

SET THEORY AND ALGEBRA

SETS

1. SETS AND SUBSETS

Set: well-defined unordered collection of distinct elements

Ex: $A = \{1, 2, 3, 4\}$

$S = \{\text{set of all students in a class}\}$

Null set: Set with no elements is called Null set, denoted as \emptyset or $\{\}$

subset: If every ele of A is also an element of ' B ' then A is subset of B .

$$A = \{1, 2, 3\} \quad B = \{1, 2\} \quad \text{Hence } (B \subseteq A)$$

Note: For every set ' A ', ' A ' and ' \emptyset ' are called Trivial subsets of ' A '.

Proper subset: Any subset of ' A ' which is not a trivial subset is called

proper subset of ' A '.

Note: If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

2. POWER SET

Denoted by $P(A)$

⇒ If ' A ' is finite set then set of all finite subsets of ' A ' is called power set of ' A '. It is denoted by $P(A)$.

Ex: $A = \{a, b\}$

subsets of $A = \emptyset, \{a\}, \{b\}, \{a, b\}$

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

⇒ If a set contains n elements ($|A|=n$) then $|P(A)| = 2^n$ elements.

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3. COMPLEMENT AND DIFFERENCE

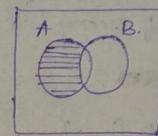
Universal set: set of all objects under discussion denoted by 'U'

Complement of a set: If 'A' is any set, then complement of 'A' denoted by \bar{A} or A^c is called

$$A^c = \{x / x \notin A \text{ and } x \in U\}$$

Difference: If 'A' and 'B' are two sets then

$$A - B = \{x / x \in A \text{ and } x \notin B\}$$



$$A = \{1, 2, 3, 4\}$$

$$B = \{3, 4, 5, 6\}$$

$$A - B = \{1, 2\}$$

Distributive law

- 1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- 2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Modular laws

- 3) $(A \cup B) \cap C = A \cup (B \cap C)$
- 4) $(A \cap B) \cup C = A \cap (B \cup C)$
- 5) $A \cup \emptyset = A$
- 6) $A \cap \emptyset = \emptyset$

6. EXAMPLE

Which of the

- a) $A - (A - B) = B$
- b) $A - (A - B) = \emptyset$



With this diagram
3 sets

$$\text{iii) } A - (A - B) = \emptyset \\ = \{2, 3\} - \{2\} = \emptyset$$

4. UNION, INTERSECTION AND SYMMETRICAL DIFFERENCE

Set Intersection: $A \cap B = \{x / x \in A \text{ and } x \in B\}$

Set Union: $A \cup B = \{x / x \in A \text{ OR } x \in B\}$

Note: If $A \cap B = \emptyset$, then 'A' and 'B' are disjoint sets.

Symmetric difference or Boolean sum: $A \Delta B = \{x / x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\}$

$$\boxed{A \Delta B = (A - B) \cup (B - A)} \\ A \Delta B = (A \cup B) - (A \cap B)$$

$$x \notin A \cap B$$

$$\text{iii) } A - (A - B) = \emptyset \\ = \{2, 3\} - \{2\} = \emptyset$$

5. LAWS OF SETS

Commutative laws: (i) $A \cup B = B \cup A$

ii) $A \cap B = B \cap A$

iii) $A \oplus B = B \oplus A$

Associative laws:

$$1) (A \cup B) \cup C = A \cup (B \cup C)$$

$$2) (A \cap B) \cap C = A \cap (B \cap C)$$

$$3) (A \oplus B) \oplus C = A \oplus (B \oplus C)$$

(2)

Distributive laws:

1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

denoted

Modular laws

3) $(A \cup B) \cap C = A \cup (B \cap C)$ iff $A \subseteq C$

4) $(A \cap B) \cup C = A \cap (B \cup C)$ iff $C \subseteq A$

5) $A \cup \emptyset = A$ 6) $A \cup U = U$, $A \cap U = A$

7) $A \cap \emptyset = \emptyset$ 8) $A \cap A^c = \emptyset$ 9) $A \cap A^c = \emptyset$

DeMorgan law

1) $(A \cup B)^c = A^c \cap B^c$

2) $(A \cap B)^c = A^c \cup B^c$

3) $A - (B \cup C) = (A - B) \cap (A - C)$

4) $A - (B \cap C) = (A - B) \cup (A - C)$

Idempotent law

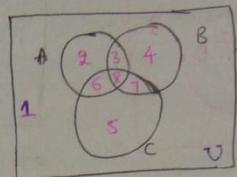
1) $A \cup A = A$

2) $B \cap B = B$

Absorption law

1) $A \cup (A \cap B) = A$

2) $A \cap (A \cup B) = A$



$$\begin{aligned} A \cup (B \cap C) &= \{2, 3, 6, 8\} \cup \{8, 7\} \\ &= \{2, 3, 6, 7, 8\} \end{aligned}$$

$$\begin{aligned} (A \cup B) \cap (A \cap C) &= \{2, 3, 4, 6, 7, 8\} \cap \\ &\quad \{2, 3, 6, 8, 7, 5\} \\ &= \{2, 3, 6, 8, 7\} \end{aligned}$$

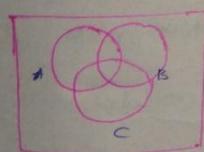
Ex. Example 1

Which of the following is not TRUE?

a) $A - (A - B) = B$ b) $(A \cap B) \cup (A \cap B^c) = A$

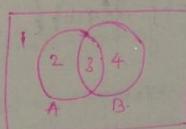
c) $A - (A - B) = (A \cap B)$ d) $B \cap (A \cup B) = A$

No. of regions = 2^n
 n = no. of sets.



Use this diagram for

3 sets



for two sets

$$\begin{aligned} \text{iii)} \quad A - (A - B) &= (A \cap B) \\ &= \{2, 3\} - \{2\} = \{3\} = A \cap B. \end{aligned}$$

(iv) $B \cap (A \cup B) = A$

= $\{3, 4\} \cap \{2, 3, 4\}$

= $\{3, 4\} \neq A$.

= False

i) $A - \{2, 3\} - \{2\} = \{3\}$

x $B = \{3, 4\}$

ii) $(A \cap B) \cup (A \cap B^c) = A$

= $\{3\} \cup \{\{2, 3\} \cap \{1, 2\}\}$

= $\{3\} \cup \{2\} = \{2, 3\}$

EB but

not A

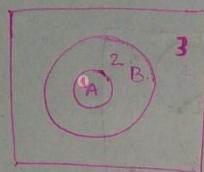
(B ∪ C)

(C ∩ C)

1. Example - 2

Which of the following is not True?

- a) If $A \subset B$, then $B^c \subset A^c$ ✓ c) An $p(A) = \emptyset$ ✓
 b) An $p(A) = A$ X d) $p(A) \cap p(p(A)) = \{\emptyset\}$ ✓



$$\begin{aligned} &\Rightarrow B^c = \{3\} \\ &A^c = \{2, 3\} \Rightarrow B^c \subset A^c \text{ (TRUE)} \end{aligned}$$

∴ An $p(A) = \emptyset$

These core elements $\leftarrow A = \{a, b\}$

These core sets $\leftarrow p(A) = \{\{\emptyset\}, \{a\}, \{b\}, \{a, b\}\}$

$$\text{An } p(A) = \{\emptyset\} \quad \cancel{\text{X}}$$

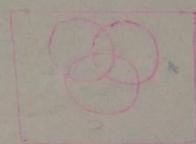
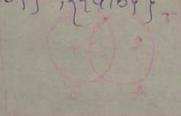
$$A \cap p(A) = \emptyset$$

Example - 2

3) $p(A) \cap p(p(A)) = \{\emptyset\}$

$$p(A) = \{\{\emptyset\}, \{a\}, \{b\}, \{a, b\}\}$$

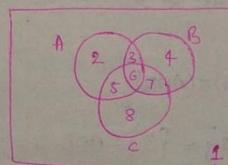
$$p(p(A)) = \{\emptyset, \{\emptyset\}, \{\{b\}\}, \{\{\{a, b\}\}\}\}$$



8. Example - 3

Which of the following is NOT TRUE?

- a) $(A-B)-C = (A-C)-B$ ✓ = TRUE
 b) $(A-B)-C = (A-C)-(B-C)$ ✓ = TRUE
 c) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$ X = FALSE
 d) $A - (B \cup C) = (A-B) \cap (A-C)$ ✓ = TRUE



$$1) (A-B) = \{2, 3, 5, 6\} - \{5, 6, 7, 4\} = \{2, 3\} - \{7, 4\} = \{2\}$$

$$2) (A-C) - B = \{2, 3\} - \{5, 6, 7, 4\} = \{2\}$$

$$② A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$$

$\{2, 3, 5, 6\} \oplus \{3, 4, 6, 7, 5, 8\}$
 present here but not here
 and present here but not here

$$\text{Now, } (A \oplus B) = (2, 5, 4, 7) \cup (2, 3, 7, 8) = (2, 3, 4, 7, 8) \neq \{2, 4, 7, 8\}$$

2. RELATION

1. INTRODUCTION

Cartesian Prod

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

⇒ Cartesian

⇒ Home LA

⇒ In general

⇒ we take a

Relation:

The Relation

⇒ If $|A| = m$

(Since Rel

contains

over the

be $2^{m \times n}$

Ex-1 Let R =

Ex-2 Let R =

Ex-3 Let A =

R: $\{(x, y) \mid$

2. RELATIONS

1. INTRODUCTION TO RELATIONS

Cartesian Product: $A = \{1, 2, 3\}$ $B = \{a, b\}$

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

Each ele is called ordered pair.

\Rightarrow Cartesian product = cross product

$$\Rightarrow \text{Here } |A \times B| = 3 \times 2 = 6.$$

\Rightarrow In general if $|A|=m$, $|B|=n$ then $(A \times B)$ contains $(m \times n)$ ordered pairs

\Rightarrow we take a Relation from cartesian product.

Relation:

The Relation is a subset of cartesian product

$$\text{For above example } R = \{(1, a), (1, b)\}$$

$$1Ra \Rightarrow (1, a) \in R$$

$$R' = \{1Ra, 2Ra, 3Ra\}$$

\Rightarrow If $|A|=m$ and $|B|=n$ then no. of Relations that can be formed = 2^{mn}

(Since Relation is subset of cartesian product, and cross product contains $(m \times n)$ elements. Now, the no. of subsets that are possible over the cartesian product set is 2^{mn} , so the no. of relations will be 2^{mn} .)

3. EXAMPLES ON REFLEXIVE RELATIONS

$$\text{Ex-1: Let } R = \{(x, y) \in Z \times Z : x < y\} = \{(1, 2), (2, 3), (4, 5), \dots\}$$

$$\text{Ex-2: Let } R = \{(x, y) \in Z \times Z : (x, y) \text{ is even}\} = \{(1, 2), (1, 4), (2, 4), (2, 6), \dots\}$$

$$\text{Ex-3: Let } A = \{1, 2, 3, 4\} \quad B = \{1, 2\}$$

$$R: \{(x, y) \in A \times B : x + y = 3\} = \{(1, 2), (2, 1)\}$$

2. REFLEXIVE RELATION

Reflexive Relation: A Relation 'R' on set 'A' is said to be Reflexive if $(aRa) \forall a \in A$

$$A = \{1, 2, 3\}$$

$$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

$R = \{(1,1), (2,2), (3,3)\}$ Hence 'R' is Reflexive Relation

$R_1 = \{(1,1), (2,2)\}$ is not Reflexive because it does not contain $(3,3)$. ($\forall x \in A \rightarrow xRx$ should be in)

$R_2 = \{(1,1), (2,2), (3,3), (1,2)\}$ = Reflexive Relation

\therefore Let $A = \{1, 2, \dots, n\} \rightarrow n$ elements

$R_n = \{(1,1), (2,2), \dots, (n,n)\} \rightarrow$ Then the smallest relation which is reflexive contains ' n ' elements.

Note: If 'R' is Reflexive then any superset of 'R' is Reflexive

The largest Reflexive Relation on 'A' is " $A \times A$ "

The smallest Reflexive Relation on 'A' contains ' n ' elements

If $|A| = n$ then largest Reflexive Relation = $n \times n$ elements.

If $|A| = n$ then smallest Reflexive Relation = n elements.

3. EXAMPLE ON REFLEXIVE RELATIONS

If $A = \{1, 2, 3, \dots, n\}$ then the no. of reflexive relations possible on 'A'?

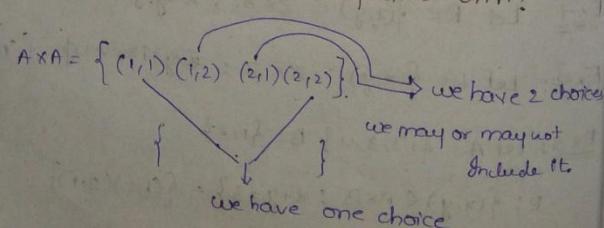
$$A = \{1, 2\}$$

$$R_1 = \{(1,1), (2,2)\}$$

$$R_2 = \{(1,1), (2,2), (1,2)\}$$

$$R_3 = \{(1,1), (2,2), (2,1)\}$$

$$R_4 = \{(1,1), (2,2), (2,1), (1,2)\} = A \times A$$



$$A = \{1, 2, 3\}$$

$$R_1 = \{(1,1), (2,2), (3,3)\}$$

\therefore If $|A|=n$

$$|A|=n$$

The NO of Rela

Non-

4. EXAMPLE 2

Now whenever an element form element 2 times.

Ex:

1) The relation

2) The relation

Real numbers

3) The Relation

collection of

4) The Relation

5) $R: \{(x,y) \in$

$$A = \{1, 2, 3\}$$

$$R_1 = \{(1,1), (2,2), (3,3)\}$$

3 3 3
3 3 3

= 2^6 combinations

∴ The no. of Reflexive relations
 $= 64.$

∴ If $|A|=n$ then the no. of Reflexive Relations on $A = 2^{n^2-n}$

$$|A|=n \Rightarrow \text{No. of Reflexive Relations on } A = 2^{n(n-1)}$$

The No. of Relations that are not Reflexive = Total Relations - Reflexive Relations

$$= 2^{n^2} - 2^{n^2-n}$$

$$\text{Non-Reflexive Relations from } A \text{ to } A = 2^{n^2} - 2^{n^2-n}$$

4. EXAMPLE 2 ON REFLEXIVE RELATIONS

Now whenever they ask if a Relation is Reflexive then try to take

an element from the relation and apply the condition between the same element 2 times.

Ex:

↪ The relation \leq is reflexive on any set of Real numbers ($x \leq x$)

↪ The relation 'is a divisor of' is reflexive on a set of non-zero Real numbers ($\forall x/x$) $\forall x$ is divisible by 'x' \Rightarrow TRUE \Rightarrow Relation is Reflexive

↪ The Relation is a subset of denoted by ' \subseteq ' is reflexive on any collection of sets $(A \subseteq A) \Rightarrow$ TRUE \Rightarrow Reflexive

↪ The Relation 'is parallel to' is reflexive on a set of all lines ($L \parallel L$)
 \Rightarrow TRUE

5. R: $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x-y \text{ is even integer}\}$ $(x-x)=0=\text{even}=$ TRUE
 \Rightarrow Reflexive

have 2 choices
may not
include it.

5. EXAMPLE 3 ON REFLEXIVE RELATIONS

(8)

⇒ There will

⇒ The min. 3
of a Irreflex

NOW, $R_5 = \{ \}$

Which of the following is false?

- If R_1 is Reflexive then every subset of R_1 is Reflexive = TRUE
- If R_1 is Reflexive, then subset of R_1 is reflexive = FALSE
- If R_1, R_2 are Reflexive then $R_1 \cap R_2$ is Reflexive = TRUE
- If R_1, R_2 are Reflexive then $R_1 \cup R_2$ is Reflexive = TRUE

④.

$$R = \{(1), 2, 3\}$$

$$\textcircled{B} \quad R = \{(1, 2, 3)\}$$

$$R_1 = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_2 = \{(1, 1)\} \rightarrow \text{NOT}$$

Reflexive

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3), (3, 1)\}$$

$\{(2, 3), (3, 1)\}$ are reflexive

⑤

$$R = \{(1, 2, 3)\}$$

$$R_1 = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)\}$$

$$R_1 \cap R_2 = \{(1, 1), (2, 2), (3, 3)\}$$

= Reflexive.

$$\textcircled{C} \quad R_1 = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)\}$$

$$R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)\}$$

= Reflexive

$$\text{Now } \{A \times A\} -$$

$$\{ \mid A \mid = n \text{ the} \}$$

7. EXAMPLE

Now, I want to

⇒ then $|A \times A|$

$$\text{let } A = \{1, 2, 3\}$$

$$A \times A = \{(1, 1)$$

$$2. (2, 1)$$

$$3. (3, 1)$$

∴ In general,

The total
Over a set

6. IRREFLEXIVE RELATIONS

Irreflexive Relations:

The relation 'R' on a set 'A' is called irreflexive if 'x' is not related to 'x' i.e. $x \not\sim x \forall x \in A$ i.e. the ordered pair $(x, x) \notin R \forall x \in A$.

Ex:-

If a Relation is Not reflexive then we think that it is irreflexive but this is wrong assumption.

$$\text{Ex: } A = \{1, 2, 3\}$$

$R_1 = \{(1, 1), (2, 2), (3, 3)\}$ Reflexive, NOT Irreflexive

$R_2 = \{\}$ NOT Reflexive, Irreflexive

$R_3 = \{(1, 1)\}$ NOT Reflexive, NOT Irreflexive

$R_4 = \{(1, 2), (2, 1)\}$ NOT Reflexive, Irreflexive

There does not exist
a relation which
is both Reflexive
and Irreflexive

③

⇒ There will be some relations which are neither reflexive nor Irreflexive.

⑨ ⑩

⇒ The min. set that is Irreflexive is \emptyset and the min cardinality of a Irreflexive Relation is "Zero".

$$\text{Now, } R_5 = A \times A$$

$$= \{(1,1), (1,2), (1,3), \\ (2,1), (2,2), (2,3) \\ (3,1), (3,2), (3,3)\}$$

Now Remove the diagonal elements
because they make the Relation Reflexive

$(2,2)(3,3)$

$(2,2)(3,3)$

$(1,1)(2,2)(3,3)$

reflexive.

$$\text{Now } \{A \times A - \text{diagonal elements}\} = \{(1,2), (1,3), \\ (2,1), (2,3), \\ (3,1), (3,2)\}$$

This is the largest
Irreflexive Relations

If $|A|=n$ then the cardinality of largest Irreflexive Relation
will be (n^2-n)

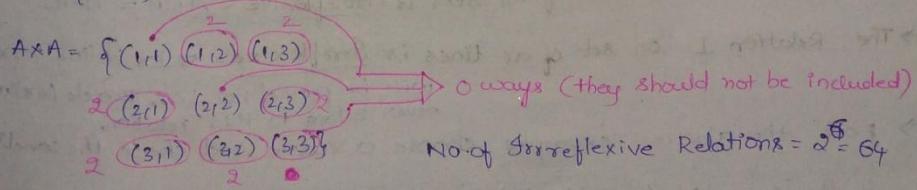
$$\left\{ \text{Total no. of elements in } A \times A = (n \times n) = n^2 \right\} - \left\{ \text{Diagonal elements} \right\}$$

7. EXAMPLE 1 ON IRREFLEXIVE RELATIONS

Now, I want to find no. of Irreflexive Relations on 'A' given $|A|=n$

⇒ then $|A \times A|$ contains n^2 elements

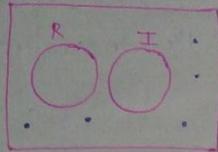
$$\text{let } A = \{1, 2, 3\}$$



∴ In general $A \times A = \{(1,1), (2,2), (3,3), \dots, (n,n), (1,2), (1,3), \dots\}$

The total no. of Irreflexive relations that can be formed
over a set of ' n ' elements is $= n^2-n$

Now, it is clear that



R = Reflexive Relations.

I = Irreflexive Relations

(10) Q

Ans
A) A = {1, 2, 3}

$$R = \{ (1, 2), (1, 3), (2, 3), (3, 1) \}$$

$$R_1 = \text{Subset} \\ = \{ (1, 2), (1, 3) \} \\ \text{TRUE}$$

Now, No. of Relations which are either reflexive or irreflexive is $2^{\frac{n(n-1)}{2}} + 2^{\frac{n(n-1)}{2}}$

\therefore No. of Relations that are either Reflexive or Irreflexive $= 2^{\frac{n(n-1)}{2}} + 2^{\frac{n(n-1)}{2}}$

$$= 2(2^{\frac{n(n-1)}{2}})$$

$$= 2^{n^2-n+1}$$

Now, No. of Relations that are neither Reflexive nor Irreflexive =
(Total) - (No. of Reflexive)

$$= 2^n - (2^{\frac{n(n-1)}{2}} + 2^{\frac{n(n-1)}{2}}) \quad \text{(Irreflexive)}$$

No. of Relations that are Neither reflexive nor Irreflexive = $2^n - (2^{\frac{n(n-1)}{2}} + 2^{\frac{n(n-1)}{2}})$

8. EXAMPLE 2 ON IRREFLEXIVE RELATIONS

→ The Relation ' $<$ ' on set of all Real numbers is Irreflexive = TRUE

→ The Relation ' C ' on set of all sets is "Irreflexive" = $A C A = \text{TRUE}$

→ The Relation ' \perp ' on set of all lines is "Irreflexive" = $x \perp x$ (A line x/x will be parallel not \perp . i.e. Two line (x/x) will never belong to this relation.)

⇒ For the above examples choose a value and check the condition

if it is valid then the Relation is Not IRReflexive if it fails then
it is Irreflexive.

9. EXAMPLE 3 ON IRREFLEXIVE RELATION

→ Which of the following is false?

→ Every subset of Irreflexive relation is irreflexive

→ Every superset of Irreflexive relation is Irreflexive

→ If R_1 is Irreflexive, R_2 is irreflexive then $R_1 R_2$ is Irreflexive

→ If R_1, R_2 are irreflexive then $R_1 R_2$ is Irreflexive

11. SYMETRY

A Relation 'R'
 $x, y \in A$ i.e. if

$$A = \{1, 2, 3\}$$

$$R_1 = \{ (1, 2), (2, 1) \}$$

$$R_2 = \{ (1, 1) \} - \text{sym}$$

$$R_5 = \{\} - \text{symmetric}$$

(12) ~~Ex~~

W.K.T. the no.

Now, set of all

The cardinality of the smallest symmetric Relation is Zero.

$R_6 = A \times A$ is symmetric \therefore The largest cardinality is " n^2 " elements and

$$A \times A =$$

The largest symmetric relation that is defined on a set 'A' is "A \times A".

12. NUMBER OF SYMMETRIC RELATIONS

$A = n$ then how many Relations are possible that are symmetric on $A \times A$

$$|A| = n \text{ then } |A \times A| = n^2$$

$$A = \{1, 2, 3\}$$

$$A \times A = \{(1,1)(2,2)(3,3), (1,2)(2,1), (1,3)(3,1), (2,3)(3,2)\}$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2$

$$= 2^6$$

$$|A| = n = \{1, 2, 3, \dots, n\}$$

$$(A \times A) = \{(1,1)(2,2)(3,3) \dots (n,n), (1,2)(2,1), (1,3)(3,1), \dots, (2,3)(3,2)\}$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $m^2 \text{ elements}$
 $m^2 \text{ elements}$
 $m^2 \text{ elements}$
 $m^2 \text{ elements}$
 $m^2 \text{ elements}$

$$= 2^n$$

\rightarrow No. of pairs
that will
be formed.

$$\therefore \text{No. of symmetric Relations} = \frac{n(n^2-n)/2}{2} = \frac{n(n^2-n)}{2}$$

$$\text{Now, } |S-R| =$$

$$\text{Now, } |R-S| =$$

$$|R-S| =$$

The rel.
Reflexi.

13. EXAMPLE 1 ON SYMMETRIC RELATIONS

$|A| = n$, now I want to find the relation between the no. of symmetric Relations and Reflexive Relations.

$$\text{Let } A = \{1, 2, 3\}$$

$$R_1 = \{(1,2)(2,1)\}$$

$$R_2 = \{(1,1)(2,2)(3,3)\}$$

$$R_3 = \{(1,1)(2,2)(3,3)(1,2)\}$$

$$R_4 = \{(1,2)\}$$

14. EXAMPLE

State which

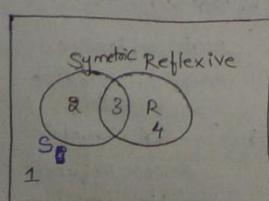
a) Every set

b) " " \Rightarrow If

c) " " \Rightarrow If R_1 a

d) " " \Rightarrow If R_1 a

e) " " \Rightarrow If R_1 a



$\mathcal{P}(A \times A)$ \Rightarrow subset of powerset (Total no. of relations).

W.K.T. the no. of symmetric Relations possible are $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2} = \frac{n(n+1)^2}{4}$

(12)

(1)
ve 1/2

Now, set of all Relations that are both Symmetric and Reflexive = Region 3.

its and

$$A \times A = \left\{ (1,1) (2,2) (3,3) \dots (n,n) \right\} = \frac{n(n+1)}{2}$$

n^2 elements
 1 way because
 the Relation should be
 reflexive (n elements).

$\frac{n^2-n}{2}$

15

$$\therefore |S \cap R| = \frac{\frac{n(n+1)}{2}}{2} = \text{Symmetric and Reflexive}$$

$$\text{Now, } |S - R| = |n(S)| - |n(S \cap R)| \\ = \frac{n(n+1)}{2} - \frac{\frac{n(n+1)}{2}}{2}$$

$$|S - R| = \text{Symmetric but not Reflexive} \\ = \frac{n(n+1)}{2} - \frac{\frac{n(n+1)}{2}}{2}$$

$$\text{Now, } |R - S| = n|R| - n|S \cap R|$$

$$|R - S| = \frac{n(n+1)}{2} - \frac{\frac{n(n+1)}{2}}{2}$$

Now the Relations that are not Symmetric
 and Reflexive = $n|S \cup R|$

$$= 2 - \left(n(S) + n(R) - n(S \cap R) \right)$$

$$= 2 - \left(\frac{n(n+1)}{2} + \frac{n(n+1)}{2} - \left(\frac{\frac{n(n+1)}{2}}{2} \right) \right)$$

The relations that are not both
 Reflexive and Symmetric

15. EXAMPLE 3 ON SYMMETRIC RELATIONS

State which of the following are TRUE and FALSE?

f Symmetric

$\{(1,2)\}$

a) Every subset of symmetric Relation is symmetric (F)

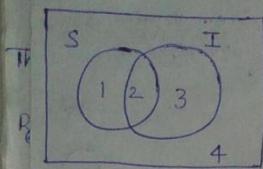
b) " Superset " " " " " (FALSE)

c) If R_1 and R_2 are symmetric then $R_1 \cap R_2$ is symmetric (TRUE)

d) " " " " " " " " " " " " " " " " " (TRUE).

e) " " " " " " " " " " " " " " " " " (TRUE).

16. RELATION BETWEEN SYMMETRIC AND IRREFLEXIVE RELATION



1) $R_1 = \{(1,1)\} \rightarrow$ Symmetric ✓ Not Reflexive

2) $R_2 = \{(1,2)(2,1)\} \rightarrow$ Symmetric ✓ Irreflexive

3) $R_3 = \{(1,2)\} \rightarrow$ Symmetric ✗ Irreflexive

4) $R_4 = \{(1,2)(1,1)\} \rightarrow$ Symmetric ✗ Irreflexive

Now, $n|S \cap I| =$ Both Symmetric and Irreflexive

$$AXA = \underbrace{\{(1,1)(2,2) \dots (n,n)\}}_x \boxed{(1,2)(2,1)} \underbrace{\dots}_{\begin{array}{c} \downarrow \\ 2 \\ \downarrow \\ 2 \\ \downarrow \\ 2 \end{array}} \underbrace{-}_{\frac{n^2-n}{2} \text{ times}} \dots$$

$$\therefore n|S \cap I| = \frac{n^2-n}{2}$$

$n(S \cup I) = n(S) + n(I) - n(S \cap I) \rightarrow$ Either Symmetric or Reflexive

\therefore The no. of Relations that are Either Symmetric or Irreflexive = $\left(\frac{n(n+1)}{2} + \frac{n^2-n}{2} \right) - \left(\frac{(n^2-n)/2}{2} \right)$

$$n(S-I) = n(S) - n(S \cap I)$$

\Rightarrow The no. of Relations that are Symmetric but not Irreflexive = $\frac{n(n+1)}{2} - \frac{(n^2-n)/2}{2}$

$$n(I-S) = n(I) - n(S \cap I)$$

\Rightarrow The no. of Relations that are but not symmetric is $= \frac{n^2-n}{2} - \frac{(n^2-n)/2}{2}$

$n(\overline{I \cup S}) =$ Relations that are neither reflexive nor symmetric = $n(U) - n(I \cup S) = \frac{n^2}{2} - n(I \cup S)$

17. ANTISYMMETRIC RELATION

(1) A Relation
if $x, y \in A$.

$$A = \{1, 2, \dots\}$$

WKT $n(S) = \frac{n(n+1)}{2}$

$$n(I) = \frac{n^2-n}{2}$$

∴ The min ca

21011709 21011709

The max ca

Now, $AXA = \{(1,1)\}$

$$(1,2)$$

$$(2,1)$$

$$(3,1)$$

Now, if $AXA =$

\therefore The max ca

REFLEXIVE RELATION

17. ANTI-SYMETRIC RELATION

(14) A Relation 'R' is said to be Anti-symmetric if $(xRy \text{ and } yRx) \Rightarrow x=y \in A$.

$$\frac{n(n+1)}{2}$$

$$A = \{1, 2, 3\}$$

$R_1 = \{(1,2), (2,1)\} \rightarrow$ NOT Anti-symmetric because we have
if $(x,y) \in R$ then $(y,x) \in R$.

$R_2 = \{(1,1)\} \rightarrow$ Anti-symmetric (This is the only exception
(the pairs of type (a,a) are allowed)).

Symmetric ✓ (we can't say symmetric &
Anti-symmetric are Compliment to each other).

$R_3 = \{(1,2), (1,1)\} \Rightarrow$ Not symmetric but it is Anti-symmetric

$R_4 = \{(1,2), (2,1), (2,3)\} \Rightarrow$ Not Anti-symmetric. $(x,y) \in R \text{ & } (y,x) \in R$.
NOT symmetric $(3,2)$ is not present.

$R_5 = \{(1,1), (2,2), (3,3)\} \rightarrow$ Both Symmetric and Anti-symmetric (Exceptional case).

$R_6 = \{\} -$ Anti-symmetric ✓

The min cardinality of Anti-symmetric Relation = 0

$R_7 = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\} -$ Longest Anti-symmetric Relation

Now, $A \times A = \{(1,1), (2,2), (3,3)\}$

$(1,2), (2,1)$

$(2,3), (3,2)$

$(3,1), (1,3)\}$

n^2 elements.

Now, if $A \times A = \underbrace{\{(1,1), (2,2), \dots, (n,n)\}}_{m^2 \text{ elements}}, \underbrace{\{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}}_{m^2 = n \text{ elements}}$.

\downarrow
choose All

choose $\frac{n^2-n}{2}$ ele

∴ The max cardinality of Anti-symmetric Relation = $\left(m + \frac{n^2-n}{2}\right)$

$$\frac{(n^2-n)/2}{2}$$

$$\frac{n^2-n}{2} - n(\text{Jus})$$

18. NUMBER OF ANTSYMETRIC RELATIONS.

$$|A|=n \quad |A \times A|=n^2$$

$$A = \{1, 2, 3\}$$

$$A \times A = \left\{ \begin{matrix} 1 \times 1 & 1 \times 2 & 1 \times 3 \\ (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{matrix} \right\}$$

= $2^3 \times 3^3$ Relations
 $= 2$ diagonal pairs $\times 3$ Non-diagonal pairs

→ Case 1: Include (1,2)

2: Include (2,1)

3: Include None

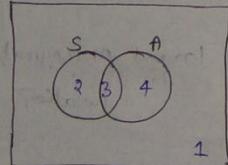
No. of Antisymmetric Relations possible on A where $|A|=n$ is

$$= \binom{n}{2} \times 3^{\frac{n(n-1)}{2}}$$

$$A \times A = \left\{ \underbrace{(1,1) \dots (2,2) \dots (3,3) \dots (n,n)}_n, \underbrace{[(1,2)(2,1)]}_{3}, \underbrace{[(3,2)(2,3)] \dots}_{3 \dots \frac{n(n-1)}{2} \text{ times}} \dots \right\}$$

$$\Rightarrow 2^n \times 3^{\frac{n(n-1)}{2}} \text{ Relations}$$

19. RELATION BETWEEN SYMMETRIC AND ANTSYMETRIC RELATIONS



$$|U| = 2^n$$

$$n(S) = \frac{n(n+1)}{2}$$

$$n(A) = 2^n \times 3^{\frac{n(n-1)}{2}}$$

$$R_1 = \{(1,2), (2,1)\} \rightarrow \text{Antisymmetric} \times \text{Symmetric}$$

$$R_2 = \{(1,2)\} \rightarrow S \checkmark \text{ ASX}$$

$$R_3 = \{\} \rightarrow \text{ASV} \quad \text{SV}$$

$$R_4 = \{(1,1)\} \rightarrow \text{ASV} \quad \text{SV}$$

$$R_5 = \{(2,1)\} \rightarrow \text{ASV} \quad \text{SX}$$

$$A \times A = \left\{ \underbrace{(1,1) \dots (n,n)}_{\text{Choose All}}, \underbrace{(1,2) \dots (2,1) \dots}_{n(n-1)/2} \dots \right\}$$

Choose All

$$= 2^n$$

The no. of Relations that will be

Both symmetric and antisymmetric

$$= 2^{\frac{n(n+1)}{2}}$$

only diagonal elements

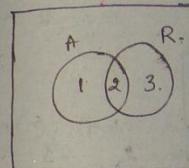
Now, the no. of relations

$$n(\text{SUA})$$

$$n(\text{SUA})$$

Similarly $n(\text{S})$

20. RELATION



$$n(A) = 2^n \times 3^{\frac{n(n-1)}{2}}$$

$$n(R) = 2^{n(n-1)/2}$$

$$n(U) = 2^{n^2}$$

$$A \times A = \{(1,1)\}$$

Both Reflexive &

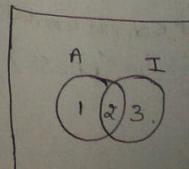
$$n(AUR) = n(A) +$$

$$n(A-R) = n(A) -$$

$$n(R-A) = n(R) -$$

$$n(\overline{AUR}) = n(U) -$$

RELATION B



$$n(U) = 2^{n^2}$$

(6)

Now, the relations that are either Symmetric / Antisymmetric are

$$n(SUA) = n(S) + n(A) - n(SNA)$$

$$n(SUA) = \binom{n(n+1)}{2} + \binom{n(n-1)/2}{2} - \binom{n^2}{2}$$

Relations

Nondiag
pairs

Similarly $n(S-A)$ and $n(A-S)$ can be found

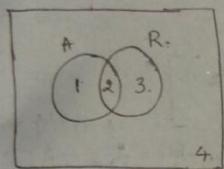
$$n(S-A) = n(S) - n(SNA)$$

$$n(A-S) = n(A) - n(SNA)$$

(7)

ver 1's

20. RELATION BETWEEN REFLEXIVE AND ANTSYMETRIC RELATIONS



$$R_1 = \{(1,1)\}$$

$$R_2 = \{(1,1), (2,2), \dots, (n,n)\}$$

$$R_3 = \{(1,1), (2,2), \dots, (n,n), (1,2), (2,1)\}$$

$$R_4 = \{(2,1), (1,2)\}$$

$$n(A) = \frac{n^2 - n}{2}$$

$$n(R) = \frac{n(n-1)}{2}$$

$$n(u) = 2^n$$

RELATIONS

Symmetric X

$$A \times A = \underbrace{\{(1,1), (2,2), \dots, (n,n)\}}_{\frac{n(n-1)}{2}} \underbrace{\{(1,2), (2,1)\}}_{\frac{n(n-1)/2}{2}} \dots \}$$

$$\text{Both Reflexive \& Antisymmetric} = 1 \times 3 = \frac{n(n-1)/2}{2} = \frac{n(n-1)}{4}$$

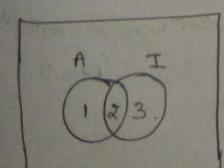
$$n(AUR) = n(A) + n(R) - n(ANR) = \text{Both Antisymmetric and Reflexive}$$

$$n(A-R) = n(A) - n(ANR) = \text{Antisymmetric but not Reflexive}$$

$$n(R-A) = n(R) - n(ANR) = \text{Reflexive but not Antisymmetric}$$

$$n(\overline{AUR}) = n(u) - n(RUA) = \text{Not Reflexive and not Antisymmetric}$$

21. RELATION BETWEEN IRREFLEXIVE AND ANTSYMETRIC RELATIONS



$$R_1 = \{(1,1)\} \Rightarrow A \checkmark \text{ IRX}$$

$$R_2 = \{(1,2)\} \Rightarrow A \checkmark \text{ IRV}$$

$$R_3 = \{(1,2)(2,1)\} \Rightarrow A \times \text{ IRV}$$

$$R_4 = \{(1,1)(1,2)(2,1)\}, A \times \text{ IRX}$$

ments

$$n(u) = 2^n$$

$$n(A) = \frac{n^2 - n}{2}$$

$$n(I) = \frac{n(n-1)}{2}$$

23. EXAMPLE

1) The Relation

2) The Relation

3) The Relation

on any set

4) The Relation

17. $x \leq y$ then

is Antisymmetric

$x \leq y$ then

therefore the

x/y (i.e.

x/y is dis-

2/4 (4)

4) Antisymmetric

24. ASYMETRIC

A Relation R'

$(y R' x) \neq x R$

$$A = \{ \}$$

$$R_1 = \{ \}$$

$$R_2 = \{ \}$$

$$R_3 = \{ \}$$

$$R_4 = \{ \}$$

$$R_5 = \{ \}$$

$$R_6 = \{ \}$$

$$R_7 = \{ \}$$

$$R_8 = \{ \}$$

$$R_9 = \{ \}$$

$$R_{10} = \{ \}$$

$$A \times A = \{(1,1) (2,2) \dots (n,n) (1,2) (2,1) (2,3) (3,2) \dots \} \quad (8)$$

The Relations that are Both Reflexive and Antisymmetric

$$A \times A = \{(1,1) (2,2) \dots (n,n) (1,2) (2,1) (2,3) (3,2) \dots \}$$

If I include this
they become reflexive
so don't include them

$$\frac{n(n-1)/2}{3}$$

1. Ques. The no. of Relations that are Both Symmetric and Irreflexive is. Anti = $\frac{(n^2-n)/2}{3}$

$$\text{Now, } n(A-I) = n(A) - n(AN\bar{I}) = \left[\begin{matrix} n(n-1)/2 & (n^2-n)/2 \\ 3 & 3 \end{matrix} \right]$$

$$n(I-A) = n(I) - n(ANI) = \text{Irreflexive but not Antisymmetric} = \frac{n(n-1)}{2} - \frac{(n^2-n)/2}{3}$$

$$n(A \cup I) = n(A) + n(I) - n(ANI) = \text{No. of Antisymmetric but not Reflexive} = \frac{(n^2-n)/2}{3}$$

$$n(\overline{A \cup I}) = n(U) - n(A \cup I) = \frac{n^2}{2} - \frac{(n^2-n)/2}{3}$$

22. ANTI-SYMMETRIC PROPERTIES.

State TRUE / FALSE

a) Every subset of Anti-symmetric relation is Anti-symmetric (TRUE)

b) " Superset " " " " " " " (FALSE)

c) Anti-symmetric relations are closed under set union. (FALSE)

d) " " " " " " " Intersection (TRUE)

e) " " " " " " " difference (TRUE)

Set Complementation-

(FALSE)

Note: .

$R = \{ \}$

23. EXAMPLES ON ANTI SYMETRIC RELATION

- (8) 1) The Relation ' \leq ' is Anti symmetric on any set of Real numbers.
- 2) The Relation ' $<$ ' is Anti symmetric on any set of Real numbers.
- 3) The Relation "is a divisor" of denoted as ' $/$ ' is an antisymmetric on any set of the Real numbers.
- 4) The Relation ' \subseteq ' (set inclusion) is Anti symmetric on any collection of sets.
5. $x \leq y$ then (y/x) will not be present and ($y \leq x$ is false) \therefore The relation is Anti symmetric.
- 6) $x < y$ then (y/x) is false, so (y/x) will not be present in relation therefore the relation is Anti symmetric.
- 7) x/y (i mean if y is divisible by x).
If x is divisor of y then y won't be divisor of x .
e.g. (4 is divisible by 2) then (4/2)(2 won't be divisible by 5).
- 8) Antisymmetric.

24. ASYMETRIC RELATIONS

A Relation ' R ' on set ' A ' is said to be Asymmetric if $(x R y)$ then $(y R x) \nvdash x, y \in A$

$$A = \{1, 2, 3\}$$

$R_1 = \{(1, 2)\}$ Asymmetric Relation (because (x, y) is present but (y, x) is not present).

$R_2 = \{(1, 2), (2, 1)\}$ Not Asymmetric (because we included x, x)

Antisymmetric (\checkmark) because (x, y) is present and (y, x) is not present and (x, x) is allowed case

Note: Diagonal elements they can be present in Antisymmetric but not in Asymmetric

$$R_3 = \{\} \text{ Asymmetric, Antisymmetric, Symmetric}$$

The min. cardinality of smallest Asymmetric Relation = '0'.

$R_4 = \{(1,2)(2,1)\} \rightarrow$ Anti-symmetric \times Asymmetric \times Symmetric

$R_5 = \{(1,2)(2,1)(3,1)(1,3)(3,2)(2,3)\}$ — This is the largest asymmetric relation.

Now, $A \times A = \{(1,1)(2,2) \dots (n,n)\}$

\downarrow \downarrow \downarrow
 $\underbrace{\quad \quad \quad}_{\text{They violate}} \quad \quad \quad$ $\boxed{(1,2)(2,1)} \quad \boxed{(3,2)(2,3)} \quad \dots$
 include one of
 $(1,2) \setminus (2,1)$
 property.

The cardinality of the largest asymmetric relation that is possible with a set with 'n' elements is $= \binom{n^2-n}{2}$

25. NO OF ASYMMETRIC RELATIONS

$$|A|=n \quad |A \times A|=n \times n$$

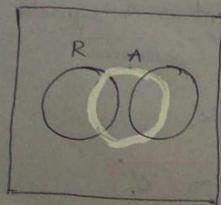
Now, $|A \times A| = \{(1,1)(2,2) \dots (n,n)\}$

\downarrow \downarrow \downarrow
 should not include $3C2$: $3C1$ $3C1$
 it $= 3 \times 3 \times 3 - \binom{n^2-n}{2}$ times.

The no. of asymmetric relations $= \binom{n^2-n}{2}$

26. REFLEXIVE AND ASYMMETRIC RELATIONS

\rightarrow If a Relation is Reflexive then it cannot be Asymmetric



$$n(A) = \frac{n(n-1)}{2}$$

$$n(R) = \frac{n(n-1)}{2}$$

$$n(A \cup R) = n(A) \cup n(R) = n(A) + n(R)$$

$$n(ANR) = 0$$

$$n(A-R) = n(A)$$

$$n(R-A) = n(R)$$

$$\begin{cases} n(\overline{R \cup A}) = n(U) - \\ n(R \cup A) \end{cases}$$

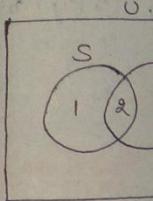
$$n(\overline{R \cup A}) = n(U) - n(R \cup A)$$

27. IRREPLE

\Rightarrow Every Assy

\Rightarrow Every Assy

28. SYMETR



$$n(S \cap A)$$

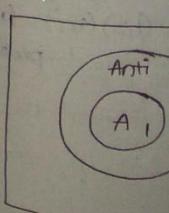
$$n(S \cup A)$$

$$n(S-A)$$

$$n(A-S)$$

29. ANTI SYME

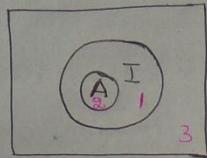
Every Asymet



27. IRREFLEXIVE AND ASYMETRIC RELATIONS

(20)

- Every Asymmetric Relation is Irreflexive but Irreflexive Relations are not necessarily Asymmetric.
- Every Irreflexive Relation is not Asymmetric.



$$n(I) = \frac{n(n-1)}{2}$$

$$n(A) = \frac{n(n-1)/2}{3}$$

$$n(I \cup A) = n(I)$$

$$n(I \cap A) = n(A)$$

$$n(I - A) = n(I) - n(A)$$

$$n(A - I) = 0.$$

$$n(\overline{I \cup A}) =$$

$$= n(U) - n(I \cup A)$$

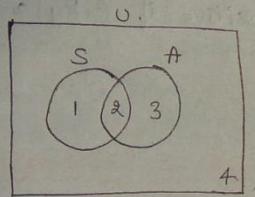
$$= n(U) - n(I)$$

(21)

28. SYMMETRIC AND ASYMMETRIC RELATIONS

(22)

$$= \left(\frac{n^2-n}{2} \right)$$



$$R_1 = \{(1,1)\} \quad S \checkmark \quad AS \times \quad n(S) = \frac{n(n-1)}{2}$$

$$R_2 = \{(1,2)\} \quad S \checkmark \quad AS \checkmark \quad n(A) = \frac{n(n-1)}{2}$$

$$R_3 = \{(1,2), (2,1)\} \quad S \checkmark \quad AS \checkmark \quad n(S \cap A) = 1$$

$$R_4 = \{(1,1), (1,2), (2,1)\} \quad S \times \quad AS \times$$

$$\text{Now, } n(S) \in n(S) \subset n(S \cap A) \Rightarrow n(S \cap A) = 1$$

$$n(S \cup A) = n(S) + n(A) - n(S \cap A)$$

$$n(\overline{S \cap A}) = n(U) - n(S \cap A)$$

$$n(S - A) = \{n(S) - n(S \cap A)\}$$

$$n(A - S) = \{n(A) - n(S \cap A)\}$$

setting

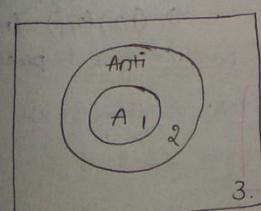
be

ence

now...

29. ANTI-SYMMETRIC AND ASYMETRIC RELATIONS.

- Every Asymmetric Relation is Anti-Symmetric.



$$R_1 = \{(1,1), (1,2)\} \quad \text{Anti-Symmetric} \checkmark \quad \text{Asymmetric} \times$$

$$R_2 = \{(1,2), (2,1)\} \quad \text{Anti-Symmetric} \times \quad \text{Asymmetric} \times$$

$$R_3 = \{(1,2)\} \quad \text{Anti-Symmetric} \checkmark \quad \text{Asymmetric} \checkmark$$

$$\text{Now, } n(A \cup \text{Anti}) = n(\text{Anti})$$

$$= n(\overline{A \cup \text{Anti}}) = n(U) - n(\text{Anti})$$

$$= n(\text{Anti} - \text{Asy}) = n(\text{Anti}) - n(\text{Asy})$$

$$= n(\text{Asy} - \text{Anti}) = 0.$$

ASYMMETRIC AND IRREFLEXIVE RELATIONS

30. PROPERTIES OF ASYMMETRIC RELATIONS

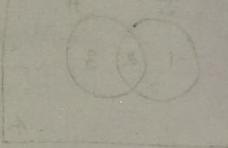
II Asymmetric Relations are closed under

- a) subset operation (TRUE).
- b) Superset operation (FALSE)
- c) union " " (FALSE)
- d) Intersection " " (TRUE)
- e) Set difference (TRUE)
- f) Complementation (FALSE)



31. TRANSITIVE RELATIONS

A Relation R on set A is said to be transitive if (xRy) and (yRz) then (xRz) $\forall x, y, z \in A$



$$R_1 = \{(\} \} - \text{Transitive}$$

$$R_2 = \{(1,1)\} - \text{Transitive}$$

$$R_3 = \{(a,b), (c,d)\} \rightarrow \text{Transitive} \quad (\text{we don't have } (a,y), (y,z))$$

$$R_4 = \{(x,y), (y,z)\} \rightarrow \text{NOT Transitive} \quad [(x,z) \text{ is absent}]$$

$$R_5 = \{(x,y), (y,z), (x,z)\} = \text{Transitive}$$

$$R_6 = \{(1,2), (2,1)\} - \text{Transitive}$$

$$R_7 = \{(1,2), (2,1)\} \rightarrow \text{Transitive} \Rightarrow (2,1), (1,1), (1,1) \rightarrow (2,1) \text{ is present}$$

$$R_8 = A \times A = \text{Transitive} \rightarrow (1,2), (2,1) = (1,1) \text{ is present}$$

The largest relation which is Transitive = $A \times A$

The smallest relation which is Transitive = $\{\}$

32. EQUIVAL

A Relation R

if R' is D

Ex: $A = \{a, b, c\}$

$R_1 = \{(a,a), (b,b), (c,c)\}$

$R_2 = \{(a,a), (b,b), (c,c)\}$

$R_3 = \{(a,a), (b,b)\}$

The smallest

\rightarrow C

The largest
and it is 4

33. EXAMPLES

Which of the
of all Real nu

a) $R_1 = \{(a,b)/a < b\}$

b) $R_2 = \{(a,b)/a > b\}$

c) $R_3 = \{(a,b)/a = b\}$

d) $R_4 = \{(a,b)/a \neq b\}$

34. POSET

Partial Ordering

A relation 'R'

order) if 'R'

Partially ordered

set 'A' with

ordered set C_P

32. EQUIVALENCE RELATION

- (22) A Relation ' R' on a set ' A ' is said to be Equivalence Relation on ' A ', if ' R' is
 1) Reflexive 2) Symmetric 3) Transitive

Ex: $A = \{a, b, c\}$

$$R_1 = \{(a,a), (b,b), (c,c)\}$$

$$R_3 = \{(a,a), (b,b), (c,c), (b,c), (c,b)\}$$

$$R_2 = \{(a,a), (b,b), (c,c), (a,b), (b,a)\} \quad R_4 = \{(a,a), (b,b), (c,c), (a,c), (c,a)\}$$

$$R_5 = \{(a,a), (b,b), (c,c), (a,b), (b,c), (a,c), (c,b), (b,a)\}$$

→ The smallest Equivalence set/Relation on set A contains ' n ' elements.

→ Contains only diagonal elements.

→ The largest Equivalence Relation on set A contains ' n^2 ' elements.
 and it is ' $A \times A$ '.

33. EXAMPLES OF EQUIVALENCE RELATIONS.

Which of the following is not an equivalence relation on a set of all Real numbers?

a) $R_1 = \{(a,b) / a-b \text{ is a integer}\}$ $\begin{array}{l} R \leftarrow \text{All diagonal ele diff=0 (integer)} \\ S \leftarrow \text{if } (a-b) \text{ is integer, } (b-a) \text{ is also integer} \\ T \leftarrow \end{array}$

b) $R_2 = \{(a,b) / a-b \text{ is } \frac{1}{5}\text{le by } 5\}$ $\begin{array}{l} R \leftarrow \\ S \leftarrow \\ T \leftarrow \end{array}$

✓ $R_3 = \{(a,b) / a-b \text{ is odd no.}\} \rightarrow$ Not Reflexive (diagonal ele diff=0=Even no.)

d) $R_4 = \{(a,b) / a-b \text{ is an even no.}\}$ $\begin{array}{l} R \leftarrow \text{All diagonal ele are present} \\ S \leftarrow \text{Symmetric} \\ T \leftarrow \text{Transitive} \end{array}$ $\begin{array}{l} 207-38 \\ \{ \} \leftarrow \text{Equivalence Relation.} \end{array}$

34. POSET

Partial Ordering Relation:

A relation ' R' on set ' A ' is said to be partial ordering relation (partial order) if ' R' is Reflexive, Antisymmetric, and Transitive.

Partially Ordered Set

A set ' A ' with a partial order ' R' defined on ' A ' is called partially ordered set (poset) and it is denoted by $[A; R]$

$$A = \{1, 2, 3\}$$

$$R_1 = \{(1,1), (2,2), (3,3)\} -$$

Reflexive ✓
Transitive ✓
Symmetric ✓
Antisymmetric ✓

EQUIVALENCE RELATION

(24)

\Rightarrow The Relation is Equivalence and partial order Relation.
 \Rightarrow This is the smallest Relation which is both partial ordering and Equivalence Relation.

Now, if I take other them, {if it is + if it is }

$B = \{(1,1), (2,2)\} \rightarrow$ cannot be Equivalent and partial ordering (Because this does not contain (3,3))

$$R_2 = \{(1,1), (2,2), (3,3)\} - (1,2) \rightarrow$$
 Not Equivalent Relation.

Reflexive ✓ Antisymmetric ✓ { partial ordering
Transitive ✓ Relation.

Now, let us consider set of all Real numbers R . Now, let the Relation be " \leq " then R' is Reflexive, Anti-symmetric, Transitive. Therefore

$[R; \leq]$ is called the partial order set / poset.

This Relation (\leq) is the partial order set on Relation ' R '.

Now, $[S; \subseteq]$ is also a poset — { Necessary condition is the Relation should be Reflexive, Anti-symmetric, Transitive.
 $[R; \sqsubseteq]$ is also a poset — }

35. TOS

Totally ordered set (Linearly ordered set or chain)

A poset $[A; R]$ is called a "Totally ordered set" if every pair of elements in 'A' are comparable, i.e. aRb or bRa & aRa

Ex: Find whether the following are totally ordered sets or not

1) If 'A' is any set of real no's then poset $[A, \leq]$ is TOS.

2) If $A = \{1, 2, 3, 4, \dots, 10\}$ then the poset $[A, \leq]$ is a TOS.

3) If $A = \{1, 2, 6, 30, 60, 300\}$ then $[A; \mid]$ is TOS [$a \mid b \Leftrightarrow a$ divides b , $b \mid a \Leftrightarrow b$ divides a]

4) If $S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ then $[S, \subseteq]$ is Not TOS.

5) If $S = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ then $[S, \subseteq]$ is TOS.

$$47. \{a\} \not\subseteq \{b\}$$

$$\{a, b\} \subseteq \{a, b, c\}$$

36. GATE QUEST

Let $A = \{a, b, c\}$ with

- a) $R_1 = \{(a,a), (c,c)\}$
- b) $R_2 = \{(a,b), (b,a)\}$
- c) $R_3 = \{(a,b), (b,a), (c,c)\}$
- d) $R_4 = \{(a,b), (b,c), (c,c)\}$

37. GATE QUESTION

Let $A = \{a, b, c, d\}$ with

$$(b,a)(b,b)(b,c)(b,d)$$

- a) R is Equivalence
- b) R is Irreflexive
- c) R is symmetric
- d) R is transitive

38. GATE QUESTION 3

Let $A =$ set of all

- i.e. $aRb \Leftrightarrow b=a^k$
- a) R is Equivalence
- b) R is partial order
- c) R is reflexive and
- d) R is total order.

$$R = \{(2,2), (2,4), (2,8), (2,16), (3,3), (3,9)\}$$

(24)
valence and
clity.

t Relation
ordering
ation.

ng (Because

Now, if I take any two Real nos., I can put the relation ' \leq ' in b/w them, {if I take 1,2 I can say $(1 \leq 2)$ } \therefore Every pair of nos. is comparable.
if I take 4,2 I can say $(2 \leq 4)$ } Real nos. is Comparable
 \Rightarrow This is "TOS".

47. $\{a\} \notin \{b\}$ \therefore This is not a "Total ordered set."

57. $\{a\} \subseteq \{a,b\}$, $\emptyset \subseteq \{a\}, \{a,b\}, \{a,b,c\}$
 $\{a,b\} \subseteq \{a,b,c\}$.

(25)

36. GATE QUESTION-1

Let $A = \{a, b, c\}$, which of the following is NOT TRUE (FALSE)?

- $R_1 = \{(a,a)(c,c)\}$ is symmetric, Antisymmetric, Transitive (TRUE)
- $R_2 = \{(a,b), (b,a)(a,c)\}$ is symmetric, Antisymmetric (FALSE) ((c,a) is not present)
- $R_3 = \{(a,b)(b,a)(c,c)\}$ is symmetric but not Antisymmetric (TRUE)
- $R_4 = \{(a,b)(b,c)(c,a)\}$ is Antisymmetric but not symmetric. (TRUE)

37. GATE QUESTION 2

Let $A = \{a, b, c, d\}$ and a relation on set A is defined as $R = \{(a,a)$

$(b,b)(b,c)(b,d)(c,a)(c,b)(c,c)(c,d)\}$. which of the following is TRUE?

- R is Equivalence Relation = NOT EQUIVALENCE (NOT REFLEXIVE ($a \neq a$))
- R is irreflexive or Antisymmetric relation FALSE (NOT IRREFLEXIVE)
- R is symmetric or Asymmetric Relation () (FALSE) $\{(a,b)\}$ is absent
- R is transitive, () ($(b,a)(a,c) \Rightarrow (b,c)$) (not present in R) (TRUE)

38. GATE QUESTION 3

Let $A = \text{set of all Real numbers}$, $R = \{(a,b) / b = a^k \text{ for some integer } k\}$

i.e. $aRb \Leftrightarrow b = a^k$

a) R is Equivalence Relation $\begin{cases} R \cup (a,a) = a = a \\ S \times T \end{cases} \therefore$ Not Equivalence

b) R' is partial order $\begin{cases} R' \cup (a,a) \\ \text{Anti-} \end{cases} \therefore$ The Given Relation is Partial Order.

c) R' is reflexive and symmetric but not Transitive

d) R' is total order.

$R = \{(2,2)(2,4)(2,8)(2,16)\dots (1,1)(3,3)(3,9)(3,27)\dots\}$.
NOT Symmetric = $(8,2)$ is not present
 $\Rightarrow 2 = 8^{1/3} \rightarrow$ but k is integer

39. GATE QUESTION 4

(26)

SUMM

Which of the following is NOT TRUE (FALSE)?

- a) If a Relation 'R' on set A is symmetric and Transitive then 'R' is Reflexive
- b) If a relation 'R' on set 'A' is irreflexive and transitive then 'R' is
Anti-symmetric
- c) If 'R' and 'S' are Anti-symmetric on 'A' then $(R \cup S), (R \cap S)$ are also Anti-symmetric
- d) If R, S are Transitive then
1. a) $A = \{1, 2, 3\}$ $A = \{ \} \rightarrow$ Not Reflexive, but not (Symmetric & Transitive). \Rightarrow Anti-symmetric
 $R = \{(1, 1)\}$ - Transitive, symmetric, not Reflexive \Rightarrow opt: FALSE.
2. b) R = TRUE. (C) FALSE. (D) TRUE

41. GATE QUESTION 5

5> Asymme

The no. of equivalence relations on set $\{1, 2, 3, 4\}$ is

- a) 15 b) 16 c) 24 d) 4: Just Remember it if a set has n elements

$$R = \{ \}$$

3 elements \rightarrow No. of Equivalence Relations

6> Transit

4 elements \rightarrow No. of Equivalence Relations = 15

= 5

7> Equival

EQUIVALENCE RELATION

8> POSET

9> TOS

SUMMARY ON RELATIONS

1) Reflexive Relation: All the diagonal ele should definitely be present.

2) Irreflexive Relation: No diagonal ele should be present if you find at least one diagonal ele then it is NOT IRREFLEXIVE.

3) Symmetric Relation: If (x,y) is present then only check for (y,x) .
(All the symmetric pairs need not be present).

$$\{ \} - \text{symmetric } \left\{ \begin{array}{l} \text{Min cardinality} = 0 \\ \text{Max cardinality} = n^2 \end{array} \right\}$$

4) Anti-symmetric Relation: Symmetric pairs should not be present but diagonal pairs are allowed (Exception case).

$$\{ \} - \text{Anti-symmetric } \left\{ \begin{array}{l} \text{Min cardinality} = 0 \\ \text{Max cardinality} = n + \frac{n^2-n}{2} \end{array} \right\}$$

5) Asymmetric Relation: → Symmetric pairs should not be present, No exception on diagonal elements also.

$$\{ \} - \text{Asymmetric } \left\{ \begin{array}{l} \text{Min cardinality} = 0 \\ \text{Max cardinality} = \frac{n^2-n}{2} \end{array} \right\}$$

6) Transitive Relation: If (a,b) is present and (b,c) is present then only check for (a,c) if (a,c) is present then Transitive else Not Transitive.

$$\Rightarrow f_{ab}(a,b) \cdot f_{bc}(b,c) \rightarrow f_{ac}(a,c). \quad R_i = \{ (a,b) | (b,c) \} = \text{Transitive}$$

$$\Rightarrow \{ \} = \text{Transitive } \left\{ \begin{array}{l} \text{Min cardinality} = 0 \\ \text{Max cardinality} = n^2(A \times A) \end{array} \right\}$$

7) Equivalence Relation: → Reflexive
→ Symmetric
→ Transitive } (TRS)

8) POSET : Reflexive, Anti-symmetric, Transitive (RAT).

9) TOS : Every pair should be comparable on the Relation given.

3. PARTIAL ORDERS AND LATTICES

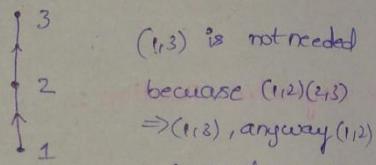
1. POSET DIAGRAM (HASSE DIAGRAM)

Let $[A; R]$ be a poset. The poset diagram is as follows

- 1) There is a vertex corresponding to each element of 'A'.
- 2) An edge between the elements 'a' and 'b' is not present in the diagram if there exists an element $x \in A$ such that (aRx) and (xRb) .
- 3) An edge b/w the elements 'a' and 'b' is present iff aRb and there is no element $x \in A$ such that (aRx) and (xRb) .

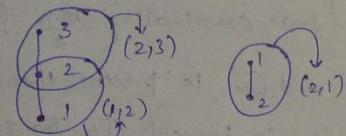
Ex: $A = \{1, 2, 3\}$

$$R = \leq = \{(1,2), (1,3), (2,3), (1,1), (2,2), (3,3)\}$$



$(1,3)$ is not needed
because $(1,2), (2,3)$
 $\Rightarrow (1,3)$, anyway $(1,2)$
 $(2,3)$ edges are
present

\Rightarrow The general convention that we assume is bottom-to-top and we don't use arrows also.

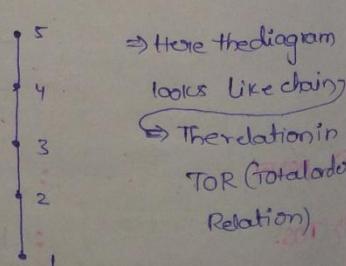


$$④ A = \{1, 2, 3\}, \text{ poset: } [A, \leq]$$

2. EXAMPLES ON POSET DIAGRAMS

$$A = \{1, 2, 3, 4, 5\} \\ R = \leq \quad \text{poset: } [A; \leq]$$

$$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$$



\Rightarrow Here the diagram looks like chain
 \Leftrightarrow The relation is TOT (Total order Relation).

$$⑤ A = \{1, 2, 3, 4, 5\}, \text{ poset: } [A, \leq]$$

3. LUB

Least upper bound
Let $[A; R]$ be such that i)
ii)

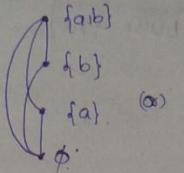
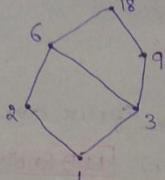
(28)

$$\text{② } S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

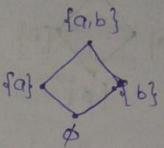
poset: $[S, \subseteq]$

$$\text{③ } A = \{1, 2, 3, 6, 18\}$$

poset: $[A, |]$



(29)



①

and (xRb)

and there

needed

$(2)(2,3)$

way $(1,2)$

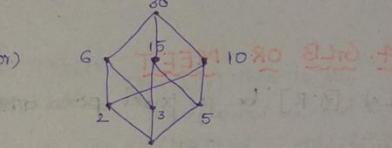
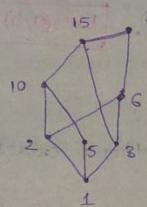
is one

present

Ans: ④

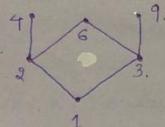
$$\text{④ } A = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

poset: $[A, |]$



$$\text{⑤ } A = \{1, 2, 3, 4, 6, 9\}$$

poset: $[A, |]$



Ans: ⑤

diagram

ke chainz

lation in

(Total order)

ation)

3. LUB

Least upper Bound (LUB or Join or Supremum)

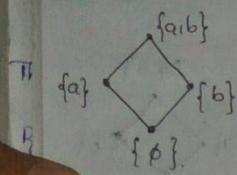
Let $[A, R]$ be a poset. For $a, b \in A$, if there exists an element $c \in A$ such that aRc and bRc ,

such that aRd and bRd ,

ii) if there exists any other element $'c'$ such that (aRd) and (bRd) then (cRd) , then ' c ' is the LUB of ' a ' and ' b '.

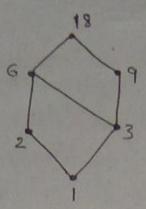


Ex:



$$\text{LUB} = \{a, b\}$$

$[D_{18}, /]$



$$\text{LUB} = \{18\}$$

⇒ In case of divisor operation $\text{LUB}(a, b) = \text{lcm}(a, b)$.

5. STANDARD E

a) $[A; \leq]$

$$\begin{matrix} \text{LUB} \\ \text{GLB} \end{matrix}$$

b) $[A; /]$

$$\begin{matrix} \text{LUB} \\ \text{GLB} \end{matrix}$$

c) $[S; \leq]$

$$\begin{matrix} \text{LUB} \\ \text{GLB} \end{matrix}$$

For a given pa

4. $[A; \leq]$

In case of \leq Relationship if we are comparing (a, b) then the LUB will be $\max(a, b)$. $\text{LUB}(a, b) = \max(a, b)$

⇒ In this diagram $\text{LUB} = \{4\}$.

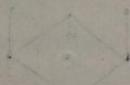
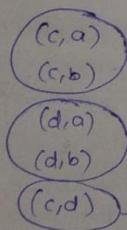
⇒ In case of set Inclusion Relation (subset or equal to Relation) then union of two sets is going to be (LUB). $\text{LUB}(a, b) = \{a\} \cup \{b\}$

4. GLB OR MEET

Let $[A; R]$ be a poset, for $a, b \in A$ if there exists an element 'c' such that

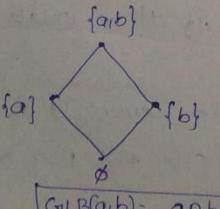
i) (cRa) and (cRb) and

ii) If there exists any other element 'd' such that (dRa) and (dRb) then (dRc) then 'c' is called the GLB of 'a' and 'b'.



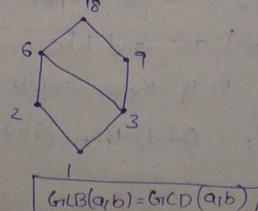
$$\text{GLB}(a, b) = \min(a, b)$$

\leq Relationship



set Inclusion

Subset / equals



$$\text{GLB}(a, b) = \text{GCD}(a, b)$$

6. LATTICE

Join semi lattice:

Meet semi lattice

Lattice: If Both

Ex: $A = \{1, 2, 3, \dots, 10\}$

S = $\{\{a\}, \{b\}, \dots\}$

A = $\{1, 2, 3, 4, \dots\}$

Subset relationship

Intersection relationship

→ To record all the

relationship between

A TOS (Total

order relationship)

join and meet

relationship

Ex: $D_6 = \{1, 2, 3, 6\}$

$$D_2 = \{1, 2, 3, 4\}$$

$$D_{30} = \{1, 2, 3, 5, 6\}$$

Every $[D_n, /]$ is a

→ If $P(A)$ denotes po

STANDARD EXAMPLES.

i) $[A; \leq]$ LUB $(a, b) = \max(a, b)$
 GLB $(a, b) = \min(a, b)$

$A = \text{set of Real numbers}$

ii) $[A; /]$ LUB $(a, b) = \text{Lcm}(a, b)$
 GLB $(a, b) = \text{GCD}(a, b)$

$A = \text{set of Real nos.}$

iii) $[S; \subseteq]$ LUB = Union
 GLB = Intersection.

b) then

for a given pair of elements the LUB, GLB may or may not exist.

6. LATTICE

Join semi lattice: If LUB exists for every pair of elements in poset.

Meet semi lattice: If GLB exists for every pair of elements in poset.

Lattice: If Both LUB and GLB " " " " " "

→ GLB exists for every pair.

such that

Ex: $A = \{1, 2, 3, \dots, 10\}$ $[A, /]$ is meet semi lattice [For $(3, 4)$, there is no.

LUB = $(\text{Lcm}(3, 4)) = 12$ and 12 is not in set, so not LUB \Rightarrow GLB].

$S = \{\{a\}, \{b\}, \{a, b\}\}$ then $[S, \subseteq]$ is join semi lattice

[GLB does not exist]

$A = \{1, 2, 3, 4\}$ $[A, \leq]$ is a lattice.

for $\{a\} \{b\}$ (Intersection of $\{a\} \{b\} = \emptyset$ (not in set)).

↓
 $\begin{array}{c} 4 \\ | \\ 3 \\ | \\ 2 \\ | \\ 1 \end{array}$
 $a \wedge b = (\max(a, b))$
 $a \vee b = \min(a, b)$

A TOS (Total Order Set) is always a lattice.

7. LATTICE EXAMPLES.

→ If A' is set of all +ve integers, then poset $[A'; /]$ is a lattice

$\begin{cases} (a, b) \Rightarrow \text{Lcm} = +ve \\ (a, b) \Rightarrow \text{GCD} = +ve \end{cases}$
 $\therefore \text{LUB/GLB exists}$

→ If 'n' is a positive integer then $D_n = \text{set of all +ve divisors of } n!$

Ex: $D_6 = \{1, 2, 3, 6\}$.

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

Every $[D_n, /]$ is a lattice.

→ If $P(A)$ denotes powerset of A then $[P(A); \subseteq]$ is a lattice.

$A = \{a, b\}$
 $P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

8. PROPERTIES OF LATTICE

BASIC PROPERTIES

10. BOUNDED

Let L be a lattice such that a

similarly if $'0'$ is called

In a lattice
Bounded lo

The following property holds good in a lattice for any 3 ele $a, b, c \in A$

i) Commutative law: $a \vee b = b \vee a$

$$a \wedge b = b \wedge a$$

ii) Associative law:

$$(a \vee b) \vee c = a \vee (b \vee c)$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

iii) Idempotent law: $a \vee a = a$

$$a \wedge a = a$$

iv) Absorption: $a \vee (a \wedge b) = a$

$$a \wedge (a \vee b) = a$$

v) Note: In a lattice $(a \vee b) = b$ iff $(a \wedge b) = a$, i.e., $b \leq a$

$$\begin{cases} V = \text{LUB} \\ A = \text{GLB} \end{cases}$$

9. DISTRIBUTIVE LATTICE AND SUBLATTICE

The lattice on which the distributive property holds is called Distributive \Rightarrow Every finite lattice, and the distributive properties are

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\begin{cases} V = \text{LUB} \\ A = \text{GLB} \end{cases}$

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $\begin{cases} V = \text{LUB} \\ A = \text{GLB} \end{cases}$

Sublattice:

Let L be a lattice $[L, \wedge, \vee]$. A subset ' M ' of ' L ' is called a sublattice of ' L ' iff

i) M is a lattice i.e. $[M, \wedge, \vee]$

ii) For any pair of elements $a, b \in M$, the LUB and GLB are same in M and L .

To find whether a lattice is distributive there are two ways

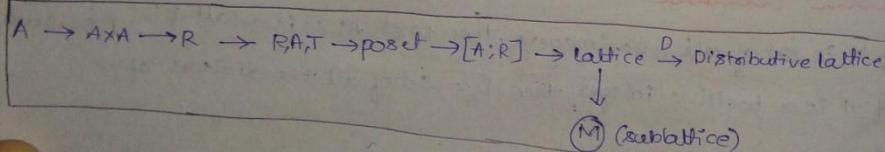
1) Take all possible Triplets and check distributive properties (Headache)

2) Use the process of concept of sublattices

\Rightarrow For a lattice

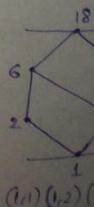
$$[I, \leq]$$

Set of Integers



$$\Rightarrow [D_{18}; \mid]$$

$$D_{18} = \{1, 2, 3, 6, 9, 18\}$$



10. BOUNDED LATTICE

Let L be a lattice with respect to \leq , if there exists an element $I \in L$ such that $(aI) \forall a \in L$, then I is called "Upper bound of Lattice L ".

Similarly if there exists an element $O \in L$, such that $(Oa) \forall a \in L$, then O is called "Lower bound of Lattice L ".

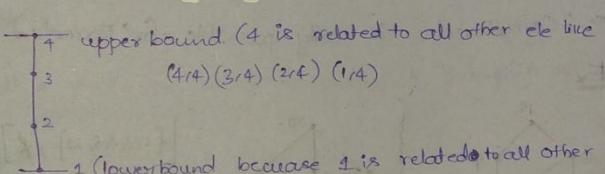
In a lattice if upper bound and lower bound exists then it is called Bounded lattice.

LUB, GLB } → Calculated on pair of elements

Upper Bound } → Calculated on entire lattice on pair of elements
Lower Bound }

Distributive → Every finite lattice is Bounded.

$$\{ \{1, 2, 3, 4\}, \leq \}$$



1 (lower bound because 1 is related to all other elements)

on a lattice

there are

stable Triplets

fibutive

(headache)

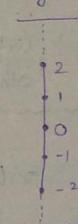
cess of /

sublattices

→ for a lattice there may/maynot be upper bound and lower bound.

$$[I, \leq]$$

Set of Integers

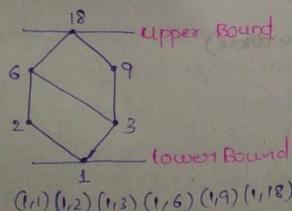


No upperbound and lowerbound.

lattice

$$\rightarrow [D_{18}; \mid]$$

$$D_{18} = \{1, 2, 3, 6, 9, 18\}$$



II. PROPERTIES OF BOUNDED LATTICE

In a Bounded lattice, the following properties holds good.

- 1) LUB of a and I i.e $a \vee I = I \{ (a, I) \}$ $\{ I = \text{Upper bound of lattice} \}$
- 2) GLB of a and I i.e $a \wedge I = a \{ (a, I) \}$ $\{ I = \text{Upper bound of lattice} \}$
- 3) LUB of a and o i.e $a \vee o = a \{ (a, o) \}$
- 4) GLB of a and o i.e $a \wedge o = a \{ (o, a) \}$

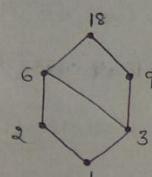
12. COMPLEMENT OF AN ELEMENT

Let L be an bounded lattice, for any element $a \in L$, if there exists

1. an element $b \in L$, such that $(a \vee b) = I$ and $(a \wedge b) = o$, then b is called 'Complement of a ' written as \bar{a} . and a, \bar{a} are complements of each other.

⇒ "Complement" is only possible for "Bounded lattice".

Ex:



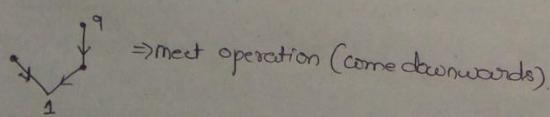
$(2, 9)$ are complement to each other

$$\Rightarrow (2 \vee 9) = \text{Join operation} = 18 \text{ (UB)}$$

$$(2 \wedge 9) = \text{meet operation} = 1 \text{ (LB)}$$

$$2 = \bar{9} \text{ or } 9 = \bar{2}$$

$(2, 9)$ are complement



$\{1, 2, 3, 5, 6, 15, 30\}; I$

$(2 \vee 3) = 6$ (Join operation)

$(2 \wedge 3) = 1$ (Meet operation)

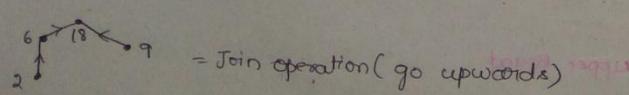
$\Rightarrow (2, 3)$ are complement of each other $\Rightarrow \bar{2}=3$ or $\bar{3}=2$

$\Rightarrow (2, 3)$ are not complement to each other because

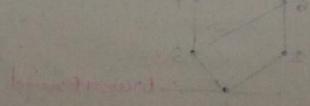
$$(2 \vee 3) = 6 \text{ (Not UB)}$$

$$(2 \wedge 3) = 1 \text{ (LB)}$$

both UB & LB.



= Join operation (go upwards)



= Meet operation (go downwards)

13. COMPLEMENT

⇒ If every lattice.

⇒ If every lattice.

⇒ In a Distributive lattice.

$\{1, 2, 3, 6\}; I$

$\{1, 2, 3, 5, 30\}; I$

⇒ In a Distributive lattice.

14. BOOLEAN ALGEBRA

Boolean Algebra

A lattice L is complemented

⇒ In Boolean Algebras Unique Complement

15. MAXIMAL AND MINIMAL ELEMENTS

Maximal element: Other element, than

Minimal element: It is called mini-

Ex: $A = \{a, b\}$.

$[P(A); \subseteq]$

$\{a\}$

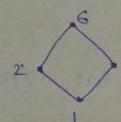
13. COMPLEMENTED LATTICE

⇒ If every element of a lattice has complement then it is called Complemented lattice.

⇒ In a complemented lattice, each element has at least one complement.

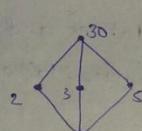
⇒ In a Distributive lattice, complement of an element if exists, is unique i.e. each element has at most one complement.

$$[\{1, 2, 3, 6\}; \leq]$$



→ Complemented lattice every ele has complement.

$$[\{1, 2, 3, 5, 30\}; \leq]$$



→ Complement lattice

⇒ In a Distributive lattice each ele has (0 complement or 1 complement).

14. BOOLEAN ALGEBRA

Boolean Algebra

A lattice L is called Boolean Algebra if it is distributive and complemented.

⇒ In Boolean Algebra every element has at most one complement, i.e. unique complement.

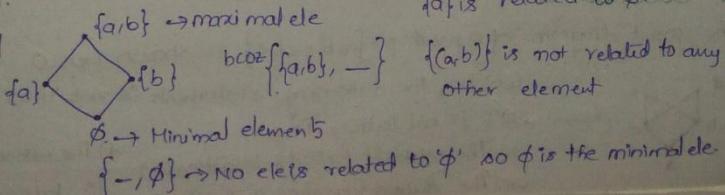
15. MAXIMAL AND MINIMAL ELEMENTS.

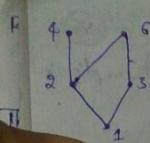
Maximal element: If in a poset, an element is not related to any other element, then it is called maximal element.

Minimal element: If in a poset, an ele is related to no element, then it is called minimal element.

Ex: $A = \{a, b\}$.

$$[P(A); \leq]$$

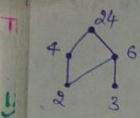




$[1, 2, 3, 4, 6]; 1]$

Maximal elements = 4, 6
Minimal elements = 1

This is not a lattice because
 $\{4, 6\}$ do not have a LUB because
and it is Meet Semilattice



Maximal elements = 4.
Minimal elements = 2, 3.

Join semilattice

The upper bound and lower bound are unique but maximal/minimal ele need not be unique.

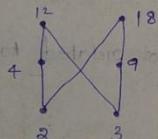
⇒ Maximal, Minimal are applicable to both lattices and posets but UB, LB can be / must be applied only for Bounded lattices.

⇒ In a poset if you have more than one maximal/minimal ele then it won't be lattice.

16. Example 1

The poset $[2, 3, 4, 9, 12, 18]; 1]$ is

- a) Join semilattice but not meet semilattice
- b) Meet semilattice but not Join SL
- c) A lattice
- d) Neither join nor meet semilattice.



Maximal ele = 12, 18

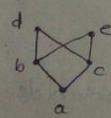
Minimal ele = 2, 3. ∴ This is not lattice

And there is no GLB for $2, 3$ and $12, 18$. They aren't LUB for $12, 18$.

Join and meet SL

17. Example 2

The poset diagram of a poset $P = \{a, b, c, d, e\}$ is shown below.



which of the following statements is not TRUE (FALSE)?

a) P is not lattice (True)

b) The subset $\{a, b, c\}$ is lattice (T)

d) The subset $\{a, b, c, e\}$ of P is lattice (F)

e) The subset $\{b, c, d, e\}$ is lattice (F)

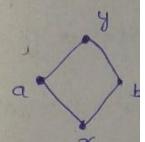
- a) TRUE because
- b) TRUE all
- c) FALSE (d is

18. Example - 3

The Hasse dia which of the follo

- a) $\{x, a, b, y\}$
- b) $\{x, a, c, y\}$
- c) $\{x, c, d, y\}$

- a) $\{x, a, b, y\}$



⇒ lattice because every pair of elem LUB, GLB exists.

⇒ For sublattice f LUB of every pa the options, Here, GLB and LUB a but in original di LUB (a, b) = c (\neq cannot be the so

- Ques:
- TRUE because (d,e) do not have least upper bound.
 - TRUE all pairs have LUB, GLB
 - FALSE (d,e) has no LUB and (b,c) do not have GLB

Ans: ①

Ques 18

Ques 36

18. Example - 3

The Hasse diagram of a lattice $L = \{x, a, b, c, d, e, y\}$ is shown below which of the following subsets of L are sublattices of L ?

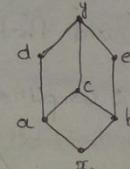
a) $\{x, a, b, y\}$ b) $\{x, a, c, y\}$

GLB = \underline{x}

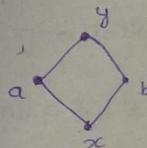
c) $\{x, a, e, y\}$ d) $\{x, d, e, y\}$

LUB = \underline{y}

e) $\{x, c, d, y\}$.

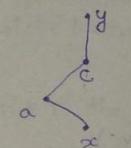


a) $\{x, a, b, y\}$



\Rightarrow lattice because for every pair of elements both exists for every pair LUB, GLB exists.

b) $\{x, a, c, y\}$.



\Rightarrow Lattice because for every pair of elements both exists for every pair LUB, GLB exists.

\Rightarrow For sublattice find the GLB, LUB of every pair of ele in the options. Here, for (a, b) the

GLB and LUB are x, y respectively



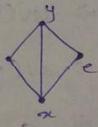
\Rightarrow lattice for (a, e) GLB = x in original diagram
LUB = y

For (a, y) GLB = a

LUB = y .

\therefore This is the sublattice

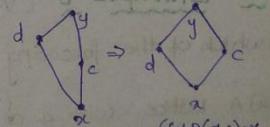
d) $\{x, d, e, y\}$.



\Rightarrow lattice

\Rightarrow sublattice also.

e) $\{x, c, d, y\}$



\Rightarrow (GLB(d, c) = x)
(LUB(d, c) = a)

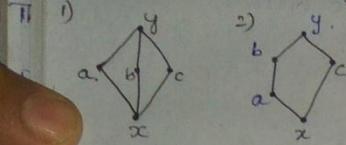
\downarrow

(original diagram)

\therefore Not a sublattice

19. Example-4 (V. Imp Question)

Which of the following lattices is not distributive



⇒ If a lattice is distributive then we should have atmost one complement for each element so in the first diagram 'a' has two complements 'b' and 'c' and therefore it cannot be distributive

⇒ In the 2nd Graph 'c' has 2 complements 'a' and 'b' so the 2nd diagram is also not distributive

Let L_1^* represent diag 1 \rightarrow L_2^* represent diag 2.

G.L.B. $\Leftarrow \wedge = \text{see down}$
L.U.B. $\Leftarrow \vee = \text{see up}$

⇒ To check whether

L_1^* are not the given lattice

⇒ In $L_1^* \Rightarrow a \vee (b \wedge c) \stackrel{?}{=} (a \vee b) \wedge (a \vee c)$

$$\begin{aligned} &\Rightarrow a \vee (a) \quad \left\{ \begin{array}{l} (y) \wedge (y) \\ = y \end{array} \right. \\ &\Rightarrow a \quad \Rightarrow (a \# y) \therefore \text{The distributive property does not hold true} \end{aligned}$$

⇒ In $L_2^* \Rightarrow a \vee (b \wedge c) \stackrel{?}{=} (a \vee b) \wedge (a \vee c)$

$$\begin{aligned} &\Rightarrow a \vee (a) \quad \left\{ \begin{array}{l} (b) \wedge (y) \\ = b \end{array} \right. \\ &\Rightarrow a \quad \Rightarrow (a \# b) \Rightarrow \text{The distributive property does not hold true for this lattice} \end{aligned}$$

20. Example-5 (V. Imp Question)

which of the following statements are not true

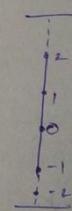
- a) A lattice with 4 or fewer elements is distributive (TRUE)
- b) Every totally ordered set is a distributive lattice (TRUE) \rightarrow
- c) Every sublattice of a distributive lattice is also distributive (TRUE)
- d) Every distributive lattice is a bounded lattice

Max=4
min=1

No isomorphic structures exist similar to L_1^* or L_2^* .

⇒ we have already seen L_1^* , L_2^* as sublattice and they are not distributive lattices. So if a lattice contains L_1^* or L_2^* as sublattice

d) $[R; \leq]$



b) $D_{81} = [1, 3, 9, 27, 81]$

For every val

~~☒☒☒☒☒~~

then that lattice won't be ~~sub~~ distributive lattice.

d) NO Relation b/w Distributive and Bounded lattices (FALSE).

Example-6

Which of the following is not distributive lattice?

one

two

butive

diagram

down

up

i) $[P(A), \subseteq]$ where $A = \{a, b, c, d\}$.

ii) $[D_{81}; \mid]$

iii) $[R; \leq]$ R is set of Real numbers

iv) $[\{1, 2, 3, 5, 30\}; \mid]$

To check whether a lattice is distributive or not we check if L_1^* and

L_2^* are not sublattices of given lattice, if they are sublattices then the given lattice is not distributive.

i) $[P(A), \subseteq]$ $A = \{a, b, c, d\}$ w.r.t the join of two elements is nothing but union and meet of two elements is nothing but intersection and we know that on a set of all sets Union is distributive over Intersection and Intersection is distributive over Union. Therefore the option a is.

Distributive

$$D_{81} = [\{1, 3, 9, 27, 81\}; \mid]$$

For every value of 'n' $[D_n; \mid]$ is always distributive



ii) $[R; \leq]$
This is a Total order Relation and w.r.t every Total order Relation is distributive.

iii) $[\{1, 2, 3, 5, 30\}; \mid]$

~~D_{30}~~

Isomorphic to L_1^* so this lattice is Not distributive. and $(2, 3)(2, 5)$ are Compliment. '2' has two complements.

Max=4
min=1

RUE)
isomorphic
dures exist
 L_1^* or L_2^* .
not
sublattice

22. Example-7

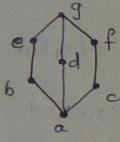
For the lattice $[D_8; \sqcup]$ which of the following is not True.

- T a) The complement of 1 = 18
- R b) The complement of 2 = 9
- T c) The complement of 3 = 6 ($3 = \overline{2}$)
- D d) The complement of 5 does not exist

$$\begin{aligned} & D_8 = \{1, 2, 3, 6, 9, 18\} \\ & \text{Answering PHT TOTALLY} \\ & \begin{array}{c} 18 \\ | \\ 6 \quad 9 \\ | \quad | \\ 2 \quad 3 \\ | \\ 1 \end{array} \\ & (6 \vee 9) = 18 \in \text{LUB} \\ & (6 \wedge 9) = 3 \text{ (Not GLB)} \} \text{ CA are not complement} \\ & \downarrow \\ & \{ \text{GLB} = 18 \text{ in the lattice} \} \end{aligned}$$

23. Example-8

For the lattice given below, how many complements does the ele 'e' have?



$$\begin{array}{l} (e,a) \\ (e,b) \\ (e,c) \\ (e,d) \\ (e,f) \\ (e,g) \end{array} \left. \right\}$$

These are the combinations possible

Now, The upper bound of lattice = g

lower bound of lattice = a.

$$\begin{array}{l} (e \wedge a) = a \\ (e \wedge b) = e \end{array} \left. \right\} (e,a) \text{ Not Complement}$$

$$\begin{array}{l} (e \wedge b) = b \\ (e \wedge c) = e \end{array} \left. \right\} \text{ Not complement.}$$

$$\begin{array}{l} (e \wedge d) = a = \text{LB of lattice} \\ (e \wedge d) = g = \text{UB of lattice} \end{array} \left. \right\} (e,d) \text{ are complement.}$$

$$\begin{array}{l} (e \wedge f) = a \\ (e \wedge f) = g \end{array} \left. \right\} (e,f) \text{ are complement}$$

Similarly (e,c) are complement to each other.

1. ALGEB

A Non-empty
Binary operat

$$2) S = \{1, -\}$$

$$* \rightarrow ?$$

$$(S, *)$$

$$2) S = \{\phi, \{ \}$$

$$* = \cup$$

Now, (S, \cup)

$$3) A = \{1, 2, 3\}$$

$R = \text{Reflexive set of all}$

$$(R, U) = A$$

$$(R, N) = A$$

$$4) (R, +) \Rightarrow "$$

$$5) (N, *) \Rightarrow "$$

$$6) (S = \{1, 2, 3\})$$

$* \rightarrow \text{mult}$

Now, $\Delta *$

$$7) [2, 1] \Rightarrow$$

4. Groups.

1. ALGEBRAIC STRUCTURES

A Non-empty set 'S' is called an Algebraic structure with respect to binary operation * if $(a * b) \in S \forall a, b \in S$ i.e. * is closure operation on 'S'.

1) $S = \{1, -1\}$.

* → \times (multiplication)

$(S, *)$ is Algebraic structure because $(1) \times (-1) = -1$ (present in 'S')

$$-1 \times -1 = 1 \quad (\in S)$$

$$1 \times 1 = 1 \quad (\in S)$$

∴ 'S' is called the Algebraic structure.

2) $S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

* = \cup (union operation)

Now $(S, *)$ is A.S. because

$$\emptyset \cup \{a\} = \{a\} \in S.$$

$$\{a\} \cup \emptyset = \{a\} \in S.$$

$$\emptyset \cup \{b\} = \{b\} \in S.$$

$$\{b\} \cup \emptyset = \{b\} \in S.$$

$$\emptyset \cup \emptyset = \emptyset \in S.$$

3) $A = \{1, 2, 3\}$

$R = \downarrow$ Reflexive Relations = $\{(1, 1), (2, 2), (3, 3)\}$.

Set of all

$$(R, \cup) = A.S.$$

$$(R, \cap) = A.S.$$

The set of all Reflexive Relations are closed under union and therefore set of all

Reflexive Relations w.r.t to Intersection/Union is an "ABELIAN STRUCTURE"

4) $(R, +) \Rightarrow$ "ABELIAN STRUCTURE" ($R = \text{Real nos.}$)

5) $(N, *) \Rightarrow$ "ABELIAN STRUCTURE" ($N = \text{Natural numbers}$)

6) $(S = \{1, 2, 3\}) \Rightarrow$ (NOT Algebraic Structure)

* → multiplication

Now, $2 * 3 = 6$ (Not in 'S'). So,

7) $[2, 1] \Rightarrow$ Not Algebraic structure.

$$2/3 = 0.6 \notin \text{Integer}$$

2. SEMI GROUP

An Algebraic structure $(S, *)$ is called a Semigroup if $(a * b) * c = a * (b * c)$ $\forall a, b, c \in S$, i.e., $*$ is associative on S .

Ex:

- 1) $(N, +) =$ Algebraic Structure (\checkmark) $\rightarrow (a+b)+c = a+(b+c)$ Natural no. = Natural No.
- 2) $(N, *) =$ Algebraic Structure (\checkmark) $\rightarrow (a * b) * c = a * (b * c) \therefore$ SEMI GROUP
- 3) $(Z, -) = Z = \{ \text{Integers Set} \}$ AS(\checkmark) $\xrightarrow{(a-b)-c = a-(b-c)}$ NOT SEMI GROUP
- 4) $(Q^*, +) = Q = \{ \text{Rational No.'s, not having '0'} \}$ AS(\checkmark) $a+(-a)=0$ $\xrightarrow{1, 2, 3} \neq$ \therefore NOT Rational.
- 5) $(Q^*, *) =$ Semi Group
- 6) $(P(A), \cup) =$ Semi Group
- 7) $(P(A), n) =$ Semi Group

3. MONOID

A semigroup $(S, *)$ is called Monoid if there exists an element $e \in S$ such that $(a * e) = (e * a) = a \forall a \in S$. ($*$ = operation defined)

The element 'e' is called Identity element of 'S' w.r.t. *.

\Rightarrow If a semi Group contains Identity element(s) then it is a Monoid.

Ex:

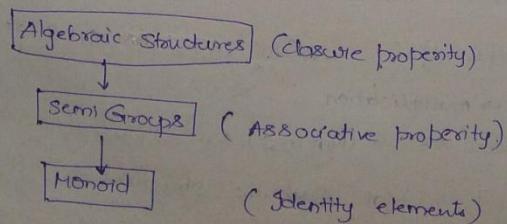
$$(N, *) = \text{Semigroup } (\checkmark) \quad a * e = a \quad \boxed{e=1} \quad \therefore (N, *) \text{ is Monoid}$$

and '1' is the identity element of N w.r.t. *.

$$(N, +) = \text{AS } (\checkmark) \quad \text{SG } (\checkmark) \quad a + e = a \quad \boxed{e=0 \neq N} \quad \therefore \text{This is not Monoid.}$$

$$(Z, +) = \text{AS } (\checkmark) \quad \text{SG } (\checkmark) \quad a + e = a \quad \boxed{e=0} \quad \therefore \text{Monoid}$$

$$(P(A), \cup) = \text{AS } (\checkmark) \quad \text{SG } (\checkmark) \quad X \cup e = X \quad \boxed{e=\emptyset \in P(A)} \quad \therefore \text{Monoid}$$



4. GROUP

A monoid $(S, *)$ each element $(b * a) = e$ then

Ex

$$(Z, +) = \text{Monoid}$$

$$(Q, \cdot) = \text{Multip}$$

$$(Q^*, \cdot) \Rightarrow Q^* = \{ \text{Set of} \}$$

note

$$(P(A), \cup) = X \cup \{\emptyset\}$$

5. ABELIAN GROUP

- \rightarrow In a Group (i) The Identity
- (ii) The Inverse
- (iii) The Inverse
- (iv) Cancellation law

$$v) (a * b)^{-1} = b^{-1} * a^{-1}$$

6. EXAMPLE 1

Abelian Group (com)

1) Group $(G, *)$ is

Ex:

$$(Z, +) = \text{Group } (\checkmark) \quad a-$$

$$(Q^*, \cdot) = \text{Group } (\checkmark) \quad a-$$

$$(M, *) = \text{Group } (\checkmark) \quad a-$$

$$m = \{ \text{Set of all} \}$$

$$a * b) * c$$

4. GROUP

A monoid $(S, *)$ with identity element 'e' is called a group if to each element $a \in S$, there exists an element $b \in S$, such that $(a * b) = (b * a) = e$ then 'b' is called Inverse of an element a , denoted by a^{-1} .

Natural No.
 $a + (b + c)$

SEMI GROUP
OT
SEMI GROUP

Ex

$$(Z, +) = \text{Monoid}(V) = a + b = b + a = 0 \text{ (Identity element)}.$$

For every ele
there is inverse

$$= \boxed{a = -b} \text{ and } (-b) \in Z \therefore \text{Group}$$

$$(Q, \cdot) \cdot = (\text{Multiplication}) = a * b = 1 \therefore \text{This is a Group}$$

Not

$$\Rightarrow \boxed{b \cdot (a) \neq 0} \text{ if } (a = 0)$$

$$(Q^*, \cdot) \Rightarrow Q^* = (\text{set of all Rational nos without } 0)$$

$$\Rightarrow \boxed{a = \frac{1}{b}} \quad \boxed{b = \frac{1}{a}} \in Q^*$$

$\cdot = \text{Multiplication}$ \Rightarrow This is a Group because the set Q^* does not have 0.

lement $e \in S$.

$$(P(A), \cup) = X \cup \{\emptyset\} = \{\emptyset\} \text{ (identity element)} \} = \text{Group.}$$

$\emptyset \in P(A)$

Monoid:

5. ABELIAN GROUP.

→ In a Group $(G, *)$ the following properties must hold good

Monoid

creating
the
juence

i) The Identity element of G is unique

ii) The Inverse of any element in G is unique

iii) The Inverse of identity element 'e' is 'e' itself

iv) Cancellation laws $(a * b) = (a * c) \Rightarrow b = c$

$(a * c) = (b * c) \Rightarrow a = b$

6. EXAMPLES

6. EXAMPLE 1

Abelian Group (commutative Group)

A group $(G, *)$ is said to be Abelian if $(a * b) = (b * a) \forall a, b \in G$.

Ex:

$(Z, +) = \text{Group}(V) \quad a + b = b + a \in Z \Rightarrow \text{Abelian Group}$

$(R^*, \cdot) = \text{Group}(V) \quad a * b = b * a \in R^* \Rightarrow \text{Abelian Group.}$

$(M_n, *) = \text{Group}(\text{set of all non singular matrices}) \rightarrow (*) = \text{matrix multiplication} \Rightarrow \text{NOT Abelian Group.}$

+ Monoid.

monoid

7. EXAMPLE 2

Which of the following is/are True?

- II D) In a group $(G, *)$ if an identity element 'e' if $a * a = a$ then $a = e$ with
- F) 2) In a Group $(G, *)$ if $x^{-1} = x \forall x \in G$ then G is Abelian Group.
- T) 3) " " " ", $(a * b)^{-1} = a^{-1} * b^{-1} \forall a, b \in G$ then G is Abelian Group.

\Rightarrow Use the formulae in 5th video. Now, $a * a = a$

- 1) $a * a = a * e$ (I can write a as $a * e^{\circ}$ because e is identity element)

$$a = e \quad \text{TRUE.}$$

$$\begin{aligned} 2) \quad x^{-1} &= x & (a * b)^{-1} &= b^{-1} * a^{-1} \\ x^{-1} &= x * e & (a * b) &= (b * a) \Rightarrow \text{Given } x^{-1} = x \text{ so } (a * b)^{-1} = (a * b) \\ & & & b^{-1} = b \\ & & & a^{-1} = a \end{aligned}$$

$$(13) \quad a^2 = a * a \quad \text{write the operation specified.}$$

$$\begin{aligned} & \Rightarrow (a * b)^2 = a^2 * b^2 \\ & = (a * b) * (a * b) \quad (a * b) * (a * b) = a * a * b * b. \quad \left| \begin{array}{l} a * (b * a) * b = \\ a * a * b * b \\ (b * a) = (a * b) \end{array} \right. \\ & = (a * a) * (b * b) = (a * a) * (b * b) = \text{LHS} = \text{RHS.} \end{aligned}$$

8. EXAMPLE - 3

- I If $A = \{1, 3, 5, 7, 9, \dots\}$ and $B = \{2, 4, 6, 8, \dots\}$ which of the following is semigroup?

a) $(A, +) =$ Algebraic Structure (\times)

b) $(A, \circ) = AS(\vee) \quad SG(\vee) \quad Monoid (\vee) \quad a * \{x\} = 1 \quad \text{Example: } a * \{x\} = 1$

c) $(B, +) = AS(\vee) \quad SG(\vee)$

d) $(B, \circ) = AS(\vee) \quad SG(\vee)$

9. EXAMPLE

Let $A = \{1, 2, 3, \dots\}$ and $a, b \in A$ which

- a) $(A, *)$ is Semigroup
- b) $(A, *)$ is monoid
- c) $(A, *)$ is group
- d) $(A, *)$ is not a semigroup

10. EXAMPLE

Let $A = \{x / 0 < x < 1\}$ is

- a) A semigroup
- b) A monoid
- c) A group
- d) Not a semigroup

$$(A, *) \Rightarrow AS$$

11. EXAMPLE

Let A' is set

$$(a * b) = \min(a, b)$$

$$(z, \cdot) \Rightarrow$$

9. EXAMPLE-4

Let $A = \{1, 2, 3, 4, \dots, \infty\}$ and a Binary operation $*$ is defined by $a*b = a^b$. Which of the following is true?

P.

in Group.

- $(A, *)$ is Semigroup but not monoid AS(✓) SG(X)
- $(A, *)$ is monoid but not group
- $(A, *)$ is Group
- $(A, *)$ is not semigroup.

$$\begin{array}{rcl} a * (b * c) &=& (a * b) * c \\ 1 & 2 & 3 \\ = 1 * (2 * 3) &=& (1^2) * 3 \\ = 8 &=& (1 * 3) = 1^3 \\ & & = 1 \end{array}$$

$(8 \neq 1)$
so not SG

Identity element)

10. EXAMPLE-5

Let $A = \{x / 0 < x \leq 1 \text{ and } x \text{ is a real number}\}$ then A' w.r.t to multiplication is

$$x \in [0, 1]$$

- A semigroup but not monoid
- A monoid but not Group
- A group
- Not a semigroup

$(A, *) \Rightarrow AS(\checkmark)$, Semigroup. (✓)

$$\begin{aligned} (a * b) * c &= a * b * c \\ 0.1 * 0.1 &= 0.1 * 0.1 \\ &= (0.1)^2 = (0.1)^2 \end{aligned}$$

Monoid (✓) Group.

$$\begin{array}{l} e=1 \\ \text{Identity element} \\ \Rightarrow x = 1/a \end{array}$$

$\frac{1}{0.1} = 10 \notin A$ Group(X). not closing

11. EXAMPLE-6

Let A' is set of all integers and Binary operation $*$ is defined by $(a * b) = \min(a, b)$, then $(A, *)$ is Semigroup.

$(z, *) \Rightarrow$ Algebraic Structure(✓)

Semigroup (✓)

$$\begin{array}{rcl} (a * b) * c &=& a * (b * c) \\ 1 & 2 & 3 \\ = 1 * 3 &=& 1 * (2 * 3) \\ = 1 &=& 1 \end{array}$$

Monoid

$$\begin{array}{l} \{a * \{e\} = a\} \\ \{e = 1\} \text{ Identity element} \\ a * e = a \\ a * e = \min(a, e) \\ \downarrow \\ 101 * 100 = 100 \neq 101 \end{array}$$

$$\begin{aligned} a * b &\neq a \Rightarrow a * ? = ? \\ \Rightarrow a * ? &= ? \\ ? &\neq a \neq z \end{aligned}$$

Semigroup. Monoid
It is Monoid but not Group

P 16. FINITE GROUPS.

A Group with finite no. of elements is called finite Groups.

$O(G)$ → order of finite Group. (No. of ele in the Group)

Eg:

1) $(\{0\}, +) = (0+0)=0$, Identity element = '0'. (Monoid) Group (v)

→ whenever a group is having only one element then that ele will be the Identity element. (v.v. Grp).

2) $(\{1\}, *) = 1*1=1 \in \text{Set (AS)}(v)$.

$(1*1)*1 = 1*(1*1)(SG)(v)$

$a*e=a = 1*e=1 \Rightarrow e \in \text{Set (monoid)}(v)$

$a*x=e \Rightarrow x=1/a \quad a=1/x \Rightarrow x=1/e \therefore (\text{Group})$

3) $(\{1, -1\}, *) = \text{Group (v)}$

= composition table

= $(SG)(v)$

= Monoid (v) ($e=1$ Identity element)

= Group.

	1	-1
1	1	-1
-1	-1	1

$\therefore (\text{AS})$

4) $\{1, \omega, \omega^2\}, *$ =

1	ω	ω^2
ω	ω	ω^2
ω^2	ω^2	1
1	ω	ω^2

$\therefore (\text{AS})$

For all the ele we are able to find inverses \therefore Group

1	ω	ω^2
ω	ω	ω^2
ω^2	ω^2	1
1	ω	ω^2

$\therefore (\text{AS})$

ω is the inverse of 1

ω^2 is the inverse of ω

ω is the inverse of ω^2

17. EXAMP

If $G_1 = \{1, 3, 4, 6\}$

a) Their inv. of

b) The Inv. of

Now, $1 \times$

3x

5

7

\therefore Each

Q2 which of

a) $\{1, 2, 3, 4\}$

b) $\{0, 1, 2, 3, 4\}$

c) $\{1, 2, 3, 4, 5\}$

d) $\{1, 2, 3, 4, 5, 6\}$

a) $\{1, 2, 3, 4, 5\}$

= QX

\Rightarrow C

\Rightarrow (AS)

c) $\{1, 2, 3, 4, 5, 6\}$

\hookrightarrow S7 (A)

\therefore Group.

EXAMPLES ON FINITE GROUPS

If $G_1 = \{1, 3, 5, 7\}$ is a group wrt \times_8 , which of the following is not true?

(a) The inv of 1 is 1

(b) The inverse of 5 is 7

(c) The Inv of 3 is 3

(d) The inverse of 7 is 7.

Now, $1 \times 1 = 1$ (1 when divided by 8 gives '1' as remainder)

$3 \times 3 = 9$ ($9 \div 8 = 1$ (belong to G_1) and 9 is identity element)

$5 \times 5 = 25$ ($25 \div 8 = 1$ (identity element))

$7 \times 7 = 49$ ($49 \div 8 = 1$ (identity element)).

∴ Each element is inverse of itself.

(a)

	1	3	5	7	
1	1	3	5	7	1 is the inverse of 1
3	3	1	7	5	$3 \times 3 = 9 \div 8 = 1$
5	5	7	1	3	$5 \times 5 = 25 \div 8 = 1$
7	7	5	3	1	$7 \times 7 = 49 \div 8 = 1$

How did I write this? $(7 \times 5) \div 8 = 35 \div 8 = 3$.

(b) Which of the following is a Group?

a) $\{1, 2, 3, 4\}$ wrt \times_6

we know that

b) $\{0, 1, 2, 3, 4, 5\}$ wrt \times_6

\oplus_m contains $0, 1, 2, \dots, (m-1)$ elements

c) $\{1, 2, 3, 4, 5, 6\}$ wrt \times_7

\otimes_m contains ' m ' elements.

d) $\{1, 2, 3, 4, 5, 6\}$ wrt \oplus_7

e) $\{1, 2, 3, 4, 5\}$ wrt \times_6

f) $\{0, 1, 2, 3, 4, 5\}$ wrt \times_6

$$= (2 \times 3) \div (6k) \Rightarrow \text{No. Inverse to '0' and we cannot get}$$

$\Rightarrow 0 \notin \text{set}$

Identity element so does not form Group.

g) $(AS) \times$.

h) $\{1, 2, 3, 4, 5, 6\}$ wrt \oplus_7

$(0, 6)$ must be present

'0' is missing.

Monoid property fails

i) $\{1, 2, 3, 4, 5, 6\}$ wrt \times_7

$\underbrace{\text{Set of all nos less}}_{7}$

than 7 and are relatively prime to 7).

j) Group.

18. EXAMPLES ON FINITE GROUPS

1) $(\{0, 1, 2, \dots, (m-1)\}, \oplus_m)$ Addition modulo m (This will always be a group)

$$a \oplus_m b = g_f(a+b) < m \Rightarrow (a+b) \text{ is the result}$$

$$g_f(a+b) > m \Rightarrow (a+b) \% m \text{ is the result}$$

$(\{0, 1, 2\}, \oplus_3) \Rightarrow$ This is a group.

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$$\{0, 1, 2\} \text{ is a set} \therefore \text{AS} \vee (\text{SG}) \vee (\text{m}) \vee (\text{C}) \vee$$

\Rightarrow For getting the identity element check the row containing $\{0, 1, 2\}$ (in the same order) then that element will be Identity element here '0' row has $\{0, 1, 2\} \therefore 0$ is the identity element

\Rightarrow Here In this case check for ds \leftarrow

0	1	2
0	0	1
1	1	2
2	2	0

$$1 \oplus_m 2 = 0 \Rightarrow 2 \text{ is the inverse of } 1$$

$$\therefore 1 \text{ is the inverse of } 2.$$

$$ax=ea$$

$$ax=a$$

$$a^{-1}=0$$

$$2) (S_m, \otimes_m) \Rightarrow [a \otimes_m b = (a \cdot b) \% m]$$

Set of all nos that are less than m and relatively prime to m .

$$S_{10} = \{1, 3, 7, 9\} \therefore S_{10} \subseteq \{1, 3, 7, 9\}$$

Two nos are said to be Relatively prime if $\text{GCD}(a, b)=1$

$$\text{Now, } \{S_{10}, \otimes_{10}\} = \text{Group.}$$

$$= \{S_{10}, \otimes_{10}\} = \{1, 3, 7, 9\}, \otimes_{10} = \text{Group.}$$

19. ORDER

Order of an

Let $(G, *)$ be
smallest pos

Ex:

$$1) \{1, -1\}$$

Now,

\Rightarrow order of

$$2) \{1, \omega, \omega^2\}$$

$$1^1 = 1$$

$$\omega^3 = 1$$

$$\omega \times \omega^2$$

$$\therefore \{1, \omega, \omega^2\}$$

\Rightarrow Order of

$$3) \{1, 3, 7, 9\}$$

Now, $0^1 =$

$$1 \oplus_3$$

$$2 \oplus_3$$

In a Gr

19. ORDER

Order of an element of a group

Let $(G, *)$ be a Group and $a \in G$, then order of element a is the smallest positive integer n such that a^n is identity element.

Ex:

1) $(\{1, -1\}, *)$ Identity element = 1

Now, $(-1)^2 = 1$ = identity element \therefore order of $(-1) = 2$

$(-1)^4 = 1$ 4 cannot be order we should always take least no.

\Rightarrow Order of Identity element is always one.

2) $(\{1, \omega, \omega^2\}, *)$ Identity element = 1

$1^1 = 1 \Rightarrow$ order of 1 = 1

$\omega^3 = 1 \Rightarrow$ order of $\omega = 3$

~~$\omega \times \omega \times \omega = 1$~~ order of $\omega^2 = 3$. Now, $(\omega^2)^3 = \omega^6 = (\omega^3)^2 = 1$

$$\therefore [O(\omega^2) = 3 \quad O(\omega) = 3 \quad O(1) = 1]$$

\Rightarrow Order of any element divides the order of group (Finite Group)

3) $(\{0, 1, 2\}, \oplus_3)$ Identity element = 0

Now, $0^1 = 0 \therefore O(0) = 1$

$$1 \oplus_3 1 \oplus_3 1 = 0 \Rightarrow [O(1) = 3] \Rightarrow [1^3 = 0]$$

$$2 \oplus_3 2 \oplus_3 2 = 0 \Rightarrow [O(2) = 3] \Rightarrow [2^3 = 0]$$

In a Group $O(a)$ and $O(a^{-1})$ are always same/equal

utive 1's

Q)

EXERCISE ON ORDER

①

utive 1's

Q)

encroaching

in the
sequence

20. EXAMPLES ON ORDER

state True/False

- T 1) In a group $(\mathbb{Z}, +)$ the order of any element except '0' does not exist.
- T 2) In the group (\mathbb{Q}^*, \cdot) where \mathbb{Q}^* is set of all Non-zero rational numbers.
- T 3) i.e. $\mathbb{Q}^* = \mathbb{Q} - \{0\}$, the order of any element except 1 does not exist.
1. 1) TRUE, because order of '0' is 1 (because '0' is Identity element)
and for other ele order does not exist
- 2) 1 is the Identity element so $O(1)=1$

$$\text{Now, } (-1)^2 = 1 \quad \therefore O(-1) = 2 \quad \therefore S_2 \text{ is FALSE.}$$

21. SUBGROUPS.

SUBGROUPS.

- Let $(G, *)$ be a group. A subset 'H' of 'G' is called a subgroup of 'G' if $(H, *)$ is a group.

Ex:

Let $(G, *)$ be a group with Identity element 'e', then $\{e\}$ and G' are the trivial subgroups of 'G'. Any subgroup which is not a trivial subgroup is called proper subgroup.

$$G = \{1, -1, i, -i\}, * \text{ then } H = \{1, -1\}, * \text{ is a proper subgroup.}$$

- Every Group is going to have atleast two subgroups if-
2> set containing Identity elements. (These are called Trivial subgroups)

The subgroups other than Trivial subgroups are called proper subgp.

22. THEOREM

Th 1: Let 'H'
iff $a * b^{-1} \in H$

Th 2: Let 'H'
of 'G' iff $(H, *)$

Th 3: Lagrange
{if 'H' is
of $O(G)$.
 $O(G) = m$
 $O(H) = n$

23. EXAMPLE

Let $G_1 = \{0, 1, 2, 3\}$
Subgroups of

- a) $H_1 = \{1, 3\}$
b) $H_2 = \{1, 5\}$
c) $H_3 = \{1, 3\}$
d) $H_4 = \{0, 2, 4\}$
e) $H_5 = \{0, 2, 3\}$

THEOREMS ON SUB GROUPS

Th1: Let 'H' be non empty subset of a group $(G, *)$. 'H' is a subgroup of G' if $a * b^{-1} \in H \forall a, b \in H$.

Th2: Let 'H' be non empty finite subset of a group $(G, *)$. 'H' is a subgroup of G' if $(a * b) \in H \forall a, b \in H$.

Lagrange's Theorem:

If 'H' is a subgroup of finite group $(G, *)$ then $\text{O}(H)$ is the divisor of $\text{O}(G)$. The converse of the above theorem need not be True.

$\begin{cases} \text{O}(G) = m \\ \text{O}(H) = n \end{cases} \Rightarrow m \text{ is divisible with/by } n. \text{ (This is what this theorem says). Ex: } \text{O}(G) = 10, \text{ O}(H) = 3 \Rightarrow H \text{ cannot be subgroup of } G \text{ because } 3 \text{ does not divide } 10.$

EXAMPLES ON SUBGROUPS - I

Let $G_1 = \{0, 1, 2, 3, 4, 5\}, \oplus_6$ is a group. Which of the following are subgroups of G_1 ?

$$i) H_1 = \{1, 3\} \Rightarrow \text{Now, } \{1, 3\} \oplus_6 (1+3) \oplus_6 = 4 \text{ mod } 6 = 4 \neq H_1$$

$$ii) H_2 = \{1, 5\} \Rightarrow \text{Now, } \{1, 5\} \oplus_6 (1+5) \oplus_6 = 0 \neq H_2$$

$$iii) H_3 = \{1, 3\} \Rightarrow \text{Now, } \{1, 3\} \oplus_6 = (4) \text{ mod } 6 = 4 \neq H_3.$$

$$iv) H_4 = \{0, 2, 4\} \Rightarrow \text{Now, } \{0, 2, 4\} \oplus_6 = \begin{cases} 2 \text{ mod } 6 = 2 \\ 0 \text{ mod } 6 = 0 \\ 4 \text{ mod } 6 = 4 \end{cases} \left. \begin{array}{l} (0+2) \text{ mod } 6 = 2 \\ (0+4) \text{ mod } 6 = 4 \\ (2+4) \text{ mod } 6 = 0 \end{array} \right\} \in H_4$$

$$v) H_5 = \{0, 2, 3, 5\}. \text{ Now, } \{0, 2, 3, 5\} \oplus_6 = \begin{cases} 0 \text{ mod } 6 = 0 \\ 2 \text{ mod } 6 = 2 \\ 3 \text{ mod } 6 = 3 \\ 5 \text{ mod } 6 = 5 \end{cases} \text{ respectively } \left. \begin{array}{l} (0+2) \text{ mod } 6 = 2 \\ (0+3) \text{ mod } 6 = 3 \\ (0+5) \text{ mod } 6 = 5 \\ (2+3) \text{ mod } 6 = 5 \\ (2+5) \text{ mod } 6 = 1 \\ (3+5) \text{ mod } 6 = 4 \end{array} \right\} \Rightarrow \text{subgroup}$$

$$\begin{aligned} &= 0 \text{ mod } 6 = 0 && (0+2) \text{ mod } 6 = 2 \\ &= 2 \text{ mod } 6 = 3 \text{ mod } 6 && (0+3) \text{ mod } 6 = 3 \\ &= 5 \text{ mod } 6 = 2, 3, 5 \text{ respectively} && (0+5) \text{ mod } 6 = 5 \\ &\Rightarrow 0, 2, 3, 5 \in H_5 && (2+3) \text{ mod } 6 = 5 \\ &&& (2+5) \text{ mod } 6 = 1 \neq H_5 \end{aligned}$$

\therefore This won't be subgroup.

(OR) "PROVE BY CONSTRUCTING COMPOSITION TABLE"

24. EXAMPLES ON SUB GROUPS - 2

$G = (\{1, 2, 3, 4, 5, 6\}, \otimes_1)$ which of the following are subgroups of 'G'?

H₁ = {1, 6} (C) H₃ = {1, 3, 5}

b) H₂ = {1, 2, 4} d) H₄ = {1, 2, 3, 5}.

$$H_1: \begin{array}{c|cc} & 1 & 6 \\ \hline 1 & 1 & 6 \\ 6 & 6 & 1 \end{array}$$

H₁ is subgroup

$$H_2: \begin{array}{c|ccc} & 1 & 2 & 4 \\ \hline 1 & 1 & 2 & 4 \\ 2 & 2 & 4 & 1 \\ 4 & 4 & 1 & 2 \end{array} \rightarrow \in H_2$$

H₂ is subgroup.

$$H_3: \begin{array}{c|ccc} & 1 & 3 & 5 \\ \hline 1 & 1 & 3 & 5 \\ 3 & 3 & 2 & 1 \\ 5 & 5 & 1 & 4 \end{array}$$

$2 \notin H_3$.
 $\therefore H_3$ is not subgroup.

$$H_4:$$

$$\begin{array}{c|ccc} & 1 & 2 & 3 & 5 \\ \hline 1 & 1 & 2 & 3 & 5 \\ 2 & 2 & 4 & 6 & 3 \\ 3 & 3 & 6 & 2 & 1 \\ 5 & 5 & 3 & 1 & 4 \end{array} \rightarrow 6 \notin H_4 \therefore H_4$$

25. EXAMPLES ON SUB GROUPS - 3

Let $(G, *)$ be a group of order p where p is prime no., then the no. of proper subgroups of 'G' is ____?

Sol: Given the order of 'G' has prime number = p (1 and p are only factors)

The subgroups of G has the order, '1' and ' p '.

\Rightarrow So 2 subgroups are possible that are the total

set $(G, *)$ and the Identity element set say $(e, *)$

But these are Trivial subgroups.

\therefore The total no. of subgroups = Total - Trivial subgroups

$$= 2 - (2)$$

$$\boxed{\text{Subgrps} = 0}$$

26. EXAM

Which of the

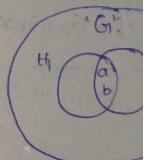
- a) The Union
- b) The Intersec
- c) The union
- d) Every subg

a) (S_8, \otimes_8)

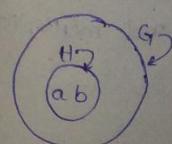
b) $(G, *)$

c) $(H, *)$

d) (G, \otimes_G)



c) FALSE (Refer)



EXAMPLES ON SUBGRAPHS - 4

S2 E1

Which of the following are not true? (FALSE)

1

$$a) (\mathbb{S}_8, \otimes_8) = (\{1, 3, 5, 7\}, \otimes_8)$$

3.
at Sab
Snowy.

$$= \text{Subgroups } H_1 = \{1, 3\} \quad H_2 = \{1, 5\}$$

$$\begin{array}{c|cc} & 1 & 3 \\ \hline 1 & 1 & 3 \\ 3 & 3 & 1 \end{array} \quad \begin{array}{c|cc} & 1 & 5 \\ \hline 1 & 1 & 5 \\ 5 & 5 & 1 \end{array}$$

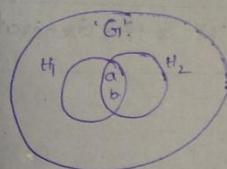
$$\begin{array}{r|rr} & 1 & 5 \\ \hline 1 & 1 & 5 \\ \hline 5 & 5 & 1 \end{array}$$

$$= \text{union of } H_1 \text{ and } H_2 = \{1, 3\} \cup \{1, 5\} = \{1, 3, 5\}$$

∴ a is false

$$\begin{array}{r}
 \begin{array}{c|ccc}
 & 1 & 3 & 5 \\
 1 & | & 1 & 3 & 5 \\
 3 & | & 3 & 1 & 7 \\
 5 & | & 5 & 7 & 4 \\
 \end{array} \\
 \text{Final Answer: } 4(4) + 7
 \end{array}$$

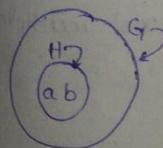
-)) TRUE. (Same above example) $\{f_1, 3\} \cap \{f_1, 5\} = \{f_1\}$



$$\text{H}_1 \cap \text{H}_2 : \text{SubGroup}$$

enerating
in the
sequence

c) FALSE (Refer(a))



Given 'G' is Abelian Group and hence H is subset of
Abelian Group is again Abelian Group.

Every subset of Abelian Group is Abelian.

27. CYCLIC GROUPS

A group $(G, *)$ is called a Cyclic Group if there exists an element $a \in G$ such that every element of 'G' can be written as a^n for some integer n . Then 'a' is called generating element/generator.

T 1) $G_1 = (\{1, -1\}, *)$ Now $\{-1\}^1 = \{1\}^1 = 1$ } The elements can be generated using $\{-1\}$ as $\{-1\}^1$ and $\{-1\}^2 = \emptyset$.

$\therefore \boxed{\{-1\} \text{ is the GENERATOR}}$

2) $G_2 = (\{1, \omega, \omega^2\}, *)$ Now, $\omega^1 = \omega$
 $\omega^2 = \omega^2$ } all the elements can be generated
 $\omega^3 = 1$ } using "ω"

$\therefore \boxed{\text{Generator} = \omega}$

3) $G_3 = (\{1, -1, i, -i\}, *)$ Now, $i^1 = i^2 = i$

$i^2 = -1$
 $(i^2)^2 = i^4 = 1$ } $\boxed{i \text{ is the Generator}}$
 $i^3 = -i$

4) $G_4 = (\{0, 1, 2, 3\}, \oplus_4)$ Now, $1^1 = 1$

$1^2 = 1+1=2$
 $1^3 = 1+1+1=3$
 $1^4 = 0 \quad (4 \bmod 4 = 0)$

Operation defined here is addition not multiplication

Now, $3^1 = 3$

$3^2 = 6 \bmod 4 = 2$
 $3^3 = 9 \bmod 4 = 1$
 $3^4 = 12 \bmod 4 = 0$

$\boxed{3 \text{ is also a Generator}}$

28. EXAMPLES

If $(G, *)$

i) a^{-1} is also

ii) The ord

Ex: $(\{0, 1, 2, 3\}, \oplus_4)$

$1^1 = 1 \quad 1^2 =$

	0	1
0	0	1
1	1	2
2	2	3
3	3	4

$3^1 = 3$
 $3^2 = 2$
 $3^3 = 1$
 $3^4 = 0$

$O(1) = 4$

$O(3) = 4$

Ex: $(S_5, \cdot_5) = (\{$

$2^1 = 2, \quad 2^2 = 4$

$3^1 = 3, \quad 3^2 = 4,$

$O(3) = 4$

$O(2) = 4$

EXAMPLES ON CYCLIC GROUPS

* an
written
generator.

If $(G, *)$ is a cyclic group with generator 'a' then
 a^{-1} is also a generator

ii) The Order of the Generator = $O(a) = \text{Order of the Group}$.

$$\text{Ex: } (\{0, 1, 2, 3\}, \oplus_4)$$

and $(-1)^2 = 1$.

be generated

$$1=1, 1^2=2, 1^3=3, 1^4=4 \pmod{4} = 0. \therefore 1 \text{ is the generator}$$

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	4	1	2

Now, the inverse of $a * a^{-1} = e$ (Identity ele)

Here 0 is the Identity element

$\therefore 1 * 0 = 0$

$\therefore 1 / 0 = 0$

\therefore check in 1 row where '0' is present

\therefore under '3' numbered column.

Inverse of 1 is '3'

$$\begin{array}{l} 3^1 = 3 \\ 3^2 = 2 \\ 3^3 = 1 \\ 3^4 = 0 \end{array} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \therefore 3 \text{ is also a Generator.}$$

1st point is satisfied

$$\therefore O(1) = 4 \quad (\text{Because } 1^4 = 0 \text{ (Identity element)})$$

$$O(3) = 4 \quad (3^4 = 0 \text{ (Identity element)})$$

generating

in the

sequence

$$\therefore (S_4, \cdot) = (\{1, 2, 3, 4\}, \otimes_4)$$

$$2^1 = 2, 2^2 = 4, 2^3 = 3, 2^4 = 1$$

$$3^1 = 3, 3^2 = 4, 3^3 = 2, 3^4 = 1$$

2 and 3 are generators

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Now, the identity element is '1'. Now

'3' is the inverse of '2'.

$$\begin{array}{l} O(3) = 4 \\ O(2) = 4 \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} = O(6) = O(5).$$

29. THEOREM ON CYCLIC GROUPS

Let $(G, *)$ be a cyclic group of order n' with generator a' then

- 1) The no. of generators in $G_1 = \phi(n)$ Euler's function of n'
- 2) a^m is also generator of G_1 if $\text{GCD}(m, n)=1$

$\phi(n) =$ The no. of numbers that are less than n' and relatively prime to n .

Now, In the
they are ∞

Now, $\phi(84)$

$\phi(84)$

Ex

Let $(G, *)$ be a cyclic group of order 8 with generator a'

- 1) No. of generators in $G_1 = ?$ 4
 - 2) Which of the following is not a generator of G_1 ?
- ~~(a)~~ a^2 (b) a^3 (c) a^5 (d) a^7

$\phi(84)$

Now, Here $n=8 \Rightarrow$ The set $\phi(n) = \{1, 3, 5, 7\}$

∴ No. of generators in $G_1 = 4$

Now, $n=8$. Now, a^m is also generator if $\text{GCD}(m, n)=1$

$\Rightarrow a^2 = \text{GCD}(2, 8) \neq 1 \rightarrow$ NOT Generator

$\Rightarrow a^3 = \text{GCD}(3, 8) = 1 \rightarrow$ Generator

$\Rightarrow a^5 = \text{GCD}(5, 8) = 1 \rightarrow$ Generator

$\Rightarrow a^7 = \text{GCD}(7, 8) = 1 \rightarrow$ Generator.

30. EXAMPLES

1) $G_1 = \{1, 2, 3, \dots\}$

2) $G_2 = \{0, 1, 2, \dots\}$

3) $G_3 = \{1, 3, 5, \dots\}$

Now, $G_1 =$

No. of

Now,

The problem with this method is as the value of n' increases then it is difficult to find the prime nos less than n' . So the product rule has been defined.

If $n = p \times q$ ($\text{If } n'$ can be written as product of two distinct prime numbers 'p' and 'q' then)

$$\phi(n) = \phi(p) \phi(q)$$



The Advantage with this method is if 'p' is a prime number

then $[\phi(p) = (p-1)] [\phi(7) = 6, \phi(11) = 10]$

Now, $\phi(77) = \phi(7) \times \phi(11)$

= $6 \times 10 = 60$.

The 2 genera

Now, In the above procedure both 'p' and 'q' should be distinct if they are same prime numbers then the formulae

$$\phi(p^n) = p^n - p^{n-1}$$

\downarrow
P should be prime

$$\phi(25) = \phi(5^2) = 5^2 - 5$$

$$\phi(25) = 20$$

Now, $\phi(84) = 2 \times 2 \times 3 \times 7$

$$\phi(84) = \phi(2^2 \times 3 \times 7)$$

$$= \phi(2^2) \times \phi(3) \times \phi(7)$$

$$= (2^2 - 2) \times (2) \times (6)$$

$$= 2 \times 2 \times 6$$

$$\begin{cases} \phi(3) = (3-1) = 2 \\ \phi(7) = (7-1) = 6 \end{cases} \quad \begin{cases} \phi(2^2) = 2^2 - 2 \\ = 2 \end{cases}$$

$$\phi(84) = 24$$

EXAMPLES ON CYCLIC GROUPS.

1) $G_1 = (\{1, 2, 3, 4, 5, 6\}, \otimes_7)$ Find all the generators of G_1, G_2, G_3 .

2) $G_2 = (\{0, 1, 2, 3, 4\}, \oplus_5)$

3) $G_3 = (\{1, 3, 5, 7\}, \otimes_8)$

Now, $G_1 = (\{1, 2, 3, 4, 5, 6\}, \otimes_7)$

No. of generators = $\phi(6) = \{1, 5\}$ = 2 generators. $\Rightarrow a^1, a^5$ are generators

Now, 1 cannot be generator because it is identity element.

2 is not generator

$$2^1 = 2$$

$$2^2 = 4$$

$$2^3 = 1$$

$$2^4 = 2 \quad 2^5 = 4 \quad \left. \begin{array}{l} 2^6 = 1 \\ 2^7 = 2 \\ 2^8 = 4 \end{array} \right\} 4, 5, 6 \text{ are not generated}$$

3 is generator.

$$3^1 = 3$$

$$3^2 = 2$$

$$3^3 = 6$$

$$3^4 = 3^2 \cdot 3^2 = 4$$

$$3^5 = 3^2 \cdot 3^3 = 35$$

$$3^6 = 3^3 \cdot 3^3 = 27$$

$$3^7 = 3^3 \cdot 3^3 = 81$$

$$3^8 = 3^3 \cdot 3^3 = 243$$

$$3^9 = 3^3 \cdot 3^3 = 729$$

$$3^{10} = 3^3 \cdot 3^3 = 2187$$

$$3^{11} = 3^3 \cdot 3^3 = 6561$$

$$3^{12} = 3^3 \cdot 3^3 = 19683$$

$$3^{13} = 3^3 \cdot 3^3 = 59049$$

$$3^{14} = 3^3 \cdot 3^3 = 177147$$

$$3^{15} = 3^3 \cdot 3^3 = 531441$$

$$3^{16} = 3^3 \cdot 3^3 = 1594323$$

$$3^{17} = 3^3 \cdot 3^3 = 4782969$$

$$3^{18} = 3^3 \cdot 3^3 = 14348907$$

$$3^{19} = 3^3 \cdot 3^3 = 4304671$$

$$3^{20} = 3^3 \cdot 3^3 = 12914013$$

$$3^{21} = 3^3 \cdot 3^3 = 38742039$$

$$3^{22} = 3^3 \cdot 3^3 = 116226117$$

$$3^{23} = 3^3 \cdot 3^3 = 348678351$$

$$3^{24} = 3^3 \cdot 3^3 = 1045935053$$

$$3^{25} = 3^3 \cdot 3^3 = 3137805161$$

$$3^{26} = 3^3 \cdot 3^3 = 9413415483$$

$$3^{27} = 3^3 \cdot 3^3 = 28239246451$$

$$3^{28} = 3^3 \cdot 3^3 = 84717739353$$

$$3^{29} = 3^3 \cdot 3^3 = 254153218059$$

$$3^{30} = 3^3 \cdot 3^3 = 762460654177$$

$$3^{31} = 3^3 \cdot 3^3 = 2287381962331$$

$$3^{32} = 3^3 \cdot 3^3 = 7062145886993$$

$$3^{33} = 3^3 \cdot 3^3 = 21186437660979$$

$$3^{34} = 3^3 \cdot 3^3 = 63559312982937$$

$$3^{35} = 3^3 \cdot 3^3 = 190677938948811$$

$$3^{36} = 3^3 \cdot 3^3 = 571933816846433$$

$$3^{37} = 3^3 \cdot 3^3 = 1715801450539299$$

$$3^{38} = 3^3 \cdot 3^3 = 5147404351617897$$

$$3^{39} = 3^3 \cdot 3^3 = 15442212054853691$$

$$3^{40} = 3^3 \cdot 3^3 = 46326636164561073$$

$$3^{41} = 3^3 \cdot 3^3 = 139079908493683219$$

$$3^{42} = 3^3 \cdot 3^3 = 417239725481049657$$

$$3^{43} = 3^3 \cdot 3^3 = 1251719176443148971$$

$$3^{44} = 3^3 \cdot 3^3 = 3755157530330446913$$

$$3^{45} = 3^3 \cdot 3^3 = 11265472691091339739$$

$$3^{46} = 3^3 \cdot 3^3 = 33796417873273919217$$

$$3^{47} = 3^3 \cdot 3^3 = 101389253619821757651$$

$$3^{48} = 3^3 \cdot 3^3 = 304167760859465272953$$

$$3^{49} = 3^3 \cdot 3^3 = 912503282578395818859$$

$$3^{50} = 3^3 \cdot 3^3 = 2737519847735187456577$$

$$3^{51} = 3^3 \cdot 3^3 = 8212559543205562369731$$

$$3^{52} = 3^3 \cdot 3^3 = 24637678633616687109193$$

$$3^{53} = 3^3 \cdot 3^3 = 73913035900850061327579$$

$$3^{54} = 3^3 \cdot 3^3 = 221739107702550184082737$$

$$3^{55} = 3^3 \cdot 3^3 = 665217323107650552248211$$

$$3^{56} = 3^3 \cdot 3^3 = 2005652069322951656744633$$

$$3^{57} = 3^3 \cdot 3^3 = 6016956208368854970233901$$

$$3^{58} = 3^3 \cdot 3^3 = 18050868625106564910701703$$

$$3^{59} = 3^3 \cdot 3^3 = 54152595875319694732105111$$

$$3^{60} = 3^3 \cdot 3^3 = 162457787626059084236315333$$

$$3^{61} = 3^3 \cdot 3^3 = 487373363878177252708946001$$

$$3^{62} = 3^3 \cdot 3^3 = 1462120091634531758126828033$$

$$3^{63} = 3^3 \cdot 3^3 = 4386360274883600274370484099$$

$$3^{64} = 3^3 \cdot 3^3 = 13159080824650800823111452297$$

$$3^{65} = 3^3 \cdot 3^3 = 39477242473952402469334356891$$

$$3^{66} = 3^3 \cdot 3^3 = 118431727421857207388003070673$$

$$3^{67} = 3^3 \cdot 3^3 = 355295182265571622164009212019$$

$$3^{68} = 3^3 \cdot 3^3 = 1065885546796714866492027636057$$

$$3^{69} = 3^3 \cdot 3^3 = 3197656640390144599476083918171$$

$$3^{70} = 3^3 \cdot 3^3 = 9592969921170433798438251754513$$

$$3^{71} = 3^3 \cdot 3^3 = 28778909763511301395314755263539$$

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$$3^{73} = 3^3 \cdot 3^3 = 259010187871601712557832797371851$$

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$$3^{89} = 3^3 \cdot 3^3 = 1130950144020303640699633557933166833$$

$$3^{90} = 3^3 \cdot 3^3 = 3492850432060910921099000673809499991$$

$$3^{91} = 3^3 \cdot 3^3 = 10478551296182732763297002021428499973$$

$$3^{92} = 3^3 \cdot 3^3 = 31435653888548208290891006064285499921$$

$$3^{93} = 3^3 \cdot 3^3 = 94306961665644624872673008192856499763$$

$$3^{94} = 3^3 \cdot 3^3 = 282920885000033874617020044578562499289$$

$$3^{95} = 3^3 \cdot 3^3 = 848762655000101623851060013735687499867$$

$$3^{96} = 3^3 \cdot 3^3 = 2546287965000304871553180041207062499601$$

$$3^{97} = 3^3 \cdot 3^3 = 7638863895000914614669560023621387499803$$

$$3^{98} = 3^3 \cdot 3^3 = 22916591685002743843988680069864162499411$$

$$3^{99} = 3^3 \cdot 3^3 = 70749775055008231531965880020592487498233$$

$$3^{100} = 3^3 \cdot 3^3 = 21224932516502470460587768006177462496699$$

$$3^{101} = 3^3 \cdot 3^3 = 63674807549507411381763304008532987496097$$

$$3^{102} = 3^3 \cdot 3^3 = 191024422648522234145290012025698962960991$$

$$3^{103} = 3^3 \cdot 3^3 = 573073267945566702435870036077096982902973$$

$$3^{104} = 3^3 \cdot 3^3 = 171922070383670010727760088123129094290881$$

$$3^{105} = 3^3 \cdot 3^3 = 515766211151010032183280064370387282902443$$

$$3^{106} = 3^3 \cdot 3^3 = 1547298633453030096549840093111141842907321$$

$$3^{107} = 3^3 \cdot 3^3 = 4641895900359100290649520089333425528721963$$

$$3^{108} = 3^3 \cdot 3^3 = 13925687701077300871948560067800276586165891$$

$$3^{109} = 3^3 \cdot 3^3 = 41777063103232002615845600203400827758517673$$

$$3^{110} = 3^3 \cdot 3^3 = 125331189310696007847536006012002433265552919$$

$$3^{111} = 3^3 \cdot 3^3 = 376093567932088002342608001836007299816658757$$

$$3^{112} = 3^3 \cdot 3^3 = 1128280703796264007027824005508021994500275271$$

$$3^{113} = 3^3 \cdot 3^3 = 3484842111388792002313464001736065983500825813$$

$$3^{114} = 3^3 \cdot 3^3 = 10454526334166376007037360052080969805004854391$$

$$3^{115} = 3^3 \cdot 3^3 = 31363579002399120023012080056740909815004563173$$

$$3^{116} = 3^3 \cdot 3^3 = 94090737007197360070036240069221729450045889519$$
</

NOW,

$$\textcircled{2} \quad G_2 = (\{0, 1, 2, 3, 4\}, \oplus_5) \quad \phi(G) = 5$$

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$$\text{No. of generators} = \phi(5) = \{4\} = \{0, 1, 2, 3, 4\} \Rightarrow a^0, a^1, a^2, a^3, a^4 \text{ are generators}$$

Now, 0 is the identity element \Rightarrow it cannot be generator

$$1^1 = 1 \quad 1^2 = 2 \quad 1^3 = 3 \quad 1^4 = 4 \quad 1^5 = 0. \quad 1 \text{ is generator.}$$

$$\begin{array}{c} 1^1 & 1^2 & 1^3 & 1^4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \boxed{1} & 2 & 3 & 4 \end{array} \quad \text{are generators.}$$

$$\begin{array}{c} 1^1 & 1^2 & 1^3 & 1^4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \boxed{1} & 2 & 3 & 4 \end{array} \quad \text{are generators}$$

$$\textcircled{3} \quad G_3 = (\{1, 3, 5, 7\}, \otimes_8)$$

Now, $\phi(4) = \{1, 3\} \therefore a^1, a^3 \text{ are the 2 generators.}$

Now 1 is identity element

$$3^1 = 3 \quad 3^2 = 1 \quad 3^3 = 3 \quad 3^4 = 1$$

$$5^1 = 5 \quad 5^2 = 1 \quad 5^3 = 5 \quad 5^4 = 1$$

$$7^1 = 7 \quad 7^2 = 1$$

~~so there are no cyclic groups of order 4.~~

$\therefore 3^1, 3^3 \text{ not generator}$

$5^1, 5^3 \text{ not generator}$

$7^1, 7^3 \text{ not generator}$

\therefore There are no generators for this group \Rightarrow The group is not cyclic group.

31. SOME POINTS ON CYCLIC GROUPS.

For cyclic groups, the following properties hold good. $\uparrow g = \text{generator}$ $\uparrow g = \text{generator}$

1) Every cyclic group is an Abelian Group $[a * b = g^n * g^m = g^{n+m} = g^m * g^n = b * a]$

2) Every Group of prime order is cyclic and so every group of prime order is Abelian Group.

3) Every subgroup of a cyclic group is also cyclic, but the generator of the subgroup need not be same as that of cyclic group.

Ex: $G = \{1, -1, i, -i\}; H = \{1, -1\}$ so, H is subgroup of G.
Generators of G are $i, -i$, Generator of H = -1

4) Let $(G, *)$ be a group of even order, then there exists atleast one element $a \in G$ ($a \neq e$) such that $a^2 = e$.

1. INTRODUCTION

A Relation
to each ele
denoted as

Range : Ran

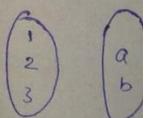
A function

$f: (A \rightarrow B)$

Range = $\{1, 2, 3, 4\}$

2. COUNTING

$f: A \rightarrow B$ $A \in$



1 has 2 choices

2 has 2 choices

3 has 2 choices

5. FUNCTIONS

INTRODUCTION TO FUNCTIONS

3, 4 are
generators
stor.

①

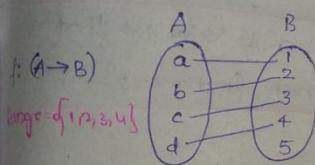
five 1's

Q7

relation 'f' from 'set A' to a 'set B' is called a function if
to each element $a \in A$, we can assign an unique element of 'B'. It is
denoted as $f: A \rightarrow B$ 'A' is domain and 'B' is co-domain.

Range: Range of function is $\{y | y \in B \text{ and } (x, y) \in f\}$ \Rightarrow range of $f \subseteq B$.

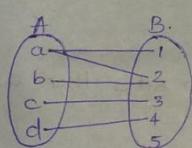
A function $f: A \rightarrow A$ is called a function on 'A'.



$$\{(a, 1)(b, 2)(c, 3)(d, 4)\}$$

Image of $a=1$
 $b=2$
 $c=3$
 $d=4$

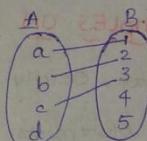
$$f: (A \rightarrow B)$$



Not a function

$$\{(a, 1)(a, 2)\}$$

Same ele is mapped to
two elements.



Not function
'd' is not mapped.

generating

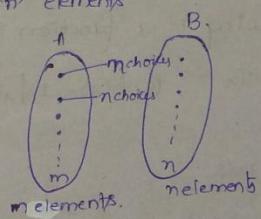
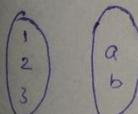
in the

sequence

COUNTING THE FUNCTIONS

$f: A \rightarrow B$ Assume 'A' has 'm' elements

'B' has 'n' elements



$n \times n \times n \times \dots$ (m times)

$$= m^m$$

The no. of functions from $A \rightarrow B$
 $= m^m$

$$= (\text{No. of ele in } B)^{\text{No. of ele in } A}$$

One

Now,

CHAPTER - 6

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The total no. of Relations from $A \rightarrow B$ which are not functions.

is



The no. of Relations that are not functions

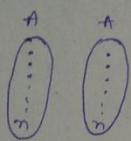
$$= [2^{mn} - (m)^m]$$

→ consider the integers.

$f_1(S)$ = The no.

$f_2(S)$ = The no. which of +

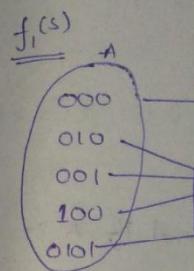
Now, if we are defining the function on the same set $f: A \rightarrow A$ then, The total no. of Relations that are not functions are



$$\text{No. of Relations} = 2^{n^2}$$

$$\text{No. of functions} = m^{n^2}$$

$$\text{No. of Relations that are not functions} = 2^{n^2} - m^{n^2}$$



3. EXAMPLES ON FUNCTIONS

If there are exactly 81 functions from set A to set B then which of the following statements is not True?

a) $|A|=4$ $|B|=3$

b) $|A|=2$ $|B|=9$

c) $|A|=1$ $|B|=81$

d) $|A|=9$ $|B|=9$.

No. of functions from $A \rightarrow B$ = $(\text{No. of ele in } A)^{\text{No. of ele in } B}$

option 1 $\Rightarrow 3^4 = 81$ ✓ function

2 $\Rightarrow 9^2 = 81$ ✓

3 $\Rightarrow 81^1 = 81$ ✓

4 $\Rightarrow 9^9 \neq 81$. \therefore option 4 is false.

Which of the following is a function if domain is set of all real nos.

a) $f(x) = \frac{1}{x}$

b) $f(x) = \sqrt{x}$

c) $b(x) = \pm \sqrt{x^2 + 1}$

d) $\phi(x) = |x|$

Domain = All Real nos.

a) $f(x) = \frac{1}{x}$ If $x=0$? then the image will not be present for 0 so this is not a function.

b) $f(x) = \sqrt{x}$ For -ve nos. square root is not possible (Not function)

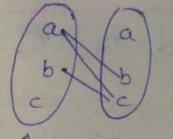
c) Not function because for every element there are two values $+x$ and $-x$.

d) function.

4. EXAMPLES

Which of the following is a function?

a) $R_1 = \{(a,b)(b,c)(c,a)\}$



NOT function

b) State True / False

S1) There exists

S2) The ~~are~~ function

S3) $f(x) = \log_e x^2$

S4) The domain

S5)

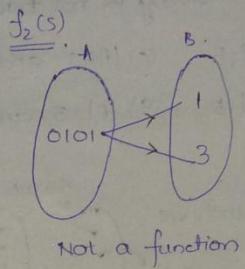
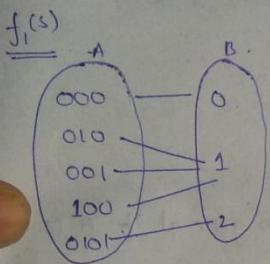
$$\begin{array}{l}
 f(x) = x \\
 f(1) = 1 \\
 f(-1) = -1 \\
 g(x) = \\
 g(1) = \\
 g(-1) =
 \end{array}$$

→ Consider the following relations from set of all bit strings to set of all integers.

$f_1(s)$ = The no. of 1's in the bit string 's'.

$f_2(s)$ = The position of a 0-bit in a bit string 's'.

which of the above relations are functions.

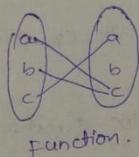
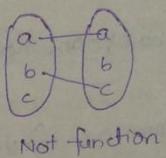
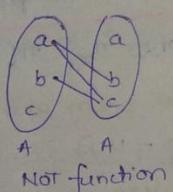


'0' is present at position '1' and '3' so it will be mapped to 2 elements in B.

4. EXAMPLES ON FUNCTIONS

Which of the following relations on set A is a function? $A = \{a, b, c\}$

- a) $R_1 = \{(a|b)(b|c)(a|c)\}$ b) $R_2 = \{(a|a)(b|c)\}$ c) $R_3 = \{(a|c)(b|c)(c|a)\}$



b) State True / False?

i) There exists equivalence Relation which is function. TRUE $\{(1,1)(2,2)(3,3)\}$.

ii) The functions $f(x) = x$, $g(x) = \sqrt{x^2}$ are identical. FALSE

iii) $f(x) = \log_e x^2$ and $g(x) = 2 \log_e x$ are identical. FALSE

iv) The domain of $f(x) = \frac{1}{\sqrt{|x|-x}}$ is $(-\infty, 0)$. $\Rightarrow f(x) = \begin{cases} 2 & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$ TRUE.

S2:

$$f(x) = x \quad g(x) = \sqrt{x^2}$$

$$f(1) = 1 \quad g(1) = 1$$

$$f(-1) = -1 \quad g(-1) = 1$$

$$\therefore (-1+1) \therefore \text{NOT IDENTICAL}$$

S3:

$$f(x) = \log_e x^2 \Rightarrow f(-1) = \log_e (-1)^2 = \log_e 1 = 0$$

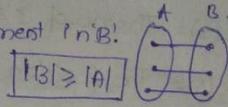
$$g(x) = 2 \log_e x = 2 \log_e (-1) = \text{does not exist.}$$

$$g(-1) = \text{does not exist.}$$

5. ONE-ONE FUNCTIONS (INJECTION)

62

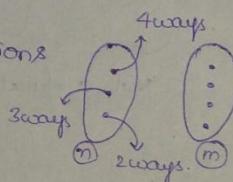
A function f from a set 'A' to set 'B' is said to be one-to-one if no two elements in 'A' are mapped to same element in 'B'.



If there are exactly 120 one-to-one functions possible from A to B then which of the following is not true?

- a) $|A|=5$ and $|B|=5$ (c) $|A|=3$ and $|B|=6$
- b) $|A|=4$ and $|B|=5$ (d) $|A|=5$ and $|B|=4$

i. No. of one-to-one functions



∴ In general if 'A' is having 'n' elements and B is having 'm' elements then

the no. of one-one function from $A \rightarrow B$ = ~~shortened no. of max. f.~~ $= (m)(m-1)(m-2)(m-3)\dots(m-n+1)$

$$= m_{P_n} = \frac{(\text{No. of ele in } B)_p}{(\text{No. of ele in } A)_p}$$

ii. If m, n are equal then the no. of one-one functions = $n_{P_n} = n!$

Now, In the Question (No. of one-one functions = $m_{P_n} = 120$)

$$\Rightarrow \text{op1: } |A|=5 \quad |B|=5 \quad = 5_{P_5} = 5! = 120. \text{ TRUE}$$

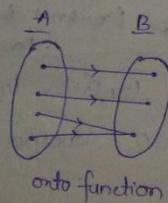
$$\Rightarrow \text{op2: } |A|=4 \quad |B|=5 \quad = 5_{P_4} = 5 \times 4 \times 3 \times 2 = 120. \text{ TRUE}$$

$$\Rightarrow \text{op3: } |A|=3 \quad \text{and} \quad |B|=6 \Rightarrow 6_{P_3} = 6 \times 5 \times 4 = 120.$$

$$\Rightarrow \text{op4: } |A|=5 \quad |B|=4 \quad (4_{P_5}) - \text{NOT possible}$$

6. ONTO FUNCTIONS

A function $f: A \rightarrow B$ is said to be onto if each element of 'B' is mapped by at least one element of 'A', i.e. Range of $f = B$.



The condition for a function to be onto is $|B| \leq |A|$. If this doesn't satisfy then func is not onto. we cannot say a func is onto if it satisfies the above condition.

If $|A|=|B|$

7. EXAMPLE

If $|A|=m$ from A to B

$$m^m$$

Ex: If $|A|=$

$$|A|=6 \Rightarrow |B|=3$$

→

Ex: If $|A|=n$

$$\Rightarrow 2^n$$

8. EXAMPLES

In how many ways so that every project is as

Ex:

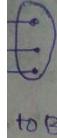
e
e
e
e
e

62

If $|A|=|B|$ then the no. of onto functions = $m!$ No 239 MAX 363 P

one

B.



1. EXAMPLES ON ONTO FUNCTIONS - 1

If $|A|=m$ and $|B|=n$, ($m > n$) then the no. of onto functions possible from A to B is.

$$n^m - \binom{m}{1}(n-1)^m + \binom{m}{2}(n-2)^m - \binom{m}{3}(n-3)^m + \dots + (-1)^m \binom{m}{m-1} 1^m$$

Ex: If $|A|=6$, $|B|=3$ then the no. of onto functions from A to B is —?

$$|A|=6 \Rightarrow m=6$$

$$|B|=3 \Rightarrow n=3$$

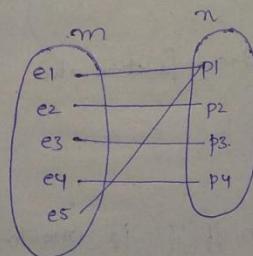
$$\Rightarrow 3^6 - \binom{6}{1}(2)^6 + \binom{6}{2}(1)^6 - \binom{6}{3}(0)^6 = 729 - 3 \times 64 + 3 \times 1 = 540$$

Ex: If $|A|=n$, $|B|=2$ ($n > 2$) then the no. of onto functions from A to B?

$$\Rightarrow 2^n - \binom{n}{1} 1^n = [2^n - 2]$$

2. EXAMPLES ON ONTO FUNCTIONS - 2

In how many ways we can assign 5 employees to 4 projects so that every employee is assigned to only one project and every project is assigned to atleast one employee?

So:

$$m=5 \quad n=4$$

$$\begin{aligned} \text{Required ways: } & 4^5 - \binom{5}{1} 3^5 + \binom{5}{2} 2^5 - \\ & \binom{5}{3} 1^5 + 0 \\ & = 1024 - (243 \times 4) + 6(32) \\ & + (-4) = 240 \end{aligned}$$

240 ways.

onto

satisfy

onto to

9. Examples on onto functions - 3

Consider the following functions on set of all integers $f(x) = x^2, g(x) = x^3$ and $h(x) = \lceil x/2 \rceil$ which of the following is TRUE?

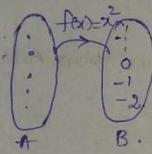
- R S1) f is one-one (FALSE) S4) g is onto (FALSE)
- T S2) f is onto (FALSE) S5) h is one-one (TRUE)
- S3) g is one-one (TRUE) S6) h is onto

$$f(x) = x^2$$

$$\begin{array}{l} f(1) = 1 \\ f(-1) = 1 \end{array}$$

Two elements are mapped to same element so NOT ONE-ONE

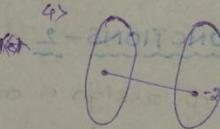
$$2) \text{ Set of Integers} = \{-\infty, \dots, 0, \dots, \infty\}$$



Square of a number cannot be negative but so Negative no.s in B cannot be mapped with any element in A' not one-one.

$$3) g(x) = x^3$$

Not one-one function



Not pre image for (-2). Not onto

$$5) h(x) = \lceil x/2 \rceil$$

$$h(1) = 1$$

$$h(2) = 1$$

Not one-one function and onto function also.

10. BIJECTION

A function $f: A \rightarrow B$ is called a Bijection if 'f' is one-to-one as well as onto.

→ If 'A' and 'B' are finite sets then Bijection from 'A' to 'B' is possible if $|A| = |B|$

→ If $|A| = |B| = n$ then No. of Bijections possible from A to B is $n!$

one-one
 $f: A \rightarrow B$
 $|A| \leq |B|$

11. EXAMPLE

Let $A = \mathbb{R}$

$$f(x) = \frac{x-2}{x-3}$$

- a) f is one
- b) f is onto

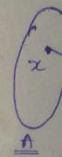
Let $f(x)$

$$= \frac{x-2}{x-3}$$

$$= (x-2)$$

$$= x^2$$

$$\Rightarrow [a]$$



12. INVERSE

Let $f: A \rightarrow B$

is called

Theorem :

- Q) Which of
- Q) $f(x)$

one-one
 $f: A \rightarrow B$
 $|A| \leq |B|$

onto function
 $f: A \rightarrow B$
 $|A| \geq |B|$

$$|A| = |B|$$

Bijection function condition.

65

five 1's

(Q)

II. EXAMPLE ON BIJECTION

Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. A function $f: A \rightarrow B$ is defined by

$f(x) = \frac{x-2}{x-3}$ which of the following is true?

NE-ONE

- a) f is one-one but not onto (C) f is bijection (X)
b) f is onto but not one-one (d) f is neither one-one or onto

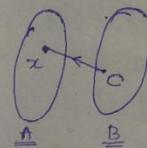
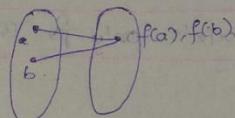
1) Let $f(a) = f(b)$

$$= \frac{a-2}{a-3} = \frac{b-2}{b-3}$$

$$= (a-2)(b-3) = (b-2)(a-3)$$

$$\Rightarrow ab - 3a - 2b + 6 = ab - 3b - 2a + 6$$

$$\Rightarrow [a=b] \text{ Not one to one}$$



$$f(x) = c$$

$$\Rightarrow \frac{x-2}{x-3} = c$$

$$\Rightarrow x-2 = cx-3c$$

$$\Rightarrow x(1+c) = -3c+2$$

$$\Rightarrow [x = \frac{2-3c}{1-c}] \text{ for every element } 'c' \text{ we can find an element } 'x' \text{ in } 'A'.$$

encountering
in the
sequence

12. INVERSE OF A FUNCTION

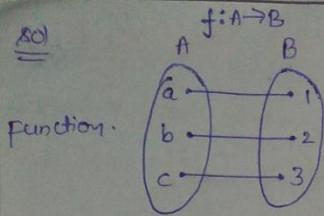
possible Let $f: A \rightarrow B$, if the inverse relation $f^{-1}: B \rightarrow A$ is a function then it is called inverse of function ' f ' and it is also denoted by $f^{-1}: B \rightarrow A$

Theorem : Inverse of $f: A \rightarrow B$ exists iff ' f ' is a bijection

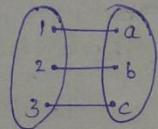
Q) which of the following functions have inverse defined on their Ranges?

- a) $f(x) = x^2$ b) $f(x) = x^3$.

F 80



$f^{-1}: B \rightarrow A$

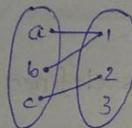


66

function $\Rightarrow f^{-1}$ exists.

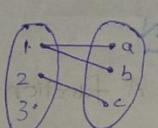
T

$f: A \rightarrow B$



function

$f^{-1}: B \rightarrow A$



Not function

For a function to have Inverse it should be one-one function and onto function.

\therefore Inverse exists only for Bijection functions.

a) $f(x) = x^2$

$\begin{cases} f(1) = 1 \\ f(-1) = 1 \end{cases}$ } Not one-one \Rightarrow Not Bijection.

b) $f(x) = x^3$. \rightarrow one-one, onto, Bijection \Rightarrow Inverse exists.

QUESTION A 30 QUESTIONS