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Winter term 23/24

Mathematics of Learning – Sheet 1 – Discussion on October 19/20th, 2023

Some information:

- The exercise sheets will be uploaded every Monday. Solution sketches will be uploaded one week later.
- You can hand in your own solutions via StudOn and we correct them - this is not mandatory. Please hand in in small groups of 2-3 students.
- For questions, please use the forum on StudOn since other students may have similar questions. If you have a more personal question about the exercises please send an email to ehsan.waiezi@fau.de or lars.weidner@fau.de respectively.

Basics [Solving linear equation systems.][1]
Solve the linear equation system $Ax = b$. A and b are given as

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}.$$

Control yourself, if your solution is right. If you need some practice, generate some random linear equation systems and solve them.

Basics [Norms.]
A mapping $\|\cdot\|$ from any (real) vector space V to the non-negative real numbers $\mathbb{R}$ is called a norm, whenever the following three properties hold:

$$\begin{aligned} &(1)\ \|v+w\| \leq \|v\| + \|w\|, \\ &(2)\ \|v\| = 0 \implies v = 0_V, \\ &(3)\ \|\lambda v\| = |\lambda|\|v\| \\ &\text{ for all } \lambda \in \mathbb{R}, v, w \in V. \end{aligned}$$

Prove for the following statements if they are true or false.

1. Let $V = \mathbb{R}^n$ for some $n \in \mathbb{N}$. The euclidean norm

$$\|v\|_2 := \sqrt{\sum_{i=1}^{n} v_i^2}$$

is a norm.

[1] There are lots of nice tutorial books for linear algebra and analysis available in our library. For a less formal introduction, you can, e.g., also consult wikipedia ;)

2. Let $V = \mathbb{R}^n$ for some $n \in \mathbb{N}$. The mapping

$$\|v\|_{\frac{1}{2}} := (\sum_{i=1}^{n} \sqrt{|v_i|})^2$$

is a norm.

3. Let V be the space of convergent sequences. The mapping

$$\|v\|_{lim} := \lim_{n\to\infty} v_n$$

is a norm.

Bonus exercise: find a mapping to the real numbers, which has the three properties and takes some negative values, or show that it is impossible.

Basics [Big-O (Landau) Notation.]

a) Express the relationships of the functions n^{1000}, 2^n, e^n, e^{n^2}, $n!$ and n^n with the help of Landau's notation (i.e., prove, if $f \in \mathcal{O}(g), f \in \Omega(g)$ or $f \in \Theta(g)$ for every pair of the functions above). Prove your statements.

b) An equivalence relation is a homogeneous relation over some set M^2, which is

1. Reflexive: $m_1 \simeq m_1$,
2. Symmetric: $m_1 \simeq m_2$ implies $m_2 \simeq m_1$,
3. Transitive: If $m_1 \simeq m_2$ and $m_2 \simeq m_3$ then $m_1 \simeq m_3$

for all $m_1, m_2, m_3 \in M$. For example, the relation $\simeq$ on $\mathbb{R}^n$, $v \simeq w$ if and only if $v_1 = w_1$ is an equivalence relation.
Prove or disprove, that $\simeq_L$ which we define as

$$f \simeq_L g \text{ if and only if } f \in \Theta(g)$$

is an equivalence relation on the set of mappings from $\mathbb{N}$ to $\mathbb{R}_{>0}$.

[2]i.e. $\simeq: M \times M \to \{0,1\}$; one would rather write $m_1 \simeq m_2$ instead of $\simeq (m_1, m_2) = 1$ and $m_1 \not\simeq m_2$ instead of $\simeq (m_1, m_2) = 0$.