Example:

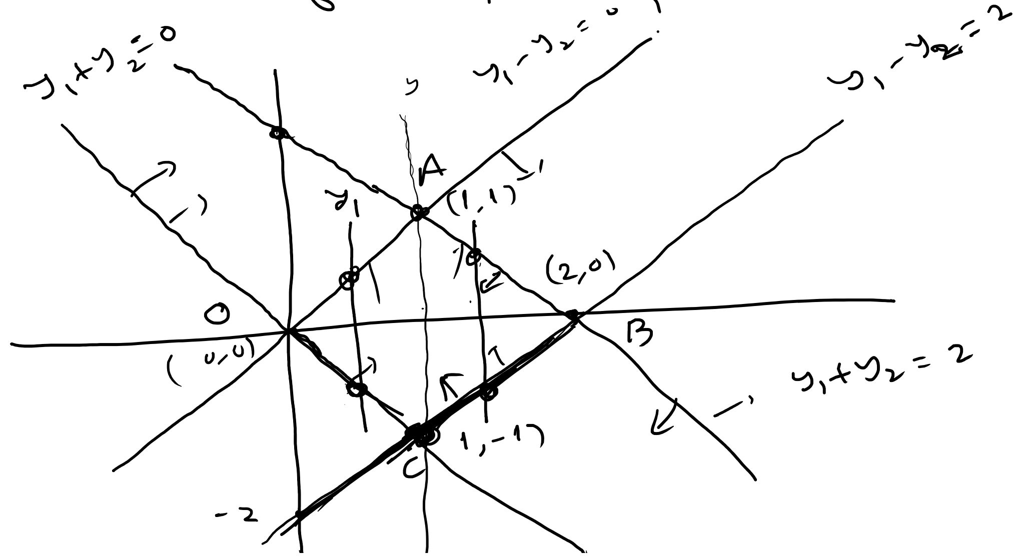
$$f(x_1, x_2) = \begin{cases} 1 & 0 < x_1 < 1 \\ & 0 < x_2 < 1 \\ 0 & \text{o.w} \end{cases}$$

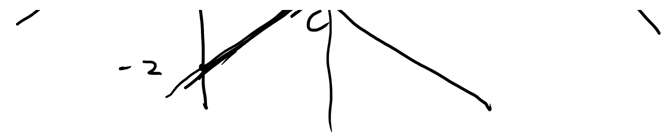
$Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$

joint PDF of Y_1, Y_2 is given by

$$w(y_1, y_2) = \begin{cases} 1/2 & (y_1, y_2) \in \left\{ (y_1, y_2) \,\middle|\, \begin{array}{l} 0 < y_1 + y_2 < 2 \\ 0 < y_1 - y_2 < 2 \end{array} \right\} \\ 0 & \text{o.w.} \end{cases}$$

What is PDF of $X_1 + X_2$?





$$W_{Y_1}(y_1) = \int_{-y_1}^{y_1} \tfrac{1}{2}\, dy_2 = y_1 \qquad 0 \le y_1 \le 1$$

$$= \int_{y_1-2}^{2-y_1} \tfrac{1}{2}\, dy_2 \qquad 1 \le y_1 \le 2$$

$$\frac{1}{2}\left(2 - y_1 - y_1 + 2\right) = 2 - y_1$$

$$= 0 \qquad \text{o.w}$$

$$W_{Y_1}(y_1) = \begin{cases} y_1 & 0 \le y_1 \le 1 \\ 2 - y_1 & 1 \le y_1 \le 2 \\ 0 & \text{o.w.} \end{cases}$$

PDF of $X_1 - X_2$ is nothing but $W_{Y_2}(y_2)$. (exercise).

<u>Remark</u>: n → Random variables

number of functions $= m < n$

Consider some $n-m$ artificial functions

Consider some non artificial functions which can solve purpose.

$\Rightarrow$ $Y_1 = \boxed{X_1 + X_2}$ ✓ $\quad Y_2 = X_1 - X_2$ $\quad$ $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$ $\Rightarrow x_1 = \frac{y_1+y_2}{2}$, $x_2 = \frac{y_1-y_2}{2}$

$Y_2 = X_2$ $\quad$ $y_2 = x_2$ $\Rightarrow \boxed{x_1 = y_1 - y_2, \; x_2 = y_2}$

Special type of Random Variables

Bernoulli Distribution:

Outcome of Experiment $\to$ Success / failure

$p \to$ probability of success

$X = 0$ if it is failure

$\quad = 1$ if it is success

$P\{X=1\} = p \qquad P\{X=0\} = 1-p$

Then, we say that X follows a Bernoulli distribution with success probability p.

Example: In coin tossing experiment

$X =$ number of heads $\begin{cases} X=0 & \text{Tail.} \\ X=1 & \text{Head.} \end{cases}$

$X \sim$ Bernoulli with $p = 1/2$

$E[X] = p, \quad Var(X) = p(1-p)$

$M(t) = p \cdot e^t + (1-p) \times t = 1-p+pe^t$

Suppose n independent trials of an experiment are performed and success probability of each trial is

p.

X = number of successes in n-trials

$x \in \{0, 1, \ldots, n\}$

$$\frac{n!}{i!\,(n-i)!} \rightarrow \text{binomial Coefficient}$$

$$P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, \ldots n$$

Then, we say that X follows a binomial distribution.

PMF of $b(n, p)$

binomial distribution with parameters n & p

$E[X] = np$, $Var(X) = np(1-p)$

Let $X_1, \ldots X_n$ are n-independent Bernoulli distribution with success probability

$Y = X_1 + X_2 + \ldots + X$

$X = X_1 + X_2 + \cdots + X_n$

number of successes in n-trials

$\Rightarrow \quad X \sim b(n, p)$

$$E[X] = \sum_{i=1}^{n} E[X_i] = np$$

$$Var[X] = \sum_{i=1}^{n} Var(X_i) = np(1-p)$$

$$M_X(t) = E\left[e^{t(X_1 + \cdots + X_n)}\right]$$

$$= E\left[e^{tX_1} \cdots e^{tX_n}\right]$$

$$= \prod_{i=1}^{n} E\left[e^{tX_i}\right]$$

$$= \prod_{i=1}^{n} M_{X_i}(t)$$

$\because X_1, \ldots, X_n$ are independent $\Rightarrow e^{tX_1}, \ldots, e^{tX_n}$ are independent

$$M_X(t) = \left(1 - p + pe^t\right)^n$$

$$E[X+Y]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (X+Y) f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f(x,y)\, dy\right) dx + \int_{-\infty}^{\infty} y \left(\int_{-\infty}^{\infty} f(x,y)\, dx\right) dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx + \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$= E[X] + E[Y]$$

$= E[X] + E[Y]$

Poisson distribution: An RV X taking values $0, 1, \ldots$ is said to follow a Poisson distribution with parameter $\lambda > 0$ if its PMF is given by

$$P\{X = k\} = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$E[X] = \lambda = Var(X)$.

$M_X(t) = e^{\lambda(e^t - 1)}$

Poisson distribution with $(\lambda = np)$ can be used an approximation of a binomial distribution when n is large & p is small.

Suppose, $n \to$ large, p – small

Define $\underline{\lambda = np}$, $X \sim b(n, p)$

$$P\{X = i\} = \frac{n!}{(n-i)!\, i!} p^i (1-p)^{n-i}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{i!} \left(\frac{\lambda}{n}\right)^i \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{\left(1-\lambda/n\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$$

If n is large.

$$\frac{n(n-1)\cdots(n-i+1)}{n^i} \approx 1 \qquad \left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}$$

$$(1-p)^i \approx 1$$

$$P\{X = i\} \approx e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$\Rightarrow \quad X \sim P(\lambda)$

<u>Example</u>: 1) Number of misprints on a page of a book.

2) The number of people in a community living to 100 years of age.