MTL-100 Lecture-3

Sequences (continued)

Sandwich Theorem (or Squeeze Theorem):

Let (a_n), (b_n) and (c_n) be three sequences such that $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$.
If $\lim_{n\to\infty} a_n = L$ & $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Proof: Let $\varepsilon > 0$ be given.
Since $\lim_{n\to\infty} a_n = L$, $\exists\, N_1 \in \mathbb{N}$ s.t.

$$|a_n - L| < \varepsilon \quad \forall\, n \geq N_1$$

$$\therefore\ n \geq N_1 \Rightarrow L-\varepsilon < a_n < L+\varepsilon \quad \text{---(i)}$$

Similarly, since $\lim_{n\to\infty} c = L$, $\exists\, N_2 \in \mathbb{N}$ s.t

$$n \geq N_2 \Rightarrow L-\varepsilon < c_n < L+\varepsilon \quad \text{---(ii)}$$

If $N = \max\{N_1, N_2\}$. Then for $n \geq N$

$$L-\varepsilon < a_n \leq b_n \leq c_n < L+\varepsilon \quad \forall n \geq N.$$

$$\Rightarrow \quad |b_n - L| < \varepsilon$$

$$\therefore \quad \lim_{n\to\infty} b_n = L.$$

Remark: The above theorem is true if $a_n \leq b_n \leq c_n$ $\forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.

Some applications of Sandwich Theorem

① $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

Pf: We know that $-1 \le \sin(n) \le 1 \ \forall n \in \mathbb{N}$

$\Rightarrow \quad -\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n} \quad \forall n \in \mathbb{N}$.

Since, $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0 = \lim_{n\to\infty} \frac{1}{n}$, by the sandwich theorem $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.

② If $b > 0$, then $\lim_{n\to\infty} b^{1/n} = 1$.

Pf: First assume $b \ge 1$.

Let $a_n = b^{1/n} - 1 \ge 0 \quad \forall n \in \mathbb{N}$.

Then $b = (1 + a_n)^n \ge 1 + n a_n$ (by Binomial theorem),

$\Rightarrow \quad a_n \le \frac{b-1}{n} \quad \forall n \in \mathbb{N}$.

$\therefore \quad 0 \leq a_n \leq \frac{b-1}{n}$

By the sandwich theorem, $\lim_{n\to\infty} a_n = 0$

But $\quad b^{1/n} = a_n + 1$

$\Rightarrow \quad \lim_{n\to\infty} b^{1/n} = 0 + 1 = 1.$

Now if $0 < b < 1$, then $\frac{1}{b} > 1$

So, by the above proof, $\lim_{n\to\infty} \left(\frac{1}{b}\right)^{1/n} = 1$

$\Rightarrow \quad \lim_{n\to\infty} b^{1/n} = 1.$

③ $\boxed{\lim_{n\to\infty} n^{1/n} = 1}$

Proof: Let $a_n = n^{1/n} - 1$

Then $a_n \geq 0$

Also, $n = (1+a_n)^n \geq \frac{n(n-1)}{2} a_n^2$ for $n \geq 2$

(by the Binomial theorem).

$\Rightarrow 0 \leq a_n \leq \sqrt{\frac{2}{n-1}}$ for all $n \geq 2$.

Since $\lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$, by the sandwich thm,

$\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} n^{1/n} = 1$

(4) $\boxed{\lim_{n\to\infty} \frac{\log n}{n} = 0}$

Proof: Let $\varepsilon > 0$ be given.

We need to show that $\exists\, N \in \mathbb{N}$ s.t.

$$-\varepsilon < \frac{\log n}{n} < \varepsilon \quad \forall n \geq N$$

$$\Leftrightarrow \quad -\varepsilon < \log n^{1/n} < \varepsilon \quad \forall n \geq N$$

$$\Leftrightarrow \quad e^{-\varepsilon} < n^{1/n} < e^{\varepsilon} \quad \forall n \geq N.$$

Now, since $\lim_{n\to\infty} n^{1/n} = 1$, $\exists\, N \in \mathbb{N}$ s.t.

$$e^{-\varepsilon} < n^{1/n} < e^{\varepsilon}.$$

(Hint: Use that $e^{\varepsilon} > 1$ and $e^{-\varepsilon} < 1$ for any $\varepsilon > 0$)

Hence, we are done.

(5) If $\alpha > 0$ is any real number, then

$$\boxed{\lim_{n\to\infty} \frac{\log n}{n^{\alpha}} = 0}$$

Proof: First, for any $n \in \mathbb{N}$ $\exists\, m \in \mathbb{N}$ s.t.

$$m \leq n^{\alpha} < m+1 \quad \text{— (i)}$$

(Use the Archimedean property and the well-ordering property).

$$\text{(i)} \Rightarrow m^{1/\alpha} \leq n < (m+1)^{1/\alpha}$$

$$\Rightarrow \log n < \frac{1}{\alpha} \log(m+1) \quad \text{— (ii)}$$

$$\therefore \quad 0 \leq \frac{\log n}{n^{\alpha}} < \frac{1}{\alpha} \frac{\log(m+1)}{m} \quad \text{(by (i) \& (ii))}$$

Now as $n \to \infty$, $m \to \infty$ (by (i))

$$\therefore \lim_{n\to\infty} \frac{1}{\alpha} \frac{\log(m+1)}{m} = \lim_{m\to\infty} \frac{1}{\alpha} \frac{\log(m+1)}{m} = 0$$

Thus, by the sandwich theorem,

$$\lim_{n\to\infty} \frac{\log n}{n^\alpha} = 0 .$$

Subsequences :

Definition: Let $(a_n)_{n=1}^{\infty}$ be a sequence and let $(n_1, n_2, n_3, \cdots)$ be a sequence of natural numbers such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $(a_{n_k})_{k=1}^{\infty}$ is called a subsequence of the sequence $(a_n)_{n=1}^{\infty}$.

$$(a_{n_k})_{k=1}^{\infty} = (a_{n_1}, a_{n_2}, a_{n_3}, \cdots)$$

Example: Let $a_n = (-1)^n$, $n \in \mathbb{N}$.

Then $(a_{2k})_{k=1}^{\infty}$ is the constant seq. $(1)_{n=1}^{\infty}$

and $(a_{2k-1})_{k=1}^{\infty}$ is the constant seq. $(-1)_{n=1}^{\infty}$

$(a_{2k})_{k=1}^{\infty}$ and $(a_{2k-1})_{k=1}^{\infty}$ are subsequences of the sequence $(a_n)_{n=1}^{\infty}$.

Theorem: If a sequence converges to L, then all its subsequences also converge to L.

Proof: Let $(a_n)_{n=1}^{\infty}$ be a sequence converging to L and let $(a_{n_k})_{k=1}^{\infty}$ be a subsequence.

Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = L$,

$\exists\, N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon \quad \forall n \geq N$ (i)

Now note that $n_k \geq k \quad \forall k \in \mathbb{N}$.

So if $k \geq N$, then $n_k \geq k \geq N$.

$\therefore \quad |a_{n_k} - L| < \varepsilon$ (by (i))

Hence, $\lim_{k\to\infty} a_{n_k} = L$.

Example Let $a_n = ((-1)^n)_{n=1}^{\infty}$.

$(a_{2k})_{k=1}^{\infty}$ converges to 1.

$(a_{2k-1})_{k=1}^{\infty}$ converges to -1.

$\therefore$ By the previous thm, the seq. $(a_n)_{n=1}^{\infty}$ does not converge.

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Divergent sequences (Infinite Limits)

Defn: A seq. $(a_n)_{n=1}^{\infty}$ is said to diverge to $+\infty$ if for any real number $M>0$, $\exists\, N\in\mathbb{N}$ s.t. $a_n > M \quad \forall\, n \geq N$.

Notation: $a_n \to +\infty$ or $\lim_{n\to\infty} a_n = +\infty$

(the terms of the sequence are eventually arbitrarily large).

Defn: A sequence $(a_n)_{n=1}^{\infty}$ is said to diverge to $-\infty$ if for any real number $M>0$, $\exists\, N\in\mathbb{N}$ s.t. $a_n < -M \quad \forall\, n\geq N$

Notation: $a_n \to -\infty$ or $\lim_{n\to\infty} a_n = -\infty$

Example: $\boxed{\lim_{n\to\infty} \log\left(\frac{1}{n}\right) = -\infty}$

Proof: Let $M > 0$ be given.

$$\log\left(\frac{1}{n}\right) < -M \iff \frac{1}{n} < e^{-M}$$
$$\iff n > e^{M}.$$

By the Arch. property, $\exists\, N \in \mathbb{N}$ s.t. $N > e^{M}$.

Then for $n \geq N$, $n > e^{M}$

$$\Rightarrow \log\left(\frac{1}{n}\right) < -M.$$

$$\therefore \lim_{n\to\infty} \log\left(\frac{1}{n}\right) = -\infty.$$

Theorem: ① If $\lim\limits_{n\to\infty} a_n = +\infty$ and $\lim\limits_{n\to\infty} b_n = +\infty$,

then $\lim\limits_{n\to\infty}(a_n+b_n) = +\infty$ & $\lim\limits_{n\to\infty}(a_n b_n) = +\infty$

② If $\lim\limits_{n\to\infty} a_n = -\infty$ and $\lim\limits_{n\to\infty} b_n = -\infty$,

then $\lim\limits_{n\to\infty}(a_n+b_n) = -\infty$ and $\lim\limits_{n\to\infty}(a_n b_n) = +\infty$.

③ If $\lim\limits_{n\to\infty} a_n = +\infty$ and $\lim\limits_{n\to\infty} b_n = L \in \mathbb{R}$

then $\lim\limits_{n\to\infty}(a_n+b_n) = +\infty$

$$\lim_{n\to\infty}(a_n b_n) = \begin{cases} +\infty & \text{if } L>0 \\ -\infty & \text{if } L<0 \end{cases}$$

(if $L=0$, not conclusive).

Proof: Left as an exercise.

Example: $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = ?$

$$\sqrt{n+1}-\sqrt{n} = \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$$

(If $\lim_{n\to\infty} a_n = \pm\infty$, then $\lim_{n\to\infty}\frac{1}{a_n} = 0$)

- Difference of two diverging sequences may converge.

Monotone Sequences

Defn: A sequence (a_n) of real numbers is said to be <u>nondecreasing</u> if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$. and <u>nonincreasing</u> if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$.
(<u>increasing seq.</u> & <u>decreasing seq.</u> if the inequalities above are <u>strict</u>.).

Examples:

① $(n^2)_{n=1}^{\infty}$, $\left(1-\frac{1}{n}\right)_{n=1}^{\infty}$ are increasing sequences.

② $\left(\frac{1}{n}\right)_{n=1}^{\infty}$, $\left(\log\frac{1}{n}\right)_{n=1}^{\infty}$ are decreasing sequences.

Defn: A sequence that is either nondecreasing or nonincreasing is called a <u>monotone sequence</u>.

Theorem:

(i) A nondecreasing sequence which is bounded above is convergent.

(ii) A nonincreasing sequence which is bounded below is convergent.

Proof: (i) Let $(a_n)_{n=1}^{\infty}$ be a nondecreasing sequence which is bounded above.

Let $a = \sup \{a_n : n \in \mathbb{N}\}$.

We claim that $\lim_{n \to \infty} a_n = a$.

First, $a_n \leq a \quad \forall n \in \mathbb{N}$.

Now if $\varepsilon > 0$, ~~the~~ $a - \varepsilon$ is not an upper bound for (a_n).

$\therefore \exists N \in \mathbb{N}$ s.t. $a_N > a - \varepsilon$

Since (a_n) is nondecreasing, for $n \geq N$

$a_n \geq a_N > a - \varepsilon$

$\therefore \quad n \geq N \Rightarrow a-\varepsilon < a_n \leq a < a+\varepsilon$

$\therefore \quad \lim_{n\to\infty} a_n = a.$

(ii) $\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$

This can be proved in a similar way as (i).

Alternatively, if we take $b_n = -a_n$.

Then since (a_n) is nonincr., (b_n) is nondecr.

Also, (a_n) is bounded below

$\Rightarrow$ (b_n) is bounded above.

By (i) $\lim_{n\to\infty} b_n = \sup\{b_n : n \in \mathbb{N}\}$

$= \sup\{-a_n : n \in \mathbb{N}\}$

$\Rightarrow \quad -\lim_{n\to\infty} a_n = -\inf\{a_n : n \in \mathbb{N}\}$

$\Rightarrow \quad \lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$

Example: If $0<b<1$, then $\lim_{n\to\infty} b^n = 0$

Proof: Let $a_n = b^n$, $n \in \mathbb{N}$.

Then $a_{n+1} = b^{n+1} = b.b^n < b^n = a_n$ $\forall n \in \mathbb{N}$

$\Rightarrow (a_n)_{n=1}^{\infty}$ is a decreasing sequence.

Also, since $b>0$, $a_n = b^n > 0$ $\forall n$.

$\therefore (a_n)_{n=1}^{\infty}$ is bounded below.

By the above thm., $\lim_{n\to\infty} b^n$ exists.

Let $L = \lim_{n\to\infty} a_n$

Then $\lim_{n\to\infty} a_{n+1} = L$

$\therefore L = \lim_{n\to\infty} b^{n+1} = b \lim_{n\to\infty} b^n = b.L$

$\Rightarrow L(1-b) = 0 \Rightarrow L = 0$ $(\because b \neq 1)$