PRS- PBL

Exponential Distribution and the Process of Radioactive Decay

SUBMITTED BY

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Exponential Distribution

In probability theory, the exponential distribution is defined as the probability distribution of time between events in the Poisson point process. The exponential distribution is considered as a special case of the gamma distribution. Also, the exponential distribution is the continuous analogue of the geometric distribution.

What is Exponential Distribution?

In Probability theory and statistics, the exponential distribution is a continuous probability distribution that often concerns the amount of time until some specific event happens. It is a process in which events happen continuously and independently at a constant average rate. The exponential distribution has the key property of being memoryless. The exponential random variable can be either more small values or fewer larger variables. For example, the amount of money spent by the customer on one trip to the supermarket follows an exponential distribution.

Exponential Distribution Formula

The continuous random variable, say X is said to have an exponential distribution, if it has the following probability density function:

$$f_X(x|\lambda) = \begin{cases} \lambda e^{-\lambda x} & for\ x > 0 \\ 0 & for\ x \leq 0 \end{cases}$$

Where λ is called the distribution rate.

Mean and Variance of Exponential Distribution:-

Mean:

The mean of the exponential distribution is calculated using the integration by parts.

$$\text{Mean} = E[X] = \int_0^\infty x\lambda e^{-\lambda x}\, dx$$

$$= \lambda\left[\left|\frac{-xe^{-\lambda x}}{\lambda}\right|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\, dx\right]$$

$$= \lambda\left[0 + \frac{1}{\lambda}\frac{-e^{-\lambda x}}{\lambda}\right]_0^\infty$$

$$= \lambda\frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda}$$

Hence, the mean of the exponential distribution is 1/λ.

Variance:

To find the variance of the exponential distribution, we need to find the second moment of the exponential distribution, and it is given by:

$$E[X^2] = \int_0^\infty x^2\lambda e^{-\lambda x} = \frac{2}{\lambda^2}$$

Hence, the variance of the continuous random variable, X is calculated as:

$$Var(X) = E(X^2) - E(X)^2$$

Now, substituting the value of mean and the second moment of the exponential distribution, we get,

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Thus, the variance of the exponential distribution is $1/\lambda^2$.

Memoryless Property of Exponential Distribution:-

The most important property of the exponential distribution is the memoryless property. This property is also applicable to the geometric distribution.

An exponentially distributed random variable "X" obeys the relation:

$P_r(X > s+t \mid X>s) = P_r(X>t)$, for all $s, t \geq 0$

Now, let us consider the the complementary cumulative distribution function:

$$P_r(X > s+t \mid X>s) = \frac{P_r(X>s+t \cap X>s)}{P_r(X>s)}$$

$$= \frac{P_r(X>s+t)}{P_r(X>s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$= P_r(X>t)$$

Hence, $P_r(X > s+t \mid X>s) = P_r(X>t)$

This property is called the memoryless property of the exponential distribution, as we don't need to remember when the process has started.

Sum of Two Independent Exponential Random Variables:-

The probability distribution function of the two independent random variables is the sum of the individual probability distribution functions.

If X1 and X2 are the two independent exponential random variables with respect to the rate parameters λ1 and λ2 respectively, then the sum of two independent exponential random variables is given by

Z = X1 + X2.

$$f_Z z = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(z - x_1) dx_1$$

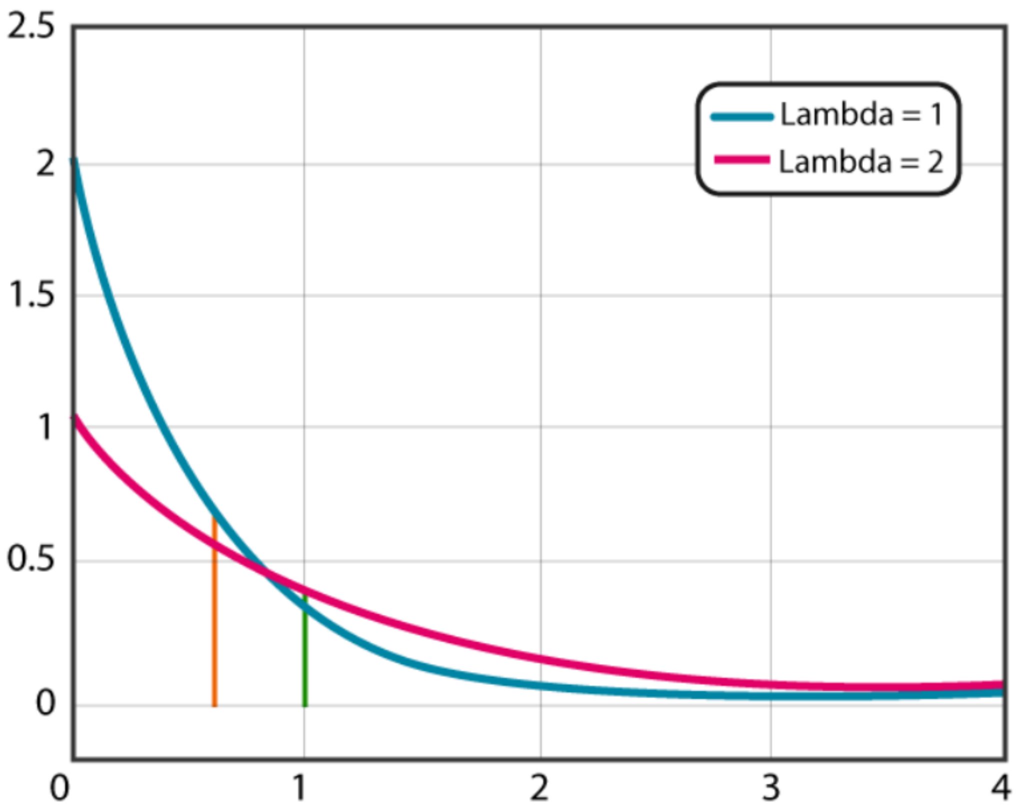
$$= \int_0^z \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 (z - x_1)} dx_1$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 z} \int_0^z e^{(\lambda_2 - \lambda_1) x_1} dx_1$$

$$= \begin{cases} \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right) & if\ \lambda_1 \neq \lambda_2 \\ \lambda^2 z e^{-\lambda z} & if\ \lambda_1 = \lambda_2 = \lambda \end{cases}$$

Exponential Distribution Graph

The exponential distribution graph is a graph of the probability density function which shows the distribution of distance or time taken between events. The two terms used in the exponential distribution graph is lambda (**λ**)and x. Here, lambda represents the events per unit time and x represents the time. The following graph shows the values for λ=1 and λ=2.



Exponential Distribution Applications:-

One of the widely used continuous distribution is the exponential distribution. It helps to determine the time elapsed between the events. It is used in a range of applications such as reliability theory, queuing theory, physics and so on. Some of the fields that are modelled by the exponential distribution are as follows:

- Exponential distribution helps to find the distance between mutations on a DNA strand
- Calculating the time until the radioactive particle decays.

- Helps on finding the height of different molecules in a gas at the stable temperature and pressure in a uniform gravitational field
- Helps to compute the monthly and annual highest values of regular rainfall and river outflow volumes

Exponential Distribution and the Process of Radioactive Decay

The problem of radioactive decay is one of the simplest examples which show the connection between deterministic models and stochastic approach. We emphasize that the stochastic model is not an alternative to the deterministic approach, rather it is a generalization: a satisfactory stochastic model gives a derivation for the deterministic model and more.

The deterministic model for radioactive decay $A \longrightarrow A^*$ is a simple linear differential equation

$$\frac{dN_A}{dt} = -kN_A$$

where N_A is the number of atom A and k is the rate of decay. To solve this ODE with initial condition $N_A(0) = N_0$, we have

$$N_A(t) = N_0 e^{-\lambda t}. \qquad (6.1)$$

If we observe the decay of atom A one by one, then one realizes that the time at which A transforming to A is random. Hence the microscopic process of radioactive decay has to be modeled by a stochastic model. Specifically, if we denote X as the lifetime of A, X is a continuous,positive random variable.How do we determine the probability distribution for X? The basic assumption for the stochastic model is that the decay of A,as a random event, is independent of the time the atom being in A.In other words, the conditional probability :*

$$Prob\{\mathbf{X} > t + h | \mathbf{X} > t\} = Prob\{\mathbf{X} > h\}.$$

As we will soon see, this assumption is sufficient to determine the probability distribution of X. It is known as the exponential distribution.

Role Of Exponential Distribution

The exponential distribution has the very important property known as memoriless:

$$Prob\{\mathbf{X} > t + h | \mathbf{X} > t\} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = Prob\{\mathbf{X} > h\}.$$

In fact, this nice property is the defining character of the exponential distribution. Consider a RV X with

$$\begin{aligned} Prob\{\mathbf{X} > t + h\} &= Prob\{\mathbf{X} > t + h | \mathbf{X} > t\} Prob\{\mathbf{X} > t\} \\ &= Prob\{\mathbf{X} > h\} Prob\{\mathbf{X} > t\}. \end{aligned}$$

Let

$$G(t) = Prob\{\mathbf{X} > t\},$$

then

$$G(t+h) = G(t)G(h)$$

$$G'(t+h) = G'(t)G(h) = G(t)G'(h)$$

$$\Longrightarrow \qquad \frac{G'(t)}{G(t)} = \frac{G'(h)}{G(h)} = -\lambda$$

$$\Longrightarrow \qquad G'(t) = -\lambda G(t)$$

i.e., $\mathbf{X}$ is exponential

$$Prob\{\mathbf{X} \le t\} = 1 - G(t) = 1 - e^{-\lambda t}.$$

Microscopic versus Macroscopic Models

What is the relation between Eqn.6.1 and Eqn.6.2. Afterall, both are mathematical models for the radioactive decay. The relation lies upon the a system of N number of independent atoms with very large *N. From Eqn.6.2, we note that at any time , the atom is still in A with probability e^-λ t and in A* with 1-e^-λ t .*

This is a binary distribution.Hence, for N° iid, we have

$$Prob\{\mathbf{N}_t = n\} = \frac{N_0!}{n!(N_0 - n)!} e^{-n\lambda t} \left(1 - e^{-\lambda t}\right)^{N_0 - n}$$

where RV $\mathbf{N}_t$ is the number of atoms being A at time t. So the expectation and variance for $\mathbf{N}_t$ are

$$E[\mathbf{N}_t] = N_0 e^{-\lambda t}, \quad Var[\mathbf{N}_t] = N_0 e^{-\lambda t}\left(1 - e^{-\lambda t}\right)$$

which indicates that Eqn. 6.1 is simply the mean of the $\mathbf{N}_t$. The stochastic model, however, also provide an estimation for the variance. It is shown that for a large N_0 on the order of 10^{20}, the relative broadness of the distribution

$$\frac{Var[\mathbf{N}_t]}{E[\mathbf{N}_t]^2} \propto \frac{1}{N_0} \approx 0.$$

This proves that the deterministic model is a very good approximation if one deals with large number of atoms, i.e., macroscopic. It also shows when studying system of only a few number of atoms (microscopic), the stochastic model is more general.

Example Problem

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Q)A container contains 13 particles, at time t=0t=0. The particles decay independently of each other and the time (unit: minutes) for a given particle's decay is a exponentially distributed random variable with expectation value 36.436.4. Let TT denote the time that has passed when the number of particles has been reduced to 12. Calculate the probability P(T>1.035714).

Solution:

$$f(x) = \frac{1}{\beta}\exp(-x/\beta), \beta > 0, 0 < x < \infty$$

Numerically, we have

$$f(x) = \frac{1}{36.4}\exp(-x/36.4)$$

We want the probability

$$P(T > 1.035714) = 1 - F(1.035714) = 1 - \int_0^{1.035714} \frac{1}{36.4}\exp(-x/36.4)dx = \exp$$
$$-1.035714/36.4 = 0.97194731$$

That is the probability that a particular particle does not decay in that time i.e 0.97194731

But there are 13 particles, we want the probability that none of the13 decay in that time
We know that their decays are independent of each other.

The *rate* of decay for one particle is $\frac{1}{36.4}$ so the rate for the first decay of 13 independent particles is $\frac{13}{36.4}$, making the probability that this does not happen by time 1.035714 be $\exp\left(-1.035714 \times \frac{13}{36.4}\right) \approx 0.6908$

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