right) + 1\left(1 - \frac{1}{2} - \frac{\epsilon_1}{2}\right)$$

$$= \epsilon_1$$

ie,

$$\boxed{U(P_\epsilon, f) = \epsilon_1}$$

Then,

$$U(P_\epsilon, f) - L(P_\epsilon, f)$$

$$= \epsilon_1 < \epsilon.$$

$\Rightarrow$ 'f' is $\boxed{\text{R- integrable}}$.

Theo: Suppose $f : [a, b] \to \mathbb{R}$ be a bounded function that has <u>finitely many</u> discontinuities then f is <u>Riemann integrable</u>.

Elementary properties:

Theo:

i) If $f \in \mathcal{R}[a,b]$ & $c \in \mathbb{R}$, then $cf \in \mathcal{R}[a,b]$. and

$$\boxed{c\int_a^b f(x)\,dx = \int_a^b c\,f(x)\,dx}$$

ii) If f & $g \in \mathcal{R}[a,b]$ then the new function, $f+g \in \mathcal{R}[a,b]$ &

$$\boxed{\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.}$$

iii) If $f, g \in \mathcal{R}[a,b]$ and $f(x) \leq g(x)$, $\forall\, x \in [a,b]$

then, $\boxed{\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx}$

iv) If $f \in \mathcal{R}[a,b]$ and 'c' is a point such that $a < c < b$, then

$\boxed{f \in \mathcal{R}[a,c]}$ & $\boxed{f \in \mathcal{R}[c,b]}$

and

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

v) If $f \in \mathcal{R}[a,b]$ then $|f| \in \mathcal{R}[a,b]$ and

$$\boxed{\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx}$$

Proof: (i) — (iv) $\longrightarrow$ Exercise.

v). First assume $f \in \mathcal{R}[a,b]$.

If P is a partition then

$$\sup_S |f(x)| - \inf_S |f(x)|$$

$$\leq \sup_S f(x) - \inf_S f(x),$$

S is any sets.

We just concentrate here in $[x_{i-1}, x_i]$.

So from above we get,

$$\sup_{[x_{i-1}, x_i]} |f|(x) - \inf_{[x_{i-1}, x_i]} |f|(x)$$

$$\leq \sup_{[x_{i-1}, x_i]} f(x) - \inf_{[x_{i-1}, x_i]} f(x)$$

Let,

$$M_i' = \sup_{[x_{i-1}, x_i]} |f|(x)$$

&

$$m_i' = \inf_{[x_{i-1}, x_i]} |f|(x)$$

Then,

$$M_i' - m_i' \leq M_i - m_i$$

which implies

$$\sum_i (M_i' - m_i') \Delta x_i \leq \sum_i (M_i - m_i) \Delta x_i$$

$$\Rightarrow \quad U(P, |f|) - L(P, |f|)$$

$$\leq \quad U(P, f) - L(P, f).$$

Since, $f \in \mathcal{R}[a, b]$, so given $\epsilon > 0 \ \exists \ P_\epsilon$ s.t

$$U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon.$$

So from the above,

$$U(P_\epsilon, |f|) - L(P_\epsilon, |f|) < \epsilon$$

$$\Rightarrow \quad |f| \in \mathcal{R}[a, b].$$

We know

$$-|f|(x) \leq f(x) \leq |f|(x)$$

Using (iii) property,

$$-\int_a^b |f|(x)\,dx \leq \int_a^b f(x)\,dx$$

$$\leq \int_a^b |f|(x)\,dx$$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f|(x)\,dx$$

$$= \int_a^b |f(x)|\,dx.$$

[Proved].