

GRAPH THEORY

①

1. GRAPH THEORY - 1

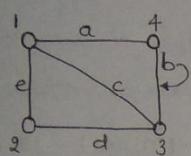
A Graph is represented as $G_1 = (V, E)$

$$V = \text{Vertices set} = \{v_1, v_2, \dots, v_n\}$$

$$E = \text{Edges set} = \{v_i \rightarrow v_j, \dots\}$$

Now, $|V| = \text{No. of vertices in the Graph} = \text{Order of the Graph}$

$|E| = \text{No. of edges} = \text{size of the Graph}$



$$G_1 = (V, E)$$

$$V = \{1, 2, 3, 4\}$$

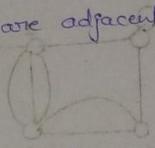
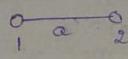
$$E = \{a, b, c, d, e\}$$

$$a = \{1, 4\} \quad d = \{2, 3\}$$

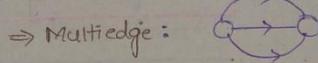
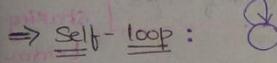
$$b = \{4, 3\} \quad e = \{1, 2\}$$

$$c = \{1, 3\}$$

\Rightarrow Adjacent Vertices: - Vertices having common edge are adjacent



\Rightarrow Adjacent edges: If two vertices have an edge in common then the edges are called adjacent edges.



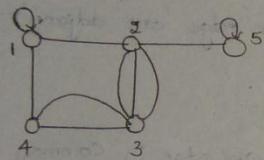
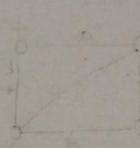
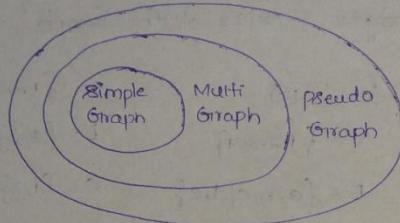
\Rightarrow Simple Graph is the one which don't have selfloops and multiedges.

| Graph | Selfloops | Multiedges |
|----------------------|-----------|------------|
| General/pseudo Graph | ✓ | ✓ |
| MultiGraph | X | ✓ |
| Simple Graph | X | X |

2. GRAPH THEORY - 2

Degree of the vertex: No. of edges incident on it (a particular vertex) is called the degree of that vertex (Counting loops twice)

⇒ Every Simple Graph is a multigraph as well as pseudo Graph. Similarly a multigraph is a pseudo Graph.



$$\deg(1) = 4 \quad \deg(4) = 3$$

$$\deg(2) = 5 \quad \deg(5) = 3$$

$$\deg(3) = 5$$

$$\therefore \text{Sum of the degrees} = 20$$

No. of edges in the Graph = 10.

$$\therefore \boxed{\text{Sum of the degrees} = 2(\text{No. of edges})}$$

Hand shaking lemma

$$\therefore \boxed{\sum_{v \in V} d(v) = 2|E|}$$

2. The no. of vertices with odd degree in a Graph is always even.

$$\sum \deg(v) = 2|E|$$

$$\sum_{\text{odd}} \deg(v) + \sum_{\text{even}} \deg(v) = \text{even}$$

even

$$\Rightarrow \sum_{\text{odd}} \deg(v) = \text{even} - \text{even} = \text{even}$$

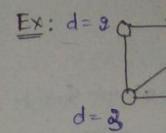
$$\therefore \boxed{\sum_{\text{odd}} \deg(v) = \text{even}}$$

∴ The no. of vertices with odd degree in a Graph is always even

3. GRAPH TH

Degree Sequence

(Descending)



→ Now, Given

a deg seq

or not

solution

HAVEL-HAKIM

① put the

② Remove -

③ Subtract

⇒ Repeat -

Stop this p

⇒ we get e

⇒ If we ge

⇒ Not eno

Ex:

1) (3, 2, 1, 1,

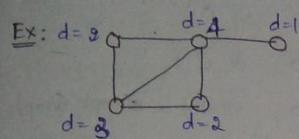
2)

3) (6, 5, 4,

⇒ ↴
Now
Sub next

3. GRAPH THEORY - 3

Degree Sequence: The Arrangement of degrees in Non-Ascending order (Descending order) is called the Degree Sequence.



$$\text{Degree Sequence} = (4, 3, 2, 2, 1)$$

Similarly

Now, Given a graph finding the degree sequence is easy but given a deg sequence finding whether there exists atleast one simple Graph or not is difficult. we use "HAVEL-HAKIMI PROCEDURE" to find solution to such problem.

HAVEL-HAKIMI PROCEDURE

① Put the degree sequence in Descending order

② Remove the highest degree (Let it be K)

③ Subtract one from the remaining K vertices

⇒ Repeat the steps ①②③

Stop this procedure when

⇒ we get all 0 entries ⇒ simple Graph exists

⇒ If we get Negative values ⇒ Simple Graph Not possible

⇒ Not enough edges ⇒ No Simple Graph.

Ex:

i) $(3, 2, 1, 1, 0) \Rightarrow$ sum of degrees should be even ⇒ Here we get '7'

∴ No such Graph exists.

ii) $(3, 2, 1, 0, 0)$

$(1, 0, 0, 0)$

⇒ Negative edge = No such Graph exists.

iii) $(6, 5, 4, 3, 3, 1)$

⇒ $\cancel{6} \quad 5 \quad 4 \quad 3 \quad 3 \quad 1$

Now cut '6' and

Subtract '1' from the next 6 nodes but there are only 5 ⇒ No Simple Graph Exists.

hand
haking
lemma

with
graph

4) Which of the following degree sequence does not correspond a simple graph

i) $(7, 6, 5, 4, 4, 3, 2, 1) \Rightarrow (\cancel{7} \ 6 \ 5 \ 4 \ 4 \ 3 \ 2 \ 1)$ ④

$$(\cancel{5} \ 4 \ 3 \ 3 \ 2 \ 1 \ 0)$$

$$(\cancel{3} \ 2 \ 2 \ 1 \ 0 \ 0)$$

$$(X \ 1 \ 0 \ 0 \ 0)$$

$$(0 \ 0 \ 0 \ 0) = \text{Simple Graph}$$

exists.

ii) $(6, 6, 6, 6, 3, 3, 2, 2) = (\cancel{6} \ 6 \ 6 \ 6 \ 3 \ 3 \ 2 \ 2)$

$$(\cancel{5} \ 5 \ 5 \ 2 \ 2 \ 1 \ 2)$$

$$(\cancel{4} \ 4 \ 1 \ 1 \ 0 \ 2)$$

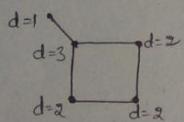
$$(3 \ 0 \ 0 \ \cancel{1} \ 1)$$

\rightarrow (ve) = No simple Graph

4. GRAPH THEORY - 4

Min degree = " δ " = Min of all the degrees of the particular Graph

Max degree = " Δ " = Max of all the degrees of the Graph is called Max deg.



$$\begin{array}{|c|} \hline \delta = 1 \\ \hline \Delta = 3 \\ \hline \end{array}$$

\Rightarrow If 'G' is a graph with V -vertices and e -edges then,

$$\boxed{\delta \leq \frac{2e}{V} \leq \Delta}$$

Proof:

$$\sum \deg(v) = 2|E|$$

\Rightarrow Replacing each degree with min. degree

$$\underbrace{\delta + \delta + \delta + \delta + \dots + \delta}_{\Rightarrow V \text{ times}} \leq 2e$$

$$\therefore V \cdot \delta \leq 2e$$

\Rightarrow Replacing with max degree we get

$$= \Delta + \Delta + \Delta + \dots + \Delta \geq 2e$$

$$= \boxed{V \cdot \Delta \geq 2e} \Rightarrow \boxed{2e \leq V \cdot \Delta}$$

$\zeta_3 =$

\Rightarrow In cycle Gra

Cycle Graph

The cycle Gra

edges: $\{v_i, v_{i+1}\}$

$\Rightarrow G$ is a G
edges in the

$$= 2e$$

$$= 2e$$

$$= e$$

5. GRAPH THE

Special Graphs

Null Graph:

$$\therefore v \cdot \delta \leq 2e \leq v \cdot \Delta$$

(4)

$$\Rightarrow \boxed{\delta \leq \frac{2e}{v} \leq \Delta}$$

(5)

\Rightarrow G is a Graph with 11 edges and min deg = 3 then what is the max no. of vertices?

Simple Graph exists.

$$\text{Now, } \delta \leq \frac{2e}{v}$$

$$\Rightarrow 3 \leq \frac{2 \times 11}{v} \Rightarrow v \leq \frac{2 \times 11}{3}$$

$$\Rightarrow v \leq 7.33$$

$$\Rightarrow \boxed{v=7}$$

Simple Graph

\Rightarrow G is a Graph with 12 vertices and max degree = 4, Then, max no. of edges in the Graph G is —?

$$= 2e \leq v \cdot \Delta$$

$$= 2e \leq 12 \cdot 4$$

$$= e \leq 24 \Rightarrow \boxed{e=24} \text{ (max no)}$$

5. GRAPH THEORY 5

Special Graphs:

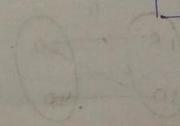
Null Graph: A graph with no edges and n-vertices is called Null Graph.

$$N_1 = \begin{bmatrix} 0 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

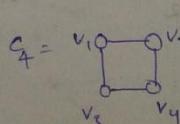
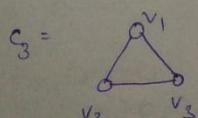
$$N_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$N_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

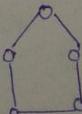


Cycle Graph

The cycle Graph is a simple Graph with n-vertices $\{v_1, v_2, \dots, v_n\}$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$

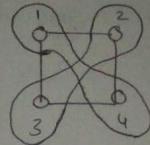
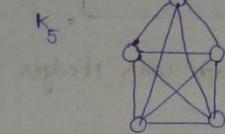
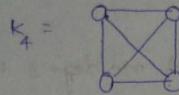
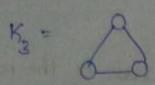


$G_5:$



\Rightarrow In cycle Graph deg of each vertex will be 2. | No. of Vertices = edges = n |

Complete Graph: Complete Graph is a Simple Graph in which every pair of vertices are adjacent, complete graph denoted by K_n : ⑥



C_n is
 C_m is

E. MAX NO. OF
→ which of the
 n -vertices

- (a) n^2 (b)

→ Bipartite Graph

Now, let us assume

| | Vertices | Edges | $\deg(v)$ |
|-------|----------|--------|-----------|
| K_n | n | nC_2 | $(n-1)$ |

⇒ Max no. of edges in simple Graph with 'n' edges = $nC_2 = \frac{n(n-1)}{2}$

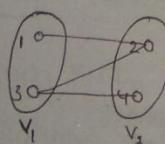
⇒ If $V = \{v_1, v_2, v_3, \dots, v_n\}$ of n -vertices then how many simple graphs are possible?

| 1 | 2 | 3 | | $(n(n-1)/2)$ |
|-------|-------|-------|-------|--------------|
| v_1 | v_2 | v_3 | | v_n |

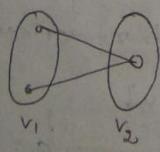
$$\therefore \text{Total possible no. of simple graphs} = \binom{\frac{n(n-1)}{2}}{2}$$

Bipartite Graphs:

A Graph $G(V, E)$ is Bipartite Graph if the vertex set can be partitioned into two sets V_1 and V_2 such that every edge is in between vertex of V_1 and vertex of V_2 .



⇒ There should not be any edges between the vertices in a vertex set. (i.e. No edge between (1, 3) and (2, 4))



Bipartite



{It is complete and cannot be Bipartite}

∴ To get th

7. COMPLETE

Complete Bipartite
Bi-partite Gra

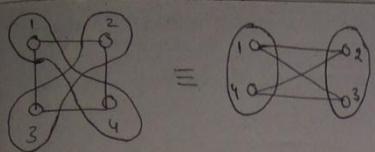
$K_{1,1} = Q$

$K_{1,2} = Q$

$K_{1,3} = Q$

very pain

(6)



\therefore The given graph is Bipartite Graph.

(7)

C_n is Bipartite if $n = \text{even}$ | A complete Graph K_n can never be
 C_n is not Bipartite if $n = \text{odd}$ | Bipartite Graph.

E. MAX NO. OF EDGES IN A COMPLETE BIPARTITE GRAPH

\Rightarrow which of the following is the max no. of edges in Bipartite graph with

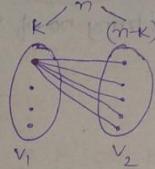
n -vertices

- (a) n^2 (b) $n^2/2$ (c) $n^2/4$ (d) $n/2$

\Rightarrow Bipartite Graph with 8 vertices.

Graphs are
possible?

Now, let us assume the n vertices are divided into K and $(n-K)$ vertices

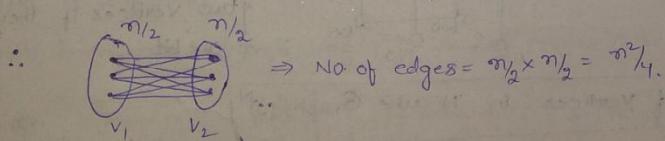


\Rightarrow To get the max no. of edges then
every vertex in V_1 should be
matched with all the vertex in V_2
 \therefore The No. of edges = $K(n-K)$

\therefore To get the max value of K , $\frac{d}{dk}(K(n-K))=0$

$$= (n-K) + K(-1) = 0$$

$$\Rightarrow K = n/2$$



7. COMPLETE BIPARTITE GRAPH AND REGULAR GRAPHS

Complete Bipartite Graph: If we try to put every possible edge in the Bi-partite Graph then it is called Complete Bi-partite Graph (K_{mn})

$$K_{1,1} = \begin{array}{c} \bullet \\ \bullet \end{array}$$

$$K_{1,2} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$$

$$K_{1,3} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

$$K_{2,1} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$$

$$K_{2,2} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

$$K_{2,3} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

$$K_{3,1} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

$$K_{3,2} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

$$K_{3,3} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

$$K_{2,3} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

| | v | e | d(v) |
|--------|-------|--------------|------------------------------|
| K_mn | $m+n$ | $m \times n$ | $\{m: vev\}$ $\{n: vev\}$ |

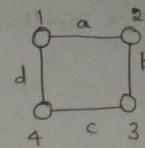
$$|V_1| = m$$

$$|V_2| = n$$

⑧

9. SUB GRAPH

A Graph +



Regular Graph:

A Graph in which every vertex has same degree is called Regular graph.

→ Every complete Graph (K_n) is always a Regular Graph.

↪ Null Graph

Ex: $N_n = 0$ (0- Regular graph)

$c_n = (2 - \text{Regular Graph})$

8. N-Cube

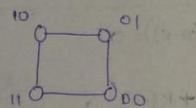
Now 1 cube means with 1 bit how many no. of vertices are possible = 2^1

1-cube =

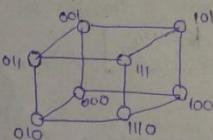


Now, 2 cube means with 2-bits how many no. of vertices are possible = 2^2

2-cube =



Similarly 3-cube =



An edge is present between two vertices if they differ by one bit.

∴ No. of Vertices in N-cube Graph = (2^N)

Deg. of Each Vertex = N

COMPLETE GRAPH GRAPH

No. of Edges = $N * 2^{N-1}$

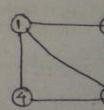
10. ADJACENCY

A Graph G

→ No prob

→ Graph ha

→ For Gra



11. ISOMORPHISM

Isomorphism

different

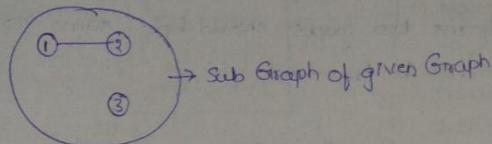
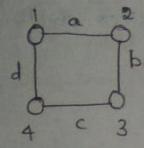
Ex:

(8)

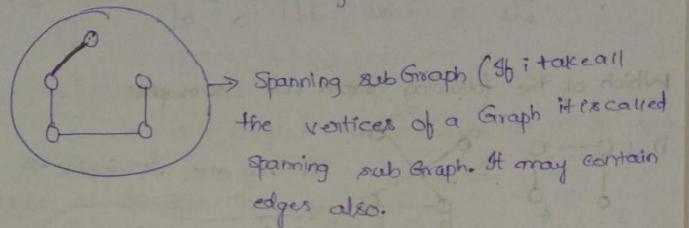
9. SUB-GRAPHS

A Graph $H(V', E')$ is a subgraph of $G(V, E)$ if $V' \subseteq V$
 $E' \subseteq E$

lled Regular
graph.



Sub Graph of given Graph
 Induced sub graph (If i take all the possible edges between the chosen vertices that are given in the Actual Graph).

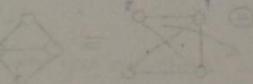


possible = $2^4 - 1$

10. ADJACENCY MATRIX

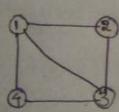
possible = $4 \cdot 2^2$
 A. Graph $G(V, E)$, $|V|=n$ it can be represented as $n \times n$ - matrix

$$A[G] = \begin{cases} 0 & \text{if } \{i, j\} \notin E \\ 1 & \text{if } \{i, j\} \in E \end{cases}$$



between
y differ by

- ⇒ No provision for parallel edges.
- ⇒ Graph has no self loops iff the matrix diagonal entries are zeros.
- ⇒ For Graph having no self loops, $\deg(v_i) = \text{sum of entries in the } i^{\text{th}} \text{ row.}$



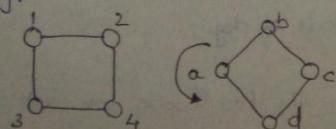
$$A[G] = \begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{array}$$

deg of (1) = 3 (NO. of 1's in that row)
 deg of (2) = 2
 deg of (3) = 2
 deg of (4) = 2

II. ISOMORPHISM (INTRODUCTION)

Isomorphism: Isomorphic graphs are the graphs that are drawn differently.

Ex:



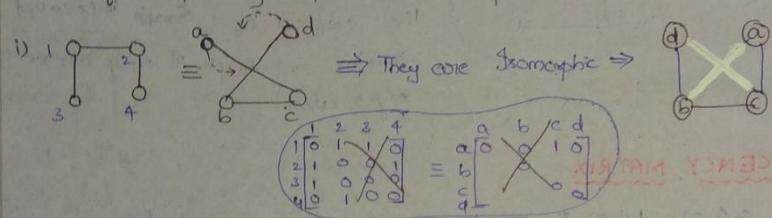
The time complexity for checking whether the Graphs are Isomorphic or not is $O(n!)$

⇒ Isomorphic means two graphs should have same no of vertices and their adjacencies should be preserved.

⇒ For the above examples, the adjacency matrix is same

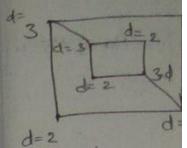
$$\begin{array}{c} \begin{matrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{matrix} \equiv \begin{matrix} a & 0 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 0 \\ d & 1 & 0 & 1 \end{matrix} \end{array} \therefore \text{The two graphs are Isomorphic}$$

Which of the following Graphs are Isomorphic

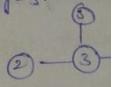


12. ISOMORPHISM

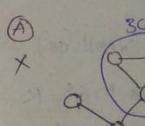
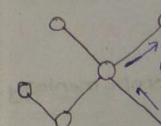
Are these G



In this graph
of deg=3 is
nodes of deg
deg=3.



→ which of



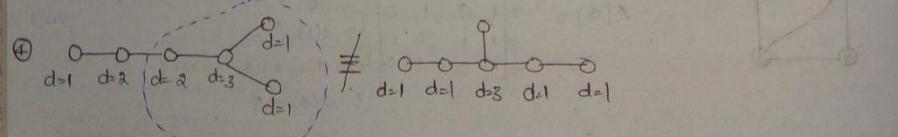
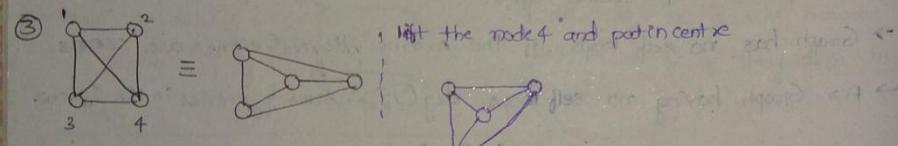
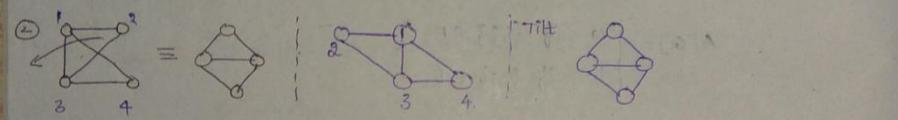
13. SELF COMPLEMENT

Complement

Let G (V, E)

such that

Ex



∴ The node having deg=3 is adjacent to nodes of deg=2 and deg=1, deg=1

Here the node of deg=3 is not adjacent to the node of deg=2

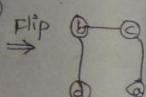
∴ The two Graphs are not Isomorphic

morphic or not

(10)

es and their

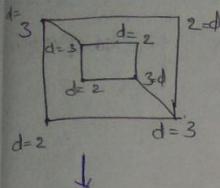
Isomorphic



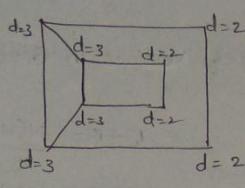
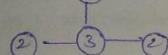
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12. ISOMORPHISM EXAMPLES

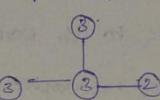
Are these Graphs Isomorphic?



In this graph every node of $\text{deg} = 3$ is adjacent to nodes of $\text{deg} = 2, \text{deg} = 2, \text{deg} = 3$.



Every node of $\text{deg} = 3$ is adjacent to nodes of $\text{deg} = 2, \text{deg} = 3, \text{deg} = 3$.

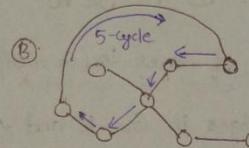
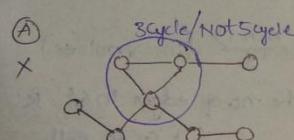
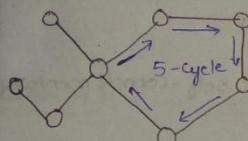


First check

- 1) No. of Vertices and Edges
 $|V(G_1)| = |V(G_2)|$
- 2) $|E(G_1)| = |E(G_2)|$
- 3) Check for Adjacency

∴ There is a conflict. These Graphs are not Isomorphic.

⇒ which of the graphs is Isomorphic to the Graph



And you can check degree sequence also.

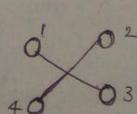
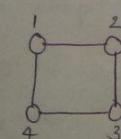
13. SELF COMPLEMENTING GRAPHS

Complement of a Graph

Let $G(V, E)$ be a simple Graph then the complement of Graph $G = G^C = (V, E^C)$

such that two vertices are adjacent in G^C if they are not adjacent in G .

Ex



No. of edges in G + No. of edges in $G^C = nC_2$ ($\because G, G^C$ forms Complete Graph)

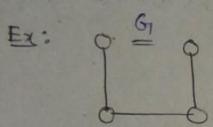
\Rightarrow If the no. of edges in $G_1 = 13$ and No. of edges in $G_1^c = 15$, then the n then.

what is the no. of vertices in G_1 ?

$$\Rightarrow \text{we know that the no. of edges in } G_1 + G_1^c = \frac{n(n-1)}{2} \quad [n = \text{No. of vertices}]$$
$$= 13 + 15 = \frac{n(n-1)}{2}$$
$$= \boxed{n = 8}$$

Self-complementing Graph

A graph which is isomorphic to its complement is called self-complementary graph. which means if G^c is isomorphic to G , then it is called Self complement graph.



$G^c \equiv G_1$ (Isomorphic Graphs)

\Rightarrow which of the following cannot be the no. of vertices in self-complementary graph. ④ 4 ⑤ 5 ⑥ 9 ⑦ 10

We know that the no. of edges in $(G_1 + G_1^c) = \frac{n(n-1)}{2}$ (n = vertices)

\Rightarrow Let the no. of edges in $G_1 = K$ and so the no. of edges in $G_1^c = K$

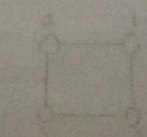
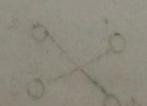
$$\therefore 2K = \frac{n(n-1)}{2}$$

(if G^c is self complementary)

$$\Rightarrow K = \frac{n(n-1)}{4}$$

∴ There K should be integer not fractions.

$\therefore n$ should be multiple of 4 or $(n-1)$ should be divisible by 4.

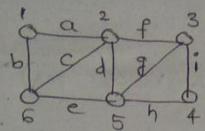


14. CONNECTED COMPONENTS

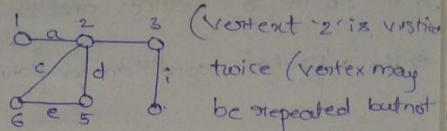
(12)

Edge Sequence: Sequence of edges starting and ending with vertex

Walk: Edge sequence in which no edge is repeated (vertices can be repeated but not the edges).



Walk



(Vertex 2 is visited twice (vertex may be repeated but not the edge)).

elementary
complement

Closed Walk: A walk in which start and end at same vertex is called closed walk. Otherwise it is open walk (vertex may be repeated).

Path: A path is an open walk in which no vertex is repeated.

Cycle: A cycle is a closed walk in which no other vertex is repeated.

Results:

1. A graph is Bipartite if every cycle in the graph is even cycle.

elementary

Connected: Two vertices are said to be connected if there exists atleast one path between them.

(8)

⇒ The maximal connected subgraph is called "component".

= K

+

elementary

2. E!

build

Results:

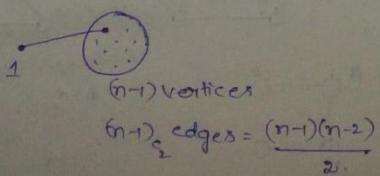
1) A simple Graph with n -vertices and k -Components has atleast $(n-k)(n-k+1)$ edges.

15. SUFFICIENT CONDITION FOR CONNECTEDNESS

The sufficient condition for connectedness is.

→ G is a simple Graph with n -vertices the (no. of edges) $\geq \frac{(n-1)(n-2)}{2}$

The 'G' is connected.



→ which of the following simple graphs are always connected.

(14)

- a) G_1 with 5 vertices and 5 edges
- b) G_1 with 6 vertices and 9 edges
- c) G_1 with 7 vertices and 13 edges
- d) G_1 with 8 vertices and 22 edges

⇒ The sufficient condition for connectedness is $(\text{no. of edges}) \geq \frac{(n-1)n}{2}$

$$(\text{No. of edges}) \geq \frac{n(n-1)}{2}$$

i) 5V, 5E

$$\Rightarrow 5 > \frac{\frac{5(5-1)}{2}}{2} \left. \begin{array}{l} \\ \end{array} \right\} \text{Disconnected Graph. (Not connected)}$$

$$= \boxed{5 > 6} \text{ false}$$

ii) 6V, 9E

$$\Rightarrow 9 > \frac{(6)(5)}{2} \left. \begin{array}{l} \\ \end{array} \right\} \text{Disconnected}$$

$$= \boxed{9 > 15} \text{ false}$$

iii) 7V, 13E

$$\Rightarrow 13 > \frac{(7)(6)}{2} \left. \begin{array}{l} \\ \end{array} \right\} \text{false}$$

iv) 8V, 22E

$$\Rightarrow 22 > \frac{8 \times 7}{2} \left. \begin{array}{l} \\ \end{array} \right\} \text{Connected}$$

$$\boxed{22 > 28}$$

16. COMPLEMENTATION AND CONNECTEDNESS

Result: At least one of the graphs G or G^c is connected.

→ which of the following is always true?

S₁: G_1 is connected, then G_1^c is disconnected

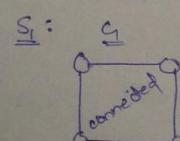
S₂: G_1 is disconnected, then G_1^c is connected.

a) only S₁ is True (Not necessarily true)

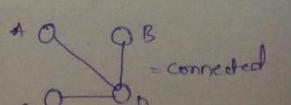
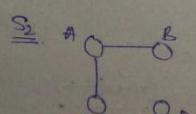
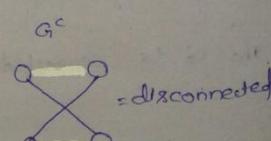
b) only S₂ is True

c) Both

d) None



| G | G^c | |
|-----|-------|----------------|
| C | C | C=connected |
| D | C | D=disconnected |
| C | D | possible cases |
| D | D | Not possible |



17. CUT VERTE

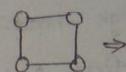
→ To make
may stems

Removal of

Removal of

CUT EDGE

the Graph

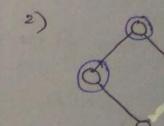
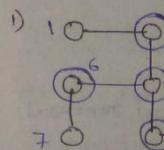


→ If there

⇒ If there

CUT VERTE

A single ve
cut vertex



→ A Graph
Biconne

17. CUT VERTEX AND CUT EDGE

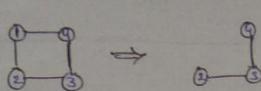
(14)

⇒ To make a Graph disconnected we may remove some edges or we may remove some vertices.

Removal of an edge :



Removal of an vertex :



CUT EDGE (OR) BRIDGE : A single edge whose removal makes the Graph dis-connected is called cut-edge.

No cut edges are present

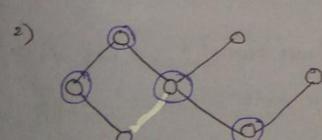
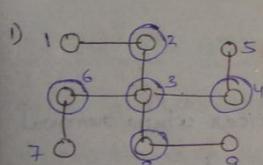
No. of cut edges = 8

⇒ If there is at least one edge is a cut edge

⇒ If there is no cycle then every edge is a cut edge

CUT VERTEX AND ARTICULATION POINT

A single vertex whose removal disconnects the connected Graph is called cut vertex or "ARTICULATION POINT".



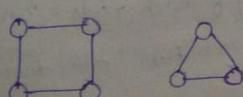
Cut vertices / Articulation points core = {2, 3, 4, 6, 8}

No. of cutvertices core (a) 3 (b) 4 (c) 5 (d) 6

⇒ A Graph having no Articulation point / cut vertex is called a

Biconnected Graph

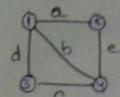
= All cycles are Bi-connected Graphs.



18. CUT SET AND EDGE CONNECTIVITY

Cut set: A set of edges whose removal disconnects the graph is called cutset. There can be more cutset for a given graph.

Ex:



$$C_1 = \{abcde\}$$

$$C_2 = \{abd\}$$

$$C_3 = \{ae\}$$

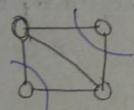
$$C_4 = \{abc\}$$

$$C_5 = \{cd\}$$

Minimum cut set

Edge-connectivity (λ):

The min. no. of edges whose removal disconnects the graph is called Edge-connectivity (λ).



$$\lambda = 2$$

⇒ If the Graph contains cycle then edge connectivity λ will always be greater than one

$$\lambda \geq 1$$

⇒ The edge connectivity is always upper bounded by min degree : $\lambda \leq \delta$

$$\text{W.K.T. } v \cdot \delta \leq 2E \Rightarrow \lambda \leq \frac{\delta E}{v}$$

$$\delta \leq \frac{2E}{v}$$

19. VERTEX CONNECTIVITY AND EDGE CONNECTIVITY

Vertex connectivity ($K = \kappa$): The min no. of vertices whose removal disconnects the graph (or) leaves trivial graph is called vertex connectivity.

→ Trivial Graph (K_1).

$$K \leq \lambda \leq \delta \leq \frac{2E}{v}$$

| G | λ | K |
|-----------|--------------|--------------|
| C_n | 2 | 2 |
| W_n | 3 | 3 |
| K_n | $(n-1)$ | $n-1$ |
| $K_{m,n}$ | $\min(m, n)$ | $\min(m, n)$ |

wheel Graph, W_n contains $(n+1)$ vertices
 W_n has $2(n-1)$ edges

deg of each node = n (which are in cycle)

hub vertex degree = $n-1$

20. EULER

In any

→ A Eu

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A

⇒ A G

more val

sol

21. EULER

⇒ If you
exacte
walk?

⇒ If you
all the

⇒ If you
. Other
Graph

⇒ If a g
(The
Result: A

20. WHITNEY'S THEOREM

In any Graph

$$K \leq \gamma \leq \delta$$

$$\text{Vertex connectivity} \leq \text{Edge connectivity} \leq \text{Mindegree}$$

Graph 18

(16)

(17)

minimum cut set

is called

edge connect
ivity than one

$$\gamma > 1$$

is upper bounded

\Rightarrow A Graph with 11 edges and min degree is 4. What is the max value of vertex connectivity?

Acc. to Whitney theorem $K \leq \gamma \leq \delta \leq \frac{2E}{V}$

$$\begin{aligned} &\Rightarrow K \leq \frac{2E}{V} \quad \delta \leq \frac{2E}{V} \Rightarrow K \leq \delta \\ &\Rightarrow K \leq \frac{2 \times 11}{7} \quad \delta \leq 4 \Rightarrow K \leq 4 \end{aligned}$$

max value
 $K=4$

\Rightarrow A Graph 'G' with 11 edges and 7 vertices is given. What is the max value of K (vertex connectivity)?

sol: $K \leq \frac{2E}{V}$

$$K \leq \frac{2 \times 11}{7}$$

$$K \leq 3.1 \Rightarrow \text{max value} = 3 \quad [K=3]$$

21. EULER'S GRAPH

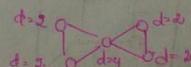
\Rightarrow If you start from a vertex and if you can travel all edges exactly once and again comeback to the same vertex then such a walk is called Euler circuit and such a Graph is called Euler Graph.

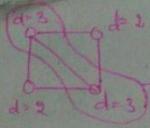
\Rightarrow If you start from a node and come back to the same node by covering all the vertices then it is called Eulerian circuit / Euler path

\Rightarrow If you start from a node and cover all the nodes and reach some other node which is final then it is called unicursal path and the Graph is called unicursal Graph

\Rightarrow If a graph has to be Euler Graph it should contain Even degree. (The vertices should have Even degree.)

Result: A multigraph is Euler Graph if its degree of every vertex is even.

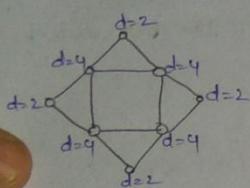




\Rightarrow so Not Euler graph. So unicursal Graph.

(18)

\Rightarrow Is this Graph a Euler Graph



All degrees are Even \therefore The Graph is Euler Graph.

Result: A Multigraph is unicursal graph if there are exactly 2 vertices of odd degree.

22. HAMILTONIAN GRAPH (H-Graph)

\Rightarrow In Euler Graph every edge should be covered and vertex can be repeated

\Rightarrow In "Hamiltonian Graph" all the vertices should be covered and nothing should be Repeated. (Need not cover all the edges)

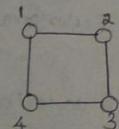
E-E (Euler Graph - Should cover all the Edges).

\Rightarrow You have to draw a cycle in the Graph such that it covers all the Vertices

\Rightarrow A Graph containing Hamiltonian Cycle is called Hamiltonian Graph.

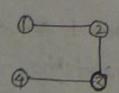
\Rightarrow A path containing all the vertices of the Graph is called "Hamiltonian Path".

Ex:-

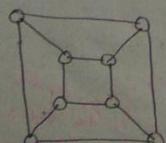


$1-2-3-4-1$ = Hamiltonian cycle

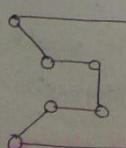
= Hamiltonian Graph (H-Graph)



$0-1-2-3-4$ = Hamiltonian path (Remove one edge from Hamiltonian cycle then you get H-path).



=



= Hamiltonian cycle = Hamiltonian path

Results:

i) If G is

1) Hami

2) Then

3) H-

\Leftrightarrow Inv

The suffi

a) Diaco

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b) Clos

of no

\Rightarrow If a

Graph

For

23. POM

Let $A[$

of Adj

Ex:-



\Rightarrow An Rep

Graph.

Results:

(18)

2 HAMILTONIAN

(7)

i) If G' is a Hamiltonian Graph with n -vertices

h.

i) Hamiltonian cycle, H -path contains n -vertices

ii) Hamiltonian cycle contains n -edges.

iii) H -path contains $(n-1)$ edges.

\Rightarrow In a Hamiltonian cycle each vertex should have $(\deg = 2)$.

The sufficient condition for Hamiltonian cycle is (Not Necessary)

a) Dirac's theorem: (Simple Graph)

kes

If $\min \deg(G) \geq n/2$ then G is Hamiltonian Graph

b) Ores Theorem: If $\deg(u) + \deg(v) \geq n$ ($\forall m \geq 3$) then for every pair of non adjacent vertices, then G is H -Graph.

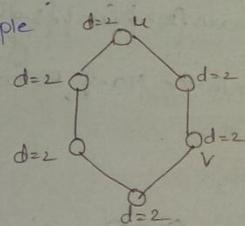
ued

hing

offices

\Rightarrow If a Graph satisfies Dirac, Ores theorem G is Hamiltonian but If a Graph doesn't satisfy above Theorems we cannot say G is Non- H Graph.

For example



Here $\delta=2$ $n=6$

$$\boxed{\delta \geq n/2} \Rightarrow 2 \geq 6/2 \Rightarrow (2 \geq 3) \text{ false}$$

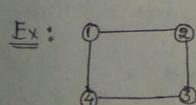
but the Graph is Hamiltonian.

$$d(u) + d(v) = 4 \quad (4 \geq n) \text{ false but } G \text{ is } H\text{-graph.}$$

Path.

23. POWERS OF ADJACENCY MATRIX

Let $A[G_1]$ be the Adjacency matrix. Then $[A(G_1)]^n$ Represents powers of Adjacency matrix.



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix}$$

Represents the no. of paths of length 1 from 1 to 2
Represents the no. of paths of length 2 from 1 to 3. $\{1-2-3, 1-4-3\}$

$\Rightarrow A^n$ Represents the no. of paths b/w two vertices of length n !

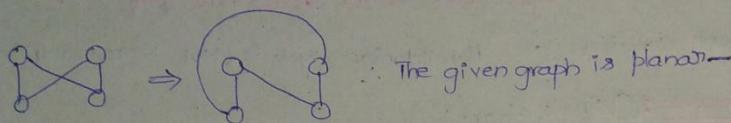
24. PLANAR GRAPHS

⇒ If we can draw a Graph on a paper without crossovers then the Graph is Planar. This concept is used in VLSI design.

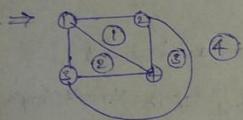
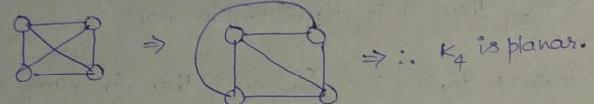
⇒ Cross over:



Ex:



$K_4 =$

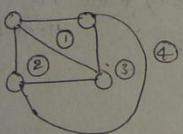


Internal Region = {1, 2, 3} [The edge that divides two Regions is called Boundary edge.]
External Region = {4}.

Here the Boundary edges are {2, 4} {1, 4} {3, 4}

Degree of the Region: Degree of the Region is the No. of Boundary edges touching it.

Ex:



| Region | $\deg(R)$ |
|--------|-----------|
| 1 | 3 |
| 2 | 3 |
| 3 | 3 |
| 4 | 3 |

= 3 edges.

BONDED CYCLES MATRIX

⇒ sum of degrees of the Regions = Twice the no. of Boundary edges.

25. EULER FORMULA FOR SIMPLE GRAPH

Let 'G' be a connected planar graph with

v = vertices

e = edges

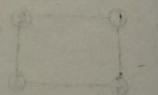
r = regions then

$$v - e + r = 2$$



$$\begin{array}{l} v=4 \\ e=5 \\ r=3 \end{array}$$

$$\begin{array}{l} v-e+r=4-5+3 \\ = 7-5 = 2. \end{array}$$



$$\rightarrow v-e+r$$

⇒ Minde

⇒ Now,

Now, $v-e$

$$\Rightarrow v-$$

$$\Rightarrow \boxed{2}$$

26. K_5

$K_5 =$



⇒ Since

27. $K_3, 3$

W.K.T.

W.K.T.

⇒ Min degree of a Region in simple Graph = 3.

⇒ Now, $\sum \text{deg}(r) = 2e$

$$= r_1 + r_2 + r_3 + \dots + r_n = 2e$$

$$= 3+4+5+\dots+r = 2e$$

$$= 3+3+3+\dots+3 \leq 2e$$

$$\Rightarrow 3r \leq 2e \Rightarrow \boxed{3r \leq 2e}$$

Now, Euler formula

$$\Rightarrow V - e + r = 2$$

$$\Rightarrow V - e + \frac{2e}{3} \geq 2$$

$$\Rightarrow e \leq 3V - 6$$

$$3r \leq 2e \Rightarrow r \leq \frac{2e}{3}$$

$$\Rightarrow r \leq \frac{2e}{3} \quad (\text{or}) \quad e \geq \frac{3r}{2}$$

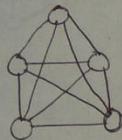
Now, $V - e + r = 2$

$$\Rightarrow V - \frac{3r}{2} + r = 2$$

$$\Rightarrow 2V - r \leq 4$$

26. K_5 IS NOT PLANAR

$K_5 =$



K_5 = Kuratowski's Graph = $K_{3,3}$ is also called Kuratowski's Graph.

$$V=5 \quad | \text{ By Euler formula } \quad V - e + r = 5 - 10 + r = 2 \\ e=10$$

$$\Rightarrow 21 \leq 2r \quad \text{Now, } 3r \leq 2e \Rightarrow r = 7$$

$$\Rightarrow 21 \leq 20 \quad \text{false} \therefore K_5 \text{ is not planar, it must not be connected}$$

⇒ Since K_5 is connected, K_5 is Not-planar Graph.

27. $K_{3,3}$ IS NOT PLANAR PART 1 AND PART 2

W.K.T. $V - e + r = 2$, Now let us assume that the min deg = k then.

W.K.T. sum of degree = $2e$ and $V - e + r = C + 1$

$$= k + k + k + \dots + k = 2e$$

$$= k |r| \leq 2e \Rightarrow \boxed{k r \leq 2e}$$

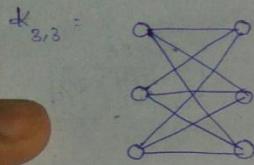
$$\Rightarrow r \leq \frac{2e}{k}$$

↪ No. of connected

Components.

$$\Rightarrow V - e + \frac{2e}{k} \geq 2 \Rightarrow e \leq \frac{k(V-2)}{k-2} \quad (\text{or}) \quad V - \frac{kr}{2} + r \leq 2$$

Now, let us check if $K_{3,3}$ is planar or not



Min. deg of any region = 4

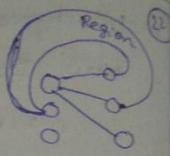
$$\text{Now, } v - e + r = 2$$

$$\Rightarrow 6 - 9 + r = 2$$

$$r = 5 \quad \text{Now } e \leq 4(4)$$

$$9 \leq 8 \quad \text{False}$$

$\therefore K_{3,3}$ is not planar



Note: There won't be any Region of deg. 3 in Bipartite Graph.

i) The Graphs that contain K_3 and K_5 as subgraphs are not planar.

ii) The Graph which is Non-planar with min. no. of edges is K_5 \therefore The

Graphs with 2, 3, 4 vertices are planar. (Simple Graphs)

iii) $K_{3,3}$ is Non-planar with min. no. of edges \therefore Any Simple Graph with edges < 8 is planar.

so, K_m is planar iff $m \leq 4$

$K_{m,n}$ is planar iff $m \leq 2$ and $n \leq 2$



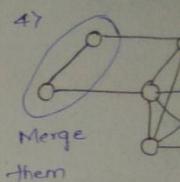
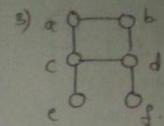
29. CHECKING FOR PLANARITY

Homomorphic Operations

\Rightarrow when ever you take a Graph and if we apply Homomorphic operations and we get another Graph then we can say that the two graphs are Homomorphic to each other.

Ex: 1)

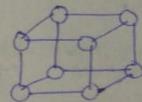
\Rightarrow Insertion of vertex of d=2 in series.



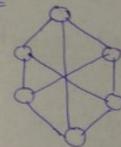
Kuratowski's

A connected

Ex:



Ex:



Ex:

i) $V = 20$, de
what are

Ex: $V = 20$
 $d(v_i) = 3$

2)

\Rightarrow Removing a vertex

22)

 Merging

23)

 Insertion and pulling.

47)

 Merge them

 mergerthem

 ∴ The given Graph is Not planar.

Kuratowski's Theorem
 A connected graph is planar iff it is not homomorphic to K_5 or $K_{3,3}$.

Ex:

 whether the Graph is Homomorphic or not?

⇒

 = planar (No cross over)

Ex:

 ⇒

 $= K_{3,3} \therefore$ NOT planar Graphs.

Ex:
 1) $V=20$, $\deg(v_i)=3$ for all $v_i \in V$ then sum of degrees $\approx 2e = 60$ then $e = 30$
 what are the no. of Regions.

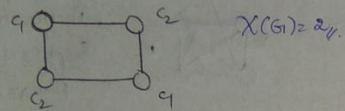
Ex:
 2d $V=20$ sum of degree $\Rightarrow 2e = 60$ | at min case each regions
 $d(v_i)=3$. $\Rightarrow e = 30$ will have min deg of 3'
 Now, $V - e + r = 2$
 $= 20 - 30 + r = 2$
 $\Rightarrow r = 12$

$3+3+\dots+3$ (20 times)
 $= 60$
 $\Rightarrow r = 12$ each vertex degree

30. GRAPH COLOURING

Graph colouring:

⇒ The min no. of colours reqd to colour all the vertices of a Graph such that no two adjacent vertices have same color is called chromatic number, $\chi(G)$.



$$\chi(G) = 2.$$

For Null Graph, $\chi(G) = 1$

$$\text{cycle } C_n = \begin{cases} 2 & \text{if } n = \text{even} \\ 3 & \text{if } n = \text{odd} \end{cases}$$

$$\text{wheel } W_n = \begin{cases} 3 & \text{if } n = \text{even} \\ 4 & \text{if } n = \text{odd} \end{cases}$$

$$\therefore \Delta_n = \{n\}$$

$$\Rightarrow \Delta_{\min} = 2$$

Any Bipartite Graph, $\chi(G) = 2$

Properties

* 'G' is a graph with n -vertices then $\chi(G) \leq n$

* ' G' is a sub-graph of Graph 'G' $\chi(G) \geq n$

* $\chi(G) \leq 1 + \Delta \rightarrow \text{max degree}$

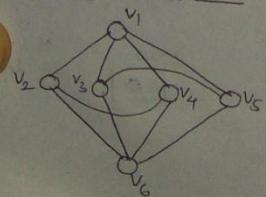
* $\chi(G) = \frac{|V|}{|\Delta| - \delta}$ | The following statements are equivalent
 i) 'G' is Bipartite
 ii) 'G' is 2-colorable
 iii) Every cycle in 'G' is even cycle.

31. GRAPH COLOURING Example-1

There is no Algo which can find the chromatic no. of a graph. (NP-Complete)

But we use some Greedy methods (Algo) like

Welsh-Powell Algorithm



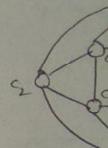
$$\begin{array}{l|l} d(v_1) = 4 & d(v_2) = d(v_3) = d(v_4) = d(v_5) = 3 \\ d(v_6) = 4 & \end{array}$$

Now, sort the vertices in descending order of the degrees. = $v_1 v_6 v_2 v_3 v_4 v_5$

| V1 | V6 | V2 | V3 | V4 | V5 |
|----|----|----|----|----|----|
| c1 | c1 | c1 | c1 | c1 | c1 |

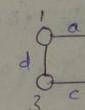
⇒ Start from core to

32. GRAPH-



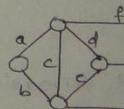
33. MATCH

Matching: S



Matching no:

Ex:



Edge cover

what are



(24)

| v_1 | v_6 | v_2 | v_3 | v_4 | v_5 |
|-------|-------|-------|-------|-------|-------|
| c_1 | c_1 | c_2 | c_2 | c_3 | c_3 |

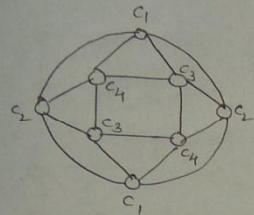
$$\chi(G) = 3.$$

(25)

Graph

⇒ Start from ' v_1 ' Now apply c_1 color to v_1 , and the vertices that are not adjacent to it, and Repeat the procedure.

32. GRAPH-COLOURING Example-2



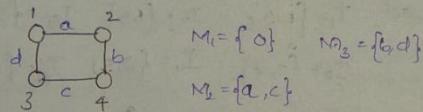
$$\chi(G) = 4$$

⇒ The Graph contains ' K_3 ' so the value of $\chi(G) \geq 3$.



33. MATCHING AND EDGE COVER

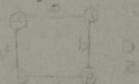
Matching: Set of Non-Adjacent edges.



$$M_1 = \{ \emptyset \}$$

$$M_2 = \{ bcd \}$$

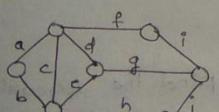
$$M_3 = \{ a,c \}$$



Matching no: $[x'(G)]$: Max no. of Non-Adjacent edges. For above Graph

$$x'(G)=2$$

Ex:

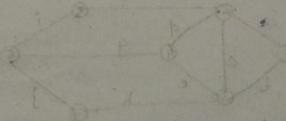


$$M_1 = \{ a, i, h \}$$

$$M_2 = \{ fig, h \}$$

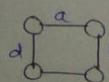
$$M_3 = \{ i, e, a \}$$

$$x'(G)=3$$



Edge cover:

What are all the edges that you should choose that covers all the vertices
(Not isolated vertex)



⇒ Edge cover = {abc} ⇒ {abcd}

= {bcd} ⇒ {acd}

= {adc} ⇒ {ab} ⇒ $a \overset{a}{\circ} b$

Edge covering No: Min no. of edges that are required to cover all the vertices is called Edge covering No. $[P'(G)]$

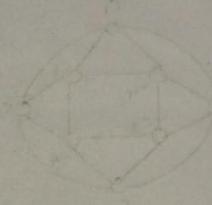
(26)

Ex:

$$P'(G) = \{ab\} + \{bc\}$$

$$P'(G) = \{2\} + 1 \text{ (for isolated vertex)} \\ \Rightarrow P'(G) = 3$$

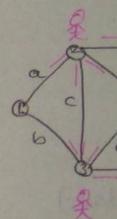
For any Graph $\alpha'(G) + P'(G) = n$



VERTEX COVER
cover.

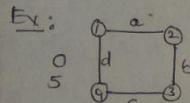
Ex:

Vertex cover
all the edges



34. VERTEX COVER AND INDEPENDENT SET

Independent set: set of non-adjacent vertices is called Independent set



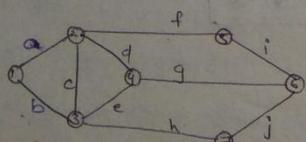
Independent set $I_1 = \{1, 3\}$, $I_3 = \{1, 5\}$, $I_5 = \{3, 5\}$
 $I_2 = \{2, 4\}$, $I_4 = \{2, 5\}$, $I_6 = \{4, 5\}$
 $I_7 = \{1, 3, 5\}$, $I_8 = \{2, 4, 5\}$

Independence no.: $\alpha(G)$

Max No. of elements in the largest Independent set is called Max No.

For the above graph $\alpha(G) = 3$ $[I_8, I_7]$

Ex:



$$I_1 = \{1, 4, 5, 7\}$$

$$I_2 = \{2, 6\}$$

{Find the decreasing
order of degree of
vertices}

{Increasing order of
vertices}

∴ we have

For any

over

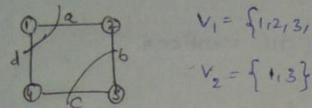
VERTEX COVER: Set of vertices that cover all the edges is called vertex cover.

(26)

cover.

(27)

Ex:



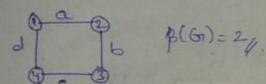
$$V_1 = \{1, 2, 3, 4\}$$

$$V_3 = \{2, 4\}$$

$$V_2 = \{1, 3\}$$

over

Vertex covering number: The min no. of vertices which can cover all the edges



$$\beta(G) = 2$$

dependent set

{3,5}

{4,5}

Max NO.

increasing
no. of
vertices

order

∴ we have removed three vertices $\therefore \beta(G) = 3$.

For any Graph, "G"

$$\alpha(G) + \beta(G) = n$$

$$\text{Independence no} + \text{vertex covering no} = n$$

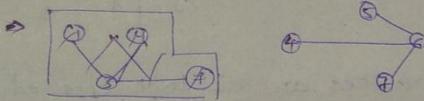
$$\text{Matching no} + \text{Edge covering no} = n \Rightarrow \omega(G) + \beta'(G) = n$$

$$\beta'(G) = n$$

→ To get the vertex covering number remove the vertex with highest degree

~~highest degree~~ ~~degree~~ ~~degree~~
 $= \{2, 3\}$ so we can remove anyone.

→ Now remove the vertex with next highest degree
 $= 5$ (In main Graph not the Graph)



→ Now Remove = 6 then all vertices get Removed.

6

5

4

3

2

1

35. PERFECT MATCHING

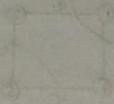
A matching which can cover all the vertices of a Graph

Ex:



$$M_1 = \{a, c\}$$

can cover all vertices



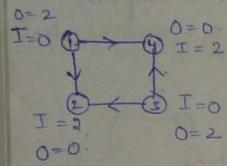
→ A graph G_i has perfect matching only if $|G_i|$ has even no. of vertices.

$$\text{* No. of perfect matchings in } K_{2n} = \frac{(2n)!}{2^n (n!)}$$

$$\Rightarrow \text{Find the no. of perfect matchings in } K_3 \Rightarrow n=3 \Rightarrow \frac{6!}{2^3 3!} = 15$$

$$\Rightarrow \text{No. of perfect matching in } K_{nn} = n!$$

36. DIRECTED GRAPH



$I = \text{Indegree}$ (No. of edges comming towards it)

$O = \text{outdegree}$ (No. of edges outgoing from it)

| V | In | out |
|-----|----|-----|
| 1 | 0 | 2 |
| 2 | 2 | 0 |
| 3 | 0 | 2 |
| 4 | 2 | 0 |
| Sum | 4 | 4 |

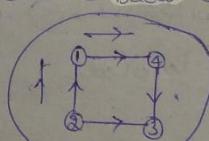
In any Graph sum of

Indegrees = sum of outdegrees

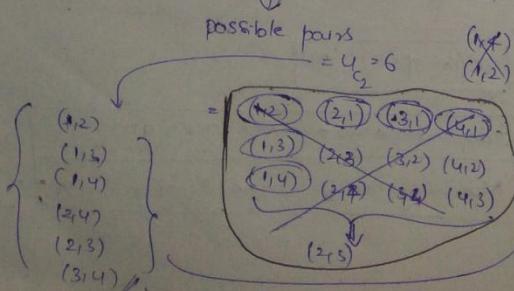
= No. of Edges

→ Two vertices are said to be connected if there exists a directed path

between them. In the below Graph ③ and ④ are connected.



→ This Graph is unilaterally connected because between every pair there exists a directed path.



| | |
|-------|-------|
| (1,2) | (1,4) |
| (1,3) | (1,5) |
| (1,4) | (2,4) |
| (2,4) | (2,5) |
| (2,3) | (3,4) |
| (3,4) | (3,5) |

