2^{nd} order homogeneous linear ODE

$$x'' + a(t)\,x'(t) + b(t)\,x(t) = 0 \quad \text{--- (1)}$$

Wroskian of two f^n's $x_1(t)$ & $x_2(t)$ is

$$W(x_1,x_2)(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = x_1(t)\,x_2'(t) - x_2(t)\,x_1'(t)$$

Thm 1 Suppose $x_1(t)$ & $x_2(t)$ are L.D. & sufficiently differentiable then $W(x_1,x_2)(t) = 0$.

Proof. Given $x_1(t)$ & $x_2(t)$ are L.D.

$\Rightarrow$ $\exists$ constants c_1, c_2 not both zero s.t.

$$c_1 x_1(t) + c_2 x_2(t) = 0 \quad \text{--- (2)}$$

$$c_1 x_1'(t) + c_2 x_2'(t) = 0 \quad \text{--- (3)}$$

Eq^n (2) & (3) can be rewritten as

$$\left\{ \underbrace{\begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right. \quad \text{--- (4)}$$

We have $C(c_1, c_2)$ is a non-trivial sol^n of (4)

(as we know that c_1 & c_2 are not both zero)

$$W(x_1, x_2)(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = 0 .$$

$Ax = 0$ has a nontrivial sol^n then $\det(A) = 0$

$\square$

If $W(x_1, x_2)(t) = 0$, does that imply that $x_1(t)$ & $x_2(t)$ are L.D. ??

NO

The converse of the statement of last thm is not true in general.

If $W(x_1, x_2)(t) = 0$ then $x_1(t)$ & $x_2(t)$ need not be L.D..

Exp:

$$x_1(t) = t|t|, \quad x_2(t) = t^2, \quad I = [-1, 1]$$

$$W(x_1, x_2)(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}$$

$$= \begin{vmatrix} t|t| & t^2 \end{vmatrix} = 0 \quad \forall t.$$

$$\begin{vmatrix} 2|t| & 2t \end{vmatrix}$$

Consider,

$$c_1 x_1(t) + c_2 x_2(t) = 0 \qquad \forall\ t \in [-1, 1]$$

i.e

$$c_1 t|t| + c_2 t^2 = 0 \qquad \forall\ t \in [-1, 1]$$

Take $t = 1$ & $t = -1$ in last eq^n, we get

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + c_2 &= 0 \end{aligned}\right\} \quad \text{sol}^n \text{ is } c_1 = 0,\ c_2 = 0$$

$\Rightarrow$ $x_1(t)$ & $x_2(t)$ are L.I.

$W(x_1, x_2)(t) \not\Rightarrow x_1, x_2$ are L.D.

$W(x_1, x_2)(t) = 0 +$?? $\Rightarrow$ x_1, x_2 are L.D.

x_1 & x_2 are sol^{n}s of a 2^{nd} order linear ODE

<u>Thm 2</u> Suppose $x_1(t)$ & $x_2(t)$ are two sol^{n}s of $x'' + a(t)x' + b(t)x = 0$ then

$\longleftrightarrow$ x_1 & x_2 are L.D.

$W(x_1, x_2)(t) = 0 \iff$ ...

Thm 3 (Abel's formula)

Suppose $x_1(t)$ & $x_2(t)$ are two sol^{n}s of ODE (1)

then $W(x_1, x_2)(t) = c\, e^{-\int a(t)\,dt}$

Proof- $x_1(t)$ & $x_2(t)$ are sol^{n}s of

$$x'' + a(t)x' + b(t)x = 0.$$

$\therefore$ We have

$$x_1'' + a(t)x_1' + b(t)x_1 = 0 \quad \text{--- (5)}$$

$$x_2'' + a(t)x_2' + b(t)x_2 = 0 \quad \text{--- (6)}$$

Eq^n (5) $\times x_2$ — Eq^n (6) $\times x_1$ yields,

$$(x_1''x_2 - x_2''x_1) + a(t)(x_1'x_2 - x_2'x_1) + b(t)\cancelto{0}{(x_1x_2 - x_2x_1)} = 0$$

$$\underbrace{(x_1''x_2 - x_2''x_1)} + a(t)\underbrace{(x_1'x_2 - x_2'x_1)} = 0 \quad \text{--- (7)}$$

We have

$W(t) = W(x_1, x_2)(t) = x_1x_2' - x_2x_1'$

Since $x_1, x_2 \to$ twice differentiable

$W(t) \to$ differentiable f^n

Upon differentiating,

$$W'(t) = x_2'' x_1 + \cancel{x_1' x_2'} - \cancel{x_2' x_1'} - x_2 x_1''$$

$$= x_2'' x_1 - x_2 x_1''$$

$\therefore$ from (7), we have

$$-W'(t) - a(t)\, W(t) = 0$$

or

$$W'(t) + a(t)\, W(t) = 0 \qquad \text{—— (8)}$$

$\to$ first order linear ODE

I.F. $\quad e^{\int a(t)\,dt}$

Upon multiplying by integrating factor the eq^n (8) can be written as

$$\frac{d}{dt}\left(e^{\int a(t)\,dt}\, W(t)\right) = 0$$

$$\Rightarrow \quad e^{\int a(t)\,dt}\, W(t) = c \quad \to \text{constant}$$

$$\Rightarrow \quad \boxed{W(t) = c\, e^{-\int a(t)\,dt}}$$

$\square$

Observations

$$W(t) = C\,\underbrace{e^{-\int a(t)\,dt}}_{\text{never zero}}$$

① If $C = 0$ then $W(t) \equiv 0$.

② If $C \neq 0$ then $W(t)$ is never zero.

Corollary 1 The Wronskian of two sol^{n}s of 2^{nd} order linear ODE is either always zero or never zero.

$W(t)$ → identically zero
→ never zero

$x_1, x_2 \rightarrow$ sol^{n}s of ODE ①

- If $\underbrace{W(x_1,x_2)(t_0) \neq 0}$ for some t_0

$\Rightarrow W(t) \neq 0 \quad \forall t$

- If $\underline{W(x_1,x_2)(t_0) = 0}$ for some t_0,

$\Rightarrow W(t) \equiv 0$.

$(W(x_1,x_2)(t) = 0 \Leftrightarrow x_1, x_2$ are L.D.$)$

Proof of Thm 2

$(\Leftarrow)$ Let x_1, x_2 are L.D.

then from thm①

$\Rightarrow \quad W(x_1, x_2)(t) = 0.$

($\Rightarrow$) To show that if

$W(x_1, x_2)(t) = 0 \Rightarrow x_1(t)$ & $x_2(t)$ are L.D.

It suffices to prove that if

$W(x_1, x_2)(t_0) = 0$ for some $t_0 \in I$

then $x_1(t)$ & $x_2(t)$ are L.D. in I.

We have

$$W(x_1, x_2)(t_0) = 0$$

$$\parallel$$

$$\begin{vmatrix} x_1(t_0) & x_2(t_0) \\ x_1'(t_0) & x_2'(t_0) \end{vmatrix} = 0 \quad \text{—} \quad (9)$$

$\Rightarrow$ the system of eq^n's

$$\underbrace{\begin{bmatrix} x_1(t_0) & x_2(t_0) \\ x_1'(t_0) & x_2'(t_0) \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{—} \quad (10)$$

has a non-trivial sol^n say $c_1 = \alpha$, $c_2 = \beta$.

Define

$$y(t) = c_1 x_1(t) + c_2 x_2(t)$$

Claim $y(t)$ is a soln of ODE ①

Since $x_1(t)$ & $x_2(t)$ are soln of ① & hence their linear combination $y(t) = c_1 x_1(t) + c_2 x_2(t)$ is also a solution of ① i.e

$y(t)$ satisfies

$$\text{IVP ①} \begin{cases} x''(t) + a(t)\, x'(t) + b(t)\, x(t) = 0 \\ x(t_0) = 0 \\ x'(t_0) = 0 \end{cases}$$

from system ⑩

$$y(t_0) = c_1 x_1(t_0) + c_2 x_2(t_0) = 0$$

$$y'(t_0) = c_1 x_1'(t_0) + c_2 x_2'(t_0) = 0$$

Thus $y(t)$ is a soln of IVP ①.

And $x(t) = 0$ is also a soln of IVP ①.

$\therefore$ By the existence & uniqueness thm, we have

$$y(t) \equiv 0$$

i.e.

$$c_1 x_1(t) + c_2 x_2(t) = 0$$

where c_1 & c_2 are non both zero

$\Rightarrow$ x_1 & x_2 are L.D.

Corollary Let $x_1(t)$ & $x_2(t)$ be sol^{n}s of ODE (1)
then show that

$x_1(t)$ & $x_2(t)$ are L.I. $\Rightarrow$ $W(x_1, x_2)(t) \neq 0$.

Thm 3 Let $x_1(t)$ & $x_2(t)$ be two linearly independent sol^{n}s of $x'' + a(t)x' + b(t)x = 0$
& let $y(t)$ be any sol^n then there exist
c_1 & c_2 s.t.

$$y(t) = c_1 x_1(t) + c_2 x_2(t). \quad (11)$$

Proof. Let $y(t)$ be any sol^n of

$$x'' + a(t)x' + b(t)x = 0. \quad (1)$$

Given $x_1(t)$ & $x_2(t)$ are sol^{n}s of (1)
& they are L.I. in I.

Using last corollary, we have

$x_1(t)$ & $x_2(t)$ are L.I. $\Rightarrow$ $\exists\ t_0 \in I$ s.t.
$W(x_1, x_2)(t_0) \neq 0$

$$\| \quad \begin{vmatrix} x_1(t_0) & x_2(t_0) \\ x_1'(t_0) & x_2'(t_0) \end{vmatrix} \neq 0$$

$\Rightarrow$ the system of eqn

$$\underbrace{\begin{bmatrix} x_1(t_0) & x_2(t_0) \\ x_1'(t_0) & x_2'(t_0) \end{bmatrix}}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(t_0) \\ y'(t_0) \end{bmatrix} \quad — \ (*)$$

$Ax=b$

has a unique soln, say $c_1 = \alpha$, $c_2 = \beta$.

Define

$$z(t) = c_1 x_1(t) + c_2 x_2(t) \quad (= \alpha x_1 + \beta x_2)$$
$$z'(t) = c_1 x_1'(t) + c_2 x_2'(t)$$

Since x_1 & x_2 are soln's of ODE (1)

$\Rightarrow$ $z(t)$ is also a soln of ODE (1).

Using (*)

$$z(t_0) = c_1 x_1(t_0) + c_2 x_2(t_0) = y(t_0) \quad ✓$$

$$z'(t_0) = c_1 x_1'(t_0) + c_2 x_2'(t_0) = y'(t_0) \quad ✓$$

Thus $y(t)$ & $z(t)$ solve the same IVP

$\therefore$ By the existence & uniqueness thm, we have

$$\boxed{y(t) \equiv z(t) = c_1 x_1(t) + c_2 x_2(t)}.$$

Observation

The sol^n $y(t)$ in (11) involves two arbitrary constants c_1 & c_2 hence it is the general sol^n of ODE (1).

$x'' + a(t)x' + b(t)x = 0$

Two linearly independent sol^{n}s of

$$x'' + a(t)x' + b(t)x = 0 \quad \text{—— (1)}$$

Let $x_1(t)$ be the sol^n of (1) which satisfies

$$x_1(0) = 1 \quad \& \quad x_1'(0) = 0,$$

& $x_2(t)$ be the sol^n of (1) which satisfies

$$x_2(0) = 0, \quad x_2'(0) = 1$$

then x_1 & x_2 are L.I. since

$$W(x_1, x_2)(0) = \begin{vmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\{x_1(t), x_2(t)\} \rightarrow$ fundamental set of sol^n

Def^n (Fundamental set of sol^{n}s of ODE (1))

Any set $\{x_1(t), x_2(t)\}$ of two linearly independent sol^n of DE (1) is said to be the fundamental set of sol^{n}s on I.

$\{x_1(t), x_2(t)\} \longrightarrow$ fundamental set of sol^{n}s of (1)

the general sol^n of (1) is

$$x(t) = c_1 x_1(t) + c_2 x_2(t) .$$