ELL100: INTRODUCTION TO ELECTRICAL ENGG.

Natural Response: Second Order Circuits

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Second order Circuits

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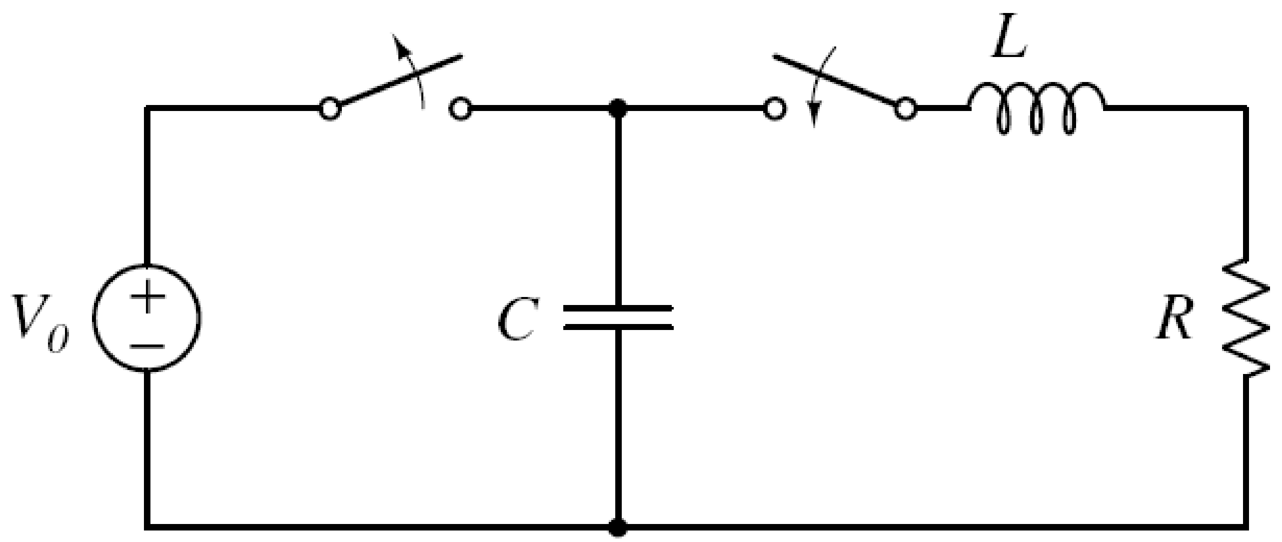
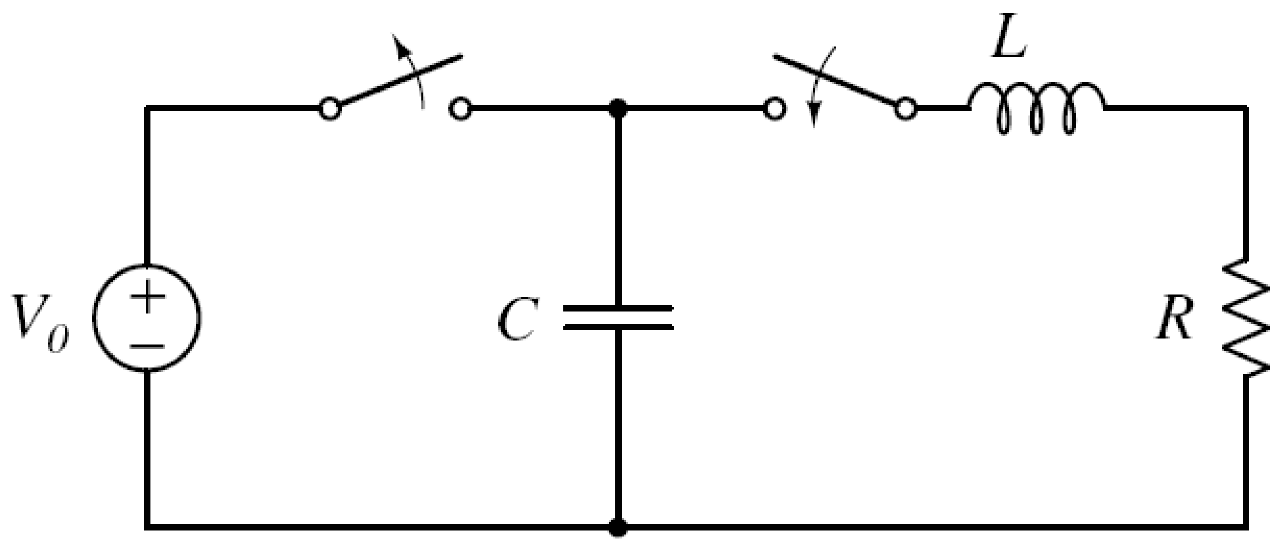


Second order Circuits

- The **state** can be represented using a second order differential equation only.



After the switch is toggled the loop equation becomes :

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = 0$$

Differentiating this equation gives

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

Solution

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- Plugging in the template into the differential equation gives

$$s^2 LAe^{st} + sRAe^{st} + \frac{1}{C}Ae^{st} = 0$$
$$\text{or } Ls^2 + Rs + \frac{1}{C} = 0$$

Solution(s)

- The quadratic equation in ‘s’ has two solutions

$$
\begin{aligned}
s_1 &= -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \\
s_2 &= -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}
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- This gives the general solution as

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

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- The current through the inductor at $t = 0^+, i(0^+) = 0$
- The voltage across the capacitor at $t = 0^+, v_c(0^+) = V_0$
  - This gives $L\frac{di}{dt}|_{t=0^+} = V_0$
- These two conditions can be used to figure out A_1 and A_2.

The Nature of the Roots

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

- The two roots s_1 and s_2 can be in one of the following configurations.
- Real Roots
  - Distinct $L = 1\ H, C = 1/3\ F, R = 4\ \Omega,\ \ s^2 + 4s + 3 = 0,\ \ s = -3, -1$

    $\xi = \frac{R}{2}\sqrt{\frac{C}{L}} > 1$

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- Real Roots
  - Distinct
  - Repeated $L = 0.5\ H, C = 0.5\ F, R = 2\ \Omega,\ \ s^2 + 4s + 4 = 0,\ \ s = -2, -2$

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- The two roots s_1 and s_2 can be in one of the following configurations.
- Real Roots
  - Distinct
  - Repeated
- Complex conjugate pairs of roots.

$$L = 1\ H, C = 1/17\ F, R = 2\ \Omega,\ \ s^2 + 2s + 17 = 0,\ \ s = -1 \pm 4j$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}} < 1$$

Real and Distinct Roots

$$L = 1\ H, C = 1/3\ F, R = 4\ \Omega,$$
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \implies s^2 + 4s + 3 = 0,\ \ s = -3, -1$$

- Thus, the solution is $i(t) = A_1 e^{-t} + A_2 e^{-3t}$

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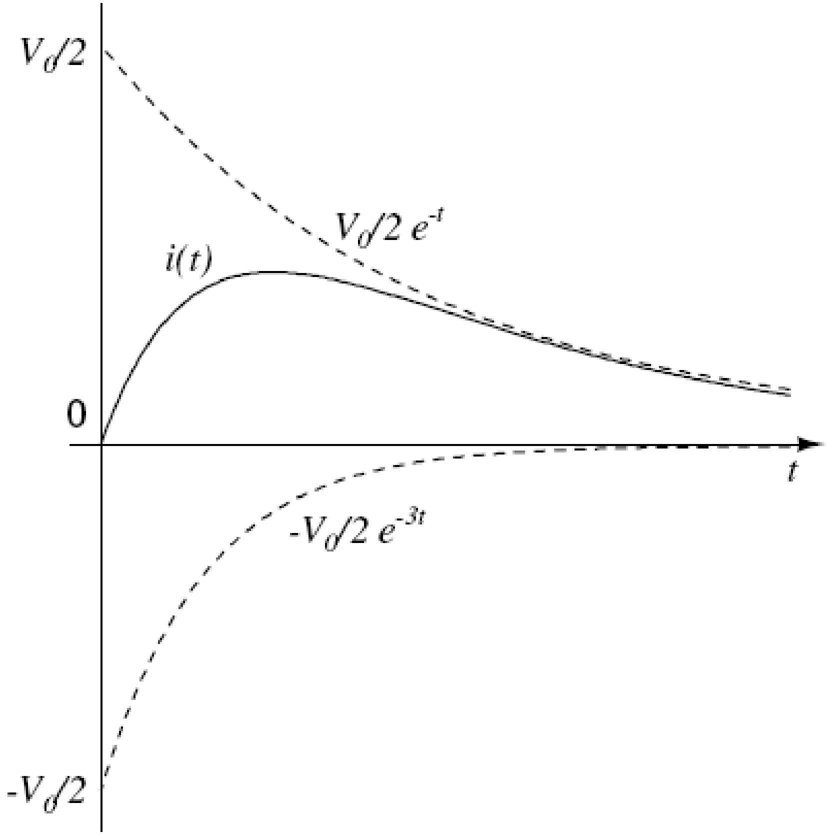
- Initial conditions give
  - Initial current $i(0^+) = 0, \implies A_1 = -A_2$
  - And $L\frac{di}{dt}|_{t=0^+} = V_0, \implies V_0 = -A_1 - 3A_2 = 2A_1$

Real and Distinct Roots: Overdamped Response

- The value of current is thus, $i(t) = \frac{V_0}{2}\left(e^{-t} - e^{-3t}\right)$



Complex Conjugate Roots

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- When $s = -\alpha \pm j\omega, i(t) = e^{-\alpha t}\left(A_1 e^{j\omega t} + A_2 e^{-j\omega t}\right)$
- With A_1 and A_2 being complex numbers.

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- When $s = -\alpha \pm j\omega, i(t) = e^{-\alpha t}\left(A_1 e^{j\omega t} + A_2 e^{-j\omega t}\right)$
- With A_1 and A_2 being complex numbers.
- Because the current is a real quantity eventually,

$$i(t) = Ae^{-\alpha t}\cos\left(\omega t + \theta\right)$$

with A, θ to be calculated from the initial conditions.

For the Example ,

$$L = 1\ H, C = 1/17\ F, R = 2\ \Omega,$$
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \implies s^2 + 2s + 17 = 0,\ \ s = -1 \pm 4j$$

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$$i(t) = Ae^{-\alpha t}\cos\left(\omega t + \theta\right) = Ae^{-t}\cos(4t + \theta)$$

- From, the initial condition

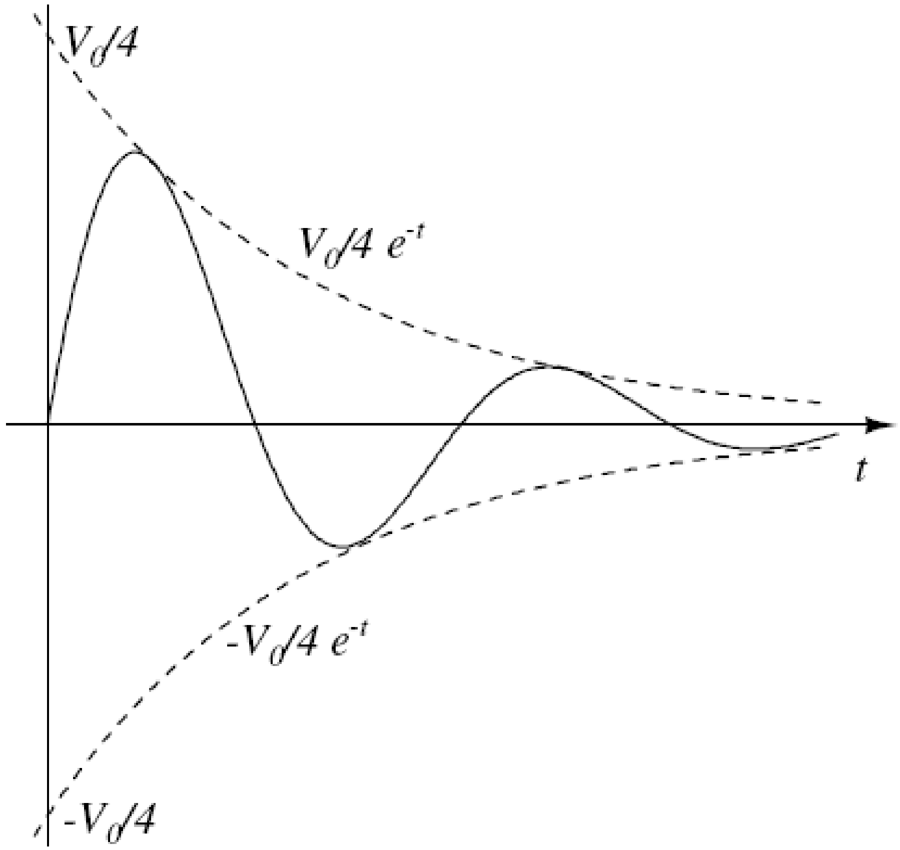
$$i(0) = A\cos(\theta) = 0, \implies \theta = -\pi/2, \implies i(t) = Ae^{-t}\sin(4t)$$

- And

$$L\frac{di}{dt} = \frac{di}{dt}|_{t=0^+} = 4A = V_0$$
$$A = \frac{V_0}{4}$$

Complex Roots: Underdamped Response

- The value of current is thus, $i(t) = \frac{V_0}{4} e^{-t} \sin(4t)$



Repeated Roots

$$L = 0.5\ H, C = 0.5\ F, R = 2\ \Omega,$$
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \implies s^2 + 4s + 4 = 0,\ \ s = -2, -2$$

- In this case the solution is NOT $i(t) = A_1 e^{st} + A_2 e^{st} = (A_1 + A_2)e^{st}$

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- In this case the solution is NOT $i(t) = A_1 e^{st} + A_2 e^{st} = (A_1 + A_2)e^{st}$
- It is not the most general solution for a second order differential equation
- The solution is of the form

$$i(t) = A_1 e^{st} + A_2 t e^{st}$$

For the Example ,

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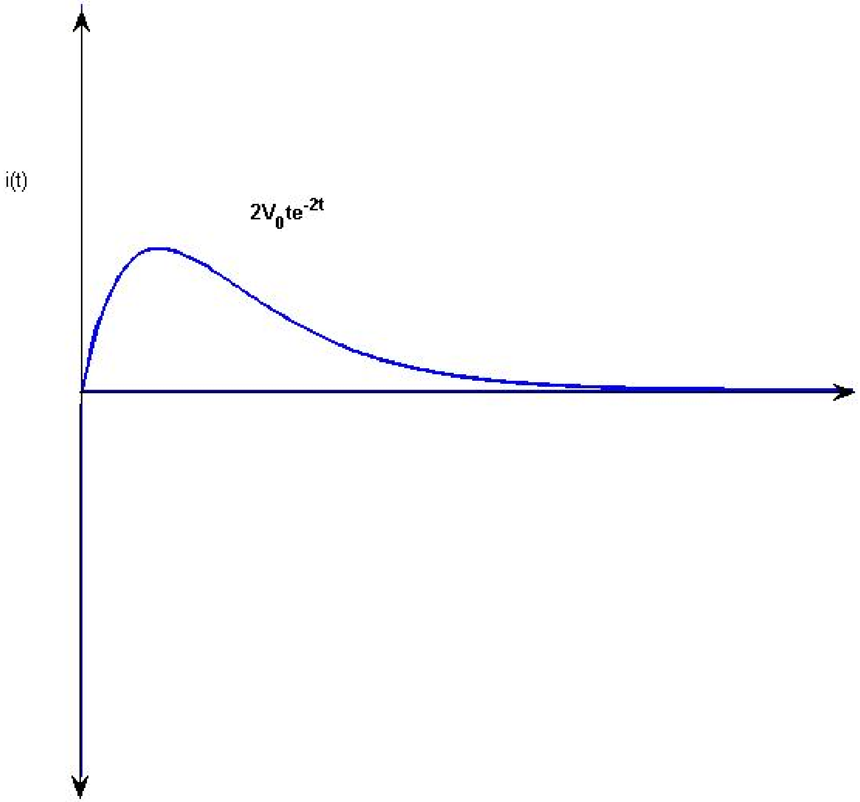
$$i(t) = A_1 e^{-2t} + A_2 t e^{-2t}$$

- From the initial condition
- At $t = 0, i(0) = 0, \implies A_1 = 0$
- Thus, $i(t) = A_2 t e^{-2t},\ \frac{di}{dt} = A_2 e^{-2t} - 2A_2 t e^{-2t}$
- At $t = 0, \frac{di}{dt} = \frac{V_0}{L} = A_2$

Repeated Roots: Critically Damped Response

- The value of current is thus, $i(t) = 2V_0 t e^{-2t}$

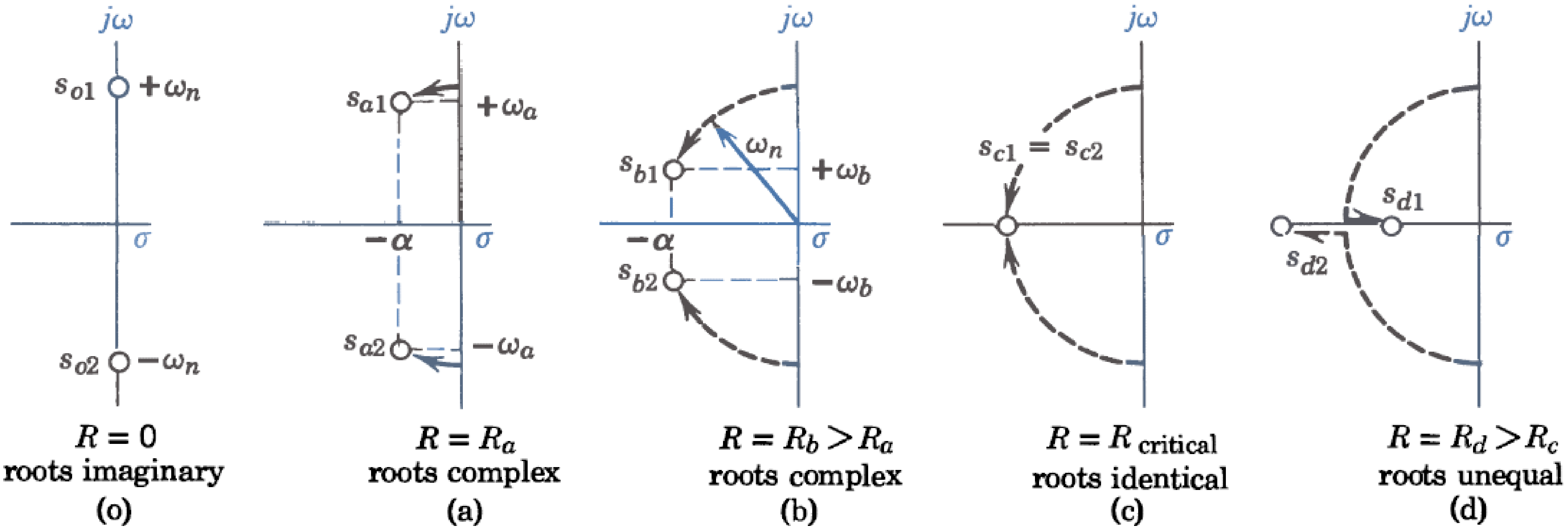


Root Locus

- The two roots s_1 and s_2 change from being imaginary for R=0 to real and distinct as R tends to infinity.

Root Locus Plot

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$



Root Locus

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

- The two roots s_1 and s_2 change from being imaginary for R=0 to real and distinct as R tends to infinity.
- R=0, roots are on the jω axis: Sustained Oscillations (Undamped)
- $0 < R < 2\sqrt{\frac{L}{C}}$, complex roots: Damped Oscillations (Underdamped)
- $R = 2\sqrt{\frac{L}{C}}$, repeated roots: No Oscillations (Critical Damping)
- $R > 2\sqrt{\frac{L}{C}}$, real and distinct roots: No Oscillations (Overdamped)