Indian Institute of Technology, Delhi
Department of Physics

`PYL 101`: Electromagnetics & Quantum Mechanics

Semester I, 2020-2021

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Vector Analysis

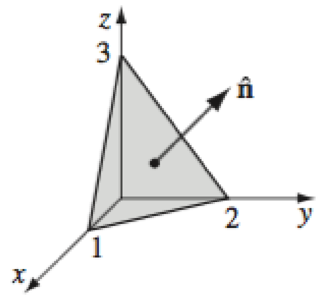
Problem Set 1

Exercise 1.
Is the cross product associative?
i.e., does,

$$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$$

hold. Demonstrate why, or why not.

Exercise 2.



Use the cross product to find the components of the unit vector $\hat{\boldsymbol{n}}$ perpendicular to the shaded plane in the given figure.

Exercise 3.
Prove the **BAC-CAB** rule by writing out both sides in component form.

Exercise 4.
Find the gradients of the following functions:

- $f(x, y, z) = x^2 + y^3 + z^4$
- $f(x, y, z) = x^2 y^3 z^4$
- $f(x, y, z) = e^x \sin(y) \ln(z)$

Exercise 5.
The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where y is the distance (in miles) north, x the distance east of the a reference point.

- Where is the top of the hill located?
- How high is the hill?
- How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of the reference point? In what direction is the slope steepest, at that point?

Exercise 6.
Sketch the vector function,

$$\boldsymbol{v} = \frac{\hat{\boldsymbol{r}}}{r^2}$$

and compute its divergence.

Exercise 7.
Draw a circle in the xy plane. At a few representative points draw the vector $\boldsymbol{v}$ tangent to the circle, pointing in the clockwise direction. By comparing adjacent vectors, determine the sign of $\frac{\partial v_x}{\partial y}$ and $\frac{\partial v_y}{\partial x}$. According to the definition of curl, what is its direction? Explain how this example illustrates the geometrical interpretation of the curl.

Exercise 8.
Construct a non-constant vector function that has zero divergence and zero curl everywhere.

Exercise 9.

- Refer to [IEDJ] and check the product rule (iv) (by calculating each term separately) for the functions

$$\boldsymbol{A} = x\hat{\boldsymbol{x}} + 2y\hat{\boldsymbol{y}} + 3z\hat{\boldsymbol{z}}; \boldsymbol{B} = 3y\hat{\boldsymbol{x}} - 2x\hat{\boldsymbol{y}}$$

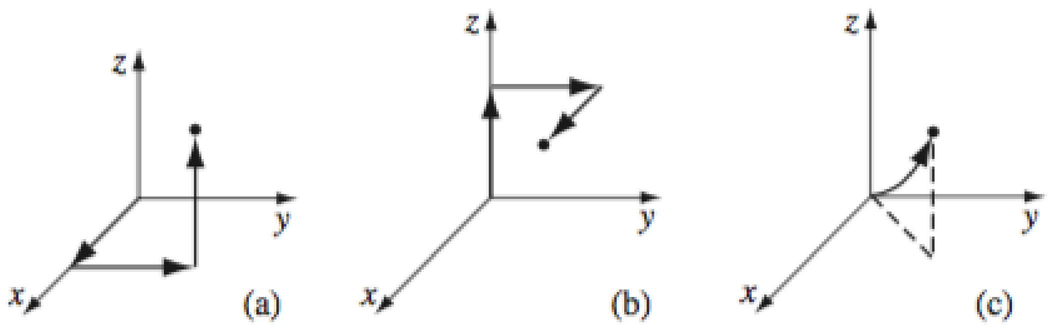
- Do the same for product rule (ii)
- Do the same for rule (vi)

Exercise 10.
Prove that the divergence of a curl is always zero.

Exercise 11.
Prove that the curl of a gradient is always zero.

Exercise 12.
Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at (0,0,0), (1,0,0), (0,1,0), and (0,0,1).

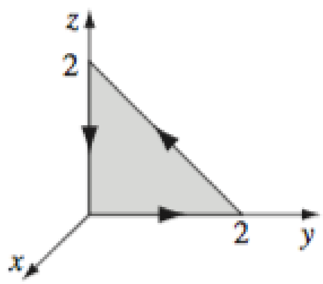
Exercise 13.
Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $a = (0, 0, 0)$, $b = (1, 1, 1)$, and the three paths given in the figure. for



- (0,0,0) $\Rightarrow$ (1,0,0) $\Rightarrow$ (1,1,0) $\Rightarrow$ (1,1,1)
- (0,0,0) $\Rightarrow$ (0,0,1) $\Rightarrow$ (0,1,1) $\Rightarrow$ (1,1,1)
- the parabolic path $z = x^2$; $y = x$.

Exercise 14.
Test the divergence theorem for the function $v = (xy)\hat{\boldsymbol{x}} + (2yz)\hat{\boldsymbol{y}} + (3zx)\hat{\boldsymbol{z}}$. Take as your volume the cube with sides of length 2.

Exercise 15.



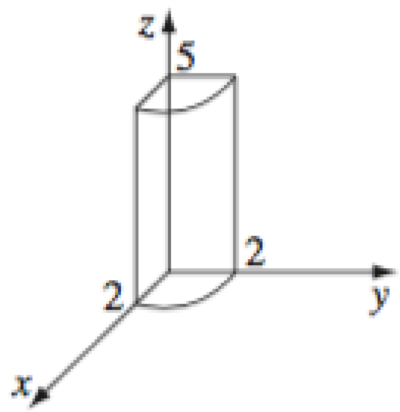
Test Stokes' theorem for the function $v = (xy)\hat{\boldsymbol{x}} + (2yz)\hat{\boldsymbol{y}} + (3zx)\hat{\boldsymbol{z}}$, using the triangular shaded area of Fig.

Exercise 16.
Compute the divergence of the function

$$\boldsymbol{v} = (r\cos\theta)\hat{\boldsymbol{r}} + (r\sin\theta)\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\hat{\boldsymbol{\phi}}$$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin.

Exercise 17.



- Find the divergence of the function

$$\boldsymbol{v} = s(2+\sin^2\phi)\hat{\boldsymbol{s}} + s\sin\phi\cos\phi\hat{\boldsymbol{\phi}} + 3z\hat{\boldsymbol{z}}$$

- Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in the figure.
- Find the curl of $\boldsymbol{v}$.

Exercise 18.
Let $\boldsymbol{F}_1 = x^2\hat{\boldsymbol{z}}$ and $\boldsymbol{F}_2 = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. Calculate the divergence and curl of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.