Performance Analysis of Communication Systems

Bit Error Rate (BER) is employed to characterize the performance of a communication system.

Transmitted Bit stream	1	0	0	1	1	1	0	1	0
Received Bit stream	1	0	0	0	1	1	0	0	0

Bit Errors occur during the communication process.

BER = Average rate of bit error for a particular scheme.

For instance, if 10,000 bits are transmitted and out of these 100 bits are received in error.

$$\text{Average BER} = \frac{\text{No. of bits in Error}}{\text{Total No. of bits Transmitted}} = \frac{100}{10,000} = \frac{n_e}{N}$$

$$= 0.01 \text{ or } 1\%$$

BER is frequently expressed as a probability of bit error (P_e).

$$\underline{0 \leq P_e \leq 0.5}$$

Naturally, the requirement of BER of a digital system depends upon the application.

- For speech transmission $\Rightarrow 10^{-2} \leq \text{BER} \leq 10^{-3}$
- For data transmission over wireless channels $\Rightarrow 10^{-5} \leq \text{BER} \leq 10^{-6}$
- For video transmission $\Rightarrow 10^{-7} \leq \text{BER} \leq 10^{-12}$
- For financial data $\Rightarrow$ $\text{BER} \geq 10^{-11}$

- In digital system, BER is often represented in terms of signal-to-noise ratio.

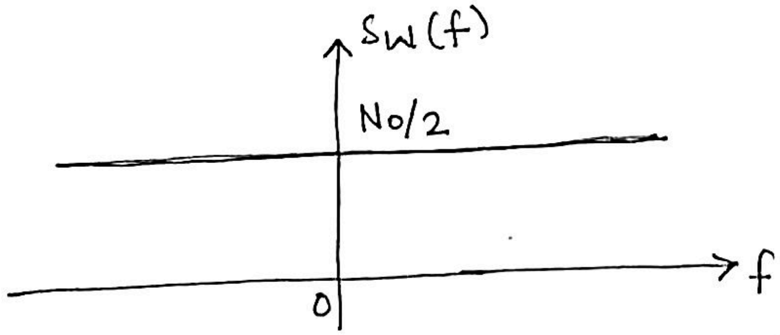
$$SNR_{ref}^{digital} = \frac{\text{Modulated energy per bit}}{\text{Noise spectral density}} = \frac{E_b}{N_o}$$

- To compare digital modulation-demodulation strategies, the objective is to determine BER performance as a function of the reference SNR, denoted by E_b/N_o. This reference model provides a frame of reference for a fair comparison of different schemes.

<u>White Noise</u> :- The noise analysis of communication systems is often based on an idealized noise process called white noise. The PSD of white noise is independent of frequency. White noise is analogous to the term "white light" in the sense that all frequency components are present in equal amounts. We denote the PSD of a white noise $w(t)$ as -

$$S_W(f) = \frac{N_o}{2} \quad \text{Watts/Hz}$$

Here, factor $\frac{1}{2}$ has been included to indicate that half the power is associated with positive frequencies and half with negative frequencies.

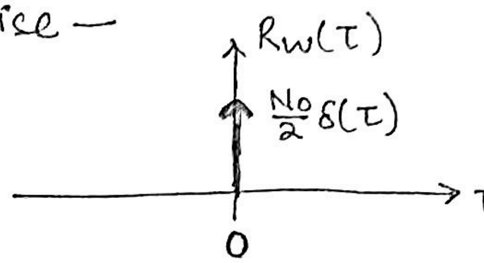


$$S_W(f) = \frac{N_o}{2}, \; -\infty < f < \infty$$

- The parameter N_o is usually measured at the input stage of receiver.

Autocorrelation function of white noise –

$$R_W(\tau) = \text{IFT}\left[S_W(f) \right]$$



- Mean value is zero.
  $\mu = 0$.

- Ideal LPFiltered white Noise

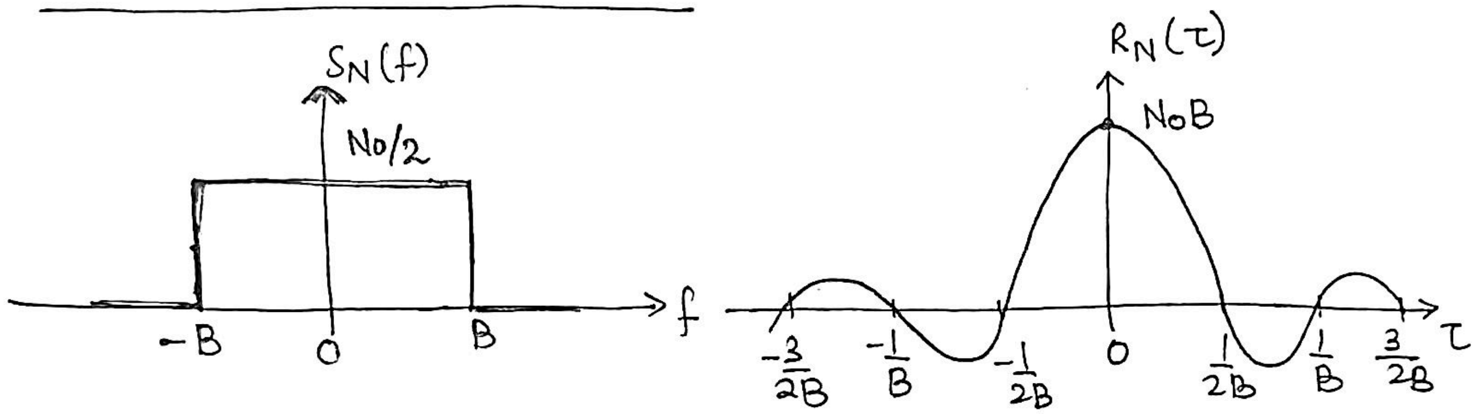
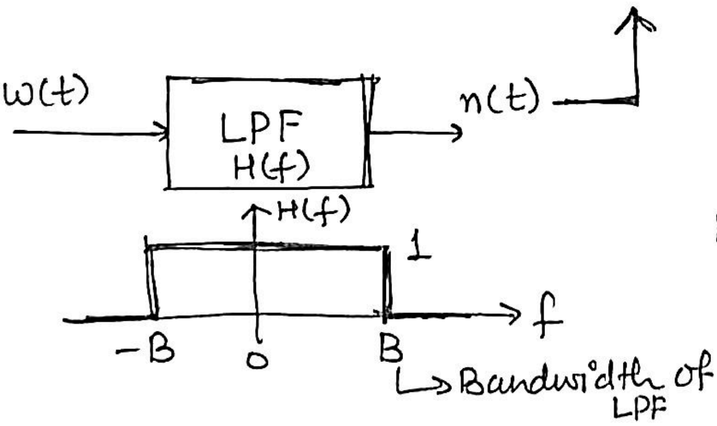


Fig:- characteristics of low-pass filtered white noise (a) PSD
(b) Autocorrelation function.

$$S_N(f) = \begin{cases} \frac{N_0}{2}, & |f| < B \\ 0, & |f| > B \end{cases}$$

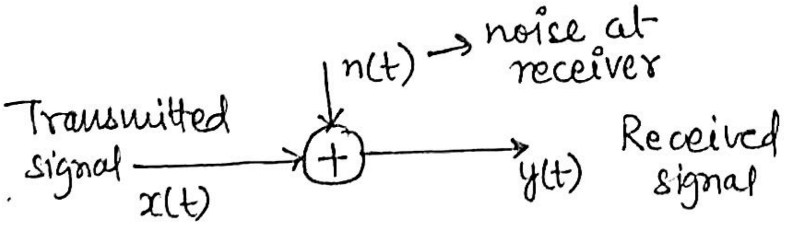


$$R_N(\tau) = \int_{-B}^{B} \frac{N_0}{2} e^{j2\pi f_c \tau} df$$

$$= N_0 B \operatorname{sinc}(2B\tau)$$

$$\tau = \pm \frac{n}{2B} \Rightarrow n = 1, 2, 3 \ldots$$

$$\text{Noise Power} = \frac{N_0}{2} \times 2B = N_0 B$$



Model of a Basic wireless system

$$y(t) = x(t) + n(t)$$

Additive white Gaussian Noise (AWGN)

PDF of Noise, $n(t)$ $\Rightarrow$ $f_N(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\downarrow$

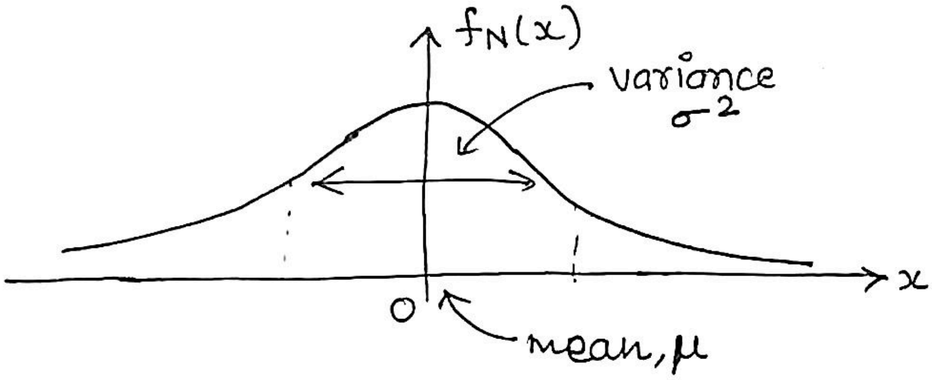
Probability density function

$\mu = 0 \Rightarrow$ Mean $\quad N(0, \sigma^2)$

$\sigma^2 =$ Variance

$\hookrightarrow$ Noise Power

$$f_N(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$



$\Rightarrow$ The Gaussian PDF is completely characterized by the mean and the variance of the distribution.

- spread of Gaussian distribution is related to σ^2.
- As σ^2 increases, the spread of Gaussian distribution increases.

Calculation of BER/SER From Constellation Diagram

We apply the union bound approximation for average probability of symbol error.

$$SER = \frac{1}{2} \times AVNB \times erfc\left(\sqrt{\frac{d_{min}^2}{4N_0}}\right)$$

AVNB = Average number of nearest neighbours.
(Nearest neighbours lie at min^m Euclidean distance)

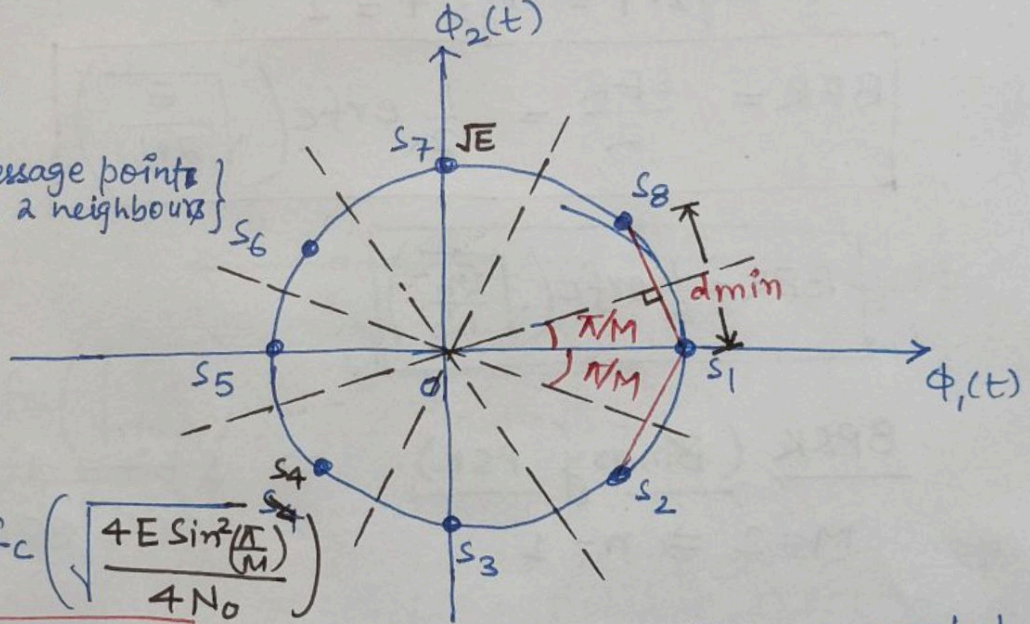
8-PSK (M=8)

$$d_{min} = 2\sqrt{E}\sin\left(\frac{\pi}{M}\right)$$

AVNB = 2 ⇒ Each message point has 2 neighbours

Ex: For $S_1 < \begin{matrix} S_2 \\ S_8 \end{matrix}$

Neighbours located at d_{min}



$$SER = \frac{1}{2} \times 2 \times erfc\left(\sqrt{\frac{4E\sin^2\left(\frac{\pi}{M}\right)}{4N_0}}\right)$$

$$\boxed{SER = erfc\left(\sqrt{\frac{E\sin^2(\pi/M)}{N_0}}\right)}$$

⇒ M=8

→ M>2

$$SER = erfc\left(\sqrt{\frac{0.146E}{N_0}}\right)$$

3 bits = 1 symbol

⇓

⇒ $M = 8 = 2^3 = 2^n$

$\boxed{n=3}$

$$\boxed{BER = \frac{SER}{3} = \frac{1}{3}erfc\left(\sqrt{\frac{0.146E}{N_0}}\right)}$$

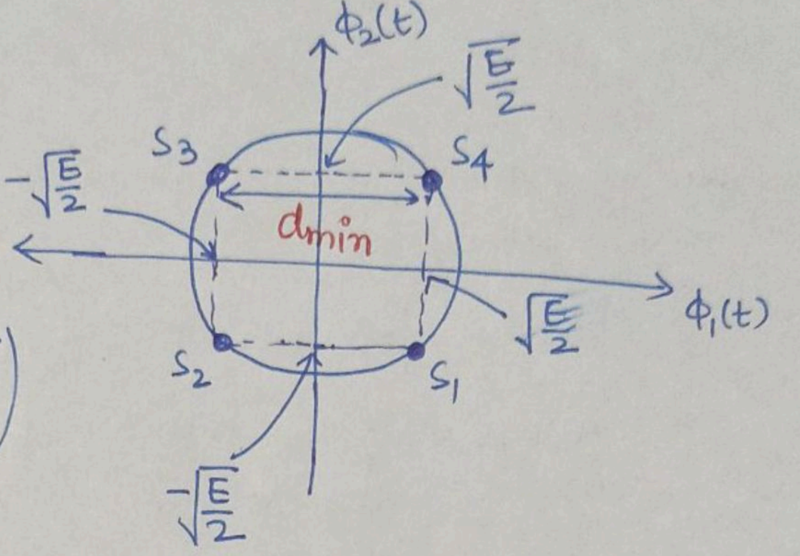
$E = 3E_b$

bit energy

symbol energy

$$\boxed{BER = \frac{1}{3}erfc\left(\sqrt{\frac{0.44E_b}{N_0}}\right)}$$

QPSK (M=4)



$d_{min} = \sqrt{\frac{E}{2}} \times 2 = \sqrt{2E}$

AVNB = 2

$SER = \frac{1}{2} \times 2 \times erfc\left(\sqrt{\frac{2E}{4N_0}}\right)$

$SER = erfc\left(\sqrt{\frac{E}{2N_0}}\right)$

$n = \log_2 M = \log_2 4 = 2 \Rightarrow$ 2 bits = 1 symbol

$$\boxed{BER = \frac{SER}{2} = \frac{1}{2} erfc\left(\sqrt{\frac{E}{2N_0}}\right)}$$

$E = 2E_b$

$$\boxed{BER = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)}$$

BPSK (Binary PSK)

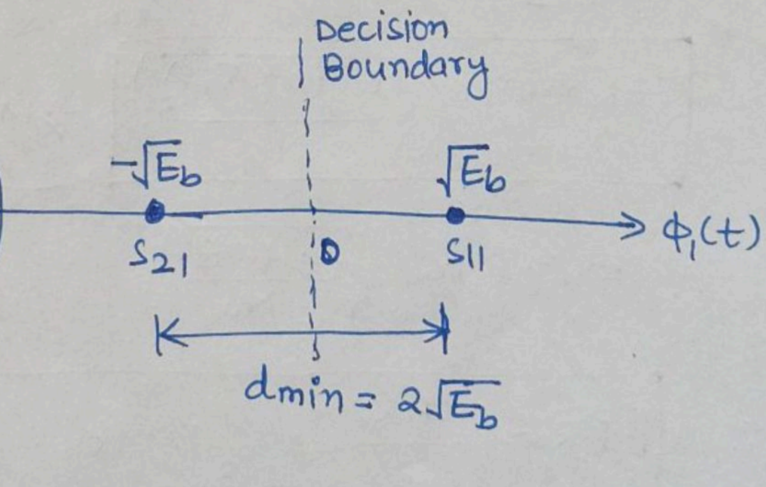
$\Rightarrow$ 1 bit = 1 symbol

$M = 2 \Rightarrow n = 1$

SER = BER and $E = E_b$

AVNB = 1

$BER = SER = \frac{1}{2} \times 1 \times erfc\left(\sqrt{\frac{4E_b}{4N_0}}\right)$

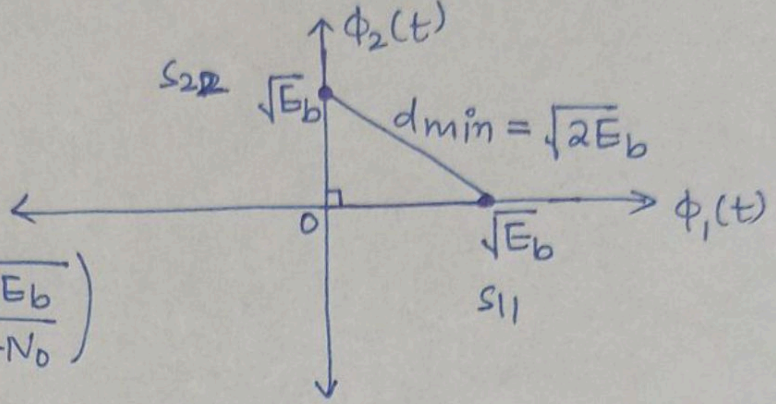


$$\boxed{BER = SER = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)}$$

BFSK

$d_{min} = \sqrt{2E_b}$

AVNB = 1



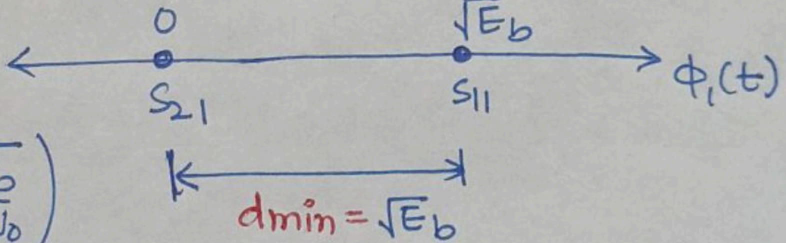
$$BER = \frac{1}{2} \times 1 \times erfc\left(\sqrt{\frac{2E_b}{4N_0}}\right)$$

$$\boxed{BER = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{2N_0}}\right)}$$

BASK

$d_{min} = \sqrt{E_b}$

AVNB = 1



$$BER = \frac{1}{2} \times 1 \times erfc\left(\sqrt{\frac{E_b}{4N_0}}\right)$$

$$\boxed{BER = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{4N_0}}\right)}$$

SER/BER → Probability of error.

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{4N_0}}\right)$$

$$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right)$$

$$P_e = Q\left(\sqrt{\frac{E_b}{2N_0}}\right)$$

Chernoff Bound $\Rightarrow Q(x) \leq \frac{1}{2} e^{-x^2/2}$

$$P_e \leq \frac{1}{2} e^{-E_b/4N_0}$$

~~In General~~

Probability of Error of M-ary QAM

$$SER = P_e^{M\text{-}QAM} = \left[4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E}{N_0(M-1)}}\right)\right]$$

$$Q(x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

$$SER = P_e^{M\text{-}QAM} = \left[4\left(1-\frac{1}{\sqrt{M}}\right) \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{3E}{2N_0(M-1)}}\right)\right]$$

$$\boxed{P_e^{M\text{-}QAM} = \left[2\left(1-\frac{1}{\sqrt{M}}\right) \mathrm{erfc}\left(\sqrt{\frac{3E}{2N_0(M-1)}}\right)\right]}$$

E = symbol energy.

Gray-coding $\Rightarrow BER^{M\text{-}QAM} = \dfrac{P_e^{M\text{-}QAM}}{\log_2 M} = \dfrac{SER^{M\text{-}QAM}}{\log_2 M}$