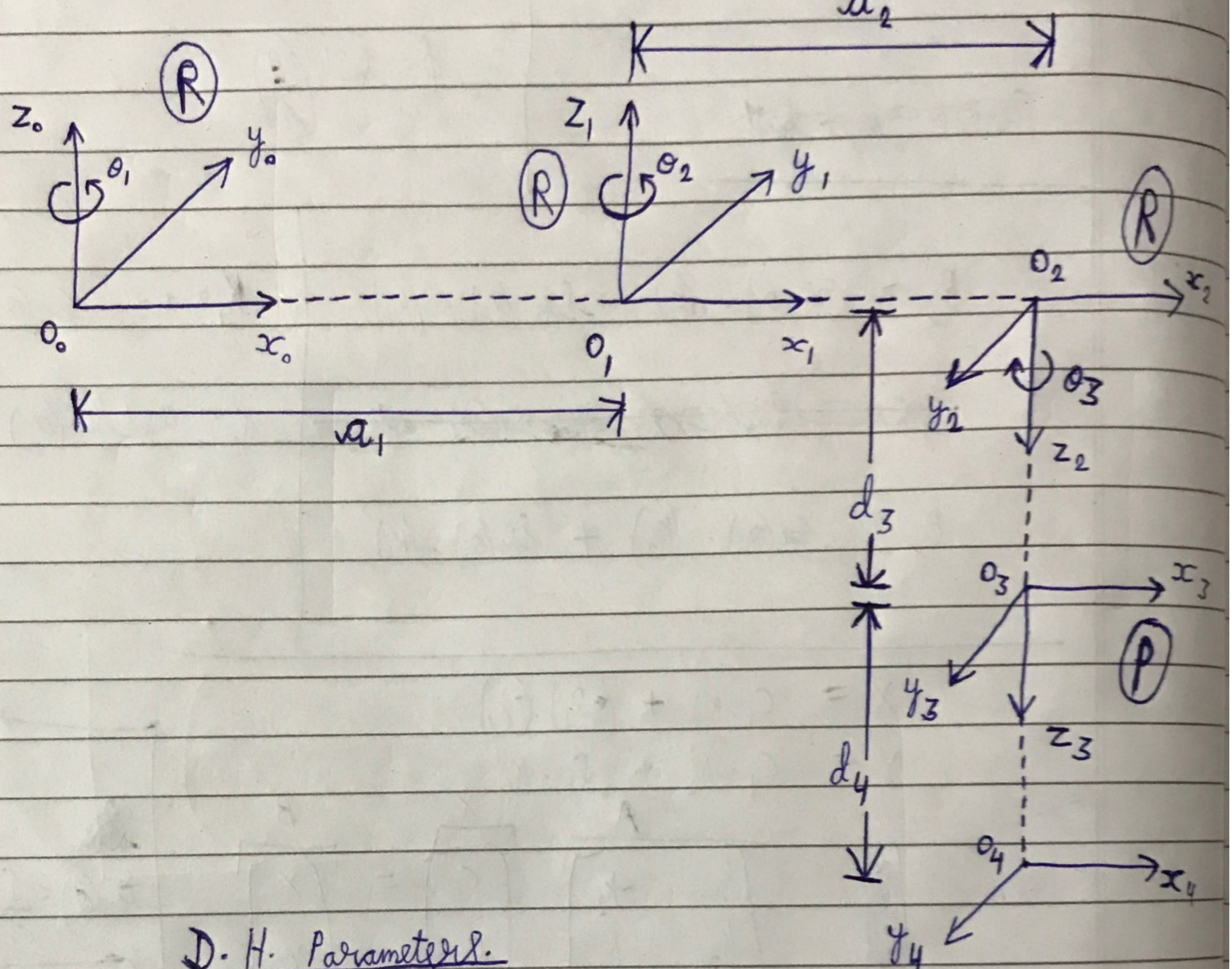
SCARA Manipulator.



D. H. Parameters.

	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	180°	0	θ_2^*
3	0	0	d_3	θ_3^*
4	0	0	d_4^*	0

$\Rightarrow (\theta_1, \theta_2, \theta_3, d_4)$

$\downarrow$

Joint Variables.

$$^0T_4(q_1, q_2, q_3, q_4) = {}^0T_1(q_1) \cdot {}^1T_2(q_2) \cdot {}^2T_3(q_3) \cdot {}^3T_4(q_4)$$

Now, $T = \begin{bmatrix} R_{(3\times3)} & t_{(3\times1)} \\ 0_{(1\times3)} & 1_{(1\times1)} \end{bmatrix} \in R^{4\times4}$, $q_i = (a_i, \alpha_i, d_i, \theta_i)$

$$R = \begin{bmatrix} C_\theta & -S_\theta C_\alpha & S_\theta S_\alpha \\ S_\theta & C_\theta C_\alpha & -C_\theta S_\alpha \\ 0 & S_\alpha & C_\alpha \end{bmatrix}$$

- $${}^0T_1(q_1) = {}^0T_1(a_1, 0, 0, \theta_1) = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & a_1\cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & a_1\sin(\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $${}^1T_2(q_2) = {}^1T_2(a_2, 180, 0, \theta_2) = \begin{bmatrix} \cos(\theta_2) & \sin(\theta_2) & 0 & a_2\cos(\theta_2) \\ \sin(\theta_2) & -\cos(\theta_2) & 0 & a_2\sin(\theta_2) \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $${}^2T_3(q_3) = {}^2T_3(0, 0, d_3, \theta_3) = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $${}^3T_4(q_4) = {}^3T_4(0, 0, d_4, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, ${}^{0}T_4 = {}^{0}T_1 \; {}^{1}T_2 \; {}^{2}T_3 \; {}^{3}T_4$

$$
{}^{0}T_4 = \begin{bmatrix} C_{12}C_3 + S_{12}S_3 & -C_{12}S_3 + S_{12}C_3 & 0 & a_1C_1 + a_2C_{12} \\ S_{12}C_3 - C_{12}S_3 & -S_{12}S_3 - C_3C_{12} & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\Rightarrow \quad {}^{0}T_4 = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 & P_x \\ \sin(\phi) & -\cos(\phi) & 0 & P_y \\ 0 & 0 & -1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\left.\begin{aligned}
\text{where, } \phi &= \theta_1 + \theta_2 - \theta_3 \\
P_x &= a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) \\
P_y &= a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) \\
P_z &= -d_3 - d_4
\end{aligned}\right] \text{Forward Kinematics}
$$

$$
\Rightarrow \text{Now, } (P_x)^2 + (P_y)^2 = a_1^2 \cos^2(\theta_1) + a_2^2 \cos^2(\theta_1+\theta_2) + 2a_1a_2 \cos(\theta_1)\cos(\theta_1+\theta_2)
$$
$$
+ a_1^2 \sin^2(\theta_1) + a_2^2 \sin^2(\theta_1+\theta_2) + 2a_1a_2 \sin(\theta_1)\sin(\theta_1+\theta_2)
$$

$$
(P_x)^2 + (P_y)^2 = a_1^2\left[\cos^2(\theta_1) + \sin^2(\theta_1)\right] + a_2^2\left[\cos^2(\theta_1+\theta_2) + \sin^2(\theta_1+\theta_2)\right]
$$
$$
+ 2a_1a_2\left[\cos(\theta_1)\cos(\theta_1+\theta_2) + \sin(\theta_1)\sin(\theta_1+\theta_2)\right]
$$

$$(P_x)^2 + (P_y)^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\cancel{\theta_1} - \cancel{\theta_1} - \theta_2)$$

$$(P_x)^2 + (P_y)^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\theta_2)$$

$$\cos(\theta_2) = \frac{(P_x)^2 + (P_y)^2 - (a_1)^2 - (a_2)^2}{2a_1a_2}$$

$$\Rightarrow \quad \text{or, } \boxed{\theta_2 = \pm \cos^{-1}\left(\frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2}\right)}$$

$\Rightarrow$ Also, $P_x = a_1\cos(\theta_1) + a_2\left[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right]$

$$P_x = \cos(\theta_1)\left[a_1 + a_2\cos(\theta_2)\right] - a_2\sin(\theta_1)\sin(\theta_2) \quad (1)$$

And, $P_y = a_1\sin(\theta_1) + a_2\sin(\theta_1 + \theta_2)$

$$P_y = a_1\sin(\theta_1) + a_2\left[\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)\right]$$

$$P_y = \sin(\theta_1)\left[a_1 + a_2\cos(\theta_2)\right] + a_2\sin(\theta_2)\cos(\theta_1) \quad (2)$$

Now let, $K_1 = a_1 + a_2\cos(\theta_2)$

$$K_2 = a_2\sin(\theta_2)$$

$$(1) \Rightarrow P_x = \cos(\theta_1)\cdot(K_1) - \sin(\theta_1)\cdot(K_2)$$

$$(2) \Rightarrow P_y = \sin(\theta_1)\cdot(K_1) + \cos(\theta_1)\cdot(K_2)$$

The above eq^n can also be written in Matrix form $\Rightarrow$

$$\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \underbrace{\begin{bmatrix} X \\ Y \end{bmatrix}_{(2\times1)}}_{R} = \underbrace{\begin{bmatrix} K_1 & -K_2 \\ K_2 & K_1 \end{bmatrix}_{(2\times2)}}_{P} \underbrace{\begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix}_{(2\times1)}}_{\theta}$$

$$\text{or,} \quad P\,\theta = R$$

$$P^{-1} P\,\theta = P^{-1} R$$

$$\theta = P^{-1} R$$

we know that, $\theta = \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix}$, $R = \begin{bmatrix} X \\ Y \end{bmatrix}$

$$P = \begin{bmatrix} K_1 & -K_2 \\ K_2 & K_1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{K_1 K_1 + K_2 K_2} \begin{bmatrix} K_1 & K_2 \\ -K_2 & K_1 \end{bmatrix}$$

$\Rightarrow$ Now, $\quad \theta = P^{-1} \cdot R$

$$\begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix} = \frac{1}{K_1^2 + K_2^2} \begin{bmatrix} K_1 & K_2 \\ -K_2 & K_1 \end{bmatrix}_{(2\times 2)} \begin{bmatrix} X \\ Y \end{bmatrix}_{(2\times 1)}$$

$$\begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix} = \frac{1}{K_1^2 + K_2^2} \begin{bmatrix} K_1 X + K_2 Y \\ -K_2 X + K_1 Y \end{bmatrix}_{(2\times 1)}$$

$$\therefore \quad \cos(\theta_1) = \frac{K_1 X + K_2 Y}{K_1^2 + K_2^2}$$

$$\sin(\theta_1) = \frac{-K_2 X + K_1 Y}{K_1^2 + K_2^2}$$

$$\text{or,} \quad \tan(\theta_1) = \frac{\sin(\theta_1)}{\cos(\theta_1)}$$

$$\tan(\theta_1) = \frac{K_1 Y - K_2 X}{K_1 X + K_2 Y}$$

$$\Rightarrow \boxed{\theta_1 = \tan^{-1}\left(\frac{K_1 Y - K_2 X}{K_1 X + K_2 Y}\right)}$$

where, $K_1 = a_1 + a_2 \cos(\theta_2)$

$K_2 = a_2 \sin(\theta_2)$

~~[illegible]~~

<u>Joint variables.</u>

- $\theta_1 = \tan^{-1}\left(\frac{K_1 Y - K_2 X}{K_1 X + K_2 Y}\right)$
- $\theta_2 = \cos^{-1}\left(\frac{X^2 + Y^2 - a_1^2 - a_2^2}{2 a_1 a_2}\right)$, eliminating the possibility of multiple solutions.
- $\theta_3 = 0$, (Given in the question)
- $d_4 = -d_3 - Z = -Z$, ($d_3 = 0 \Rightarrow$ Given in question)