Markov chain :

A software engineer goes to his workplace everyday by bike or car. He never goes by bike on 2 consecutive days. But if he goes by car on a day, then he is equally likely to go by car or bike the next day. Find the transition matrix for the chain of mode of transport he uses. If car is used on the first day of the week, find the probability that

① Bike is used

② Car is used on the fifth day.

Let a_1 be using bike and a_2 - car be the 2 states of given Markov chain.

Transition matrix,

$$P = \begin{array}{c c} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

P_{11} = Probability of using bike on a day, given bike was used on previous day = 0

P_{12} = Probability of using car on a day, given bike was used on previous day = 1

P_{21} = Probability of using bike, given car was used on previous day = 1/2

P_{22} = Probability of using car given car was used on previous day = 1/2.

Req: $P_{21}^{(4)} = P(X_{r+4} = a_1 \mid X_r = a_2)$

If car is used on first day of the week, probability of using bike on fifth day is $P_{21}^{(4)}$

IIIrd car on 5th day, given car is used on first day is $p^{(4)}_{22}$

We have $P^{(4)} = P^4$

$$\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$\frac{1}{2} + \frac{1}{4}$

$$P^4 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix}$$

$\therefore P^{(4)}_{21} = \frac{5}{16}$ $\quad P^{(4)}_{22} = \frac{11}{16}$

2) Consider $P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$ ST it is a regular stochastic matrix.

Since all entries in P are non negative and row sums are 1, given matrix is a stochastic matrix.

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/8 & 5/16 & 9/16 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1/8 & 5/16 & 9/16 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 5/32 & 41/64 & 13/64 \\ 1/8 & 5/16 & 9/16 \end{bmatrix}$$

All entries are strictly non zero.

$\therefore$ Regular.

3) PT transition matrix corresponding to $\{X_t \mid t \in T\}$ is irreducible (Lregular)

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

4) Find

P =

Find the fixed probability vector for a regular stochastic matrix $P = \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix}$ from a markov chain.

Let $Q = [x \quad y]$ be the fixed probability vector for P.

where $x \geq 0$, $y \geq 0$ and $x + y = 1$.

Then $QP = Q$

$$[x \quad y] \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix} = [x \quad y]$$

$$\Rightarrow \begin{bmatrix} \frac{x}{3} + \frac{y}{4} & \frac{2x}{3} + \frac{3y}{4} \end{bmatrix} = [x \quad y]$$

$$\Rightarrow \frac{x}{3} + \frac{y}{4} = x \quad \text{and} \quad \frac{2x}{3} + \frac{3y}{4} = y$$

Since $x + y = 1 \Rightarrow x = 1 - y$

$$\frac{(1-y)}{3} + \frac{y}{4} = (1-y)$$

$$4(1-y) + 3y = 12(1-y)$$

$$4 - 4y + 3y - 12 + 12y = 0$$

$$11y = 8$$

$$y = 8/11$$

$$\frac{2x}{3} + \frac{3(1-x)}{4} = 1 - x$$

$$8x + 9(1-x) = 12(1-x)$$

$$8x + 9 - 9x - 12 + 12x = 0$$

$$11x = 3$$

$$x = 3/11$$

$$\therefore Q = [3/11, \ 8/11]$$

5) Find the fixed probability vector of regular stochastic process

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$ from a given MC $\{X_t \mid t \in T\}$

Let $Q = [x \quad y \quad z]$ be fixed probability vector where

$x, y, z \geq 0$ and $x+y+z = 1$.

Then $QP = Q$

$$[x \quad y \quad z] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x \quad y \quad z]$$

$\frac{1}{2} z = x$

$x + \frac{z}{2} = y$

$y = z$

$z = 1 - x - y$

$(1 - x - y) = 2x$

$1 = 3x + y$

$y = 1 - x - y$

$x + 2y = 1$

$x + 2y = 1$

$6x + 2y = 2$

$x = 1/5$

$y = 2/5$

$z = 2/5$

$\therefore Q = \left[\frac{1}{5} \quad \frac{2}{5} \quad \frac{2}{5}\right]$

Find the fixed probability vector for regular stochastic matrix

$$P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$\left(4/11 \quad 4/11 \quad 3/11\right)$

Let $\{X_t \mid t \in T\}$ be a Markov chain with a transition probability matrix $P = [p_{ij}]$. Then the probability distribution of $X_r, X_{r+1}, \ldots X_{r+n}$ can be determined in terms of the transition probabilities p_{kj} and the initial probability distribution of X_r.

For our convenience take $r=0$. Then $P(X_0, X_1 \ldots X_n)$

$$P(X_0 = k, X_1 = j, X_2 = i, \ldots X_{n-1} = b, X_n = a).$$

$$= P(X_n = a, X_{n-1} = b \ldots X_2 = i, X_1 = j, X_0 = k)$$

$$= P(X_n = a \mid X_{n-1} = b, X_{n-2} = c \ldots X_2 = i, X_1 = j, X_0 = k)\, P(X_{n-1} = b, X_{n-2} = c \ldots X_2 = i, X_1 = j, X_0 = k)$$

$$= P(X_n = a \mid X_{n-1} = b)\, P(X_{n-1} = b \mid X_{n-2} = c \ldots X_2 = i, X_1 = j, X_0 = k)\, P(X_{n-2} = c, \ldots X_2 = i, X_1 = j, X_0 = k)$$

$$= P(X_n = a \mid X_{n-1} = b)\, P(X_{n-1} = b \mid X_{n-2} = c) \ldots P(X_1 = j \mid X_0 = k)\, P(X_0 = k)$$

$$= P_{ba} P_{cb} \ldots P_{kj}\, P(X_0 = k)$$

$$\therefore P(X_r = k, X_{r+1} = j \ldots X_{r+n} = a) = P_{ba} P_{cb} \ldots P_{kj}\, P(X_r = k)$$

For a Markov chain we represent the states by vertices, called nodes and one step transition between 2 states are denoted by directed arcs and one step transition ~~probab~~ between same states by directed loops. Such graphs are called stochastic graph or digraph of a Markov chain.

Theorem CHAPMAN - KOLOMOGROV EQUATION.

Let $\{X_n ; n \geq 0\}$ be the given Markov chain with state space S. Then for any $n \geq 0$ and $m \geq 0$, and $i, j \in S$ then

$$P_{ij}^{(m+n)} = \sum_{k \in S} P_{ik}^{(n)} P_{kj}^{(m)}$$

$$P_{ij}^{(m+n)} = P\{X_{m+n} = j \mid X_0 = i\}$$

$$= \sum_{k \in S} P\{X_{m+n} = j, X_n = k \mid X_0 = i\}$$

$$= \sum_{k \in S} \frac{P\{X_{m+n} = j, X_n = k, X_0 = i\}}{P\{X_0 = i\}} \quad \text{(cond. prob)}$$

$$= \sum_{k \in S} \frac{P\{X_{m+n} = j \mid X_n = k, X_0 = i\} P\{X_n = k, X_0 = i\}}{P\{X_0 = i\}}$$

$$= \sum_{k \in S} \frac{P\{X_{m+n} = j \mid X_n = k\} P\{X_n = k, X_0 = i\}}{P\{X_0 = i\}}$$

$$= \sum_{k \in S} P_{kj}^{(m)} \frac{P\{X_n = k, X_0 = i\}}{P\{X_0 = i\}}$$

$$= \sum_{k \in S} P_{kj}^{(m)} P(X_n = k \mid X_0 = i)$$

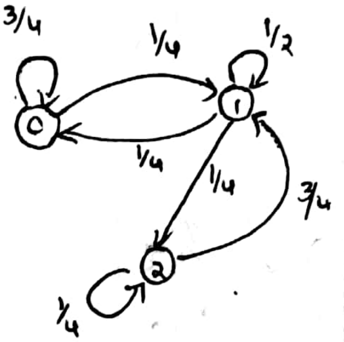
$$= \sum_{k \in S} P_{kj}^{(m)} P_{ik}^{(n)}$$

1) Let $\{X_n \mid n \geq 0\}$ be the given Markov chain with 3 states $i = 0, 1, 2$ and transition probability matrix
$$P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$
Find the following probabilities. Assume that the initial distribution is equally likely for the 3 states. Compute (i) $P\{X_1 = 1 \mid X_0 = 2\}$ (ii) $P\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\}$,

(iii) $P(X_2 = 2, X_1 = 1, X_0 = 2)$

iv) Draw the stochastic graph. v) $P\{X_2 = 2, X_1 = 1 \mid X_0 = 2\}$

Given $P\{X_0 = i\} = 1/3$ for $i = 0, 1, 2$.



$$P\{X_1 = 1 \mid X_0 = 2\} = p_{21} = 3/4$$

$$P\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\} = P\{X_3 = 1 \mid X_2 = 2\}\, P\{X_2 = 2 \mid X_1 = 1\}$$
$$P\{X_1 = 1 \mid X_0 = 2\}\, P\{X_0 = 2\}$$
$$= p_{21}\, p_{12}\, p_{21}\, P\{X_0 = 2\} = \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3} = \frac{3}{64}$$

$$P\{X_2 = 2, X_1 = 1, X_0 = 2\} = P\{X_2 = 2 \mid X_1 = 1\}\, P\{X_1 = 1 \mid X_0 = 2\}\, P\{X_0 = 2\}$$
$$= p_{12}\, p_{21}\, P\{X_0 = 2\} = \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{16}$$

v) $$P^2 = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 5/8 & 5/16 & 1/16 \\ 5/16 & 1/2 & 3/16 \\ 3/16 & 9/16 & 1/4 \end{bmatrix}$$

Req: Reach 2 in 2 steps and 1 in 1 step from 2.

$$P\{X_2 = 2, X_1 = 1 \mid X_0 = 2\} = P\{X_2 = 2 \mid X_0 = 2\}\, P\{X_1 = 1 \mid X_0 = 2\}$$
$$= p^{(2)}\, p_{21}^{(1)} = \frac{1}{4} \times \frac{3}{4} = 3/16$$

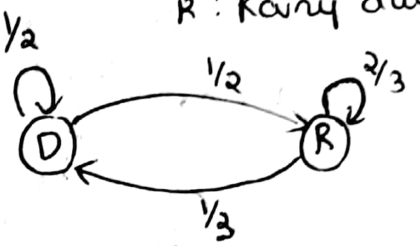
2) Suppose that the probability of a dry day following a rainy day is $1/3$ and the probability of a rainy day following a dry day is $1/2$. Given that May 1^{st} is a dry day, find the probability that (i) May 3^{rd} is a dry day (ii) May 4^{th} is a rainy day.

Rain pattern forms a Markov chain with 2 states

D : Dry day

R : Rainy day

$$\begin{array}{c|cc} & D & R \\ \hline D & 1/2 & 1/2 \\ R & 1/3 & 2/3 \end{array}$$



Req.

$$P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 5/12 & 7/12 \\ 5/18 & 11/18 \end{bmatrix}$$

$\frac{1}{4}+\frac{1}{6} = \frac{3+2}{12}$

$\frac{1}{4}+\frac{1}{3} =$

$\frac{1}{6}+\frac{2}{9} = \frac{3+2}{18}$

$\frac{1}{6}+\frac{4}{9} = \frac{3+8}{18} =$

$$P^3 = \begin{bmatrix} 5/12 & 7/12 \\ 5/18 & 11/18 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} =$$

Req $P(X_2 = D \mid X_0 = D) = P_{DD}^{(2)}$

$P(X_3 = R \mid X_0 = D) = P_{DR}^{(3)}$.

3) Let $\{X_n ; n \geq 0\}$ be a Markov chain with $P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$

Find i) $P\{X_2 = 1 \mid X_0 = 0\}$

ii) $P\{X_2 = 2\}$

iii) $P\{X_0 = 1 \mid X_2 = 2\}$

iv) $P\{X_2 = 1, X_0 = 0\}$

$$P^2 = \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{1}{4} \end{bmatrix}$$

i) $P\{X_2 = 1 \mid X_0 = 0\} = P_{01}^{(2)} = \frac{5}{16}$

ii) $P\{X_2 = 2\} = \sum_{i=0}^{2} P(X_2 = 2 \mid X_0 = i)\, P(X_0 = i)$

$= P(X_2 = 2 \mid X_0 = 0)\, P(X_0 = 0) + P(X_2 = 2 \mid X_0 = 1)\, P(X_0 = 1) + P(X_2 = 2 \mid X_0 = 2)\, P(X_0 = 2)$

$= \left[P_{02}^{(2)} + P_{12}^{(2)} + P_{22}^{(2)}\right] \frac{1}{3}$

$= \frac{1}{6}$

iii) $P\{X_0 = 1 \mid X_2 = 2\} = \dfrac{P\{X_2 = 2 \mid X_0 = 1\}\; P\{X_0 = 1\}}{P\{X_2 = 2\}}$ $\qquad P(E_1 \mid A) = \dfrac{P(A \mid E_1) P(E_1)}{P(A)}$

$= \dfrac{P_{12}^{(2)} \times \frac{1}{3}}{\frac{1}{6}} = \frac{1}{16} \times 6 = \frac{3}{8}$

iv) $P\{X_2 = 1, X_0 = 0\} = P\{X_2 = 1 \mid X_0 = 0)\; P\{X_0 = 0\}$ $\qquad P(A \cap B) = P(A \mid B)\, P(B)$

$= P_{01}^{(2)} \frac{1}{3}$

$= \frac{5}{16} \times \frac{1}{3} = \frac{5}{48}$

Classification of states:

Let $\{X_n ; n \geq 0\}$ be the given Markov chain with state space $S = \{j\}$ ∋' $j = 0, 1, 2 \ldots$ and with transition probability matrix $P = [p_{ij}]$. Then

① Reachable state (or accessible)

If $p_{ij}^{(n)} > 0$ for some $n \geq 1$ then we say state j is reachable from state i denoted by $i \longrightarrow j$.

state j is not reachable from i if $p_{ij}^{(n)} = 0$ $\forall\, n \in N$.

② Irreducible state :

If every state ~~can~~ is ~~be~~ reachable from any other state (in any no. of transitions), the Markov chain is said to be irreducible.

closed state space:

If C is a set of states such that no states outside C can be reached from any state in C, then C is said to be closed. ie, If C is a closed set then for any $j \in C$ and $k \notin C$ $p_{jk} = 0$.

Absorbing state;

State j is said to be an absorbing state if $p_{ij} = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}$

Return state

i is said to be return state if $p_{ii}^{(n)} > 0$ for some $n \geq 1$, i - state of a Markov chain

Period of a return state

Let $d_i = \gcd\{m;\ p_{ii}^{(m)} > 0\}$. A state i of a Markov chain is said to be periodic with period d_i if $d_i > 1$.

If $d_i = 1$ then state i is aperiodic.

A Markov chain is said to be periodic if every state is periodic.

First return time probability (or recurrence time probability):

The probability that the Markov chain having started from i returns to i for the first time at the n^{th} step is denoted by $f_{ii}^{(n)} = P\{X_n = i,\ X_m \neq i \text{ for } i = 1,2 \ldots n-1 \mid X_0 = i\}$.

The pair $\{n, f_{ii}^{(n)}\}$ for $n = 1, 2, 3 \ldots$ is the probability distribution of the recurrence time of the state i.

Let $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, then the return state i is said to be certain or peristent state or recurrent state.

The mean recurrence time of a state i is defined as

$$\mu_{ii} = \sum_{n=1}^{\infty} n\, f_{ii}^{(n)}$$

Transient state (or non recurrent state):

A state i is said to be transient (or non certain) if $F_{ii} < 1$.

Null persistent state: A persistent state i is called a null state if $\mu_{ii} = \infty$. It is said to be non null if $\mu_{ii} < \infty$

(i.e. it is finite)

Ergodic state

A persistent, non null and an aperiodic state i is called an ergodic state.

A Markov chain is said to be ergodic if all of its states are ergodic.

Essential state:

A state i is said to be essential if it communicates with any state from which it is accessible.

Theorem 1:

A state i is (i) persistent iff $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$

(ii) transient iff $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$

Theorem 2:

If a Markov chain is irreducible then all its states are of the same type.

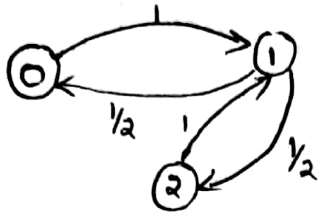
Theorem 3:

In a finite Markov chain, it is impossible to have all the states transient. Further if it is irreducible then all its state are persistent and non null.

Theorem 4:

A finite state Markov which is irreducible and aperiodic is ergodic.

1) Find the nature of the states of a Markov chain with tpm $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$ where the state space $S = \{0, 1, 2\}$. Also draw stochastic graph.



From P, $p_{00}^{(1)} = p_{11}^{(1)} = p_{22}^{(1)} = 0$

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$p_{ii}^{(2)} > 0 \quad \forall i = 0,1,2$

$$P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = P$$

$|||^{ly}$ $P^4 = P^2$

$\therefore P^{2n} = P^2$

$P^{2n+1} = P$

wkt $d_i = \gcd\{m \mid p_{ii}^{(m)} > 0\} = \gcd\{2, 4, 6 \ldots\} = 2 > 1$

$\therefore$ All states $(i = 0, 1, 2)$ are periodic with period $d_i = 2$.

Here we have $p_{ij}^{(n)} > 0$ for some $n \geq 1$.

$\therefore$ Markov chain is irreducible.

$\therefore$ All the states in the Markov chain are persistent and non null.

(since the state space is finite)

ie, All the states are persistent, non null and periodic.

$\therefore$ All states are not ergodic $\Rightarrow$ Markov chain is not ergodic.

Ergodic state

A persistent, non null j are ~~equ~~ reachable to each other

an ergodic stat to communicate with each others,

A Markov $i \leftrightarrow j$. This means $i \to j$ and $j \to i$.

The relation 'communication' between states ie, $i \leftrightarrow j$ is an equivalence relation.

ie, Relation communication is

1. Reflexive $i \leftrightarrow i$
2. Symmetric ie, if $i \leftrightarrow j$ then $j \leftrightarrow i$
3. Transitive ie, if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$.

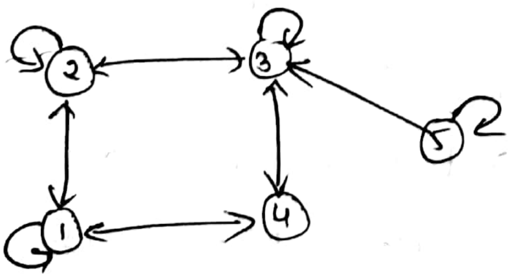
Therefore, the accessibility relation (communication) divides the state space into mutually disjoint equivalence classes within ~~which~~ each class all states communicate with each other.

Theorem:

Markov chain is irreducible if there is only one equivalence class (ie, every state is reachable from every other state).

Ex:

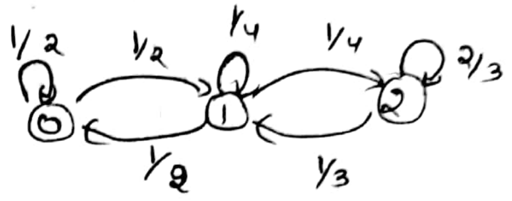
(i) Consider the Markov chain with stochastic graph



Here the equivalence classes are $\{1, 2, 3, 4\}$ and $\{5\}$

Since the Markov chain has 2 equivalence classes, it is not irreducible.

(ii) Consider the Markov chain



with state space $S = \{0, 1, 2\}$ stochastic graph

and $P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix}$.

Equivalence class: $\{0, 1, 2\}$

$\therefore$ Markov chain is irreducible.

<u>Result</u>: For a finite state Markov chain either all states in a class are transient or all are recurrent.

Let $\{X_n ; n \geq 0\}$ be a Markov chain having $S = \{1, 2, 3, 4\}$ with transition probability matrix $= \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$

Then find (i) stochastic graph of markov chain.

(ii) Find $f_{ii}^{(n)}$ for $i \in S$

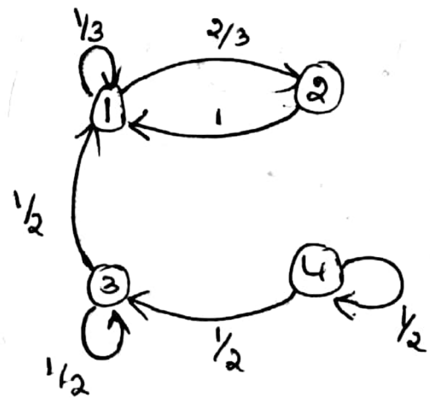
(iii) F_{ii} for $i \in S$

(iv) μ_{ii} for $i \in S$

(v) Verify whether the states $i = 1, 2$ are ergodic or not.

(vi) Find the equivalence class.

(i)



(ii) when $i=1$ $\quad f_{11}^{(1)} = 1/3 \quad f_{11}^{(2)} = 2/3 \times 1 = 2/3 \quad f_{11}^{(3)} = 0$

$$f_{11}^{(n)} = 0 \quad \forall \; n \geq 3$$

$i=2, \quad f_{22}^{(1)} = 0 \quad f_{22}^{(2)} = 1 \cdot \frac{2}{3} = \frac{2}{3} \quad f_{22}^{(3)} = 1 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$

$$f_{22}^{(4)} = 1 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{27} \quad \cdots$$

$i=3, \quad f_{33}^{(1)} = 1/2 \quad f_{33}^{(2)} = 0 \quad f_{33}^{(3)} = 0 \quad \therefore f_{33}^{(n)} = 0 \quad \forall \; n \geq 2$

$i=4, \quad f_{44}^{(1)} = 1/2 \quad f_{44}^{(2)} = 0 \quad f_{44}^{(3)} = 0 \quad \therefore f_{44}^{(n)} = 0 \quad \forall \; n \geq 2$

iii) when $i=1$, $\quad F_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)} = \frac{1}{3} + \frac{2}{3} + 0 = 1 \Rightarrow i=1$ is persistent

$i=2 \quad F_{22} = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots = \frac{2/3}{1-1/3} = 1 \Rightarrow i=2$ is persistent

$i=3 \quad F_{33} = \sum_{n=1}^{\infty} f_{33}^{(n)} = 1/2 + 0 = 1/2 < 1 \Rightarrow i=3$ is not persistent

$i=4 \quad F_{44} = \frac{1}{2} < 1 \Rightarrow i=4$ is not persistent.

$\therefore i = 3, 4$ are transient states.

iv) $\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$

$i=1, \quad \mu_{11} = \sum_{n=1}^{\infty} n f_{11}^{(n)} = 1\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) = \frac{1+4}{3} = \frac{5}{3} < \infty$

$\therefore i=1$ is non null.

$i=2 \quad \mu_{22} = \sum_{n=1}^{\infty} n f_{22}^{(n)} = 1(0) + 2\left(\frac{2}{3}\right) + 3\left(\frac{2}{9}\right) + 4\left(\frac{2}{27}\right) + \cdots$

$$= \frac{2}{3}\left[2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3^2} + 4 \cdot \frac{1}{3^3} + \cdots\right]$$

$$\cancel{= \frac{2}{3}\left[2 + 3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3^2} + \cdots\right]}$$

$$\cancel{= \frac{2}{3} \sum_{n=2}^{\infty} n \left(\frac{1}{3}\right)^{n-2} = \frac{2}{3} \sum_{n=1}^{\infty}}$$

$2\left[\left(\frac{2}{3}\right)^{-2} - 1\right]$

$= 2\left[\frac{9-4}{4}\right]$

$(3^2) + \dots$

$$= 2\left[\left(1 - \tfrac{1}{3}\right)^{-2} - 1\right] = \frac{5}{2} < \infty$$

$$\cancel{= 2\left(\tfrac{3}{2}\right) - 2}$$

$\therefore i = 2$ is non null

$i = 3$

$\mu_{33} = 1\left(\frac{1}{2}\right) < \infty$ $\quad$ $i = 3$ is non null

$i = 4$. $\mu_{44} = \frac{1}{2} < \infty$

$\therefore i = 4$ is non null.

v) When $i = 1$, $d_1 = \gcd\{m \mid f_{11}^{(m)} > 0\} = \gcd\{1, 2\} = 1$

$\therefore i = 1$ is aperiodic.

We have $i = 1$ is persistent, non null and aperiodic

$\Rightarrow i = 1$ is ergodic.

$i = 2$, $d_2 = \gcd\{2, 3, 4 \dots\} = 1$ $\quad$ (2, 3 you can divide gcd = 1)

$\therefore i = 2$ is aperiodic.

As $i = 2$ is persistent, non null and aperiodic.

$\therefore i = 2$ is ergodic.

vi) $\{1, 2\}$, $\{3\}$, $\{4\}$ are equivalence classes.

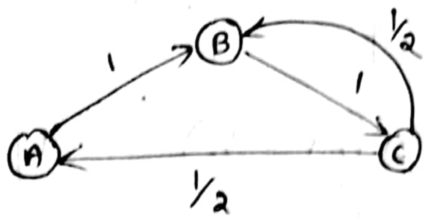
More than one equivalence classes.

$\therefore$ Markov chain is not irreducible.

2) 3 boys A, B, C are throwing a ball to each other. A always throws ball to B and B always to C. But C is as likely to throw ball to B as he is to A. ST process is Markovian. Find tpm, stochastic graph. Classify

the states. Write the equivalence classes.

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$



Equivalence class = $\{0, 1, 2\}$

$\therefore$ Irreducible.

$\underline{i=1}$ $\quad f_{11}^{(1)} = 0$

$f_{11}^{(2)} = 1 \cdot \frac{1}{2} = \frac{1}{2}$

$f_{11}^{(3)} = \frac{1}{2}$

$\vdots$

$d_1 = \gcd\{2, 3 \ldots\} = 1$

$\Rightarrow i=1 \therefore i$ is Aperiodic

In finite state markov chain, if it is irreducible all states are persistent and non null.

All states are persistent, non null, aperiodic.

$\therefore$ Ergodic Markov chain.

3) Let $\{X_n ; n = 1, 2, 3 \ldots\}$ be a Markov chain with state space $S = \{0, 1, 2\}$ and tpm is given by $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix}$.

i) Whether the Markov chain is Ergodic?

ii) Find μ_{ii} for $i = 0, 1, 2$.

Solutions of boundary value problems in PDE:

Consider the equation $AU_{xx} + 2BU_{xy} + CU_{yy} + F(u, x, y, u_x, u_y) = 0$

(General form of 2nd order PDE) —— ①

Then ① is said to be parabolic if $AC - B^2 = 0$

elliptic if $AC - B^2 > 0$

hyperbolic if $AC - B^2 < 0$

Ex: Consider the 1D heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Here $A = c^2$. $B = 0$ $C = 0$ $\Rightarrow AC - B^2 = 0.$

$\therefore$ Parabolic.

Consider 2D Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$A = 1$ $B = 0$ $C = 1$ $\Rightarrow AC - B^2 = 1 > 0$

$\therefore$ Elliptic

Consider 1D wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$A = c^2$ $B = 0$ $C = -1$ $\Rightarrow AC - B^2 = -c^2 < 0$

$\therefore$ Hyperbolic

Solution of Laplace equation and Poisson equation:

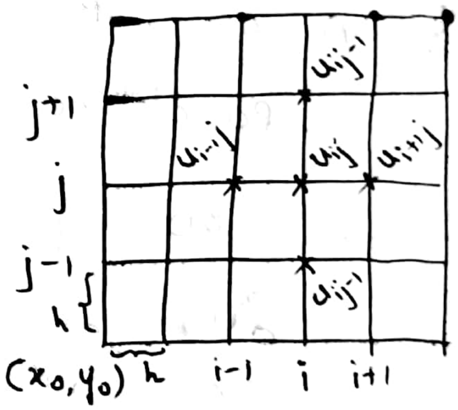
Consider the 2D Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——— (1)}$$

To solve the Laplace equation and Poisson equation in a rectangular region r in an XY plane, we divide the region R into small squares by drawing lines parallel to the sides of rectangular region.

Let h be the side of each square. Let (x_0, y_0) be an arbitrary corner point and $x_i = x_0 + ih$, $y_j = y_0 + jh$.

Take $u_{ij} = u(x_i, y_j)$



$$x_{i+1} = x_0 + (i+1)h$$
$$= (x_0 + ih) + h$$
$$= x_i + h$$

|||ly $y_{1+j} = y_j + h$

We have $u_{i+1j} = u(x_{i+1}, y_j)$

$$= u(x_i + h, y_j)$$

∴ By Taylor series expansion for fn of 2 variables,

$$u_{i+1j} = u(x_i, y_j) + \frac{1}{1!} h \frac{\partial u(x_i, y_j)}{\partial x_i} + \frac{1}{2!} h^2 \frac{\partial^2 u(x_i, y_j)}{\partial x_i^2} + \dots$$

$$= u_{ij} + h \frac{\partial x_{ij}}{\partial x_i} + \frac{h^2}{2} \frac{\partial^2 x_i y_j}{\partial x_i^2} + \dots \quad \text{——— (2)}$$

|||ly, $$u_{i-1j} = u(x_i, y_j) - \frac{1}{1!} h \frac{\partial u(x_i, y_j)}{\partial x_i} + \frac{h^2}{2!} \frac{\partial^2 u(x_i, y_j)}{\partial x_i^2} + \dots \quad \text{——— (3)}$$

(2) + (3) ⇒ $$u_{i+1j} + u_{i-1j} = 2u(x_i, y_j) + \frac{h^2}{2!} \frac{\partial^2 u(x_i, y_j)}{\partial x_i^2} + \text{Higher powers of } h$$

Since h is small, ignoring higher power we get

$$u_{i+1j} + u_{i-1j} = 2u_{ij} + h^2 \frac{\partial^2 u_{ij}}{\partial x_i^2}$$

$$\Rightarrow \frac{\partial^2 u_{ij}}{\partial x_i^2} = \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2}$$

|||ly, $$\frac{\partial^2 u_{ij}}{\partial y_j^2} = \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2}$$

$$① \Rightarrow \frac{\partial^2 u_{ij}}{\partial x_i^2} + \frac{\partial^2 u_{ij}}{\partial y_j^2} = 0$$

$$\Rightarrow \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2} + \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2} = 0$$

$$\Rightarrow 4u_{ij} = u_{i+1j} + u_{i-1j} + u_{ij+1} + u_{ij-1}$$

$$\Rightarrow u_{ij} = \frac{1}{4}\left[u_{i+1j} + u_{i-1j} + u_{ij+1} + u_{ij-1}\right]$$ called std five pt formula

Consider Poisson eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$

For $x = x_i$, $y = y_j$ we get $\frac{\partial^2 u_{ij}}{\partial x_i^2} + \frac{\partial^2 u_{ij}}{\partial y_j^2} = f(x_i, y_j)$

Using finite diff formula,

$$\frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2} + \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2} = f(x_i, y_j)$$

$$\Rightarrow u_{ij} = \frac{1}{4}\left[u_{i+1j} + u_{i-1j} + u_{ij+1} + u_{ij-1} - h^2 f(x_i, y_j)\right]$$

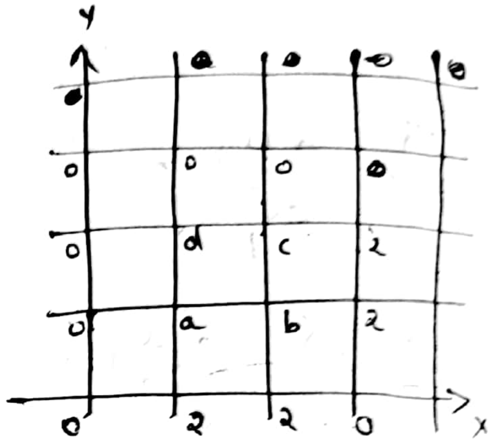
1) Using the step size $h = 1/3$ solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in $0<x<1$, $0<y<1$

with the boundary condition $u(x,1) = (0,y) = 0$

$u(1,y) = 9(y-y^2)$

using finite difference method.

$u(x,0) = 9(x, x^2)$



Using five point formula,

$u_{11} = a = \frac{1}{4}[b+d+2]$

$u_{21} = b = \frac{1}{4}[a+c+4]$

$u_{22} = c = \frac{1}{4}[b+d+2]$

$u_{12} = d = \frac{1}{4}[a+c]$

$a = c$

$\therefore \; a = \frac{1}{4}(b+d+2)$

$~~b = (a+2)\frac{1}{2}~~$

$~~\Rightarrow a = 2b-2~~$

$b = \frac{1}{4}(2a+4)$

$d = \frac{1}{4}(2a)$ $~~2d = a~~$

$a = c = 1$

$b = 3/2$

$d = 1/2$

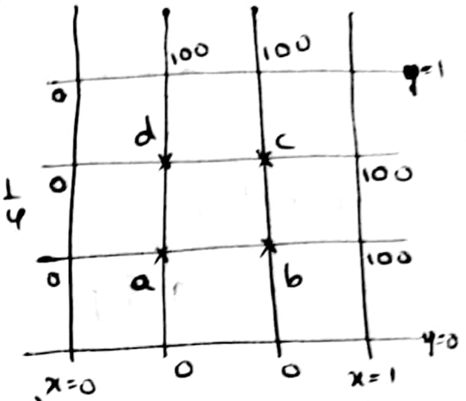
2) Using finite difference method taking $h = 1/3$, solve

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -81xy$ in $0<x<1$, $0<y<1$ with the

boundary conditions $u(0,y) = u(x,0) = 0$

$u(1,y) = u(x,1) = 100$

$$u_{ij} = \frac{1}{4}\left[u_{i+1j} + u_{i-1j} + u_{ij-1} + u_{ij+1} - h^2 f(x_i, y_j)\right]$$



$$u_{11} = a = \frac{1}{4}\left(b + d - \frac{1}{9} f\left(\tfrac{1}{3}, \tfrac{1}{3}\right)\right) = (b + d + 1)\frac{1}{4}$$

$$u_{21} = b = \frac{1}{4}\left(a + c + 100 - \frac{1}{9} f\left(\tfrac{2}{3}, \tfrac{1}{3}\right)\right) = (a + c + 102)\frac{1}{4}$$

$$u_{22} = c = \frac{1}{4}\left(200 + b + d - \frac{1}{9}\left(\tfrac{2}{3}, \tfrac{2}{3}\right)\right) = (b + d + 204)\frac{1}{4}$$

$$u_{12} = d = \frac{1}{4}\left(a + c + 100 - \frac{1}{9}\left(\tfrac{1}{3}, \tfrac{2}{3}\right)\right) = \frac{1}{4}(a + c + 102)$$

$+\frac{81}{9} \times \frac{2}{3} \times \frac{2}{3}$

$\Rightarrow b = d.$

$$\therefore \quad a = \frac{1}{4}(2b + 1)$$

$$b = \frac{1}{4}(a + c + 102)$$

$$c = \frac{1}{4}(2b + 204)$$

$a = 25.79$

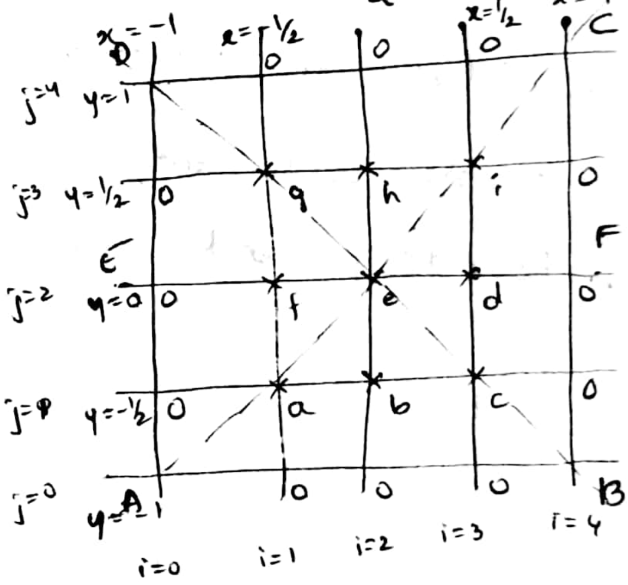
$b = 51.08$

$c = 76.54$

3) Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1$ in $|x| < 1$ and $|y| < 1$

Condition $u(\pm 1, y) = u(x, \pm 1) = 0$

$h = 1/2$



$$a = \frac{1}{4}\left(b + f + \frac{1}{4}\right)$$

$$b = \frac{1}{4}\left(a + c + e + \frac{1}{4}\right)$$

$$c = \frac{1}{4}\left(b + d + \frac{1}{4}\right)$$

$$d = \frac{1}{4}\left(i + e + c + \frac{1}{4}\right)$$

$$e = \frac{1}{4}\left(h + f + b + d + \frac{1}{4}\right)$$

$$f = \frac{1}{4}\left(a + e + g + \frac{1}{4}\right)$$

$$g = \frac{1}{4}\left(h + f + \frac{1}{4}\right)$$

$$h = \frac{1}{4}\left(i + e + g + \frac{1}{4}\right)$$

$$i = \frac{1}{4}\left(d + h + \frac{1}{4}\right)$$

Symmetry about 2 diagonals

By symmetry about diagonal BD,

$a = i$, $f = h$, $b = d$

symmetry about line EF ($y=0$ line)

$a = g$ $\quad b = h \quad c = i$

$\therefore$ By symmetry it is enough to determine a, b, e.

$$a = \frac{1}{4}\left[2b + \frac{1}{4}\right]$$

$$b = \frac{1}{4}\left[2a + e + \frac{1}{4}\right]$$

$h + b + f + d$
$b + b + b + b$

$$e = \frac{1}{4}\left[4b + \frac{1}{4}\right]$$

$a = 11/64$

$b = 7/32$

$c = 9/32$

$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

4) Solve the equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with side $x=0$ & $y=0$, $x=3$ & $y=3$ with $u=0$ on the boundary and mesh length $= 1$.

Solution of 1D heat equation:

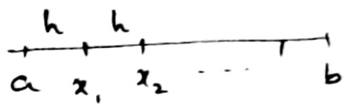
Consider the boundary value problem

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)} \qquad \text{where } a < x < b, \; t > 0$$

under the condition $u(x, 0) = f(x)$ —— (2)

$$\left.\begin{array}{l} u(a, t) = \phi(t) \\ u(b, t) = \psi(t) \end{array}\right\} \text{—— (3)}$$

Divide the interval $[a, b]$ into n equal parts each of length $h = \frac{b-a}{n}$.



Let $x_i = a + ih$ for $i = 0, 1, 2 \dots n$

Let k be the time interval size, and let $t_j = jk$

Let $u_{ij} = u(x_i, t_j)$ Then by Taylor series expansion,

$$u_{ij+1} = u(x_i, t_{j+1}) = u(x_i, (j+1)k) = u(x_i, t_j + k)$$

$$= u_{ij} + \frac{1}{1!} k \frac{\partial u_{ij}}{\partial t_j} + \frac{1}{2!} k^2 \frac{\partial^2 u_{ij}}{\partial t_j^2} + \dots \quad \text{—— (4)}$$

In (4) as k is very small, neglect terms containing higher powers of k.

we get $u_{ij+1} = u_{ij} + k \frac{\partial u_{ij}}{\partial t_j} \Rightarrow \frac{\partial u_{ij}}{\partial t_j} = \frac{u_{ij+1} - u_{ij}}{k}$ —— (5)

We have $$\frac{\partial^2 u_{ij}}{\partial x_i^2} = \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2} \quad \text{—— (6)}$$

Put $x = x_i$ and $t = t_j$ in (1).

We get $$\frac{\partial u_{ij}}{\partial t_j} = c^2 \frac{\partial^2 u_{ij}}{\partial x_i^2}$$

From ⑤ and ⑥ we get,

$$\left[\frac{u_{ij+1} - u_{ij}}{k}\right] = c^2\left[\frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2}\right]$$

$$\Rightarrow u_{ij+1} - u_{ij} = \frac{kc^2}{h^2}\left[u_{i+1j} - 2u_{ij} + u_{i-1j}\right]$$

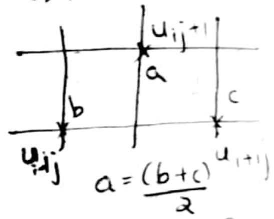
Let $\lambda = \frac{kc^2}{h^2}$

$$u_{ij+1} - u_{ij} = \lambda\left[u_{i+1j} - 2u_{ij} + u_{i-1j}\right]$$

$$u_{ij+1} = \lambda(u_{i+1j} + u_{i-1j}) + (1-2\lambda)u_{ij} \quad \text{—— ⑦}$$

⑦ is called the explicit finite difference scheme by Schmidt.

<u>Note</u>: In ⑦ parameter λ should satisfy $0 < \lambda < 1$.



When $\lambda = \frac{1}{2}$ we get $u_{ij+1} = \frac{u_{i+1j} + u_{i-1j}}{2}$

From initial condition ② we get $u(x,0) = u(x_i, 0) = u_{i0} = f(x_i)$

From boundary conditions ③ we get $u(a,t) = u(x_0, t_j) = u_{0j} = \phi(t_j)$

and $u(b,t) = u(x_n, t_j) = u_{nj} = \psi(t_j)$

1) Compute the values of $u(x,t)$ for 4 time steps with $h = 1/4$ given that $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ where $0 < x < 1$ and $t > 0$ under the conditions $u(x,0) = 100(x - x^2)$, $u(0,t) = u(1,t) = 0$.

Take $\lambda = 1/2$.

Given $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \Rightarrow c^2 = 1$

$h = 1/4$

$$\lambda = \frac{kc^2}{h^2} \Rightarrow k = \frac{h^2}{2c^2} = \frac{1}{16 \times 2} = \frac{1}{32}$$

Given initial condition $u(x,0) = 100(x - x^2) = f(x)$

$$\Rightarrow u_{i0} = 100(x_i - x_i)^2$$

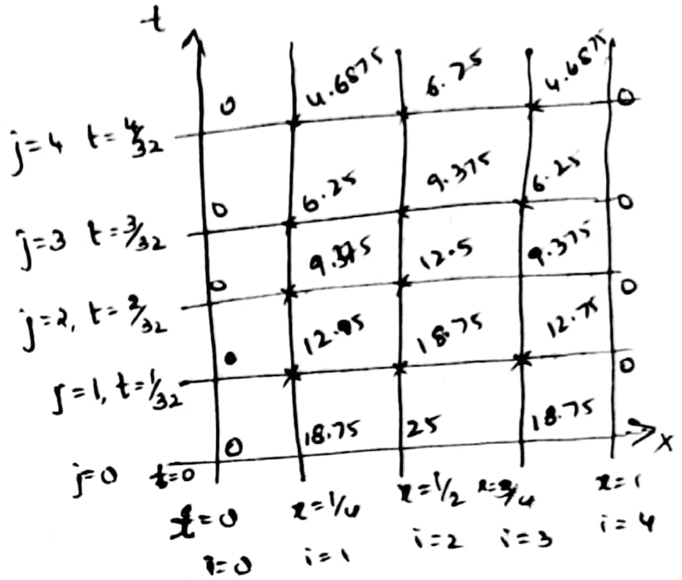
From boundary conditions, $u(0,t) = u_{0j} = 0$

$u(1,t) = u_{nj} = 0$

$u_{10} = 100\left(\frac{1}{4} - \frac{1}{16}\right) = 18.75$

$u_{20} = 25$

$u_{30} = 18.75$



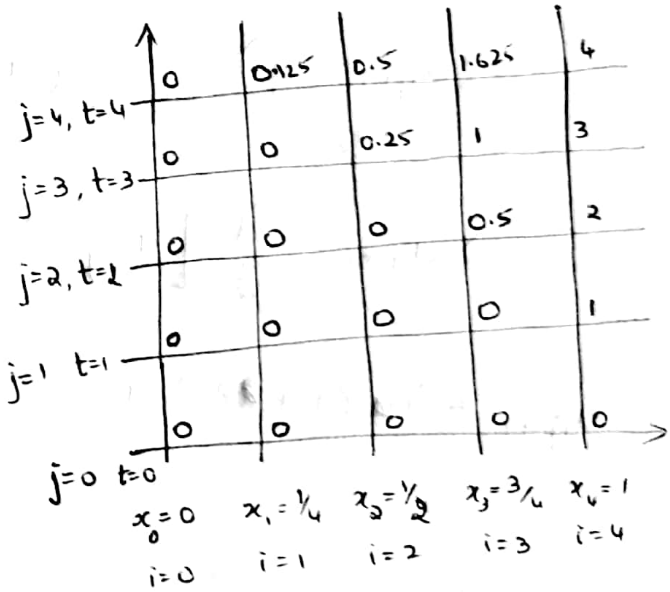
By schmidt's scheme,

$$u_{ij+1} = \frac{u_{i+1j} + u_{i-1j}}{2}$$

2) Using schmidt's finite difference scheme with $h = 1/4$ and $\lambda = 1/2$ solve $32\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $0 < x < 1$ $t > 0$. under the conditions $u(x,0) = 0$ $u(0,t) = 0$ and $u(1,t) = t$. Compute u for 4 time steps.

$$\frac{\partial u}{\partial t} = \frac{1}{32}\frac{\partial^2 u}{\partial x^2} \Rightarrow c^2 = \frac{1}{32}$$

$$k = \frac{h^2\lambda}{c^2} = \frac{1}{16} \times \frac{1}{2} \times 32 = 1$$



$u(x,0) = 0 \Rightarrow u(x_i, 0) = 0$

$\Rightarrow u_{i0} = 0 = f_i$

$u(0,t) = 0 \Rightarrow u(0, t_j) = 0$

$\Rightarrow u_{0j} = 0 = \phi_j$

$$u_{ij+1} = \frac{u_{i+1j} + u_{i-1j}}{2}$$

$u(1,t) = t \Rightarrow u(1, t_j) = t_j$

$\Rightarrow u_{1j} = t_j = \psi_j$

CRANK NICOLSON METHOD:

In Schmidt's finite difference scheme we have ~~eqn~~

$$\frac{u_{ij+1} - u_{ij}}{k} = \frac{c^2}{h^2}\left(u_{i-1j} - 2u_{ij} + u_{i+1j}\right) \quad \text{--- (1)}$$

The LHS of (1) is the forward difference approximation of $\frac{\partial u}{\partial t}$.

If we use the backward difference of $\frac{\partial u}{\partial t}$ we get

$$\frac{u_{ij} - u_{ij-1}}{k} = \frac{c^2}{h^2}\left(u_{i-1j} - 2u_{ij} + u_{i+1j}\right) \quad \text{--- (2)}$$

Replacing j by j+1 in (2),

$$\frac{u_{ij+1} - u_{ij}}{k} = \frac{c^2}{h^2}\left(u_{i-1j+1} - 2u_{ij+1} + u_{i+1j+1}\right) \quad \text{--- (3)}$$

(1) + (3):

$$\frac{2(u_{ij+1} - u_{ij})}{k} = \frac{c^2}{h^2}\left(u_{i-1j} - 2u_{ij} + u_{i+1j} + u_{i-1j+1} - 2u_{ij+1} + u_{i+1j+1}\right)$$

$$\Rightarrow u_{ij+1} = \frac{\lambda}{2}\left[u_{i-1j} - 2u_{ij} + u_{i+1j} + u_{i-1j+1} - 2u_{ij+1} + u_{i+1j+1}\right]$$

$$\Rightarrow u_{ij+1} + \cancel{2}(1+\lambda)u_{ij+1} - \lambda u_{i+1j+1} = \lambda u_{i-1j} + 2(1-\lambda)u_{ij} + \lambda u_{i+1j} \quad \text{--- (4)}$$

(4) is finite difference scheme for Crank Nicolson method.

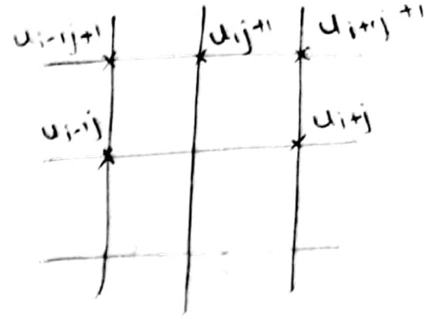
Here the parameter λ has no restriction.

When $\lambda = 1$, (4) becomes.

$$u_{ij+1} + 4u_{ij+1} - u_{i+1j+1} = u_{i-1j} - u_{ij} + u_{i+1j}$$

Scanned by CamScanner

$$-u_{i-1j+1} + 4u_{ij+1} - u_{i+1j+1} = u_{i-1j} + u_{i+1j}$$



1) Solve the heat equation by Crank Nicolson method.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0<x<1, \ t>0, \text{ under the conditions}$$

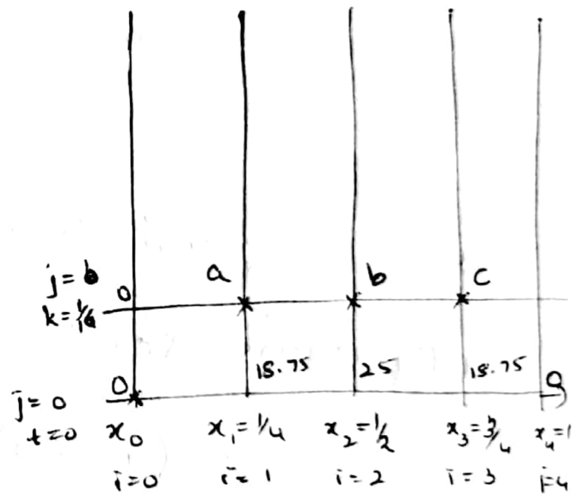
$$u(x,0) = 100(x-x^2)$$

$$u(0,t) = u(1,t) = 0$$

Compute u for 1 time step with $h = 1/4$. Given $\lambda = 1$.

$$\lambda = \frac{kc^2}{h}$$

$$\Rightarrow k = \frac{h^2\lambda}{c^2} = \frac{1}{4^2} = \frac{1}{16}$$



$$u(x_1,0) = 18.75$$

$$u(x_2,0) = 25$$

$$u(x_3,0) = 18.75$$

$$u(x_4,0) = 0$$

Symmetry about x_2

$\Rightarrow a = c.$

$$0 + 4a - b = 25 + 0$$

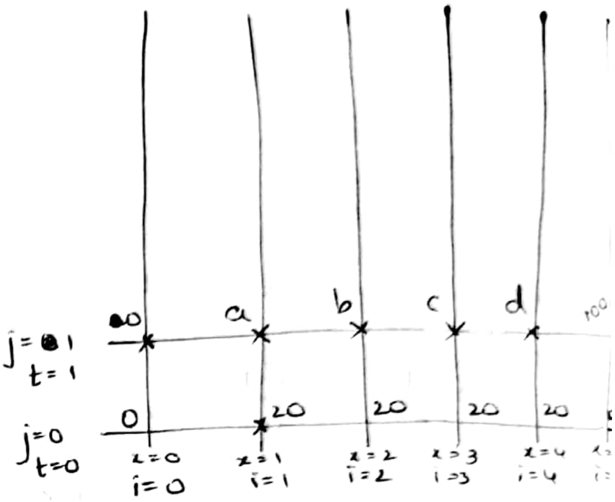
$$-a + 4b - a = 18.75 + 18.75$$

$$4a - b = 25 \qquad a = c = 9.82$$

$$-2a + 4b = 37.5 \qquad b = 14.29$$

Scanned by CamScanner

2) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $0 < x < 5$ and $t > 0$ under conditions $u(x,0) = 20$ and $u(0,t) = 0$, $u(5,t) = 100$. Compute u for 1 time step by Crank Nicolson method with $h = 1$ and $\lambda = 1$



$$\lambda = \frac{kc^2}{h^2} \Rightarrow k = 1$$

$$4a - b = 20$$

$$4b - a - c = 40$$

$$4c - b - d = 40$$

$$4d - c - 100 = 120 \Rightarrow 4d - c = 220$$

$$\begin{aligned} 4a - b \quad\quad\quad &= 20 \\ -a + 4b - c \quad &= 40 \\ -b + 4c - d &= 40 \\ -c + 4d &= 220 \end{aligned}$$

There is diagonal dominance.

$\therefore$ By Gauss Seidal method:

$$a = \frac{1}{4}(b + 20)$$

$$b = \frac{1}{4}(40 + a + c)$$

$$c = \frac{1}{4}(b + d + 40)$$

$$d = \frac{1}{4}(220 + c)$$

Iteration 1: $b = 0$, $c = 0$, $d = 0$

$a^{(1)} = \frac{1}{4}(20) = 5$ $\quad\quad$ $b^{(1)} = \frac{1}{4}(40 + 5) = 11.25$

$c^{(1)} = \frac{1}{4}(11.25 + 40) = 12.812$

$d^{(1)} = \frac{1}{4}(220 + 12.812) = 58.203$

Iteration 2: $a^{(2)} = \frac{1}{4}(20 + 11.25) = 7.812$

$b^{(2)} = \frac{1}{4}(40 + 7.812) = 16.953$

$c^{(2)} = \frac{1}{4}(11.953 + 58.203 + 40) = 27.539$

$d^{(2)} = \frac{1}{4}(220 + 27.539) = 61.885$

Continuing we get $a = 9.8$

$b = 20.19$

$c = 30.72$

$d = 59.92$

3) Using schmidt's scheme with $\lambda = 1/2$ solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ under the condition where $0 < x < 1$ and $t > 0$ $u(x,0) = 100 \sin \pi x$, $u(0,t) = u(1,t) = 0$. Take $h = 1/4$ and compute u for 4 time steps.

Solution of one dimensional wave equation:

Consider the boundary value problem. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $a<x<b$, $t>0$ —— ①

under the conditions $u(x,0) = f(x)$,

$\frac{\partial u}{\partial t}(x,0) = g(x)$ —— ②

Boundary conditions: $\left.\begin{array}{l} u(a,t) = \phi(x) \\ u(b,t) = \psi(x) \end{array}\right\}$ —— ③

Divide $[a,b]$ into n equal parts, each of width $h = \frac{b-a}{n}$

where $x_i = a + ih$ for $i = 0,1,2 \ldots n$.

Replace x by x_i and t by t_j in ①. We get,

$$\frac{\partial^2 u_{ij}}{\partial t_j^2} = c^2 \frac{\partial^2 u_{ij}}{\partial x_i^2} \quad \text{—— ④}$$

We have $\frac{\partial^2 u_{ij}}{\partial t_j^2} = \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{k^2}$ where k is the time interval size with $t_j = jk$.

Also we have $\frac{\partial^2 u_{ij}}{\partial x_i^2} = \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2}$

$$\therefore ④ \Rightarrow u_{ij+1} - 2u_{ij} + u_{ij-1} = \frac{k^2c^2}{h^2}\left(u_{i+1j} - 2u_{ij} + u_{i-1j}\right)$$

If $k = \frac{h}{c}$, we have $u_{ij+1} = u_{i+1j} + u_{i-1j} - u_{ij-1}$ —— ⑤

⑤ This is called the finite difference scheme for wave eqn.

② $\Rightarrow u(x_i,0) = f(x_i) \Rightarrow u_{i0} = f_i$ $\quad \frac{\partial u}{\partial x}(x_i,0) = g(x_i) \Rightarrow \frac{\partial u_{i0}}{\partial x} = g_i$

$u(a,t) = u(x_0,t_j) = \phi(x_i) = u_{0j} = \phi(x_i)$

$u(b,t) = \psi(x) \Rightarrow u(x_n,t_j) = \psi(x_i) \Rightarrow u_{nj} = \psi_i$

put $j=0$ in (5)

$$u_{i1} = u_{i+1\,0} + u_{i-1\,0} - u_{i\,-1}$$

$$= f_{i+1} + f_{i-1} - [u(x_i, t_{-1})]$$

$$= f_{i+1} + f_{i-1} - [u(x_i, t_0 - k)]$$

$$= f_{i+1} + f_{i-1} - \left[u_{i0} - k\frac{\partial u_{i0}}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 u_{i0}}{\partial t^2} + \dots\right]$$

$$= f_{i+1} + f_{i-1} - \left(f_i - kq_i + \frac{k^2}{2!} c^2 \frac{\partial^2 u_{i0}}{\partial x^2} + \dots\right]$$

$$= f_{i+1} + f_{i-1} - f_i + kq_i - \frac{h^2}{2}\left[\frac{u_{i+1\,0} - 2u_{i0} + u_{i-1\,0}}{h^2}\right]$$

$$= f_{i+1} + f_{i-1} - f_i + kq_i - \frac{1}{2}\left[f_{i+1} - 2f_i + f_{i-1}\right]$$

$u(x_i, t_{j+1}) = u(x_i, (j+1)k)$

$= u(x_i, jk+k)$

$= u(x_i, t_j + k)$

$u(x_i, t_{-1}) = u(x_i, (-1)k)$

$= u(x_i, t_0 - k)$

$\frac{\partial^2 u_{ij}}{\partial t^2} = c^2 \frac{\partial^2 u_{i0}}{\partial x^2}$

$$\Rightarrow u_{i1} = \frac{1}{2}(f_{i+1} + f_{i-1}) + kq_i$$

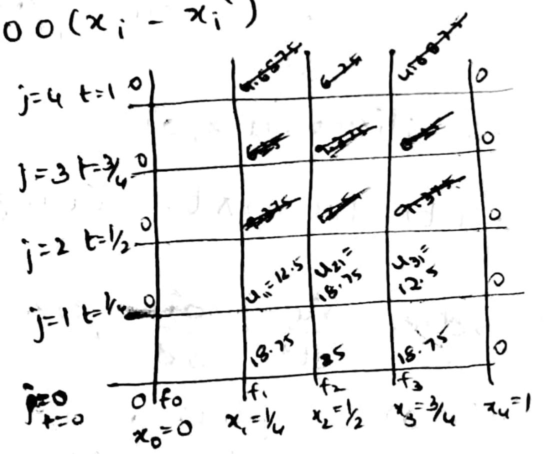
1) solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ for $0<x<1$ and $t>0$. with initial condition $u(x,0) = 100(x-x^2)$ and $\frac{\partial u}{\partial t}(x,0) = 0$ and $u(0,t) = u(1,t) = 0$. Take $h = 1/4$ and compute u for 4 time steps.

Here $c=1 \Rightarrow k = \frac{h}{c} = \frac{1}{4}$

$u(x,0) = f(x) = 100(x-x^2) \Rightarrow u_{i0} = f_i = 100(x_i - x_i^2)$

$\frac{\partial u}{\partial t}(x,0) \Rightarrow \frac{\partial u_{i0}}{\partial t} = q_i = 0$

$u_{i1} = \frac{1}{2}(f_{i+1} + f_{i-1}) + kq_i = \frac{1}{2}(f_{i+1} + f_{i-1})$



$u_{11} = \frac{1}{2}(f_2 + f_0) = 12.5$ $u_{21} = 18.75$

$u_{31} = 12.5$

$u_{ij+1} = u_{i+1j} + u_{i-1j} - u_{ij-1}$

$i=1, j=1$ $\quad u_{12} = u_{21} + u_{01} - u_{10}$

$\quad = \overset{18.75}{} + 0 - 18.75 =$

$i=2, j=1$ $\quad u_{22} = u_{31} + u_{11} - u_{20} = 0$

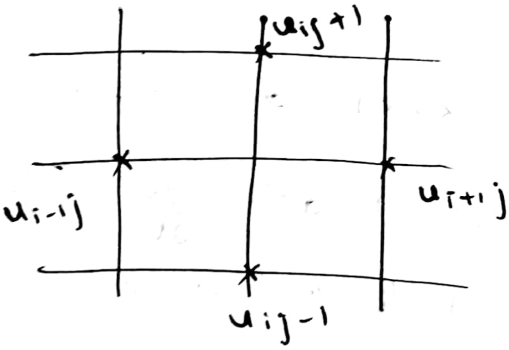
$i=3, j=1$ $\quad u_{32} = u_{41} + u_{21} - u_{30} = 0$

$u_{13} = -12.5$

$u_{23} = -18.75$

$u_{33} = -12.5$

$u_{14} = -18.75 \qquad u_{24} = -25 \qquad u_{34} = -18.75$



$u_{ij+1} = u_{i-1j} + u_{i+1j} - u_{ij-1}$

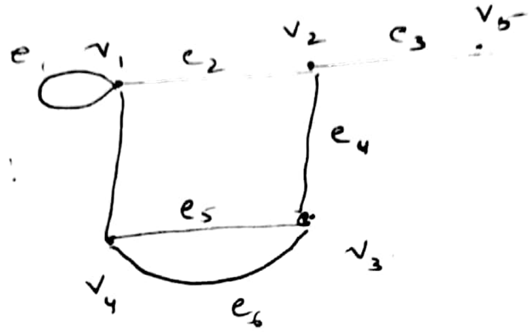
2) Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0<x<1$ and $t>0$ under the conditions $u(x,0)=0$. $\frac{\partial u}{\partial t}(x,0) = 0$ $\quad u(0,t)=0$ ~~$u(1,t)=100$~~ $u(1,t) = 100\sin\pi t$. Compute u for 4 time steps with $h = 1/4$.

3) Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0<x<2$, $t>0$, under the conditions $u(x,0)=0$, $\frac{\partial u}{\partial t}(x,0) = 100(2x - x^2)$, $u(0,t) = u(2,t) = 0$. $h = 1/2$. Compute u for 4 time steps.

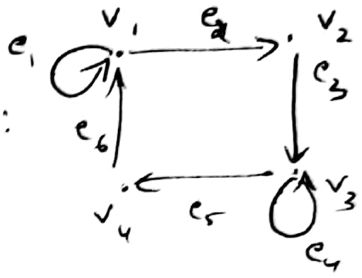
Graph theory

Graph: A graph $G = (V(G), E(G))$ consists of 2 finite sets $V(G)$ - vertex set and $E(G)$ - edge set such that each edge $e \in E$ is assigned an unordered pair (u, v) called the end vertices of e.

G_1 :



G_2 :



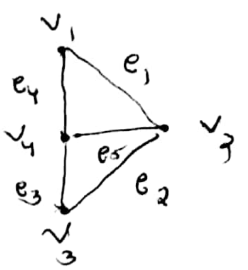
In a graph G, vertex joined to itself by an edge, is called a loop. 2 edges with same end vertices are called parallel edges. Ex: e_5, e_6 in G_1.

Graph G is called simple if it has no loops and no parallel edges.

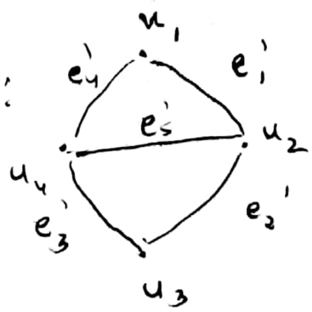
Isomorphism of graphs:

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be 2 graphs. Then we say that G_1 is isomorphic to G_2 if there is one-one correspondance b/w the vertex set V_1 and V_2 and edge set E_1 and E_2.

Ex: ① G_1 :



G_2 :



$v_1 \to u_1$	$e_1 \to e_1'$
$v_2 \to u_2$	$e_2 \to e_2'$
$v_3 \to u_3$	$e_3 \to e_3'$
$v_4 \to u_4$	$e_4 \to e_4'$
	$e_5 \to e_5'$

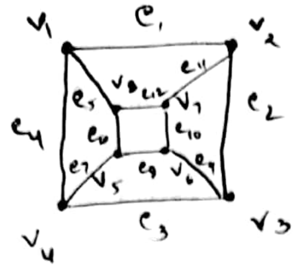
$V_1 \to V_2$ } Both one-one
$E_1 \to E_2$ } and onto.

$\therefore G_1 \simeq G_2$

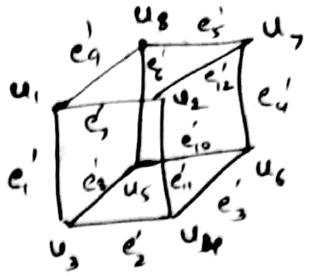
Scanned by CamScanner

(2)

G_1:



G_2:



V_1	V_2
v_1	u_8
v_2	u_1
v_3	u_6
v_4	u_5
v_5	u_3
v_6	u_4
v_7	u_2
v_8	u_1

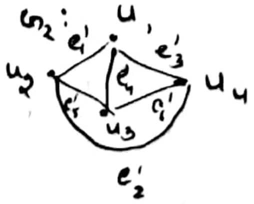
E_1	E_2
e_1	e_5'
e_2	e_4'
e_3	e_{10}'
e_4	e_6'
e_5	e_9'
e_6	e_1'
e_7	e_8'
e_8	e_2'
e_9	e_3'
e_{10}	e_{11}'
e_{11}	e_{12}'
e_{12}	e_7'

$\therefore G_1 \simeq G_2$

Check whether the following pairs of graph are isomorphic or not.

(1) G_1:

G_2:



$G_1 \simeq G_2$

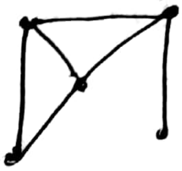
$v_1 \to u_1$

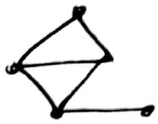
$v_2 \to u_2$

$v_3 \to u_3$

$v_4 \to u_4$

②



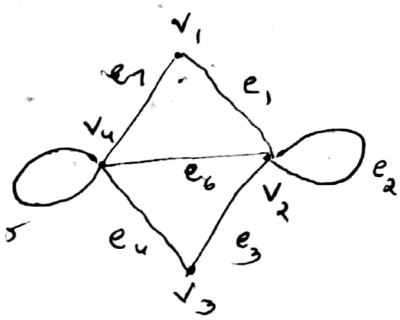


Let G be a graph and edge ~~e~~ e is said to be incident with vertex b if b is an end vertex of e.

To edges e and f which are incident with ~~a~~ a common vertex v, then e and f are called adjacent

Let b be a vertex of graph G. Then the degree of the vertex be is the number of edges of G incident with b, counting each loop twice.

G:



~~degree~~ $d(v_1) = 2$

$d(v_2) = 5$

$d(v_3) = 2$

$d(v_4) = 5$

First theorem:

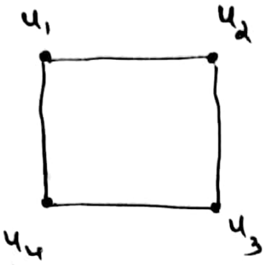
Let G be a graph with e edges and n vertices $v_1, v_2 \ldots v_n$. Then $\sum_{i=1}^{n} d(v_i) = 2e$.

A vertex of a graph is called odd or even, depending on whether its degree is odd or even.

Result: In a graph there is an even number of odd vertices

Let G be a graph. If for some integer k, $d(v) = k$ for every vertex $v \in V(G)$. Then G is called k regular.

A regular graph G is the one that is k regular for some k.



$d(u_i) = 2 \quad \forall \; i = 1, 2, 3, 4$

$\therefore G$ is 2- regular.

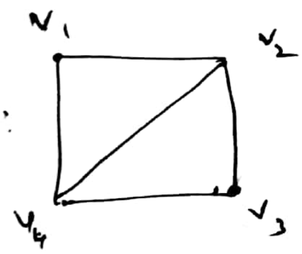
$\therefore G$ is regular.

Subgraph of a graph:

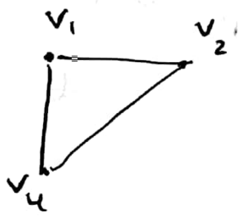
Let $G = (V(G), E(G))$ be a graph. Then a graph $H = (V(H), E(H))$ is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

A spanning subgraph of G is a subgraph ~~of~~ H such that $V(H) = V(G)$.

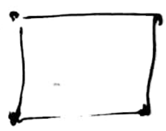
G_e:



H_1:



H_2:



H_2 is a spanning subgraph too.

Complete graph:

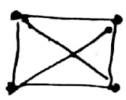
A simple graph G in which each pair of distinct vertices is joined by an edge is called a complete graph. A complete graph with n vertices is denoted by K_n.

K_1 is called an isolated graph. •

K_2

K_3

K_4:



...

Remark: Given any 2 complete graphs with same number of vertices, they are isomorphic.

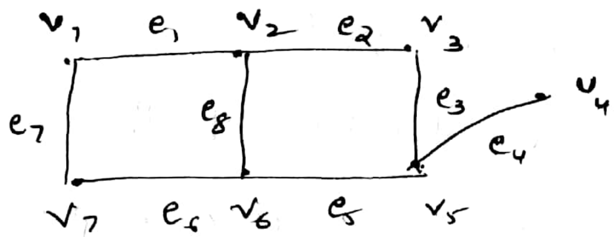
Bipartite graph:

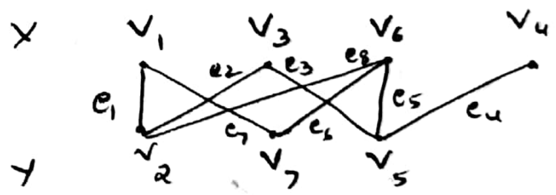
Let G be a graph with vertex set V(G). If V(G) can be partitioned into 2 non empty subsets in such a way that ($X \cup Y = V$ and $X \cap Y = \phi$) each edge of G has one end in X and the other end in Y, then G is called a bipartite graph.

A complete bipartite graph G is a simple bipartite ~~graph~~ graph with bipartition $V = X \cup Y$ in which every vertex in X is joined by every vertex in Y.

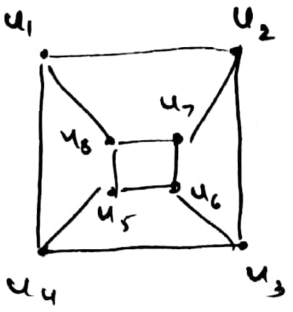
If X has m vertices and Y has n vertices then the complete bipartite graph is denoted by $K_{m,n}$.

Let G be





Bipartite but not complete.



X u_1 u_7 u_3 u_5

Y u_2 u_4 u_8 u_6

Let G be a graph. A walk in G is a finite sequence $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_n$ whose terms are alternatively vertices and edges. Here v_0 is called the origin of the walk and v_n is called the terminus of the walk.

The number of edges in a walk is called the length of the walk.

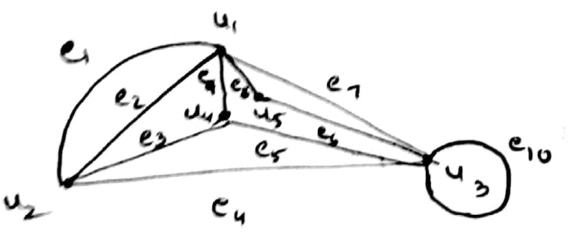
A trivial walk is the one containing no edges.

Let G be a graph and (u, v) be two vertices in G. Then a ~~[illegible]~~ $u-v$ walk is said to closed or open if $u=v$ or $u \neq v$, resp^{ly}.

In a graph G if the edges in a $u-v$ walk are distinct then it is called a trail. If the vertices of a $u-v$ walk are distinct it is called a path.

A path with n vertices is denoted by P_n. It is clear that length of a path ~~n~~ is $(n-1)$. Any 2 paths with same number of vertices are isomorphic.

Ex:



Walk: $u_1 - u_3$ walk: $u_1 e_1 u_2 e_2 u_1 e_9 u_4 e_5 v_3$

Trail: $u_1 - u_3$ trail: $u_1 e_1 u_2 e_3 u_4 e_9 u_1 e_8 u_5 e_6 u_3$.

Path: $u_1 - u_3$ path: $u_1 e_1 u_2 e_3 u_4 e_5 u_3$

In a graph G vertex u is said to be connected to vertex v if there exists a u-v path.

A graph G is said to be connected if every 2 of its vertices are connected, otherwise it is called a disconnected graph.

Let G be a graph and u be a vertex in G. Let c(u) denote the set of all vertices in G which are connected to u. Then the subgraph of G obtained by the vertex set c(u) is called a connected component of G or component of G.

The number of connected components of a graph G is denoted by $\omega(G)$.

A non trivial closed trail in a graph G with distinct internal vertices is called a cycle.

A cycle of length k (with k edges) is called a k-cycle.

A k-cycle is said to be odd or even if k is odd or even.

2 cycles of same length are isomorphic.

A n-cycle with n-vertices is denoted by C_n.

A graph G is said to be acyclic if it contains no cycles.

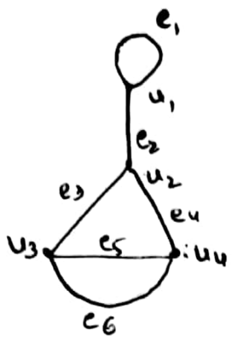
A connected, acyclic graph is called a tree.

The matrix representation of a graph:

Let G be a graph with n vertices $v_1, v_2 \dots v_n$. Then the adjacency matrix of G is a $n \times n$ matrix denoted by $A(G)$

$A(G) = [a_{ij}]_{n\times n}$ where a_{ij} is the number of edges joining the vertex v_i to vertex v_j.

G:



$$A(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Incidence matrix:

Let G be a graph with n vertices $v_1, v_2 \dots v_n$ and k edges $e_1, e_2 \dots e_k$. The incidence matrix of G is a $n \times k$ matrix denoted by $M(G) = [a_{ij}]_{n\times k}$ where a_{ij} is the number of times that a vertex v_i is incident with e_j

ie, $$a_{ij} = \begin{cases} 0 & \text{if } v_i \text{ is not an end of } e_j \\ 1 & \text{if } v_i \text{ is an end of non loop } e_j. \\ 2 & \text{if } v_i \text{ is an end of a loop } e_j \end{cases}$$

$$M(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Theorem: Given any 2 vertices u and v of a graph G. Every u-v walk contains a u-v path. ~~then after some deletion of vertices and edges if necessary~~ that is given any walk $W: u\, e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v$ then after deletion of some vertices and edges if necessary we can find P of W which is a u-v path.

Pf: Let $W: u = v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k = v$

If $u = v$ then W is closed. $P = u$ is a trivial path.

Then the result is true.

suppose $u \neq v$.

Let the vertices of W be in the order $u = v_0 v_1 \dots v_{k-1} v_k = v$

If none of the vertices of G occurs in W more than once then W itself is a $u-v$ path.

suppose that there are vertices of G that occur in W twice or more. Then there exist distinct i, j with $i < j$ such that $v_i = v_j$.

Delete the vertices $(v_{i+1}, v_{i+2} \dots v_{j-1}, v_j)$ and the preceding edges from W: We obtain a $u-v$ walk W say W_1 having less no. of vertices than W.

If the vertices of W_1 are not repeating then W_1 will be a $u-v$ path contained in W.

If not repeat the same deletion procedure till we get a $u-v$ path contained in W.

<u>Result</u>:

Let G be a non empty graph with atleast 2 vertices. Then G is bipartite iff G has no odd cycles.

Tree eg : G_1:

G_2:

which is impossible as T is a tree.

If v is not a vertex present in P, then we get a new path $P' = u_0 u_1 \ldots u_n v$ of T with length $n+1$ which is the contradiction to P is the longest path in T.

$\Rightarrow$ There is no such edge e incident to u_0.

$\therefore$ The only possibility is $d(u_0) = 1$.

Similarly we can prove that $d(u_n) = 1$.

<u>Corollary:</u>

Any tree T with atleast 2 vertices has more than one vertex of degree 1.

<u>Theorem</u> :

If T is a tree with n vertices, then it has precisely $n-1$ edges.

<u>Pf</u>: We prove this result by applying induction on no. of vertices n.

When $n=1$, T is a tree with one vertex.

$\therefore$ No. of edges $= 0 = (n-1)$ $\because$ Tree has no loops.

$\Rightarrow$ Result is true for $n=1$.

Assume that result is true for $n=k$.

ie, if a tree has k vertices then it has $(k-1)$ edges.

To prove that the result is true for $n=k+1$.

Assume that T is a tree with $(k+1)$ vertices.

We have T has atleast one vertex of degree one.

Let u be one such vertex with $d(u) = 1$.

Let $e = uv$ be an edge incident on u in T.

If x and y are vertices in T both different from u then any path P joining x to y does not go through the vertex u. Therefore the subgraph $T-u$ is connected.

If C is a cycle in $T-u$ then C is a cycle in T, which is impossible as T is a tree.

$\therefore$ $T-u$ is acyclic $\Rightarrow$ $T-u$ is connected and acyclic

$\Rightarrow$ $T-u$ is a tree with k vertices.

$\therefore$ By assumption, $T-u$ has $(k-1)$ edges.

Since $T-u$ is a subgraph of T by deleting 1 edge $e = uv$, and $T-u$ has $k-1$ edges, T has k edges.

$\Rightarrow$ Result is true for $n = k+1$

$\therefore$ MI result is true for any $n \in \mathbb{N}$.

<u>Result:</u>
Let G be an acyclic graph with n vertices and k connected components (ie, $\omega(G) = k$). Then G has $n-k$ edges.

Let G be an acyclic graph with n vertices.
Let $C_1, C_2 \dots C_k$ be the k connected components of G.
Suppose that each C_i for $i = 1, 2, 3 \dots k$ has n_i vertices

Then $n = n_1 + n_2 + \dots + n_k$

Since each C_i is connected and acyclic $\Rightarrow$ C_i is a tree.

with n_i vertices.

$\Rightarrow C_i$ has $n_i - 1$ edges.

sum of edges of connected components

$\therefore$ No. of edges of $G = (n_1 - 1) + (n_2 - 1) + \ldots (n_k - 1)$

$= (n_1 + n_2 + \ldots n_k) - k$

$= \underline{(n - k) \text{ edges}}$

Scanned by CamScanner