Types of Isolated singularities:

$$f(z) = \cdots + \frac{b_n}{(z-z_0)^n} + \cdots \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

- Poles of order m: $\quad b_n = 0 \;\; \forall \;\; n > m$

  $m = 1$ is referred to as Simple pole.

- Removable singularity: $\quad b_n = 0 \;\; \forall \;\; n \geq 1$

  e.g. $\frac{\sin z}{z}$ has removable singularity at $z = 0$

- Essential Singularity: An infinite number of b_n's are non vanishing.

  $e^{1/z}$ has essential singularity at $z = 0$

Some comments/results on behaviour near isolated singularities

- If $f(z)$ has a removable singularity at $z = z_0$ then $f(z)$ is bounded in the deleted neighbourhood $0 < |z - z_0| < \epsilon$.

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{m=0}^{\infty} a_m (z-z_0)^m \quad \text{w/} \quad b_n = 0 \;\; \forall \;\; n = 1, 2, \ldots$$

$$\longrightarrow \sum_{m=0}^{\infty} a_m (z-z_0)^m \quad \forall \quad 0 < |z - z_0| < \epsilon$$

which is bounded.

⋆ We will state a couple of theorems with proof now

1. If $f(z)$ is analytic & bounded in a deleted neighbourhood of z_0 $(0 < |z - z_0| < \epsilon)$ then $f(z)$ is either analytic or can only have a removable singularity at $z = z_0$.

2. Great Picard theorem: If $f(z)$ is an analytic function with an singularity at z_0 then in any punctured neighborhood of z_0, $f(z)$ takes all possible complex values, with at most a single exception, infinitely many times.

<u>Zero of order m</u> : An analytic function $f(z)$ has a zero of order m at $z = z_0$ if

$$f(z) = (z - z_0)^m g(z) \quad \text{s.t. } g(z_0) \neq 0 \ \& \ g(z) \text{ is analytic at } z_0.$$

Equivalently

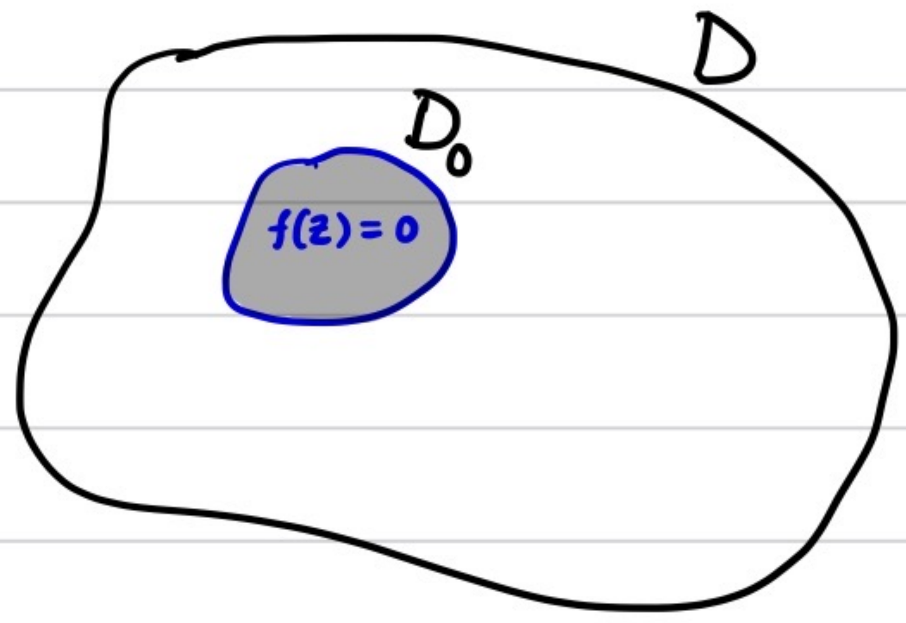
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \text{with} \quad f^{(n)}(z_0) = 0 \ \forall\, n = 0, 1, 2, \dots m-1$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!} (z - z_0)^{n+m}$$

$$= \underbrace{(z - z_0)^m \left[\sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!} (z - z_0)^n \right]}_{g(z)} \quad \Rightarrow \quad g(z_0) = \frac{f^{(m)}(z_0)}{m!}$$

- If $f(z)$ is an analytic function in some domain D and $f(z) = 0$ in some subdomain D_0 or a line segment L completely inside D, then $f(z) = 0 \ \forall\, z \in D$.

Let $z_0 \in D_0 \subset D$, then $\forall\, z \in D$



$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0) (z - z_0)^n$$

Now $\quad f(z) = 0 \quad \forall\, z \in D_0$

$$\Rightarrow \quad f^{(n)}(z_0) = 0 \quad \forall\, n = 0, 1, 2, \dots$$

But since the Taylor series is valid $\forall\, z \in D$, we get

$$f(z) = 0 \quad \forall\, z \in D.$$

Multivalued functions & Branch cuts :

Consider $f(z) = z^{1/n}$ n is positive integer

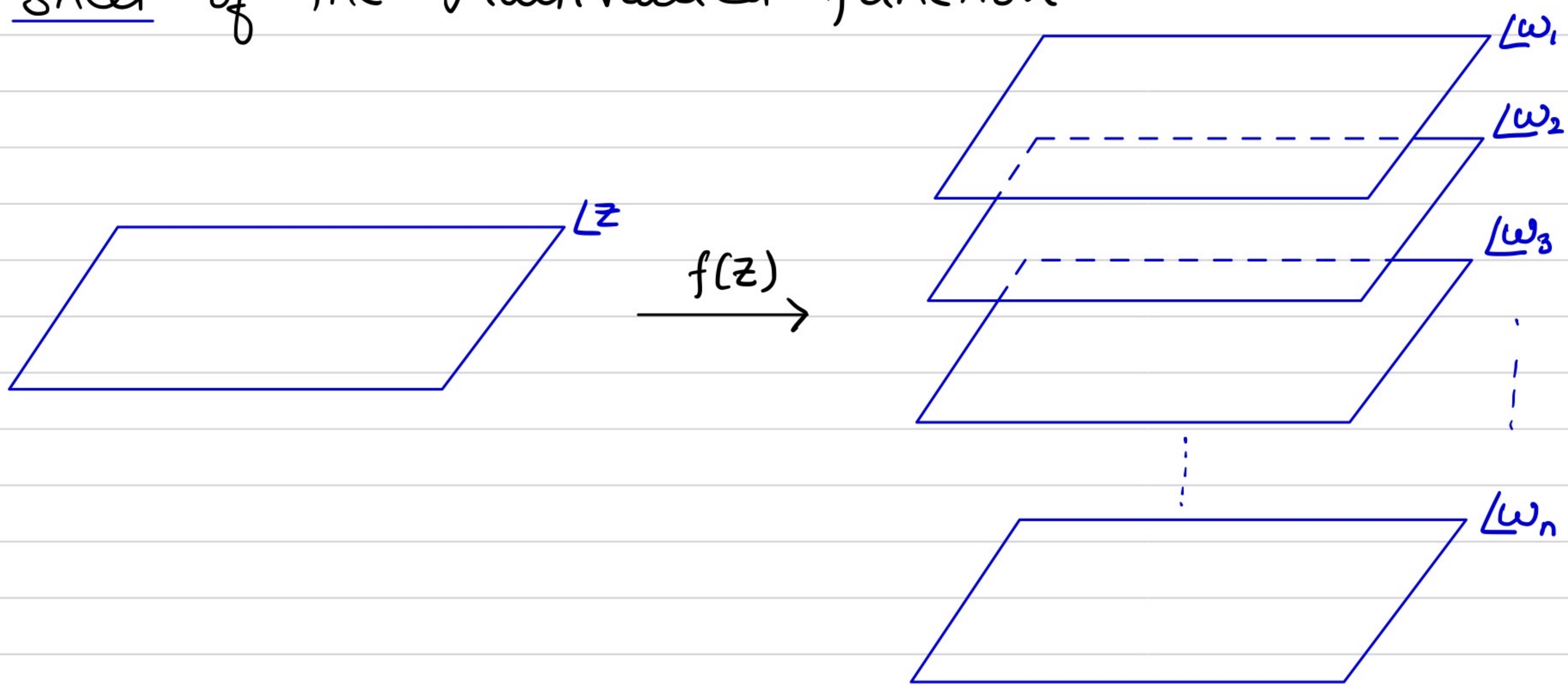
using $z \equiv r e^{i(\theta + 2\pi m)}$ $m \in \mathbb{Z}$

$$f(z) = r^{1/n} \cdot e^{i\left(\frac{\theta + 2\pi m}{n}\right)}$$

Thus $f(z)$ takes n distinct values for $m = 0, 1, 2, \dots (n-1)$ all of which correspond to same value of z.

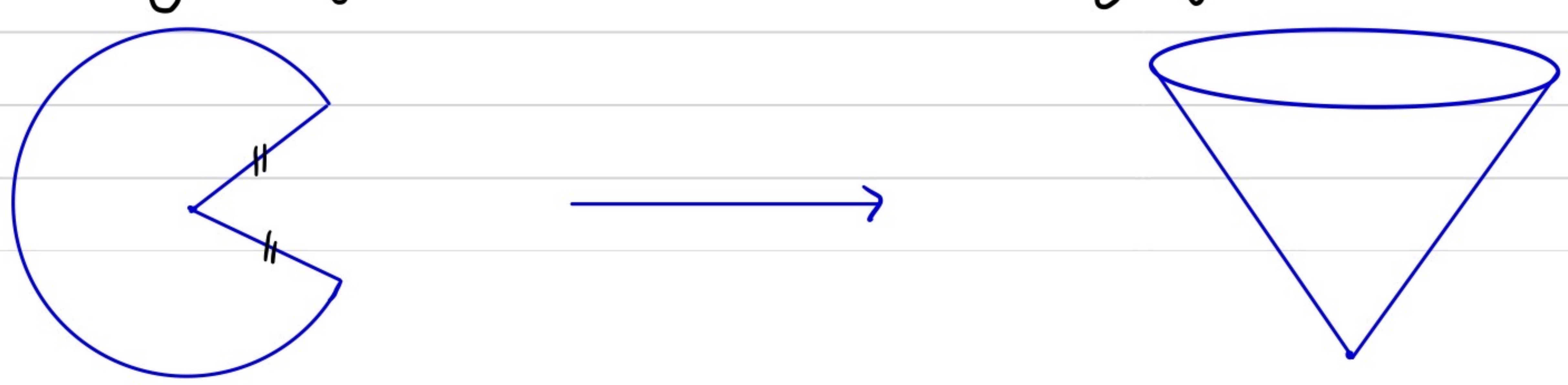
$\rightarrow$ Multivalued function

- In visual representation of z & $\omega = f(z)$ in two different complex planes, Each of these n distinct values can be represented in a distinct "plane" called a Branch or Riemann sheet of the Multivalued function



where $\omega_m = r^{1/n} e^{i\left(\frac{\theta + 2\pi m}{n}\right)}$ $m = 1, 2, \dots n.$

- This visual representation is not accurate since as θ runs from 0 to 2π the phase of any ω_m only runs over an interval $2\pi/n$ so each Riemann sheet here should be a cone of angle $2\pi/n$ instead of full complex plane



- Further notice that, starting from a particular branch, as we move around the point $z=0$ in a closed path

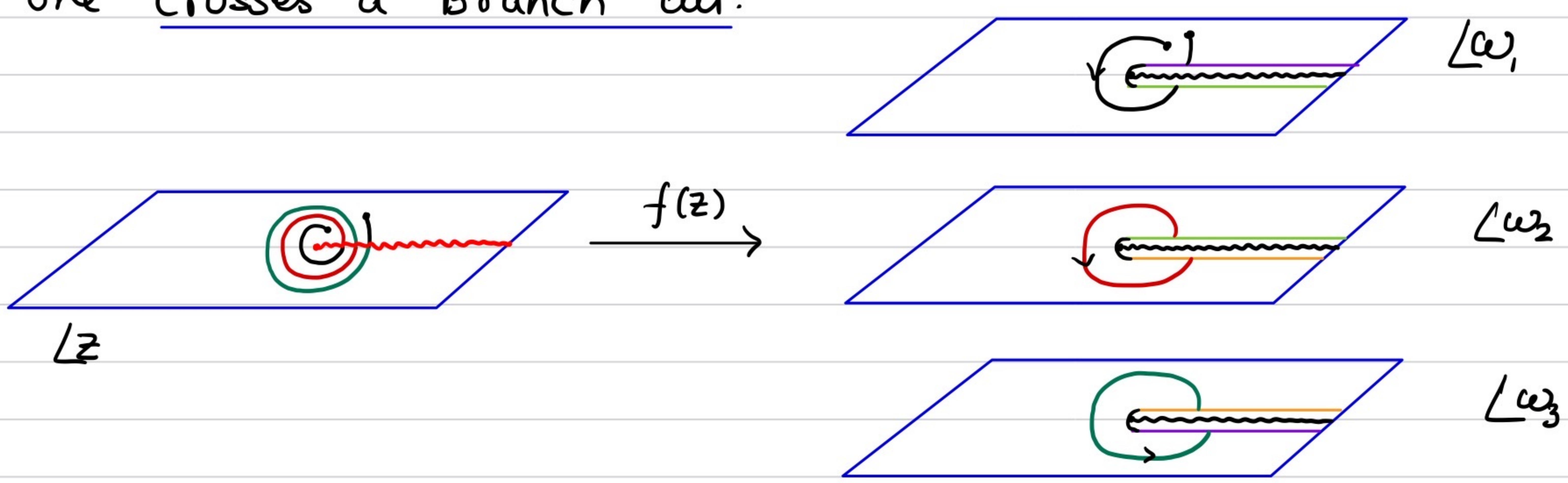
$$z = re^{i\theta} \qquad \theta \text{ goes from } \theta \text{ to } \theta+2\pi$$

$$f(re^{i\theta}) = r^{1/n} e^{i\frac{\theta+2\pi m}{n}}$$

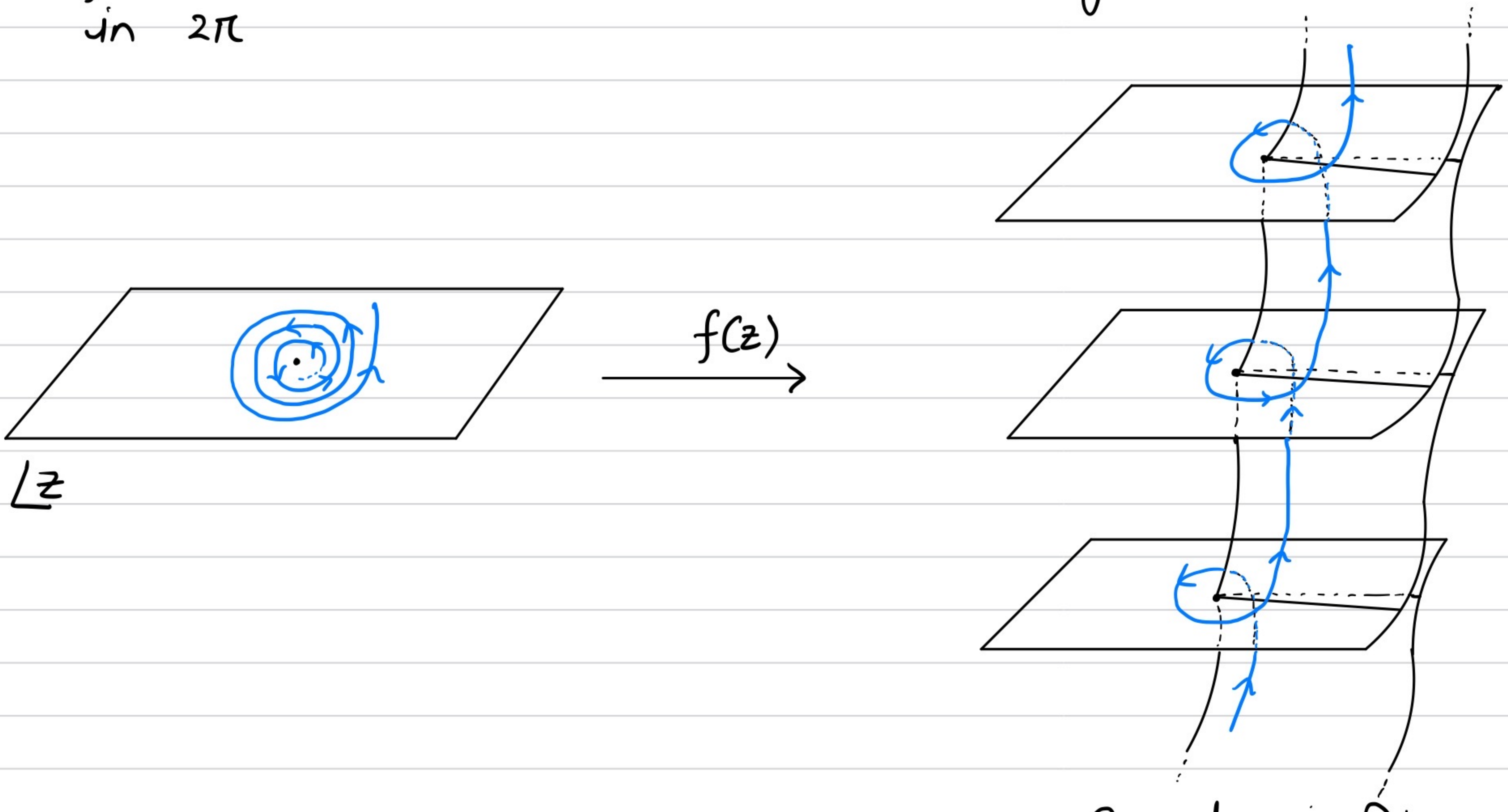
$$f(re^{i(\theta+2\pi)}) = r^{1/n} \cdot e^{i\frac{\theta+2\pi(m+1)}{n}}$$

value of $f(z)$ moves from one branch to next branch.

Such points are referred to as Branch points of Multivalued functions. A way of representing this property is to draw a Branch cut connecting the different Branches. The value of the function jumps when one crosses a Branch cut.



Another way of representing the function value moving from one Branch to another as θ goes around in 2π



Branches or Riemann sheets of $f(z)$

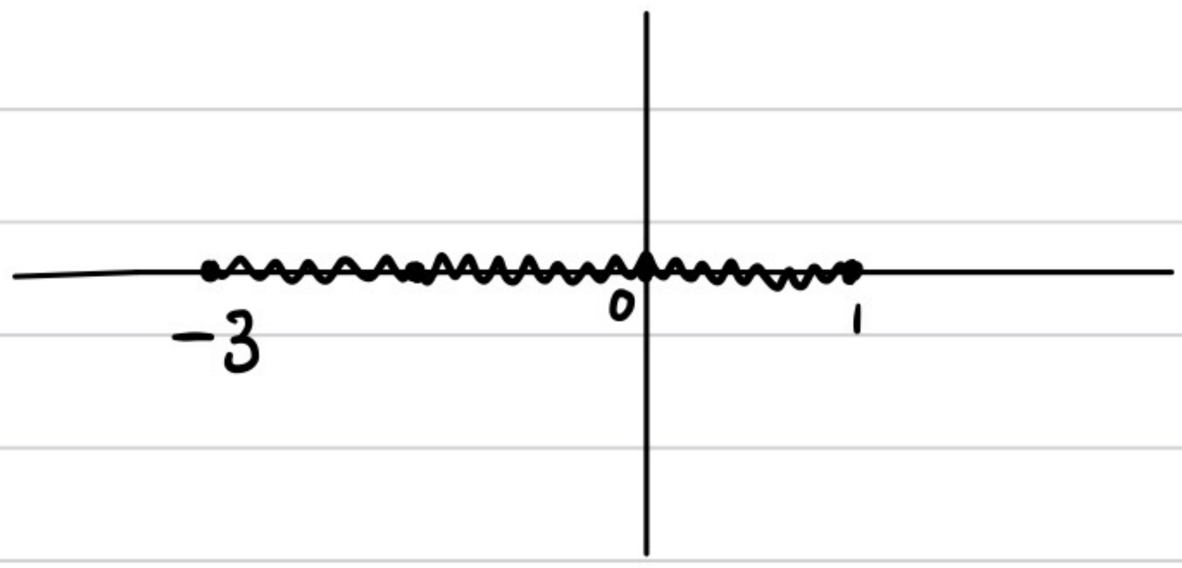
Comments :

- ⋆ Branch points of a multivalued function are fixed & uniquely determined
- ⋆ Branch cut is a choice & can be choosen in many ways.

Example :

$f(z) = \sqrt{(z-1)(z+3)}$ has two Branch points $z = -3, +1$

different choices of Branch cuts



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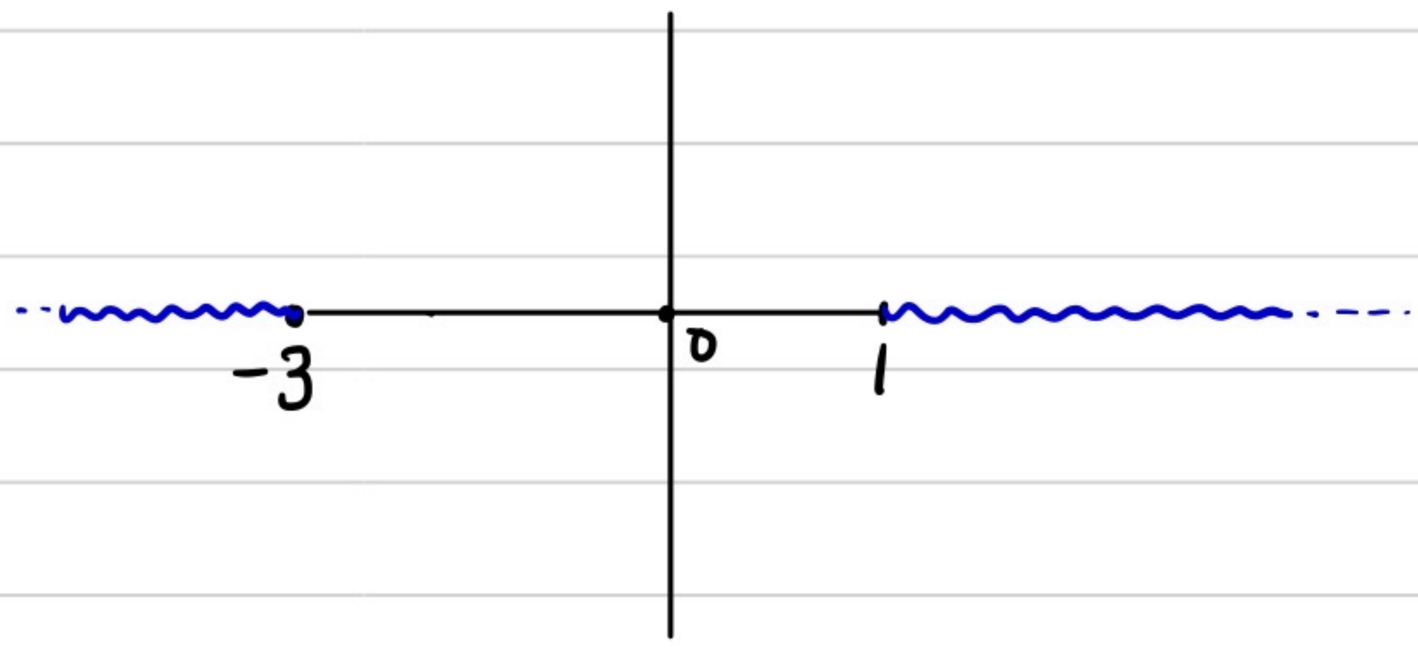
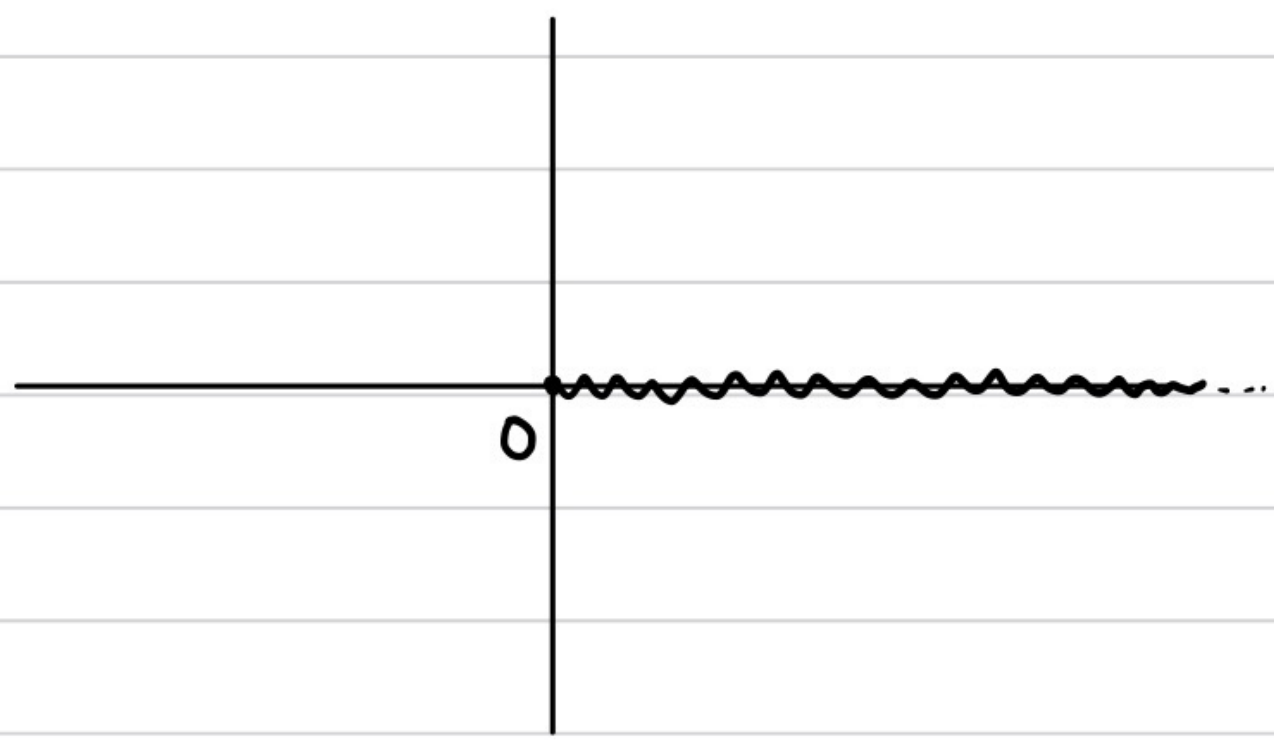


or others

- $f(z) = Log(z) = Log(r) + i(\theta + 2\pi n)$ has $z = 0$ as the Branch point & has infinitely many branches correspon-ding to $n \in (\cdots -3, -2, 1, 0, 1, 2, 3, \cdots)$.



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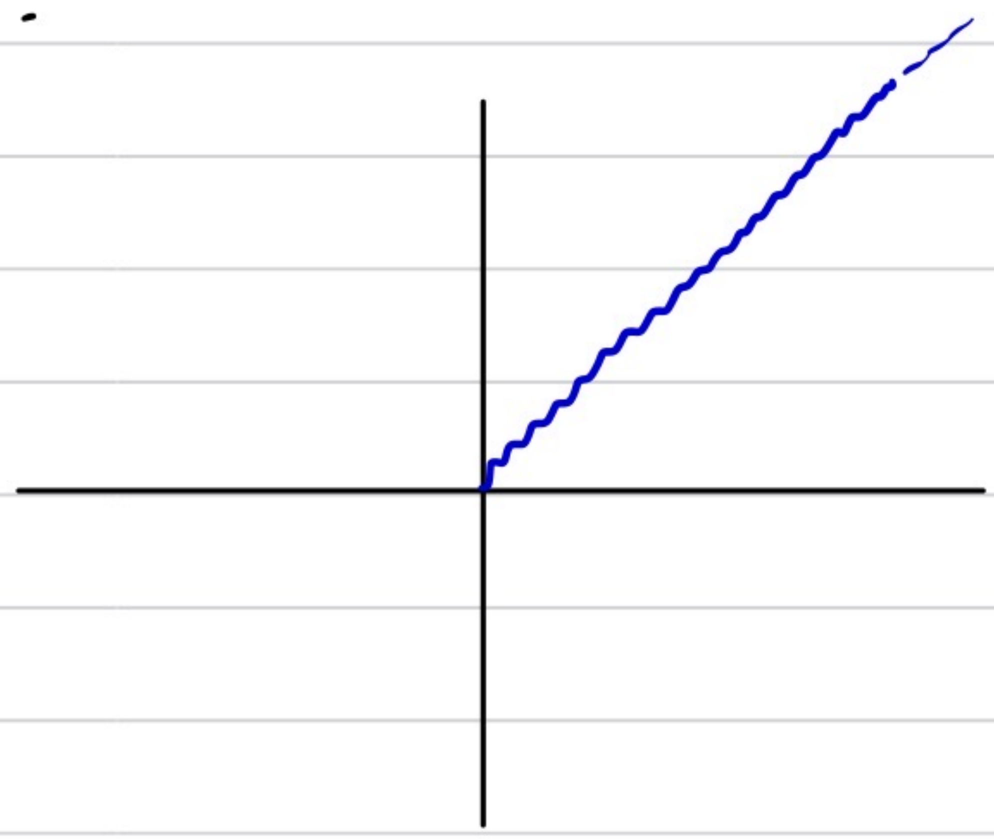

or others

- Notice that though these multivalued functions consider-ed above ($\sqrt{(z-1)(z+3)}$, $Log\, z$) are non analytic at the branch points, they are analytic away from these branch points as long as we stay on a fixed branch.

e.g. $$Log\, z = Log\left(r e^{i(\theta + 2\pi m)}\right) = \underbrace{\frac{1}{2} Log\,(x^2+y^2)}_{u} + i\underbrace{\left(\tan^{-1}\left(\frac{y}{x}\right) + 2\pi m\right)}_{v}$$

It can be easily verified that u & v above indeed satisfy the Cauchy-Riemann equations.

Ex : Show this ⤴ assuming m is constant ($\partial_x m = 0 = \partial_y m$)