Lecture on 05/12/2020

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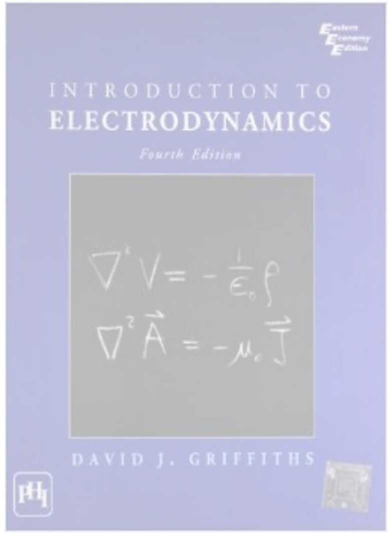
PYL101 course:

Electromagnetics & Quantum Mechanics

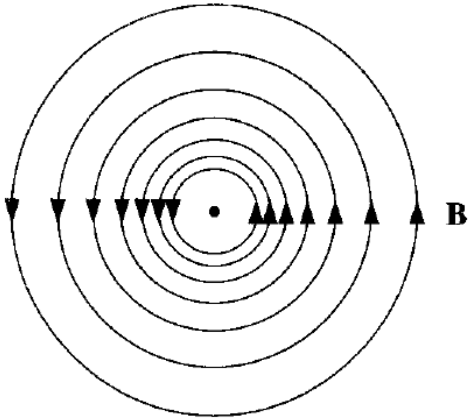
- ➢ *Next few classes, we will discuss the following topics:*
- ➢ *Magnetostatics (ch5)*
- ➢ *Magnetic fields in matter (ch6)*
- ➢ *Electrodynamics (ch7)*
- ➢ *Continuity equation and Poynting's theorem (ch8)*



The divergence and curl of B:

Home work: find the magnetic field at certain distance from a long straight wire carrying a steady current?, see example.5.5

Now, let's see in the below figure the magnetic field lines, when the current is coming out of the page…



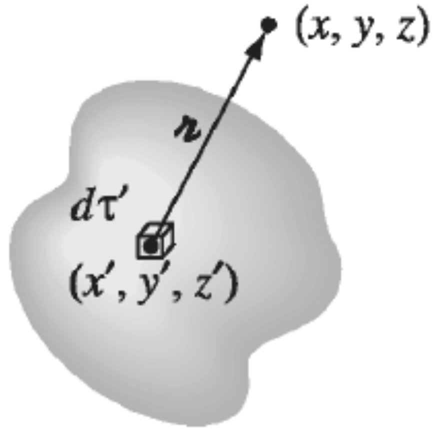
What you will see here is that this field has a nonzero curl…

We have learned the Biot-Savart law, write for the general case of a volume current…

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau' \quad .....(1)$$

The divergence and curl of B:

The above equation (1) gives the magnetic field at a point r = (x, y, z) in terms of an integral over the current distribution J (x', y', z')



Please note below:

$$\mathbf{B} \text{ is a function of } (x, y, z),$$

$$\mathbf{J} \text{ is a function of } (x', y', z'),$$

$$\mathscr{r} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

The integration is over the primed coordinates;
the divergence and the curl of B are with respect to the unprimed coordinates.

The divergence and curl of B:

Let’s apply the divergence, and use the product rule…

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

Because J doesn't depend on the unprimed variables

zero zero

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \cancel{(\nabla \times \mathbf{J})} - \mathbf{J} \cdot \cancel{\left(\nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)}$$

So, what we get is... $\nabla \cdot \mathbf{B} = 0.$

Now, let’s apply the curl and again product rule…

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

see, ch1 zero

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \mathbf{J} \cancel{\left(\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)} - \cancel{(\mathbf{J} \cdot \nabla)} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

So, what we get is..

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta^3 (\mathbf{r} - \mathbf{r}') \, d\tau' = \mu_0 \mathbf{J}(\mathbf{r})$$

The divergence and curl of B:

The equation for the curl of B is called the Ampere's law..

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$ in differential form

Let's convert to integral form by applying Stokes' theorem..

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

and we can write as..

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$ integral form

where, I_{enc} is the total current passing through the surface..

So, now we can compare, Electrostatics and Magnetostatics:

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho, \quad \text{(Gauss's law)} \qquad \nabla \cdot \mathbf{B} = 0, \quad \text{(no name);}$$

$$\nabla \times \mathbf{E} = 0, \quad \text{(no name).} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \text{(Ampère's law).}$$

Ampere's law: Integral to differential form:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Here I_{enc} is the total current enclosed by the integration path

where,

$$I_{enc} = \int \vec{J} \cdot d\vec{a}$$

Here, flow of charge is represented by a volume current density (***J***).

- Integral is taken over the surface bounded by the loop.

Applying Stokes' theorem to the LHS: $\int_s (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_p \vec{v} \cdot d\vec{l}$

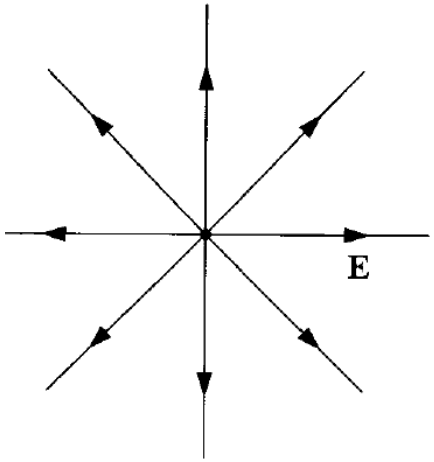
$$\int \left(\vec{\nabla} \times \vec{B}\right) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

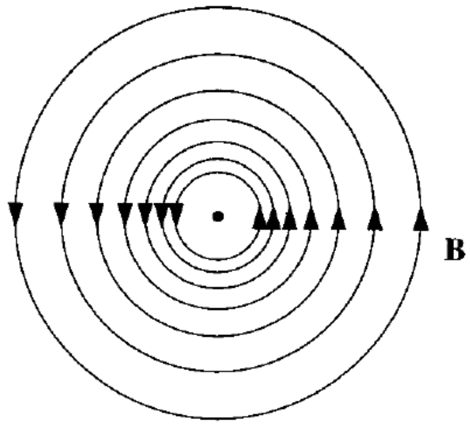
Ampere's law in differential form

What we learn?:

The electric field diverges away from a +ve charge; the magnetic field line curls Around a current….



(a) Electrostatic field of a point charge



(b) Magnetostatic field of a long wire

Electric field lines originate from +ve charge and end on –ve.. Magnetic field lines do not begin or end anywhere…. they either form closed loops or extend out to infinity…

In other words, there are no point sources for B, as there are for E, i.e., there exists no magnetic analog to electric charge This is the physical content of the statement, $\nabla . B = 0$

There are no magnetic monopoles, it takes a moving charge to produce B and another moving charge to feel B…

Ch.6: What is the auxiliary field H?

- ✧ The magnetic fields generated by currents and calculated from Ampere's Law or the Biot-Savart Law are characterized by the "**B**" measured in Tesla……
- ✧ But when the generated fields pass through magnetic materials, which themselves contribute internal magnetic fields, ambiguities can arise about what part of the field comes from the external currents and what comes from the material itself…..
- ✧ It has been common practice to define another magnetic field quantity, usually called the "Auxiliary field" designated by **H**……..

Ch.6: How to define the auxiliary field H?

✧ As discussed, the Ampere's Law can be written as:

$$\frac{1}{\mu_o}(\vec{\nabla}\times\vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + \vec{\nabla}\times\vec{M}$$

where, **J** is the total current, which consists of

(i) given current or free current (controlled externally) $\mathbf{J_f}$

(ii) the bound current $\mathbf{J_b}$ due to magnetized material

we can write as: $\nabla\times(\frac{1}{\mu_o}\vec{B}-\vec{M}) = \vec{J}_f$ where, $\vec{H} = \frac{1}{\mu_0}\vec{B}-\vec{M}$

which means: $\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f}$ is the Ampere's law for $\vec{\mathrm{H}}$

✧ And, $\vec{\mathrm{H}}$ is called the Auxiliary field…

✧ Also, integral form of Ampere's law: ⇨ $\oint \mathbf{H}\cdot d\mathbf{l} = I_{f_{enc}}$

Relation between B and H?

❖ The relationship for B (Tesla) can be written as:

$$\vec{B} = \mu_0(\vec{H} + \vec{M})$$

- **H** and **M** have the same units, Amperes/meter.
- **B** is sometimes called the <u>magnetic flux density</u> or the magnetic induction.
- **H** is called auxiliary field or <u>magnetic intensity</u> or magnetizing field.

(analogous to $\vec{D}$ in Electrostatics.)

✧ Just as **D** allowed us to write Gauss's law in terms of the free charge alone, **H** permits us to express Ampere's law in terms of the free current alone.

$$\vec{\nabla} \cdot \vec{D} = \rho_{free} \qquad \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f \qquad \oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{enc}}$$

Ch.6: Linear and nonlinear magnetic media:

We have seen that para- and dia-magnetic materials get magnetized (M) when exposed to magnetic field…

✧ When field is removed, M disappears.

✧ In the small field limit….

If, **M** is proportional to field: linear media

$\vec{M} \alpha \vec{H} \Rightarrow \vec{M} = \chi_m \vec{H}$, χ_m = magnetic susceptibility (dimensionless)

(+ve for para-, -ve for dia-)

$$\Rightarrow \vec{B} = \mu_o(\vec{H} + \vec{M}) = \mu_o(1 + \chi_m)\vec{H}$$

$\vec{B} = \mu \vec{H}$, where, $\mu = \mu_o(1 + \chi_m)$ is the permeability of the material

In vacuum, there is no matter to magnetize: $\chi_m = 0$, $\mu = \mu_o$

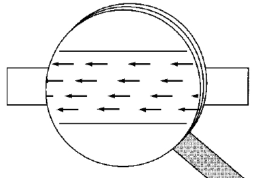
✧ If, **M** is not proportional to **B**:

Nonlinear media (Ex: ferromagnetic materials)….

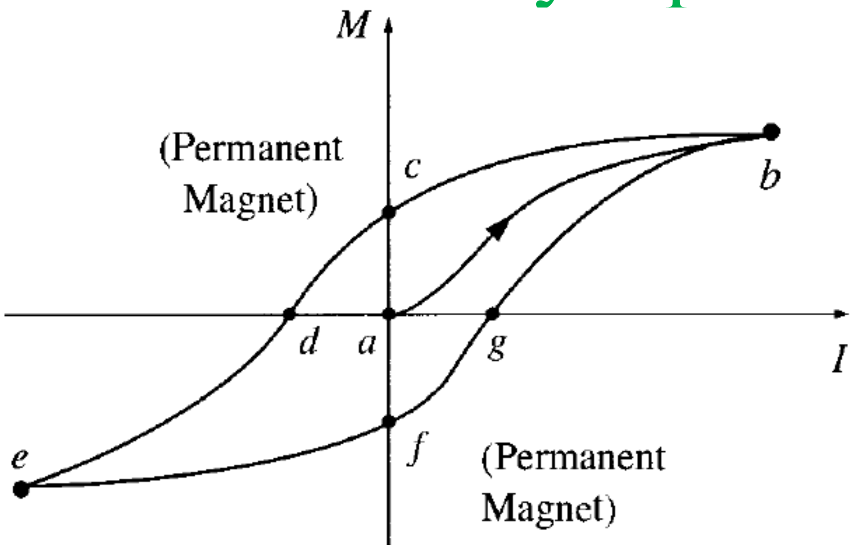
Ch.6: Nonlinear magnetic media:

In a linear medium the alignment of atomic dipoles is maintained by a magnetic field applied from outside…

Ferromagnets are different, which require no external field to sustain the magnetization..



Ferromagnetism involves the magnetic dipoles/domains associated with the unpaired spins and interactions between nearby dipoles….quantum mechanics…later..



M-H magnetization curve of FM,

I ↑, H ↑, domains move and M ↑, reach the saturation point *b*,

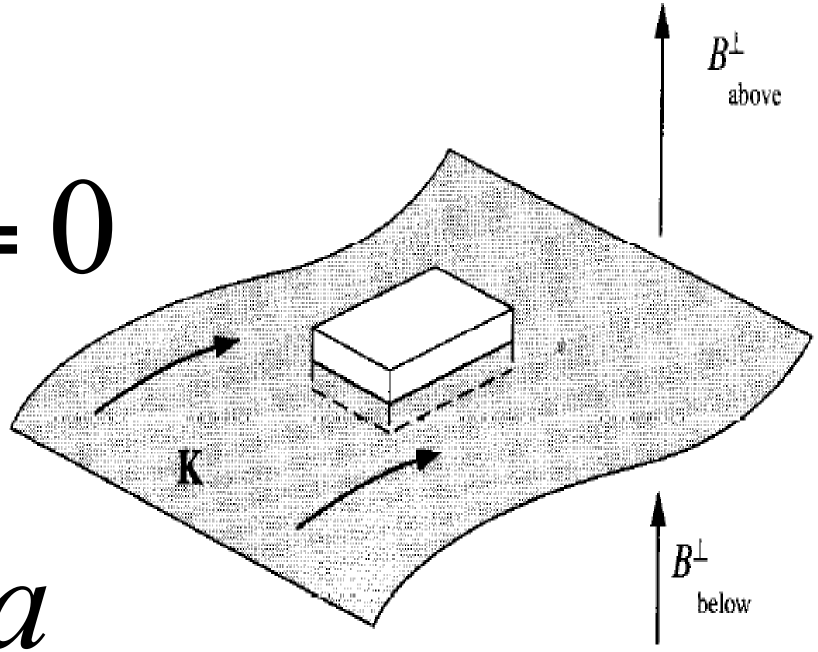
M ↓ to 0 at point *d*,

define the hysteresis loop

Ch.5: Boundary Conditions for Magnetostatics:

✧ Magnetic field is discontinuous at a surface current….

$$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \oint \vec{B}\cdot d\vec{a}=0$$



✧ Use the following:

$$\int(\nabla\cdot B)d\tau=\oint_s B\cdot da$$

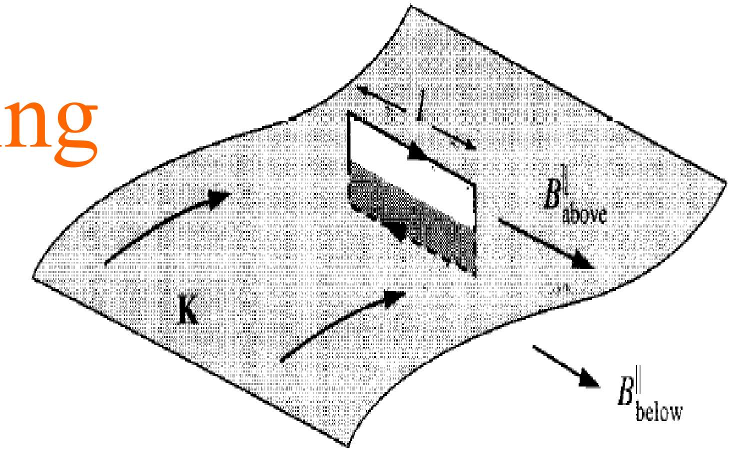
✧ Applying to a wafer-thin pillbox (just below and above), we get -

$$B^{\perp}_{above}A - B^{\perp}_{below}A = 0 \quad \Rightarrow \quad B^{\perp}_{above} = B^{\perp}_{below}$$

Ch.5: Boundary Conditions for Magnetostatics:

✧ For tangential components,

consider an Amperian loop running perpendicular to the current



$$\oint \vec{B}\cdot d\vec{l} = \left(B^{\parallel}_{above} - B^{\parallel}_{below}\right) l$$

$$= \mu_o I_{enc} = \mu_o K l \Rightarrow B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_o K$$

The component of $\vec{B}$ parallel to the surface, but perpendicular to the current is discontinuous by the amount $\mu_o K$

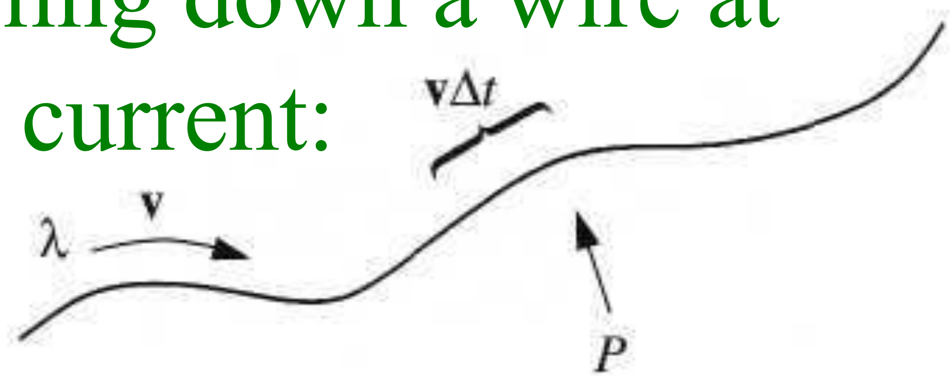
$$\boxed{\vec{B}_{above} - \vec{B}_{below} = \mu_o\left(\vec{K}\times\hat{n}\right)}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface, pointing "upward."

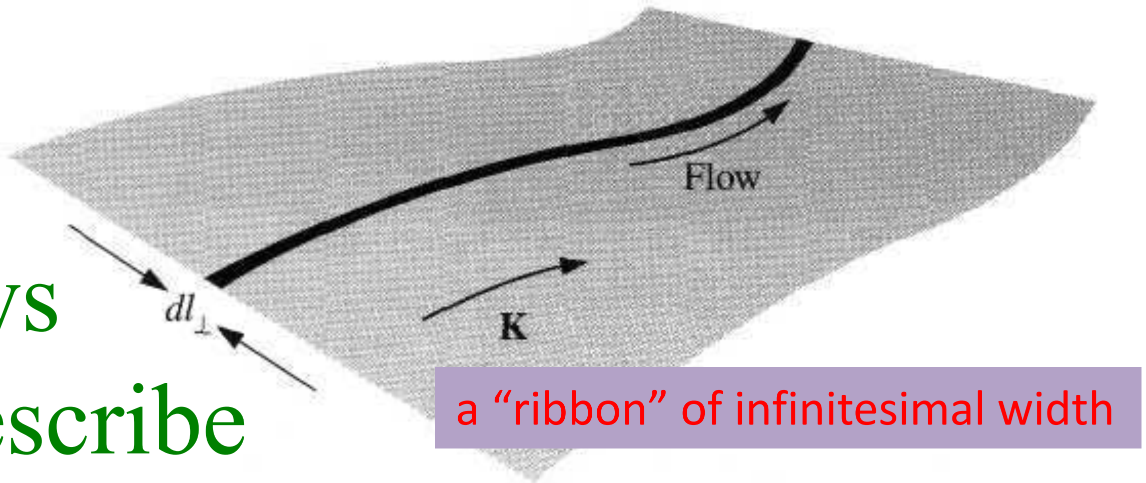
Ch.5: Currents: Surface K, & Volume J:

✧ A line charge λ traveling down a wire at speed v constitutes a current:

$$\mathbf{I} = \lambda \mathbf{v}$$



✧ This is because a segment of length vΔt, carrying charge λvΔt, passes point P in a time interval Δt.



✧ When charge flows over a surface, we describe it by the surface current density **K**:

Ch.5: Currents: Line λ, Surface K, & Volume J:

✧ If the current in this ribbon is d**I**, the surface current density is: $\mathbf{K} \equiv \dfrac{d\mathbf{I}}{dl_\perp}$ where **K** is the current per unit width-perpendicular-to-flow.

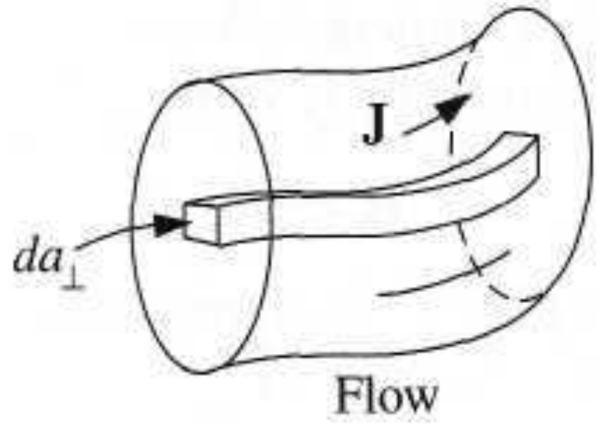
➢ if the mobile surface charge density is σ and its velocity is **v**, then $\mathbf{K} = \sigma \mathbf{v}$

✧ When the flow of charge is distributed throughout a **3D region**, we describe it by the volume current density, **J**:

✧ In this case, let's consider a "tube" of infinitesimal cross section, running parallel to the flow:

Ch.5: Currents: Line λ, Surface K, & Volume J:

✧ If the current in this tube is $d\mathbf{I}$, the volume current density is:



$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp}$$

where, J is the current per unit area-perpendicular-to-flow.

➢ If the mobile volume charge density is ρ and the velocity is **v**, then we get.. $\mathbf{J} = \rho\mathbf{v}$

<u>Equation of continuity:</u>

❖ The current crossing a surface S can be written as:

$$I = \int_S J\, da_\perp = \int_S \mathbf{J} \cdot d\mathbf{a}$$

$$= -\frac{\partial}{\partial t} Q(\text{contained by } S)$$

Ch.5: Equation of continuity:

✧ The divergence theorem: $$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau$$

✧ The total charge per unit time leaving a volume τ can be represented by the volume integral of the charge density:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} \right) d\tau$$

✧ Because charge (local) is conserved, i.e., neither created nor destroyed, means it hold only if,

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

✧ It is called as continuity equation…