31. Determine the dimensions of $Sym_n(\mathbb{R})$ and $Skew_n(\mathbb{R})$, and show that

$$\dim[Sym_n(\mathbb{R})] + \dim[Skew_n(\mathbb{R})] = \dim[M_n(\mathbb{R})].$$

For Problems 32–34, a subspace S of a vector space V is given. Determine a basis for S and extend your basis for S to obtain a basis for V.

32. $V = \mathbb{R}^3$, S is the subspace consisting of all points lying on the plane with Cartesian equation

$$x + 4y - 3z = 0.$$

33. $V = M_2(\mathbb{R})$, S is the subspace consisting of all matrices of the form

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

34. $V = P_2$, S is the subspace consisting of all polynomials of the form $(2a_1+a_2)x^2+(a_1+a_2)x+(3a_1-a_2)$.

35. Let S be a basis for P_{n-1}. Prove that $S \cup \{x^n\}$ is a basis for P_n.

36. Generalize the previous problem as follows. Let S be a basis for P_{n-1}, and let p be any polynomial of degree n. Prove that $S \cup \{p\}$ is a basis for P_n.

37. **(a)** What is the dimension of $\mathbb{C}^n$ as a real vector space? Determine a basis.

(b) What is the dimension of $\mathbb{C}^n$ as a complex vector space? Determine a basis.

4.7 Change of Basis

Throughout this section, we restrict our attention to vector spaces that are *finite-dimensional*. If we have a (finite) basis for such a vector space V, then, since the vectors in a basis span V, any vector in V can be expressed as a linear combination of the basis vectors. The next theorem establishes that there is only one way in which we can do this.

Theorem 4.7.1 If V is a vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then every vector $\mathbf{v} \in V$ can be written **uniquely** as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

Proof Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ span V, every vector $\mathbf{v} \in V$ can be expressed as

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n, \tag{4.7.1}$$

for some scalars $a_1, a_2, \dots, a_n$. Suppose also that

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n, \tag{4.7.2}$$

for some scalars $b_1, b_2, \dots, b_n$. We will show that $a_i = b_i$ for each i, which will prove the uniqueness assertion of this theorem. Subtracting Equation (4.7.2) from Equation (4.7.1) yields

$$(a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n = \mathbf{0}. \tag{4.7.3}$$

But $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent, and so Equation (4.7.3) implies that

$$a_1 - b_1 = 0, \qquad a_2 - b_2 = 0, \qquad \dots, \qquad a_n - b_n = 0.$$

That is, $a_i = b_i$ for each $i = 1, 2, \dots, n$. ■

Remark The converse of Theorem 4.7.1 is also true. That is, if every vector $\mathbf{v}$ in a vector space V can be written uniquely as a linear combination of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V. The proof of this fact is left as an exercise (Problem 38).

Up to this point, we have not paid particular attention to the order in which the vectors of a basis are listed. However, in the remainder of this section, this will become an important consideration. By an **ordered basis** for a vector space, we mean a basis in which we are keeping track of the order in which the basis vectors are listed.

DEFINITION 4.7.2

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an ordered basis for V and $\mathbf{v}$ is a vector in V, then the scalars $c_1, c_2, \dots, c_n$ in the unique n-tuple $(c_1, c_2, \dots, c_n)$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

are called the **components of v relative to the ordered basis** $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. We denote the *column* vector consisting of the components of $\mathbf{v}$ relative to the ordered basis B by $[\mathbf{v}]_B$, and we call $[\mathbf{v}]_B$ the **component vector of v relative to** B.

Example 4.7.3

Determine the components of the vector $\mathbf{v} = (1, 7)$ relative to the ordered basis $B = \{(1, 2), (3, 1)\}$.

Solution: If we let $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (3, 1)$, then since these vectors are not collinear, $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$. We must determine constants c_1, c_2 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{v}.$$

We write

$$c_1(1, 2) + c_2(3, 1) = (1, 7).$$

This requires that

$$c_1 + 3c_2 = 1 \qquad \text{and} \qquad 2c_1 + c_2 = 7.$$

The solution to this system is $(4, -1)$, which gives the components of $\mathbf{v}$ relative to the ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$. (See Figure 4.7.1.) Thus,

$$\mathbf{v} = 4\mathbf{v}_1 - \mathbf{v}_2.$$

Therefore, we have

$$[\mathbf{v}]_B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}. \qquad \square$$

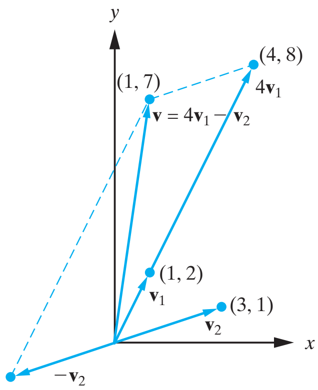

Figure 4.7.1: The components of the vector $\mathbf{v} = (1, 7)$ relative to the basis $\{(1, 2), (3, 1)\}$.

Remark In the preceding example, the component vector of $\mathbf{v} = (1, 7)$ relative to the ordered basis $B' = \{(3, 1), (1, 2)\}$ is

$$[\mathbf{v}]_{B'} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

Thus, even though the bases B and B' contain the same vectors, the fact that the vectors are listed in different order affects the components of the vectors in the vector space.

Example 4.7.4

In P_2, determine the component vector of $p(x) = 5 + 7x - 3x^2$ relative to the following:

(a) The standard (ordered) basis $B = \{1, x, x^2\}$.

(b) The ordered basis $C = \{1 + x, 2 + 3x, 5 + x + x^2\}$.

Solution:

(a) The given polynomial is already written as a linear combination of the standard basis vectors. Consequently, the components of $p(x) = 5 + 7x - 3x^2$ relative to the standard basis B are 5, 7, and -3. We write

$$[p(x)]_B = \begin{bmatrix} 5 \\ 7 \\ -3 \end{bmatrix}.$$

(b) The components of $p(x) = 5 + 7x - 3x^2$ relative to the ordered basis

$$C = \{1 + x, 2 + 3x, 5 + x + x^2\}$$

are c_1, c_2, and c_3, where

$$c_1(1 + x) + c_2(2 + 3x) + c_3(5 + x + x^2) = 5 + 7x - 3x^2.$$

That is,

$$(c_1 + 2c_2 + 5c_3) + (c_1 + 3c_2 + c_3)x + c_3x^2 = 5 + 7x - 3x^2.$$

Hence, c_1, c_2, and c_3 satisfy

$$\begin{aligned} c_1 + 2c_2 + 5c_3 &= 5, \\ c_1 + 3c_2 + c_3 &= 7, \\ c_3 &= -3. \end{aligned}$$

The augmented matrix of this system has reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -3 \end{bmatrix},$$

so that the system has solution $(40, -10, -3)$, which gives the required components. Hence, we can write

$$5 + 7x - 3x^2 = 40(1 + x) - 10(2 + 3x) - 3(5 + x + x^2).$$

Therefore,

$$[p(x)]_C = \begin{bmatrix} 40 \\ -10 \\ -3 \end{bmatrix}. \qquad \square$$

Change-of-Basis Matrix

The preceding example naturally motivates the following question: If we are given two different ordered bases for an n-dimensional vector space V, say

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \quad \text{and} \quad C = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}, \tag{4.7.4}$$

and a vector $\mathbf{v}$ in V, how are $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$ related? In practical terms, we may know the components of $\mathbf{v}$ relative to B and wish to know the components of $\mathbf{v}$ relative to a different ordered basis C. This question actually arises quite often, since different bases are advantageous in different circumstances, so it is useful to be able to convert components of a vector relative to one basis to components relative to another basis. The tool we need in order to do this efficiently is the change-of-basis matrix. Before we describe this matrix, we pause to record the linearity properties satisfied by the components of a vector. These properties will facilitate the discussion that follows.

Lemma 4.7.5 Let V be a vector space with ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, let $\mathbf{x}$ and $\mathbf{y}$ be vectors in V, and let c be a scalar. Then we have

(a) $[\mathbf{x}+\mathbf{y}]_B = [\mathbf{x}]_B + [\mathbf{y}]_B$.

(b) $[c\mathbf{x}]_B = c[\mathbf{x}]_B$.

Proof Write

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \qquad \text{and} \qquad \mathbf{y} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n,$$

so that

$$\mathbf{x}+\mathbf{y} = (a_1+b_1)\mathbf{v}_1 + (a_2+b_2)\mathbf{v}_2 + \cdots + (a_n+b_n)\mathbf{v}_n.$$

Hence,

$$[\mathbf{x}+\mathbf{y}]_B = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [\mathbf{x}]_B + [\mathbf{y}]_B,$$

which establishes (a). The proof of (b) is left as an exercise (Problem 37). ■

DEFINITION 4.7.6

Let V be an n-dimensional vector space with ordered bases B and C given in (4.7.4). We define the **change-of-basis matrix from B to C** by

$$P_{C\leftarrow B} = \Big[[\mathbf{v}_1]_C, [\mathbf{v}_2]_C, \dots, [\mathbf{v}_n]_C \Big]. \tag{4.7.5}$$

In words, we determine the components of each vector in the "old basis" B with respect the "new basis" C and write the component vectors in the columns of the change-of-basis matrix.

Remark Of course, there is also a change-of-basis matrix from C to B, given by

$$P_{B\leftarrow C} = \Big[[\mathbf{w}_1]_B, [\mathbf{w}_2]_B, \dots, [\mathbf{w}_n]_B \Big].$$

We will see shortly that the matrices $P_{B\leftarrow C}$ and $P_{C\leftarrow B}$ are intimately related.

Our first order of business at this point is to see why the matrix in (4.7.5) converts the components of a vector relative to B into components relative to C. Let $\mathbf{v}$ be a vector in V and write

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Then

$$[\mathbf{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Hence, using Theorem 2.2.9 and Lemma 4.7.5, we have

$$P_{C\leftarrow B}[\mathbf{v}]_B = a_1[\mathbf{v}_1]_C + a_2[\mathbf{v}_2]_C + \cdots + a_n[\mathbf{v}_n]_C = [a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n]_C = [\mathbf{v}]_C.$$

This calculation shows that premultiplying the component vector of $\mathbf{v}$ relative to B by the change of basis matrix $P_{C\leftarrow B}$ yields the component vector of $\mathbf{v}$ relative to C:

$$\boxed{[\mathbf{v}]_C = P_{C\leftarrow B}[\mathbf{v}]_B.} \tag{4.7.6}$$

Example 4.7.7 Let $V = \mathbb{R}^2$, $B = \{(1, 2), (3, 4)\}$, $C = \{(7, 3), (4, 2)\}$, and $\mathbf{v} = (1, 0)$. It is routine to verify that B and C are bases for V.

(a) Determine $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$.

(b) Find $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$.

(c) Use (4.7.6) to compute $[\mathbf{v}]_C$, and compare your answer with (a).

Solution:

(a) Solving $(1, 0) = a_1(1, 2) + a_2(3, 4)$, we find $a_1 = -2$ and $a_2 = 1$. Hence,

$$[\mathbf{v}]_B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Likewise, setting $(1, 0) = b_1(7, 3) + b_2(4, 2)$, we find $b_1 = 1$ and $b_2 = -1.5$. Hence,

$$[\mathbf{v}]_C = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}.$$

(b) A short calculation shows that

$$[(1, 2)]_C = \begin{bmatrix} -3 \\ 5.5 \end{bmatrix} \quad \text{and} \quad [(3, 4)]_C = \begin{bmatrix} -5 \\ 9.5 \end{bmatrix}.$$

Thus, we have

$$P_{C\leftarrow B} = \begin{bmatrix} -3 & -5 \\ 5.5 & 9.5 \end{bmatrix}.$$

Likewise, another short calculation shows that

$$[(7, 3)]_B = \begin{bmatrix} -9.5 \\ 5.5 \end{bmatrix} \quad \text{and} \quad [(4, 2)]_B = \begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

Hence,

$$P_{B\leftarrow C} = \begin{bmatrix} -9.5 & -5 \\ 5.5 & 3 \end{bmatrix}.$$

(c) We compute as follows:

$$P_{C\leftarrow B}[\mathbf{v}]_B = \begin{bmatrix} -3 & -5 \\ 5.5 & 9.5 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} = [\mathbf{v}]_C,$$

as we found in part (a). □

The reader may have noticed a close resemblance between the two matrices $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$ computed in part (b) of the preceding example. In fact, a brief calculation shows that

$$P_{C\leftarrow B}P_{B\leftarrow C} = I_2 = P_{B\leftarrow C}P_{C\leftarrow B}.$$

The two change-of-basis matrices are inverses of each other. This turns out to be always true. To see why, consider again Equation (4.7.6). If we premultiply both sides of (4.7.6) by the matrix $P_{B\leftarrow C}$, we get

$$P_{B\leftarrow C}[\mathbf{v}]_C = P_{B\leftarrow C}P_{C\leftarrow B}[\mathbf{v}]_B. \tag{4.7.7}$$

Rearranging the roles of B and C in (4.7.6), the left side of (4.7.7) is simply $[\mathbf{v}]_B$. Thus,

$$P_{B\leftarrow C}P_{C\leftarrow B}[\mathbf{v}]_B = [\mathbf{v}]_B.$$

Since this is true for any vector $[\mathbf{v}]_B$ in $\mathbb{R}^n$, this implies that

$$P_{B\leftarrow C}P_{C\leftarrow B} = I_n,$$

the $n \times n$ identity matrix. Likewise, a similar calculation shows that

$$P_{C\leftarrow B}P_{B\leftarrow C} = I_n.$$

Thus, we have proved that

$$\boxed{\text{The matrices } P_{C\leftarrow B} \text{ and } P_{B\leftarrow C} \text{ are inverses of one another.}}$$

Example 4.7.8 Let $V = P_2$, and let $B = \{1, 1+x, 1+x+x^2\}$, and $C = \{2+x+x^2, x+x^2, x\}$. It is routine to verify that B and C are bases for V. Find the change-of-basis matrix from B to C, and use it to calculate the change-of-basis matrix from C to B.

Solution: We set $1 = a_1(2+x+x^2) + a_2(x+x^2) + a_3x$. With a quick calculation, we find that $a_1 = 0.5$, $a_2 = -0.5$, and $a_3 = 0$. Next, we set $1 + x = b_1(2+x+x^2) + b_2(x+x^2) + b_3x$, and we find that $b_1 = 0.5$, $b_2 = -0.5$, and $b_3 = 1$. Finally, we set $1 + x + x^2 = c_1(2+x+x^2) + c_2(x+x^2) + c_3x$, from which it follows that $c_1 = 0.5$, $c_2 = 0.5$, and $c_3 = 0$. Hence, we have

$$P_{C\leftarrow B} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus, we have

$$P_{B\leftarrow C} = (P_{C\leftarrow B})^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$ □

In much the same way that we showed above that the matrices $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ are inverses of one another, we can make the following observation.

Theorem 4.7.9 Let V be a vector space with ordered bases A, B, and C. Then

$$P_{C \leftarrow A} = P_{C \leftarrow B} P_{B \leftarrow A}. \tag{4.7.8}$$

Proof Using (4.7.6), for every $\mathbf{v} \in V$, we have

$$P_{C \leftarrow B} P_{B \leftarrow A} [\mathbf{v}]_A = P_{C \leftarrow B} [\mathbf{v}]_B = [\mathbf{v}]_C = P_{C \leftarrow A} [\mathbf{v}]_A,$$

so that premultiplication of $[\mathbf{v}]_A$ by either matrix in (4.7.8) yields the same result. Hence, the matrices on either side of (4.7.8) are the same. ■

We conclude this section by using Theorem 4.7.9 to show how an arbitrary change-of-basis matrix $P_{C \leftarrow B}$ in $\mathbb{R}^n$ can be expressed as a product of change-of-basis matrices involving the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $C = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be arbitrary ordered bases for $\mathbb{R}^n$. Since $[\mathbf{v}]_E = \mathbf{v}$ for all column vectors $\mathbf{v}$ in $\mathbb{R}^n$, the matrices

$$P_{E \leftarrow B} = [[\mathbf{v}_1]_E, [\mathbf{v}_2]_E, \dots, [\mathbf{v}_n]_E] = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$$

and

$$P_{E \leftarrow C} = [[\mathbf{w}_1]_E, [\mathbf{w}_2]_E, \dots, [\mathbf{w}_n]_E] = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n]$$

can be written down immediately. Using these matrices, together with Theorem 4.7.9, we can compute the arbitrary change-of-basis matrix $P_{C \leftarrow B}$ with ease:

$$P_{C \leftarrow B} = P_{C \leftarrow E} P_{E \leftarrow B} = (P_{E \leftarrow C})^{-1} P_{E \leftarrow B}.$$

Exercises for 4.7

Key Terms

Ordered basis, Components of a vector relative to an ordered basis, Change-of-basis matrix.

Skills

- Be able to find the components of a vector relative to a given ordered basis for a vector space V.
- Be able to compute the change-of-basis matrix for a vector space V from one ordered basis B to another ordered basis C.
- Be able to use the change-of-basis matrix from B to C to determine the components of a vector relative to C from the components of the vector relative to B.
- Be familiar with the relationship between the two change-of-basis matrices $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$.

True-False Review

For Questions 1–8, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. Every vector in a finite-dimensional vector space V can be expressed uniquely as a linear combination of vectors comprising a basis for V.

2. The change-of-basis matrix $P_{B \leftarrow C}$ acts on the component vector of a vector $\mathbf{v}$ relative to the basis C and produces the component vector of $\mathbf{v}$ relative to the basis B.

3. A change-of-basis matrix is always a square matrix.

4. A change-of-basis matrix is always invertible.

5. For any vectors $\mathbf{v}$ and $\mathbf{w}$ in a finite-dimensional vector space V with basis B, we have $[\mathbf{v}-\mathbf{w}]_B = [\mathbf{v}]_B - [\mathbf{w}]_B$.

6. If the bases B and C for a vector space V contain the same set of vectors, then $[\mathbf{v}]_B = [\mathbf{v}]_C$ for every vector $\mathbf{v}$ in V.

7. If B and C are bases for a finite-dimensional vector space V, and $\mathbf{v}$ and $\mathbf{w}$ are in V such that $[\mathbf{v}]_B = [\mathbf{w}]_C$, then $\mathbf{v} = \mathbf{w}$.

8. The matrix $P_{B\leftarrow B}$ is the identity matrix for any basis B for V.

Problems

For Problems 1–13, determine the component vector of the given vector in the vector space V relative to the given ordered basis B.

1. $V = \mathbb{R}^2$; $B = \{(2,-2), (1,4)\}$; $\mathbf{v} = (5,-10)$.

2. $V = \mathbb{R}^2$; $B = \{(-1,3), (3,2)\}$; $\mathbf{v} = (8,-2)$.

3. $V = \mathbb{R}^3$; $B = \{(1,0,1), (1,1,-1), (2,0,1)\}$; $\mathbf{v} = (-9,1,-8)$.

4. $V = \mathbb{R}^3$; $B = \{(1,-6,3), (0,5,-1), (3,-1,-1)\}$; $\mathbf{v} = (1,7,7)$.

5. $V = \mathbb{R}^3$; $B = \{(3,-1,-1), (1,-6,3), (0,5,-1)\}$; $\mathbf{v} = (1,7,7)$.

6. $V = \mathbb{R}^3$; $B = \{(-1,0,0), (0,0,-3), (0,-2,0)\}$; $\mathbf{v} = (5,5,5)$.

7. $V = P_2$; $B = \{x^2+x, 2+2x, 1\}$; $p(x) = -4x^2 + 2x + 6$.

8. $V = P_2$; $B = \{5-3x, 1, 1+2x^2\}$; $p(x) = 15 - 18x - 30x^2$.

9. $V = P_3$; $B = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$; $p(x) = 4 - x + x^2 - 2x^3$.

10. $V = P_3$; $B = \{x^3+x^2, x^3-1, x^3+1, x^3+x\}$; $p(x) = 8 + x + 6x^2 + 9x^3$.

11. $V = M_2(\mathbb{R})$;
$B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$;
$A = \begin{bmatrix} -3 & -2 \\ -1 & 2 \end{bmatrix}$.

12. $V = M_2(\mathbb{R})$;
$B = \left\{ \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \right\}$;
$A = \begin{bmatrix} -10 & 16 \\ -15 & -14 \end{bmatrix}$.

13. $V = M_2(\mathbb{R})$;
$B = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \right\}$;
$A = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

14. Let $\mathbf{v}_1 = (0,6,3)$, $\mathbf{v}_2 = (3,0,3)$, and $\mathbf{v}_3 = (6,-3,0)$. Determine the component vector of an arbitrary vector $\mathbf{v} = (x,y,z)$ relative to the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

15. Let $p_1(x) = 1 + x$, $p_2(x) = x(x-1)$, and $p_3(x) = 1 + 2x^2$. Determine the component vector of an arbitrary polynomial $p(x) = a_0 + a_1x + a_2x^2$ relative to the ordered basis $\{p_1, p_2, p_3\}$.

For Problems 16–25, find the change-of-basis matrix $P_{C\leftarrow B}$ from the given ordered basis B to the given ordered basis C of the vector space V.

16. $V = \mathbb{R}^2$; $B = \{(9,2), (4,-3)\}$; $C = \{(2,1), (-3,1)\}$.

17. $V = \mathbb{R}^2$; $B = \{(-5,-3), (4,28)\}$; $C = \{(6,2), (1,-1)\}$.

18. $V = \mathbb{R}^3$; $B = \{(2,-5,0), (3,0,5), (8,-2,-9)\}$; $C = \{(1,-1,1), (2,0,1), (0,1,3)\}$.

19. $V = \mathbb{R}^3$; $B = \{(-7,4,4), (4,2,-1), (-7,5,0)\}$; $C = \{(1,1,0), (0,1,1), (3,-1,-1)\}$.

20. $V = P_1$; $B = \{7-4x, 5x\}$; $C = \{1-2x, 2+x\}$.

21. $V = P_2$; $B = \{-4+x-6x^2, 6+2x^2, -6-2x+4x^2\}$; $C = \{1-x+3x^2, 2, 3+x^2\}$.

22. $V = P_3$;
$B = \{-2+3x+4x^2-x^3, 3x+5x^2+2x^3, -5x^2-5x^3, 4+4x+4x^2\}$; $C = \{1-x^3, 1+x, x+x^2, x^2+x^3\}$.

23. $V = P_2$; $B = \{2+x^2, -1-6x+8x^2, -7-3x-9x^2\}$; $C = \{1+x, -x+x^2, 1+2x^2\}$.

24. $V = M_2(\mathbb{R})$;
$B = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix} \right\}$;
$C = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

25. $V = M_2(\mathbb{R})$; $B = \{E_{12}, E_{22}, E_{21}, E_{11}\}$;
$C = \{E_{22}, E_{11}, E_{21}, E_{12}\}$.

For Problems 26–31, find the change-of-basis matrix $P_{B \leftarrow C}$ from the given basis C to the given basis B of the vector space V.

26. V, B, and C from Problem 16.

27. V, B, and C from Problem 17.

28. V, B, and C from Problem 18.

29. V, B, and C from Problem 20.

30. V, B, and C from Problem 22.

31. V, B, and C from Problem 25.

For Problems 32–36, verify Equation (4.7.6) for the given vector.

32. $\mathbf{v} = (-5, 3)$; V, B, and C from Problem 16.

33. $\mathbf{v} = (-1, 2, 0)$; V, B, and C from Problem 19.

34. $p(x) = 6 - 4x$; V, B, and C from Problem 20.

35. $p(x) = 5 - x + 3x^2$; V, B, and C from Problem 21.

36. $A = \begin{bmatrix} -1 & -1 \\ -4 & 5 \end{bmatrix}$; V, B, and C from Problem 24.

37. Prove part (b) of Lemma 4.7.5.

38. Prove that if every vector $\mathbf{v}$ in a vector space V can be written uniquely as a linear combination of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V.

39. Show that if B is a basis for a finite-dimensional vector space V, and C is a basis obtained by reordering the vectors in B, then the matrices $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ each contain exactly one 1 in each row and column, and zeros elsewhere.

4.8 Row Space and Column Space

In this section, we consider two vector spaces that can be associated with any $m \times n$ matrix. For simplicity, we will assume that the matrices have real entries, although the results that we establish can easily be extended to matrices with complex entries.

Row Space

Let $A = [a_{ij}]$ be an $m \times n$ real matrix. The row vectors of this matrix are row n-vectors, and therefore they can be associated with vectors in $\mathbb{R}^n$. The subspace of $\mathbb{R}^n$ spanned by these vectors is called the **row space** of A and denoted rowspace(A). For example, if

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 9 & -7 \end{bmatrix},$$

then

$$\text{rowspace}(A) = \text{span}\{(2, -1, 3), (5, 9, -7)\}.$$

For a general $m \times n$ matrix A, how can we obtain a basis for rowspace(A)? By its very definition, the row space of A is spanned by the row vectors of A, but these may not be linearly independent, hence the row vectors of A do not necessarily form a basis for rowspace(A). We wish to determine a systematic and efficient method for obtaining a basis for the row space. Perhaps not surprisingly, it involves the use of elementary row operations.

If we perform elementary row operations on A, then we are merely taking linear combinations of vectors in rowspace(A), and we therefore might suspect that the row space of the resulting matrix coincides with the row space of A. This is the content of the following theorem.

Theorem 4.8.1 If A and B are row-equivalent matrices, then

$$\text{rowspace}(A) = \text{rowspace}(B).$$