Ways to Integrate Functions without Using Standard Integration Formulas

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Abstract

Computing integrals can be a tedious task, especially when using standard integration formulas. This article explores alternative methods for integration, such as the Method of Exhaustion, Riemann sums, and the Risch Algorithm. However, these methods still rely on traditional integration techniques. We propose a series formula for integrating functions that eliminates the need for standard integration methods. The formula is derived as follows:

$$\int f(x)\,dx = xf(x) - \frac{x^2 f'(x)}{2!} + \frac{x^3 f''(x)}{3!} - \frac{x^4 f'''(x)}{4!} + \dots$$

This formula does not include the symbol of integration and provides the desired integration of $f(x)$.

1 Introduction

Integrating functions without using standard integration formulas can be challenging. In this paper, we explore alternative methods that can compute definite or indefinite integrals of functions without relying on traditional integration techniques such as substitution, integration by parts, and other antiderivative formulas.

The methods we will discuss include:

1.1 Geometrical Methods

These methods help in computing definite integrals of functions.

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1.1.1 Method of Exhaustion

The Method of Exhaustion is a historical method used to calculate the area of curves by splitting them into different shapes.

1.1.2 Riemann Sum

The Riemann Sum divides the curve into rectangles, where the height varies according to the value of the function at each point, and the width tends to zero. The sum of the areas of these rectangles provides the required definite integral.

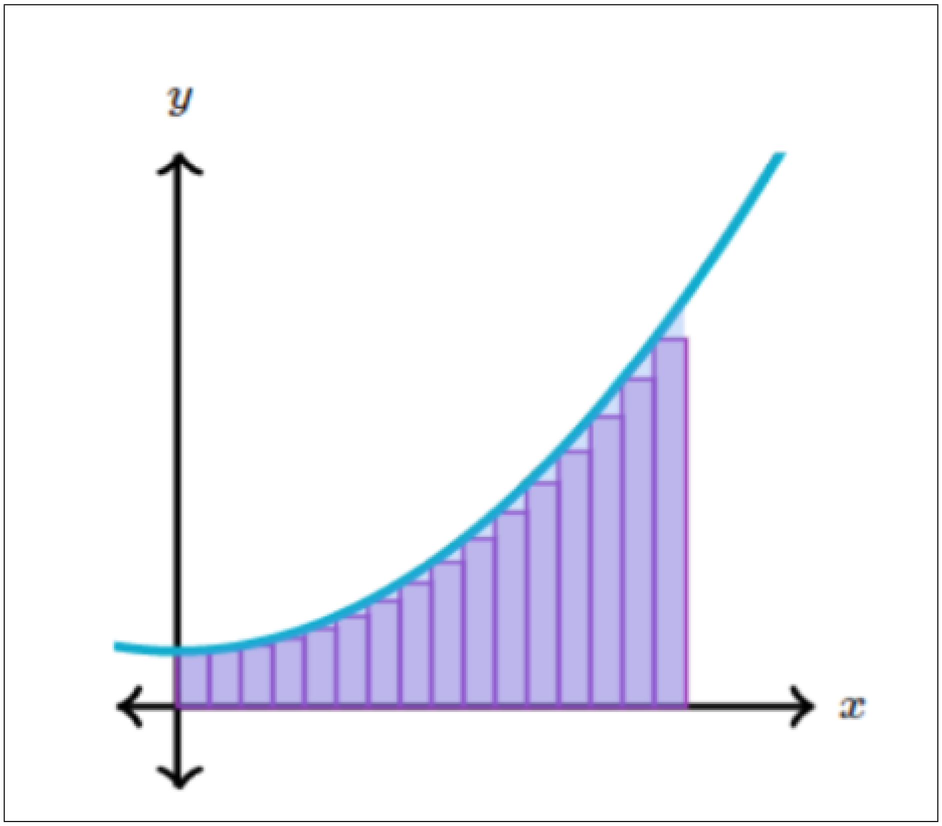

Figure 1: Riemann Sum

1.2 Risch Algorithm

The Risch Algorithm is a method of indefinite integration used in computer algebra systems to find antiderivatives. It is named after the American mathematician Robert Henry Risch, who developed it in 1968. The algorithm determines whether a given integral is elementary and, if so, returns a closed-form result for the integral. While the algorithm handles logarithmic, exponential, and algebraic extensions, the case of algebraic extensions is complex and not fully implemented in any computer algebra system.

To understand the Risch Algorithm in detail, one needs to study the subject extensively, as it involves a significant amount of material. Integration Calculator [5] is a website that employs the Risch Algorithm to perform step-by-step indefinite integrations. The website consists of more than 17,000 lines of code dedicated solely to the integration process.

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2 Integrating Functions using Differentiation

We will now present a method to perform integrations without explicitly using integration symbols. The prerequisite for this method is to have a differentiation module, such as SymPy in Python.

The series formula for integration is as follows:

$$\int f(x)\,dx = xf(x) - \frac{x^2 f'(x)}{2!} + \frac{x^3 f''(x)}{3!} - \frac{x^4 f'''(x)}{4!} + \dots$$

Note that this is an infinite series that yields the final integration of a function. The series does not require performing integration at any point to calculate $\int f(x)$. We can utilize this formula to obtain the desired integrations in our code without explicitly performing integration.

The pattern of the series can be described as follows:

$$\int f(x)\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x^n f^{(n-1)})}{n!}$$

In the numerator, the powers of x increase by 1 in each term, while $f(x)$ is differentiated $n-1$ times. Every alternate term has a negative sign. The denominator comprises factorials starting from 1, 2, 3, and so on.

Now, let's proceed to the code implementation using Python 3.9.

3 Code in Python 3.9

Please find below the Python code that applies the series formula for integration. It does not rely on standard integration methods.

```
from sympy import *
x = symbols('x')
n = 0
expr = (1/(2*pi)**(1/2))*exp((-(x**2)/2))
expr_saved = expr
upperlimit = 1
lowerlimit = 0
termlist = []
sum_indefinite = 0
sum_definite = 0
while True:
    n += 1
    if n == 1:
        termlist.append((x*(expr_saved)))
        sum_indefinite = termlist[0]
        sum_definite = (limit(sum_indefinite,x,upperlimit) - limit(sum_indefinite,x,lowerlin
    else:
        expr = diff(expr, x)
        if expr != 0:
            z = (((-1)**(n+1))*(x**n)*expr)/factorial(n)
            termlist.append(z)
            prev_sum_indefinite = sum_indefinite
            sum_indefinite += z
            sum_definite = (limit(sum_indefinite,x,upperlimit) - limit(sum_indefinite,x,lowe
            if sum_indefinite == prev_sum_indefinite:
                print('Break due to repetition')
                break
            else:
                print(termlist)
                print(sum_definite)
        else:
            print('Break due to differentiation = 0')
            break
print(termlist)
print(poly(sum_indefinite))
print(sum_definite)
```

4 Results and Discussion

4.1 $f(x) = x^2$

```
[x**3, -x**3, x**3/3]
x**3/3
291.666666666667
```

Figure 2: Result for x^2

here, $1/3 * x * * 3$ means $\frac{x^3}{3}$ and the definite integral is 291.66 for the limits lower limit 5 to upper limit 10 which is the correct result.

4.2 $f(x) = x^{100}$

```
[x**101, -50*x**101, 1650*x**101, -40425*x**101,
x**101/101
9.90099009900990e+98
```

Figure 3: Result for x^{100}

which is required result

4.3 $f(x) = x^2 + 1$

```
[x*(x**2 + 1), -x**3, x**3/3]
-2*x**3/3 + x*(x**2 + 1)
296.666666666667
```

Figure 4: Result for $f(x) = x^2 + 1$

which is required result

4.4 $f(x) = x^{1/2}$

In this case the code does not end up with a specific value but instead, it is approaching to correct value.

```
13.6291637408261
[x**1.5, -0.25*x**1.5, -0.041666666
13.6291588930592
[x**1.5, -0.25*x**1.5, -0.041666666
13.6291540900134
```

Figure 5: Result for $f(x) = x^{1/2}$

And it does not give indefinite Integration for this case, because to get indefinite integration the function must end up at a constant(while differentiation again and again in terms) and then to zero so that all terms after that are zero, and the code can stop running at that point.

4.5 $f(x) = \ln x$

```
[x*log(x), -x/2, -x/6, -x/12,
9.98478131881035
[x*log(x), -x/2, -x/6, -x/12,
9.98477383720761
[x*log(x), -x/2, -x/6, -x/12,
9.98476637387496
```

Figure 6: Result for $f(x) = \ln x$

In this case also, the answer tends towards the correct answer.
The indefinite integration does not come, because the differentiation never comes out to be equal to 0.
$f(x) = \ln x, \quad f'(x) = \frac{1}{x}, \quad f''(x) = \frac{-1}{x^2}, \dots$

4.6 $f(x) = \frac{\sin x}{x}$

This also approaches to correct value

```
[sin(x), -x**2*(cos(x)/x
0.108416349244988
[sin(x), -x**2*(cos(x)/x
0.108416349343085
```

Figure 7: Result for $f(x) = \frac{\sin x}{x}$

4.7 $f(x) = e^{x^2}$

This approaches to correct value for limit 0 to 1 or smaller limit.

In case of limit 5 to 10, the sum of terms keeps alternating between positive and negative, maybe it will approach some value in future, don't know.

```
[x*exp(x**2), -x**3*exp(x**2), x**3*(4
1.46265174590718
[x*exp(x**2), -x**3*exp(x**2), x**3*(4
1.46265174590718
```

Figure 8: Result for $f(x) = e^{x^2}$ for limit 0 to 1

```
[x*exp(x**2), -x**3*exp(x**2), x**3*(
-4.53498516374967e+98
[x*exp(x**2), -x**3*exp(x**2), x**3*(
1.94398237821602e+99
```

Figure 9: Result for $f(x) = e^{x^2}$ for limit 5 to 10

4.8 $f(x) = \frac{1}{x}$

Now in the case of negative powers, strange things start happening if we use 1/x in our given formula all denominators and numerators with x gets cancelled out. Leaving a series of

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

```
[1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10,
0
[1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10,
```

Figure 10: Result for $f(x) = \frac{1}{x}$

Does it mean? $\ln x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ So, graph of $\ln x$ is the break point for this formula to do integrations, because for further negative powers, the series never ends. For example

4.9 $f(x) = \frac{1}{x^2}$

```
[1/x, 1/x, 1/x, 1/x, 1/x, 1/x, 1/x,
-95.6000000000000
[1/x, 1/x, 1/x, 1/x, 1/x, 1/x, 1/x,
-95.7000000000000
```

Figure 11: Result for $f(x) = \frac{1}{x^2}$

$\frac{1}{x} + \frac{1}{x} + \frac{1}{x} + \ldots$ Does it mean? $\frac{-1}{x} = \frac{1}{x} + \frac{1}{x} + \frac{1}{x} + \ldots$ And the value keeps on increasing.

4.10 $f(x) = \frac{1}{x^3}$

```
-1126.15500000000
[x**(-2), 3/(2*x**2), 2/x**2, 5/(2*x**2), 3/x**2, 7/
-1131.97500000000
[x**(-2), 3/(2*x**2), 2/x**2, 5/(2*x**2), 3/x**2, 7/
```

Figure 12: Result for $f(x) = \frac{1}{x^3}$

The value keeps on increasing as expected from the series to infinity

The reason for this depends on the function and the series in the formula formed if the function integration comes out to be uniformly converging which the above functions are not, so they are not integrable using our series formula for integration.

Some Application : -

4.11 Binomial Theorum using this series

$\int f(x)\,dx = xf(x) - \frac{x^2 f'(x)}{2!} + \frac{x^3 f''(x)}{3!} - \frac{x^4 f'''(x)}{4!} + \dots$

Let $f(x) = n(1+x)^{n-1}$
putting in the series, we have

$\int n(1+x)^{n-1}\,dx \ =$
$(1+x)^n =$
$x * n(1+x)^{n-1}$
$-\, n(n-1)(1+x)^{n-2} * \frac{x^2}{2!}$
$+\, n(n-1)(n-2)(1+x)^{n-3} * \frac{x^3}{3!}$
$-n(n-1)(n-2)(n-3)(1+x)^{n-4} * \frac{x^4}{4!} \dots$

if we give this equation in the code for $n = 1, 2, 3, 4 \dots$

```
a = 3
expr = a*((1+x)**(a-1))
```

Figure 13: Here, a is used instead of n

```
[x]
Poly(x, x, domain='ZZ')
5.00000000000000
```

Figure 14: Result for $a = 1$

```
break due to diff = 0
[x*(2*x + 2), -x**2]
Poly(x**2 + 2*x, x, domain='ZZ')
85.0000000000000
```

Figure 15: Result for $a = 2$

```
break due to diff = 0
[3*x*(x + 1)**2, -x**2*(6*x + 6)/2, x**3]
Poly(x**3 + 3*x**2 + 3*x, x, domain='QQ')
1115.00000000000
```

Figure 16: Result for $a = 3$

```
break due to diff = 0
[4*x*(x + 1)**3, -6*x**2*(x + 1)**2, x**3*(24*x + 24)/6, -x**4]
Poly(x**4 + 4*x**3 + 6*x**2 + 4*x, x, domain='QQ')
13345.0000000000
```

Figure 17: Result for $a = 4$

and so on.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Figure 18: Pascal's Triangle

And Since, we are doing integration, the constant that is missing here is 1. So, This formula -

$\int n(1+x)^{n-1}\,dx \ =$
$(1+x)^n =$
$x * n(1+x)^{n-1}$
$-\, n(n-1)(1+x)^{n-2} * \frac{x^2}{2!}$
$+\, n(n-1)(n-2)(1+x)^{n-3} * \frac{x^3}{3!}$
$-n(n-1)(n-2)(n-3)(1+x)^{n-4} * \frac{x^4}{4!} \dots$

Adding 1 to the results gives the required expansion.

So, $1 + \int n(1+x)^{n-1}\,dx \ =$
$(1+x)^n =$
$1 + x * n(1+x)^{n-1}$
$-\, n(n-1)(1+x)^{n-2} * \frac{x^2}{2!}$
$+\, n(n-1)(n-2)(1+x)^{n-3} * \frac{x^3}{3!}$
$-n(n-1)(n-2)(n-3)(1+x)^{n-4} * \frac{x^4}{4!} \dots$

gives the expansion of $(1+x)^n$, but it does not give directly all the pascal coefficients directly, it needs some simplification to reach the result once, values of n are substituted.
For other Binomials like $(2+x)^n$ C becomes 2^n.
Therefore, the final expansion for getting the same result as the binomial theorem is $(a+x)^n =$
$x * n(1+x)^{n-1}$
$- n(n-1)(1+x)^{n-2} * \frac{x^2}{2!}$
$+ n(n-1)(n-2)(1+x)^{n-3} * \frac{x^3}{3!}$
$-n(n-1)(n-2)(n-3)(1+x)^{n-4} * \frac{x^4}{4!} \ldots$
$+ a^n$
which gives the same values as the binomial theorem.

4.12 Taylor Expansions

Both, Taylor Expansion

$$f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \ldots + \frac{(x-a)^n f^{(n)}(a)}{n!}$$

and the series

$$\int f(x)\,dx = xf(x) - \frac{x^2 f'(x)}{2!} + \frac{x^3 f''(x)}{3!} - \frac{x^4 f'''(x)}{4!} + \ldots \text{ [SR equation]}$$

might look similar and you may think that we can use Taylor instead to do the same integrations, but at a = 0, or Taylor Expansion about any other constant, the co-efficient of variable x, x^2, etc. will be different so both are different series.

5 Conclusion

The proposed series formula provides an alternative method for integrating functions without using standard integration formulas. The code implementation in Python demonstrates the application of this formula, yielding correct results for various functions. By avoiding traditional integration techniques, this approach offers a unique perspective on integration and may be particularly useful in cases where standard integration methods are challenging to apply.

Further research can be done to explore the convergence and accuracy of the series formula in different scenarios. Additionally, investigating its limitations and comparing it with other non-standard integration methods would provide a comprehensive understanding of its applicability and effectiveness.

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