[1]

GAME THEORY

Definition 0.1. *A competitive situation is called* ***game****. The term* ***Game*** *represents a conflict between two or more groups.*

A situation is said to be a game if it possess the following properties;

1. The number of competitors is finite.
2. There is a conflict of interest between the participants.
3. Each of the participants has a finite set of possible course of action.
4. The rules governing these choices are specified and known to all the players, a play of the results when each of the player chooses a single course of action from the list of courses available to him / her.
5. The outcome of the game is affected by the choices made by all the players.
6. The outcome of all specific set of choices by all of the players is known in advance and numerically defined.

Definition 0.2. ***Strategy*** *of a player is the decision rule he/she uses for making a choice from his / her list of course of action.*

Strategies are classified as

1. *Pure strategy: A strategy is called* ***pure*** *if one knows in advance of the pay that it is certain to be adopted irrespective of the strategy the other players might choose.*
2. *Mixed strategy: The optimal strategy mixture for each player may be determined by assigning to each strategy, its probability of being chosen. These strategies so determined are called* ***mixed strategy*** *because they are probabilistic combinations of the available choices of the strategy.*

Mixed strategy is usually denoted by $S = \begin{pmatrix} X_1 & X_2 & X_3 & ... & X_n \\ p_1 & p_2 & p_3 & ... & p_n \end{pmatrix}$ *where* p_j *is the probability of choosing the course of plan* X_j *such that for all* $p_j \geq 0$ *and* $p_1 + p_2 + p_3 + ... + p_n = 1$.

Definition 0.3. ***Payoff*** *is the outcome of playing the game. A* ***payoff matrix*** *is a matrix showing the amount received by the player named at the left hand side after all possible plays of the game. The payment is made by the player named at the top of the table.*

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If a player A has $m-$courses of action and player B has $n-$courses, then a payoff matrix can be constructed as below

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ \cdots \\ \cdots \\ \\ A_m \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & \cdots & B_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots\cdots\cdots & & & & \\ \cdots\cdots\cdots & & & & \\ & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

If we need to write the payoff matrix of player B, it will be the negative of the payoff matrix of player A.

TYPES OF GAMES

1. **Two - person games and n-person games:** In a game the players may have many possible choices open to them for each play of the game but the number of players remain only two, such games are called ***2-person games***.In case of more than two persons, the game is called ***n-person game.***

2. **Zero sum game**In a game, if the profit and loss of the two players is zero then it is called **zero sum game.**

3. **Two - person zero sum game** In a two person game if the sum of the gain and loss of the players is equal to zero, such games are called two-person zero sum game.

MAXIMIN - MINIMAX PRINCIPLE

Definition 0.4. *A **saddle point** is a position in the payoff matrix where the maximum of the row minima coincides with the minimum of column maxima. The payoff at the saddle point is called **value of the game.***

Notation 0.1. *We shall denote Maximin by $\underline{\nu}$; Minimax by $\bar{\nu}$ and the value of the game by ν.*

Definition 0.5. *1. A game is said to be **fair** if $\underline{\nu} = \bar{\nu} = 0$*

*2. A game is said to be **strictly determinable** if maximin=minimax$\neq 0$*

Games without saddle point (Mixed strategy)

Two-person zero sum game without saddle point

Consider a 2×2 two-person zero sum game without any saddle point having a payoff matrix for player A,

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array}$$

then the optimum strategy for player A and B are $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$, $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$ where $p_1 = \dfrac{a_{22} - a_{21}}{(a_{11} - a_{22}) - (a_{12} + a_{21})}; p_1 + p_2 = 1$

$q_1 = \dfrac{a_{22} - a_{12}}{(a_{11} - a_{22}) - (a_{12} + a_{21})}; q_1 + q_2 = 1$ and value of the game is, $\nu = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} - a_{22}) - (a_{12} + a_{21})}$

GRAPHICAL METHOD FOR $2 \times n$ or $m \times 2$ GAMES

Consider the following $2 \times n$ games

$$\text{Player A} \begin{pmatrix} & B_1 & B_2 & B_3 & \dots & B_n \\ A_1 & a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ A_2 & a_{21} & a_{22} & a_{23} & \dots & a_{2n} \end{pmatrix}$$

Let the mixed strategy for player A be given by $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$ where $p_1 + p_2 = 1, p_1 \geq 0; p_2 \geq 0$.

For each of the pure strategies available to B, expected payoff for player A would be as follows;

B's pure move	A's expected payoff $E(p)$
B_1	$E_1(p) = a_{11}p_1 + a_{21}p_2$
B_2	$E_2(p) = a_{12}p_1 + a_{22}p_2$
...	.....
B_n	$E_n(p) = a_{1n}p_1 + a_{2n}p_2$

The player B would like to choose that pure move B_j aganist S_A for which $E_j(p)$ is a minimum for $j = 1, 2, ..., n$.

Denote the minimum expected payoff for A by $\nu = \min(E_j(p))$ for $j = 1, 2, ..., n$

The objective of player A is to select p_1 and p_2 in such a way that ν is large as possible. This can be achieved by plotting the straight lines

$$E_j(p) = a_{1j}p_1 + a_{2j}p_2 = a_{1j}p_1 + a_{2j}(1 - p_1) = (a_{1j} - a_{2j})p_1 + a_{2j}$$

for $j = 1, 2, ..., n$ is a linear function of p_1.

The highest point on the lower boundary (or lower envelope) will give the maximum value among the minimum expected payoffs on the lower envelope and the optimum value of the probabilities p_1 and p_2.

Now the two strategies of player B corresponding to those lines which passes through the maximum point can be determined. These two strategies of player B will help us to reduce the size of the game to 2×2 game.

Remark 0.1. *Using the similar procedure as discussed above, we can reduce the size of a $m \times 2$ game to a 2×2 game and get the minimum point which will be the lowest point on the upper boundary (or upper envelope).*

DOMINANCE PROPERTY

In certain games, one of the pure strategies of either player is always inferior to atleast one of the remaining ones. The superior strategies are said to dominate the inferior ones. In such cases, using the dominance we reduce the size of the payoff matrix by deleting those strategies which are dominated by others.

The general rule for dominance are;

1. If all the elements of k^{th} row are less than or equal to the corresponding elements of any other row, say r^{th} row then k^{th} row is dominated by the r^{th} row. So delete the k^{th} row.

2. If all the elements of k^{th} column are greater than or equal to the corresponding elements of any other column, say r^{th} column then k^{th} column dominates r^{th} column. So delete the k^{th} column.

3. If some linear combinations of some rows dominates i^{th} row, then the i^{th} row will be deleted. Similar arguments follow for column.