Quantum Error Correcting Codes
FQC Online Classes 2020

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Quantum Error Correcting Codes (QECC)

- **Introduction**
  - A quantum error-correcting code (QECC) can be viewed as a mapping of k qubits into n qubits, where n > k.
  - The k qubits are the "logical qubits" or "encoded qubits" that we wish to protect from errors.
  - The additional n – k qubits allow us to store the k logical qubits in a redundant fashion so that the encoded information is not easily damaged.
- **How does the classical error correction works?**
  - The simplest example of a classical error-correcting code is a repetition code: the bit we wish to protect by 3 copies of the bit.
  - We encode logical 0 by 000 and 1 by 111

Quantum Error Correcting Codes (QECC)

- Now an error may occur that causes one of the three bits to flip.
- Each bit is assumed to flip ($0 \longleftrightarrow 1$) independently with probability p.
- Thus a bit 0 sent through the channel will be received as 0 with probability 1-p and will be received as 1 with probability p.
- To reduce channel errors, we may invoke the majority vote.

Quantum Error Correcting Codes (QECC)

- When 000 is sent through this channel, it will be received as 000 with probability $(1 - p)^3$, as 100, 010 or 001 with probability $3p\,(1 - p)^2$, as 011, 101, or 110 with probability $3p^2(1 - p)$ and finally as 111 with probability p^3.

- Note that summation of all the probabilities is 1.

- By taking the majority vote, we correctly reproduce the desired result 0 with probability $p_0 = (1 - p)^3 + 3p\,(1 - p)^2 = (1 - p)^2\,(1 + 2p)$ and fail with probability $p_1 = 3p^2(1 - p) + p^3 = (3 – 2p)\,p^2$. We obtain $p_0 \gg p_1$ for sufficiently small $p \geq 0$. In fact, we find $p_0 = 0.972$ and $p_1 = 0.028$ for $p = 0.1$. The success probability p_0 increases as p approaches 0.

Quantum Error Correcting Codes (QECC)

- **Three Challenges in QECCs**
  - No cloning: Duplicating quantum states to get repetition code is impossible.
  - Errors are continuous: A continuum of different errors may occur on a single qubit. Therefore, determination of which error occurred requires infinite precision and infinite resources.
  - Measurement destroys quantum information: Measurement destroys quantum information and makes recovery impossible if quantum state is destroyed.

Three-Qubit Bit-Flip Code

- The single bit flip error is described by:

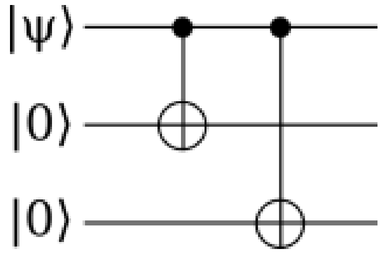
$$|0\rangle \to |1\rangle$$
$$|1\rangle \to |0\rangle$$

The Bit flip error is equivalent to action of the X-gate.

- Suppose Alice wants to send qubit $\Psi = a|0\rangle + b|1\rangle$ to Bob through a noisy quantum channel
- Encoding: Alice uses 3 qubits to encode one qubit. Similar to the repetition code, we will append two ancillary bits $|00\rangle$ to Ψ, giving us Ψ_1.

Three-Qubit Bit-Flip Code

- A circuit performing this encoding is shown below:



$$\begin{aligned}\Psi_0 &\equiv (a\,|0\rangle + b\,|1\rangle)\,|0\rangle\,|0\rangle \\ &\equiv a\,|000\rangle + b\,|100\rangle \\ \Psi_1 &\equiv a\,|000\rangle + b\,|111\rangle\end{aligned}$$

Three-Qubit Bit-Flip Code

- Error detection: Assuming that only a single qubit flip can occur during transmission, Bob will receive one of these 4 possible states

$$\Phi_0 \equiv a\,|000\rangle + b\,|111\rangle \quad // \text{ No error}$$

$$\Phi_1 \equiv a\,|001\rangle + b\,|110\rangle \quad // \; 1^{st} \text{ qubit flip}$$

$$\Phi_2 \equiv a\,|010\rangle + b\,|101\rangle \quad // \; 2^{nd} \text{ qubit flip}$$

$$\Phi_3 \equiv a\,|100\rangle + b\,|011\rangle \quad // \; 3^{rd} \text{ qubit flip}$$

- Now Bob has to extract from the received state which error occurred during the qubit transmission. For this purpose, Bob introduces two ancillary qubits in the state $|00\rangle$ as shown in Fig. 10.1.

Three-Qubit Bit-Flip Code

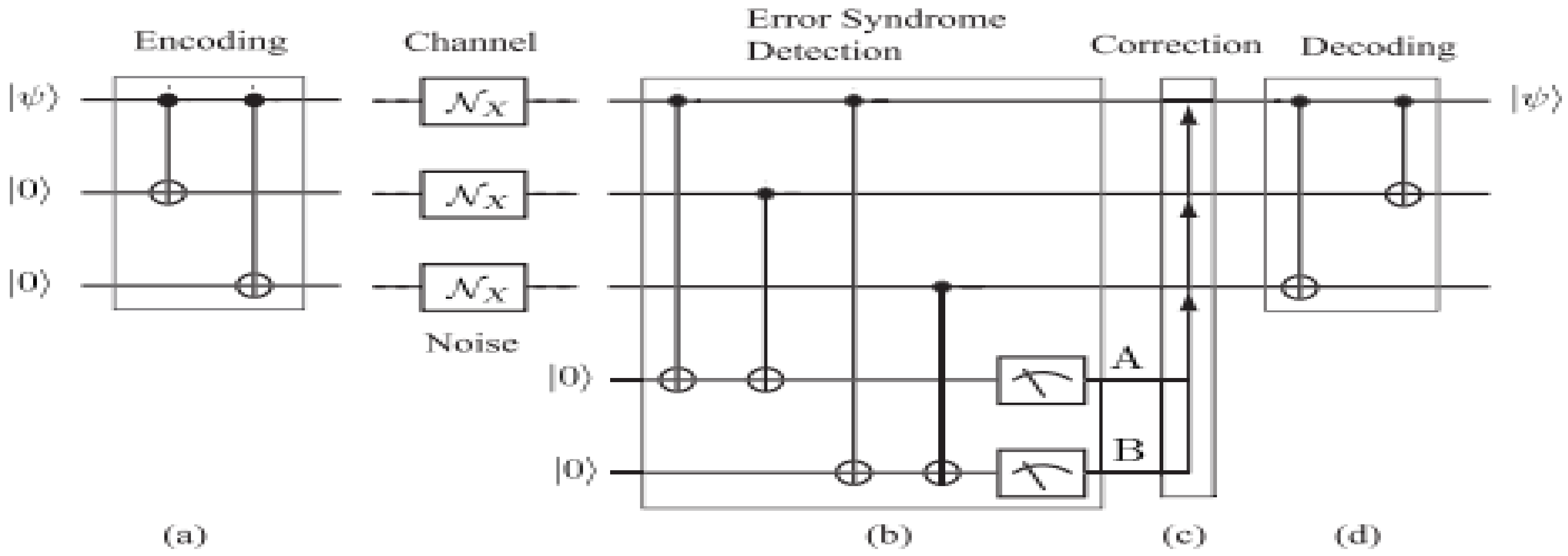


FIGURE 10.1
Quantum circuits to (a) encode, (b) detect bit-flip error syndrome, (c) make correction to a relevant qubit and (d) decode. The gate $\mathcal{N}_X$ stands for the bit-flip noise. The circuit (a) belongs to Alice, while the circuits (b), (c) and (d) belong to Bob.

Three-Qubit Bit-Flip Code

- Bob applies four CNOT operations whose control bits are the encoded qubits while the target qubits are Bob's two ancillary qubits. Let $|x_1 x_2 x_3\rangle$ be a basis vector Bob has received and let *A* (*B*) be the output state of the first (second) ancillary qubit. It is seen from Fig. 10.1 (b) that $A = x_1 \oplus x_2$ and $B = x_1 \oplus x_3$.
- Suppose $a|100\rangle + b|011\rangle$ is the received logical qubit. Then error extracting sequence transforms the ancillary qubits as

$$(a|100\rangle + b|011\rangle)|00\rangle \to a|10011\rangle + b|01111\rangle = (a|100\rangle + b|011\rangle)|11\rangle.$$

- Both of the ancillary qubits are flipped since $x_1 \oplus x_2 = x_1 \oplus x_3 = 1$ for both $|100\rangle$ and $|011\rangle$. The set of two bits is called error syndrome and it tells Bob in which physical qubit the error occurred during transmission

Three-Qubit Bit-Flip Code

- The states after error extraction is made is given below:

State after error syndrome extraction
$(a\|000\rangle + b\|111\rangle)\|00\rangle$
$(a\|100\rangle + b\|011\rangle)\|11\rangle$
$(a\|010\rangle + b\|101\rangle)\|10\rangle$
$(a\|001\rangle + b\|110\rangle)\|01\rangle$

Three-Qubit Bit-Flip Code

- Error recovery: Once the *error syndrome* is detected, error recovery is very simple.
- Suppose $(a|100\rangle + b|011\rangle)|11\rangle$ is the state after error syndrome extraction.
- Now Bob measures his two ancillary qubits and obtains two bits 11 of classical information.
- In this case it tells us that a bit-flip error occurred on qubit number 3 (or 11 in binary). So, Bob corrects the error by applying X-gate to qubit number 3.
- $$a|100\rangle + b|011\rangle \longrightarrow a|000\rangle + b|111\rangle$$

Now, by applying the inverse of the encoding procedure he recovers $a|0\rangle + b|1\rangle$

$$a|000\rangle + b|111\rangle \longrightarrow a|0\rangle + b|1\rangle\ |00\rangle$$

Three-Qubit Bit-Flip Code

- The same procedure works in the case that the second or third qubit experiences a bit-flip as well.

Error Syndrome	**Correction to be made**
(00)	Do Nothing
(01)	Apply X gate to 1st qubit (X_1)
(10)	Apply X gate to 2nd qubit (X_2)
(11)	Apply X gate to 3rd qubit (X_3)

Phase-Flip QECC

- Now we will see, how to handle only a single qubit phase flip which might creep in during transmission. Phase flip error is equivalent to action of Pauli matrix *Z*

$$a|0\rangle + b|1\rangle \to a|0\rangle - b|1\rangle$$

- Here Hadamard gate comes to our rescue again. The idea is that phase flip error in the $|+\rangle, |-\rangle$ basis is same as bit flip error

$$|+\rangle \to |-\rangle$$
$$|-\rangle \to |+\rangle$$

This simple idea gives the whole algorithm for phase flip errors:

Phase-Flip QECC

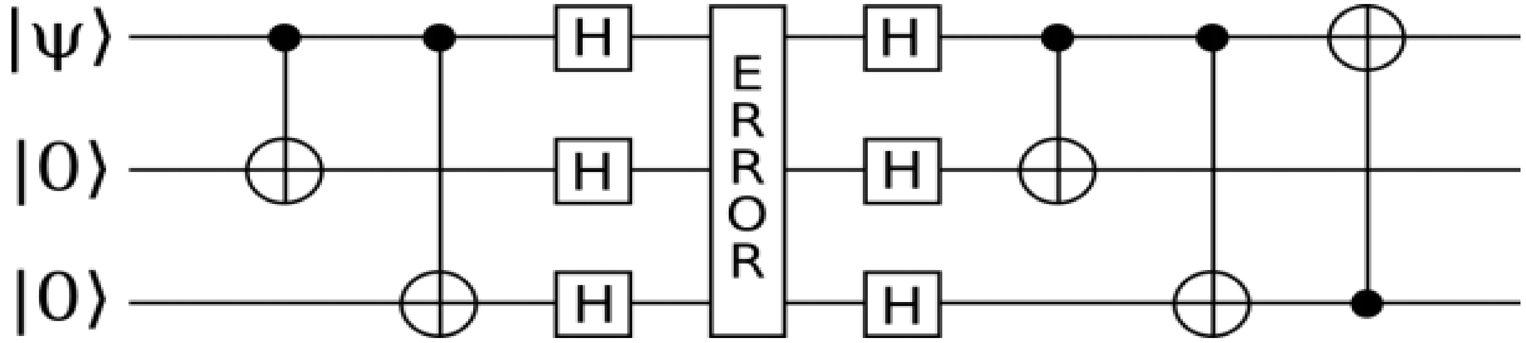


Figure 4: Phase Flip Encoding and Decoding [Wik17]

- Append ancilla bits: $a|0\rangle + b|1\rangle \rightarrow a|000\rangle + b|111\rangle$
- Apply Hadamard: $a|000\rangle + b|111\rangle \rightarrow a|+++\rangle + b|---\rangle$
- Apply bit flip algorithm to detect and correct a bit flip error in the $|+\rangle, |-\rangle$ basis.
- Apply Hadamard again to recover original message in $|0\rangle, |1\rangle$ basis.

Shor's Nine-qubit Code

- The Shor code is such a code which uses 9 qubits to encode 1 qubit. Essentially, the Shor code is a combination of bit flip and phase flip codes. We first encode a qubit using the encoding procedure of the phase flip code:

$$|0\rangle \rightarrow |+++\rangle, |1\rangle \rightarrow |---\rangle$$

- This is followed by encoding each of these 3 qubits according to the bit flip encoding procedure:

$$|+\rangle \rightarrow (|000\rangle + |111\rangle)/\sqrt{2}$$
$$|-\rangle \rightarrow (|000\rangle - |111\rangle)/\sqrt{2}$$

Thus the final codewords are:

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$
$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Shor's Nine-qubit Code

- The Fig. 5 below shows Shor's nine-qubit encoding

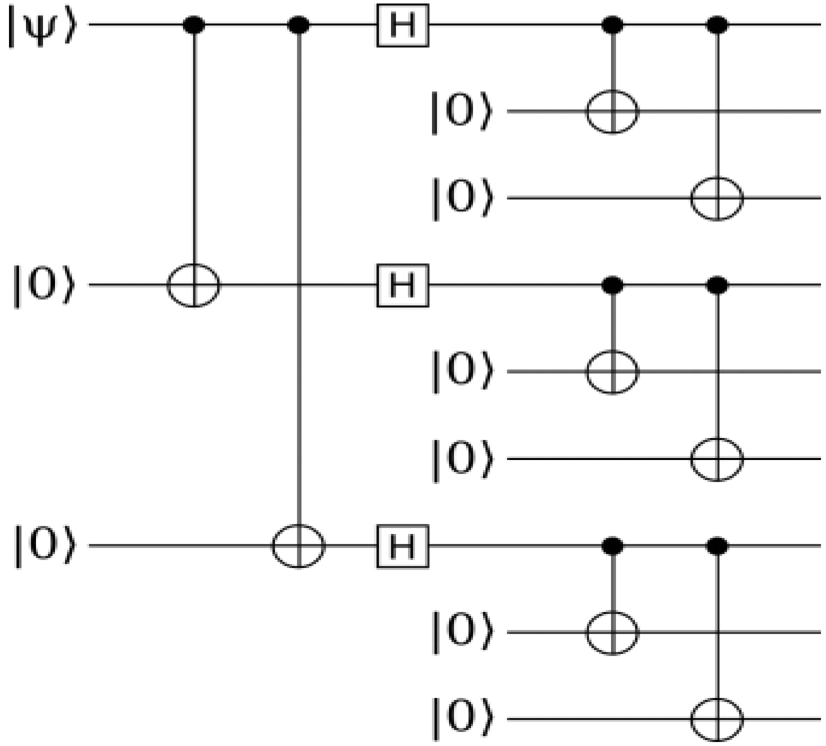


Figure 5: Shor Code Encoding [Wik17]

Shor's Nine-qubit Code

- One can show that shor's code detects and corrects the following errors:
    - Bit flip error on a single qubit.
    - Phase flip error on a single qubit.
    - Combined bit and phase flip error on a single qubit.