ELL101: INTRODUCTION TO ELECTRICAL ENG.

Superposition Theorem

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Linearity

A mathematical equation is said to be linear if the following properties hold.

- Homogeneity
- Additivity

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Homogeneity (Scaling)

Homogeneity exists if input of a system is multiplied by a constant, then the output should also be multiplying by the same constant.

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Homogeneity (Scaling)

Check the following equation for homogeneity.

$$y = 7x$$

$$x = 1 \rightarrow y = 7$$

$$x = 2 \rightarrow y = 14$$

Now for homogenity to hold, scaling should hold for y, that is, y has a value of 7 when x = 1. If we increase x by a factor of 2 when we should be able to multiply y by the same factor and get the same answer when we substitute into the right side of the equation for x = 2.

$$ny = 7(nx)$$

$$nx \rightarrow ny$$

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Homogeneity (Scaling)

Check homogenity for the following equation.

$$y = 5x + 2$$

Solution

$$nx \nrightarrow ny$$

Does not follow the property of homogeneity

Additivity

The additivity property is equivalent to the statement that the response of a system to a sum of inputs is the same as the responses of the system when each input is applied separately and the individual responses summed (added together).

Additivity

Let us check the additivity for y = 7x.

$$x = x_1 \rightarrow y = 7x_1 = y_1$$

$$x = x_2 \rightarrow y = 7x_2 = y_2$$

$$x = x_1 + x_2 \rightarrow y = 7(x_1 + x_2)$$

$$x = x_1 + x_2 \rightarrow y = 7x_1 + 7x_2 = y_1 + y_2$$

The equation follows the property of additivity.

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Additivity

Let us check the additivity for y = 5x+2.

$$x = x_1 \rightarrow y = 5x_1 + 2 = y_1$$

$$x = x_2 \rightarrow y = 5x_2 + 2 = y_2$$

$$x = x_1 + x_2 \rightarrow y = 5(x_1 + x_2) + 2$$

$$x = x_1 + x_2 \rightarrow y = 5x_1 + 5x_2 + 2 \neq y_1 + y_2 = 5x_1 + 5x_2 + 2 + 2$$

The equation does not follow the property of additivity.

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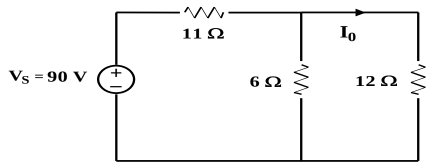
Example: For the given circuit, use the concept of linearity (homogeneity or scaling) to find the current I_0.



- Let $I_0 = 1$A
- Then voltage drop across 12Ω resistance is 12 V.
- As 12Ω and 6Ω resistances are in parallel, 12 V is also the voltage drop for 6Ω resistance.
- Hence, the current in 6Ω resistance is 2A
- So, total current in 11Ω resistance will be 1A+2A (by KCL)
- Voltage drop at 11Ω resistance will be 11x3=33 V
- By KVL $V_s = 33 + 12 = 45\ V$
- However, $V_s = 90$V is given.
- So from linearity $I_0 = 2$A

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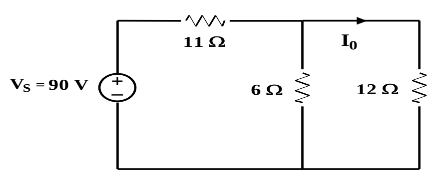

Check:

Using KVL in the first mesh:

$$11I_1 + 6(I_1 - I_0) = \mathrm{V_s}$$

Using KVL in the second mesh:

$$12I_0 + 6(I_0 - I_1) = 0$$

Solving:

$$I_0 = \frac{V_s \times 6}{18 \times 17 - 6 \times 6} = \frac{V_s}{45}$$

For $V_s = 90$, $I_0 = 2\ A$

Since

$$V_s = 45I_0$$

It is a linear relation between V_s and I_0.

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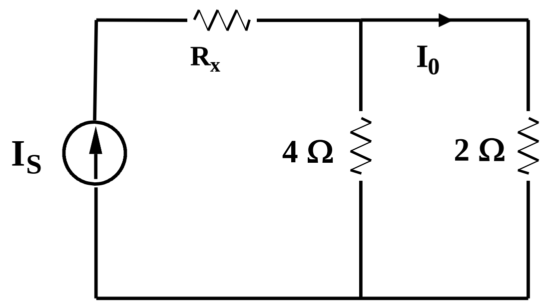
Example: In the circuit shown below it is known that I_0 = 4 A when I_S = 6 A. Find I_0 when I_S = 18 A.



As $I_{S\ NEW} = 3xI_{S\ OLD}$ we conclude $I_{0\ NEW} = 3xI_{0\ OLD.}$
Thus, $I_{0\ NEW}$ = 3x4 = 12 A.

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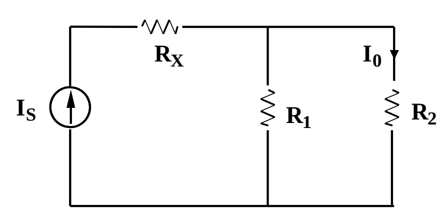
Let us use the current splitting rule (current division) to write the following equation:

$$I_0 = \frac{(I_S)(R_1)}{(R_1 + R_2)} = (K)(I_s)$$



The relation between I_0 and I_s is linear.

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Superposition

Let inputs f_1 and f_2 be applied to a system y such that,

$$y = k_1f_1 + k_2f_2$$

Where k_1 and k_2 are constants of the systems.

Let f_1 act alone so that, $y = y_1 = k_1f_1$
Let f_2 act alone so that, $y = y_2 = k_2f_2$

The property of superposition states that if f_1 and f_2
Are applied together, the output y will be,

$$y = y_1 + y_2 = k_1f_1 + k_2f_2$$

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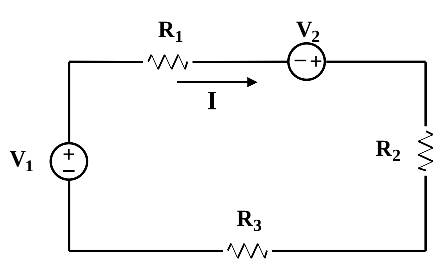
Example

Consider the circuit below that contains two voltage sources.



Use principle of superposition for finding the value of I

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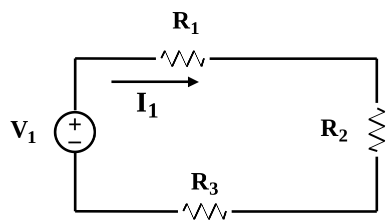
Let us excite the circuit by a single voltage source at a time.
Other source remains inactive, i.e., produces 0 V, means short circuit.

Source V1 active, V2 inactive



Current I_1 is produced by V_1

Observe that:

$$V_1 = (R_1 + R_2 + R_3)I_1$$

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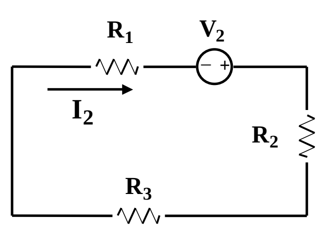
Source V2 active, V1 inactive



Current I_2 is produced by V_2

Observe that:

$$V_2 = (R_1 + R_2 + R_3)I_2$$

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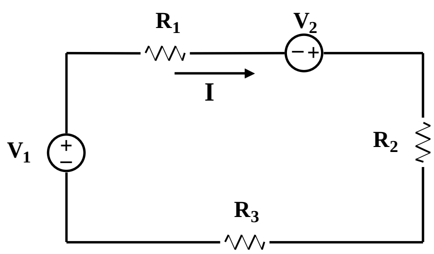
Let us calculate the circuit current by both voltage sources jointly.



Applying KVL:

$$V_1 + V_2 = (R_1 + R_2 + R_3)I$$

We have observed that:

$$V_1 = (R_1 + R_2 + R_3)I_1$$
$$V_2 = (R_1 + R_2 + R_3)I_2$$
$$V_1 + V_2 = (R_1 + R_2 + R_3)\underbrace{(I_1 + I_2)}_{I}$$

Superposition is applicable!

$$I_1 = V_1/(R_1 + R_2 + R_3)$$
$$I_2 = V_2/(R_1 + R_2 + R_3)$$
$$I = I_1 + I_2$$

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Superposition Theorem

- Definition: The current or voltage of an element in a linear bilateral circuit is equal to the algebraic sum of the currents and voltages produced independently by each source.
- This principle applies because of the linear relationship between current and voltage.
- Conditions to be met for applying superposition theorem
  - All the elements must be linear
  - All the components must be bilateral
  - Passive components must be used
  - Active components must not be there

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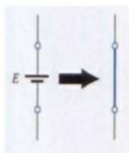
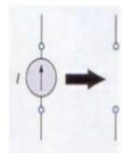
Superposition Theorem

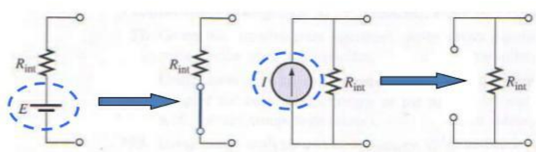
Procedure to apply Superposition theorem

- Consider only one source to be active at a time
- Remove ideal voltage source with short-circuit, and ideal current source with open-circui

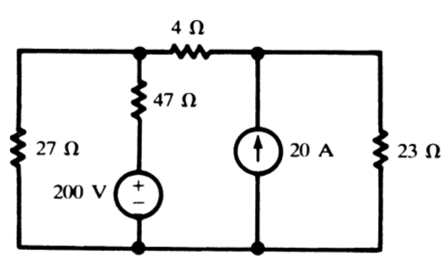




- Remove practical voltage source with short-circuit with internal resistance in series, and practical current source with open-circuit with internal resistance in parallel

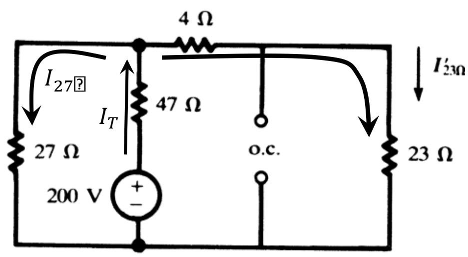


Example



Compute the current in the 23 Ω resistor by applying the superposition principle.

Make the current source inactive



$$R_{eq} = 47 + \frac{(27)(4 + 23)}{54} = 60.5\ \Omega$$

$$I_T = \frac{200}{60.5} = 3.31\ \text{A}$$

$$I'_{23\Omega} = \left(\frac{27}{54}\right)(3.31) = 1.65\ \text{A}$$ Current division

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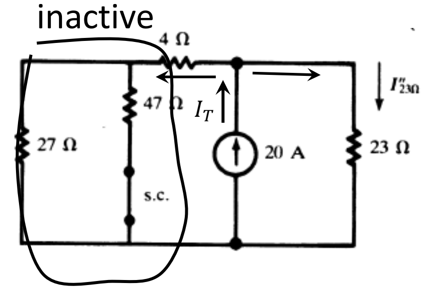
Make the voltage source inactive



$$R_{eq} = 4 + \frac{(27)(47)}{74} = 21.15\ \Omega$$

$$I''_{23\Omega} = \left(\frac{21.15}{21.15 + 23}\right)(20) = 9.58\ \text{A}$$ Current division

$$I_{23\Omega} = I'_{23\Omega} + I''_{23\Omega} = 11.23\ \text{A}$$

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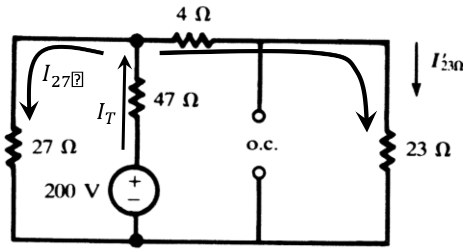
Make the current source inactive



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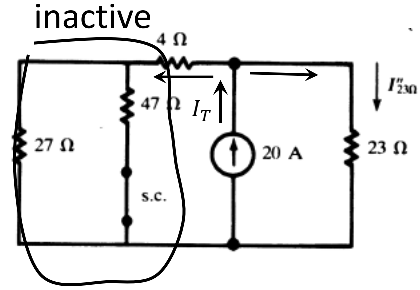
Make the voltage source inactive



$$R_{eq} = 4 + \frac{(27)(47)}{74} = 21.15\ \Omega$$

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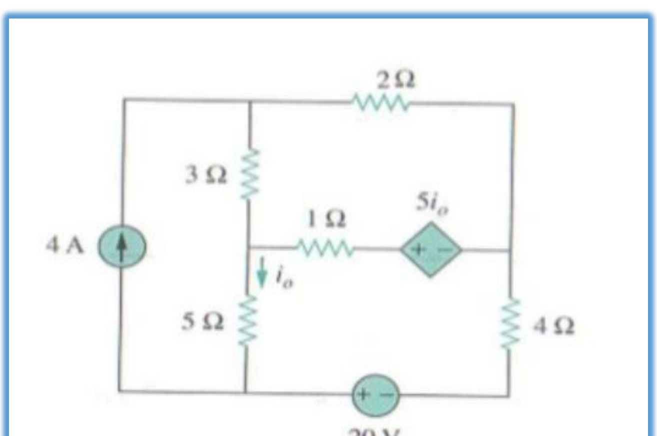
$$I_{23\Omega} = I'_{23\Omega} + I''_{23\Omega} = 11.23\ \text{A}$$

When there is Dependent source in the circuit

Example 3: Find the current i_0 in the given circuit



Solution: Part 1: Short ckt 20V voltage source

In loop 1

$i_1 = 4A$

In loop 2, apply KVL

$3(i_2 - i_1) + 2i_2 - 5i_0' + 1(i_2 - i_3) = 0$

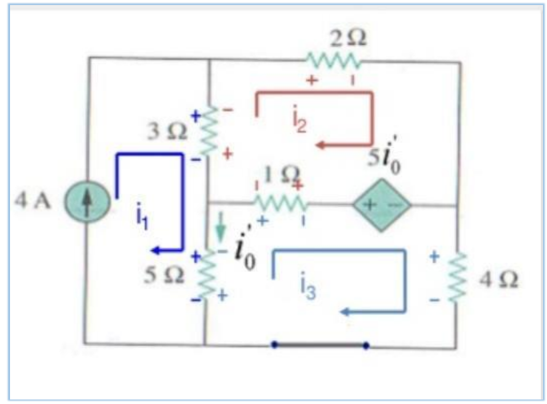
In loop3, apply KVL

$5(i_3 - i_1) + 1(i_3 - i_2) + 5\,i_0' + 4\,i_3 = 0$

Also, $i_0' = i_1 - i_3$

Solving above equations we get

$i_0' = 52/17$ A



Solution: Part 2: Open ckt 4A current source

In loop 4, apply KVL

$3i_4 + 2i_4 - 5i_0'' + 1(i_4 - i_5) = 0$

In loop 5, apply KVL

$5\,i_5 + 4\,i_5 + 5i_0'' + 1(i_5 - i_4) = 20$

Also, $i_0'' = -\,i_5$

Solving these equations, we get

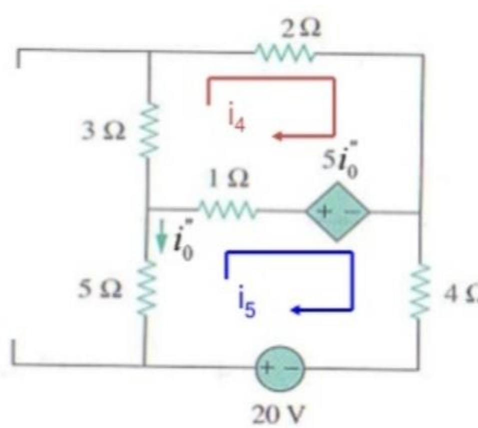
$i_0'' = -60/17A$

Applying superposition theorem,

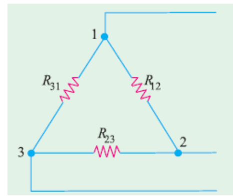
$i_0 = i_0' + i_0'' = 52/17 - 60/17$ A

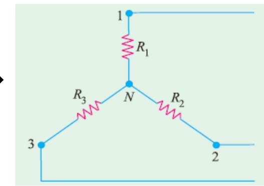
$= -8/17$

Note: Superposition theorem is not applicable in case of power calculation



Proof of Delta-Star Conversion





UNDER NO LOAD SCENARIO:

For Delta resistance between nodes 1 and 2 will be: $R_{12}||(R_{31}+R_{23})$

For Star resistance between nodes 1 and 2 will be: R_1+R_2

So R_1+R_2=$R_{12}||(R_{31}+R_{23})=\frac{R_{12}(R_{31}+R_{23})}{R_{12}+R_{31}+R_{23}}$ (i)

Similarly:

R_2+R_3=$R_{23}||(R_{31}+R_{12})=\frac{R_{23}(R_{31}+R_{12})}{R_{12}+R_{31}+R_{23}}$ (ii) (between nodes 2 and 3 for Delta and Star)

R_1+R_3=$R_{31}||(R_{12}+R_{23})=\frac{R_{31}(R_{12}+R_{23})}{R_{12}+R_{31}+R_{23}}$ (iii) (between nodes 1 and 3 for Delta and Star)

Now, subtracting ***(ii)*** from ***(i)*** and adding the result to ***(iii)***, we get: $R_1=\frac{R_{12}R_{31}}{R_{12}+R_{31}+R_{23}}$

Subtracting ***(iii)*** from ***(ii)*** and adding the result to ***(i)***, we get: $R_2=\frac{R_{12}R_{23}}{R_{12}+R_{31}+R_{23}}$

Subtracting ***(i)*** from ***(iii)*** and adding the result to ***(ii)***, we get: $R_3=\frac{R_{31}R_{23}}{R_{12}+R_{31}+R_{23}}$

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Proof of Star-Delta Conversion

$$R_1=\frac{R_{12}R_{31}}{R_{12}+R_{23}+R_{31}};\ R_2=\frac{R_{23}R_{12}}{R_{12}+R_{23}+R_{31}}\text{ and }R_3=\frac{R_{31}R_{23}}{R_{12}+R_{23}+R_{31}}$$

$$R_1R_2+R_2R_3+R_1R_3=\frac{R_{12}^2R_{31}R_{23}+R_{12}R_{23}^2R_{31}+R_{12}R_{23}R_{31}^2}{(R_{12}+R_{31}+R_{23})(R_{12}+R_{31}+R_{23})}$$

$$R_1R_2+R_2R_3+R_1R_3=\frac{R_{12}R_{31}R_{23}}{(R_{12}+R_{31}+R_{23})}\frac{(R_{12}+R_{23}+R_{31})}{(R_{12}+R_{31}+R_{23})}$$

$$R_1R_2+R_2R_3+R_1R_3=R_3\,R_{12}$$

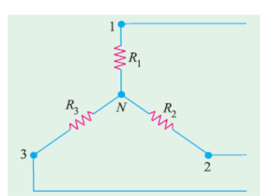
$$R_{12}=\frac{R_1R_2+R_2R_3+R_1R_3}{R_3}$$

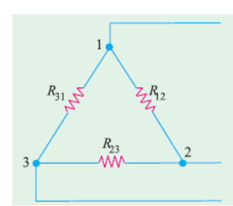
- Similarly, $R_1R_2+R_2R_3+R_1R_3=R_2\,R_{31}$

$$R_{31}=\frac{R_1R_2+R_2R_3+R_1R_3}{R_2}$$

$$R_1R_2+R_2R_3+R_1R_3=R_3\,R_{12}$$

$$R_{12}=\frac{R_1R_2+R_2R_3+R_1R_3}{R_3}$$





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