MULTIVARIABLE CALCULUS
LECTURE 20

It is now known to science that there are many more dimensions than the classical four. Scientists say that these don't normally impinge on the world because the extra dimensions are very small and curve in on themselves, and that since reality is fractal most of it is tucked inside itself. This means either that the universe is more full of wonders than we can hope to understand or, more probably, that scientists make things up as they go along. $\sim$ Terry Pratchett

1. Functions of Several Variable

1.1. **Motivation.**
Multivariable calculus (also known as multivariate calculus) is the extension of calculus in one variable to calculus with functions of several variables: the differentiation and integration of functions involving several variables, rather than just one.

We have already studied function of *one* variable and their calculus. You developed knowledge of calculus of functions of type

$$y = f(x) \quad , x \in I,$$

where I is an interval in $\mathbb{R}$. In these lectures, we extend these ideas to functions of *many variables*. In particular, we will learn limits, continuity, derivatives and their properties.

A function of two variables (x, y) maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The set D is called the domain of the function. The function of type:

$$z = f(x, y) \quad , (x, y) \in D,$$

where $D \subset \mathbb{R}^2$ is the domain of f and f is real valued. We write

$$f : D(\subset \mathbb{R}^2) \to \mathbb{R}.$$

Let us illustrate with the following examples :

- $f(x, y) = x \cos y + xy \sin x$, $D = \mathbb{R}^2$.
- $f(x, y) = \frac{1}{2x-y}$, $D = \{(x, y) \in \mathbb{R}^2 : y \neq 2x\}$.
- $f(x, y) = \sqrt{x + y}$, $D = \{(x, y) \in \mathbb{R}^2 : x + y \geq 0\}$.

1.2. **Graph of functions of two variables : Visualising and Sketching.** As a first step to understand functions of *two* variables, we wish to develop some methods for visualising and sketching their graphs. This function has two independent variables (x and y) and one dependent variable z. When graphing a function $y = f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x, y) in the plane, and every point in the plane has an ordered pair (x, y) associated with it. With a function of two variables, each ordered pair (x, y) the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x, y, z). The graph of a function $z = f(x, y)$ two variables is called a surface. The set of points in the plane where a function has a constant value $f(x, y) = c$ is called a *level curve of f.* The set of all points $(x, y, f(x, y))$ in the space, for (x, y) in the domain of f, is called graph of f.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x, y) coordinate system laying flat. Then, every point in the domain of the function f has a unique z-value associated with it. If z is positive, then the graphed point is located above the xy-plane, if z is negative, then the graphed point is located below the xy-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f. Later you will learn more about this.

We have now examined functions of more than one variable and seen how to graph them. Now we begin with the Calculus of functions for more than one variables. To begin with, let us recall the definition of limits in one variable.

Definition 1.1. Let f be a real valued function defined on (a, b) except possibly at $c \in (a, b)$. We say that the limit of f in $L \in \mathbb{R}$ as x approaches to c (or at $x = c$), written $\lim_{x\to c} f(x) = L$, if for every sequence $\{x_n\}$ in (a, b) with $x_n \neq a$ for all n and $x_n \to c$, we have $f(x_n) \to L$, as $n \to \infty$. Equivalently, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in (a, b)$ there holds

$$|f(x) - L| < \varepsilon.$$

Limit of a Function of Two Variables Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition 1.2. Consider a point (a, b) in $\mathbb{R}^2$. A $r > 0$ disk centered at point (a, b) is defined to be an open disk of radius $r > 0$ centered at point (a, b), denoted by $B_r(a, b)$, i.e.,

$$B_r(a, b) := \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < r\}. \tag{1}$$

Remark 1.1. The idea of a $r > 0$ disk appears in the definition of the limit of a function of two variables. If r is small, then all the points (a, b) in the r disk are close to (a, b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as

$$a - r < x < a + r, \quad i.e, \quad |x - a| < r.$$

In more than one dimension, we use a $r > 0$ disk.

Remark 1.2. A point (a, b) is said to be an *interior point* of a subset A of $\mathbb{R}^2$, if there exists $r > 0$ such that $B_r(a, b) \subset A$. A subset D is called *open* if each point of D is an interior point of D. For example $B_r(a, b)$.
A subset is said to be closed if its complement is an open subset of $\mathbb{R}^2$. For example

$$B_r(a, b) := \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} \leq r\}.$$

Definition 1.3. (Limit of a function of two variables) Let D be an open set in $\mathbb{R}^2$, $(a,b) \in D$ and f be a real valued function function defined on D except possibly at (a,b). Then limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ (depending on $\varepsilon > 0$) such that for all points (x,y) in a δ disk around (a,b) except possibly for (a,b) itself, the value of $f(x,y)$ no more than $\varepsilon > 0$ away from L.
In symbols, we write the following: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $(x,y) \in \mathbb{R}^2$ with

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \Rightarrow \quad |f(x,y) - L| < \varepsilon$$

Remark 1.3. Let f be a real valued function defined on $D \subset \mathbb{R}^2$, except possibly at (a,b). We say that the limit of f is $L \in \mathbb{R}$ as (x,y) approaches to (a,b) (or at $(x,y) = (a,b)$), written $\lim_{(x,y)\to(a,b)} f(x) = L$, if for every sequence $\{(x_n, y_n)\}$ in D with $(x_n, y_n) \neq (a,b)$ for all n and $(x_n, y_n) \to (a,b)$, we have $f(x_n, y_n) \to L$, as $n \to \infty$.

Remark 1.4. One can show as in the case of one variable calculus, the limit is unique. This means that the limit is independent of the path $(x_n, y_n) \to (a,b)$ or $(x,y) \to (a,b)$. This plays an important role in the existence of limits. We shall illustrate in the examples.

1.3. **Examples.**

(1) Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$,

$$lim_{(x,y)\to(1,1)} \frac{x^2 y}{x^4 + y^2} = \frac{1^2.1}{1+1} = 1.$$

(2) Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$,

$$lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x - y} = \lim_{(x,y)\to(0,0)} x^2 + xy + y^2 = 0.$$

(3) Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$,

$$lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{x - y} = 0$$

(4) Let f is defined by

$$f(x,y) = \frac{4xy^2}{x^2 + y^2}, \quad D = \mathbb{R}^2 \setminus \{(0,0)\}.$$

Compute the limit at $(x,y) = (0,0)$, using $\varepsilon-\delta$ definition. One can see immediately, using AM-GM inequality,

$$|f(x,y) - 0| = \frac{4|xy^2|}{x^2 + y^2} \leq \frac{2(x^2 + y^2)(\sqrt{x^2 + y^2})}{x^2 + y^2} = 2\sqrt{x^2 + y^2}.$$

Therefore $|f(x,y) - 0| < \varepsilon$, whenever $\sqrt{x^2 + y^2} < \frac{\varepsilon}{2} = \delta$.
Hence the limit $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(5) Let f is defined by

$$f(x,y) = \frac{x^3}{x^2+y^2}, \quad D = \mathbb{R}^2 \setminus \{(0,0)\}.$$

Compute the limit at $(x,y) = (0,0)$. We shall use polar coordinate : $x = r\cos\theta, y = r\sin\theta$, with $r > 0$, $0 \leq \theta < 2\pi$. Then

$$|f(x,y) - 0| := |f(r,\theta) - 0| = |r\cos\theta| \leq r \to 0, \quad \text{as } r \to 0.$$

1.4. **Limits that fail to exist : NOT Independent of path.**

Example 1.1. *Evaluate the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}.$$

Consider the path : $(x,y) \sim (0,y)$, i.e., the y-axis, along this path the limit,

$$\lim_{(0,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = 0.$$

Consider the path : $(x,y) \sim (x,0)$, i.e., the x-axis, along this path the limit,

$$\lim_{(x,0)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = 0.$$

So it is natural to think that the limit is 0. BUT this is not true. Consider the path : $(x,y) \sim (x,x)$, along this path the limit,

$$\lim_{(x,x)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \frac{1}{4}.$$

Hence the limit does not exist.

Example 1.2. *Evaluate the limit*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Consider the path : $(x,y) \sim (0,y)$, i.e., the y-axis, along this path the limit,

$$\lim_{(0,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0.$$

Consider the path : $(x,y) \sim (x,0)$, i.e., the x-axis, along this path the limit,

$$\lim_{(x,0)\to(0,0)} \frac{xy}{x^2+y^2} = 0.$$

Consider the path : $(x,y) \sim (x,mx)$, $m \in \mathbb{R}$, along this path the limit,

$$\lim_{(x,mx)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{m}{1+m^2} \neq 0 \quad (if m \neq 0).$$

It depends on m. Hence the limit does not exist.

Example 1.3. *Evaluate the limit*

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}.$$

Consider the path : $(x,y) \sim (x,mx)$, $m \in \mathbb{R}$, *along this path the limit,*

$$\lim_{(x,mx)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{(x,mx)\to(0,0)} \frac{2mx}{x^2+m^2} = 0.$$

Consider the path : $(x,y) \sim (x,mx^2)$, $m \in \mathbb{R}$, *along this path the limit,*

$$\lim_{(x,mx^2)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \frac{2m}{1+m^2} \neq 0 \quad (if m \neq 0).$$

It depends on m*. Hence the limit does not exist.*