Formula Sheet

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August 2020

1 Mathematics

1.1 spherical

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\varphi\sin\theta\right) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

1.2 cylindrical

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial\varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial\left(\rho A_\varphi\right)}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right)\hat{\mathbf{z}}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

1.3 Identities

1.3.1 Product Rules

$$\nabla(fg) = f\,\nabla g + g\,\nabla f$$

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + (\nabla f)\cdot\mathbf{A}$$

$$\nabla\times(f\mathbf{A}) = f\nabla\times\mathbf{A} + (\nabla f)\times\mathbf{A}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) \;=\; (\mathbf{A}\cdot\nabla)\mathbf{B} \,+\, (\mathbf{B}\cdot\nabla)\mathbf{A} \,+\, \mathbf{A}\times(\nabla\times\mathbf{B}) \,+\, \mathbf{B}\times(\nabla\times\mathbf{A})$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) \;=\; (\nabla\times\mathbf{A})\cdot\mathbf{B} \,-\, \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{A}(\nabla\cdot\mathbf{B}) \,-\, \mathbf{B}(\nabla\cdot\mathbf{A}) \,+\, (\mathbf{B}\cdot\nabla)\mathbf{A} \,-\, (\mathbf{A}\cdot\nabla)\mathbf{B}$$

1.3.2 Others

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla f) = \mathbf{0}$$

$$\nabla\times(\nabla\times\mathbf{A}) \;=\; \nabla(\nabla\cdot\mathbf{A}) \,-\, \nabla^2\mathbf{A}$$

$$\oiint_{\partial V}\mathbf{A}\cdot d\mathbf{S} \;=\; \iiint_V (\nabla\cdot\mathbf{A})\,dV$$

$$\oint_{\partial S}\mathbf{A}\cdot d\boldsymbol{\ell} \;=\; \iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

2 Maxwell's equations

2.1 Vacuum

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)$$

2.2 Medium

$$\nabla\cdot\mathbf{D} = \rho_f$$

$$\nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

2.3 Auxiliary

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$$

$$\rho_b = -\nabla \cdot \boldsymbol{P}$$

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}$$

3 Magnetostatics

- Surface current density = $\boldsymbol{K} = \sigma \boldsymbol{v} = \frac{d\boldsymbol{I}}{dl_\perp}$
- Hence Magnetic Force = $\boldsymbol{F}_{\text{mag}} = \int (\boldsymbol{K} \times \boldsymbol{B}) da$
- Volume current desnity = $\boldsymbol{J} = \frac{d\boldsymbol{I}}{da_\perp} = \rho \boldsymbol{v}$
- Hence Magnetic Force = $\boldsymbol{F}_{\text{mag}} = \int (\boldsymbol{J} \times \boldsymbol{B}) d\tau$
- Continuity equation : $\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}$
- Biot-Savart Law : $\boldsymbol{B}(\boldsymbol{r}) = \dfrac{\mu_0}{4\pi} I \displaystyle\int \frac{d\boldsymbol{l'} \times \hat{\boldsymbol{s}}}{s^2}$
- Vector Poisson's equation : $\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$
- Coulomb Gauge : $\nabla \cdot \boldsymbol{A} = 0$
- $\boldsymbol{A} = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\boldsymbol{J}(\boldsymbol{r'})}{s} d\tau'$
- Far Field vector potential of dipole $\boldsymbol{m}$: $\boldsymbol{A}(\boldsymbol{r}) = \dfrac{\mu_0}{4\pi} \dfrac{\boldsymbol{m} \times \hat{\boldsymbol{s}}}{s^2}$
- Magnetic Far field of dipole placed along $\hat{\boldsymbol{z}}$: $\boldsymbol{B}_{dip} = \frac{\mu_0}{4\pi} \frac{|\boldsymbol{m}|}{|r|^3} \left(2\cos\theta \hat{\boldsymbol{r}} + \sin\theta \hat{\boldsymbol{\theta}}\right)$

4 Electrodynamics

- $\int_{\mathcal{S}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a} = \frac{d}{dt} \int_{\mathcal{S}} \boldsymbol{B} \cdot d\boldsymbol{a} + \oint_{\mathcal{C}} (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l}$
- Poynting Theorem :

$$\int_{\mathcal{V}} (\boldsymbol{J} \cdot \boldsymbol{E}) d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \frac{1}{2} \left(\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} \right) d\tau - \oint_{\mathcal{S}} \boldsymbol{S} \cdot d\boldsymbol{a}$$

- Poynting Vector : $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = \frac{1}{\mu_0} (\boldsymbol{E} \times \boldsymbol{B})$

5 Electro-Magnetic Waves

- Phase Velocity : $v = \frac{\omega_0}{k_0}$
- Group velocity : $v_g = \frac{\partial \omega(k)}{\partial k}$
- For Plane waves :
- $\boldsymbol{B_0} = \frac{(\hat{\boldsymbol{k}} \times \boldsymbol{E_0})}{\omega}$
- Reflection and Refraction
  - $R = |r|^2$
  - $T = 1 - R = \frac{n_2 \cos \theta_{\mathrm{t}}}{n_1 \cos \theta_{\mathrm{i}}} |t|^2$
  - $k_1 \sin \theta_1 = k_2 \sin \theta_2 = k_1 \sin \theta_3$
  - **p-polarization** : Electric Field parallel to PoI
    * $r_{\mathrm{p}} = \frac{n_2 \cos \theta_{\mathrm{i}} - n_1 \cos \theta_{\mathrm{t}}}{n_2 \cos \theta_{\mathrm{i}} + n_1 \cos \theta_{\mathrm{t}}}$
    * $t_{\mathrm{p}} = \frac{2 n_1 \cos \theta_{\mathrm{i}}}{n_2 \cos \theta_{\mathrm{i}} + n_1 \cos \theta_{\mathrm{t}}}$
  - **s-polarization** : Electric Field perpendicular to PoI
    * $r_{\mathrm{s}} = \frac{n_1 \cos \theta_{\mathrm{i}} - n_2 \cos \theta_{\mathrm{t}}}{n_1 \cos \theta_{\mathrm{i}} + n_2 \cos \theta_{\mathrm{t}}}$
    * $t_{\mathrm{s}} = \frac{2 n_1 \cos \theta_{\mathrm{i}}}{n_1 \cos \theta_{\mathrm{i}} + n_2 \cos \theta_{\mathrm{t}}}$