COL 352 Introduction to Automata and Theory of Computation

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Lecture 30: Rice's Theorem (Part 2)

Rice's theorem

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Let P be a non-trivial property of Turing recognizable languages. Let $\mathcal{L}_P = \{M \mid L(M) \in P\}$. Then $\mathcal{L}_P$ is undecidable.

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Write down x on the tape.

Run M on w

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[1]We will remove this assumption later.

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Conclude undecidibility of $\mathcal{L}_P$.

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- **Observation:** If a machine M does not halt on input w then any final state is "useless".
- Given an input M, x for HALT, construct M_x that halts on every input (final state is useful!) if and only if M halts on x.

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