Lecture 3

§ Gaussian Elimination.

Ex:

$$x_1 + 4x_2 - 3x_3 = 2$$
$$3x_1 - 2x_2 - x_3 = -1$$
$$-x_1 + 10x_2 - 5x_3 = 3.$$

Augmented matrix
$$\left[\begin{array}{ccc|c} 1 & 4 & -3 & 2 \\ 3 & -2 & -1 & -1 \\ -1 & 10 & -5 & 3 \end{array}\right]$$

$R_2 \to R_2 - 3R_1 \quad R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 4 & -3 & 2 \\ 0 & -14 & 8 & -7 \\ 0 & 14 & -8 & 5 \end{array}\right]$$

$R_3 \to R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 4 & -3 & 2 \\ 0 & -14 & 8 & -7 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$$x_1 + 4x_2 - 3x_3 = 2$$
$$-14x_2 + 8x_3 = -7$$

No solution! $\boxed{0 = -2}$

Remark: If at any point of solving the echelon form there is a row of the form $[0 \cdots 0 \mid c]$ then the system will always have no solution

§ Gauss-Jordan Elimination.

① Multiply each nonzero row by the reciprocal of the pivot so that we end up with 1 as the leading term in each nonzero row

② Use row operations to introduce zeros in the entries above the pivot positions.

Ex:

$$
\begin{aligned}
2x_1 - 2x_2 - 6x_3 + x_4 &= 3 \\
-x_1 + x_2 + 3x_3 - x_4 &= -3 \\
x_1 - 2x_2 - x_3 + x_4 &= 2
\end{aligned}
$$

Ans:

$$
\left[\begin{array}{cccc|c}
2 & -2 & -6 & 1 & 3 \\
-1 & 1 & 3 & -1 & -3 \\
1 & -2 & -1 & 1 & 2
\end{array}\right]
$$

Echelon form:

$$
\left[\begin{array}{cccc|c}
1 & -2 & -1 & 1 & 2 \\
0 & -1 & 2 & 0 & -1 \\
0 & 0 & 0 & -1 & -3
\end{array}\right]
$$

$R_2 \to R_2(-1)$

$R_3 \to R_3(-1)$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 \to R_1 + 2R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & -5 & 1 & 4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 \to R_1 - R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$x_4 = 3$ $\qquad$ $x_3 = s$ free parameter

$x_2 - 2x_3 = 1 \Rightarrow x_2 = 1 + 2s$

$x_1 - 5x_3 = 1 \Rightarrow x_1 = 1 + 5s$.

Def: A matrix is in reduced Echelon form (or reduced row echelon form) if

1. it is in Echelon form
2. All the pivot positions are 1
3. The only nonzero term in a pivot col is in the pivot position

Theorem: A given matrix is equivalent to a UNIQUE matrix that is in reduced echelon form.

Ex:

$$x_1 - 2x_2 - 3x_3 = -1$$
$$x_1 - x_2 - 2x_3 = 1$$
$$-x_1 + 3x_2 + 5x_3 = 2$$

Augmented System

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 1 & -1 & -2 & 1 \\ -1 & 3 & 5 & 2 \end{array}\right]$$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Echelon form.

$$R_2 \to R_2 - R_3$$

$$R_1 \to R_1 + 3R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$R_1 \to R_1 + 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

reduced row Echelon form

$x_1 = 2$

$x_2 = 3$

$x_3 = -1$

§ Homogeneous Linear System:

$A\underline{x} = \underline{0}$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \qquad (*)$$

always have a soln: $x_1 = x_2 \cdots = x_n = 0$

trivial solution

If there are additional solutions, they are called nontrivial solution.

Ex:

$$\begin{aligned} 2x_1 - 6x_2 - x_3 + 8x_4 &= 0 \\ x_1 - 3x_2 - x_3 + 6x_4 &= 0 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

Aug: $\left[\begin{array}{cccc|c} 2 & -6 & -1 & 8 & 0 \\ 1 & -3 & -1 & 6 & 0 \\ -1 & 3 & -1 & 2 & 0 \end{array}\right]$

$R_1 \leftrightarrow R_2$ $\left[\begin{array}{cccc|c} 1 & -3 & -1 & 6 & 0 \\ 2 & -6 & -1 & 8 & 0 \\ -1 & 3 & -1 & 2 & 0 \end{array}\right]$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$\left[\begin{array}{cccc|c} 1 & -3 & -1 & 6 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & -2 & 8 & 0 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{cccc|c} 1 & -3 & -1 & 6 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - 3x_2 + 2x_4 = 0$

$x_3 - 4x_4 = 0$

$x_2 = s_1 \qquad x_4 = s_2$

$x_3 = 4s_2 \qquad x_1 = 3s_1 - 2s_2$

Proof of why a linear system has only 3 choices,

(1) no soln (2) ∞ many sol (3) unique sol.

$A\underline{x} = b$ be the linear system.

Compute the Augmented matrix $\longrightarrow$ Echelon form.

(a) The system has a row of the form $[0 \cdots 0 \vdots c]$ &

its corresponding equation is of the form

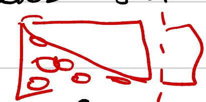
$0 = c$

$\Rightarrow$ system has no solutions

$$\left[\begin{array}{cccc|c} \ddots & & & & \\ 0 & 0 & \cdots & 0 & c \end{array}\right]$$

If (1a) does not occur, then ~~the~~ either of the following has to happen

(b) The Echelon form has no free parameters $\Rightarrow$ it has a unique sol



(2) The Echelon form has one or more free parameters & hence ∞ many solution.

non pivot cols.

Def: (1) For a system of equations a variable that appears in the first term in atleast one equation is called a leading variable

(2) For a system in echelon form, any variable that is not a leading variable is called a "free variable".

(3) Rank of a matrix = number of nonzero rows in the echelon form.

rank of
$$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2$$
← zero row.

$$\text{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3.$$

Theorem: If $A\underline{x} = b$ is a linear system of equations with n variables $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

then the number of free variables $= n - \text{rank } A$.

Theorem: $A\underline{x}=0$ is a homogeneous linear system.

~~If~~ rank $A<n \iff A\underline{x}=0$ has a nontrivial sol.

Theorem: $A\underline{x}=\underline{b}$ is a homogeneous system of linear equations with fewer equations than unknowns $(m<n)$ has at least one free variable & hence ∞ many solutions. (Atleast one solution)

Remark: If a system of linear equations has a $0=0$ equation then the system need not always have ∞ many solutions. It can have no solutions or unique solution.

$A\underline{x}=b$

Echelon form
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
Unique sol.

Echelon
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
No solution.