Quantum Information Theory
FQC Online Classes 2020

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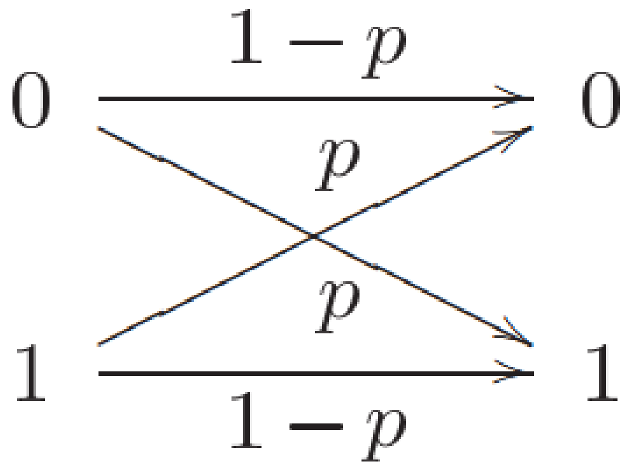
Quantum Noise

- To understand noise in quantum systems it is helpful to build same intuition by understanding noise in classical systems.
- Imagine a bit is being used on a hard disk attached to an ordinary classical computer. The bit is in the state 0 or 1.
- Let p be a probability for the bit to flip, and 1 – p be a probability for the bit to remain same.



After a long time a bit on a hard disk drive may flip with probability p.

Quantum Noise

- Suppose p_0 and p_1 are the initial probabilities that the bit is in the states 0 and 1, respectively. Let q_0 and q_1 be the corresponding probabilities after the noise has occurred.
- In the matrix notation, for bit in hard disk we have

$$\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}$$

- For a single stage process the output probabilities $\vec{q}$ are related to the input probabilities $\vec{p}$ by the equation

$$\vec{q} = E\vec{p},$$

- Where E is a matrix of transition probabilities which we shall refer to as the evolution matrix

Properties of evolution matrix E

- Positivity: All elements E_{ij} non-negative real numbers
- Completeness: Sum of all elements in a column $\sum_i E_{ij} = 1$

Quantum Operations

- Let ρ is the initial quantum state and ρ' is the output (final) quantum state. A general quantum operation is defined as

$$\rho' = \mathcal{E}(\rho)$$

The map $\mathcal{E}$ in this equation is a *quantum operation.*

- Two simple examples of quantum operations are unitary transformations and measurements.

Examples:

- Unitary transformation $\mathcal{E}(\rho) = U\rho U^\dagger$
- Measurement $\mathcal{E}_m(\rho) = M_m \rho M_m^\dagger$

Bit flip and Phase flip channels

- Bit flip: The bit flip channel flips the state of a qubit from |0> to |1> (and vice versa with probability 1-p. It has operation elements

$$E_0 = \sqrt{p}I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \sqrt{1-p}X = \sqrt{1-p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Phase flip: With probability 1-p the state "phase" of the qubit is flipped |1> ↔ -|1>. The phase flip channel has operation elements

$$E_0 = \sqrt{p}I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \sqrt{1-p}Z = \sqrt{1-p}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Bit-Phase flip channel

- Bit-Phase flip: With probability there is a bit flip and a phase flip. It has operation elements

$$E_0 = \sqrt{p}I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \sqrt{1-p}Y = \sqrt{1-p}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Quantum Information Measures

- How close are two quantum states?
- Trace Distance: The trace distance between quantum states ρ and σ is defined as

$$D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}|\rho - \sigma|$$

where $|A| = \sqrt{A^\dagger A}$

- The trace of matrix is defined to be the sum of its diagonal elements
- Fidelity: A second measure of distance quantum states is the fidelity. The fidelity of states ρ and σ defined to be

$$F(\rho,\sigma) = \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$$

Stabilizer Codes

- Stabilizer codes, sometimes known as additive quantum codes, are an important class of quantum codes whose construction is analogous to classical linear codes.
- In order to understand stabilizer codes it is useful to first develop the stabilizer formalism.
- The stabilizer formalism
  - The central idea of the stabilizer formalism is easily illustrated by an example.
  - Consider EPR state of two qubits:

$$|\Psi> = \frac{|00> + |11>}{\sqrt{2}}$$

  - It is easy to verify that this state is satisfies the identities: (Here X and Z are quantum gates)

$$XX\,|\Psi> = |\Psi> \quad \text{and} \quad ZZ\,|\Psi> = |\Psi>$$

  - We say that the state $|\Psi>$ is stabilized by the operators XX and ZZ

Example 1

1. Show that the quantum state $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$ is stabilized by {I, Y}

Answer: We have to show that $I|\Psi\rangle = |\Psi\rangle$ and $Y|\Psi\rangle = |\Psi\rangle$

$$I|\Psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) = |\Psi\rangle$$

and

$$Y|\Psi\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) = |\Psi\rangle$$

Example 2

2. Show that the quantum state $|\Psi> = \frac{1}{\sqrt{2}}\left(|01> - |10>\right)$ is stabilized by {-XX, -ZZ}

Answer: We have to show that $-XX|\Psi> = |\Psi>$ and $-ZZ|\Psi> = |\Psi>$

$$-XX|\Psi> = -\frac{1}{\sqrt{2}}\left(|10> - |01>\right) = \frac{1}{\sqrt{2}}\left(|01> - |10>\right) = |\Psi>$$

and

$$-ZZ|\Psi> = -\frac{1}{\sqrt{2}}\left(-|01> + |10>\right) = \frac{1}{\sqrt{2}}\left(|01> - |10>\right) = |\Psi>$$