Antiderivative : Let $f(z)$ be a continuous function on some domain D such that there exist another function $F(z)$ satisfying

$$F'(z) = \frac{d}{dz} F(z) = f(z) \quad \forall \; z \in D$$

then $F(z)$ is called the Antiderivative of $f(z)$.

- $F(z)$ may not always exist.
- $F(z)$, when it exists, is unique upto additive constant.
- Clearly $F(z)$ is an analytic function.

Theorem : Given a continuous function $f(z)$ (on domain D), the following statements are equivalent.

1. $f(z)$ has an Antiderivative $F(z)$ on D.
2. $\displaystyle {}_{C}\!\int_{z_1}^{z_2} f(z)\, dz = F(z_2) - F(z_1)$ independent of the contour $C \in D$
3. $\displaystyle \oint_C f(z)\, dz = 0$ where C is a closed contour inside D.

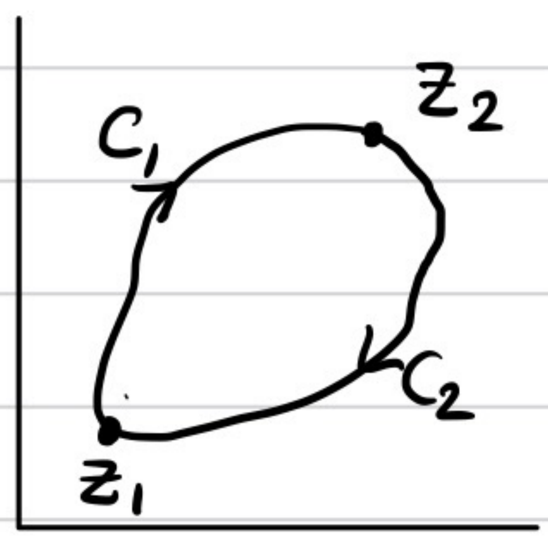
$1. \to 2.$ $F'(z) = f(z)$ then

$$ {}_{C}\!\int_{z_1}^{z_2} f(z)\, dz = {}_{C}\!\int_{z_1}^{z_2} dz\, F'(z) \equiv \int_{t_1}^{t_2} dt\, \frac{dz}{dt} \cdot \frac{d}{dz} F(z) = \int_{t_1}^{t_2} dt\, \frac{d}{dt} F(z(t))$$

$$= F(z(t_2)) - F(z(t_1)) = F(z_2) - F(z_1)$$

$2 \to 3.$

$$\oint_C f(z) dz = \oint_{C_1+C_2} f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$

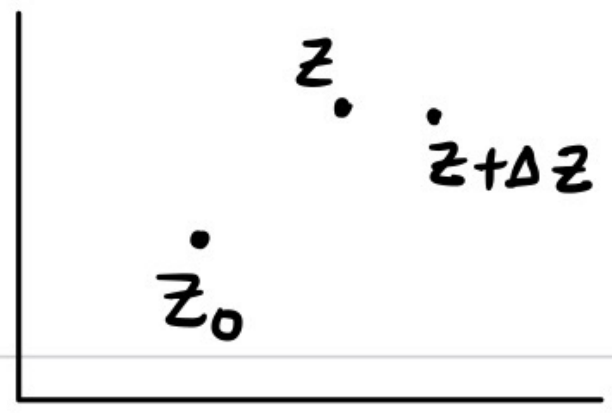


$$= \int_{C_1} f(z)\, dz - \int_{-C_2} f(z) dz = \left(F(z_2) - F(z_1)\right) - \left(F(z_2) - F(z_1)\right)$$

$$= 0$$

$3. \to 1.$ $\displaystyle F(z) = \int_{z_0}^{z} dw\, f(w)$ Since the integral is independent of path $F(z)$ only depends on z & not the contour.

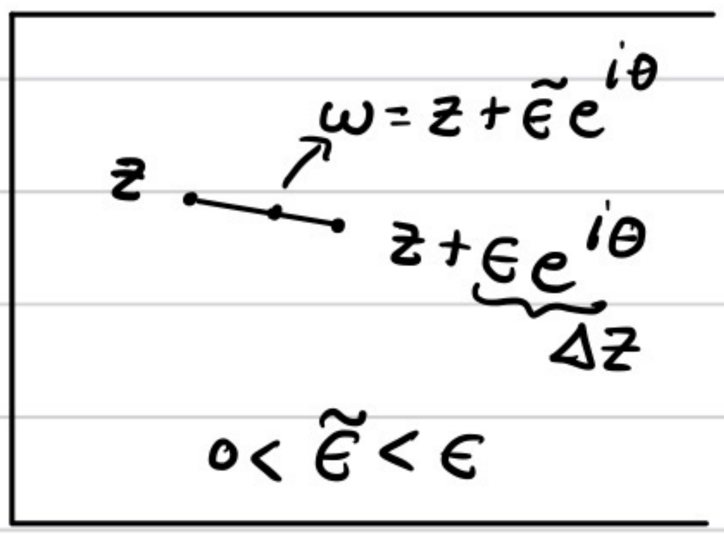
$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z}$$



$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} dw\, f(w) - \int_{z_0}^{z} dw\, f(w) \right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{z}^{z+\Delta z} dw\, f(w) = \lim_{\epsilon \to 0} \frac{1}{\epsilon e^{i\theta}} \int_{z}^{z+\epsilon e^{i\theta}} dw\, f(w) \qquad \text{—— E1}$$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} dw \left(f(w) - f(z) \right)$$



$$= \frac{1}{\epsilon \cancel{e^{i\theta}}} \int_0^{\epsilon} d\tilde{\epsilon} \cdot \cancel{e^{i\theta}} \cdot \left(f(z+\tilde{\epsilon} e^{i\theta}) - f(z) \right)$$

$$= \frac{1}{\epsilon} \int_0^{\epsilon} d\tilde{\epsilon} \left(f(z+\tilde{\epsilon} e^{i\theta}) - f(z) \right) \qquad \text{—— E2}$$

Now continuity of $f(z)$ implies : $|f(z+\Delta z) - f(z)| < \delta \;\; \forall \; |\Delta z| < \epsilon$

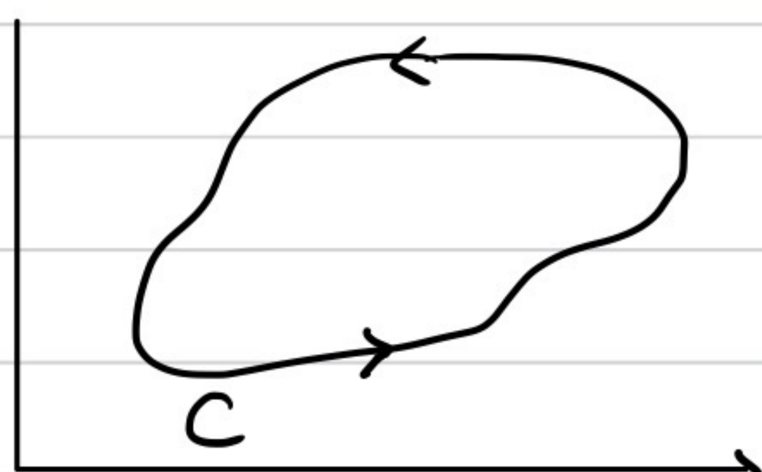
Using this in E2 along with $\left| \int dt\, f(t) \right| \leqslant \int dt\, |f(t)|$

$$\Rightarrow \quad \left| \frac{F(z+\epsilon e^{i\theta}) - F(z)}{\epsilon e^{i\theta}} - f(z) \right| < \frac{1}{\epsilon} \int_0^{\epsilon} d\tilde{\epsilon}\, \delta = \delta$$

$$\Rightarrow \quad F'(z) \equiv \boxed{\lim_{\epsilon \to 0} \left(\frac{F(z+\epsilon e^{i\theta}) - F(z)}{\epsilon e^{i\theta}} \right) = f(z)}$$

<u>Cauchy-Goursat theorem</u> : If $f(z)$ is analytic at all points inside a simple closed contour C, then

$$\oint_C f(z)\, dz = 0$$



$$\oint_C f(z)\, dz = \oint_C dt\, z'(t)\, f(z(t)) = \oint_C dt \left(x'(t) + i\, y'(t) \right) (u + iv)$$

$$= \oint_C dt \left[x' u(x,y) - y' v(x,y) + i \left(y' u(x,y) + x' v(x,y) \right) \right]$$

$$= \oint_C (dx\, u - dy\, v) + i \oint_C (dx\, v + dy . u) \qquad \left\{ \vec{dl} = \hat{i}\, dx(t) + \hat{j}\, dy(t) \right.$$

$$= \oint \vec{dl} . (u\hat{i} - v\hat{j}) + i \oint \vec{dl} . (v\hat{i} + u\hat{j})$$

$$= \int ds\ \vec{\nabla} \times (u\hat{i} - v\hat{j}) + i \int ds\ \vec{\nabla} \times (v\hat{i} + u\hat{j})$$

$$\vec{\nabla} \times \vec{V} \quad (x\text{-}y \text{ plane})$$

$$\hat{i}\nabla_x + \hat{j}\nabla_y$$

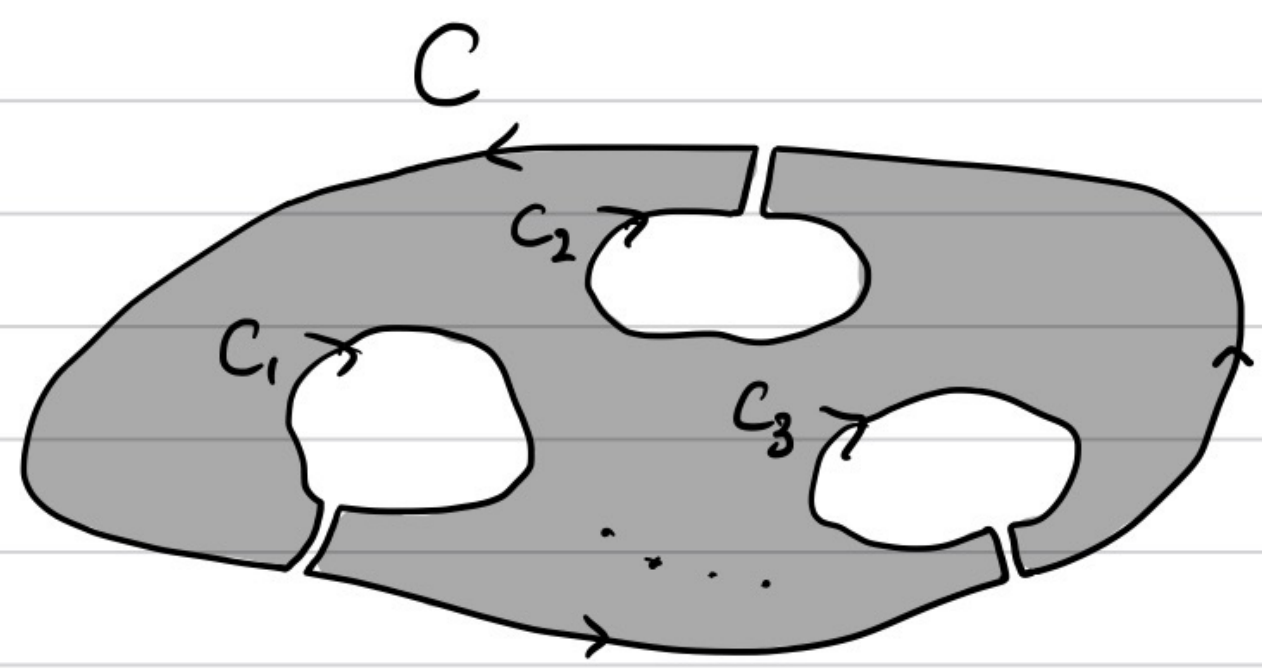
$$(\nabla_x V_y - \nabla_y V_x)$$

$$= \int ds \cancelto{0}{(-\partial_x v - \partial_y u)} + i \int ds \cancelto{0}{(\partial_x u - \partial_y v)}$$

Cauchy Riemann eqns.

$$= 0$$

More generally, If $f(z)$ is analytic in the shaded region and on the contours $\{C, C_1, C_2, \dots C_n\}$



(C runs counterclockwise while $(C_1, C_2, \dots C_n)$ are clockwise)

then

$$\boxed{\oint_C dz\, f(z) + \sum_{i=1}^{n} \oint_{C_i} dz\, f(z) = 0}$$