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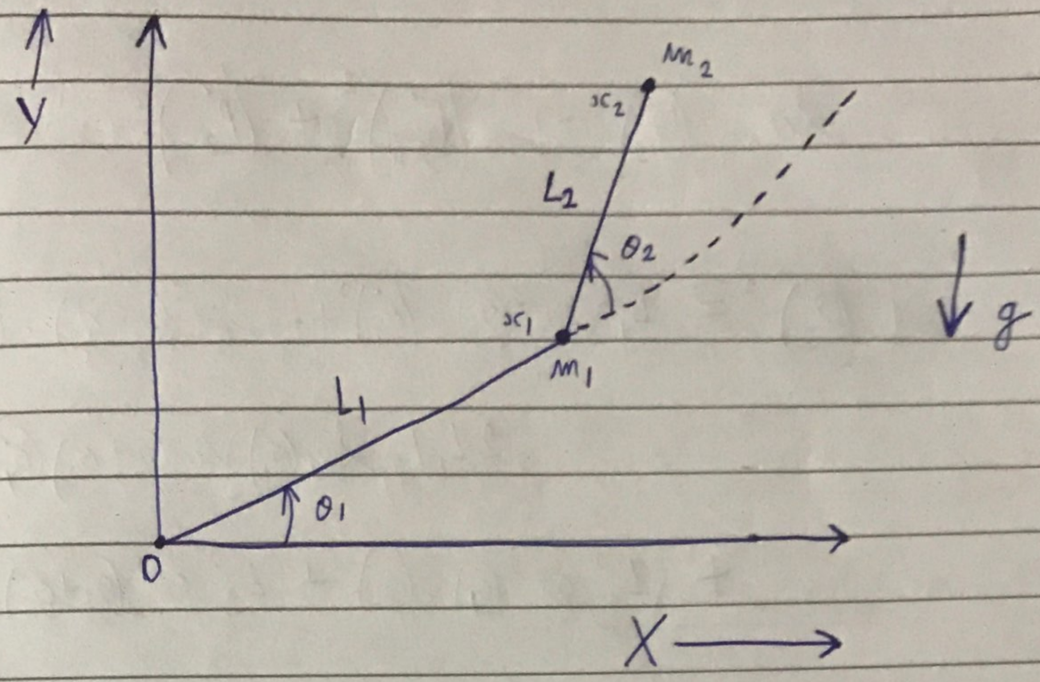

• $P_1 = \text{Point } x_1 = \left(L_1 \cos(\theta_1),\ L_1 \sin(\theta_1)\right)$

also, $(v_1)^2 = (\dot{P}_{1x})^2 + (\dot{P}_{1y})^2$

$$(v_1)^2 = \left[-L_1 \sin(\theta_1) \cdot (\dot{\theta}_1)\right]^2 + \left[L_1 \cos(\theta_1) \cdot (\dot{\theta}_1)\right]^2$$

$$(v_1)^2 = (L_1 \dot{\theta}_1)^2 \left[\sin^2(\theta_1) + \cos^2(\theta_1)\right]$$

$$v_1^2 = L_1^2 (\dot{\theta}_1)^2$$

Now, $K.E_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 L_1^2 (\dot{\theta}_1)^2$ — (A)

• $P_2 = \text{Point } x_2 = \left(L_1 \cos(\theta_1) + L_2 \cos(\theta_1+\theta_2),\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1+\theta_2)\right)$

$$v_{2x} = \dot{P}_{2x} = -L_1 \sin(\theta_1)(\dot{\theta}_1) - L_2 \sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)$$

$$v_{2y} = \dot{P}_{2y} = L_1 \cos(\theta_1)(\dot{\theta}_1) + L_2 \cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)$$

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So, $(v_2)^2 = (\dot{P}_{2x})^2 + (\dot{P}_{2y})^2$

$$(v_2)^2 = L_1^2 \sin^2(\theta_1) \cdot (\dot{\theta}_1)^2 + L_2^2 \sin^2(\theta_1+\theta_2) \cdot (\dot{\theta}_1+\dot{\theta}_2)^2 +$$

$$2L_1L_2 \sin(\theta_1) \sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)(\dot{\theta}_1)$$

$$+ L_1^2 \cos^2(\theta_1) \cdot (\dot{\theta}_1)^2 + L_2^2 \cos^2(\theta_1+\theta_2) \cdot (\dot{\theta}_1+\dot{\theta}_2)^2 +$$

$$2L_1L_2 \cos(\theta_1) \cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)(\dot{\theta}_1)$$

$$(v_2)^2 = L_1^2(\dot{\theta}_1)^2 \left[\cos^2(\theta_1) + \sin^2(\theta_1)\right] +$$

$$L_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2 \left[\sin^2(\theta_1+\theta_2) + \cos^2(\theta_1+\theta_2)\right] +$$

$$2L_1L_2(\dot{\theta}_1)(\dot{\theta}_1+\dot{\theta}_2)\left[\cos(\theta_1)\cos(\theta_1+\theta_2) + \sin(\theta_1)\sin(\theta_1+\theta_2)\right]$$

$$(v_2)^2 = L_1^2(\dot{\theta}_1)^2 + L_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2L_1L_2(\dot{\theta}_1^2+\dot{\theta}_1\dot{\theta}_2)\cos(\theta_2)$$

$\left[\text{because, } \cos(A)\cos(B) + \sin(A)\sin(B) = \cos(A-B)\right]$

Now, $K.E._2 = \frac{1}{2} m_2 v_2^2$

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$$K.E._2 = \frac{1}{2} m_2 L_1^2 (\dot{\theta}_1)^2 + \frac{1}{2} m_2 L_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$
$$+ m_2 L_1 L_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos(\theta_2) \quad — (B)$$

Now, $K.E. = K.E_1 + K.E_2$

$$\left.\begin{aligned} \text{Total } K.E. = \frac{1}{2} m_1 L_1^2 (\dot{\theta}_1)^2 + \frac{1}{2} m_2 L_1^2 (\dot{\theta}_1)^2 + \frac{1}{2} m_2 L_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ + m_2 L_1 L_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos(\theta_2) \end{aligned}\right] — (1)$$

Similarly, $P.E._1 = m_1 g L_1 \sin(\theta_1)$

and, $P.E._2 = m_2 g L_1 \sin(\theta_1) + m_2 g L_2 \sin(\theta_1 + \theta_2)$

$$\left.\begin{aligned} \text{So, Total } P.E. = m_1 g L_1 \sin(\theta_1) + m_2 g L_1 \sin(\theta_1) \\ + m_2 g L_2 \sin(\theta_1 + \theta_2) \end{aligned}\right] — (2)$$

We know that, $L = K - P$ (L-E equation)

$$\left.\begin{aligned} L = &\frac{1}{2}(m_1+m_2)L_1^2(\dot{\theta}_1)^2 + \frac{1}{2}m_2 L_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 \\ &+ m_2 L_1 L_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)\cos(\theta_2) \\ &- (m_1+m_2) g L_1 \sin(\theta_1) - m_2 g L_2 \sin(\theta_1+\theta_2) \end{aligned}\right] \text{— } (3)$$

- Also, $\underbrace{\Gamma}_{\text{external torque input}} = \underbrace{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}}_{(2\times1)} = \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)}_{(2\times1)} - \underbrace{\frac{\partial L}{\partial q}}_{(2\times1)}$

- We have, $q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}_{(2\times1)}$ → Both revolute joints.

Now, $$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \dfrac{\partial L}{\partial \dot{\theta}_1} \\[2ex] \dfrac{\partial L}{\partial \dot{\theta}_2} \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} (m_1+m_2)L_1^2\dot{\theta}_1 + m_2L_2^2(\dot{\theta}_1+\dot{\theta}_2)(1) + m_2L_1L_2(2\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_2) \\[2ex] m_2L_2^2(\dot{\theta}_1+\dot{\theta}_2)(1) + m_2L_1L_2\dot{\theta}_1\cos(\theta_2) \end{bmatrix}_{(2\times1)}$$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} (m_1+m_2) L_1^2 \ddot{\theta}_1 + m_2 L_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 L_1 L_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) \cos(\theta_2) \\ - m_2 L_1 L_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \sin(\theta_2) \\ \\ m_2 L_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_2) - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2) \end{bmatrix}_{(2\times 1)}$$

and, $$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial \theta_1} \\ \frac{\partial L}{\partial \theta_2} \end{bmatrix}$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1+m_2) g L_1 \cos(\theta_1) - m_2 g L_2 \cos(\theta_1+\theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 L_1 L_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \sin(\theta_2) - m_2 g L_2 \cos(\theta_1+\theta_2)$$

Finally, $$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

and, $$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

$$\tau_1 = \ddot{\theta}_1 \left[(m_1 + m_2) L_1^2 + m_2 L_2^2 + 2 L_1 L_2 m_2 \cos\theta_2 \right] +$$

$$\ddot{\theta}_2 \left[(m_2 L_2^2 + m_2 L_1 L_2 \cos(\theta_2)) \right] - L_1 L_2 m_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \sin(\theta_2)$$

$$+ m_2 g L_2 \cos(\theta_1 + \theta_2) + (m_1 + m_2) g L_1 \cos(\theta_1)$$

and, $\tau_2 = \ddot{\theta}_1 \left[m_2 L_2^2 + m_2 a_1 a_2 \cos(\theta_2) \right] + m_2 L_2^2 \ddot{\theta}_2 +$

$$m_2 L_1 L_2 \dot{\theta}_1^2 \sin(\theta_2) + m_2 g L_2 \cos(\theta_1 + \theta_2)$$

Finally, $\underbrace{\Gamma}_{\substack{\text{External} \\ \text{torque input}}} = \underbrace{M(q)}_{\substack{\text{Mass} \\ \text{Matrix}}} \cdot \ddot{q} + \underbrace{C(q, \dot{q})}_{\substack{\text{Coriolis} \\ \text{term.}}} + \underbrace{G(q)}_{\text{Gravity}}$

$$\text{or, } \Gamma = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \underbrace{M(q)}_{(2,2)} \cdot \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{(2\times1)} + \underbrace{C(q, \dot{q})}_{(2\times1)} + \underbrace{G(q)}_{(2\times1)}$$

after comparing τ_1, τ_2 and this eq^n $\Longrightarrow$

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- $$M(q) = \begin{bmatrix} (m_1+m_2)L_1^2 + m_2 L_2^2 + 2m_2 L_1 L_2 \cos(\theta_2) & m_2 L_2^2 + m_2 L_1 L_2 \cos(\theta_2) \\ m_2 L_2^2 + m_2 L_1 L_2 \cos(\theta_2) & m_2 L_2^2 \end{bmatrix}_{(2\times 2)}$$

- $$C(q,\dot{q}) = \begin{bmatrix} -m_2 L_1 L_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \sin(\theta_2) \\ m_2 L_1 L_2 (\dot{\theta}_1)^2 \sin(\theta_2) \end{bmatrix}_{(2\times 1)}$$

- $$G(q) = \begin{bmatrix} (m_1+m_2)\, g L_1 \cos(\theta_1) + m_2 g L_2 \cos(\theta_1+\theta_2) \\ m_2 g L_2 \cos(\theta_1+\theta_2) \end{bmatrix}_{(2\times 1)}$$