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STOCHASTIC PROCESS / RANDOM PROCESS

1 RANDOM PROCESS

A stochastic process (or Random process) is a probabilistic model used for characterizing the random signals.

The concept of random process is an extension of the idea of a random variable to include time.

Definition 1.1. *Let S be the sample space of a random experiment. A Stochastic process is a mapping $X : S \to \mathbb{R}$ which assigns every outcome $s \in S$ a real valued function of time $x(t, s)$ i.e., $X(s) = x(t, s)$. The collection of all such time functions is denoted by $X(t, s)$, is called a random process.*

Notation 1.1. *We denote a random process by $X(t, s) = \{x(t, s)\} = \{X_t; t \in T\}$*

Example 1.1. *Consider $S = \{\sin t, \cos t\}$ then define X by $X(H) = X(t, H) = \sin t$ and $X(T) = X(t, T) = \cos t$ then the set $X(t, s) = \{\sin t, \cos t\}$ is a random process.*

Example 1.2. *Consider the random experiment "throwing a die". Let X_n denote the outcome of the n^{th} throw where $n \geq 1$.Then $\{X_n; n \geq 1\}$ is a stochastic process, which is called* ***Bernoulli process****.*

Example 1.3. *(ii) Let $N(t)$ denotes the number of customers arriving to a bank in an interval of duration "t" minutes then $N(t)$ for each $t > 0$ is a random variable with Poisson distribution. So $\{N(t); t \geq 0\}$ is a stochastic process.*

A one dimensional stochastic process can be classified into one of the following four types;

1. Discrete time discrete state space (ie; both $X(t)$ and t are discrete)
2. Continuous time discrete state space (ie; $X(t)$ is discrete and t is continuous)
3. Continuous time continuous state space (ie; both $X(t)$ and t is continuous)
4. Discrete time continuous state space (ie; t is discrete and $X(t)$ is continuous)

Process with independent increments

If for every $t_1, t_2, ..., t_n$ with $t_1 < t_2 < ... < t_n$ the random variables $X(t_1) - X(t_2), X(t_3) - X(t_2), ..., X(t_n) - X(t_{n-1})$ are independent then $\{X_t; t \in T\}$ is said to be a process with independent increments.

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The **mean** of the random process $X(t)$ is the expected value of the random variable $X(t)$ at any time t i.e., $\mu_X(t) = E\left[X(t)\right]$ for $-\infty < t < \infty$.

Mean is a function of time t so it is called a mean function.

Correlation coefficient: The correlation coefficient of two random variables

X and Y is given by $\rho = \dfrac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}} = \dfrac{COV(X,Y)}{\sqrt{V(X)V(Y)}}$.

Note: Random variables X and Y are uncorrelated if the covariance

$COV(X, Y) = 0$.

Auto Correlation Function(ACF)

The **auto correlation function** of a stochastic process $\{X_t; t \in T\}$ is defined as,

$R_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}$

The **average power of** $X(t)$ is defined as $R(t,t) = E\{X(t)X(t)\} = E\{X^2(t)\}$.

Auto covariance is defined as $C(t_1, t_2) = R_{XX}(t_1, t_2) - E\{X(t_1)\}E\{X(t_2)\}$.

So, $C(t,t) = R(t,t) - E\{X(t)\}E\{X(t)\} = (E\{X(t)\})^2$

$= E\{X^2(t)\} - (E\{X(t)\})^2 = V\{X(t)\}$.

Stationary process

A stochastic process $\{X_t; t \in T\}$ is said to be a stationary process of order "n" if for arbitrary $t_1, t_2, ..., t_n$ the joint p.d.f of $(X(t_1), X(t_2), X(t_3), ..., X(t_n))$ and $(X(t_1+h), X(t_2+h), X(t_3+h), ..., X(t_n+h))$ are same for every $h > 0$.

In particular, if we need to say that a random process is said to be first order stationary if its first order probability density function does not change with a shift in time origin. i.e., $f_X(x, t_1) = f_X(x, t_1 + h)$ for any time t_1 and any real number h.

Remark 1.1. *A first order stationary random process has a constant mean.*

Strictly stationary stochastic process

The stationary process is said to be **strictly stationary stochastic process** (SSS) if it is stationary of order "n" for any $n \geq 1$.

The stationarity of a process is the probabilistic structure of the process, is invariant under the translation of the time axis. For SSS the statistics properties of the associated joint distribution of $(X(t_1+h), X(t_2+h), X(t_3+h), ..., X(t_n+h))$ is independent of "h".

Wide Sense Stationary

A stochastic process $\{X_t; t \in T\}$ with finite second order momemts is called wide sense stationary (WSS) [or covariance stationary or weakly stationary] if its mean $E\{X(t)\} = m$, a constant and its auto correlation $R(t_1, t_2) = E\{X(t_1)X(t_2)\}$ depends only on the difference $|t_1 - t_2|$ for every t_1, t_2.

Evolutionary Process

A process which is not stationary in any sense is called an evolutionary process.

2 MARKOV PROCESS

Markov process is a special class of random process, which represent the simplest generalization of independent process which allow the outcome at any instant to depend only on the outcome that precedes it and not on the earlier ones. i.e., in a Markov process the past has no influence on the future if the present is specified.

Definition 2.1. *A random process $X(t)$ is said to be a **Markov process** if for any $t_1, t_2, ..., t_n$*

$$P\left[X(t_n) \le x_n | X(t_{n-1}) \le x_{n-1}, X(t_{n-2}) \le x_{n-2}, ...X(t_1) \le x_1\right] = P\left[X(t_n) \le x_n | X(t_{n-1}) \le x_{n-1}\right]$$

i.e., the conditional probability distribution of $X(t_n)$ for given values of $X(t_1), X(t_2), ..., X(t_{n-1})$ depends only on $X(t_{n-1})$.

Remark 2.1. *The above definition of Markov Process will hold for continuous Markov process. But we are discussing about discrete state Markov process known as **Markov Chains** where the system can occupy only a finite or countable number of states.*

Definition 2.2. ***Markov Chains:*** *Let $X(t)$ be a Markov process with states $X(t_i) = X_i$ where $t_0 < t_1 < t_2 < ... < t_n$. If for all n,*

$$P\left[X_n = a_n | X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, ...X_0 = a_0\right] = P\left[X_n = a_n | X_{n-1} = a_{n-1}\right]$$

then the sequence of random variables $\{X_n\}$ is called a Markov Chain for $n = 0, 1, 2, 3, ...$ and $a_1, a_2, ..., a_n$ are the states of the Markov Chain.

Definition 2.3. ***Transition probabilities and Transition Matrix:*** *Let $A = \{a_1, a_2, ..., a_n\}$ be the state space of a Markov chain, and for any two states a_i, a_j, let p_{ij} denote the conditional probability that $X_{r+1} = a_j$ given that $X_r = a_i$.*

$$p_{ij} = P\left[X_{r+1} = a_j | X_r = a_i\right]$$

where $X_{r+1} \& X_r = a_i$ are any two successive random variables present in the process.The probabilities p_{ij} are called ***transition probabilities****.*

The square matrix P whose elements are the transition probabilities p_{ij} is called a ***transition probability matrix or transition matrix****.*

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ \cdots\cdots\cdots\cdots & & & & \\ \cdots\cdots\cdots & & & & \\ & & & & \\ p_{m1} & p_{m2} & p_{m3} & \cdots & p_{mn} \end{pmatrix}$$

Remark 2.2. *In P it is clear that, $p_{ij} \geq 0$ for all $i, j = 1, 2, 3, .., n$ and $\Sigma_{j=1}^{n} p_{ij} = 1$ for $i = 1, 2, 3.., n$.*

Definition 2.4. ***Stochastic Matrix*** *A transition probability matrix $P = [p_{ij}]$ with the above two properties is called a* ***stochastic matrix.***

Definition 2.5. *A row matrix of the form $Q = \begin{pmatrix} q_1 & q_2 & q_3 & \ldots & q_n \end{pmatrix}$ where each $q_i \geq 0$ and $\Sigma_{i=1}^{n} q_i = 1$ is called a* ***probability vector of order n.***

Remark 2.3. *Every row of a stochastic matrix is a probability vector.*

Definition 2.6. *Let $\{X_r\}$ be a Markov chain.*

1. ***Higher transition probabilities:*** *Given that a random variable X_r in the Markov chain takes the value a_i then for some positive integer $k \geq 1$ the probability*

$$p_{ij}^{(k)} = P\left[X_{r+k} = a_j | X_r = a_i\right]$$

   Here, $p_{ij}^{(k)}$ is called the ***k-step transition probabilities.***

2. *It is clear that $p_{ij} = p_{ij}^{(1)}$ are the* ***1-step transition probabilities.***

3. *The matrix $P^{(k)} = \left[p_{ij}^{(k)}\right]$ is called the* ***k-step transition matrix.***

4. *$P^{(k)}$ is identically equal to P^k, the k^{th} power of P. i.e., the k-step transition matrix is precisely equal to the k^{th} power of the 1-step transition matrix.*

$$P^{(k)} = \left[p_{ij}^{(k)}\right] = [p_{ij}]^k = P^{(k)}$$

5. *A Stochastic matrix P is said to be* ***regular*** *if all the entries in some positive integral power of P are positive.*

   i.e., the transition matrix $P = [p_{ij}]$ is regular if and only if $p_{ij}^{(k)} > 0$ for all possible values of i and j and for some $k \geq 1$.

6. *A Markov chain is said to be irreducible if its transition matrix is a* ***regular stochastic matrix.***

7. *For a given regular stochastic matrix P of order m, if there exists a probability vector Q of order m such that $QP = Q$ then Q is called a* ***fixed probability vector*** *of P. Such vector Q exists and is unique.*

3 Classification of States

Let $A = \{a_1, a_2, ..., a_n\}$ be the state space of a Markov chain $\{X_r\}$. When $\{X_r\} = a_i$ which means that the chain is in the $a_i$state (i.e., visiting state a_i state or in the i^{th} state) at the r^{th} step.

Note 1. *p_{ij} denotes the probability that the system (or Chain) makes a transition (or moves) from the state i to j in one-step.*

Similarly, $p_{ij}^{(k)}$ denotes the probability that the system (or Chain) makes a transition (or moves) from the state i to j in k-steps.

Definition 3.1.

1. ***Absorbing State:*** *A state i is said to be an absorbing state if the transition probabilities p_{ij} are such that*

$$p_{ij} = \begin{cases} 1, & \textit{for } i = j \\ 0, & \textit{else where} \end{cases}$$

2. ***Transient State:*** *A state i is said to be a transient state if the chain is in this state at some step and there is a chance (i.e., there is a non-zero probability) that it will not return to that state.*

   For example, if we model a Markov chain all the states except the final state are transient states.

3. ***Recurrent State:*** *A state i is said to be a recurrent state if starting from the state i the chain does eventually return to the same state (i.e., the probability of the return is 1).*

4. ***Periodic State:*** *Let i be a recurrent state so that $p_{ii}^k > 0$ for some $k \geq 1$. Let $d_i = \gcd\{k \in \mathbb{Z}^+; p_{ii}^k > 0\}$. Then d_i is called the* ***period*** *of te state i.*

   The recurrent state i is said to be ***periodic*** *if $d_i > 1$ and* ***aperiodic*** *if $d_i = 1$*

5. *All the states of an irreducible Markov Chain are of same type.*

6. *If one of the state in a irreducible Markov chain is aperiodic then all the states are aperiodic then such Markov chains are called an* ***aperiodic Markov chain.***