Introduction to Z transforms

Discrete-Time Fourier Transform

- Representing signals as linear combination of basic signals $e^{j\Omega n}$

$$x[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$$

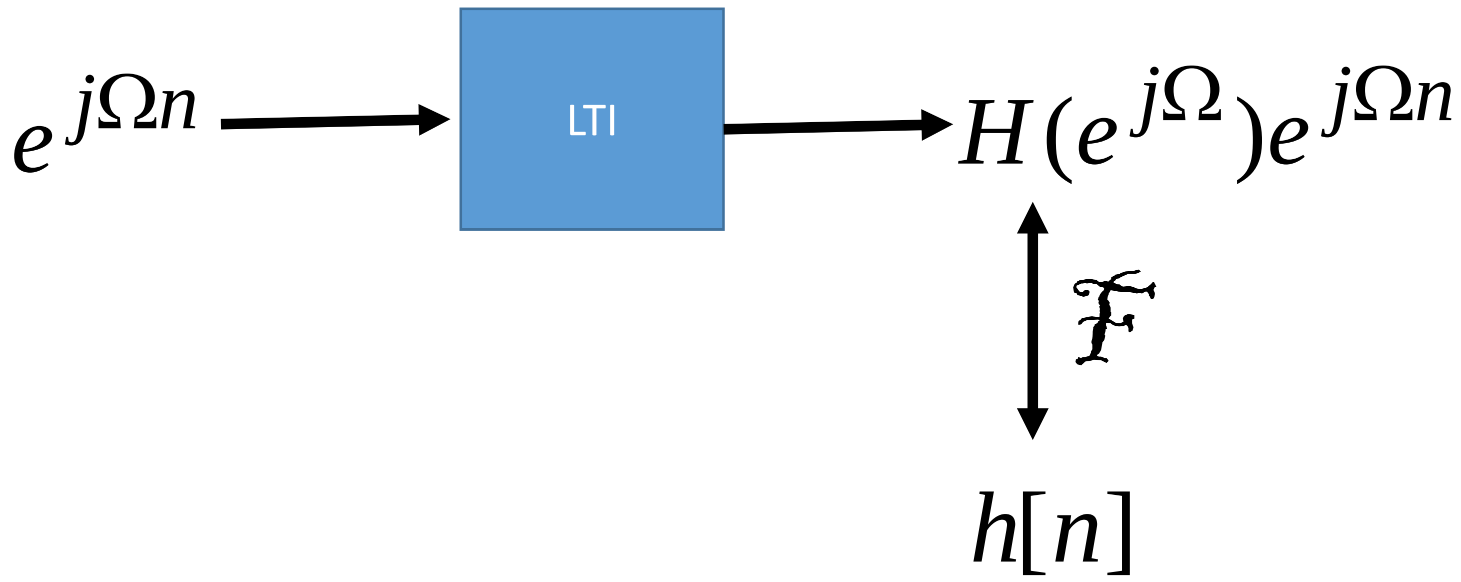
Synthesis equation

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

Analysis equation

LTI systems

- Impulse response h[n]



LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}}$$

$$z = re^{j\Omega}$$

LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow \sum_{k=-\infty}^{\infty} h[k] z^{n-k}$$

$$z = re^{j\Omega}$$

LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$$z = re^{j\Omega}$$

LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$$z = re^{j\Omega}$$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

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$$\boxed{H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}}$$

LTI systems

- Impulse response h[n]

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow z^n H(z)$$

$$z = re^{j\Omega}$$

$$\boxed{H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}}$$

Applications of Z transforms

- Scale transformations (images with different resolutions)
- Solving Difference equations with initial conditions

Z-Transform

Bilateral

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

Connection between Z and Fourier Transform

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$z = re^{j\Omega}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\boxed{X(z)|_{r=1} = \mathcal{F}\{x[n]\}}$$

Connection between Z and Fourier Transform

$$\boxed{X(z)|_{r=1} = X(e^{j\Omega})}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\Omega n}$$

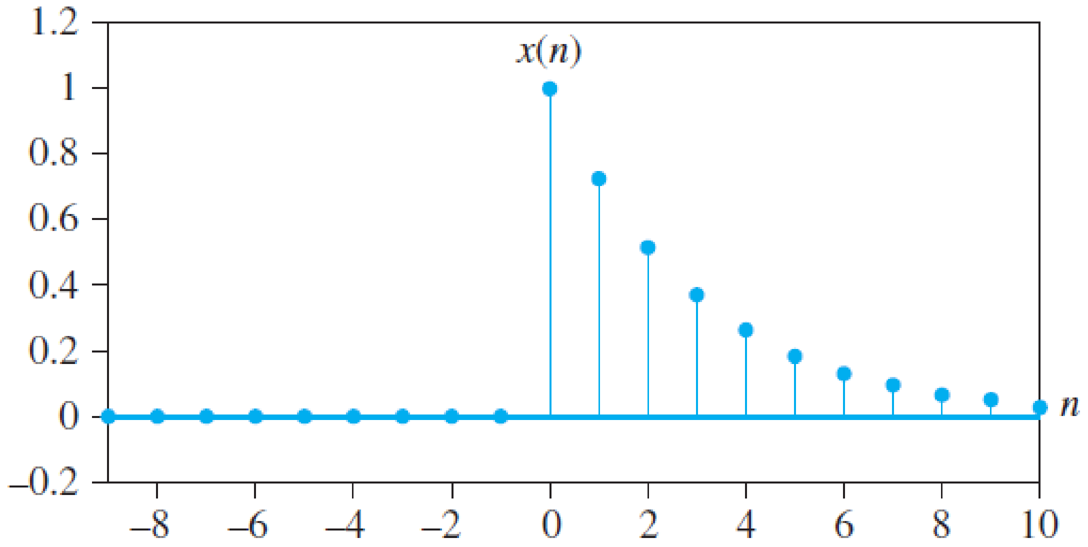
$$\boxed{X(z) = \mathcal{F}\{x[n] r^{-n}\}}$$

$\mathcal{Z}$ may converge when
$\mathcal{F}$ does not

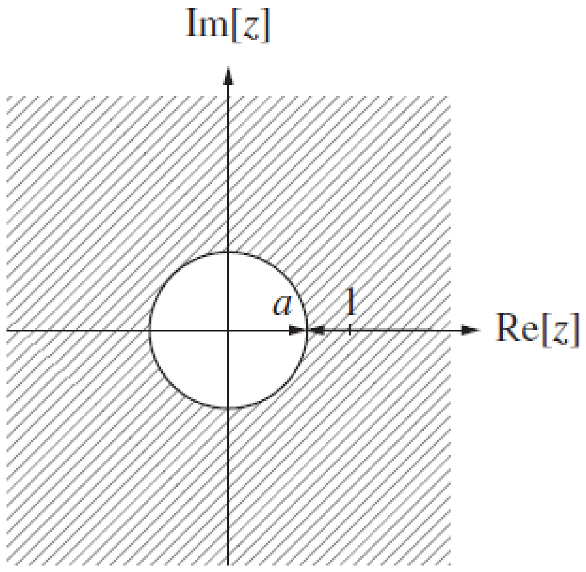
Causal Exponential Function

Find the z-transform of $x(n) = a^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$



(a)

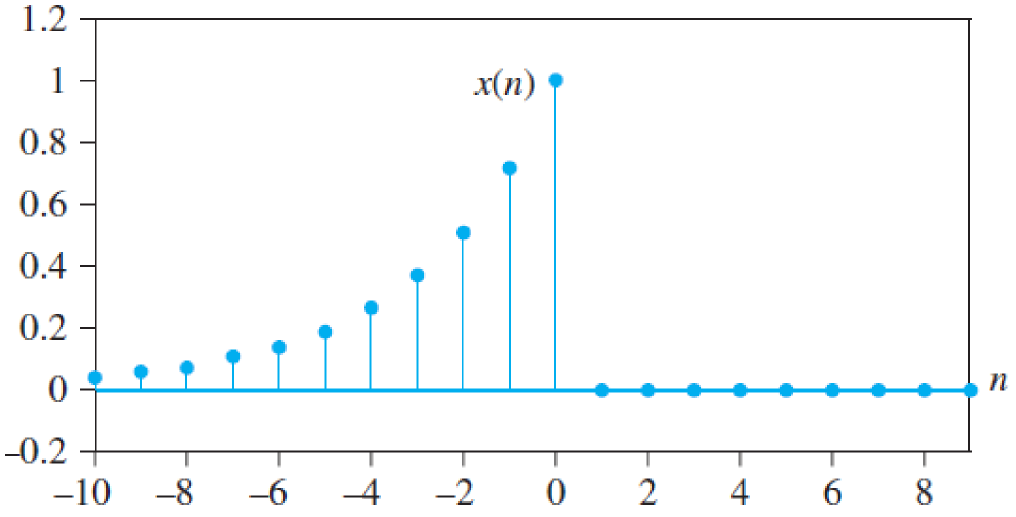


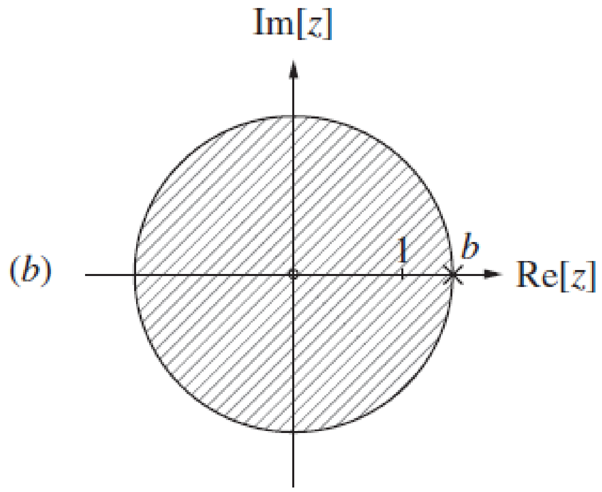
X(z) exists outside a circle with radius |a|.

Anticausal Exponential Function

Find the z-transform of $x(n) = b^n u(-n)$.

$$X(z) = \sum_{n=-\infty}^{0} b^n z^{-n} = \sum_{n=0}^{\infty} (b^{-1}z)^n = \frac{1}{1 - b^{-1}z}, \quad |z| < |b|$$





X(z) exists inside a circle with radius |b|.

Anticausal Exponential Function

Find the z-transform of $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-1}^{-\infty} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = -\{\sum_{n=0}^{\infty} a^{-n} z^n - 1\}$$

$$X(z) = \{1 - \sum_{n=0}^{\infty} a^{-n} z^n\} = 1 - \frac{1}{1 - a^{-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{a^{-1}z}{a^{-1}z - 1}$$

$$X(z) = \frac{a^{-1}z}{a^{-1}z - 1} = \frac{z}{z - a} = \frac{1}{1 - az^{-1}} \qquad |z| < |a|$$

Exponential Function

Z-transform of

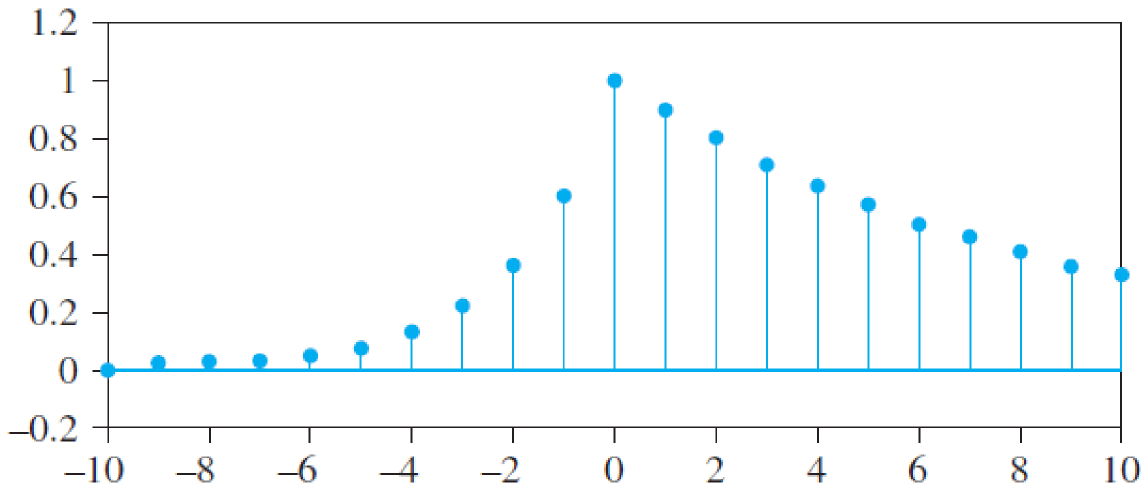
$x[n] = -a^n u[-n-1]$ $\qquad X(z) = \dfrac{1}{1 - az^{-1}}$ $\qquad |z|<|a|$

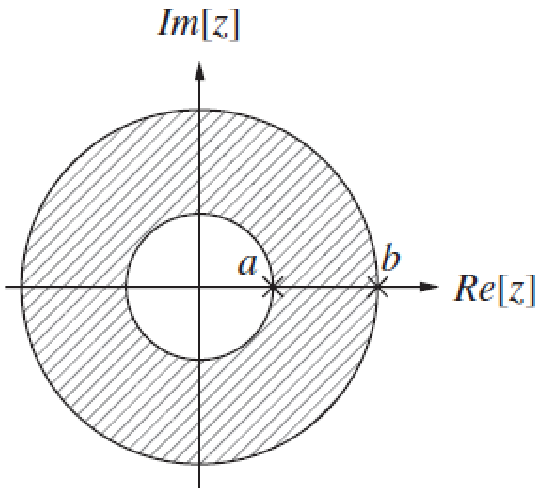
$x[n] = a^n u[n]$ $\qquad X(z) = \dfrac{1}{1 - az^{-1}}$ $\qquad |z| > |a|$

Two-Sided Exponential Function

$$x(n) = \begin{cases} a^n, & n > 1 \\ b^n, & n \leq 0 \end{cases}$$

$$X(z) = \frac{(a-b)z^{-1}}{(1-az^{-1})(1-bz^{-1})}, \quad |a| < |z| < |b|$$





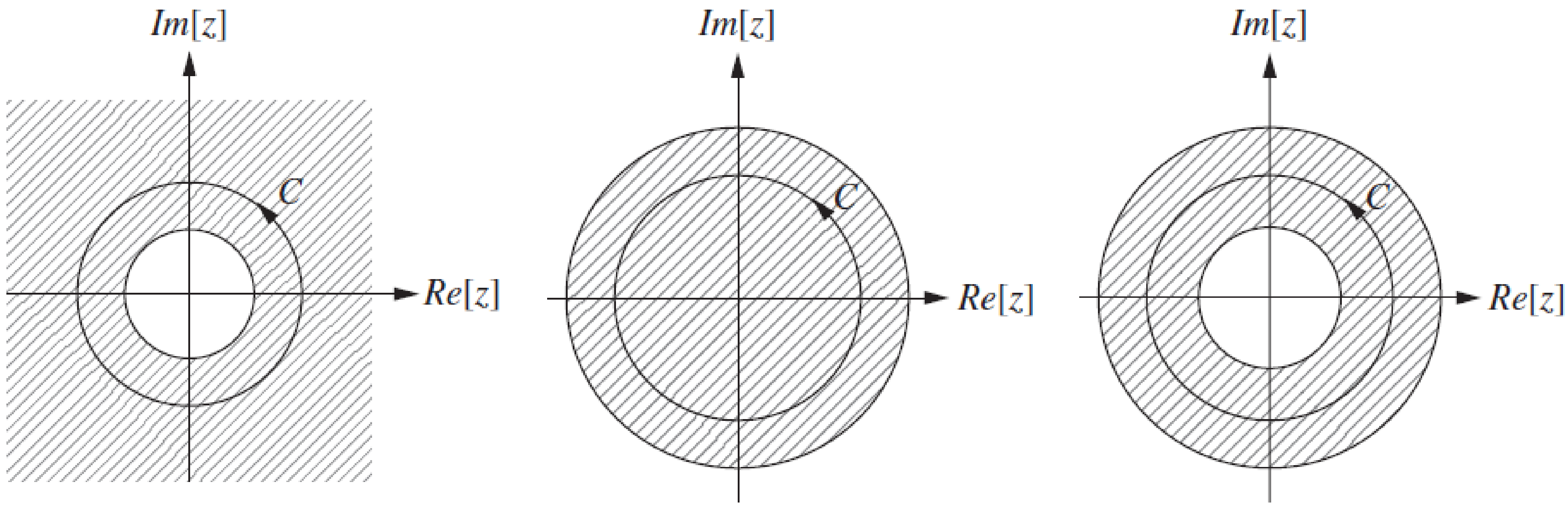
X(z) exists within the ring $|a| < |z| < |b|$

Annoying points in ROC

- If x[n] is non-zero (even for one sample) for positive n, ROC cannot include Zero
- If x[n] is non-zero (even for one sample) for negative n, ROC cannot include Infinity.

ROC Summary

- The ROC of a causal function is outside the circle bordering the outermost pole (Fig. a).
- The ROC of an anticausal function is inside the circle bordering the innermost pole (Fig b).
- The ROC of a two-sided function is an annular ring containing no poles (Fig. c).



(a) ROC of a causal function (b) ROC of an anticausal function (c) ROC of a two-sided function

ROC Summary

- The ROC of a right-sided non-causal function is outside the circle bordering the outermost pole, except at z = Infinity
- The ROC of a left sided non-anticausal function is inside the circle bordering the innermost pole, except at z = 0

Causal and Stable Discrete-time LTI system

Choose the right option
I) All poles lie in right-half plane
II) All poles lie in left-half plane
III) All poles lie inside r=1 circle
IV) All poles lie outside r=1 circle

Causal and Stable LTI system

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I) All poles lie in right-half plane
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III) All poles lie inside r=1 circle
IV) All poles lie outside r=1 circle

Causal and stable LTI system

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$= \ldots\ldots + h[-2]z^{2} + h[-1]z + h[0] + h[1]z^{-1} + h[2]z^{-2} + \ldots\ldots\ldots$$

If $h[n]$ is causal, it can be made convergent by choosing magnitude of Z (that is, r) large, so that $\sum_{n=-\infty}^{\infty} h[n] z^{-n}$ becomes smaller.

For stability $\sum_{n=-\infty}^{\infty} |h[n]|$ should be absolutely summable, that is r =1 must be in ROC.

Properties of Z Transforms

- **Linearity**

$$ax[n]+by[n] \leftrightarrow aX(z)+bY(z) \qquad \supset (R_1 \cap R_2)$$

- **Time-shift**

$$x[n-n_0] \leftrightarrow z^{-n_o} X(z) \qquad R,$$

$$except\ for\ possible\ +/-\ of\ 0/\infty$$

- **Convolution**

$$x[n]*y[n] \leftrightarrow X(z)Y(z) \qquad \supset (R_1 \cap R_2)$$

Properties of Z Transforms

- **Modulation**

$$z_0{}^n x[n] \leftrightarrow X(z / z_o) \qquad R|z_o|$$

- **Differentiation in z domain**

$$nx[n] \leftrightarrow -z\frac{d}{dz}(X(z)) \qquad R$$

Check yourself:

$x[n] = 0, n < 0$

$$\lim_{z \to \infty} X(z) = ?$$

Check yourself:

$x[n] = 0, n < 0$

$$\lim_{z \to \infty} X(z) = ?$$

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + ...$$

$$Lim\ z \to \infty X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + ... = \mathrm{x}[0]$$

Check yourself:

$x[n] = 0, n < 0$

$$\lim_{z \to \infty} X(z) = \mathrm{x}[0] \text{ (initial-value theorem)}$$

If x[0] is finite, then the number of zeros cannot be greater than the number of poles

Check yourself:

$x[n] = 0, n < 0$

$$\mathrm{X}(z) = \sum_{n=-\infty}^{\infty} \mathrm{x[n]z}^{-n} \qquad \mathrm{X}_1(z) = \sum_{n=-\infty}^{\infty} \mathrm{x[n+1]z}^{-n} = z \sum_{m=-\infty}^{\infty} \mathrm{x[m]z}^{-m} = zX(z)$$

$$(\mathrm{z}-1)\,\mathrm{X}(z) = \sum_{n=-\infty}^{\infty} \{\mathrm{x[n+1]} - \mathrm{x[n]}\}z^{-n}$$

$$\mathrm{Lt}_{z\to 1}(\mathrm{z}-1)\,\mathrm{X}(z) = Lt_{k\to\infty} Lt_{z\to 1} \sum_{n=-\infty}^{k} \{\mathrm{x[n+1]} - \mathrm{x[n]}\}z^{-n} = Lt_{k\to\infty} x[k] = x[\infty]$$

Check yourself:

$x[n] = 0, n < 0$

$$\lim_{z \to 1}(z-1)X(z) = \mathrm{x}[\infty] \text{ (final-value theorem)}$$

x[∞] is finite, and the order of poles at z = 1 is not more than 1.

Check yourself

$$X(z) = \log(1 + az^{-1}) \quad |z| > |a|$$

Find x[n]

Example of Inverse by Expansion

Find inverse z-transform of

$$X(z) = \frac{1 - 2z^{-1} + 4z^{-2}}{(1 - z^{-1})(1 + 2z^{-1})}, \quad |z| > 2$$

Example of Inverse by Expansion

Find inverse z-transform of

$$X(z) = \frac{1 - 2z^{-1} + 4z^{-2}}{(1 - z^{-1})(1 + 2z^{-1})}, \quad |z| > 2$$

Solution. **We first reduce the degree of the numerator by extracting the constant -2 from it, then expand into fractions**

$$X(z) = -2 + \frac{3}{(1 - z^{-1})(1 + 2z^{-1})}, \quad |z| > 2$$

$$= -2 + \frac{A}{1 - z^{-1}} + \frac{B}{1 + 2z^{-1}}, \quad |z| > 2$$

$$= -2 + \frac{1}{1 - z^{-1}} + \frac{2}{1 + 2z^{-1}}, \quad |z| > 2$$

$$x(n) = -2d(n) + u(n) + 2(-2)^n u(n)$$

Application to Difference Equations

Consider the difference equation (with x(n) known for all n)

$$y(n) + a_1 y(n-1) \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) \cdots + b_M x(n-M)$$

Taking the bilateral z-transform of both sides and making use of shift property, we find

$$Y(z) = \frac{b_0 + b_1 z^{-1} \cdots + b_M z^{-M}}{1 + a_1 z^{-1} \cdots + a_N z^{-N}} X(z)$$

Let

$$H(z) = \frac{b_0 + b_1 z^{-1} \cdots + b_M z^{-M}}{1 + a_1 z^{-1} \cdots + a_N z^{-N}}$$

Then

$$Y(z) = H(z)X(z)$$

Filter (what kind of filter is this?)

$$y[n]-2r\cos\theta y[n-1]+r^2y[n-2]=x[n]$$

Which filter?
i) All-pass
ii) Band-pass
iii) Low-pass
iv) High-pass

Filter (where are the poles?)

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

$$Y(z) - 2r\cos\theta\, z^{-1} Y(z) + r^2 z^{-2} Y(z) = X(z)$$

$$Y(z)\{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}\} = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}}$$

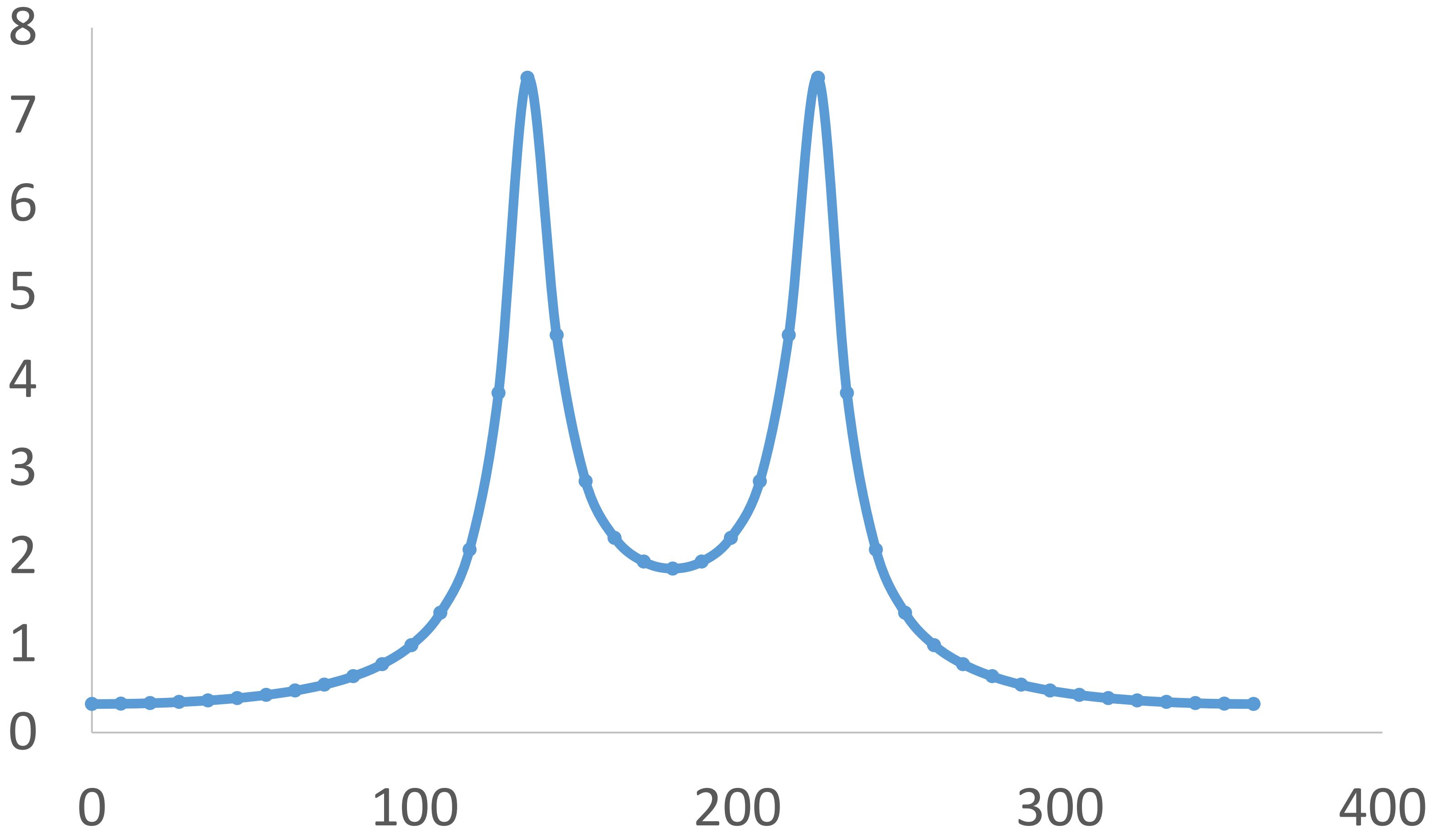
Filter

$$y[n] - 2r\cos\theta y[n-1] + r^2 y[n-2] = x[n]$$

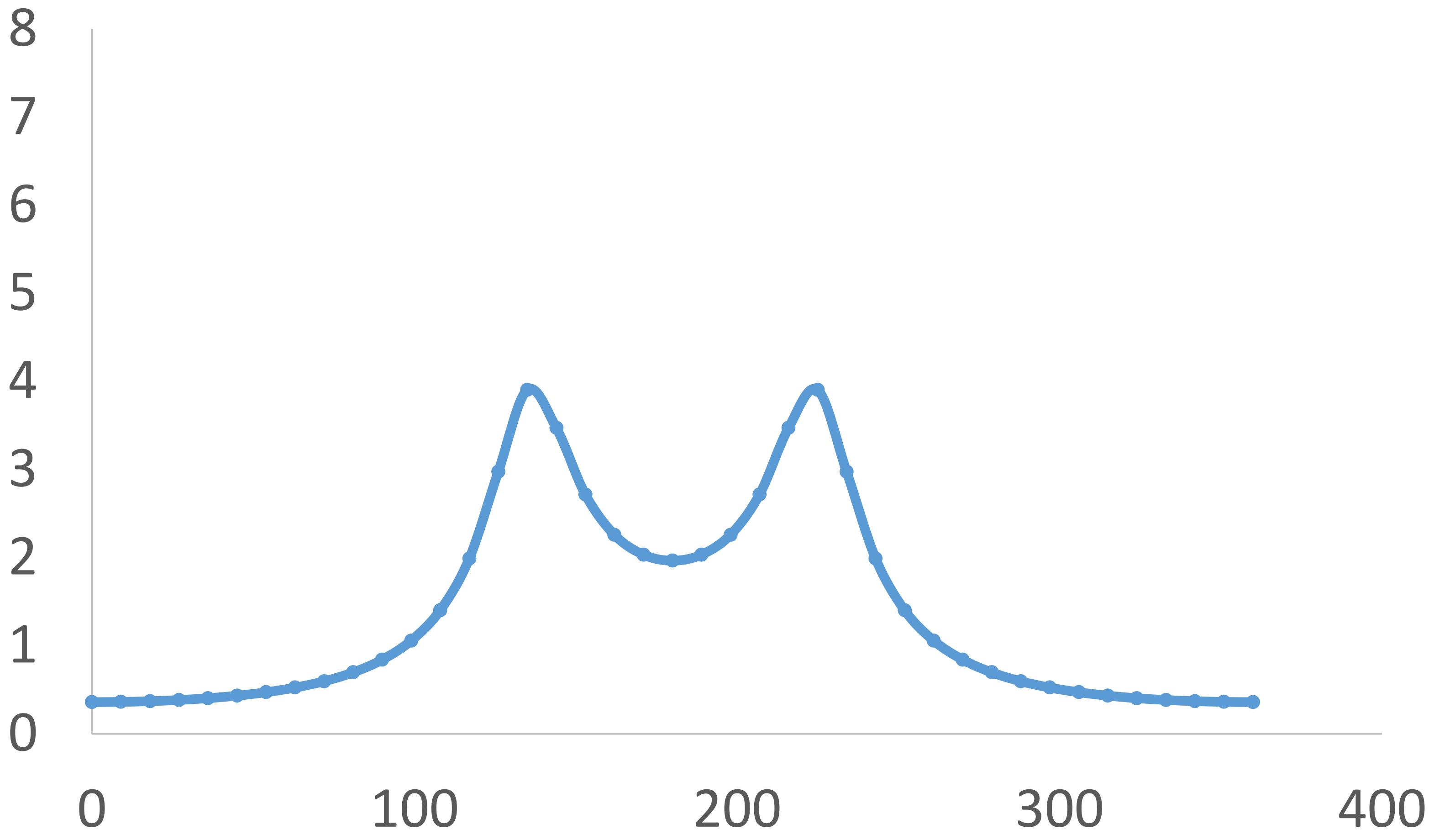
$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

$$\frac{Y(z)}{X(z)} = \frac{z^2}{z^2 - 2r\cos\theta z + r^2}$$

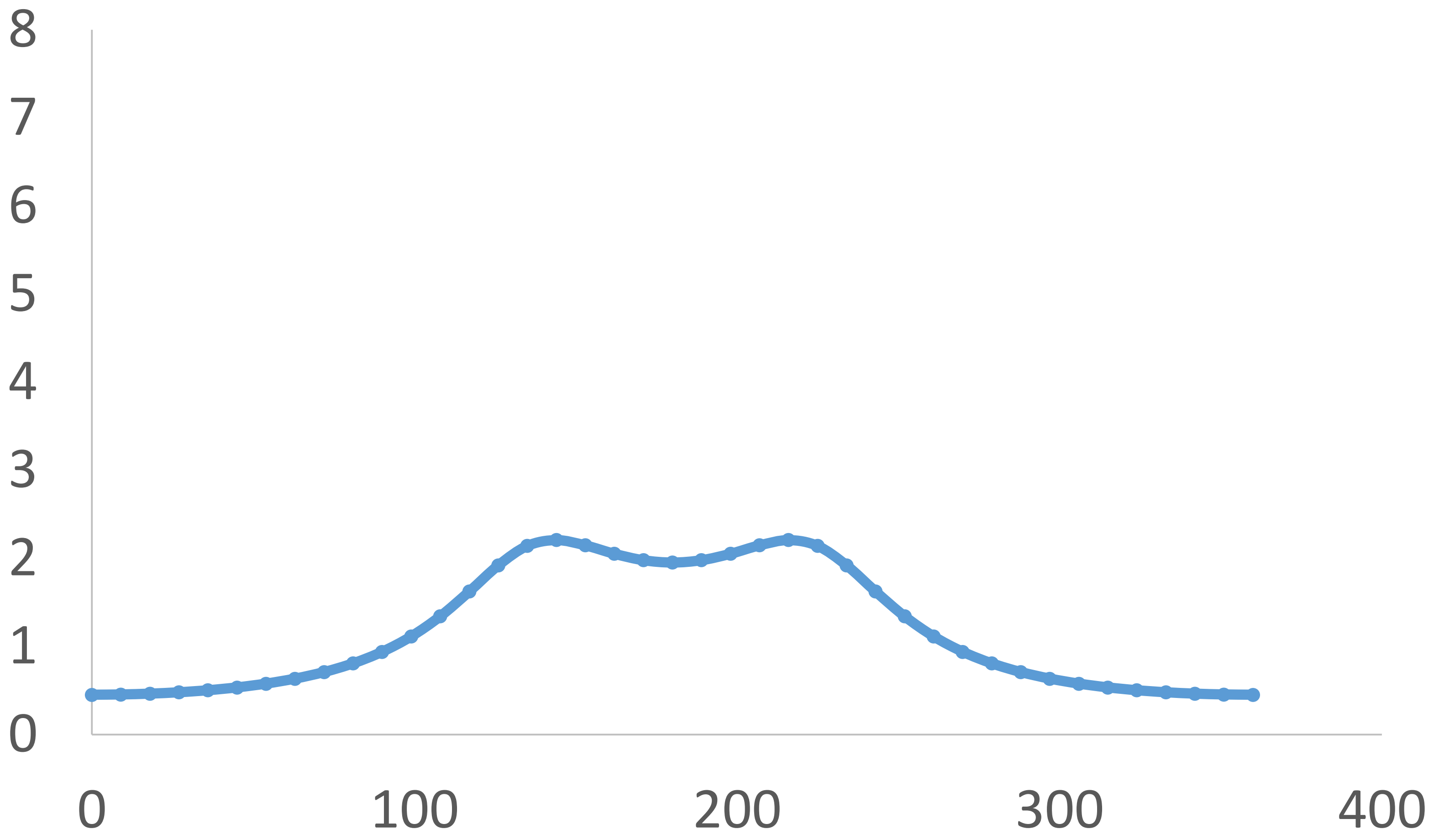
Changing r = 0.9



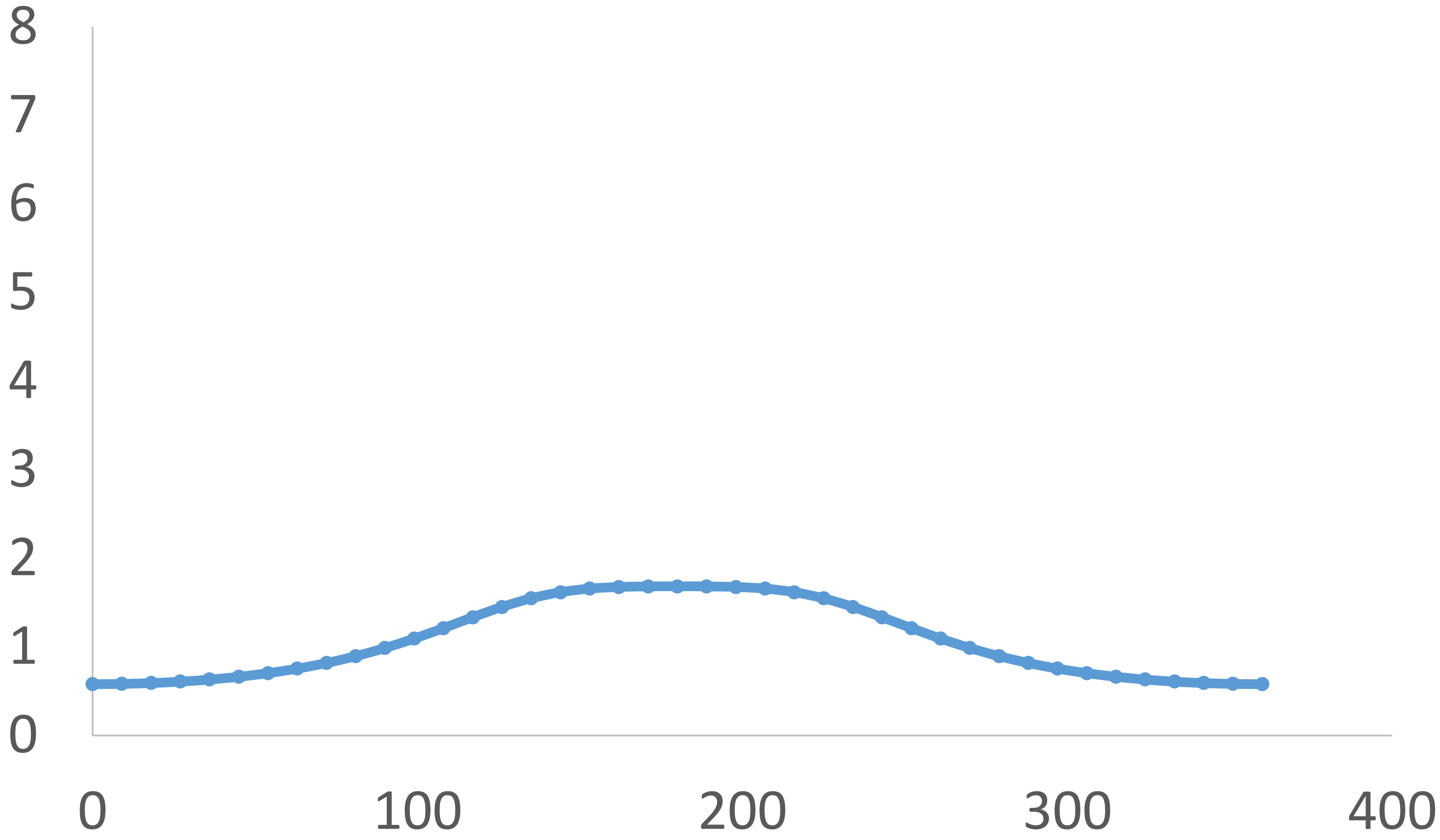
Changing r = 0.8



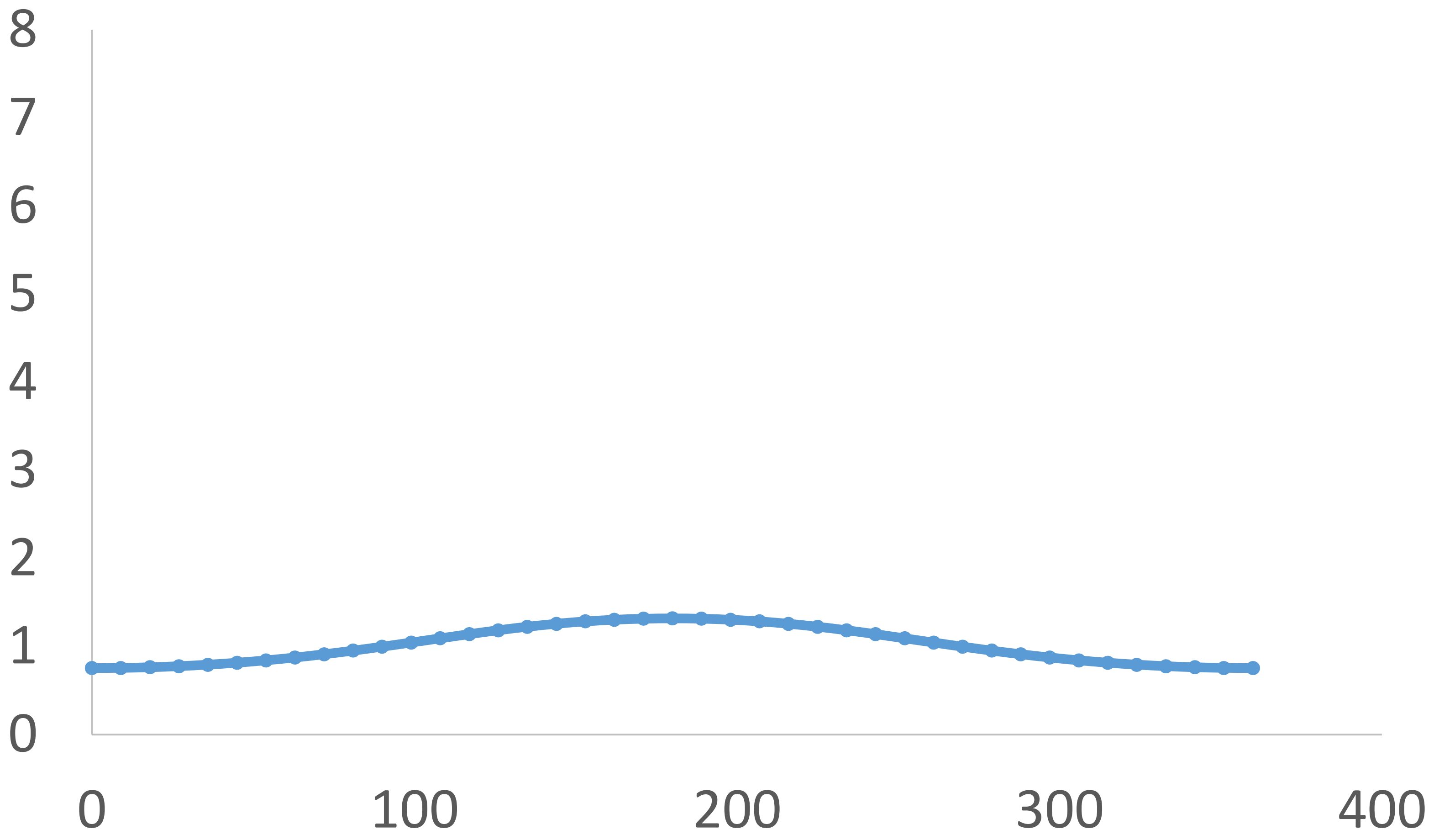
Changing r = 0.6



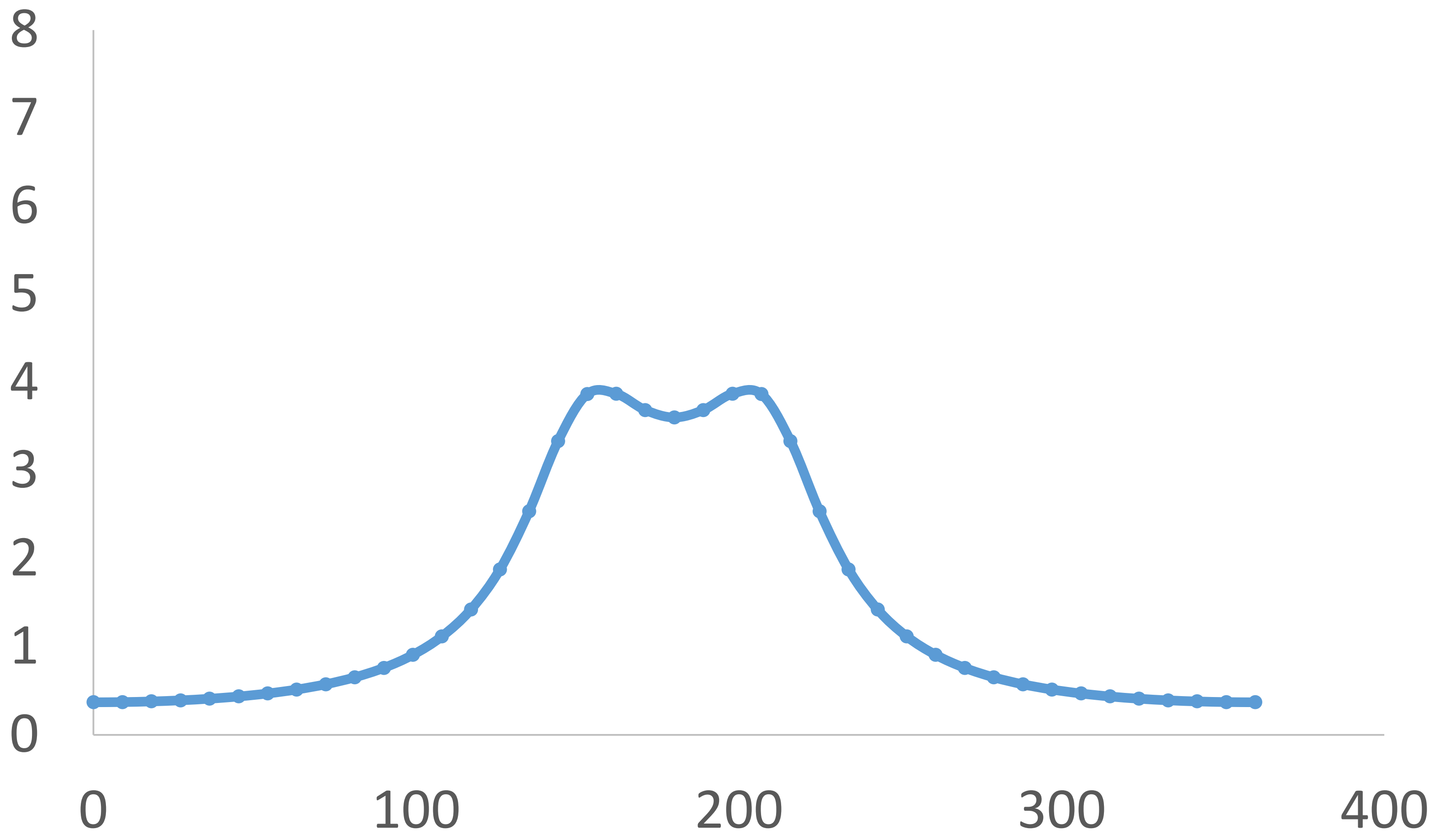
Changing r = 0.4



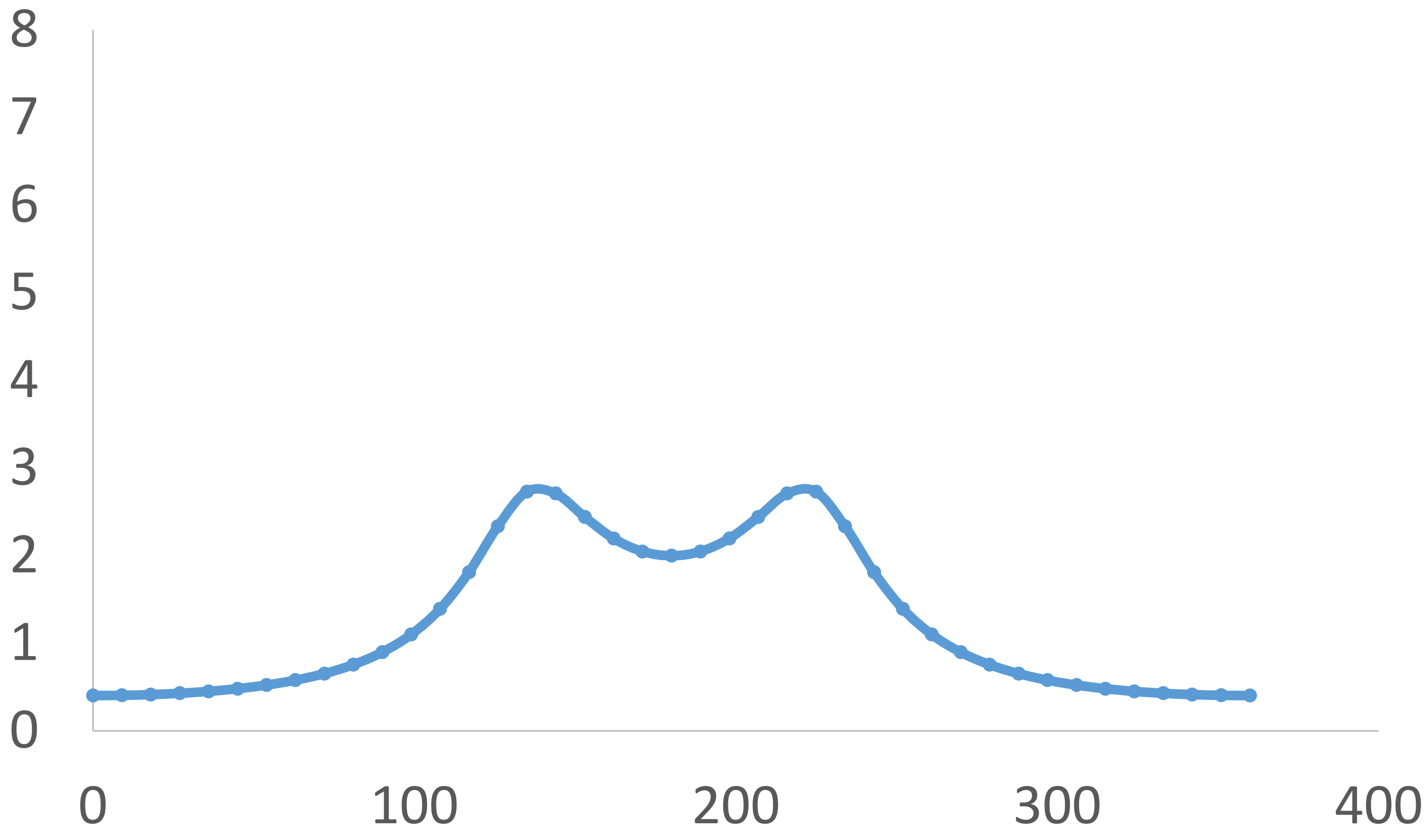
Changing r = 0.2



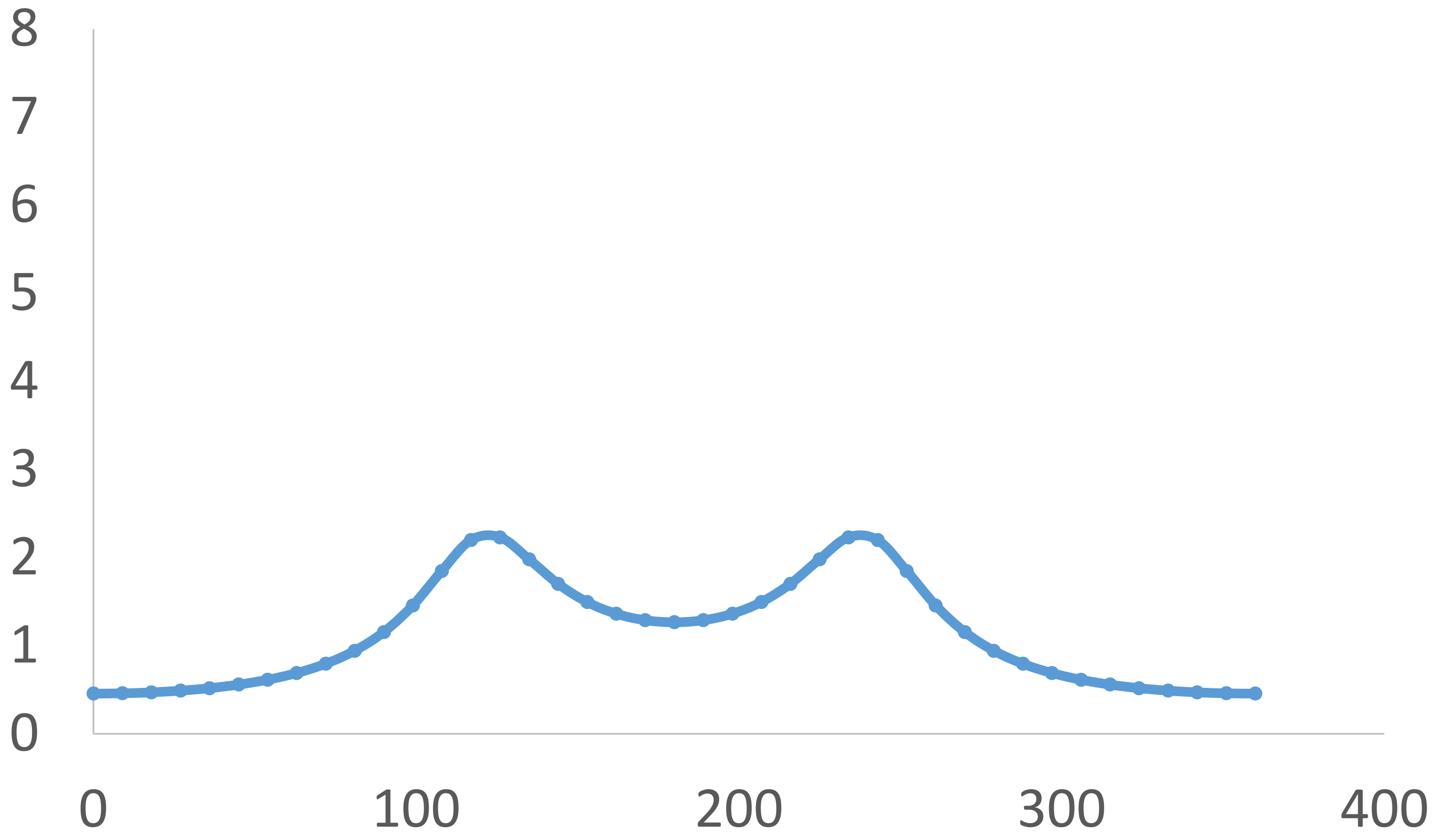
Changing theta = 150



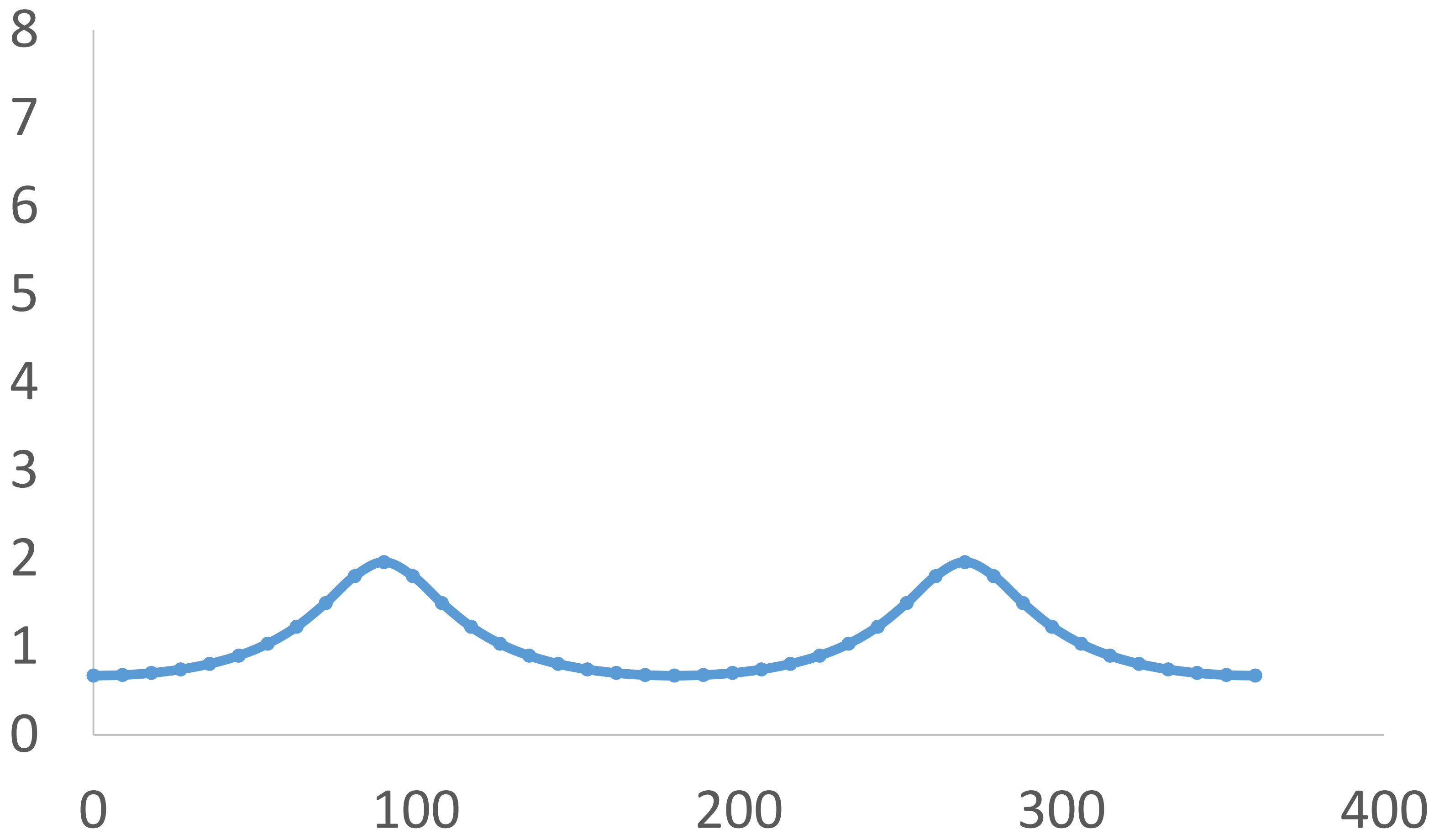
Changing theta = 135



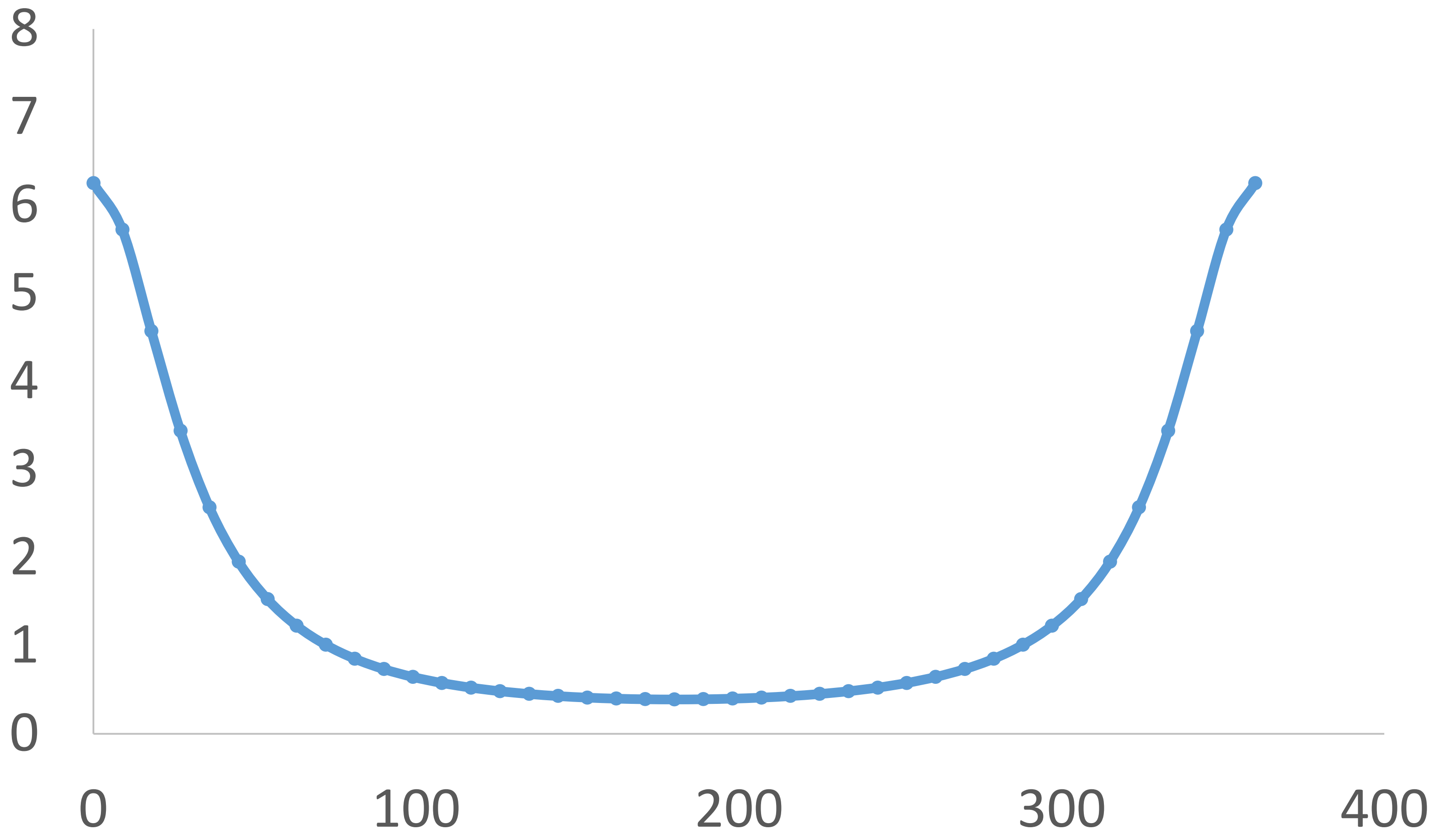
Changing theta = 120



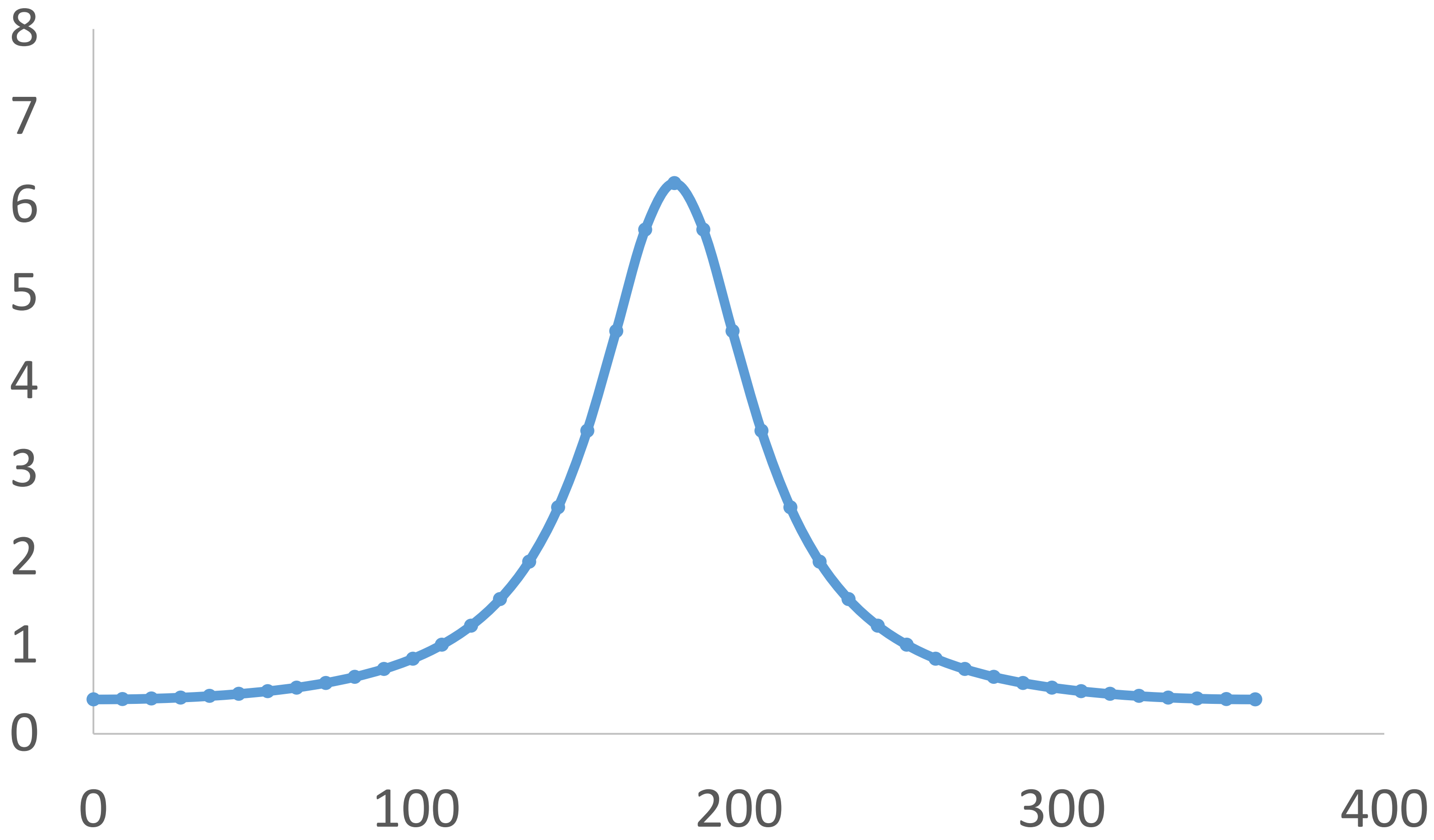
Changing theta = 90



Changing theta = 0



Changing theta = 180



Unilateral Z transform

$$X_u(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x[n] \leftrightarrow X_u(z) \qquad x[n-1] \leftrightarrow ?$$

$$Y_u(z) = \sum_{n=0}^{\infty} x[\mathrm{n}-1] z^{-n} = \sum_{r=-1}^{\infty} x[\mathrm{r}] z^{-(\mathrm{r}+1)} = z^{-1} \sum_{n=-1}^{\infty} x[\mathrm{n}] z^{-n}$$

$$= z^{-1} \left\{ \sum_{n=0}^{\infty} x[\mathrm{n}] z^{-n} + x[-1] \mathrm{z} \right\} = z^{-1} \left\{ X_u(z) + x[-1] \mathrm{z} \right\} = z^{-1} X_u(z) + x[-1]$$

Unilateral Z transform

$$X_u(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x[n-1] \leftrightarrow z^{-1} X_u(z) + x[-1]$$

$$x[n-2] \leftrightarrow ?$$

Unilateral Z transform

$$X_u(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x[n-1] \leftrightarrow z^{-1} X_u(z) + x[-1]$$

$$x[n-2] \leftrightarrow z^{-2} X_u(z) + z^{-1} x[-1] + x[-2]$$

Unilateral Z transform

$$x[n-2] \leftrightarrow z^{-2}X_u(z) + z^{-1}x[-1] + x[-2]$$

$$Y_u(z) = \sum_{n=0}^{\infty} x[\mathrm{n}-2]z^{-n} = \sum_{r=-2}^{\infty} x[\mathrm{r}]z^{-(\mathrm{r}+2)} = z^{-2}\sum_{n=-2}^{\infty} x[\mathrm{n}]z^{-n}$$

$$= z^{-2}\left\{\sum_{n=0}^{\infty} x[\mathrm{n}]z^{-n} + x[-1]\mathrm{z} + x[-2]\mathrm{z}^{2}\right\} = z^{-2}\left\{X_u(z) + x[-1]\mathrm{z} + +x[-2]\mathrm{z}^{2}\right\}$$

$$= z^{-2}X_u(z) + x[-1]z^{-1} + x[-2]$$

Solving DE with initial conditions

- $y[n] - y[n-1] = x[n] \qquad x[n] = \alpha u[n] \qquad \mathrm{y}[-1] = \beta$

Solving DE with initial conditions

- $y[n] - y[n-1] = x[n]$ $\quad x[n] = \alpha u[n]$ $\quad y[-1] = \beta$

$$Y_u(z) - \{z^{-1}Y_u(z) + y[-1]\} = X_u(z)$$

$$Y_u(z)\{1 - z^{-1}\} = \frac{\alpha}{1 - z^{-1}} + \beta$$

$$Y_u(z) = \frac{\alpha}{(1 - z^{-1})^2} + \frac{\beta}{1 - z^{-1}}$$

Solving DE with initial conditions

- $y[n] - y[n-1] = x[n] \qquad x[n] = \alpha u[n] \qquad \mathrm{y}[-1] = \beta$

$$Y_u(z) - \{z^{-1}Y_u(z) + \mathrm{y}[-1]\} = X_u(z)$$

$$Y_u(z)\{1 - z^{-1}\} = \frac{\alpha}{1 - z^{-1}} + \beta$$

$$Y_u(z) = \frac{\alpha}{(1 - z^{-1})^2} + \frac{\beta}{1 - z^{-1}}$$

$$y[n] = \alpha(n+1)u[n+1] + \beta u[n]$$
$$n \geq 0$$

Solving DE with initial conditions

$$Z\{\alpha u[n]\} = \frac{\alpha}{1 - z^{-1}}$$

$$Z\{\alpha n u[n]\} = -z \frac{d}{dz}\left(\frac{\alpha}{1 - z^{-1}}\right) = \frac{\alpha z^{-1}}{(1 - z^{-1})^2}$$

$$Z\{\alpha (n+1) u[n+1]\} = \frac{\alpha}{(1 - z^{-1})^2}$$