MTL100 - Calculus

Tutorial 1

Department of Mathematics
Indian Institute of Technology Delhi

Question 1

Question 1(i)

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By definition $A + B = \{a + b : a \in A, b \in B\}$.

- Let us assume that $\inf(A) = \alpha, \inf(B) = \beta$ and $\inf(A+B) = \gamma$. Then we need to prove that $\alpha + \beta = \gamma$.

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- Let us assume that $\inf(A) = \alpha, \inf(B) = \beta$ and $\inf(A+B) = \gamma$. Then we need to prove that $\alpha + \beta = \gamma$.
- Since α and β are lower bounds of A and B respectively, we have that $\alpha \leq a$ and $\beta \leq b\ \forall\ a \in A, b \in B$.

Question 1(i) Contd...

- It implies that for any $c = a + b \in A + B, \alpha + \beta \leq a + b$.
- Therefore, $\alpha + \beta$ is a lower bound of $A + B$.
- Since γ is the greatest lower bound of $A + B$, we have that $\alpha + \beta \leq \gamma$.

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- Therefore, $\alpha + \beta$ is a lower bound of $A + B$.
- Since γ is the greatest lower bound of $A + B$, we have that $\alpha + \beta \leq \gamma$.

Now, we prove the reverse inequality $\alpha + \beta \geq \gamma$.

- Let $\epsilon > 0$ be a real number.
- Since α and β are infimum of A and B, respectively, it follows from the definition of infimum that we can find some $x \in A$ and $y \in B$ such that $x < \alpha + \dfrac{\epsilon}{2}$ and $y < \beta + \dfrac{\epsilon}{2}$.
- It further implies that $x + y < \alpha + \beta + \epsilon$ and since γ is a lower bound of $A + B$, we have $\gamma \leq x + y < \alpha + \beta + \epsilon$.
- As $\epsilon > 0$ was arbitrary, we get that $\gamma \leq \alpha + \beta$.
- Thus $\alpha + \beta = \gamma$.

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Thus one can say that for any $\epsilon > 0 \; \exists \; z \in S$ such that $u - \epsilon < z$.

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By definition $A + B = \{a + b : a \in A, b \in B\}$.

- Suppose $\sup(A) = \alpha, \sup(B) = \beta$ and $\sup(A+B) = \gamma$. Then we need to prove that $\alpha + \beta = \gamma$.

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- u is an upper bound of S.
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Thus one can say that for any $\epsilon > 0 \ \exists \ z \in S$ such that $u - \epsilon < z$.
By definition $A + B = \{a + b : a \in A, b \in B\}$.

- Suppose $\sup(A) = \alpha, \sup(B) = \beta$ and $\sup(A+B) = \gamma$. Then we need to prove that $\alpha + \beta = \gamma$.
- Since α and β are upper bounds of A and B respectively, we have that $\alpha \geq a$ and $\beta \geq b \ \forall \ a \in A, b \in B$.

Question 1.(ii) Contd...

- Thus for any $c = a + b \in A + B, \alpha + \beta \geq a + b$, which implies that $\alpha + \beta$ is an upper bound of $A + B$.
- Since γ is the least upper bound of $A + B, \alpha + \beta \geq \gamma$.

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- Thus for any $c = a + b \in A + B, \alpha + \beta \geq a + b$, which implies that $\alpha + \beta$ is an upper bound of $A + B$.
- Since γ is the least upper bound of $A + B, \alpha + \beta \geq \gamma$.

Now, we prove the reverse inequality $\boldsymbol{\alpha + \beta \leq \gamma}$.

- Let $\epsilon > 0$ be any real number.
- Since α and β are the supremum of A and B, respectively, we can find some $x \in A$ and $y \in B$ such that $x > \alpha - \dfrac{\epsilon}{2}$ and $y > \beta - \dfrac{\epsilon}{2}$.
- It implies that $x + y > \alpha + \beta - \epsilon$. Since γ is also an upper bound of $A + B$, we get that $\gamma \geq x + y > \alpha + \beta - \epsilon$, and so $\gamma + \epsilon > \alpha + \beta$.
- Since $\epsilon > 0$ was arbitrary, we get that $\gamma \geq \alpha + \beta$.
- Thus $\alpha + \beta = \gamma$ and the result follows.

Question 2

Question

Let $r \in \mathbb{R}$. Prove that there exists a sequence $\{x_n\}$ of rational numbers such that $\lim_{n\to\infty} x_n = r$.

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Solution:

- If r itself is a rational number, then we can take $\{x_n\}$ to be a constant sequence i.e. $x_n = r \ \forall \ n \in \mathbb{N}$.
- Otherwise, it follows from the density property of rational numbers that for any two real numbers a and $b \in \mathbb{R}$, with $a < b$, there exists a rational number q such that $a < q < b$.
- Then for the numbers $a = r - 1$ and $b = r + 1$, we have a rational number x_1 such that $r - 1 < x_1 < r + 1$.

Question 2 Contd...

- Now for $a = r - \frac{1}{2}$ and $b = r + \frac{1}{2}$, we have a rational number x_2 such that $r - \frac{1}{2} < x_2 < r + \frac{1}{2}$.
- Similarly, we have rational numbers $x_3, x_4, \ldots, x_n$ such that for any $n \in \mathbb{N}, r - \frac{1}{n} < x_n < r + \frac{1}{n}$.

Question 2 Contd...

- Now for $a = r - \frac{1}{2}$ and $b = r + \frac{1}{2}$, we have a rational number x_2 such that $r - \frac{1}{2} < x_2 < r + \frac{1}{2}$.
- Similarly, we have rational numbers $x_3, x_4, \ldots, x_n$ such that for any $n \in \mathbb{N}, r - \frac{1}{n} < x_n < r + \frac{1}{n}$.
- Now consider the sequence $\{x_n\}$ of rational numbers. Let us check whether the sequence $\{x_n\}$ converges to r or not.
- Note that $r - \frac{1}{n} < x_n < r + \frac{1}{n}$ implies that $|x_n - r| < \frac{1}{n} \ \forall \ n \in \mathbb{N}$.
- It follows from the Archimedean Property that for any $\epsilon > 0 \ \exists$ some $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.
- Hence $\forall \ n \geq N, \frac{1}{n} \leq \frac{1}{N} < \epsilon$ and so $|x_n - r| < \frac{1}{n} < \epsilon \ \forall \ n \geq N$. Therefore, $\{x_n\}$ converges to r.

Question 3

Question 3(i)

Suppose $\{a_n\}$ is a sequence of real numbers such that its subsequences $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n-1}\}_{n=1}^{\infty}$ both converge to the same limit. Then show that$\{a_n\}$ converges to the same limit.

Solution:

- We are given that both subsequences $\{a_{2n}\}$ and $\{a_{2n-1}\}$ converge to the same limit, say, L. Let $\epsilon > 0$ be given.
- Then there exists some $N_1 \in \mathbb{N}$ such that

$$|a_{2n} - L| < \epsilon \text{ for all } n > N_1.$$

- Similarly, there exists some $N_2 \in \mathbb{N}$ such that

$$|a_{2n-1} - L| < \epsilon \text{ for all } n > N_2.$$

Question 3(i) contd...

- Let $N = \max\{2N_1, 2N_2 - 1\}$. Assume that $n > N$ so that $n > 2N_1$ and $n > 2N_2 - 1$.

Question 3(i) contd...

- Let $N = \max\{2N_1, 2N_2 - 1\}$. Assume that $n > N$ so that $n > 2N_1$ and $n > 2N_2 - 1$.
- If n is even, then $n = 2m$ for some integer m. Hence $n = 2m > 2N_1$ implies that $m > N_1$ and so

$$|a_n - L| = |a_{2m} - L| < \epsilon$$

- If n is odd, then $n = 2k - 1$ for some integer k. Then we have $2k - 1 > 2N_2 - 1$, which further implies that $k > N_2$. Hence

$$|a_n - L| = |a_{2k-1} - L| < \epsilon.$$

Question 3(i) contd...

- Let $N = \max\{2N_1, 2N_2 - 1\}$. Assume that $n > N$ so that $n > 2N_1$ and $n > 2N_2 - 1$.
- If n is even, then $n = 2m$ for some integer m. Hence $n = 2m > 2N_1$ implies that $m > N_1$ and so

$$|a_n - L| = |a_{2m} - L| < \epsilon$$

- If n is odd, then $n = 2k - 1$ for some integer k. Then we have $2k - 1 > 2N_2 - 1$, which further implies that $k > N_2$. Hence

$$|a_n - L| = |a_{2k-1} - L| < \epsilon.$$

- Thus $|a_n - L| < \epsilon$, whenever $n > N$ and so $\{a_n\}$ converges to L.

Question 3

Question 3(ii)

If $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is another sequence which converges to 0, show that the product sequence also converges to 0. What can you say about the product sequence, if $\{b_n\}$ converges, but to a non-zero point?

Solution:

- Since $\{a_n\}$ is a bounded sequence, there exists a real number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.
- Since $\{b_n\}$ converges to zero, it implies that for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|b_n - 0| = |b_n| < \frac{\epsilon}{M} \text{ for all } n > N$$

Question 3(ii) Contd...

Now note that $|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| \leq M|b_n| < \epsilon \ \forall \ \ n > N$.
Hence $\{a_n b_n\}$ converges to 0.

Now note that $|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| \leq M|b_n| < \epsilon \ \forall \ \ n > N$.
Hence $\{a_n b_n\}$ converges to 0.
For the second part, observe that

- If we choose $a_n = (-1)^n$ and $b_n = 1$, then, clearly, a_n is bounded and b_n converges to 1 $(\neq 0)$. But in this case $a_n b_n = (-1)^n$, which doesn't converge.
- Further, if $a_n = 1$ and $b_n = 1$ for all n, then $a_n b_n = 1$. So $\{a_n b_n\}$ converges to 1.
  Hence $\{a_n b_n\}$ may or may not be convergent, if $\{b_n\}$ converges to a non-zero limit.

Question 4

Question

Let $\{a_n\}$ be a sequence of real numbers. Define the sequence $\{s_n\}$ by $s_n = \frac{1}{n}\sum_{i=1}^{n} a_i$.

(i) If $\{a_n\}$ is monotone and bounded, then show that $\{s_n\}$ is also monotone and bounded.

(ii) If $\{a_n\}$ converges to a, then show that the sequence $\{s_n\}$ also converges to a.

Question 4

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Let $\{a_n\}$ be a sequence of real numbers. Define the sequence $\{s_n\}$ by $s_n = \frac{1}{n}\sum_{i=1}^{n} a_i$.

(i) If $\{a_n\}$ is monotone and bounded, then show that $\{s_n\}$ is also monotone and bounded.

(ii) If $\{a_n\}$ converges to a, then show that the sequence $\{s_n\}$ also converges to a.

Solution: (i) Assume that $\{a_n\}$ is monotone and bounded. We show that the sequence $\{s_n\}$ is monotone and bounded.

Question 4(i) Contd...

First, we show that $\{s_n\}$ is a monotone sequence. For $\forall n \in \mathbb{N}$, consider

$$
\begin{aligned}
s_{n+1} - s_n &= \sum_{i=1}^{n+1} \frac{a_i}{n+1} - \sum_{i=1}^{n} \frac{a_i}{n} \\
&= \frac{a_1 + \cdots + a_{n+1}}{n+1} - \frac{a_1 + \cdots + a_n}{n} \\
&= \frac{n(a_1 + \cdots + a_{n+1}) - (n+1)(a_1 + \cdots + a_n)}{n(n+1)} \\
&= \frac{na_{n+1} - a_1 - \cdots - a_n}{n(n+1)} \\
&= \frac{(a_{n+1} - a_1) + \cdots + (a_{n+1} - a_n)}{n(n+1)}.
\end{aligned}
$$

Since $\{a_n\}$ is a monotone sequence, $\{a_n\}$ is either a nondecreasing sequence or a nonincreasing sequence.

- If $\{a_n\}$ is nondecreasing, i.e. $\forall n \in \mathbb{N}, a_n \leq a_{n+1}$, then we get that $s_{n+1} - s_n \geq 0$, $\forall n \in \mathbb{N}$, and hence, $\{s_n\}$ is a nondecreasing sequence.
- If $\{a_n\}$ is nonincreasing, i.e. $\forall n \in \mathbb{N}, a_n \geq a_{n+1}$, then we get that $s_{n+1} - s_n \leq 0$, $\forall n \in \mathbb{N}$, and hence, $\{s_n\}$ is nonincreasing sequence.
- Hence, $\{s_n\}$ is a monotone sequence.

Next, we show that $\{s_n\}$ is a bounded sequence.

- Since $\{a_n\}$ is a bounded sequence, $\exists\ M > 0$ such that

$$|a_n| \leq M, \ \forall n \in \mathbb{N}.$$

- For $n \in \mathbb{N}$, consider

$$\begin{aligned} |s_n| &= \left| \frac{a_1 + a_2 + \cdots + a_n}{n} \right| \\ &\leq \frac{|a_1| + |a_2| + \cdots + |a_n|}{n} && \{\textit{by triangle inequality}\} \\ &\leq \frac{nM}{n} = M. && \{|a_n| \leq M, \ \forall n \in \mathbb{N}\} \end{aligned}$$

- Hence, $\{s_n\}$ is a bounded sequence.

Question 4(ii)

(ii) If $\{a_n\}$ converges to a, then $\{s_n\}$ also converges to a.
Let $\epsilon > 0$ be arbitrary. Consider,

$$\begin{aligned} |s_n - a| &= \left| \sum_{i=1}^{n} \frac{a_i}{n} - a \right| \\ &= \left| \frac{a_1 + \cdots + a_n}{n} - a \right| \\ &= \left| \frac{a_1 + \cdots + a_n - na}{n} \right| \\ &= \left| \frac{(a_1 - a) + \cdots + (a_n - a)}{n} \right| \leq \sum_{i=1}^{n} \frac{|a_i - a|}{n}. \end{aligned} \tag{1}$$

Since a_n converges to a, $\exists\ n_0 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$, $\forall n \geq n_0$. Let $\alpha = max\{|a_1 - a|, |a_2 - a|, \ldots, |a_{n_0 - 1} - a|\}$.

Question 4(ii) Contd...

Now, for $n \geq n_0$, we have

$$\begin{aligned}
|s_n - a| &\leq \sum_{i=1}^{n_0-1} \frac{|a_i - a|}{n} + \sum_{i=n_0}^{n} \frac{|a_i - a|}{n} \qquad \{\text{by Eq. (1)}\} \\
&\leq \frac{\alpha(n_0 - 1)}{n} + (n - n_0 + 1)\frac{\epsilon}{2n} \\
&< \frac{\alpha n_0}{n} + \frac{\epsilon}{2}. \qquad (2)
\end{aligned}$$

By Archimedian principle, $\exists n_1 \in \mathbb{N}$ such that $\frac{2\alpha n_0}{\epsilon} < n_1$. Thus, for $n \geq n_1$, we have $\frac{\alpha n_0}{n} < \frac{\epsilon}{2}$. By Eq. (2), we have

$$|s_n - a| < \epsilon, \quad \text{for } n \geq \max\{n_0, n_1\}.$$

Hence, $\{s_n\}$ converges to a.

Question 5

Question

Let $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and let $t_n = \max\{a_n, b_n\}$, $s_n = \min\{a_n, b_n\}$. Show that $\{t_n\}$ and $\{s_n\}$ are convergent and

$$\lim_{n\to\infty} t_n = \max\{a, b\}, \quad \lim_{n\to\infty} s_n = \min\{a, b\}.$$

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$$\lim_{n\to\infty} t_n = \max\{a, b\}, \quad \lim_{n\to\infty} s_n = \min\{a, b\}.$$

Solution: Observe that for any two real number x and y, we have

$$\max\{x, y\} := \frac{x+y}{2} + \frac{|x-y|}{2},$$

and

$$\min\{x, y\} := \frac{x+y}{2} - \frac{|x-y|}{2}.$$

Question 5 Contd...

Recall result: Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$. Then, the following holds:

1. $\lim\limits_{n\to\infty} (x_n + y_n) = x + y$,
2. $\lim\limits_{n\to\infty} (x_n - y_n) = x - y$,
3. $\lim\limits_{n\to\infty} (cx_n) = cx$, $\forall c \in \mathbb{R}$,
4. $\lim\limits_{n\to\infty} |x_n| = |x|$.

Question 5 Contd...

Using this result, we get

$$\begin{aligned}\lim_{n\to\infty} t_n &= \lim_{n\to\infty}\left(\frac{a_n+b_n}{2}+\frac{|a_n-b_n|}{2}\right)\\ &= \lim_{n\to\infty}\frac{a_n+b_n}{2}+\lim_{n\to\infty}\frac{|a_n-b_n|}{2}\\ &= \frac{a+b}{2}+\frac{|a-b|}{2}\\ &= \max\{a,b\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\frac{a_n+b_n}{2}-\lim_{n\to\infty}\frac{|a_n-b_n|}{2}\\ &= \frac{a+b}{2}-\frac{|a-b|}{2}\\ &= \min\{a,b\}.\end{aligned}$$

Question 6

Question

If $a_1 \geq 0$ and for $n \geq 1$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, then show that the sequence $\{a_n\}_{n\geq 2}$ is nonincreasing and bounded. Also, find its limit.

Question 6

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Solution:

- Clearly, $a_n > 0$ for all n.
- Note that for all $n \in \mathbb{N}$,

$$\begin{aligned} a_n - a_{n+1} &= a_n - \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \\ &= \frac{a_n}{2} - \frac{1}{a_n} = \frac{a_n^2 - 2}{2a_n}. \end{aligned}$$

- For all $n \geq 2$, $a_n - a_{n+1} \geq 0$ if and only if $a_n^2 - 2 \geq 0$, since $a_n > 0$.
- Therefore, $\{a_n\}$ is nonincreasing if and only if $a_n \geq \sqrt{2}$ for all $n \geq 2$.

Question 6 Contd...

- Now, for $n \geq 1$,

$$\begin{array}{rrcl} & a_{n+1} & \geq & \sqrt{2} \\ \text{if and only if} & \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) & \geq & \sqrt{2} \\ \text{if and only if} & a_n^2 - 2\sqrt{2}a_n + 2 & \geq & 0 \\ \text{if and only if} & (a_n - \sqrt{2})^2 & \geq & 0. \end{array}$$

- Since $(a_n - \sqrt{2})^2 \geq 0$, $\{a_n\}_{n=2}^{\infty}$ is a nonincreasing sequence.
- $\{a_n\}$ is bounded below by $\min\{a_1, \sqrt{2}\}$ and bounded above by $\max\{a_1, a_2\}$.
- Hence, $\{a_n\}$ is a bounded sequence.

Question 6 Contd...

- We know that a bounded monotone sequence is convergent. Therefore, $\{a_n\}$ is a convergent sequence. So, let $L = \lim_{n\to\infty} a_n$.
- Note that $L \geq \sqrt{2}$ and $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \lim_{n\to\infty} \frac{a_n}{2} + \lim_{n\to\infty} \frac{1}{a_n}$.
- Therefore, we get

$$L = \frac{L}{2} + \frac{1}{L} \text{ which implies that } L^2 = 2.$$

- Since $a_n > 0$, $\forall n \in \mathbb{N}$, we have $\lim_{n\to\infty} a_n \geq 0$.
- Hence, $L = \sqrt{2}$

Question 7

Question

For $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = \dfrac{1}{4}(x_n^2 + 3)$ for all $n \geq 1$. Examine the convergence of the sequence $\{x_n\}$ for different values of a. Also, find $\lim\limits_{n\to\infty} x_n$ whenever it exists.

Question 7

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Solution:

- If the sequence $\{x_n\}$ converges, the limit of the sequence $l = \lim_{n\to\infty} x_n$ satisfies $l^2 - 4l + 3 = 0$. (why?)
- So the only possible limits are $l = 1$ or $l = 3$.
- We have,

$$x_2 - x_1 = \frac{1}{4}(a-1)(a-3)$$

and

$$x_{n+1} - x_n = \frac{1}{4}(x_n^2 - x_{n-1}^2) \quad \text{for all } n \geq 2. \tag{3}$$

Question 7 Contd...

Depending upon the values of a, we will discuss the following cases:

- **Case I** $(a > 3)$
  If $a > 3$ then $x_2 > x_1$ and consequently, $x_{n+1} > x_n$ for all n. If the sequence $\{x_n\}$ converges, we have
  $$l = \lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\} \geq x_1 = a > 3,$$
  which is impossible as the possible limits are 1 or 3.
- **Case II** (a=3)
  If $a = 3$, then $x_n = 3$ for all $n \in \mathbb{N}$. In this case $\{x_n\}$ is a constant sequence converging to 3.

- **Case III** $(1 < a < 3)$
  In this case,

$$x_2 - x_1 = \frac{1}{4}(a-1)(a-3) < 0 \implies x_2 < x_1.$$

Then from (3) we have $x_{n+1} < x_n$ for all $n \in \mathbb{N}$, that is, $\{x_n\}$ is a non-increasing sequence. Observe that $x_n > 1$ and by induction, we get

$$x_{n+1} - 1 = \frac{1}{4}(x_n^2 - 1) > 0 \text{ for all } n \in \mathbb{N}. \tag{4}$$

Hence $\{x_n\}$ converges and converges to 1 as,

$$l = \lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\} \le x_1 = a < 3.$$

Question 7 Contd...

- **Case IV** $(0 \leq a \leq 1)$
  In this case, we get $x_2 \geq x_1$ and subsequently, $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. So $\{x_n\}$ is a non-decreasing sequence bounded above by 1 (see (4)). Hence $\{x_n\}$ converges to 1.
- **Case V** $(a < 0)$
  Note that, if $a < 0$, we have $-a > 0$. Since x_2 is same irrespective of whether we choose $x_1 = a$ or $x_1 = -a$, we can replace $x_1 = -a$ and proceed as the above cases. So we can conclude the following:
  - for $-1 \leq a \leq 0, x_n \to 1$,
  - for $-3 < a < -1, x_n \to 1$,
  - for $a = -3, x_n \to 3$,
  - for $a < -3$, $\{x_n\}$ does not converge.

Question 8

Question

Prove or disprove that the sequence $\sum_{k=0}^{n} \frac{1}{(n+k)^2}$ converges to 0.

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Solution : Recall the following theorem:

Theorem (Sandwich theorem)

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

- Note that, $\forall\ n$,

$$\frac{1}{n^2} \leq \sum_{k=0}^{n} \frac{1}{(n+k)^2} \leq \frac{n+1}{n^2}.$$

- Since

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 \text{ and } \lim_{n\to\infty} \frac{n+1}{n^2} = 0,$$

by Sandwich theorem, $\sum_{k=0}^{n} \frac{1}{(n+k)^2} \to 0.$

Question 9

Question

Check if the following sequences are Cauchy sequences or not.

(a) $a_n = \sum_{k=1}^{n} \frac{1}{k!}$ for $n \in \mathbb{N}$

(b) $a_1 = 1, a_{n+1} = (1 + \frac{(-1)^n}{2^n})a_n$ for $n \in \mathbb{N}$

(c) $x_1 = a, x_2 = b, x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n \geq 3$

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Solution (a): Without loss of generality let us assume that, $n \geq m$. Then,

$$\begin{aligned}
|a_n - a_m| &= \left| \sum_{k=m+1}^{n} \frac{1}{k!} \right| \\
&= \sum_{k=m+1}^{n} \frac{1}{k!} \leq \sum_{k=m+1}^{n} \frac{1}{2^{k-1}} \quad (\text{since } k! \geq 2^{k-1} \text{ for each } k) \\
&= \frac{1}{2^m} \sum_{k=0}^{n-m} \frac{1}{2^k} \leq \frac{1}{2^m} \times 2.
\end{aligned}$$

- Now for $\epsilon > 0$,

$$\frac{1}{2^{m-1}} < \epsilon \iff \frac{1}{\epsilon} < 2^{m-1} \iff \frac{2}{\epsilon} < 2^m \iff m > \log_2\left(\frac{2}{\epsilon}\right).$$

- Choose $n_0 = \left\lceil \log_2\left(\frac{2}{\epsilon}\right)\right\rceil$. Then for all $n, m \geq n_0$ we have $|a_n - a_m| < \epsilon$.
- Hence $\{a_n\}$ is a Cauchy sequence.

Question 9(b)

Solution (b): $a_1 = 1, a_{n+1} = \left(1 + \frac{(-1)^n}{2^n}\right) a_n$ for $n \in \mathbb{N}$.

$$
\begin{aligned}
a_{n+1} &= \left(1 + \frac{(-1)^n}{2^n}\right)\left(1 + \frac{(-1)^{n-1}}{2^{n-1}}\right)\cdots\left(1 + \frac{-1}{2}\right) a_1 \\
&\leq \left(1 + \frac{1}{2^n}\right)\left(1 + \frac{1}{2^{n-1}}\right)\cdots\left(1 + \frac{1}{2}\right) \\
&\leq \left(\frac{1}{n}\left(n + \sum_{i=1}^{n} \frac{1}{2^i}\right)\right)^n \text{(By AM-GM inequality)} \\
&\leq \left(1 + \frac{1}{n}\right)^n < 3 \quad \textit{(why?)}
\end{aligned}
$$

Note that,

$$
|a_{n+1} - a_n| = \left| a_n + \frac{(-1)^n}{2^n} a_n - a_n \right| \leq \frac{1}{2^n} a_n < \frac{3}{2^n}.
$$

Question 9(b) Contd...

- Then, for $n > m$,

$$\begin{aligned} |a_n - a_m| &\leq |a_n - a_{n-1}| + \cdots + |a_{m+1} - a_m| \\ &\leq \frac{3}{2^{n-1}} + \frac{3}{2^{n-2}} + \cdots + \frac{3}{2^m} \\ &\leq \frac{3}{2^m}\left(\frac{1}{2^{(n-1)-m}} + \cdots + \frac{1}{2} + 1\right) \\ &\leq \frac{3}{2^m} \times 2 \\ &= \frac{3}{2^{m-1}}. \end{aligned}$$

- Now for $\epsilon > 0$,

$$\frac{3}{2^{m-1}} < \epsilon \iff \frac{6}{\epsilon} < 2^m \iff m > \log_2\left(\frac{6}{\epsilon}\right).$$

- Choose, $n_0 = \left\lceil \log_2\left(\frac{6}{\epsilon}\right) \right\rceil$. Then $|a_n - a_m| < \epsilon$ for all $n, m \geq n_0$.

Question 9(c)

Solution(c): $x_1 = a, x_2 = b, x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n \geq 3$.

$$\begin{aligned}
|x_n - x_{n-1}| &= \left|\frac{1}{2}(x_{n-1} + x_{n-2}) - x_{n-1}\right| \\
&\leq \frac{1}{2}|x_{n-1} - x_{n-2}| \\
&\leq \frac{1}{2^{n-2}}|x_2 - x_1| \\
&\leq \frac{1}{2^{n-2}}|b - a|.
\end{aligned}$$

For $n \geq m$,

$$\begin{aligned}
|x_n - x_m| &\leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \\
&\leq \frac{1}{2^{n-2}}|b - a| + \cdots + \frac{1}{2^{m-1}}|b - a|
\end{aligned}$$

$$= \frac{1}{2^{m-1}}|b-a|\left(\frac{1}{2^{(n-2)-(m-1)}}+\frac{1}{2^{(n-3)-(m-1)}}+\cdots+\frac{1}{2}+1\right)$$
$$\leq \frac{1}{2^{m-1}}|b-a| \times 2.$$

Let $\epsilon > 0$.

- For $a \neq b$ we have,

$$\frac{1}{2^{m-2}}|b-a| < \epsilon \iff \frac{1}{2^m} < \frac{\epsilon}{4|b-a|} \iff m > \log_2\left(\frac{4|b-a|}{\epsilon}\right).$$

- For $a = b$ we have,

$$\frac{1}{2^{m-2}}|b-a| = 0 < \epsilon \text{ for all } m.$$

Question 9(c) Contd...

- Choose n_0 as,

$$n_0 = \begin{cases} \left\lceil \log_2 \left(\frac{4|b-a|}{\epsilon} \right) \right\rceil & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases}.$$

Then we have, $|x_n - x_m| < \epsilon$ for all $n, m \geq n_0$.

Question no. 10

Question

Find the limit superior and the limit inferior for the sequence

$$\left\{(-1)^n(1+\frac{1}{n})\right\}_{n=1}^{n=\infty}$$

.

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.

Solution:

- Recall the definition of limit superior,

$$\limsup_{n\to\infty} a_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} a_n.$$

- Denote, $\alpha_k = \sup\limits_{n\geq k} a_n = \sup\{(-1)^n(1+\frac{1}{n}) : n \geq k\}$.

Question 10 Contd...

- Then, some terms of the sequence $\{\alpha_k\}$ are following:

$$\alpha_1 = \sup_{n\geq 1} a_n = 1 + 1/2$$

$$\alpha_2 = \sup_{n\geq 2} a_n = 1 + 1/2$$

$$\alpha_3 = \sup_{n\geq 3} a_n = 1 + 1/4$$

$$\alpha_4 = \sup_{n\geq 4} a_n = 1 + 1/4$$

$$\vdots$$

$$\alpha_k = \sup_{n\geq k} a_n = \begin{cases} 1 + \frac{1}{k+1}, & k \text{ is odd} \\ 1 + \frac{1}{k}, & k \text{ is even} \end{cases}$$

- Thus $\inf_{k\in\mathbb{N}} \alpha_k = 1$ (why?), and hence $\limsup_{n\to\infty} a_n = 1$.

Question 10 Contd...

- Recall the definition of limit inferior,

$$\liminf_{n\to\infty} a_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} a_n.$$

- Denote, $\beta_k = \inf_{n\geq k} a_n = \inf\{(-1)^n(1+\frac{1}{n}) : n \geq k\}$.
- Then, some terms of the sequence $\{\beta_k\}$ are following:

$$\beta_1 = \inf_{n\geq 1} a_n = -(1+1)$$

$$\beta_2 = \inf_{n\geq 2} a_n = -(1+1/3)$$

$$\beta_3 = \inf_{n\geq 3} a_n = -(1+1/3)$$

$$\beta_4 = \inf_{n\geq 4} a_n = -(1+1/5)$$

$$\vdots$$

Question 10 Contd...

- In general,

$$\beta_k = \inf_{n \geq k} a_n = \begin{cases} -(1 + \frac{1}{k}), & k \text{ is odd} \\ -(1 + \frac{1}{k+1}), & k \text{ is even} \end{cases}.$$

- Hence,

$$\liminf_{n \to \infty} a_n = \sup_{k \in \mathbb{N}} \ \beta_k = -1.$$