Technical Electrodynamics for iCE

(in English language)

Prof. Dr.-Ing. Irina Munteanu

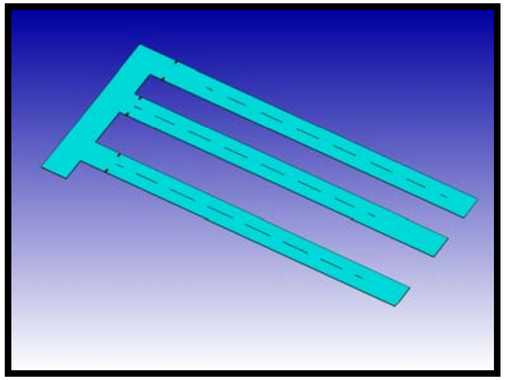
Institut für Theorie Elektromagnetischer Felder (TEMF)

Graduate School of Computational Engineering (GSC)

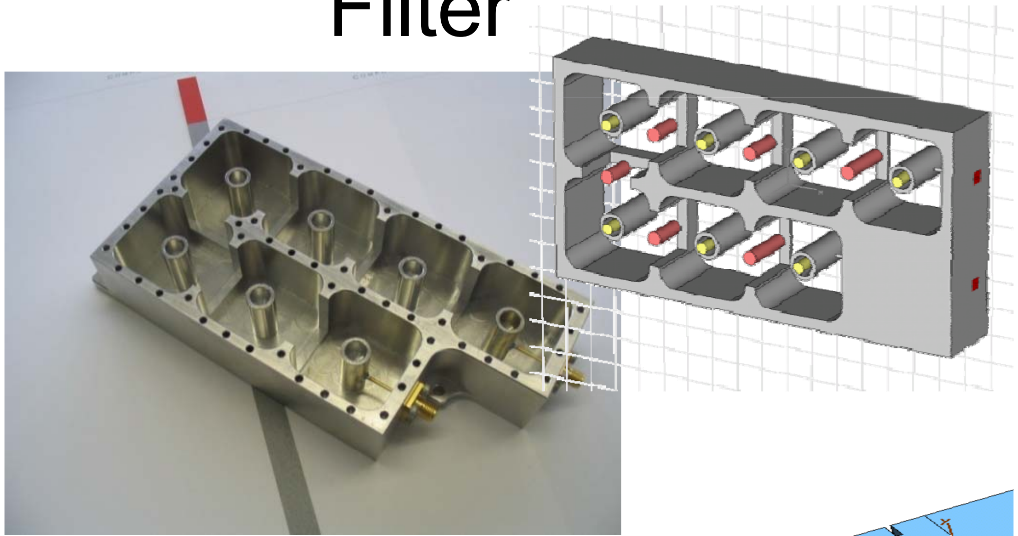
Waveguides

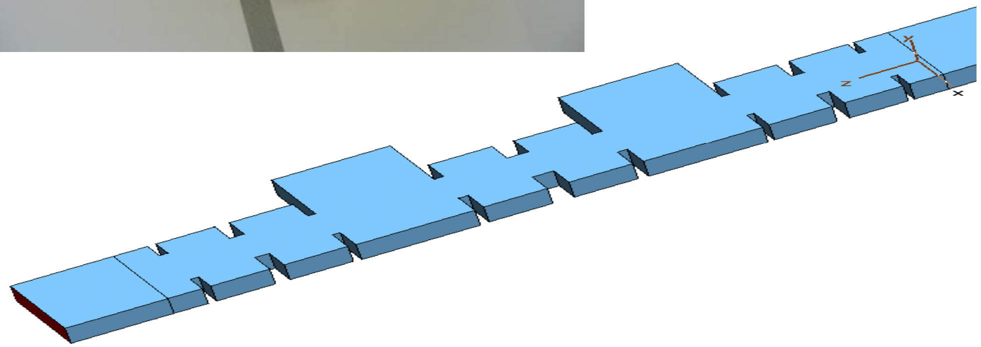
Waveguide structures: applications

Multiplexer

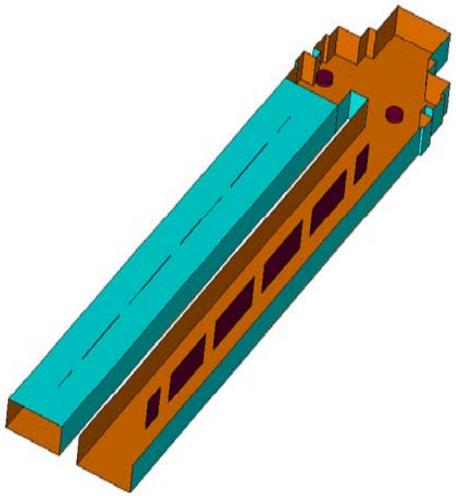


Filter

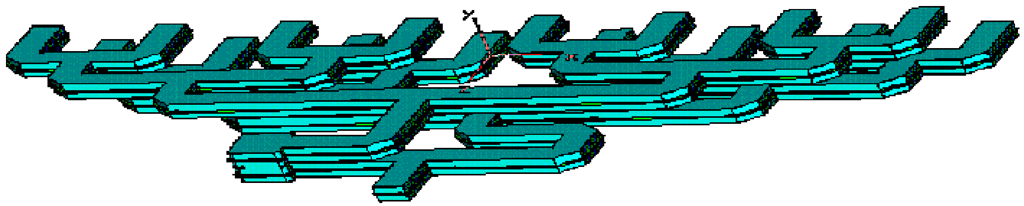




Diplexer



16-way power divider

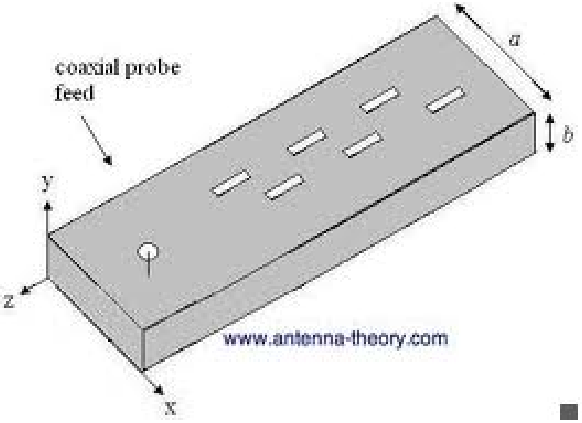


Source: CST AG

Waveguide structures: applications

Source: Microwave Engineering Services Corp.

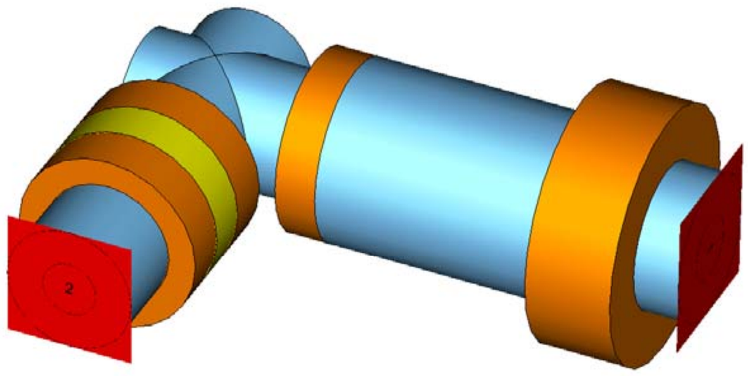
Source: Quinstar.com



Source: antennatheory.com

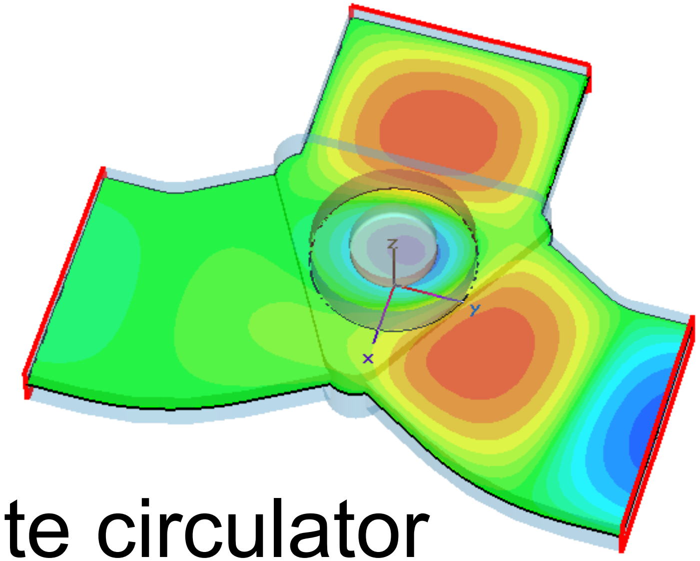
Source: fairviewmicrowawe.com

Waveguide structures: applications



Coaxial connectors

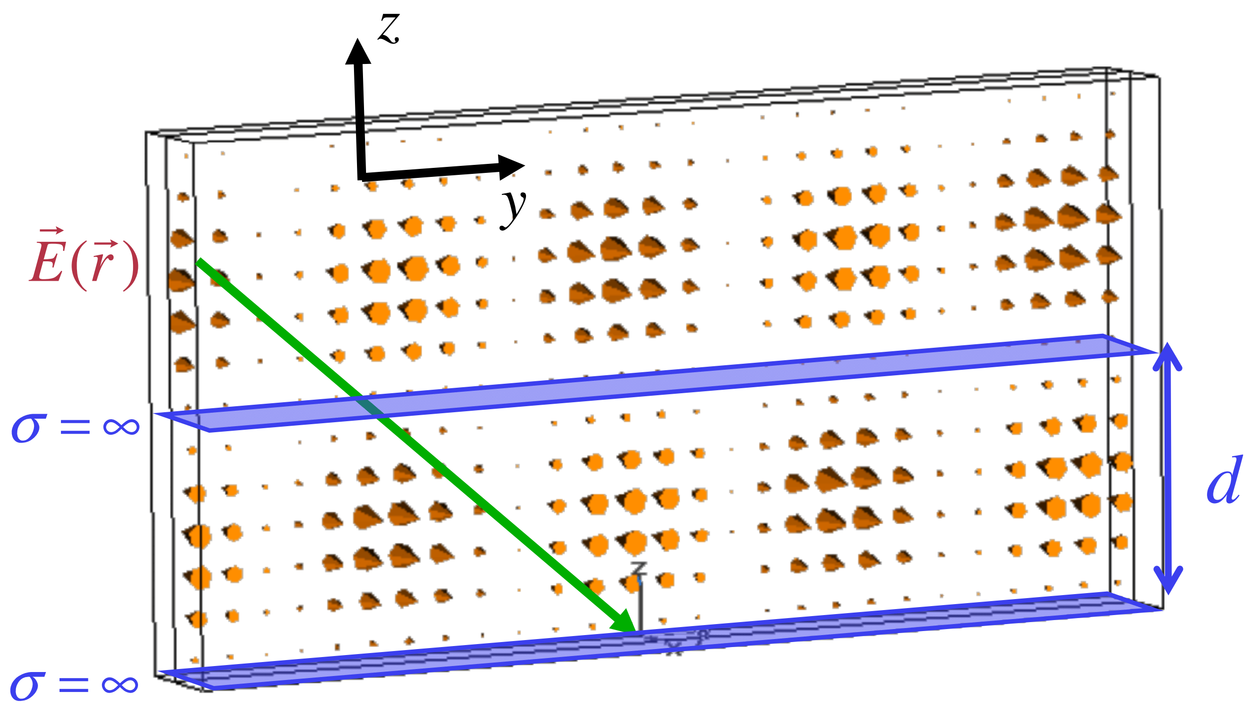
Source: CST AG



Ferrite circulator

Source: CST AG

- **Transmit power from a source to a load for antennas, magnetrons, power division, …**
- **Able to transfer large powers → e.g. microwave oven**
- **Mechanically robust → used in safety-critical applications (e.g. space applications such as satellites)**



$$Z_2 = 0 \quad \rightarrow \quad r_s = -1$$

$$E_x(y,z) \propto e^{-jk_1 \sin\phi_i y} \cdot \sin(k_1 \cos\phi_i \, z)$$

$$k_1 = \frac{\omega}{c}$$

transversal:

$$\sin(k_1 \cos\phi_i \, d) = 0 \quad \Rightarrow \quad \frac{\omega}{c}\cos\phi_i \, d = n\pi$$

longitudinal:

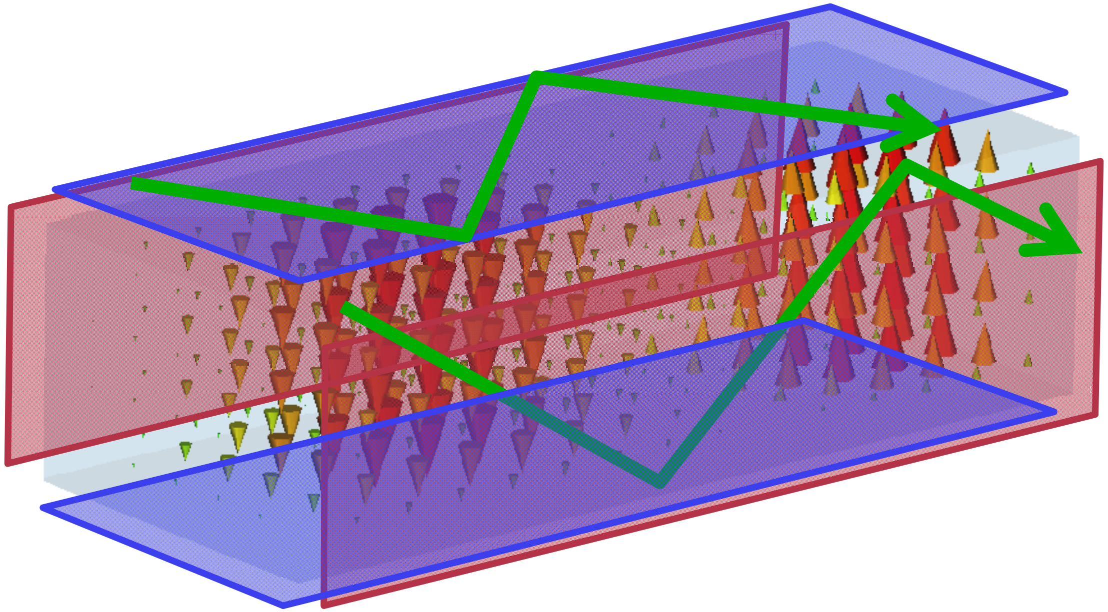
$$\beta_y = k_1 \sin\phi_i = \frac{\omega}{c}\sin\phi_i$$

$$\Rightarrow \quad \boxed{\beta_y^2 + \left(\frac{n\pi}{d}\right)^2 = \left(\frac{\omega}{c}\right)^2}$$

dispersion relation of a parallel plate waveguide

From plane waves to guided waves

- ☞ Oblique incidence of plane waves at metallic half plane
  - standing wave part transversal to the metallic plane
- ☞ Repeat this principle in 2nd direction:



➡ (Rectangular) Hollow Waveguide

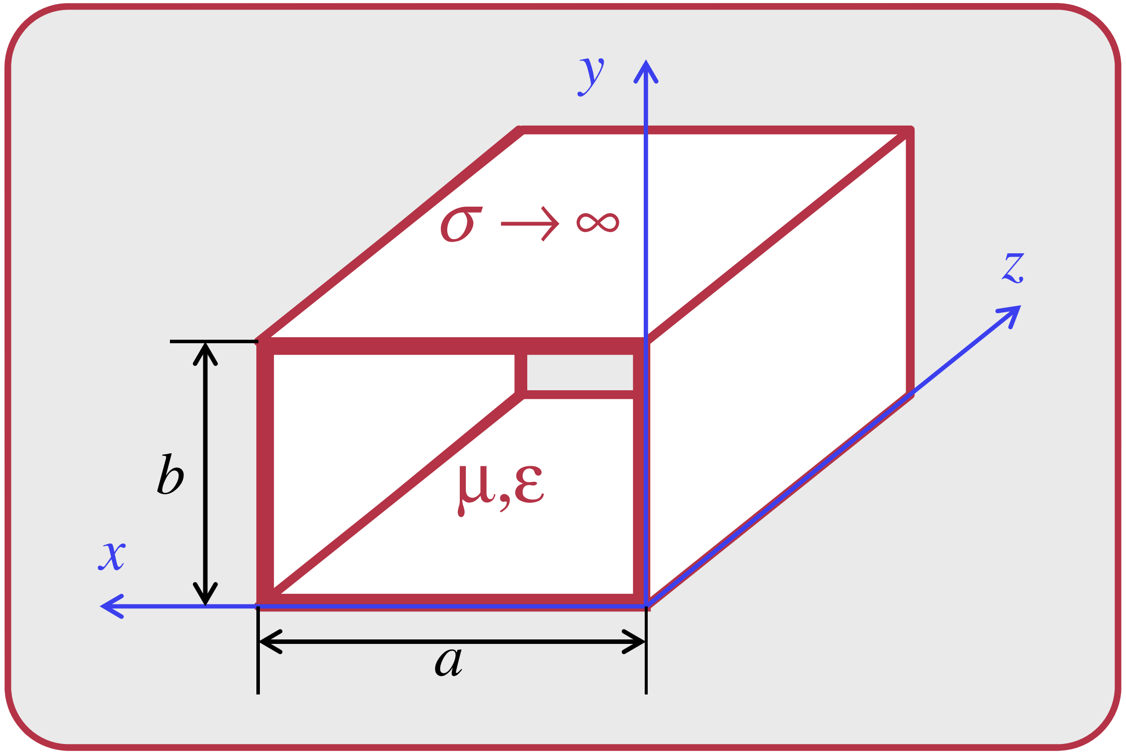
- ☞ Modified dispersion characteristics

Rectangular Waveguides

- **Rectangular waveguide:** $a>b$, perfectly conducting walls

☞ **2 Wave types:**

- **TE (‚transversal electric')** $E_z = 0$
- **TM (‚transversal magnetic')** $H_z = 0$



(Lecture Notes pp. 49-57)

- **Wave equation for vector potential (homogeneous medium):**

$$\mathrm{div}\,\vec{E} = 0 \Rightarrow \vec{E} = \mathrm{curl}\,\vec{A}$$

(TE-case, TM: equivalent)

$$\mathrm{curl}\,\vec{E} = \mathrm{curl}\,\mathrm{curl}\,\vec{A} = \mathrm{grad}\,\mathrm{div}\,\vec{A} - \Delta\vec{A} = -j\omega\mu\vec{H}$$

(Faraday's law)

$$\mathrm{curl}\,\vec{H} = j\omega\varepsilon\vec{E} = j\omega\varepsilon\,\mathrm{curl}\,\vec{A}$$

(Ampere's law)

$$\Rightarrow \quad \vec{H} = j\omega\varepsilon\vec{A} - \mathrm{grad}\,\phi$$

(integration constant)

$$\mathrm{grad}\,\mathrm{div}\,\vec{A} - \Delta\vec{A} = \omega^2\mu\varepsilon\vec{A} + j\omega\mu\,\mathrm{grad}\,\phi$$

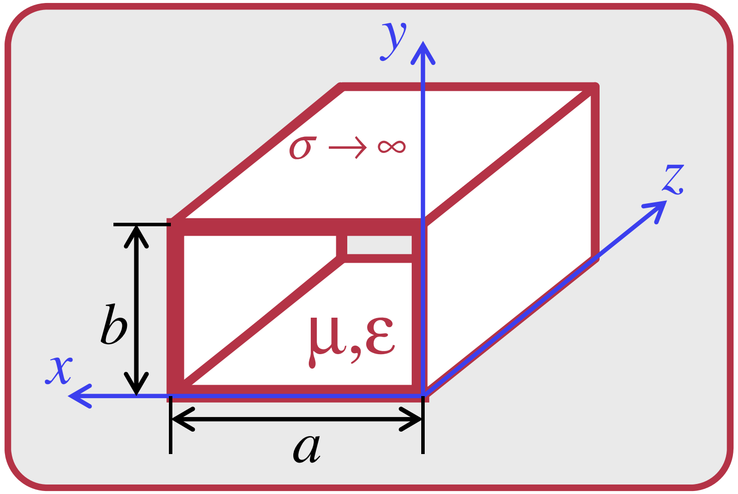
$$\text{set:} \quad \mathrm{div}\,\vec{A} = j\omega\mu\phi$$

‚Lorentz gauge'

$$\Rightarrow \boxed{\Delta\vec{A} + k^2\vec{A} = 0 \qquad k^2 = \omega^2\mu\varepsilon}$$

Rectangular Waveguides

- 2 Wave types:
  - **TE:** $E_z = 0$
  - **TM:** $H_z = 0$
  - **Vector potential approach:** $\vec{A}(\vec{r}) = A_z(x,y,z)\,\vec{e}_z$



(Lecture Notes p. 49)

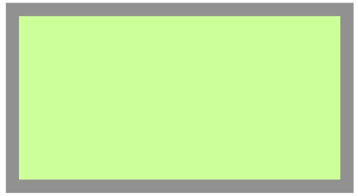
$$\vec{E} = \mathrm{curl}\,\vec{A} = \mathrm{curl}\begin{pmatrix}0\\0\\A_z\end{pmatrix} = \begin{pmatrix}*\\*\\0\end{pmatrix} \;\Rightarrow\; \text{TE-case}$$

$$\vec{H} = \mathrm{curl}\,\vec{A} = \mathrm{curl}\begin{pmatrix}0\\0\\A_z\end{pmatrix} = \begin{pmatrix}*\\*\\0\end{pmatrix} \;\Rightarrow\; \text{TM-case}$$

formulate and solve wave equation for $\vec{A}(\vec{r})$

Rectangular Waveguides

- **Solutions of the wave equation (Summary):** Rectangular WG



$$\Delta\vec{A} + k^2\vec{A} = 0 \qquad k^2 = \omega^2\mu\varepsilon$$

$$\rightarrow \Delta A_z + k^2 A_z = 0 \qquad \vec{A} = \vec{e}_z A_z$$

☞ product approach: $A_z(x,y,z) = f(x)\cdot g(y)\cdot h(z)$

$$(\ldots)\ \boxed{A_z(x,y,z) = \left\{\begin{matrix}\sin(k_x x)\\ \cos(k_x x)\end{matrix}\right\}\cdot\left\{\begin{matrix}\sin(k_y y)\\ \cos(k_y y)\end{matrix}\right\}\cdot\left\{\begin{matrix}e^{+jk_z z}\\ e^{-jk_z z}\end{matrix}\right\}}$$

superposition: $\left(C_1\sin(k_x x) + C_2\cos(k_x x)\right)\cdot\ldots$

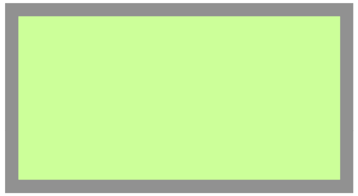
☞ *separation relation*: $k_x^2 + k_y^2 + k_z^2 = k^2$ (insert Az in wave eqn.)

☞ To determine unknown coefficients: boundary conditions for fields !

Rectangular Waveguides

- **Derivation of the solutions of the wave equation:**

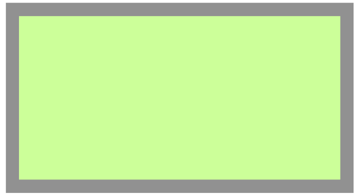
Rectangular WG



Rectangular Waveguides

- **Solutions of the wave equation:**

Rectangular WG



$$\Delta\vec{A} + k^2\vec{A} = 0 \qquad k^2 = \omega^2\mu\varepsilon$$

$$\rightarrow \Delta A_z + k^2 A_z = 0 \quad \vec{A} = \vec{e}_z A_z$$

$$\boxed{\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} + k^2 A_z = 0}$$

(A_z complex variable)

1. Product approach: $A_z(x,y,z) = f(x)\cdot g(y)\cdot h(z)$

2. Replace in the equation and divide by A_z :

$$\underbrace{\frac{1}{f}\frac{\partial^2 f}{\partial x^2}}_{-k_x^2} + \underbrace{\frac{1}{g}\frac{\partial^2 g}{\partial y^2}}_{-k_y^2} + \underbrace{\frac{1}{h}\frac{\partial^2 h}{\partial z^2}}_{-k_z^2} + k^2 = 0$$

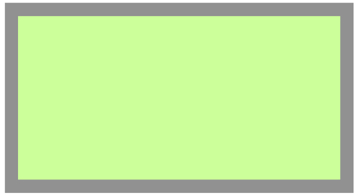
$$k_x^2 + k_y^2 + k_z^2 = k^2$$

!

Rectangular Waveguides

Solutions of the wave equation:

Rectangular WG



- Remember the solution of the wave equation:

$$\frac{\partial^2 \underline{E}_x}{\partial z^2} + k^2 \underline{E}_x = 0 \Leftrightarrow \quad \frac{1}{\underline{E}_x}\frac{\partial^2 \underline{E}_x}{\partial z^2} + k^2 = 0$$

$$\Rightarrow \underline{E}_x = \underline{C}_1 e^{-jkz} + \underline{C}_2 e^{jkz}$$

- By using $e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$ the solution can be rewritten as

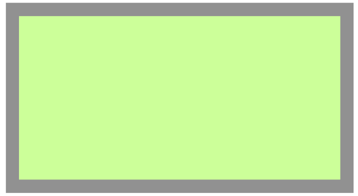
$$\underline{E}_x = \underline{C}_1(\cos kz - \sin kz) + \underline{C}_2(\cos kz + \sin kz) = \underline{D}_1 \cos kz + \underline{D}_2 \sin kz$$

$$\text{with } \underline{D}_1 = \underline{C}_1 + \underline{C}_2, \underline{D}_2 = \underline{C}_1 - \underline{C}_2$$

- The form with exponential is useful for describing a propagation, the form with cos/sin for describing standing waves

Solutions of the wave equation:

Rectangular WG



☞ *separation relation*: $k_x^2 + k_y^2 + k_z^2 = k^2$

3. All equations are of the same form and have the solutions:

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = -k_x^2 \Rightarrow f(x) = A_1 \cos(k_x x) + A_2 \sin(k_x x)$$

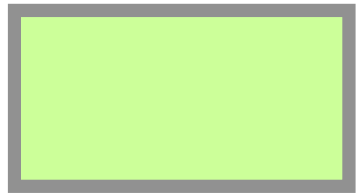
$$\frac{1}{g}\frac{\partial^2 g}{\partial y^2} = -k_y^2 \Rightarrow g(y) = B_1 \cos(k_y y) + B_2 \sin(k_y y)$$

$$\frac{1}{h}\frac{\partial^2 h}{\partial z^2} = -k_z^2 \Rightarrow h(z) = C_1 e^{jk_z z} + C_2 e^{-jk_z z}$$

Rectangular Waveguides

Solutions of the wave equation:

Rectangular WG



5. Final solution:

$$f(x) = A_1 \cos(k_x x) + A_2 \sin(k_x x)$$

$$g(y) = B_1 \cos(k_y y) + B_2 \sin(k_y y)$$

$$h(z) = C_1 e^{jk_z z} + C_2 e^{-jk_z z}$$

$$A_z(x,y,z) = f(x) \cdot g(y) \cdot h(z)$$

☞ Compact notation:

$$\boxed{A_z(x,y,z) = \underbrace{\begin{Bmatrix} \sin(k_x x) \\ \cos(k_x x) \end{Bmatrix}}_{\text{superposition: } \left(A_2 \sin(k_x x) + A_1 \cos(k_x x)\right) \cdot \ldots} \cdot \begin{Bmatrix} \sin(k_y y) \\ \cos(k_y y) \end{Bmatrix} \cdot \begin{Bmatrix} e^{+jk_z z} \\ e^{-jk_z z} \end{Bmatrix}}$$

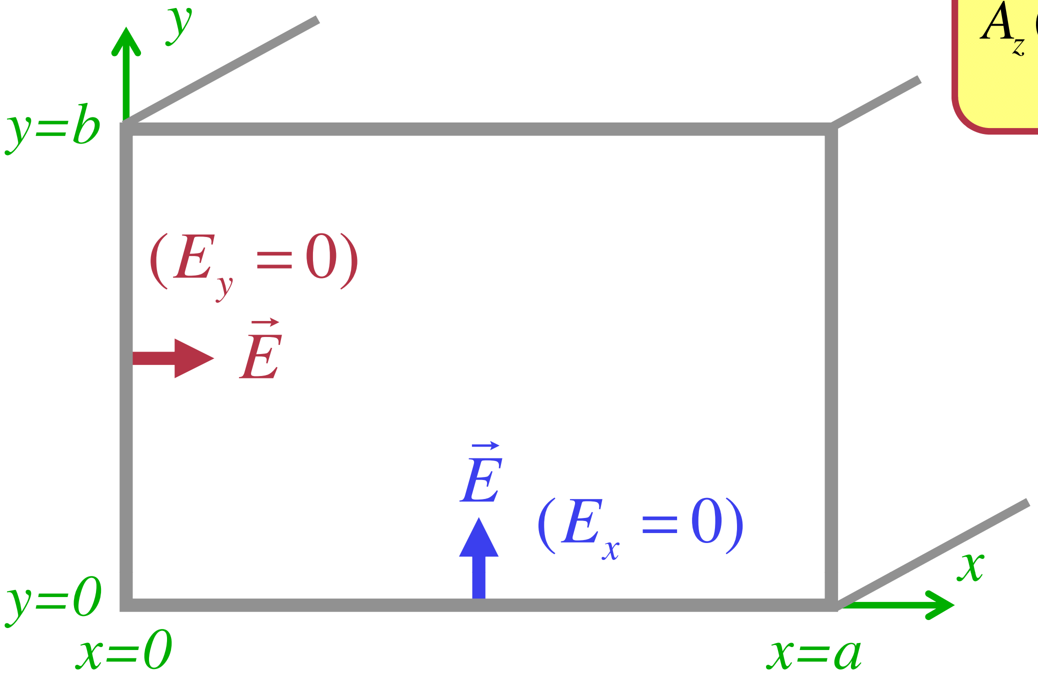
☞ To determine unknown coefficients: boundary conditions for fields !

- **Imposing the boundary conditions**

Rectangular Waveguides

- **TE-solution:**

$$A_z(x,y,z)=\begin{Bmatrix}\sin(k_x x)\\ \cos(k_x x)\end{Bmatrix}\cdot\begin{Bmatrix}\sin(k_y y)\\ \cos(k_y y)\end{Bmatrix}\cdot\begin{Bmatrix}e^{+jk_z z}\\ e^{-jk_z z}\end{Bmatrix}$$



$$\vec{E}=\mathrm{curl}\,\vec{A}=(\ldots E_x\ldots)\,\mathbf{e}_x+(\ldots E_y\ldots)\,\mathbf{e}_y+(\ldots\,0\,\ldots)\,\mathbf{e}_z$$

$$\vec{E}_x\sim\frac{\partial A}{\partial y},\vec{E}_y\sim\frac{\partial A}{\partial x}$$

Continuity (1): $E_x(x\quad,y=0)=0$

$E_y(x=0,y\quad)=0$

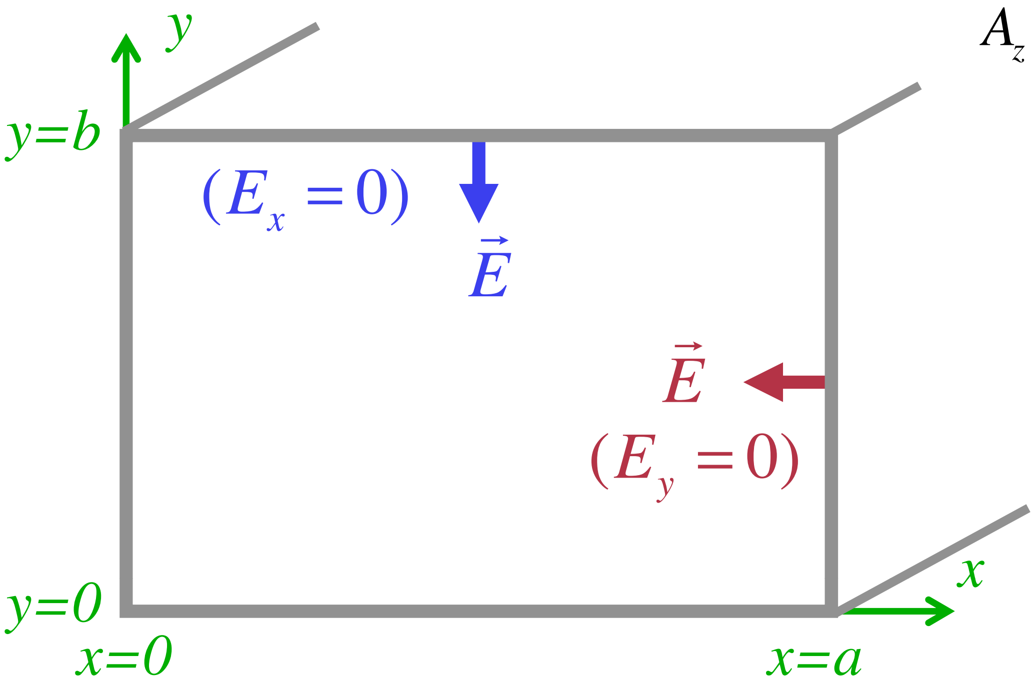
$(\ldots)$

$$A_z(x,y,z)=\begin{Bmatrix}\cancel{\sin(k_x x)}\\ \cos(k_x x)\end{Bmatrix}\cdot\begin{Bmatrix}\cancel{\sin(k_y y)}\\ \cos(k_y y)\end{Bmatrix}\cdot\begin{Bmatrix}e^{+jk_z z}\\ e^{-jk_z z}\end{Bmatrix}$$

(Lecture Notes pp. 51-52)

Rectangular Waveguides

- **TE-solution (...):**



$$A_z(x,y,z) = \left\{ \begin{matrix} \cancel{\sin(k_x x)} \\ \cos(k_x x) \end{matrix} \right\} \cdot \left\{ \begin{matrix} \cancel{\sin(k_y y)} \\ \cos(k_y y) \end{matrix} \right\} \cdot \left\{ \begin{matrix} e^{+jk_z z} \\ e^{-jk_z z} \end{matrix} \right\}$$

$$E_x(x,y) \propto \cos(k_x x) \cdot \sin(k_y y) k_y$$

$$E_y(x,y) \propto \sin(k_x x) \cdot \cos(k_y y) k_x$$

Continuity (2): $E_x(x \quad , y = b) = 0$

$E_y(x = a, y \quad) = 0$

$\Rightarrow$ (...) $\quad k_y = \dfrac{n\,\pi}{b}$

$k_x = \dfrac{m\,\pi}{a}$

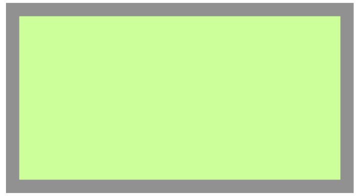
(transversal separation constants)

(Lecture Notes p. 52)

Rectangular Waveguides

- **Calculation of the field components – TE modes**

Rectangular WG



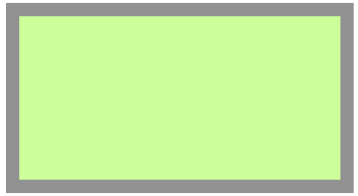
$$\vec{E} = curl\vec{A}$$

$$\vec{H} = -\frac{1}{j\omega\mu} curl\,\vec{E}$$

Rectangular Waveguides

- **Field components**

Rectangular WG



☞ TE modes

$$E_x = -C\frac{n\pi}{b}\cos(m\pi x/a)\sin(n\pi y/b)e^{\pm jk_z z}$$

$$E_y = +C\frac{m\pi}{a}\sin(m\pi x/a)\cos(n\pi y/b)e^{\pm jk_z z}$$

$$E_z = 0$$

$$H_x = \pm C\frac{k_z}{\omega\mu}\frac{m\pi}{a}\sin(m\pi x/a)\cos(n\pi y/b)e^{\pm jk_z z}$$

$$H_y = \pm C\frac{k_z}{\omega\mu}\frac{n\pi}{b}\cos(m\pi x/a)\sin(n\pi y/b)e^{\pm jk_z z}$$

$$H_z = -C\frac{k^2 - k_z^2}{j\omega\mu}\cos(m\pi x/a)\cos(n\pi y/b)e^{\pm jk_z z}$$

☞ TM modes - similar

$$E_x = \pm C\frac{k_z}{\omega\varepsilon}\frac{m\pi}{a}\cos(m\pi x/a)\sin(n\pi y/b)e^{\pm jk_z z}$$

$$E_y = \pm C\frac{k_z}{\omega\varepsilon}\frac{n\pi}{b}\sin(m\pi x/a)\cos(n\pi y/b)e^{\pm jk_z z}$$

$$E_z = +C\frac{k^2 - k_z^2}{j\omega\varepsilon}\sin(m\pi x/a)\sin(n\pi y/b)e^{\pm jk_z z}$$

$$H_x = +C\frac{n\pi}{b}\sin(m\pi x/a)\cos(n\pi y/b)e^{\pm jk_z z}$$

$$H_y = -C\frac{m\pi}{a}\cos(m\pi x/a)\sin(n\pi y/b)e^{\pm jk_z z}$$

$$H_z = 0$$

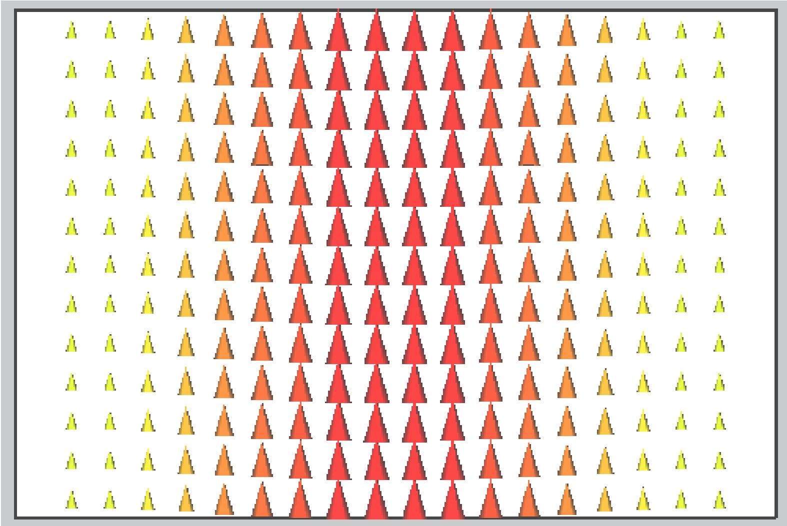
Rectangular Waveguides

- **Infinite set of solutions (‚*Waveguide Modes*'):**

$$A_z(x,y,z) = A_0 \cos(k_x x)\cdot\cos(k_y y)\cdot\begin{Bmatrix} e^{+jk_z z} \\ e^{-jk_z z}\end{Bmatrix}$$

$$E_x(x,y) \propto \cos(k_x x)\cdot\sin(k_y y)$$

$$E_y(x,y) \propto \sin(k_x x)\cdot\cos(k_y y)$$



$$k_x = \frac{m\,\pi}{a} \qquad m = 0,1,2,\ldots$$

$$k_y = \frac{n\,\pi}{b} \qquad n = 0,1,2,\ldots$$

$$k_z^2 = k^2 - (k_x^2 + k_y^2)$$

$$= \omega^2\mu\varepsilon - (k_x^2 + k_y^2)$$

$$(m,n) = \{ (1,0), (2,0), (3,0),\ldots,$$
$$(0,1), (1,1), (2,1), (3,1),\ldots.$$
$$(0,2),(1,2), (2,2), (3,2),\ldots,$$
$$\ldots\}$$

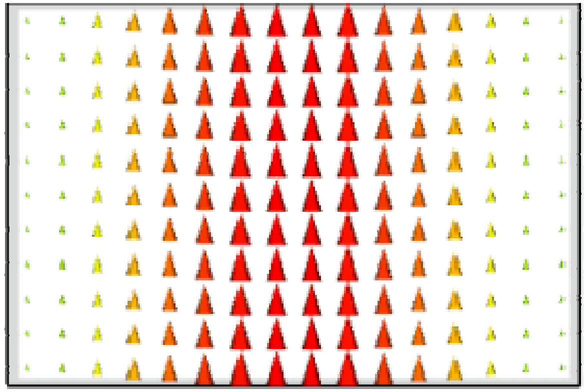
$$(m = n = 0:\ \text{all zero})$$

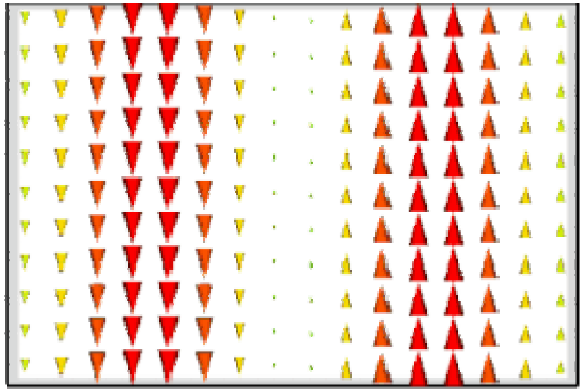
Rectangular Waveguides

m=0 m=1 m=2

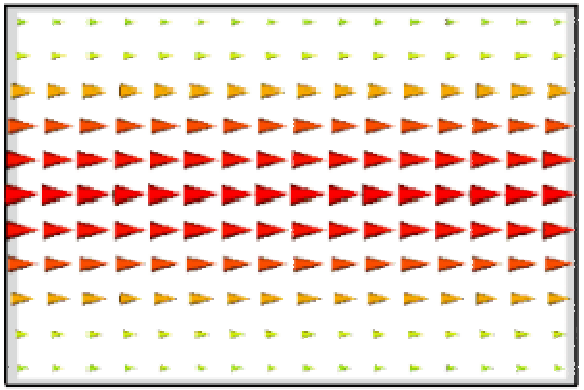
n=0

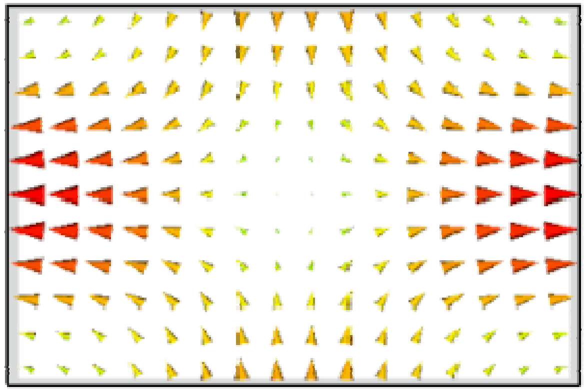
TE-modes,
electric field

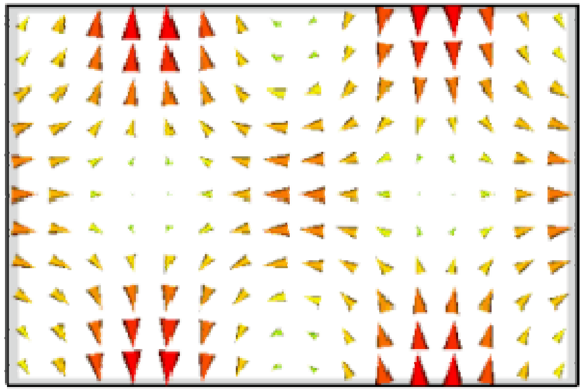




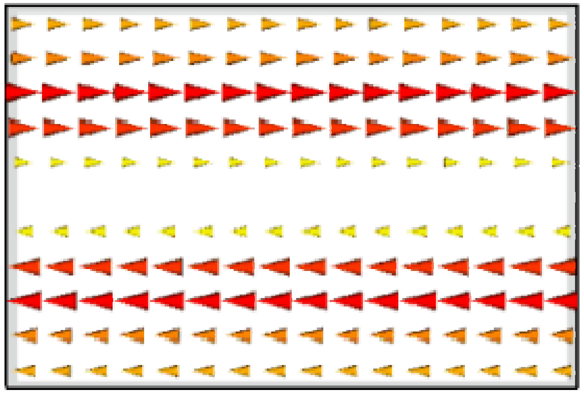
n=1

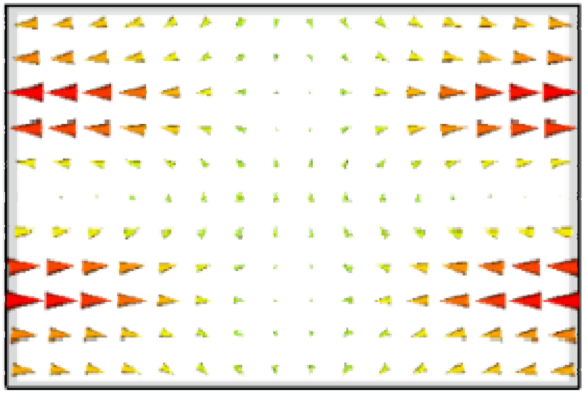






n=2





(...)

Number of half-periods:
m: along x
n: along y

Waveguide modes

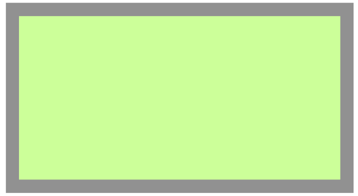
- ☞ Solution of the wave equation in a hollow guide define field patterns for the electric (and the magnetic) field
- ☞ This pattern propagates in longitudinal direction
- ☞ Total field may be a superposition of:
  - different modes
  - modes propagating in +/- z-direction
- ☞ Two types of modes:
  - TE (‚transversal electric‘) and TM (‚transversal magnetic‘)
  - sometimes also referred to as H_z and E_z-modes
- ☞ In rectangular guides:
  - modes are determined by lengths a and b

Waveguides
(dispersion relation)

Rectangular Waveguides

- **Dispersion relation:**

Rectangular WG



$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k_z^2 = \omega^2\mu\varepsilon$$

periodicity in x

periodicity in y

wave propagation in waveguide depends on mode type !

$$k_z = \sqrt{\omega^2\mu\varepsilon - \left(\frac{(m\pi)^2}{a^2} + \frac{(n\pi)^2}{b^2}\right)}$$

$$= \frac{2\pi}{c}\sqrt{f^2 - f_c^2}$$

$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

f_c = *cutoff-frequency*

$$k_z = \begin{cases} \text{real (wave propagation)} & f > f_c \\ \text{imag. (attenuation)} & f < f_c \end{cases}$$

(hint: see degenerated modes)

Two very important formulas

- **Cutoff frequency**

$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- **Propagation constant in the waveguide (≠ $\omega\sqrt{\mu\varepsilon}$!)**

$$k_z = \sqrt{\omega^2\mu\varepsilon - \left(\frac{(m\pi)^2}{a^2} + \frac{(n\pi)^2}{b^2}\right)} = \sqrt{\left(\frac{2\pi f}{c}\right)^2 - \left(\frac{(m\pi)^2}{a^2} + \frac{(n\pi)^2}{b^2}\right)} =$$

$$= \frac{2\pi}{c}\sqrt{f^2 - f_c^2} \qquad \text{at cutoff } k_z = 0$$

- **Fundamental mode: the mode with the smallest cutoff frequency – TE10**

Rectangular Waveguides

- **Dispersion diagram:**

  ☞ Plot k_z^2 as a function of f^2

- **Formula:**

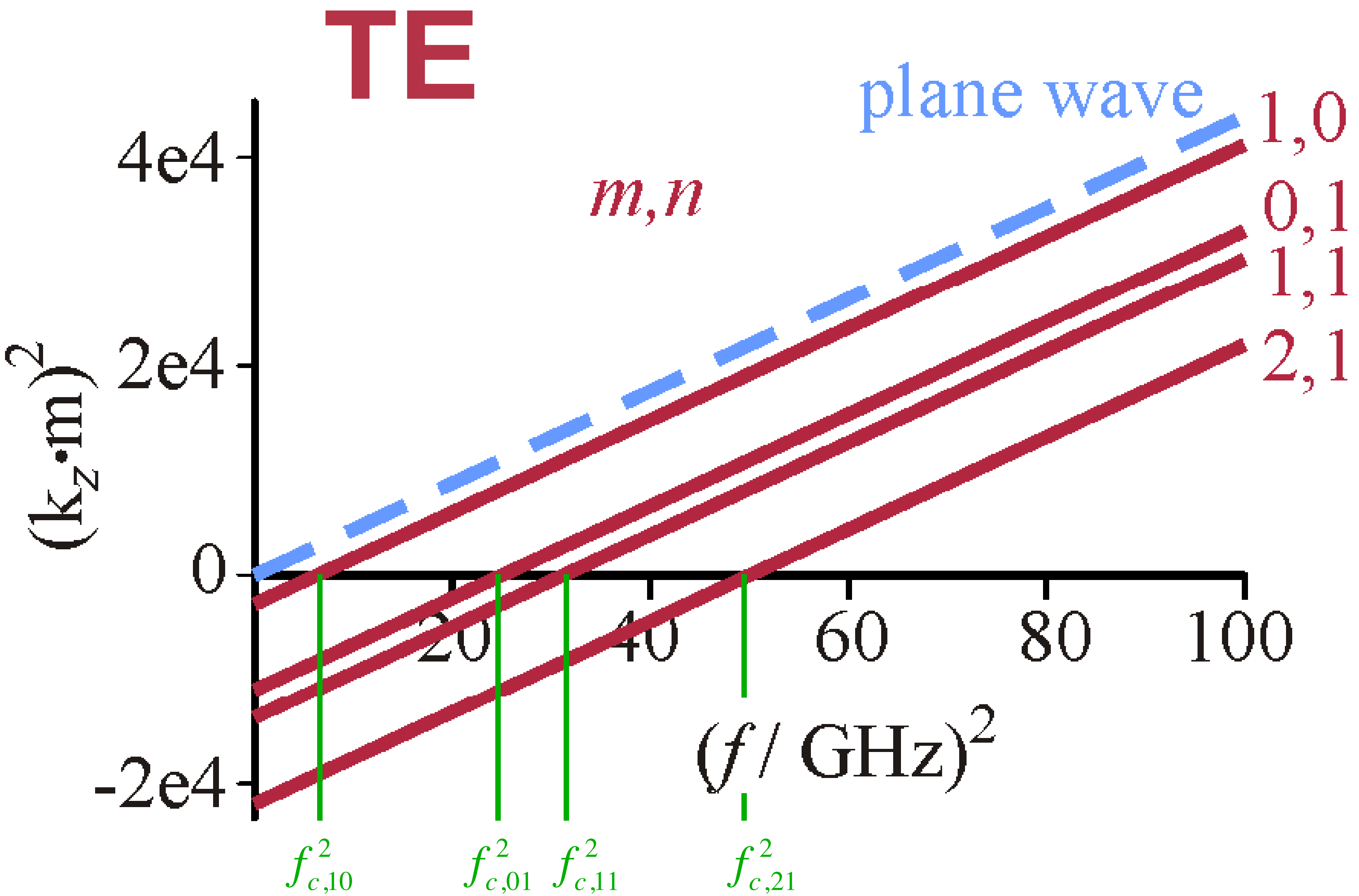
$$k_z^2(f^2) = (2\pi / c)^2 - \left(f^2 - f_c^2\right)$$

- **Alternative:**

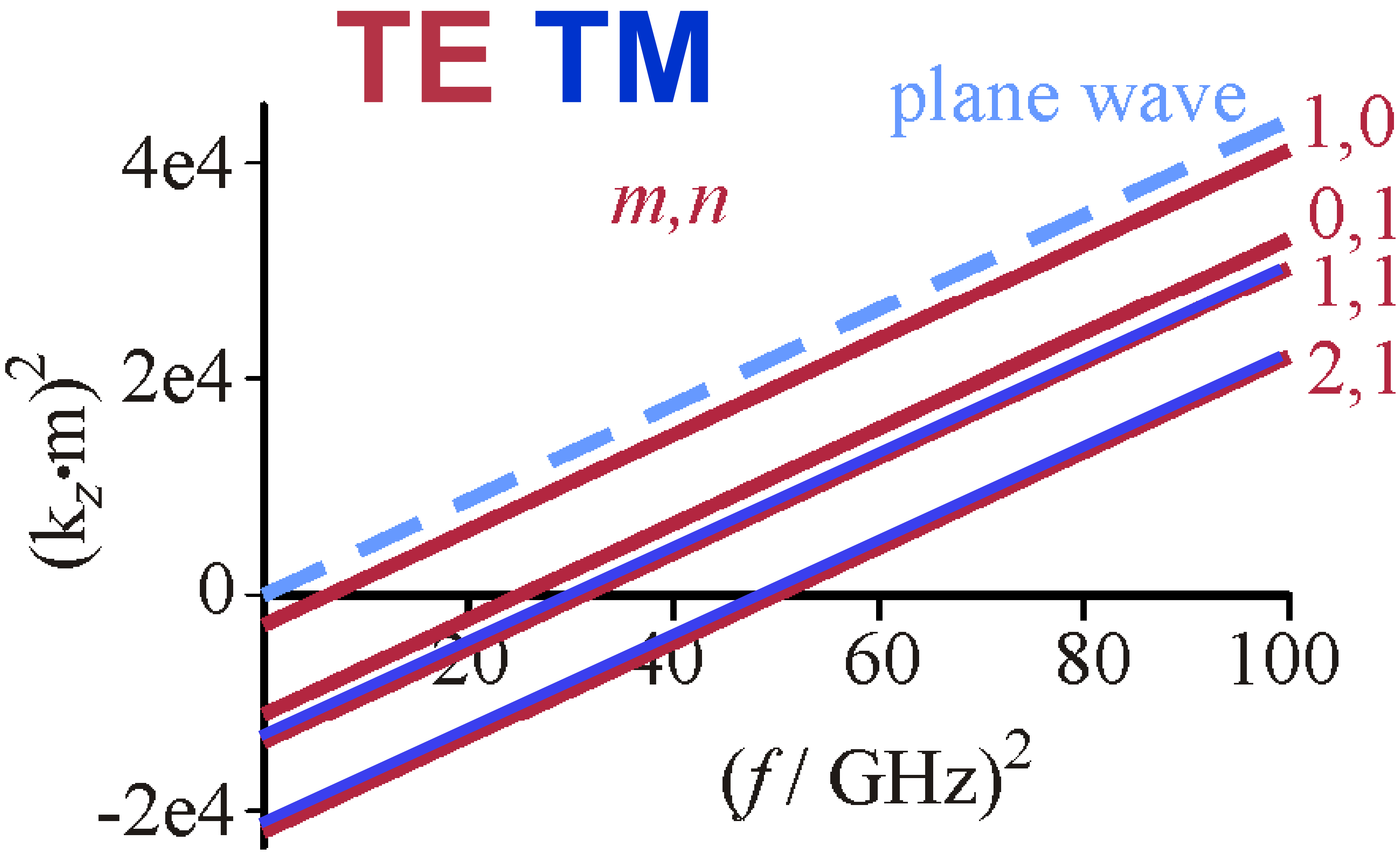
  ☞ Plot $|k_z|$ as a function of f

Rectangular Waveguides

- **Dispersion diagram:** $k_z^2(f^2) = \left(2\pi / c\right)^2 - \left(f^2 - f_c^2\right)$



- **Dispersion diagram:** $k_z^2(f^2)$



Rectangular Waveguides

- **Some modes can have the same cutoff frequency, although they have completely different mode patterns**

- **These are called *degenerated* modes**

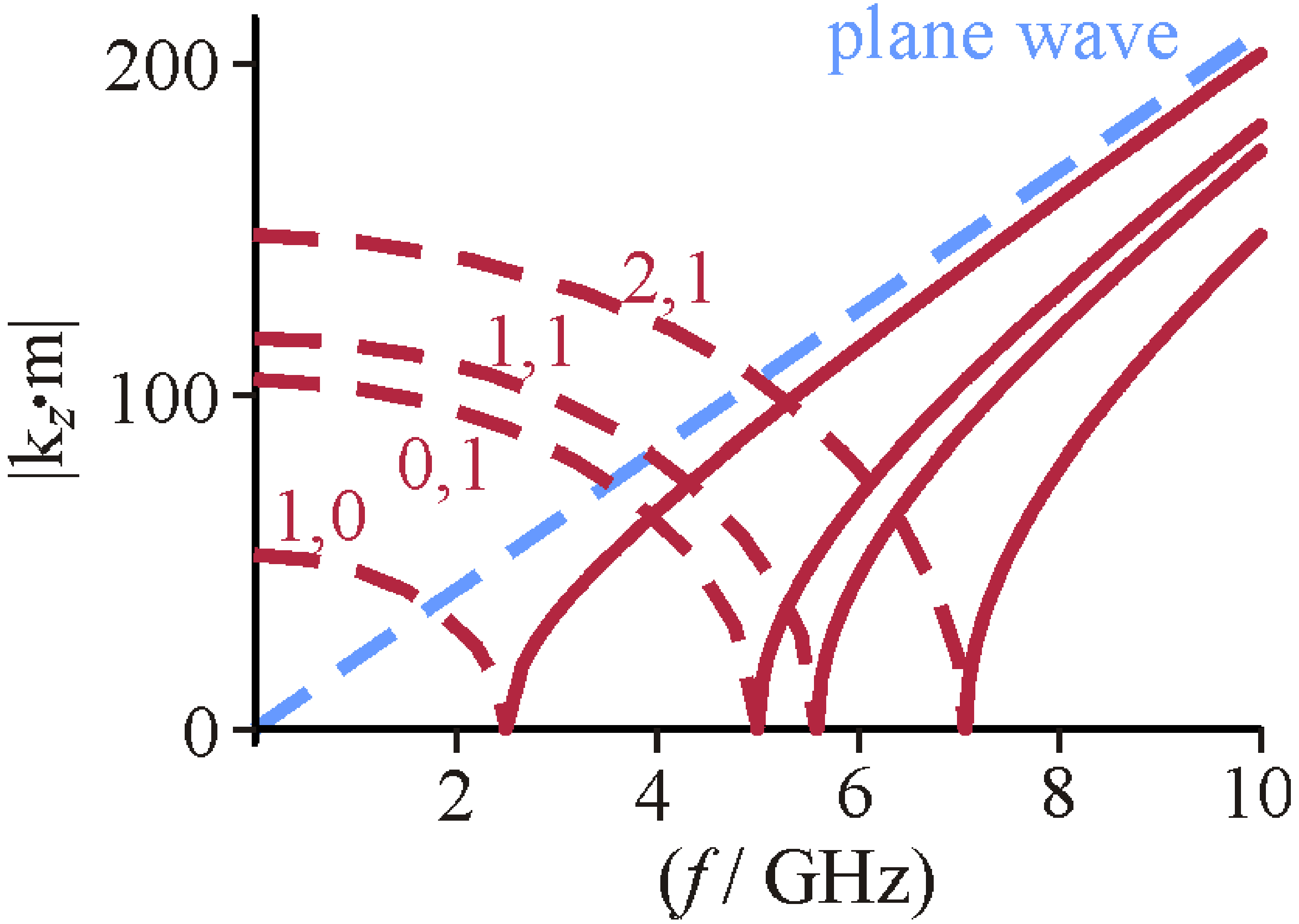
- **Above cutoff** $k_z^2(f^2)$ **is positive → *propagating* modes**

$$E_x = f(x,y)e^{\pm jk_z z}$$

- **Below cutoff,** $k_z^2(f^2)$ **is negative → *evanescent* modes (cannot propagate, field attenuation along the waveguide)**

$$E_x = f(x,y)e^{-k_z z}$$

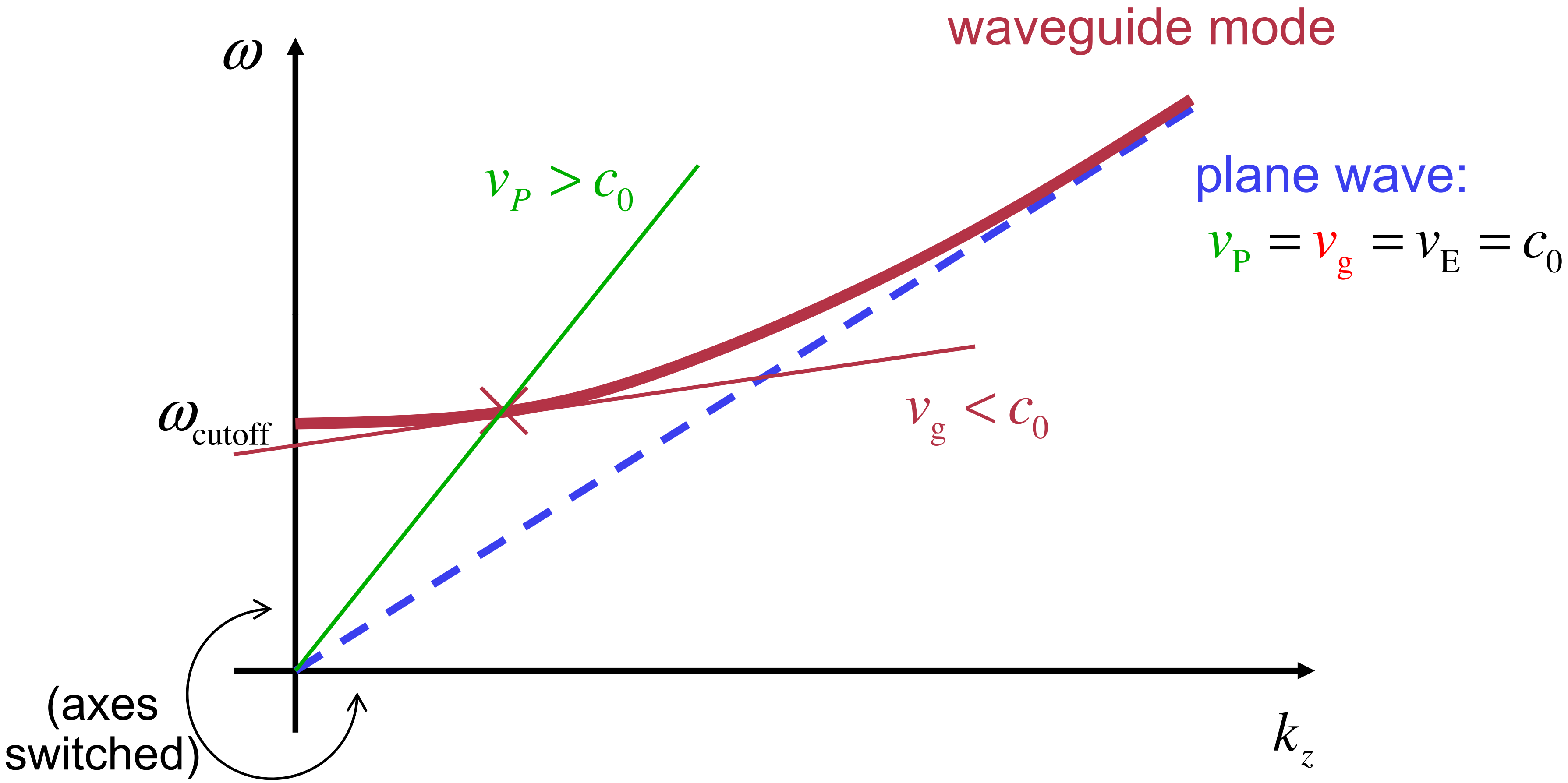
- **Dispersion diagram:** $\left|k_z(f)\right|$



Dispersion Diagram

- **Phase and group velocity in waveguides:**

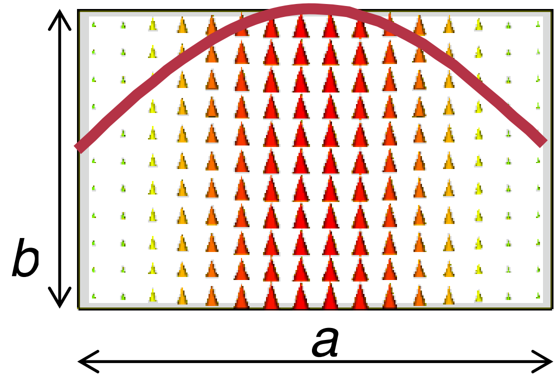
$$v_\mathrm{P} = \frac{\omega}{k_\mathrm{z}} \qquad v_\mathrm{g} = \frac{\mathrm{d}\omega(k_\mathrm{z})}{\mathrm{d}k_\mathrm{z}} \qquad \vec{v}_\mathrm{E} = \frac{\mathrm{Re}\{\vec{\underline{S}}(\vec{r})\}}{\overline{w}(\vec{r})}$$



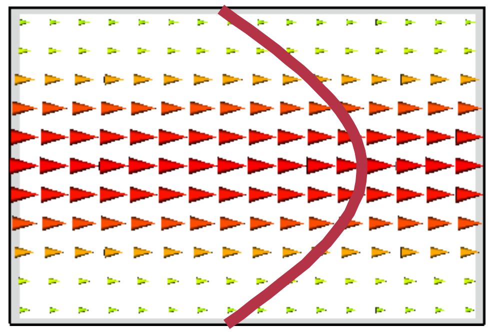
Size of Rectangular Waveguides:

smallest cutoff-frequencies (cutoff-wavelength):

$$f_{c,10} = \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{c}{2a} \qquad \lambda_{c,10} = \frac{c}{f_{c,10}} = 2a$$



$$f_{c,01} = \frac{c}{2}\sqrt{\left(\frac{0}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = \frac{c}{2b} \qquad \lambda_{c,01} = \frac{c}{f_{c,01}} = 2b$$



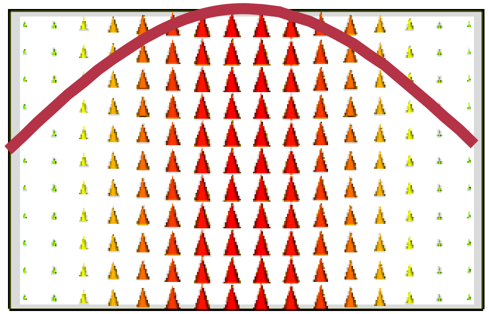
mode with smallest cutoff:

$$\frac{\lambda_c}{2} = \max(a,b) \qquad f_c = \frac{c}{\lambda_c} = \frac{c}{2\,\max(a,b)}$$

Waveguides

$$\frac{\lambda_c}{2} = \max(a,b)$$

$$f_c = \frac{c}{\lambda_c} = \frac{c}{2\max(a,b)}$$



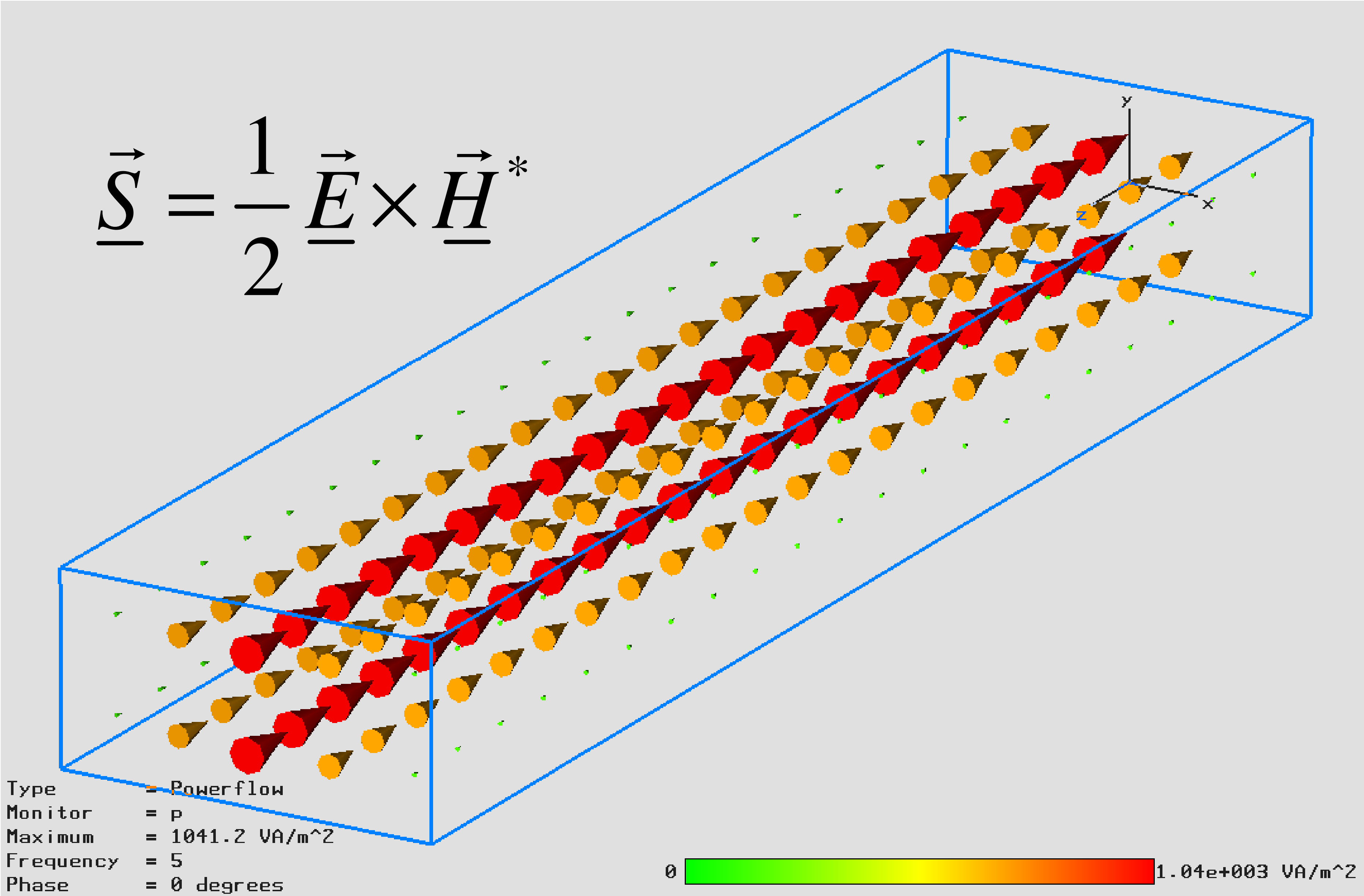
- Examples:

f_{cutoff}	a
1 MHz	150 m
10 MHz	15 m
100 MHz	1.5 m
0.91 GHz	165.1 mm x 82.55 mm („WR 650“)
1 GHz	15 cm
10 GHz	1.5 cm

too large for most applications !

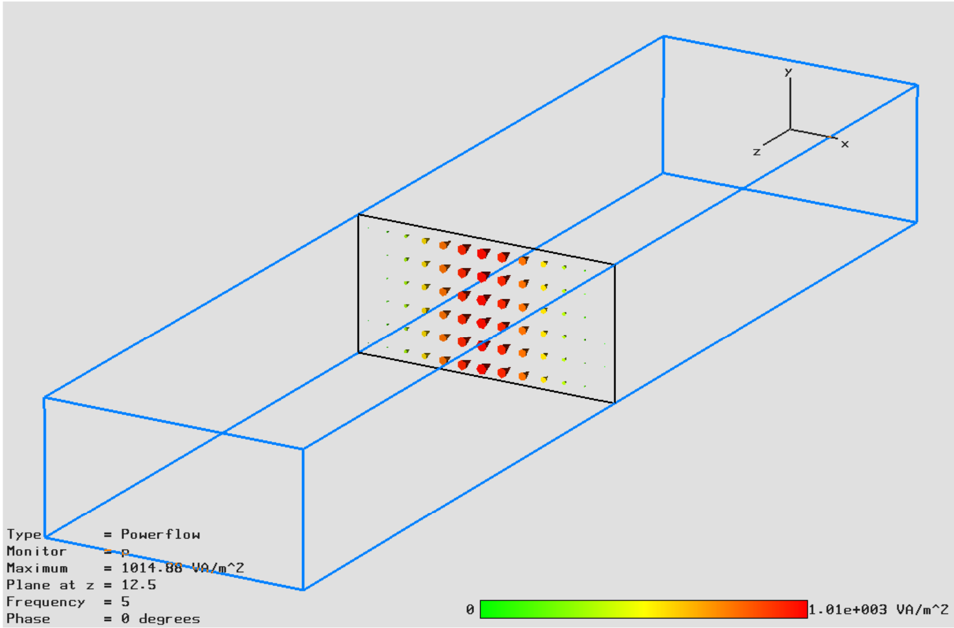
Waveguides

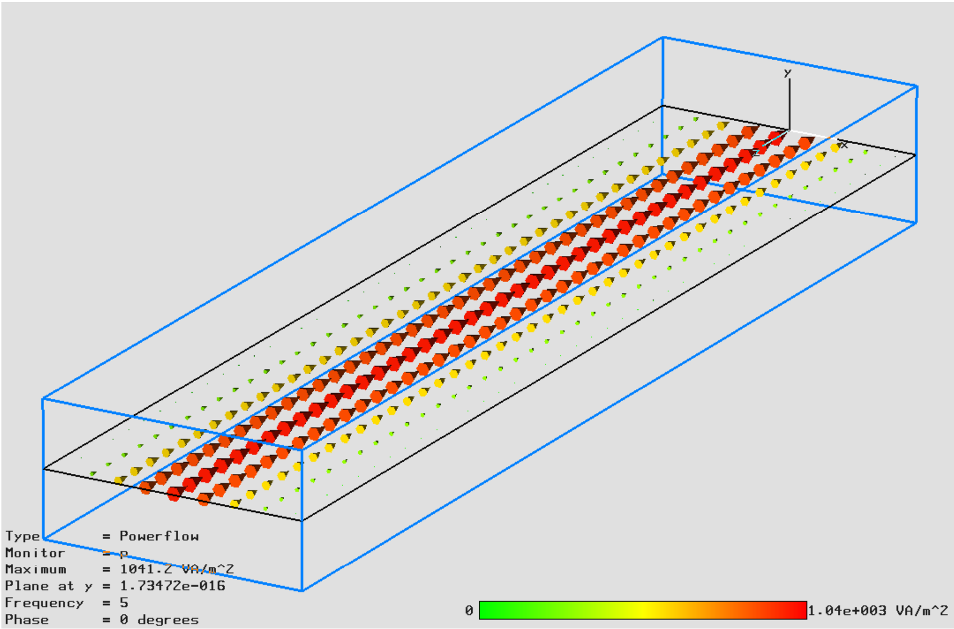
$$\underline{\vec{S}} = \frac{1}{2}\underline{\vec{E}}\times\underline{\vec{H}}^{*}$$

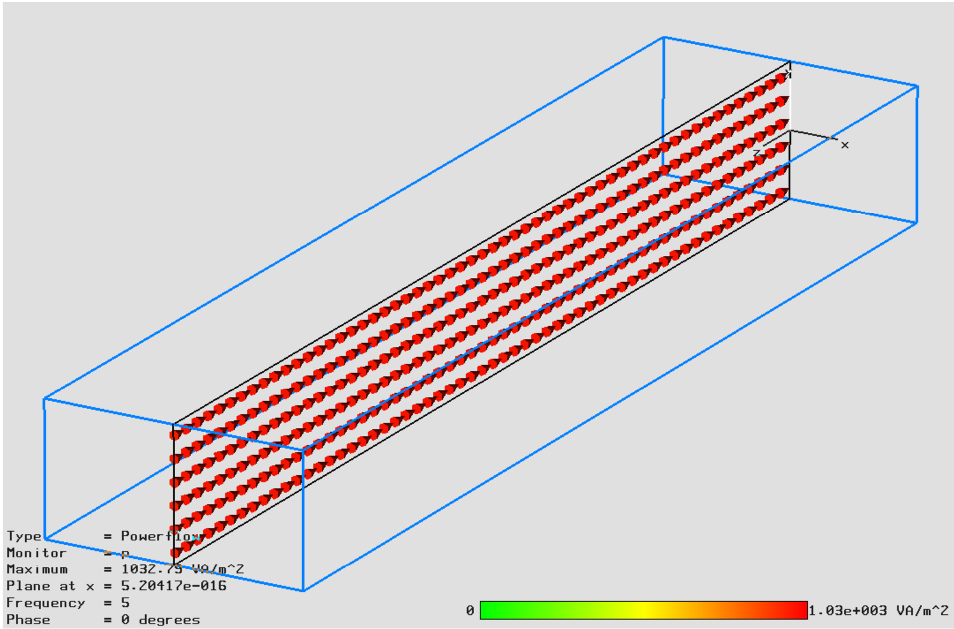


Waveguides

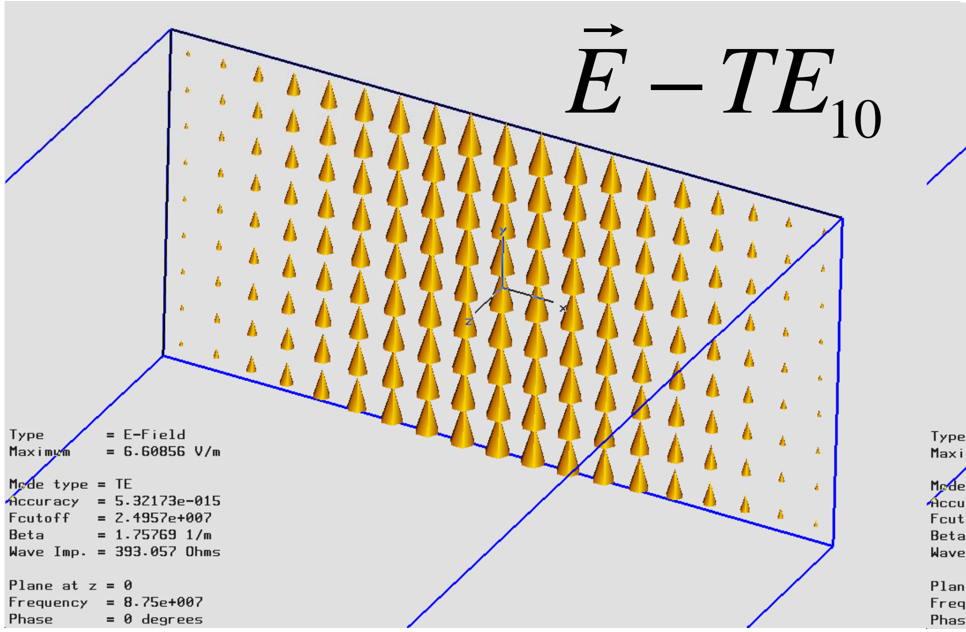
$$\underline{\vec{S}} = \frac{1}{2}\underline{\vec{E}} \times \underline{\vec{H}}^{*}$$

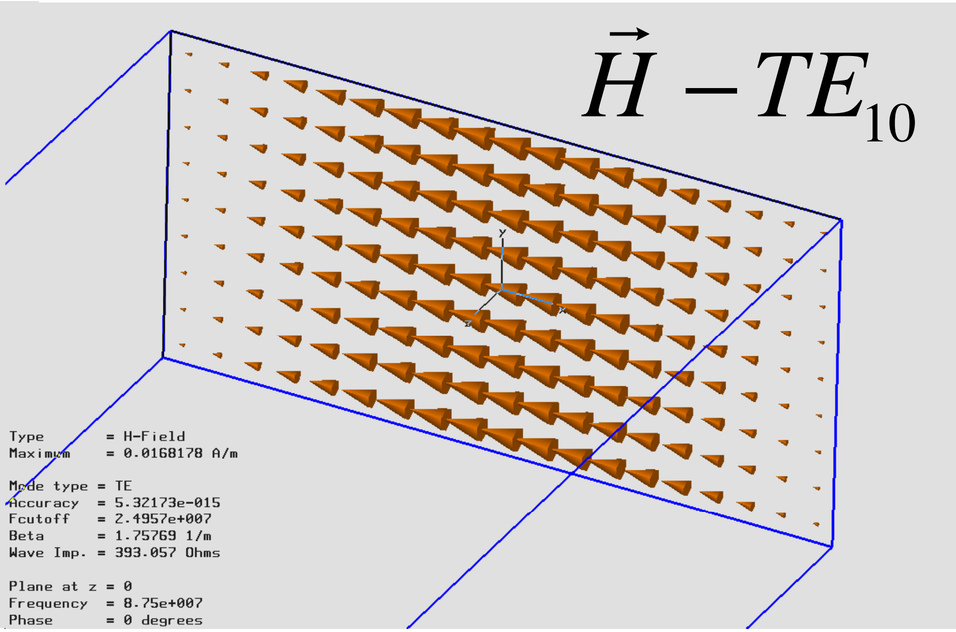


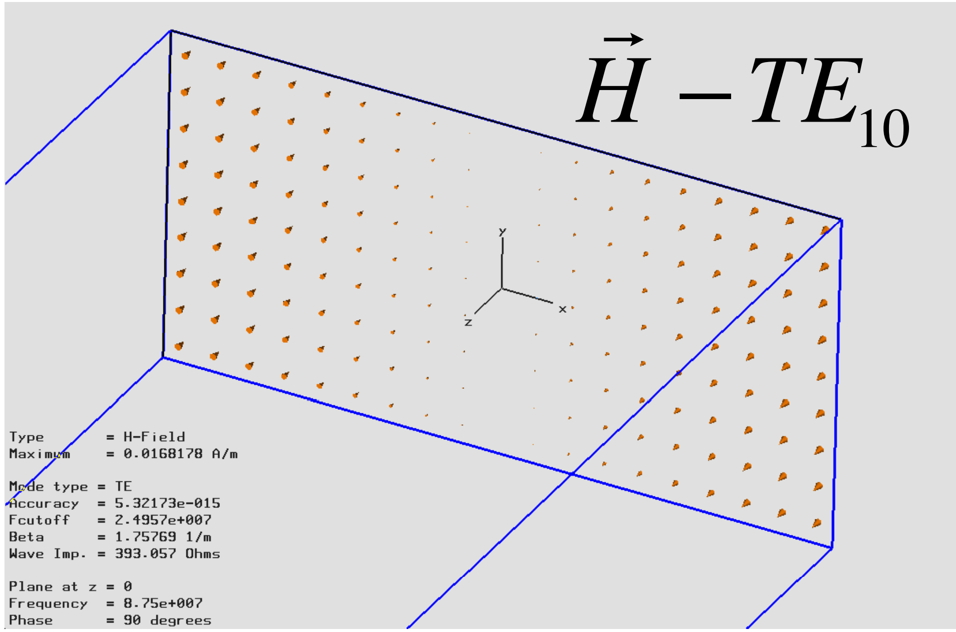




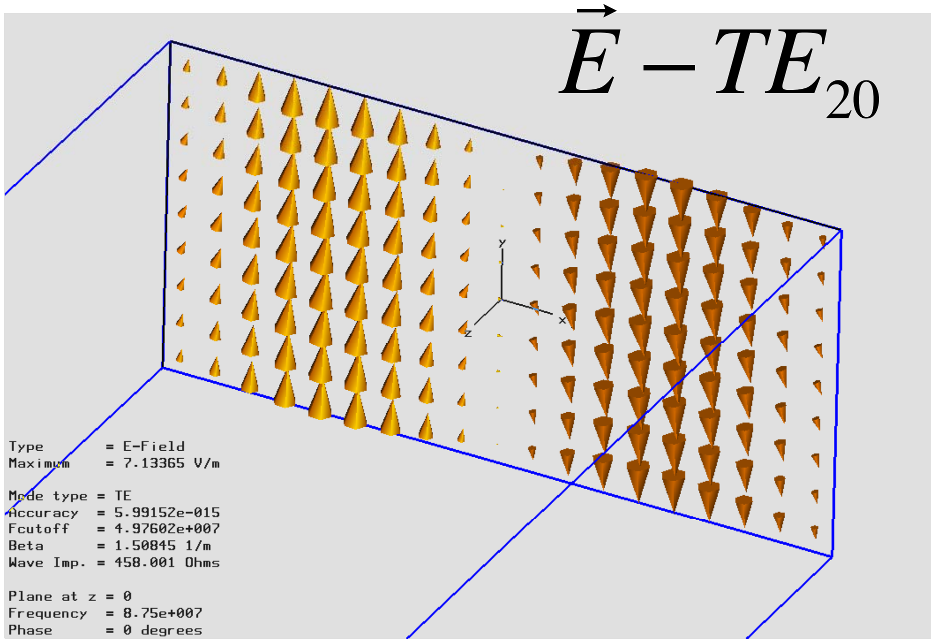
First Modes

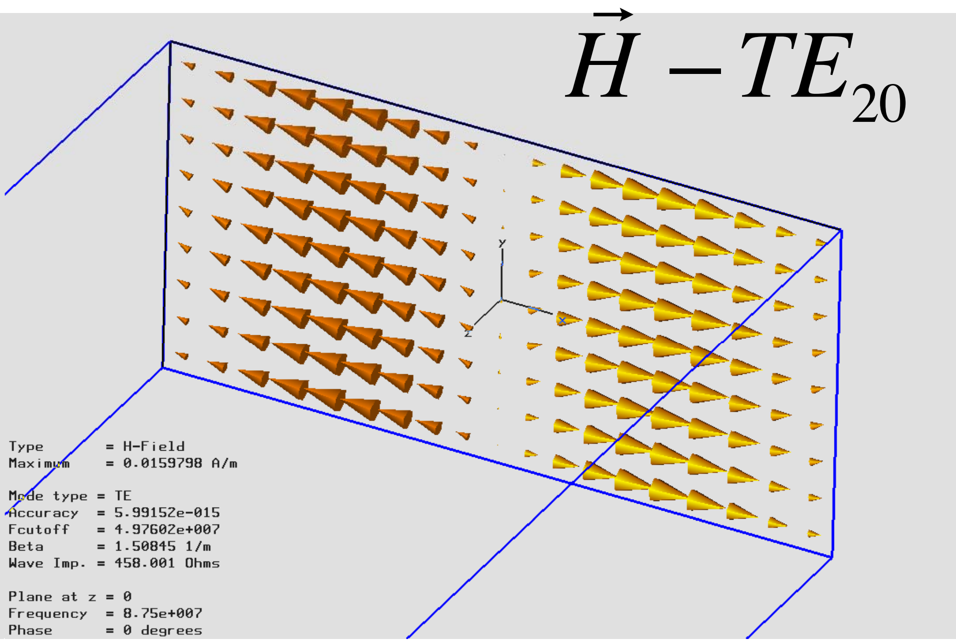

$\vec{E} - TE_{10}$
Type = E-Field
Maximum = 6.60856 V/m
Mode type = TE
Accuracy = 5.32173e-015
Fcutoff = 2.4957e+007
Beta = 1.75769 1/m
Wave Imp. = 393.057 Ohms
Plane at z = 0
Frequency = 8.75e+007
Phase = 0 degrees


$\vec{H} - TE_{10}$
Type = H-Field
Maximum = 0.0168178 A/m
Mode type = TE
Accuracy = 5.32173e-015
Fcutoff = 2.4957e+007
Beta = 1.75769 1/m
Wave Imp. = 393.057 Ohms
Plane at z = 0
Frequency = 8.75e+007
Phase = 0 degrees


$\vec{H} - TE_{10}$
Type = H-Field
Maximum = 0.0168178 A/m
Mode type = TE
Accuracy = 5.32173e-015
Fcutoff = 2.4957e+007
Beta = 1.75769 1/m
Wave Imp. = 393.057 Ohms
Plane at z = 0
Frequency = 8.75e+007
Phase = 90 degrees

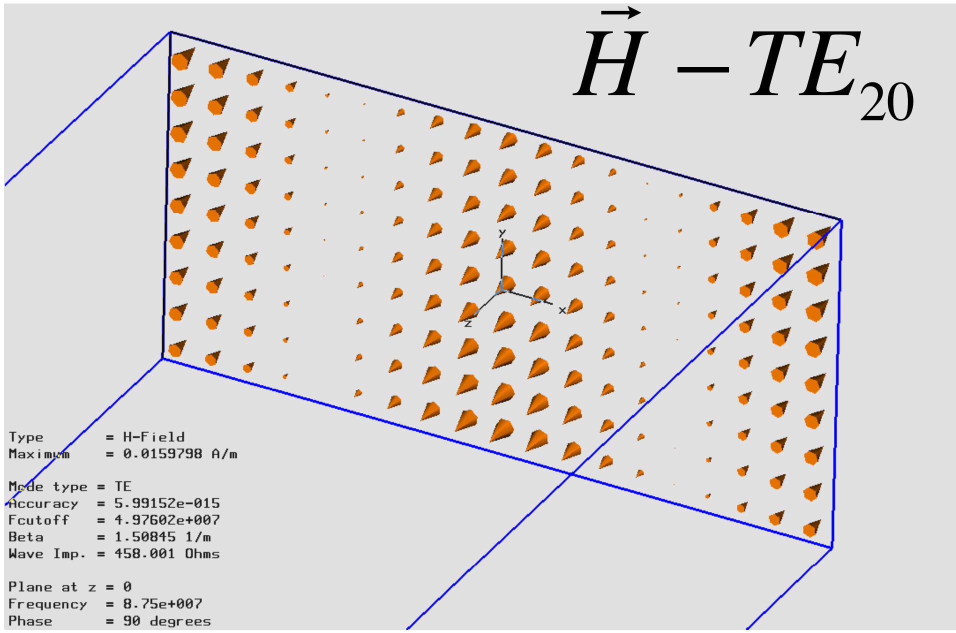
First Modes



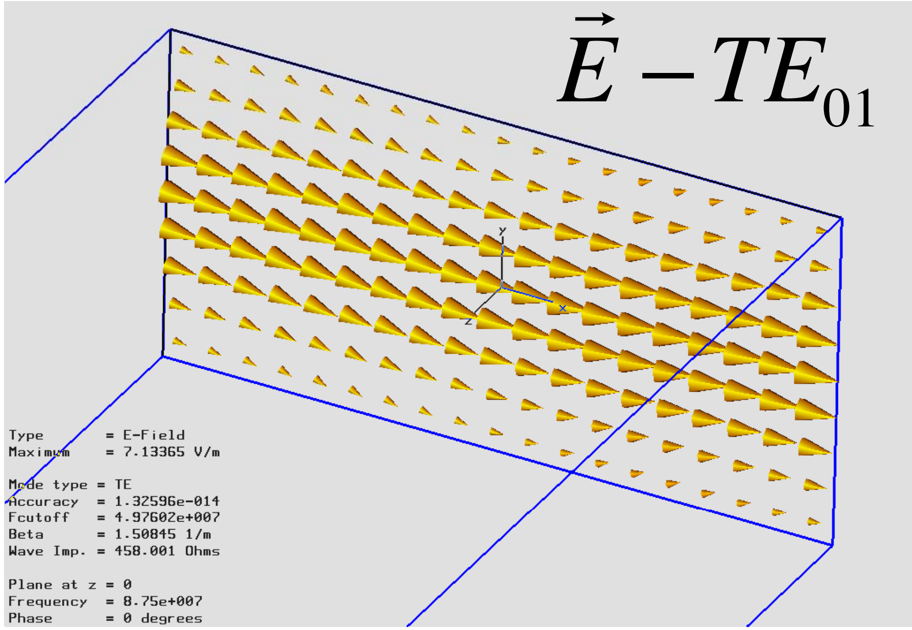


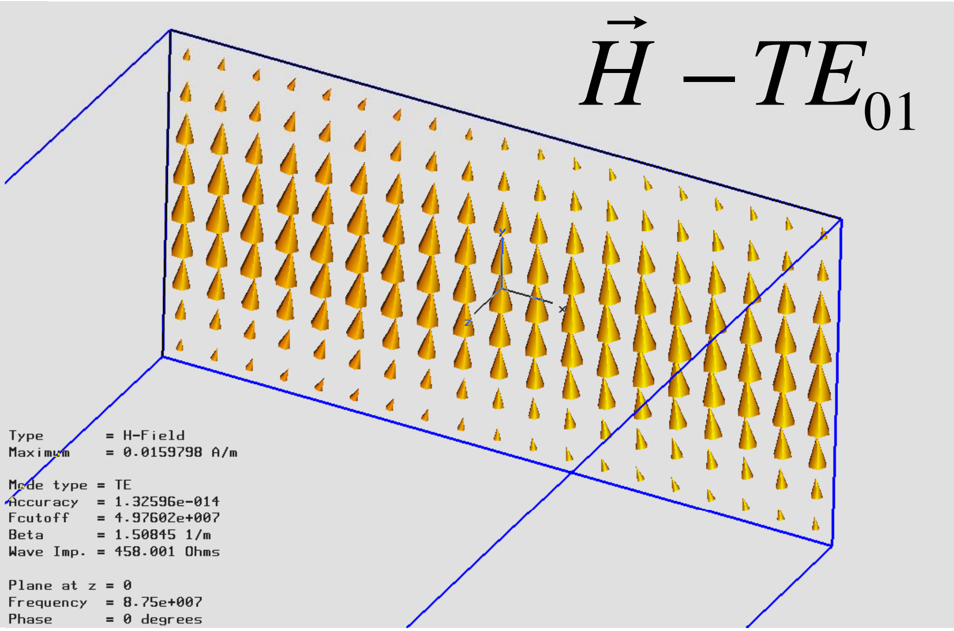
m=2, n=0

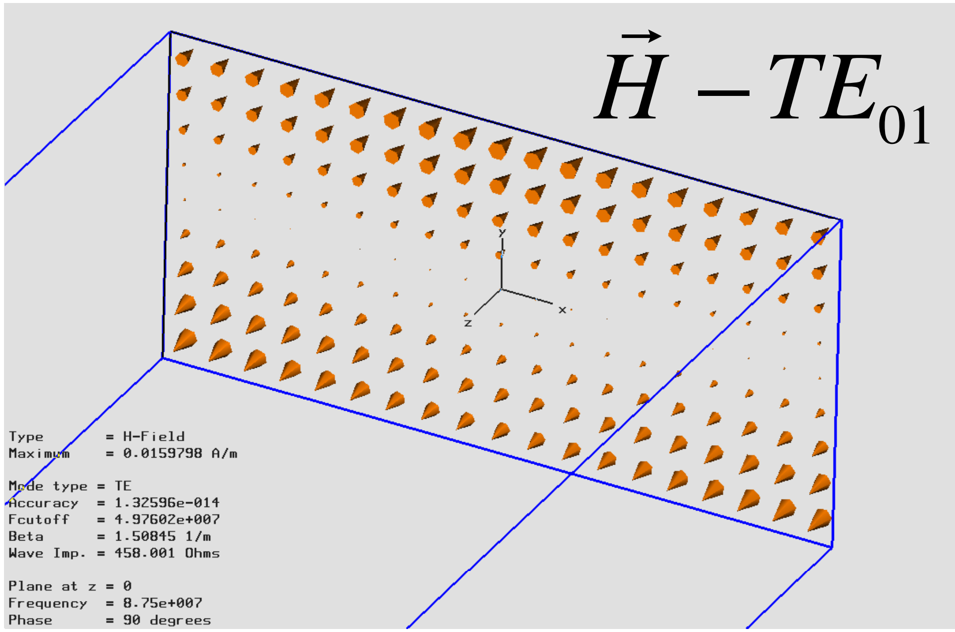
Number of half-periods:
m: along x
n: along y



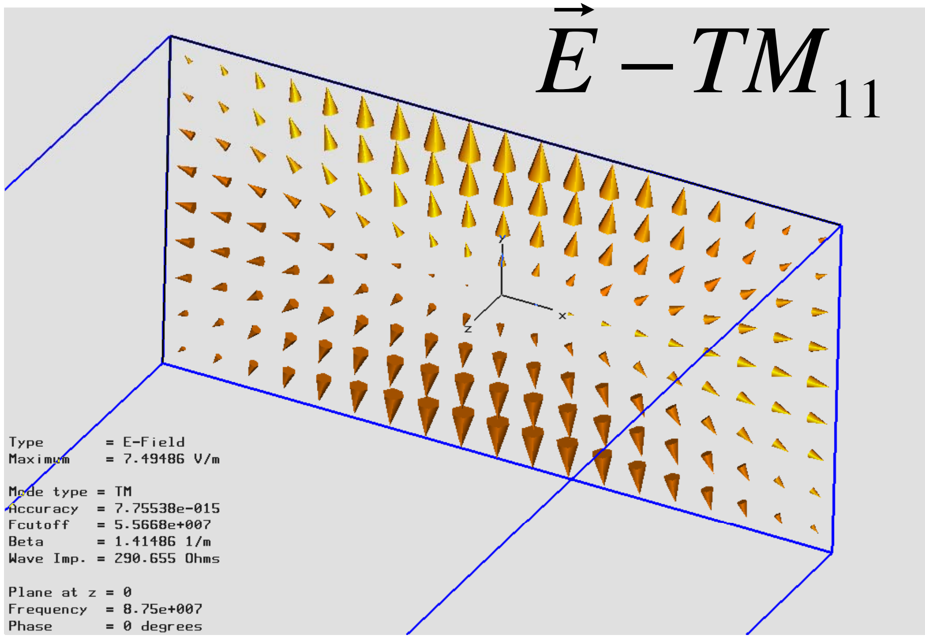
First Modes

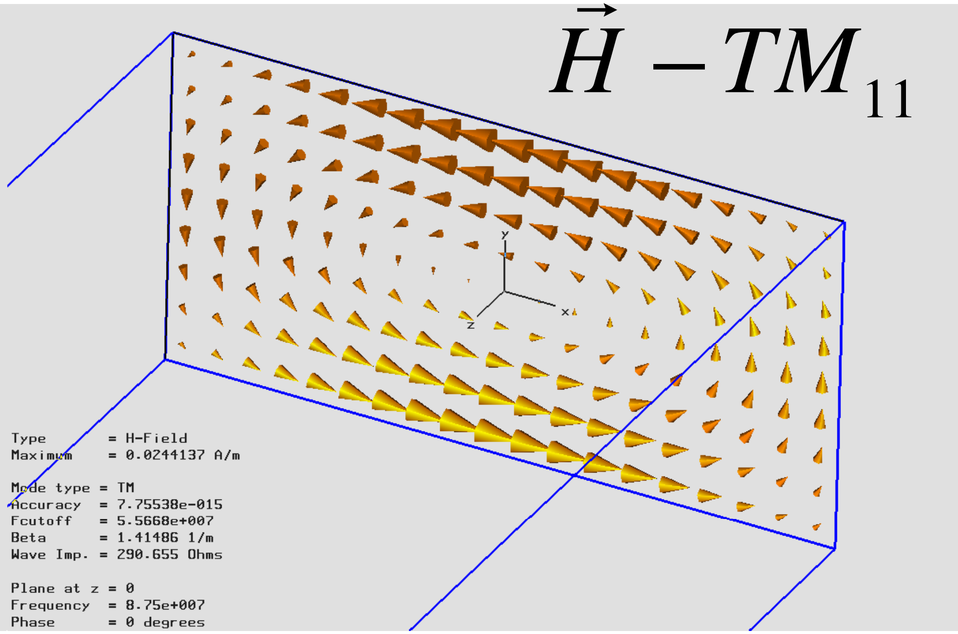

$\vec{E} - TE_{01}$
Type = E-Field
Maximum = 7.13365 V/m
Mode type = TE
Accuracy = 1.32596e-014
Fcutoff = 4.97602e+007
Beta = 1.50845 1/m
Wave Imp. = 458.001 Ohms
Plane at z = 0
Frequency = 8.75e+007
Phase = 0 degrees

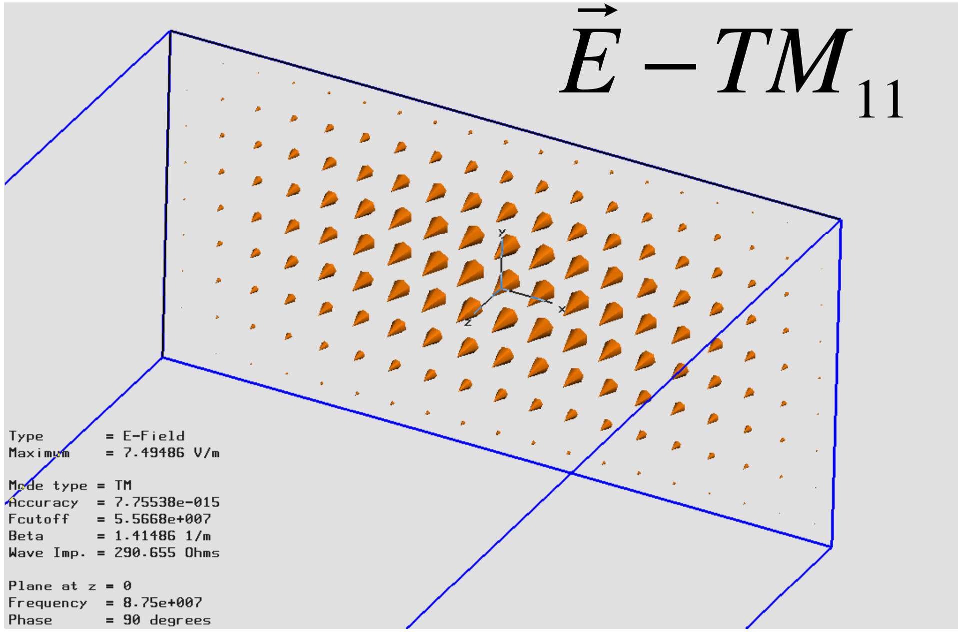

$\vec{H} - TE_{01}$
Type = H-Field
Maximum = 0.0159798 A/m
Mode type = TE
Accuracy = 1.32596e-014
Fcutoff = 4.97602e+007
Beta = 1.50845 1/m
Wave Imp. = 458.001 Ohms
Plane at z = 0
Frequency = 8.75e+007
Phase = 0 degrees


$\vec{H} - TE_{01}$
Type = H-Field
Maximum = 0.0159798 A/m
Mode type = TE
Accuracy = 1.32596e-014
Fcutoff = 4.97602e+007
Beta = 1.50845 1/m
Wave Imp. = 458.001 Ohms
Plane at z = 0
Frequency = 8.75e+007
Phase = 90 degrees

First Modes

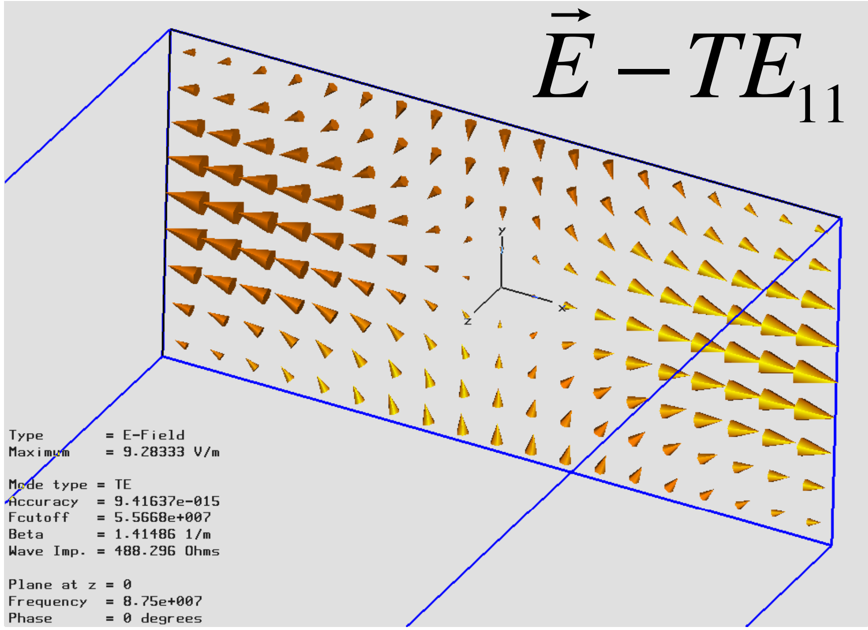

$\vec{E} - TM_{11}$
Type = E-Field
Maximum = 7.49486 V/m
Mode type = TM
Accuracy = 7.75538e-015
Fcutoff = 5.5668e+007
Beta = 1.41486 1/m
Wave Imp. = 290.655 Ohms
Plane at z = 0
Frequency = 8.75e+007
Phase = 0 degrees


$\vec{H} - TM_{11}$
Type = H-Field
Maximum = 0.0244137 A/m
Mode type = TM
Accuracy = 7.75538e-015
Fcutoff = 5.5668e+007
Beta = 1.41486 1/m
Wave Imp. = 290.655 Ohms
Plane at z = 0
Frequency = 8.75e+007
Phase = 0 degrees

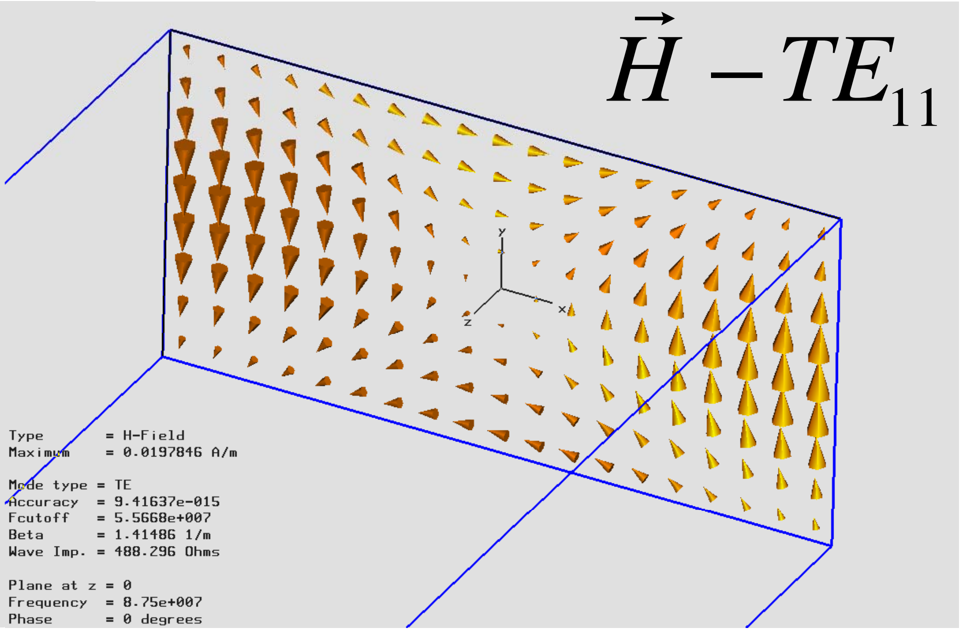

$\vec{E} - TM_{11}$
Type = E-Field
Maximum = 7.49486 V/m
Mode type = TM
Accuracy = 7.75538e-015
Fcutoff = 5.5668e+007
Beta = 1.41486 1/m
Wave Imp. = 290.655 Ohms
Plane at z = 0
Frequency = 8.75e+007
Phase = 90 degrees

First Modes

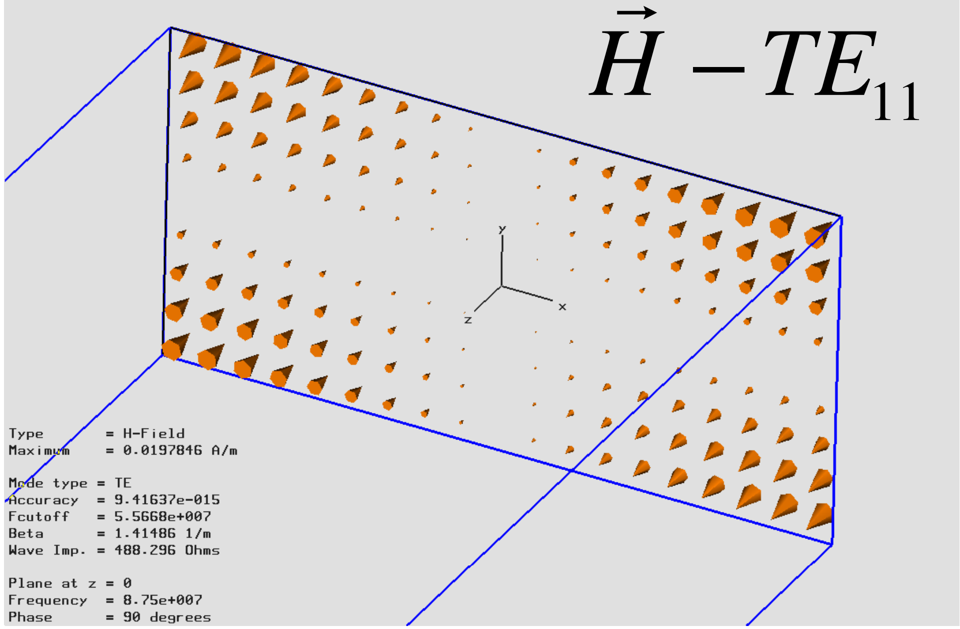
$\vec{E} - TE_{11}$



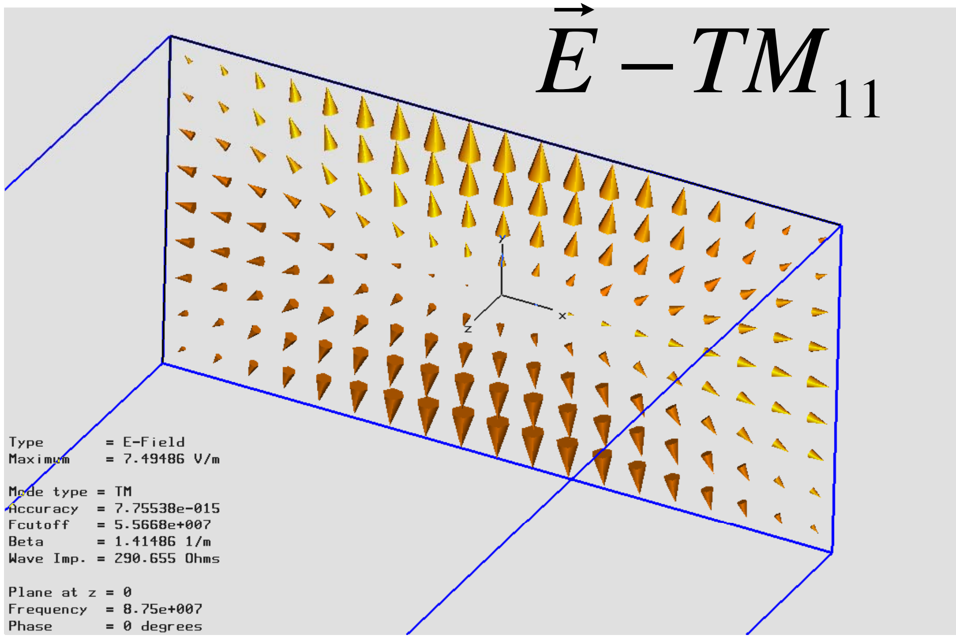
$\vec{H} - TE_{11}$

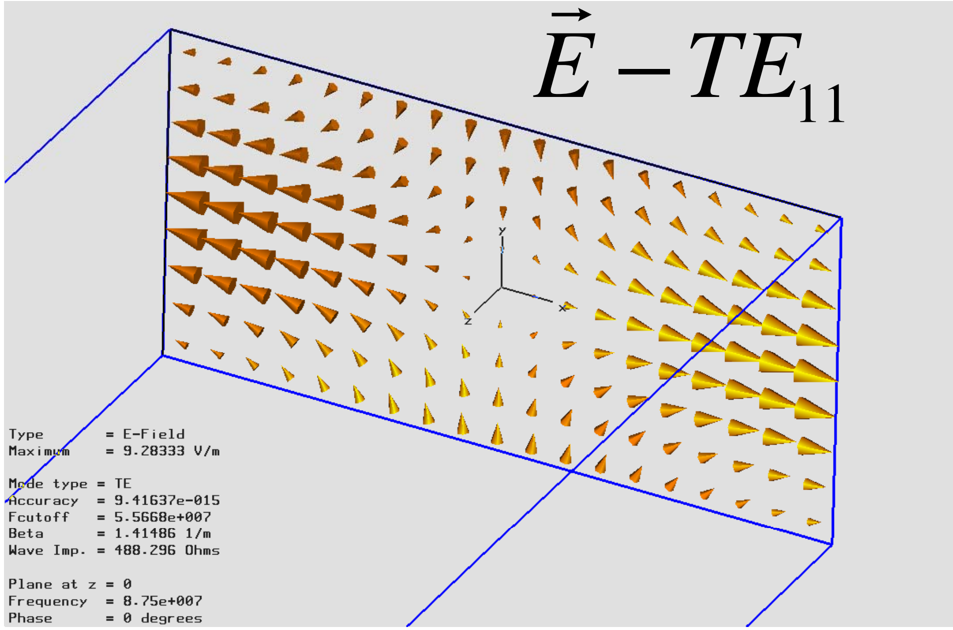


$\vec{H} - TE_{11}$



Degenerated modes



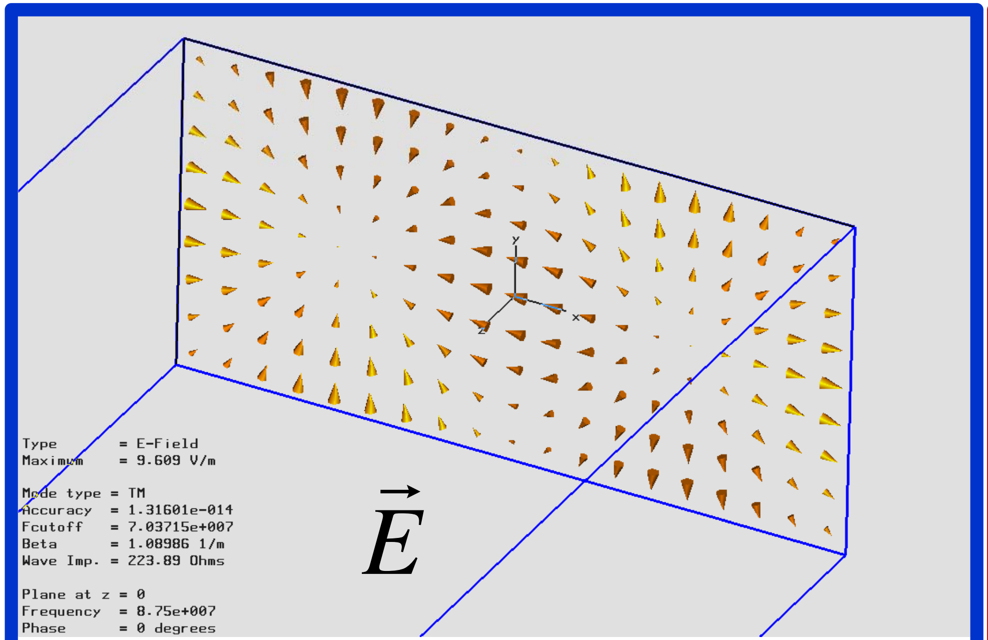


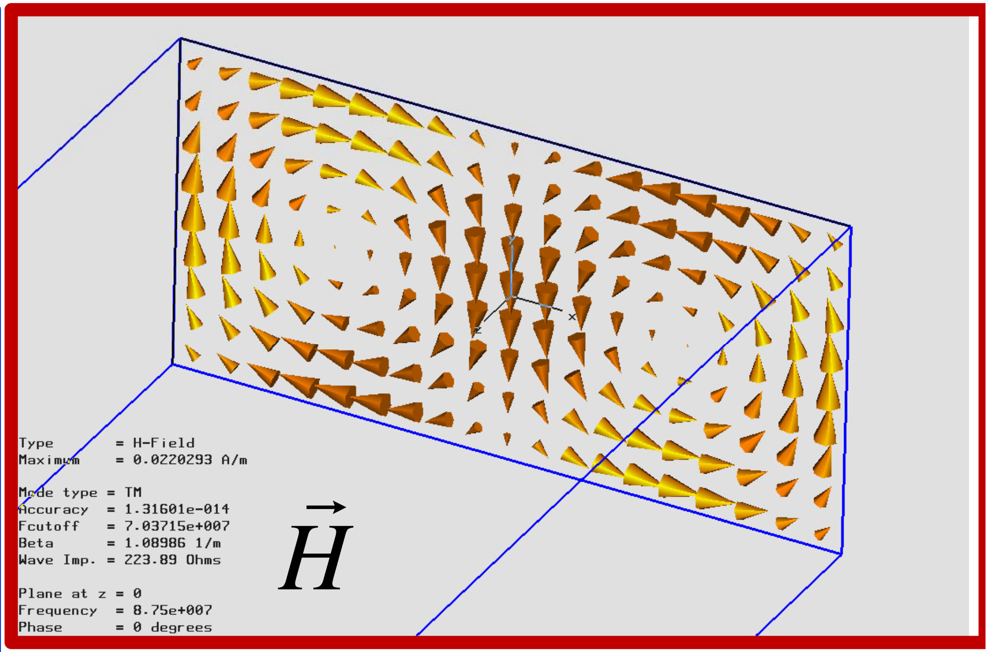
$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- **For this particular waveguide, these modes have the same cutoff frequency – "degenerated modes"**

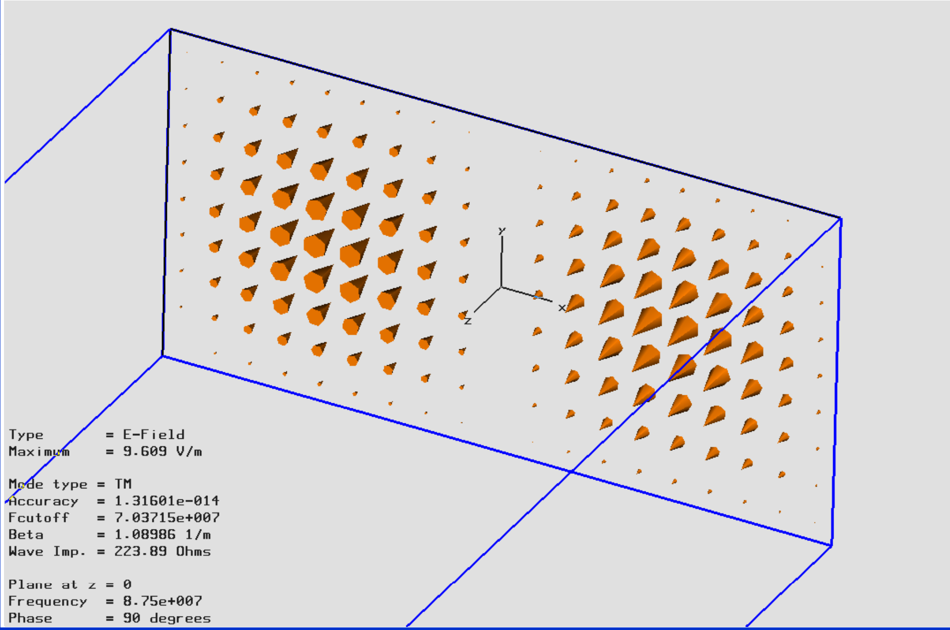
First Modes

xy





z

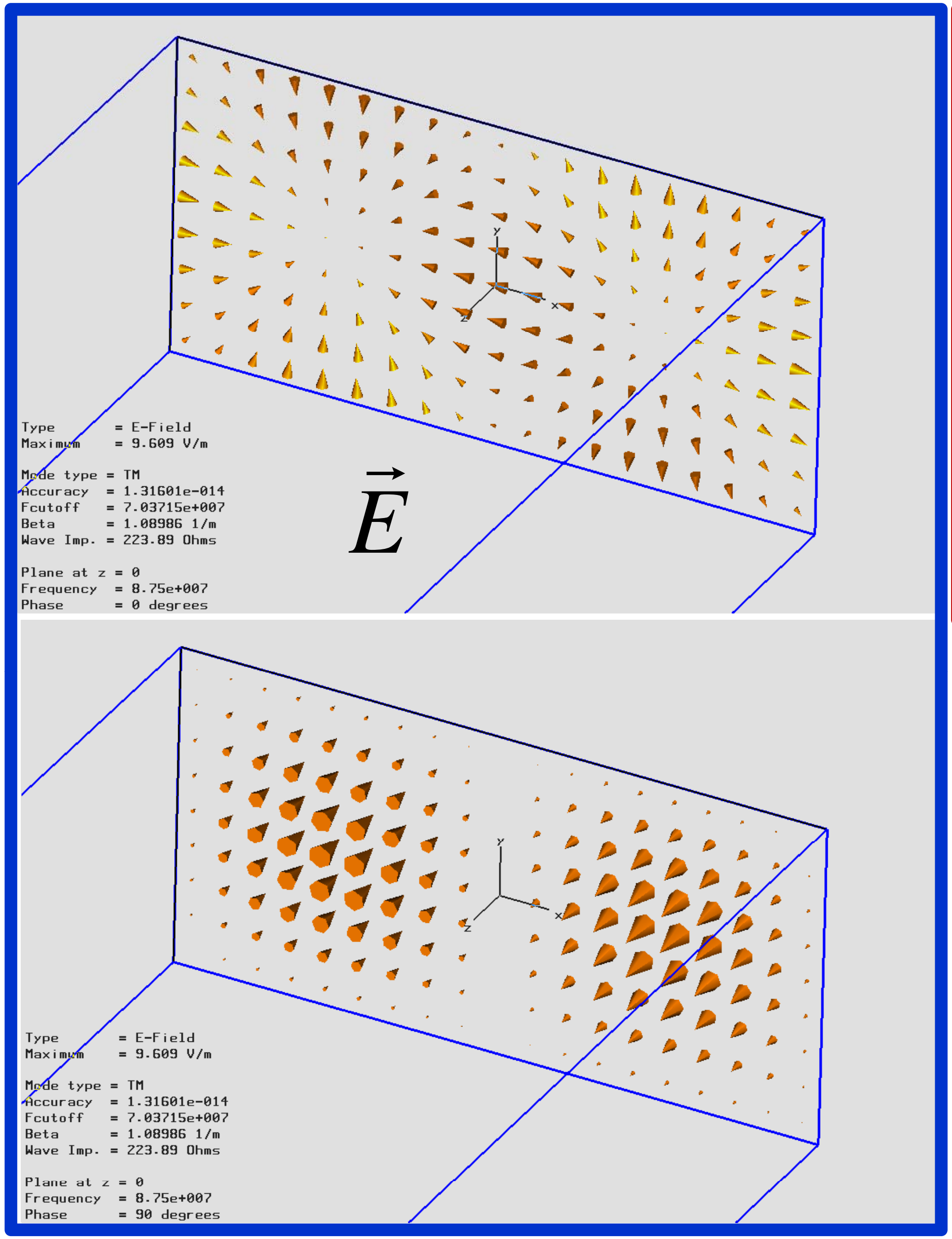


These field plots all correspond to the same mode

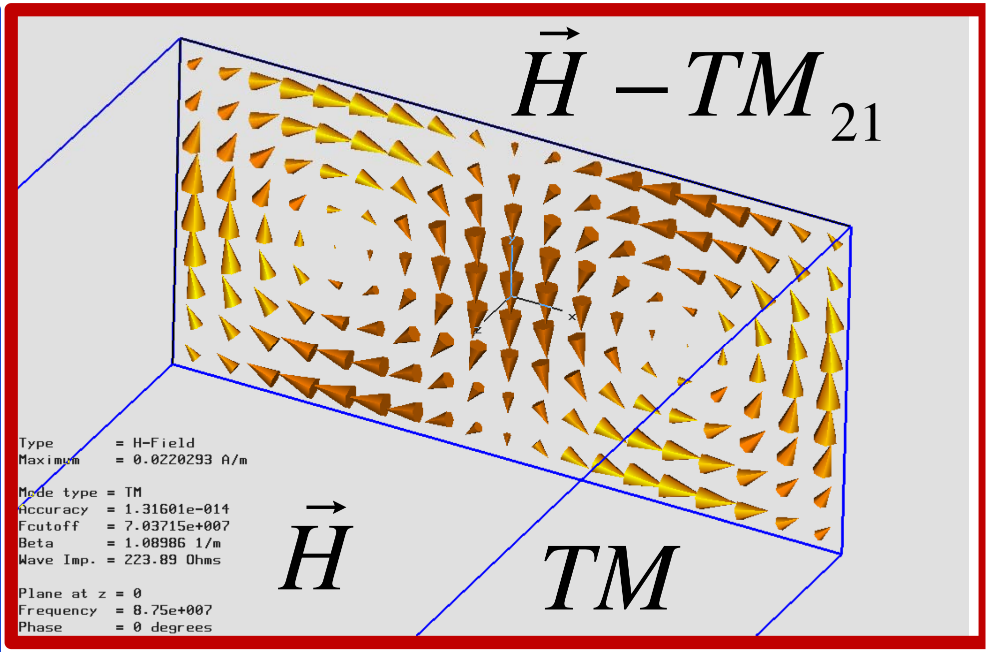
Which mode is this?

First Modes

xy



z

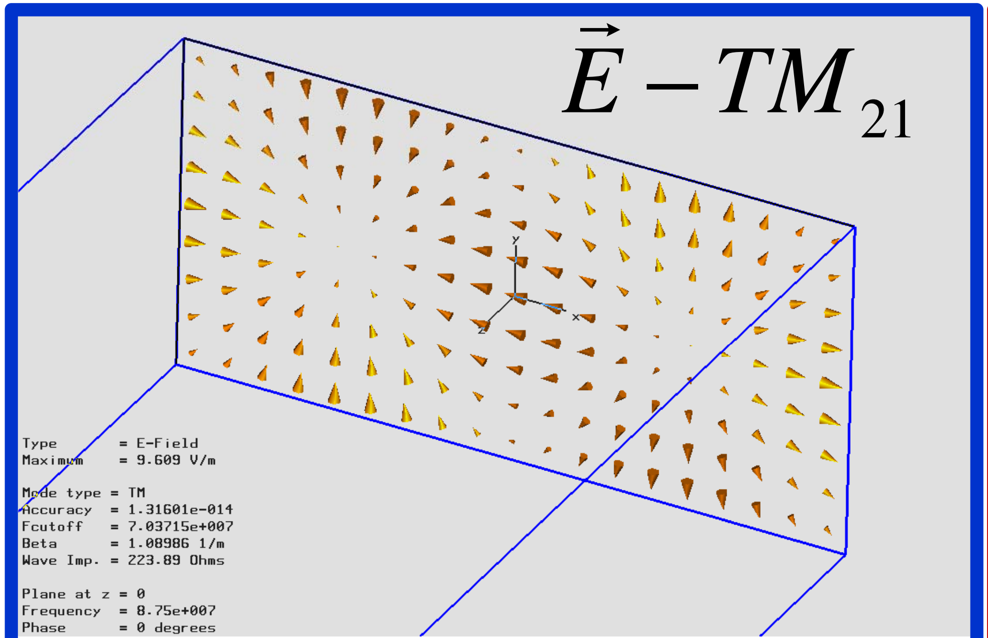


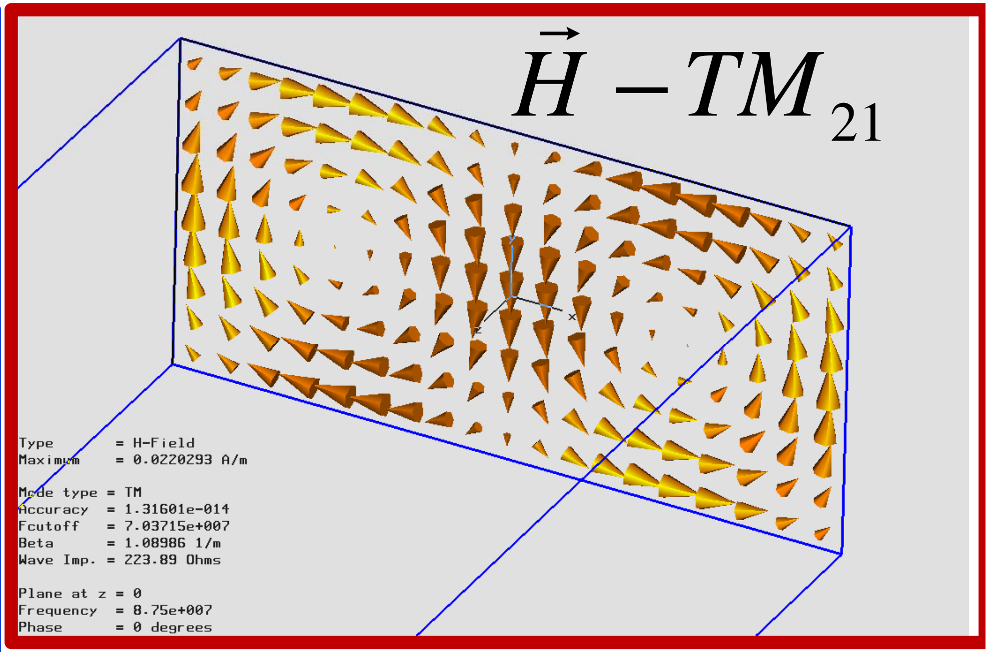
These field plots all correspond to the same mode

Which mode is this?

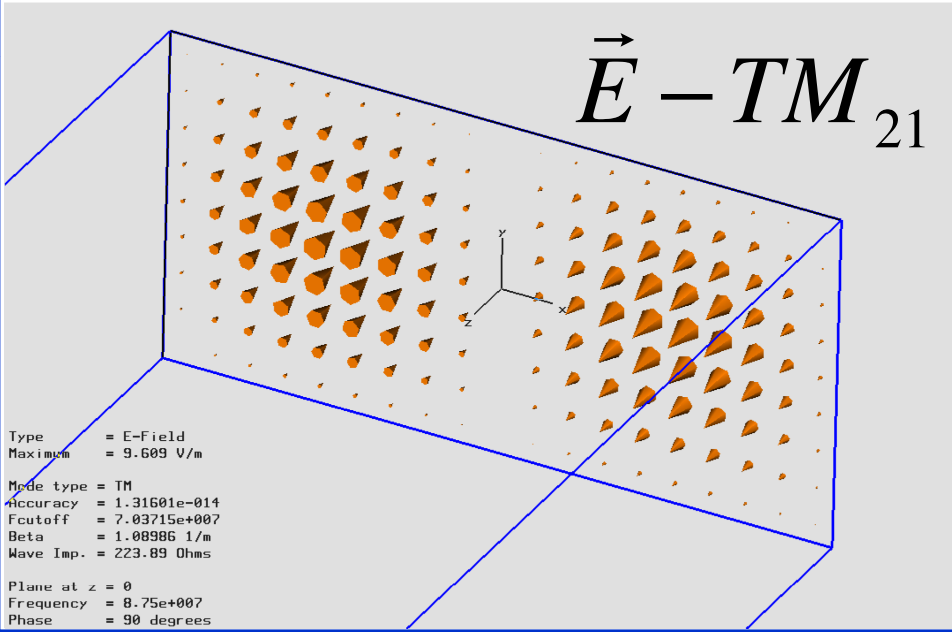
First Modes

xy





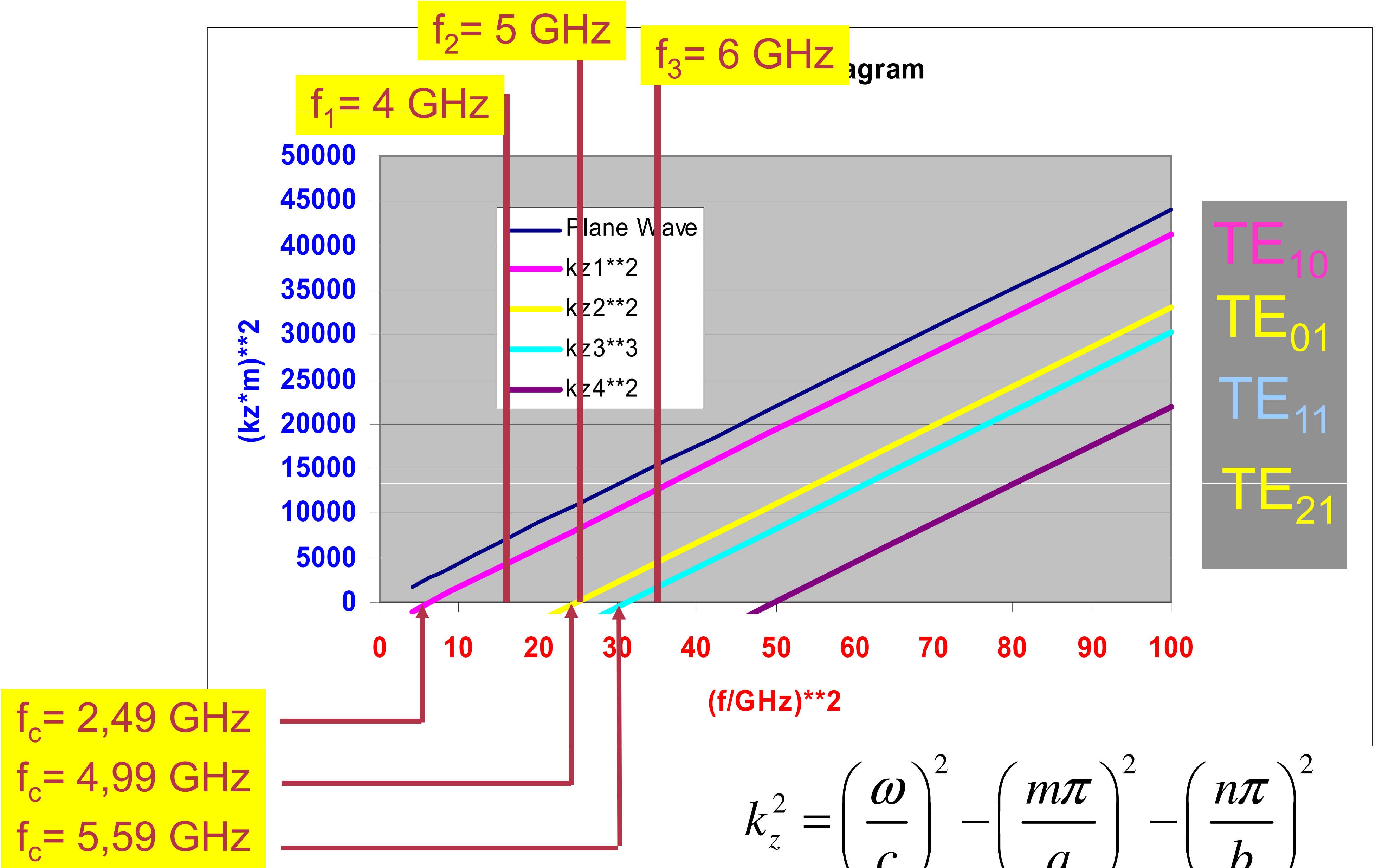
z



These field plots all correspond to the same mode

Which mode is this?

Waveguide Bend



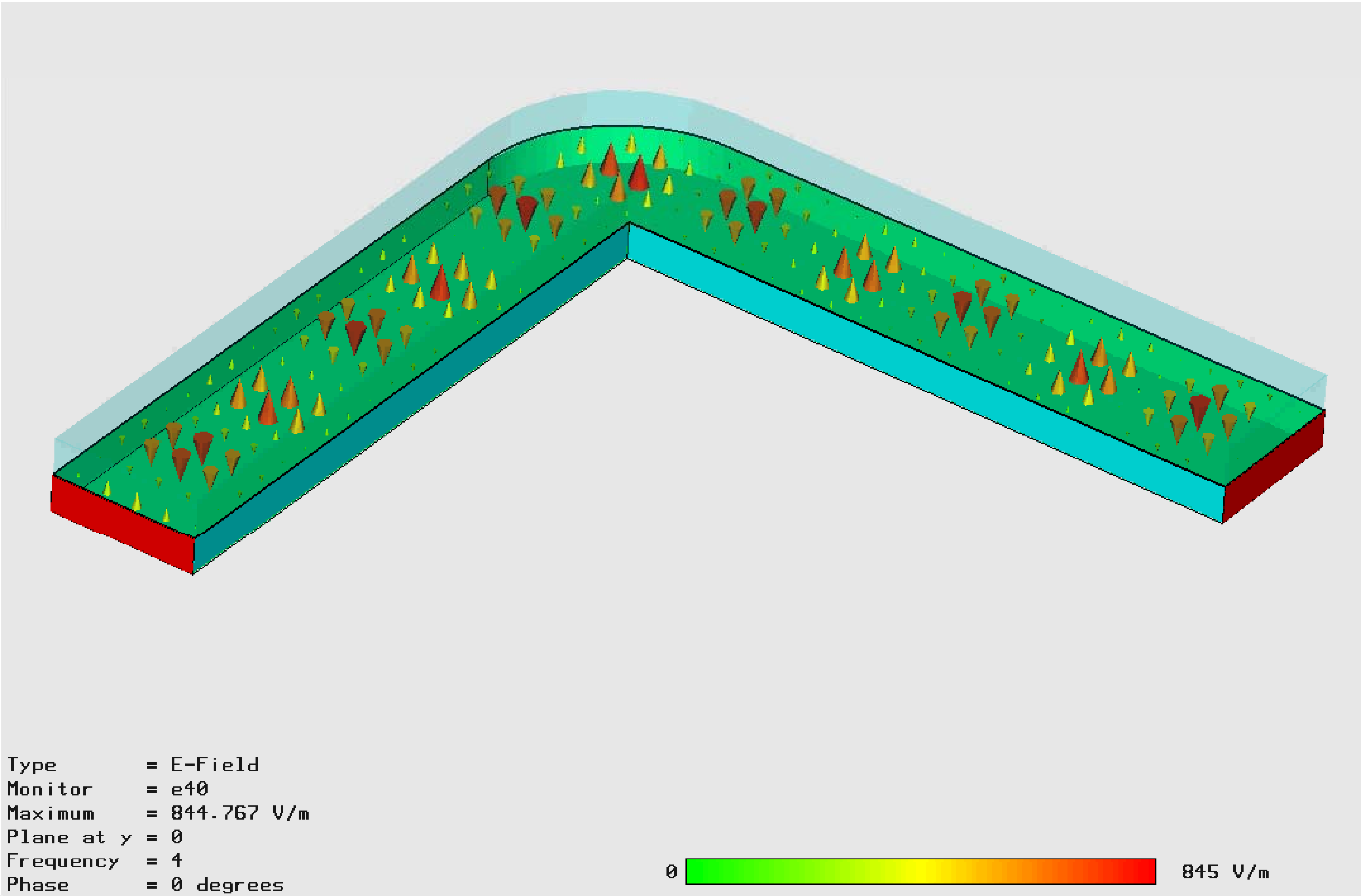
$$k_z^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

Waveguide Bend

f = 4 GHz

Input-Mode = TE_{10} Reflection = -26 dB

$\vec{E}$

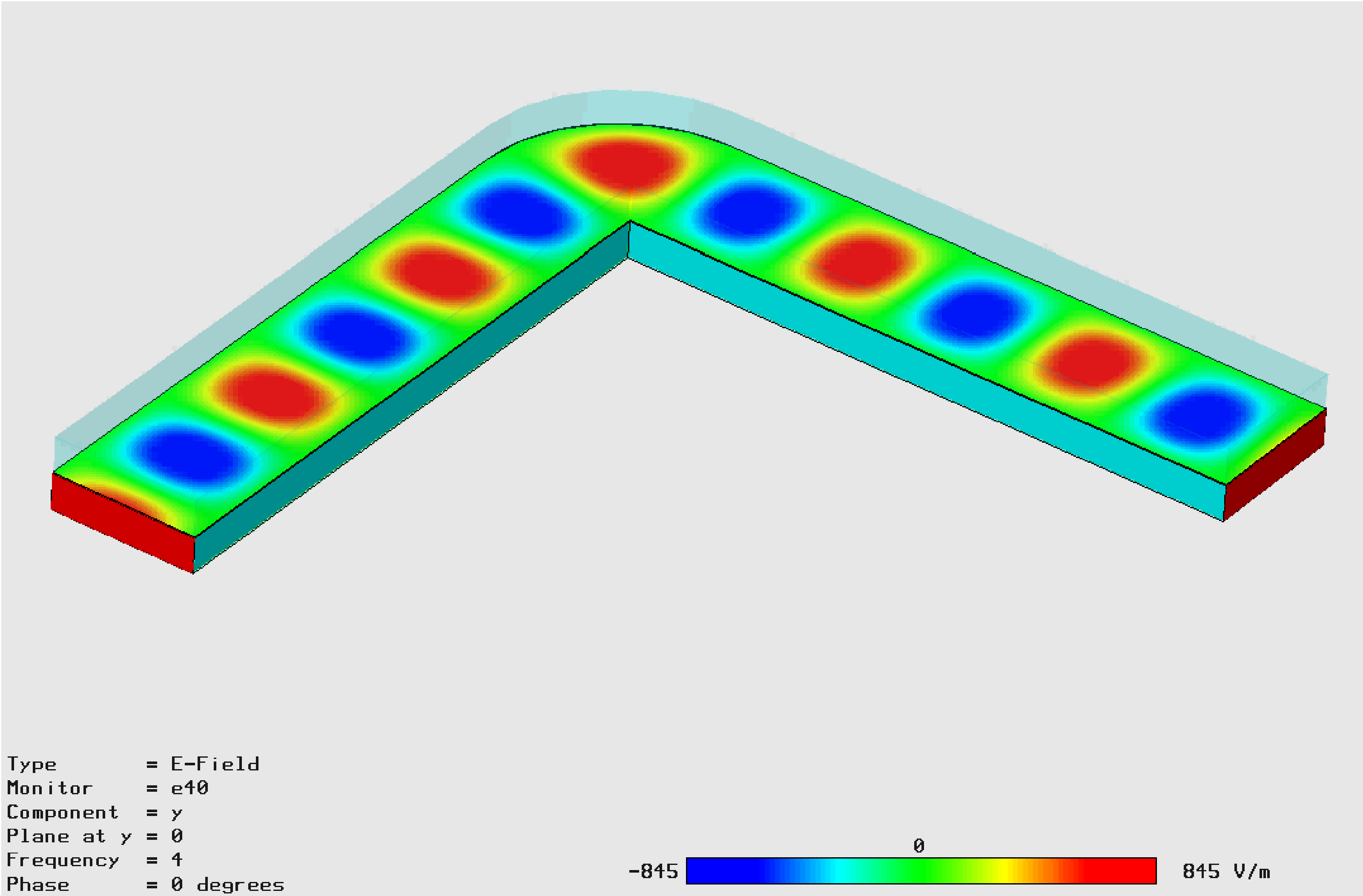


Waveguide Bend

f = 4 GHz

Input-Mode = TE_{10} Reflection = -26 dB

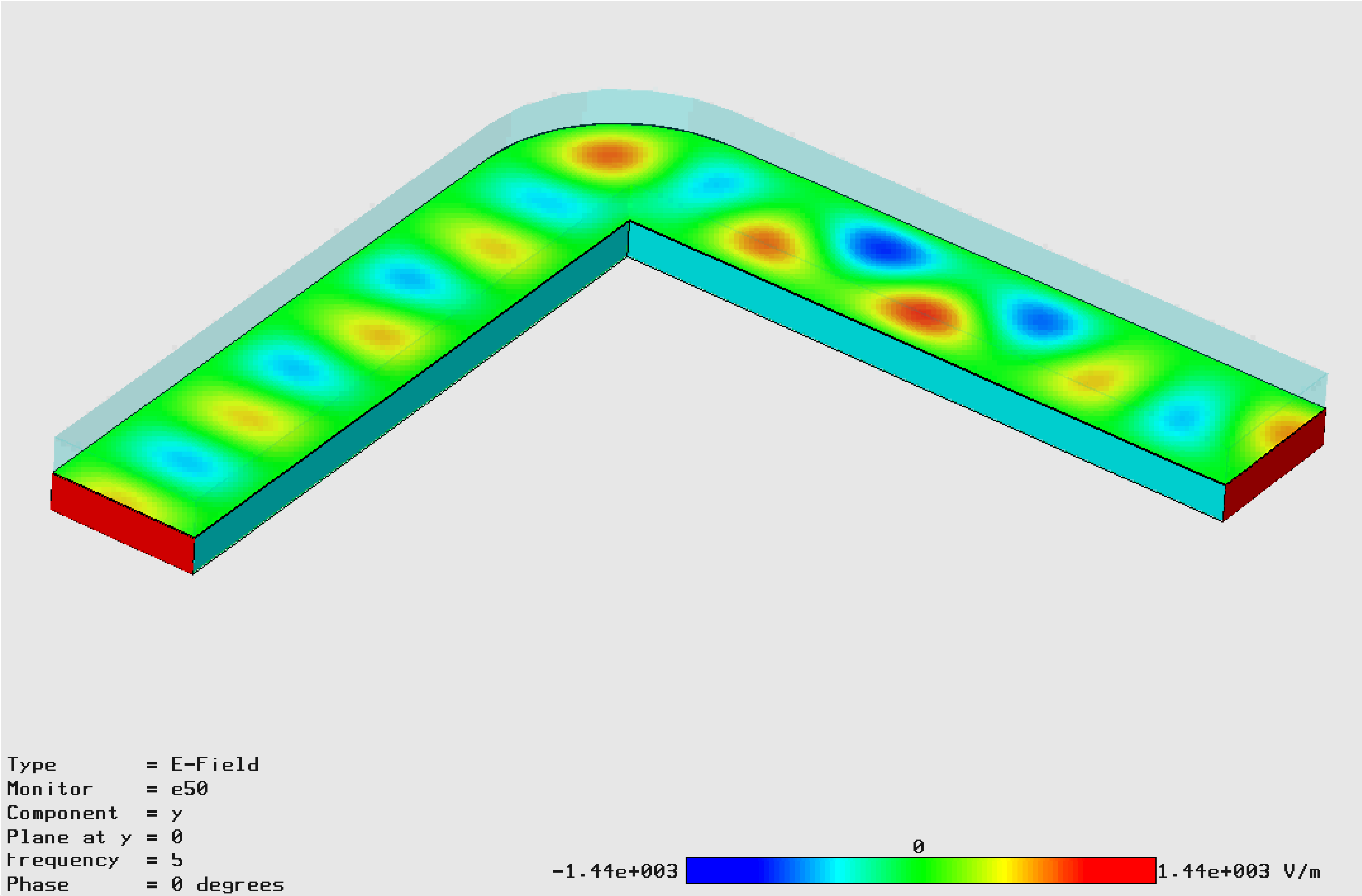
E_y



f = 5 GHz

Input-Mode = TE_{10} - 15% of energy in wrong mode

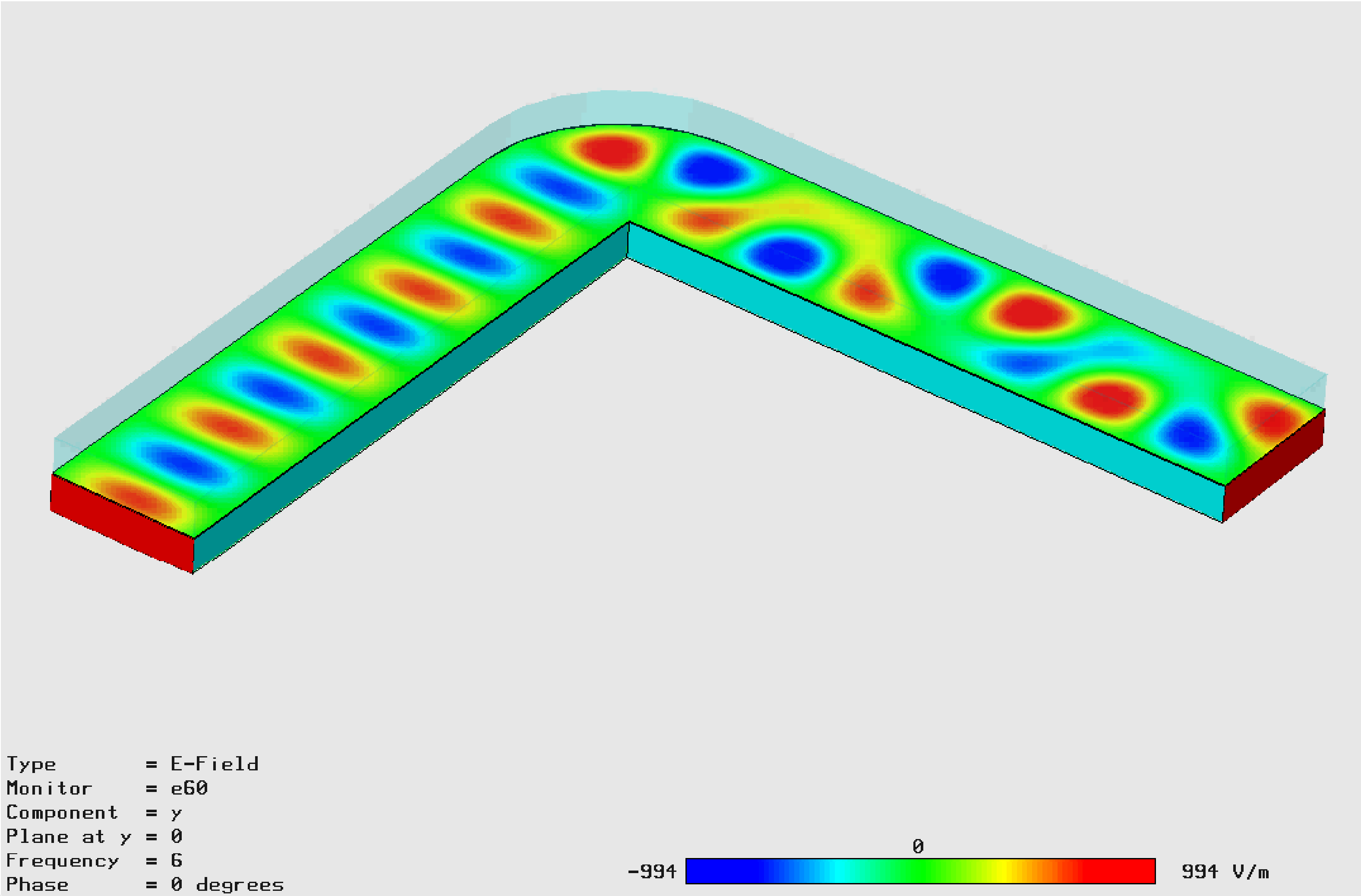
E_y



f = 6 GHz

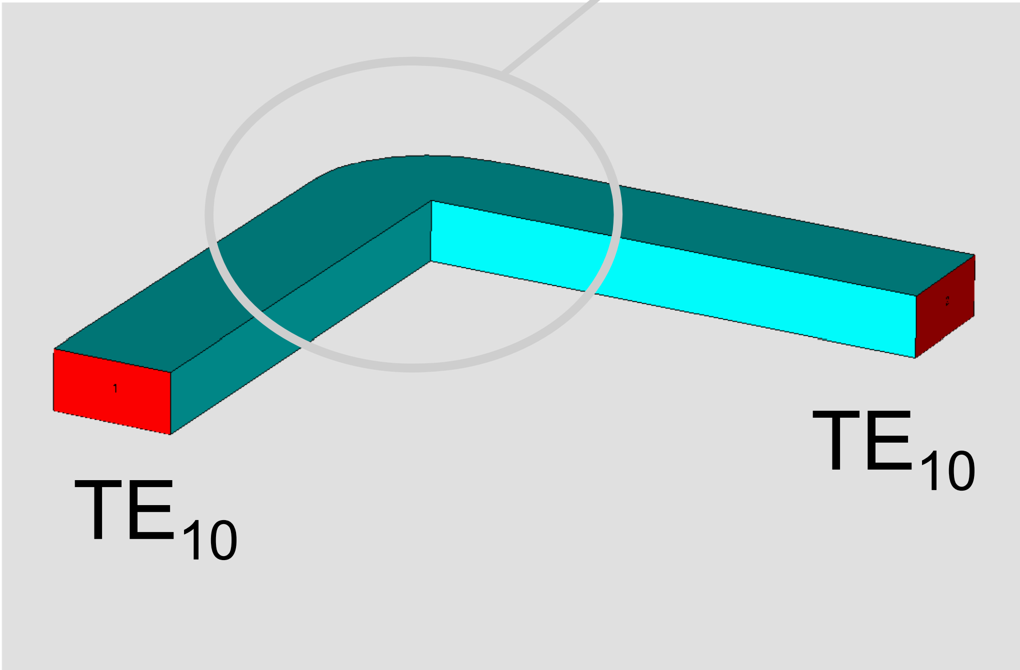
Input-Mode = TE_{10} - 75% of energy in wrong mode

E_y



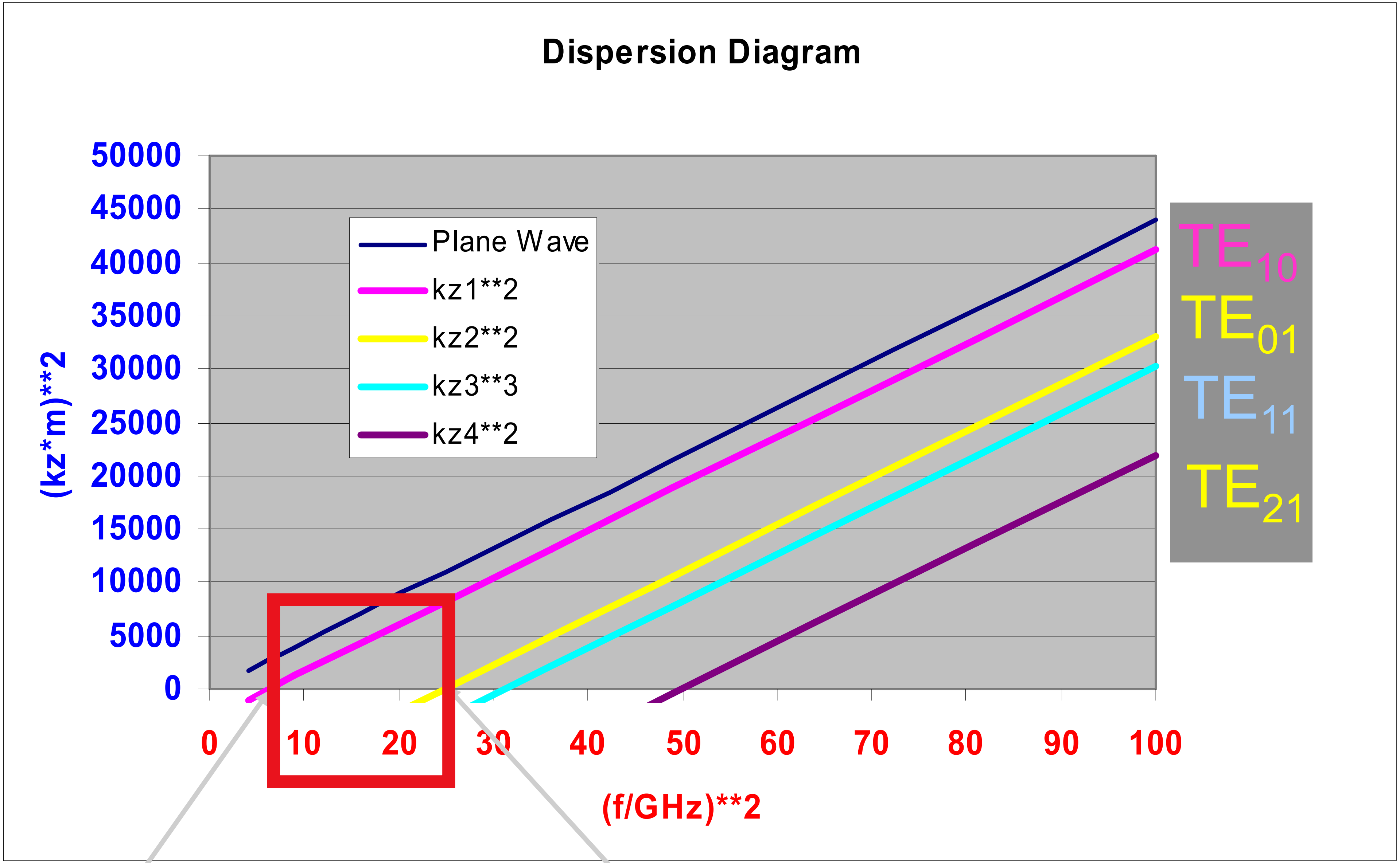
Waveguide Bend

$$\sum_{1}^{\infty} \vec{E}_{mn}\,;\vec{H}_{mn}$$



At the *sharp* corner, many higher modes can appear

Usable Frequency Band



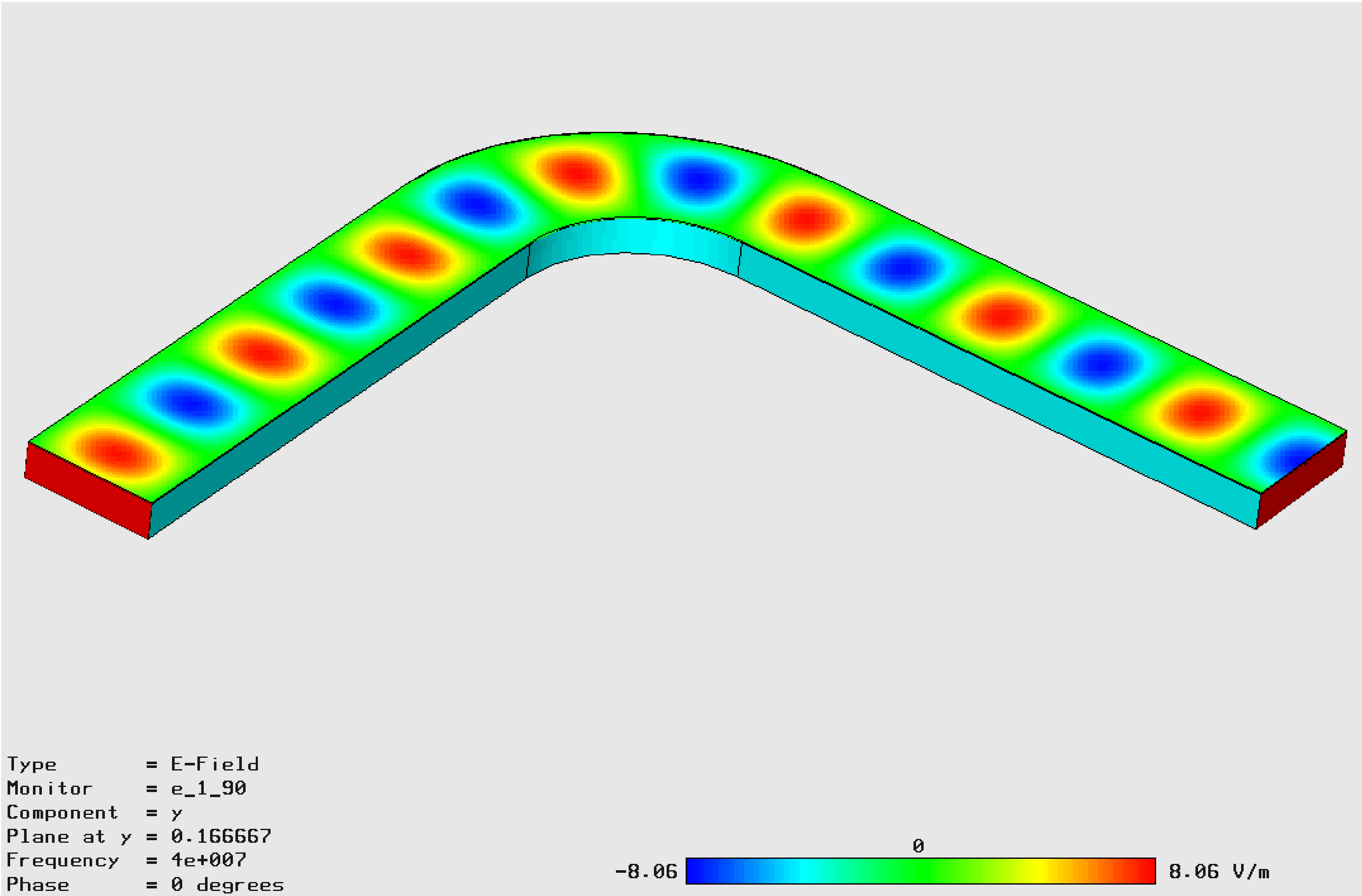
2,49 GHz = f_{c1} < f < f_{c2}= 4,99 GHz

Better Geometry

f = 4 GHz

Input-Mode = TE_{10} Reflection = -50 dB

E_y

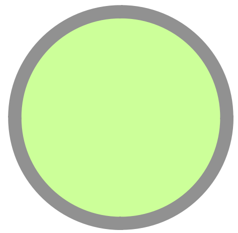


Circular Waveguides

Circular Waveguides

- **Solution of the wave equation (Summary):**

Circular WG



$$\Delta\vec{A} + k^2\vec{A} = 0 \qquad k^2 = \omega^2\mu\varepsilon$$

☞ Vector potential: $\vec{A} = \vec{e}_z A_z$

☞ $$\frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r}\frac{\partial A_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \varphi^2} + \frac{\partial^2 A_z}{\partial z^2} + k^2 A_z = 0$$ (cylindrical coordinates)

☞ Product approach: $A_z(r,\varphi,z) = f(r)\cdot g(\varphi)\cdot h(z)$

$$\Rightarrow \quad \boxed{A_z(r,\varphi,z) = \left\{\begin{matrix} J_\mu(Kr) \\ N_\mu(Kr) \end{matrix}\right\}\cdot\left\{\begin{matrix} \sin(\mu\varphi) \\ \cos(\mu\varphi) \end{matrix}\right\}\cdot\left\{\begin{matrix} e^{+jk_z z} \\ e^{-jk_z z} \end{matrix}\right\}}$$

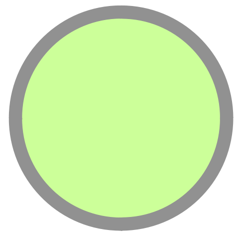
☞ *Separation relation*: $K^2 + k_z^2 = k^2$

(Lecture Notes pp. 57-61)

Circular Waveguides

- **Derivation of the solutions of the wave equation:**

Circular WG

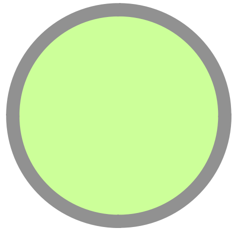


Circular Waveguides

- **Solution of the wave equation:**

$$\frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r}\frac{\partial A_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \varphi^2} + \frac{\partial^2 A_z}{\partial z^2} + k^2 A_z = 0$$

Circular WG



☞ Product approach: $A_z(r,\varphi,z) = f(r)\cdot g(\varphi)\cdot h(z)$

☞ Insert this into the equation, divide by A_z

$$\frac{1}{f}\frac{\partial^2 f}{\partial r^2} + \frac{1}{fr}\frac{\partial f}{\partial r} + \frac{1}{r^2}\underbrace{\frac{1}{g}\frac{\partial^2 g}{\partial \varphi^2}}_{-\mu^2} + \underbrace{\frac{1}{h}\frac{\partial^2 h}{\partial z^2}}_{-k_z^2} + k^2 = 0$$

(cylindrical coordinates)

$$\Rightarrow \quad g(\varphi) = A_1 \sin(\mu\varphi) + A_2 \cos(\mu\varphi)$$

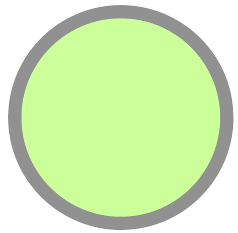
$$h(z) = C_1 e^{jk_z z} + C_2 e^{-jk_z z}$$

(Lecture Notes pp. 57-61)

Circular Waveguides

- **Solution of the wave equation:**

Circular WG



☞ $$\underbrace{\frac{1}{f}\frac{\partial^2 f}{\partial r^2}+\frac{1}{fr}\frac{\partial f}{\partial r}-\frac{1}{r^2}\mu^2}_{-K^2}-k_z^2+k^2=0$$

☞ Separation relation: $K^2+k_z^2=k^2$

☞ After multiplication with f:

$$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\left(k^2-k_z^2-\frac{\mu^2}{r^2}\right)f=0$$

$$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\left(K^2-\frac{\mu^2}{r^2}\right)f=0$$

Bessel differential equation

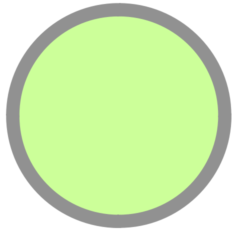
$$\boxed{f(r)=B_1 J_\mu(Kr)+B_2 N_\mu(Kr)}$$

(Lecture Notes pp. 57-61)

Circular Waveguides

- **Solution of the wave equation:**

Circular WG



$$A_z(r,\varphi,z) = \begin{Bmatrix} J_\mu(Kr) \\ N_\mu(Kr) \end{Bmatrix} \cdot \begin{Bmatrix} \sin(\mu\varphi) \\ \cos(\mu\varphi) \end{Bmatrix} \cdot \begin{Bmatrix} e^{+jk_z z} \\ e^{-jk_z z} \end{Bmatrix}$$

☞ *Separation relation*: $K^2 + k_z^2 = k^2$

(Lecture Notes pp. 57-61)

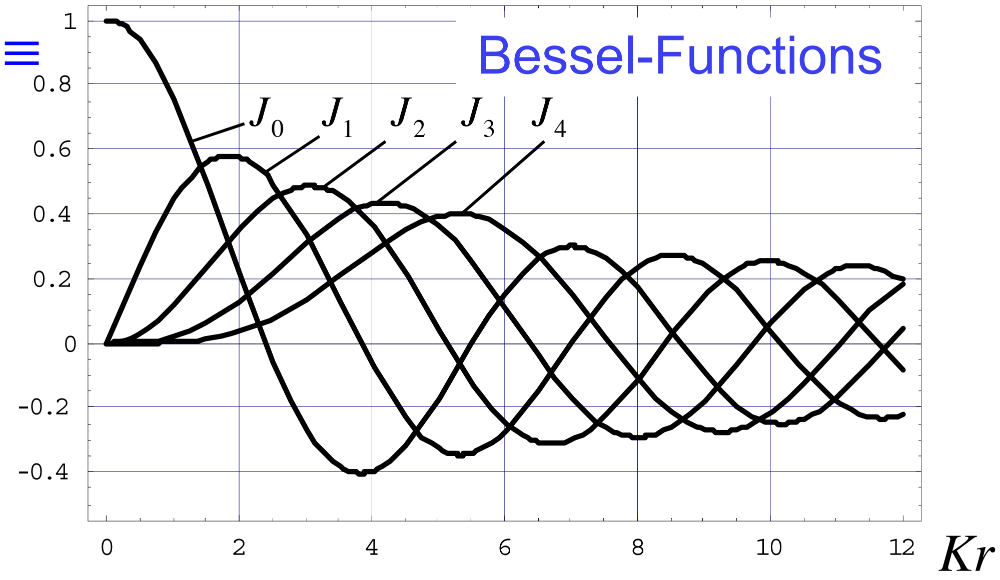
Cylindrical functions

- **Cylindrical functions:**

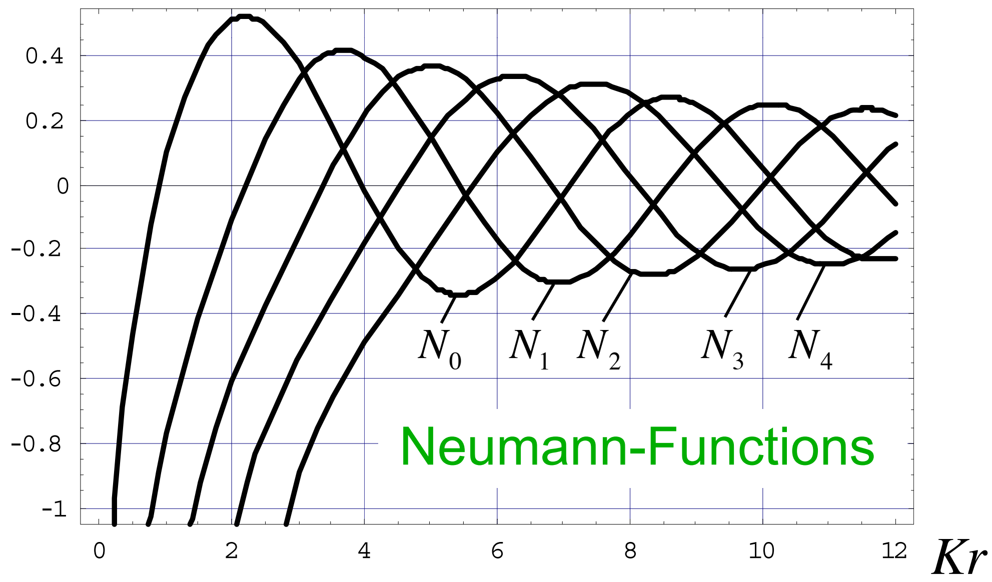
  - μ couples azimuthal and radial dependency

  - $\mu \in \mathbb{N}_0$ in cylindrical systems with $0 \le \varphi < 2\pi$

$$J_\mu(Kr) \equiv J_m(Kr)$$



$$N_\mu(Kr)$$



  - If the z-axis is included only the functions of the first form (Bessel) may be considered

$$\lim_{r \to 0} N_\mu(Kr) = -\infty \quad \forall \mu$$

$$\Rightarrow A \sim J_\mu(Kr)$$

Circular Waveguides

$$\mu = m \qquad A_z(r,\varphi,z) = \begin{Bmatrix} J_m(Kr) \\ N_m(Kr) \end{Bmatrix} \cdot \begin{Bmatrix} \sin(m\varphi) \\ \cos(m\varphi) \end{Bmatrix} \cdot \begin{Bmatrix} e^{+jk_z z} \\ e^{-jk_z z} \end{Bmatrix}$$

- **φ dependency: linear combination of sine and cosine functions**
  - ☞ Rotational symmetry: choose origin for φ such that one single function is sufficient
  - ☞ Sine function is 0 for m = 0

$$\Rightarrow A \sim \cos(m\varphi)$$

- **z dependency: consider first forward-propagating waves**

$$\Rightarrow A \sim e^{-jk_z z}$$

$$A_z(r,\varphi,z) = C\, J_m(Kr) \cos(m\varphi)\, e^{-jk_z z}$$

Circular Waveguides

- **TE-modes**
  - Field vectors:

$$\vec{E} = \operatorname{curl} \vec{A} \qquad \vec{H} = \frac{-1}{j\omega\mu} \operatorname{curl} \vec{E}$$

$$A_z(r,\varphi,z) = C\, J_m(Kr) \cos(m\varphi)\, e^{-jk_z z}$$

  - Components:

$$E_r = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = -C\frac{m}{r} J_m(Kr)\sin m\varphi\, e^{-jk_z z}$$

$$E_\varphi = -\frac{\partial A_z}{\partial r} = -C\,K\,J_m^{'}(Kr)\cos m\varphi\, e^{-jk_z z}$$

$$H_r = \frac{1}{j\omega\mu}\frac{\partial E_\varphi}{\partial z} = -C\frac{k_z}{\omega\mu} K\,J_m^{'}(Kr)\cos m\varphi\, e^{-jk_z z}$$

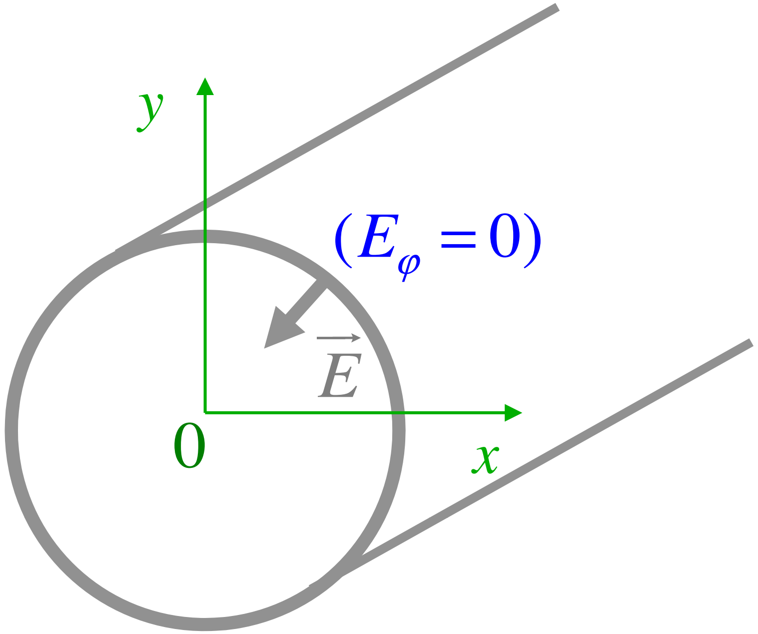
$$H_\varphi = \frac{-1}{j\omega\mu}\frac{\partial E_r}{\partial z} = -C\frac{k_z}{\omega\mu}\frac{m}{r} J_m(Kr)\sin m\varphi\, e^{-jk_z z}$$

$$H_z = -\frac{K^2}{j\omega\mu} A_z = -C\frac{K^2}{j\omega\mu} J_m(Kr)\cos m\varphi\, e^{-jk_z z}$$

Circular Waveguides

- **TE-solution:**

$$E_\varphi = -C\ K\ J_m^{'}(Kr)\cos m\varphi\ e^{-jk_z z} = 0$$



$$J_m^{'}(Kr) = 0 \quad for \quad r = a \quad \rightarrow \quad K = \frac{j_{mn}^{'}}{a}$$

$j_{mn}^{'}$: n-th root of $J_m^{'}(Kr)$

$j_{mn}^{'}$		m			
		0	1	2	3
	1	3.832	1.841	3.054	4.201
n	2	7.016	5.331	6.706	8.015
	3	10.173	8.536	9.969	11.346

Continuity:

$$E_\varphi(r = a, \varphi, z) = 0$$

Fundamental mode

- **Fundamental mode for the circular waveguide: TE11**

- **Cutoff angular frequency:**

$$k_z = \sqrt{k^2 - K^2} = 0 \quad \Rightarrow K = k = \frac{\omega}{c}$$

$$\Rightarrow \omega = Kc$$

- **for TE modes:**

$$\omega_{c_{mn}} = Kc = \frac{j'_{mn}}{a} c$$

- **Cutoff angular frequency for the fundamental mode:**

$$\omega_{c_{11}} = \frac{j'_{11}}{a} c$$

TM-modes

- Field vectors:

$$\vec{H} = \operatorname{curl}\vec{A} \qquad \vec{E} = \frac{1}{j\omega\varepsilon}\operatorname{curl}\vec{H}$$

- Components:

$$H_r = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = -C\frac{m}{r}J_m(Kr)\sin m\varphi\, e^{-jk_z z}$$

$$H_\varphi = -\frac{\partial A_z}{\partial r} = -C\,K\,J_m'(Kr)\cos m\varphi\, e^{-jk_z z}$$

$$E_r = \frac{-1}{j\omega\varepsilon}\frac{\partial H_\varphi}{\partial z} = -C\frac{k_z}{\omega\varepsilon}K\,J_m'(Kr)\cos m\varphi\, e^{-jk_z z}$$

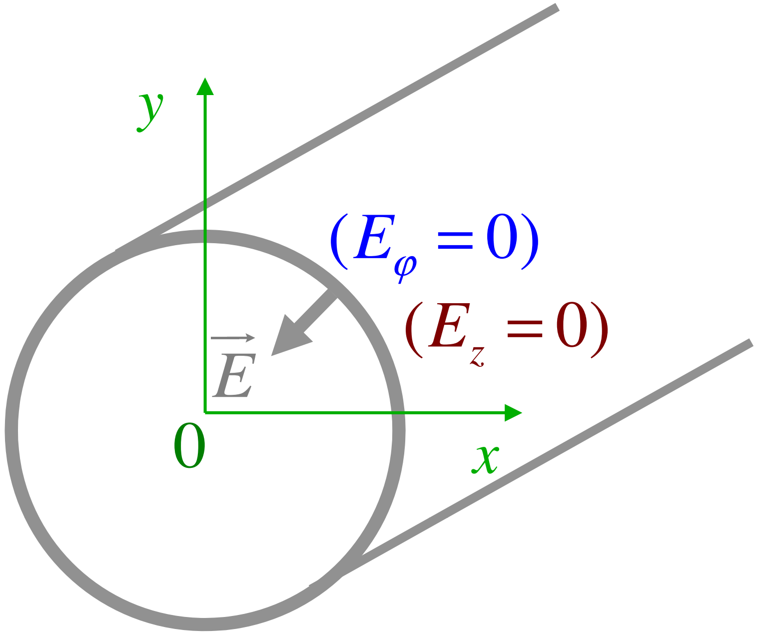
$$E_\varphi = \frac{1}{j\omega\varepsilon}\frac{\partial H_r}{\partial z} = C\frac{k_z}{\omega\varepsilon}\frac{m}{r}J_m(Kr)\sin m\varphi\, e^{-jk_z z}$$

$$E_z = \frac{K^2}{j\omega\varepsilon}A_z = C\frac{K^2}{j\omega\varepsilon}J_m(Kr)\cos m\varphi\, e^{-jk_z z}$$

TM-solution:

$$E_\varphi = C\frac{k_z}{\omega\varepsilon}\frac{m}{r}J_m(Kr)\sin m\varphi\, e^{-jk_z z} = 0$$

$$E_z = C\frac{K^2}{j\omega\varepsilon}J_m(Kr)\cos m\varphi\, e^{-jk_z z} = 0$$



$$J_m(Kr) = 0 \quad for \quad r = a \quad \rightarrow \quad K = \frac{j_{mn}}{a}$$

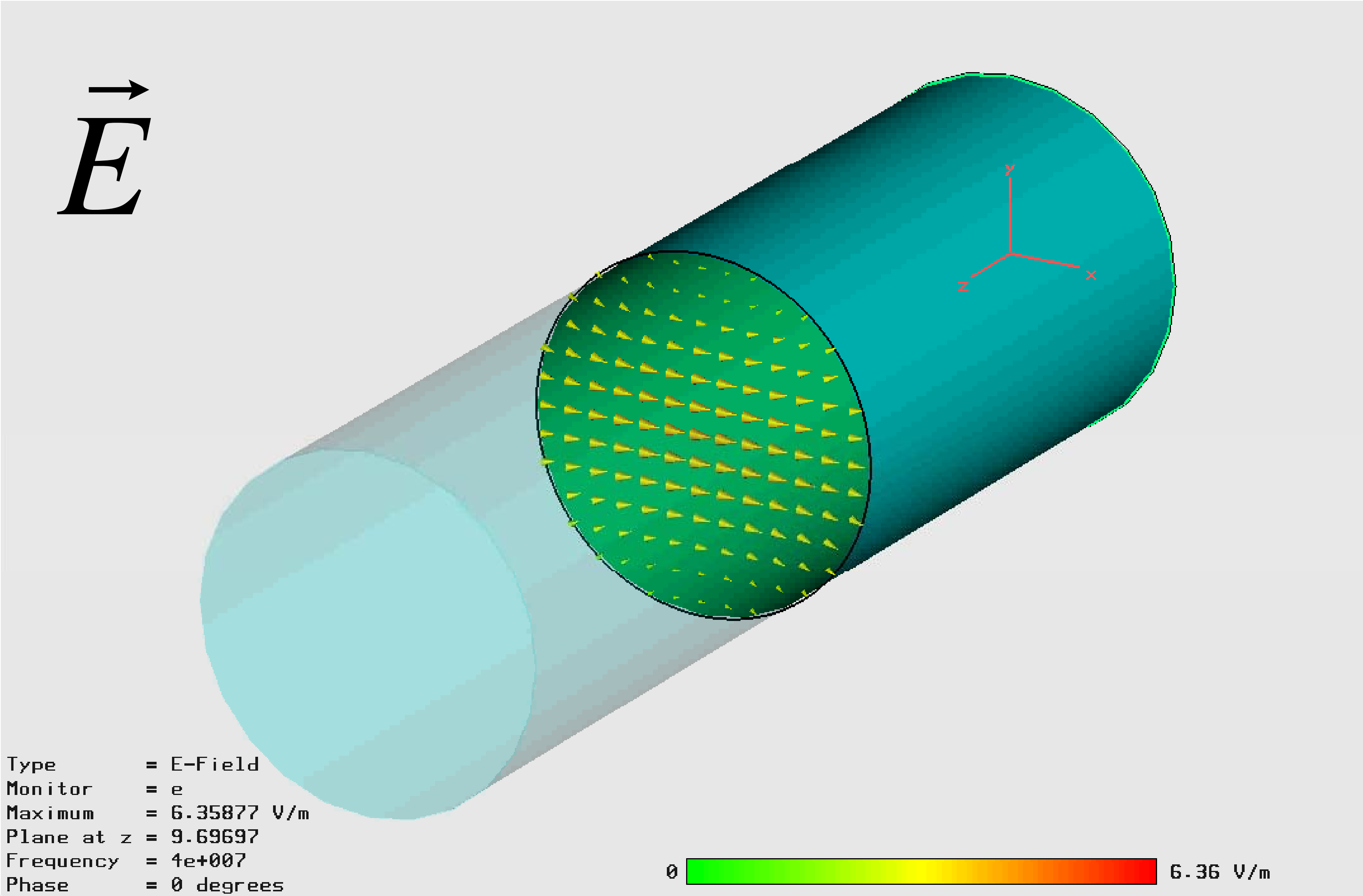
j_{mn}		m			
		0	1	2	3
	1	2.405	3.832	5.136	6.380
n	2	5.520	7.016	8.417	9.761
	3	8.654	10.173	11.620	13.015

Continuity:

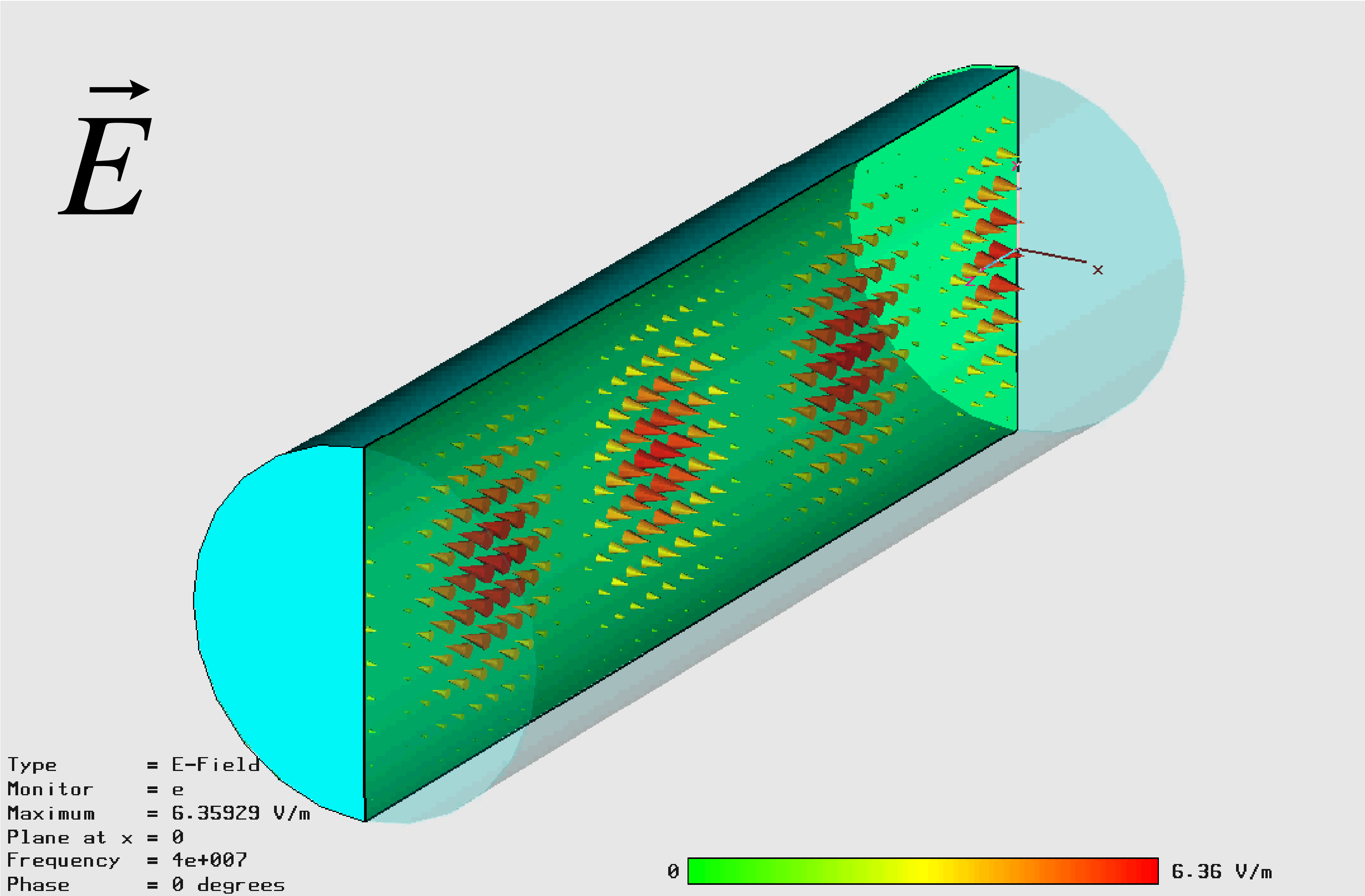
$E_\varphi(r = a, \varphi, z) = 0$

$E_z(r = a, \varphi, z) = 0$

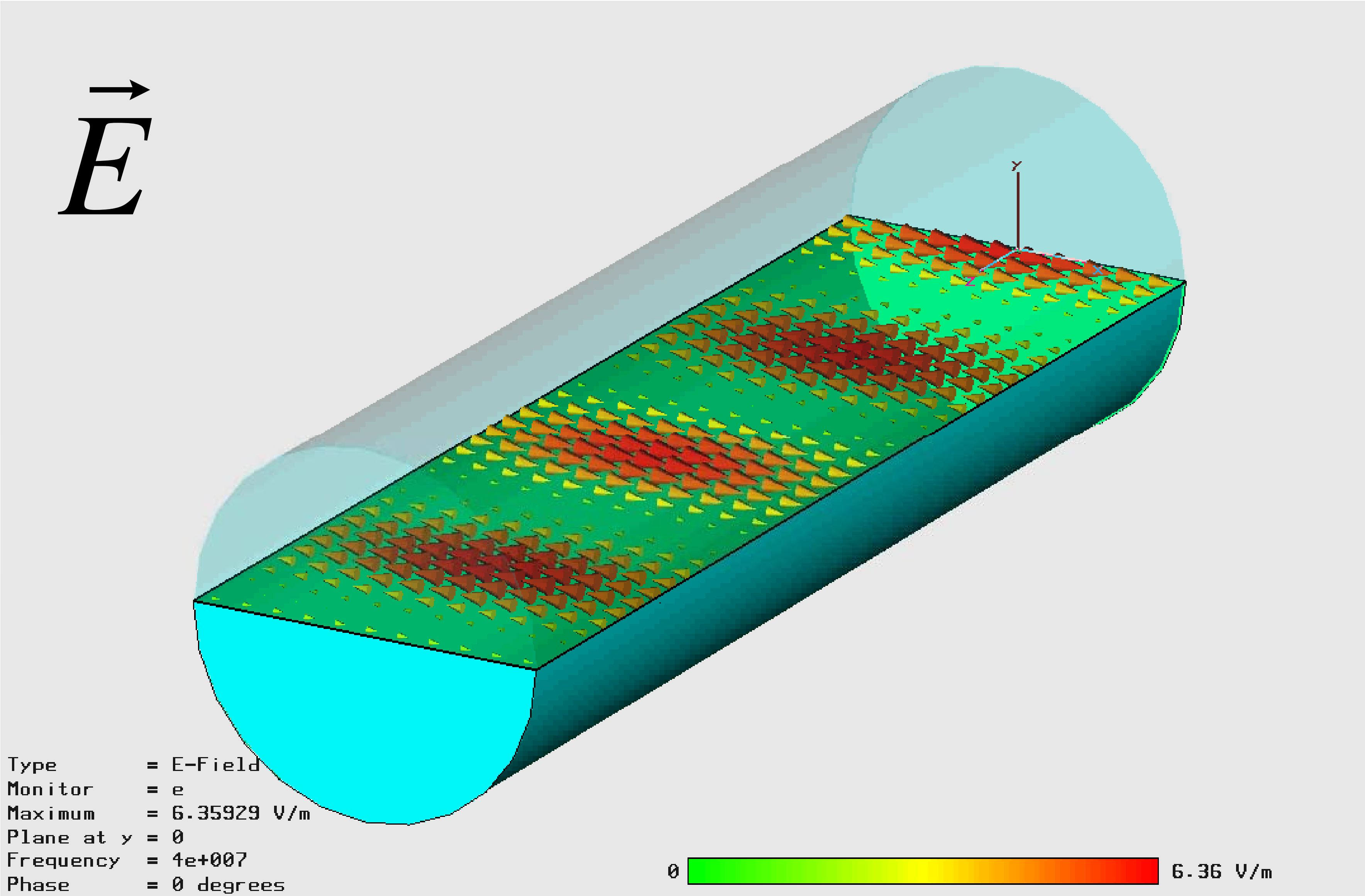
- **Electric Field of TE_{11}-mode (fundamental mode):**



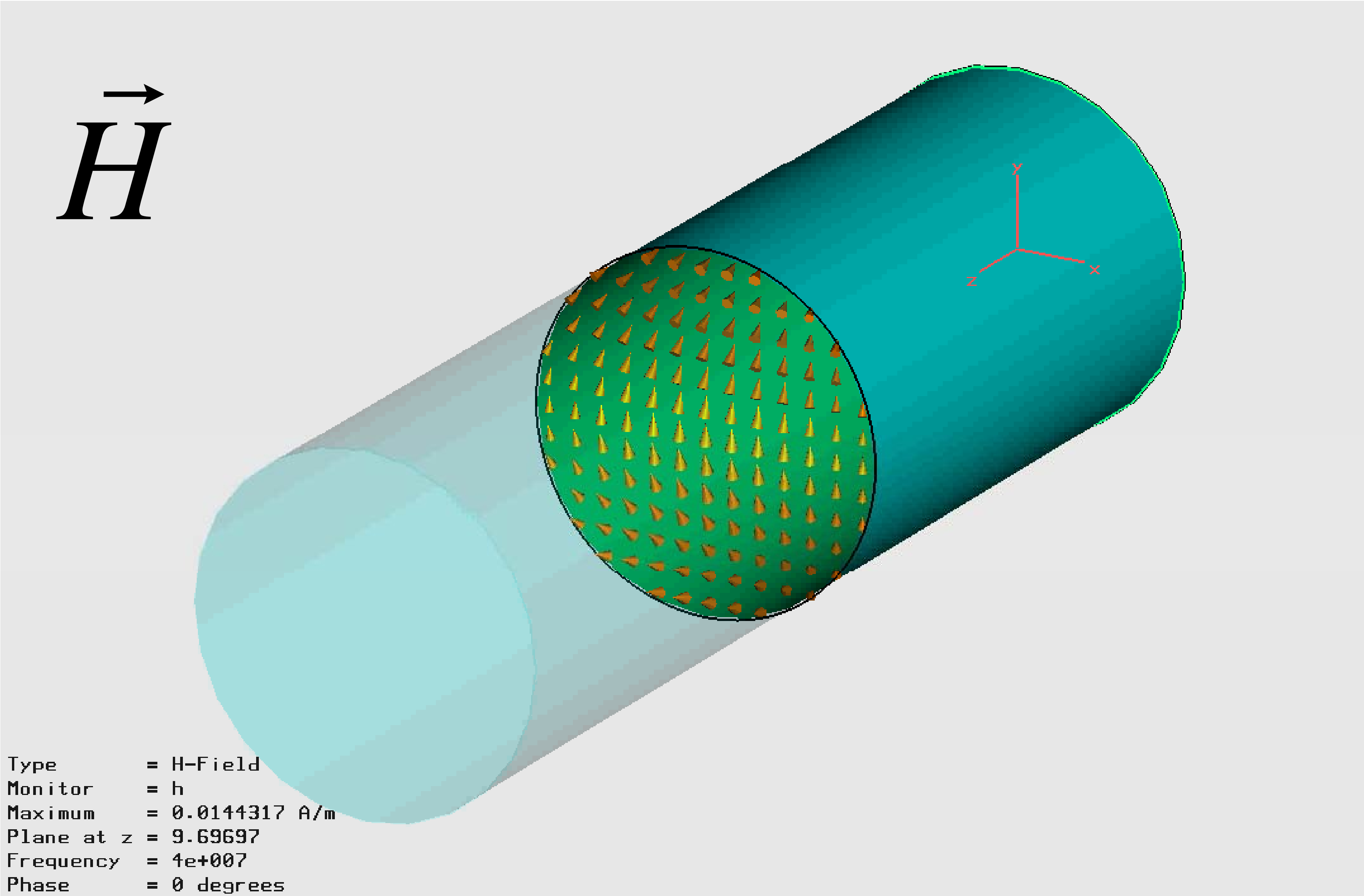
- **Electric Field of TE_{11}-mode (fundamental mode):**



- **Electric Field of TE_{11}-mode (fundamental mode):**

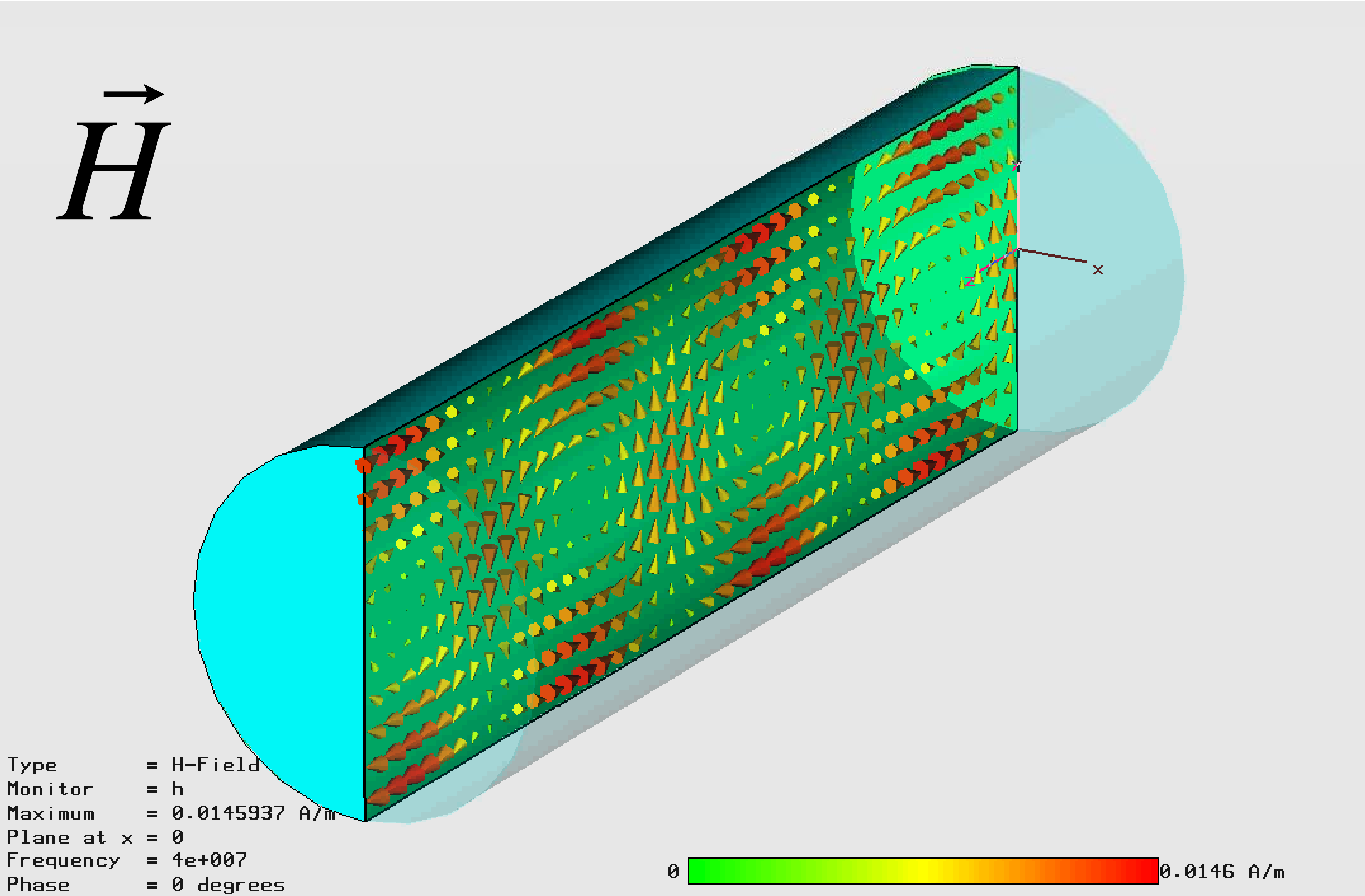


- **Magnetic Field of TE_{11}-mode (fundamental mode):**

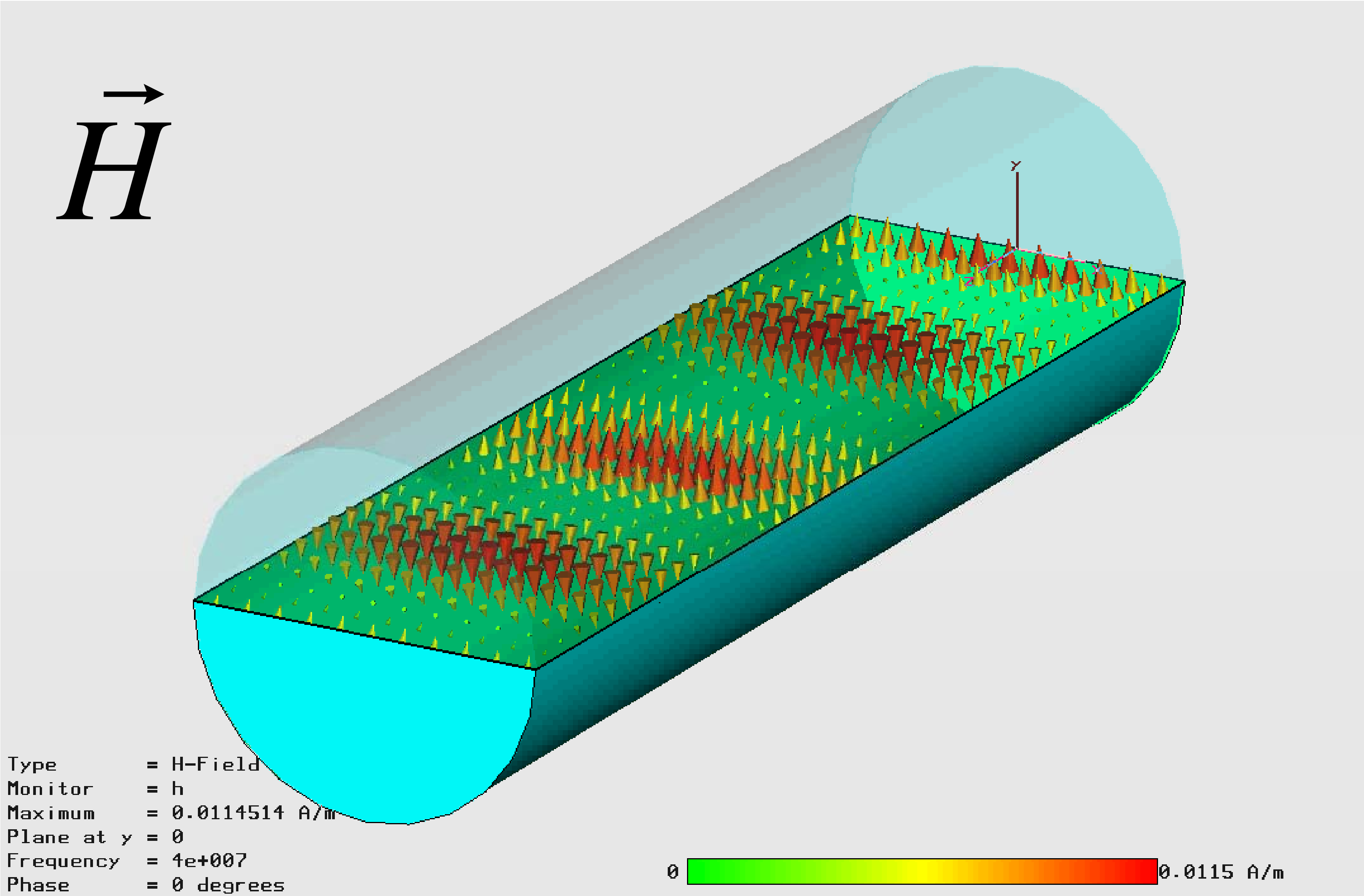


Circular Waveguides

- **Magnetic Field of TE_{11}-mode (fundamental mode):**

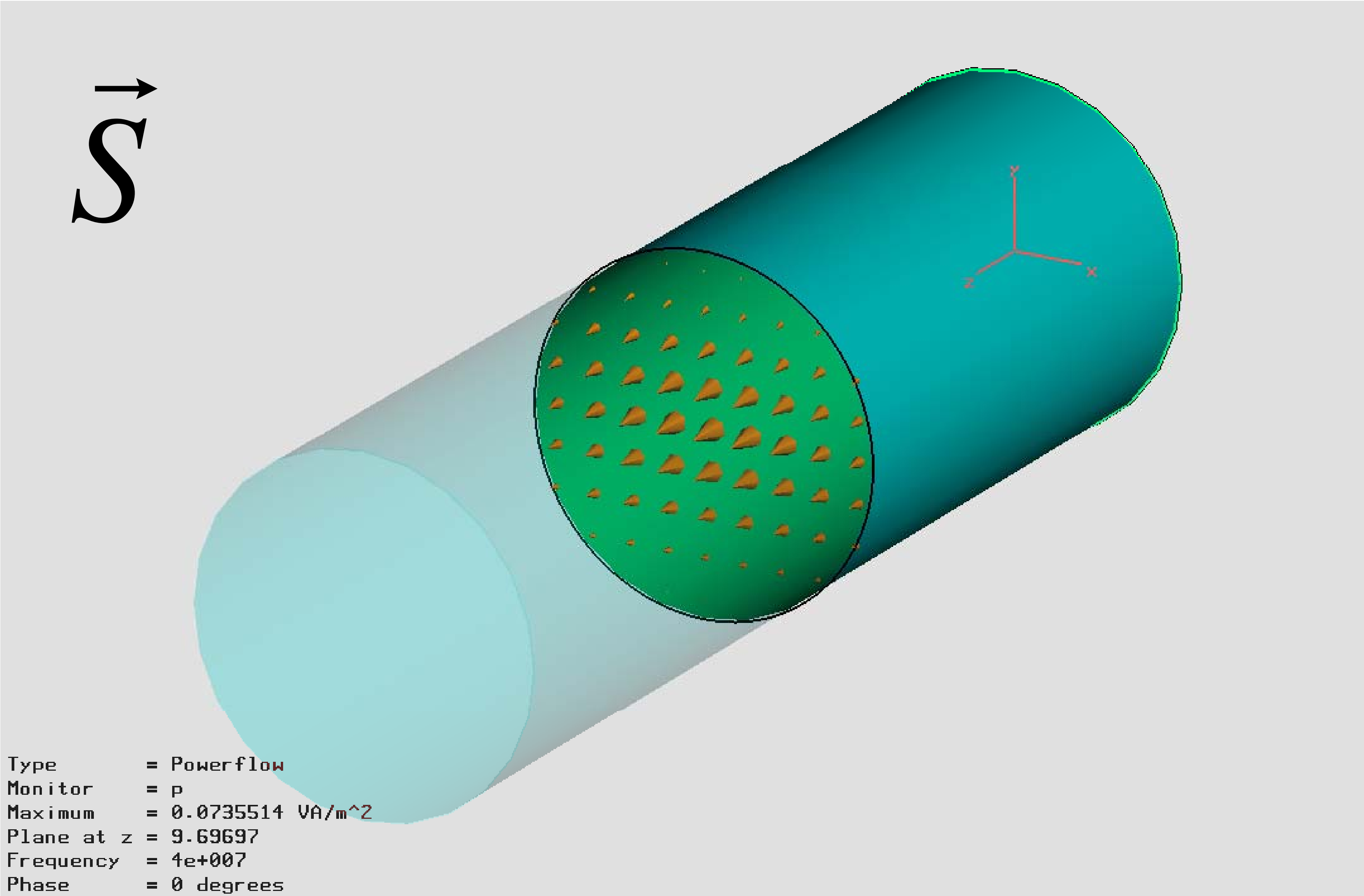


- **Magnetic Field of TE_{11}-mode (fundamental mode):**

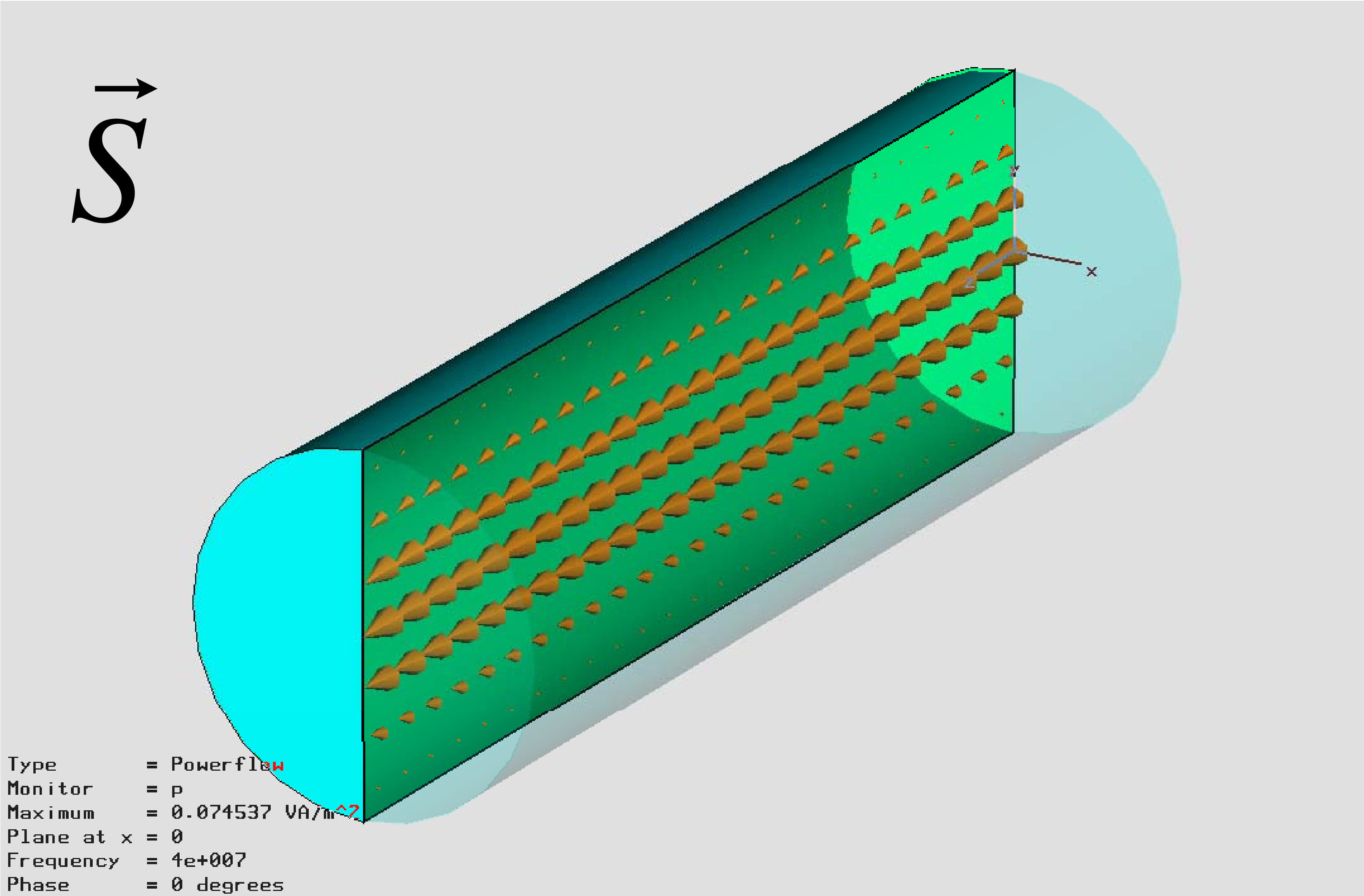


Circular Waveguides

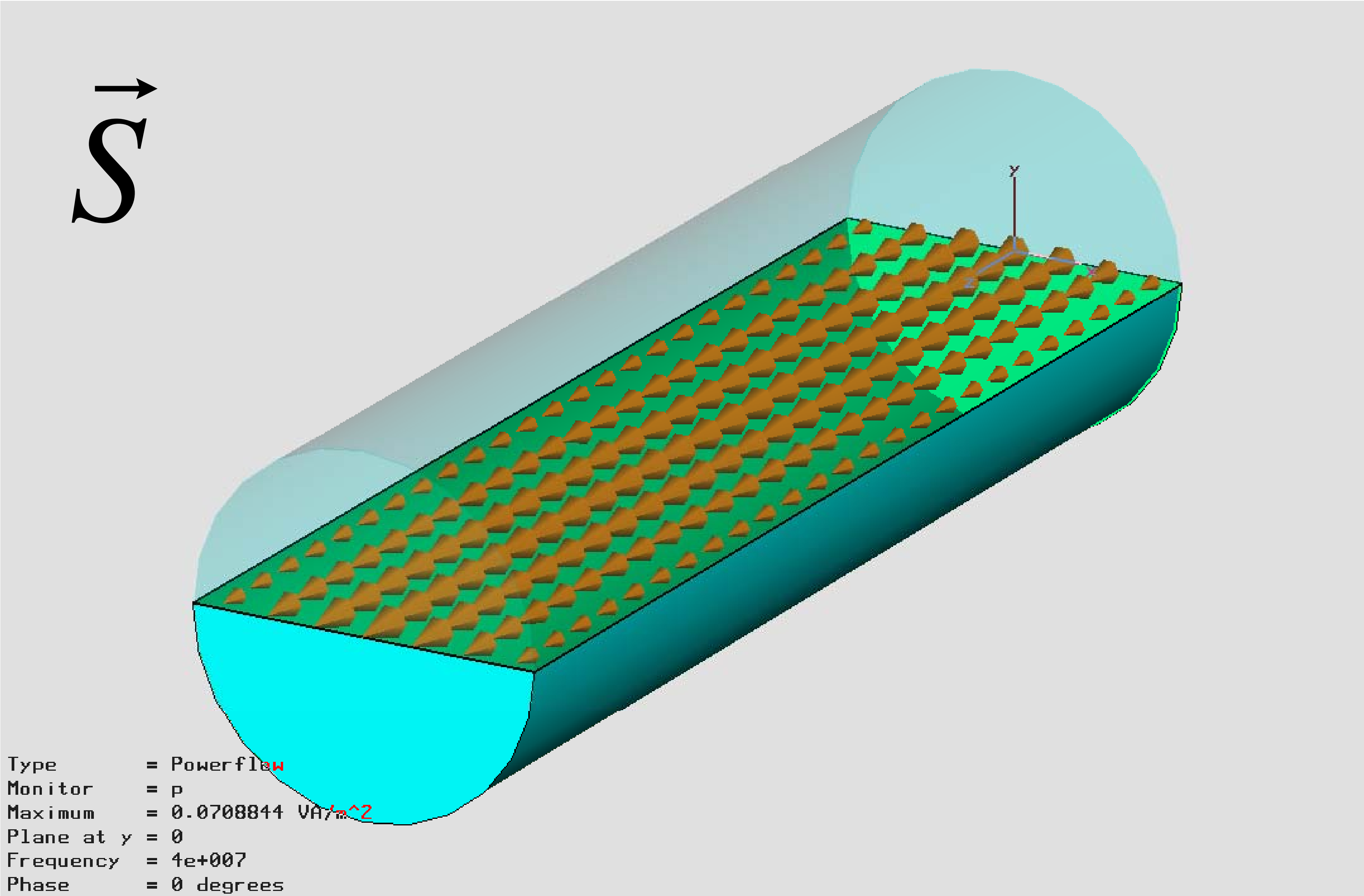
- **Poynting vector of TE_{11}-mode (fundamental mode):**



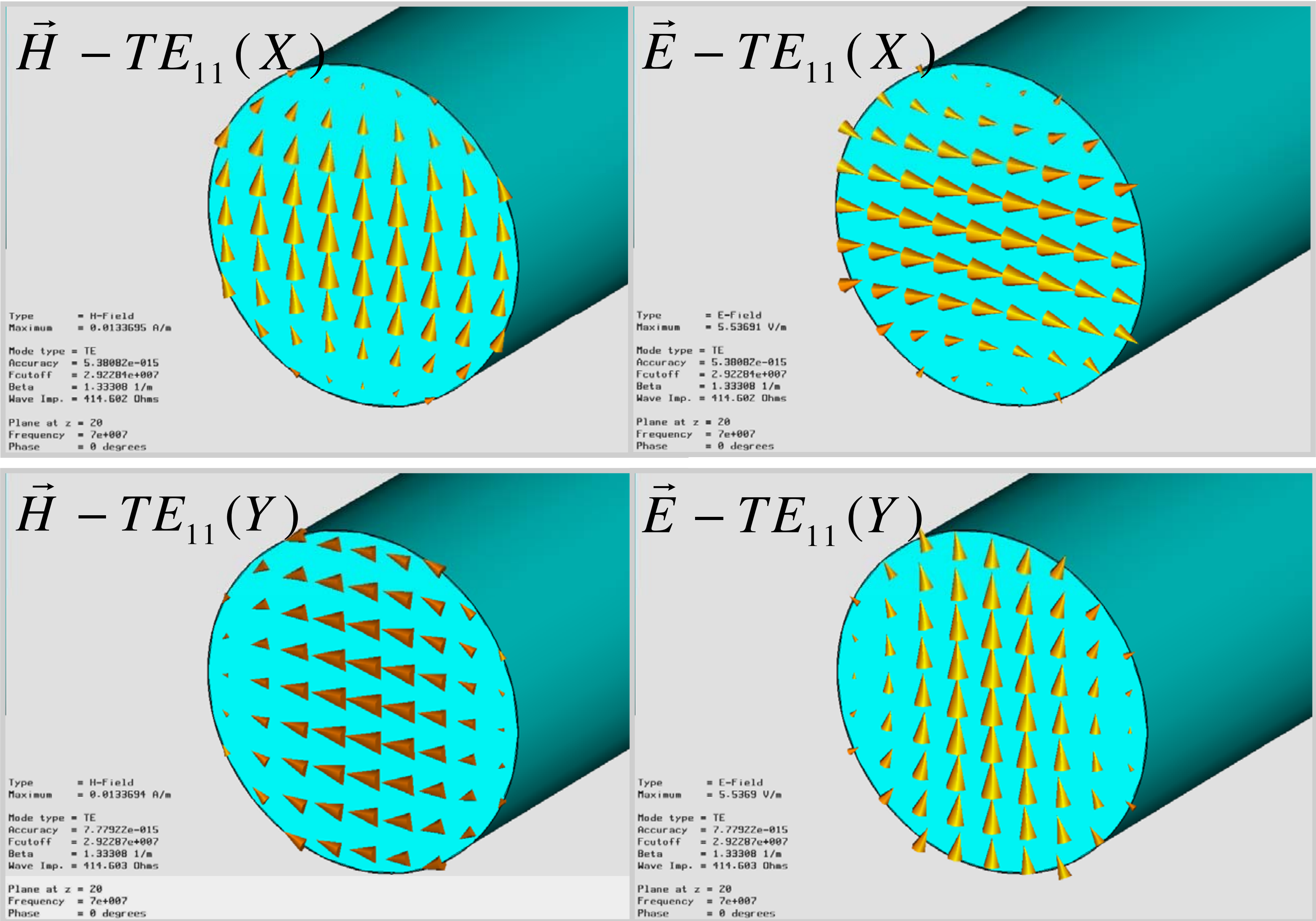
- **Poynting vector of TE_{11}-mode (fundamental mode):**



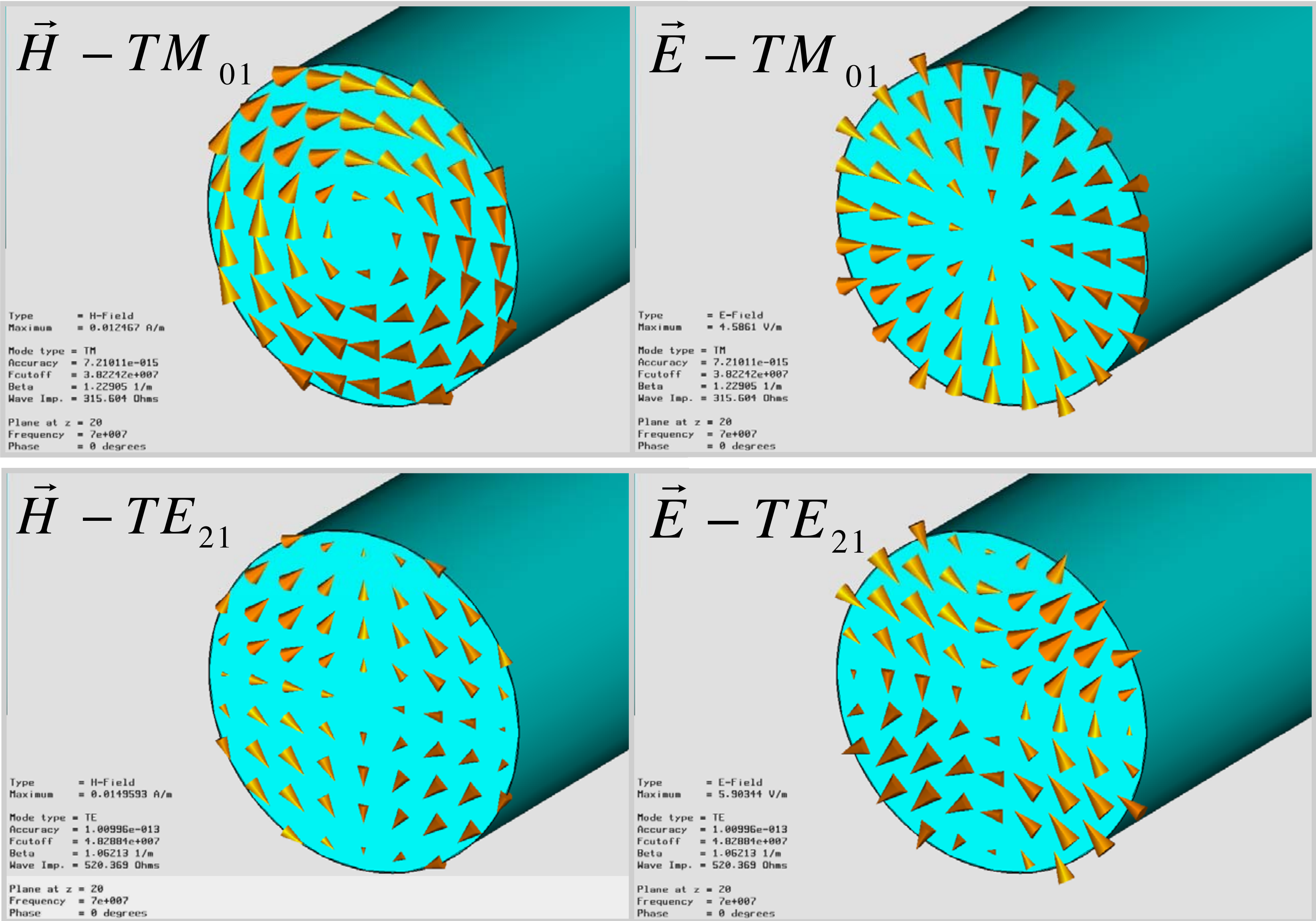
- **Poynting vector of TE_{11}-mode (fundamental mode):**



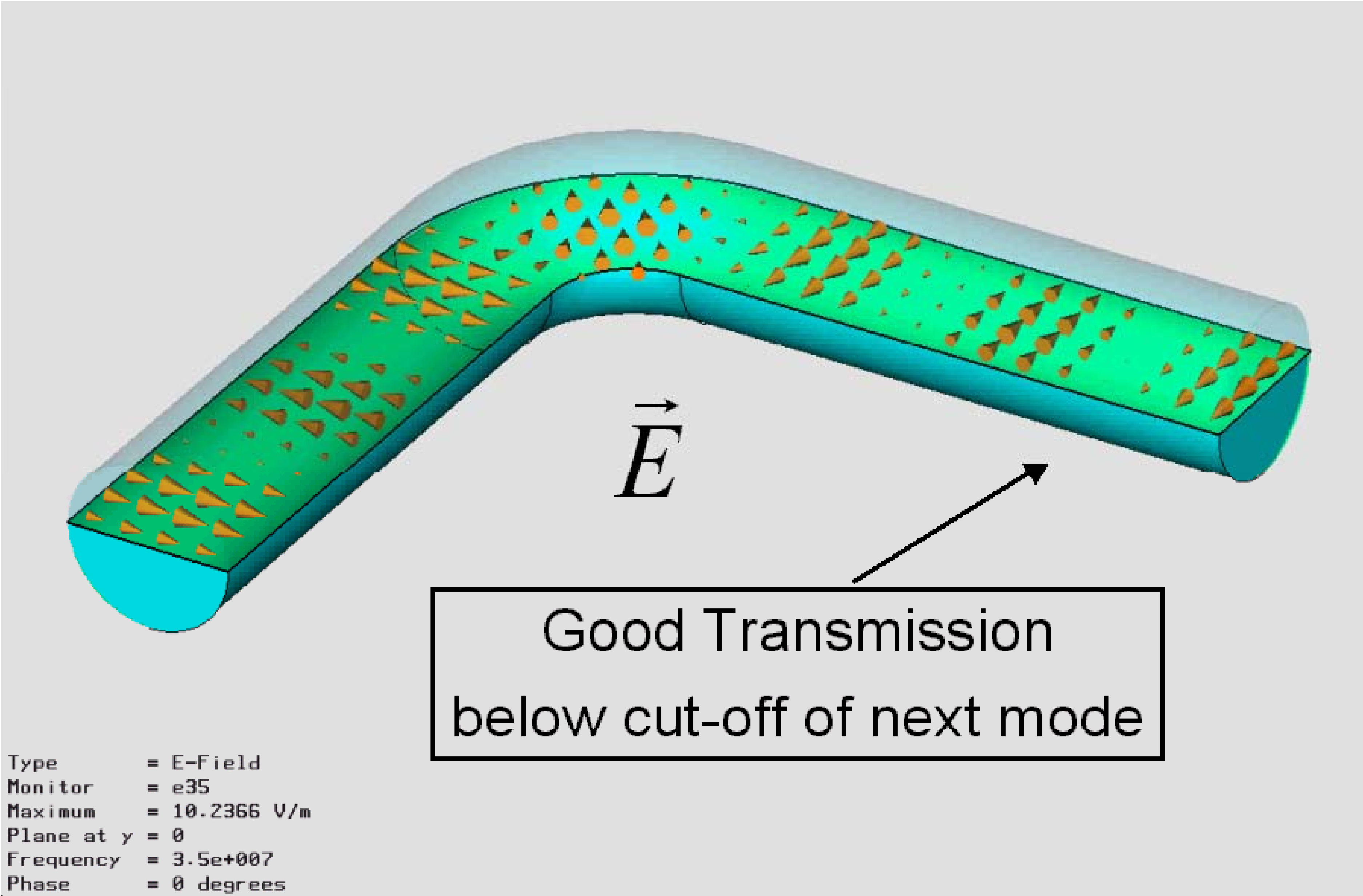
- **Magnetic and electric Fields of TE_{11}-mode:**



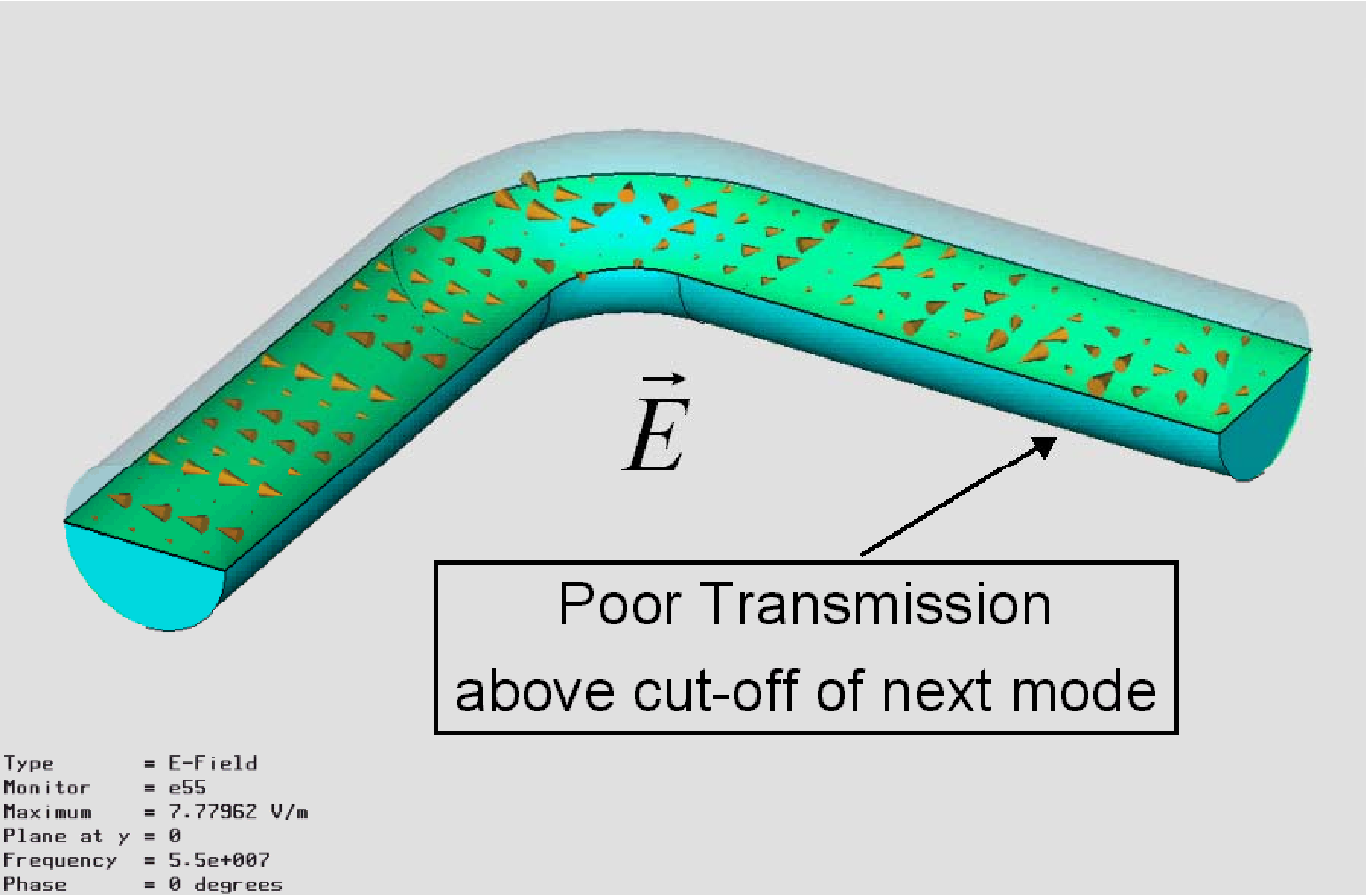
- **Magnetic and electric Fields of higher order modes :**



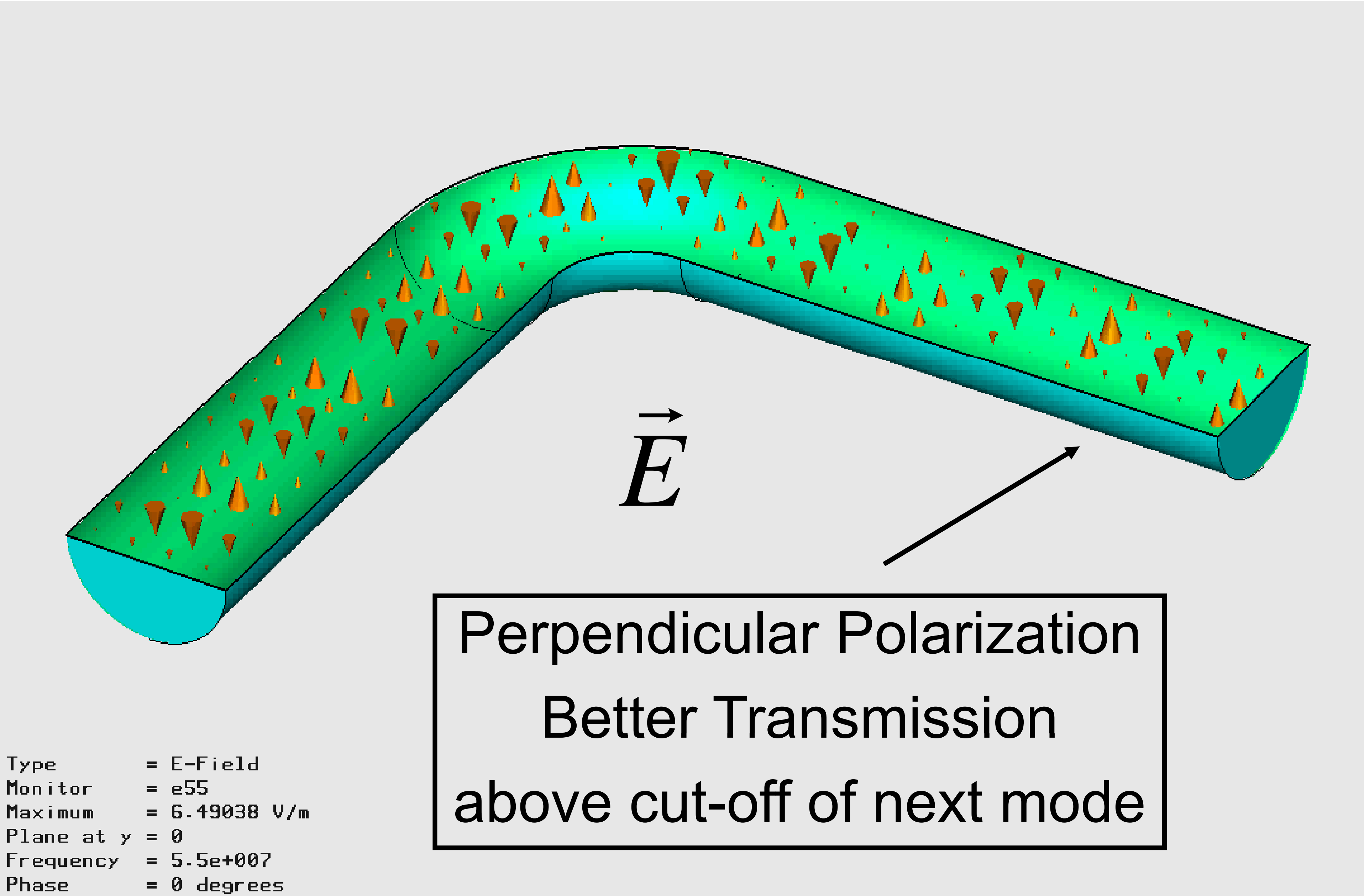
H11-Y-3.5GHz


$\vec{E}$
Good Transmission
below cut-off of next mode
Type = E-Field
Monitor = e35
Maximum = 10.2366 V/m
Plane at y = 0
Frequency = 3.5e+007
Phase = 0 degrees

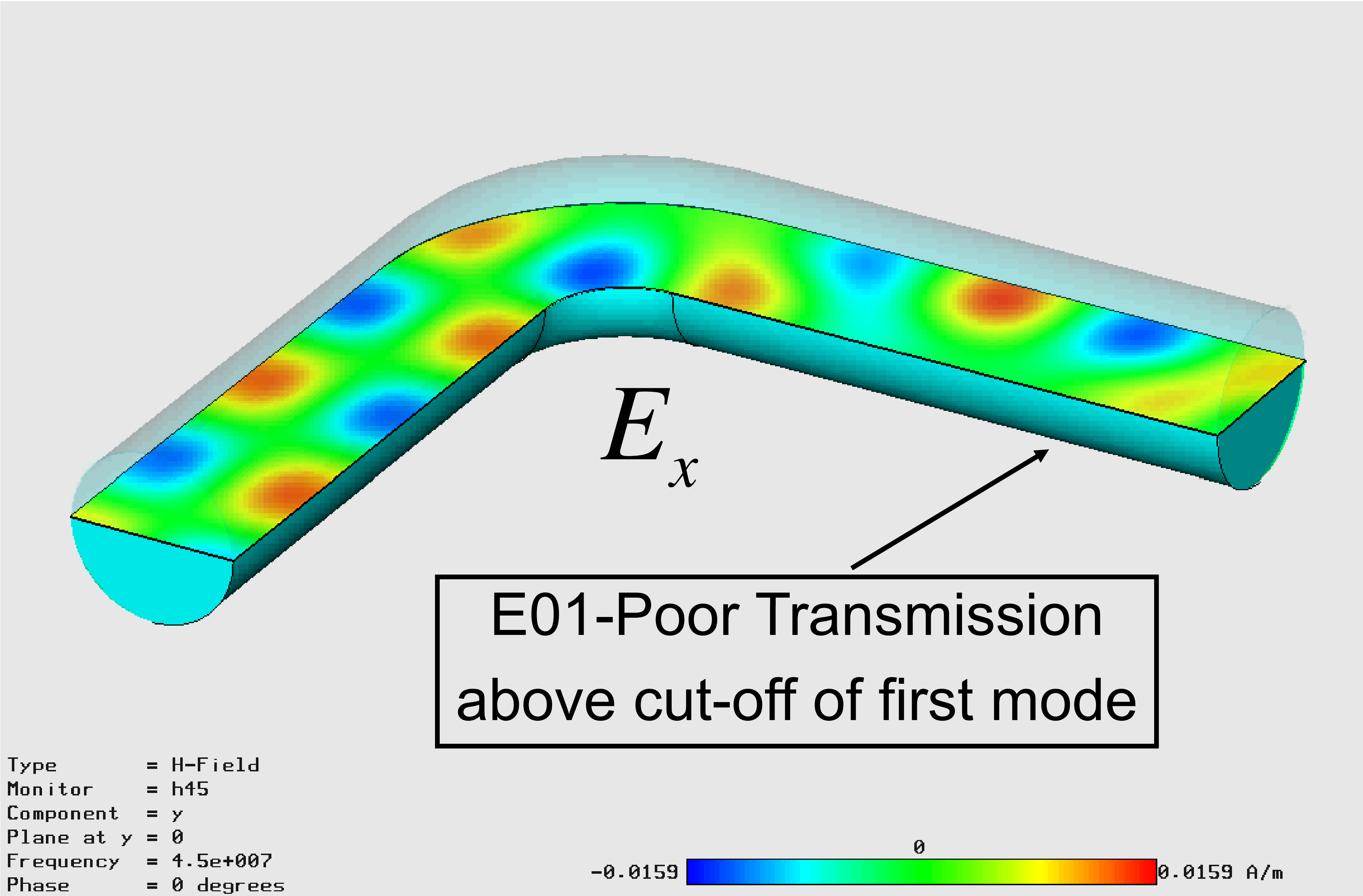
H11-Y-5.5GHz


$\vec{E}$
Poor Transmission
above cut-off of next mode
Type = E-Field
Monitor = e55
Maximum = 7.77962 V/m
Plane at y = 0
Frequency = 5.5e+007
Phase = 0 degrees

H11-X-5.5GHz

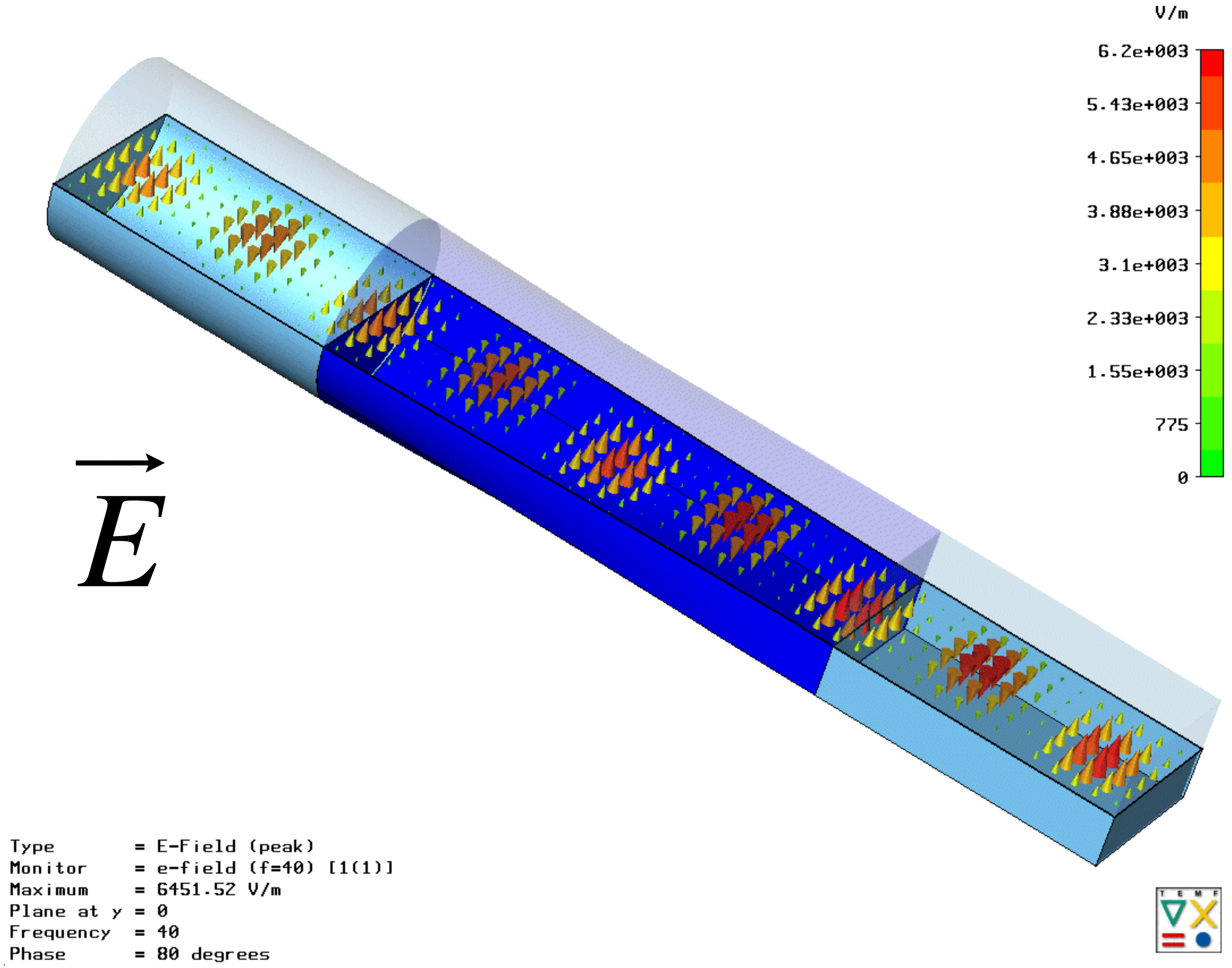

$\vec{E}$
Perpendicular Polarization
Better Transmission
above cut-off of next mode
Type = E-Field
Monitor = e55
Maximum = 6.49038 V/m
Plane at y = 0
Frequency = 5.5e+007
Phase = 0 degrees

E01-4.5 GHz

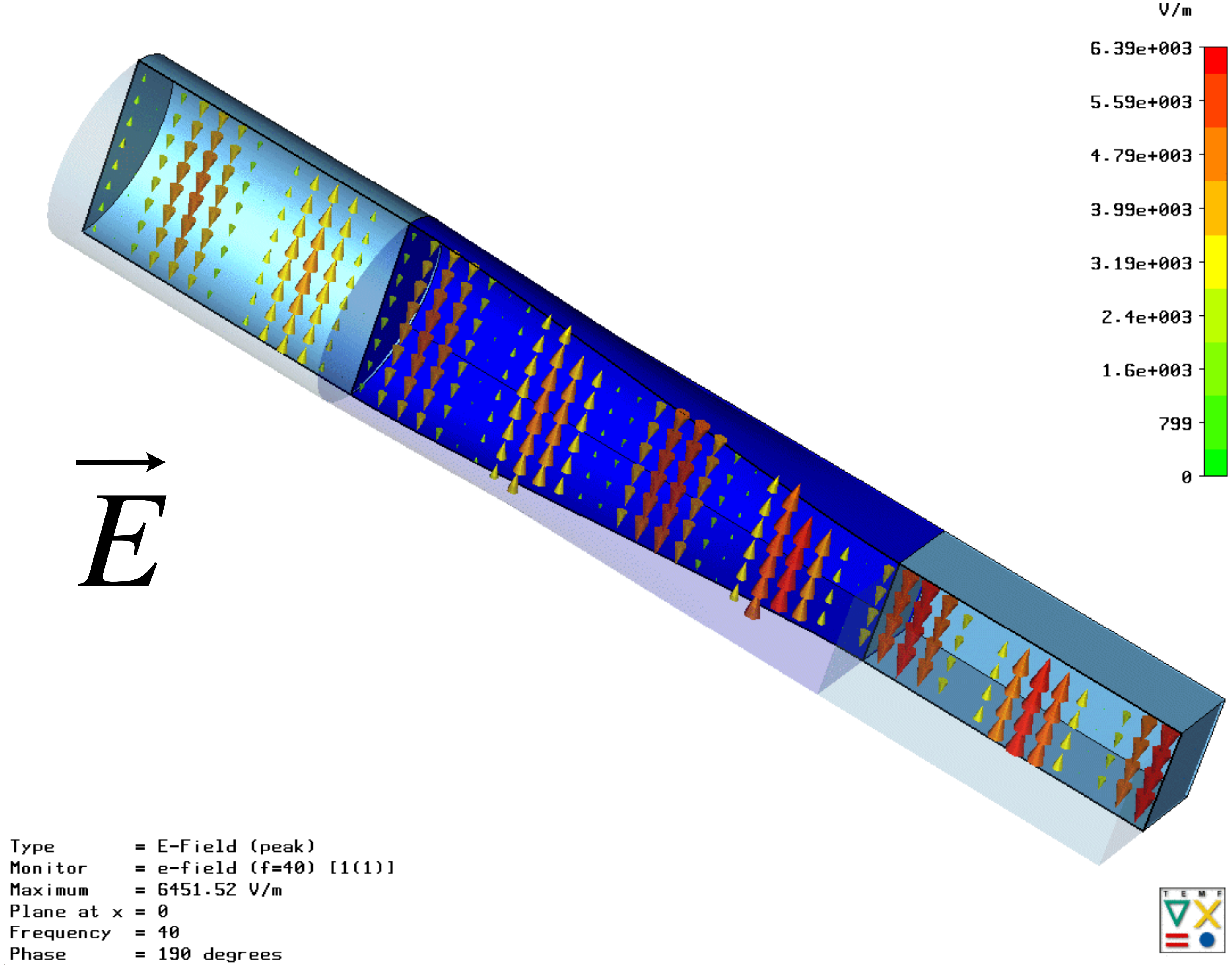

E_x
E01-Poor Transmission
above cut-off of first mode
Type = H-Field
Monitor = h45
Component = y
Plane at y = 0
Frequency = 4.5e+007
Phase = 0 degrees
0
-0.0159
0.0159 A/m

Rectangular-Circular Transition

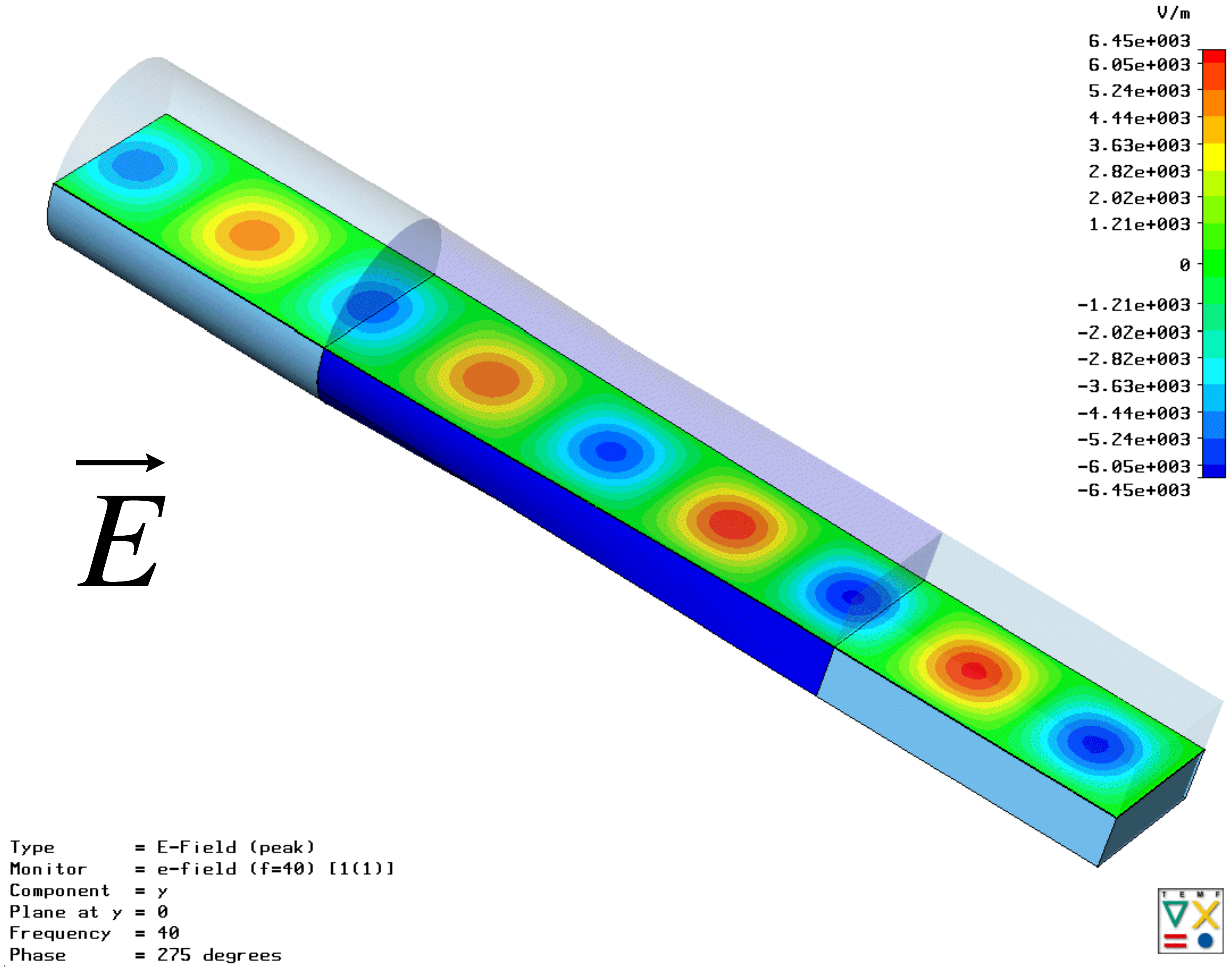
- **Transition from rectangular to circular waveguides:**



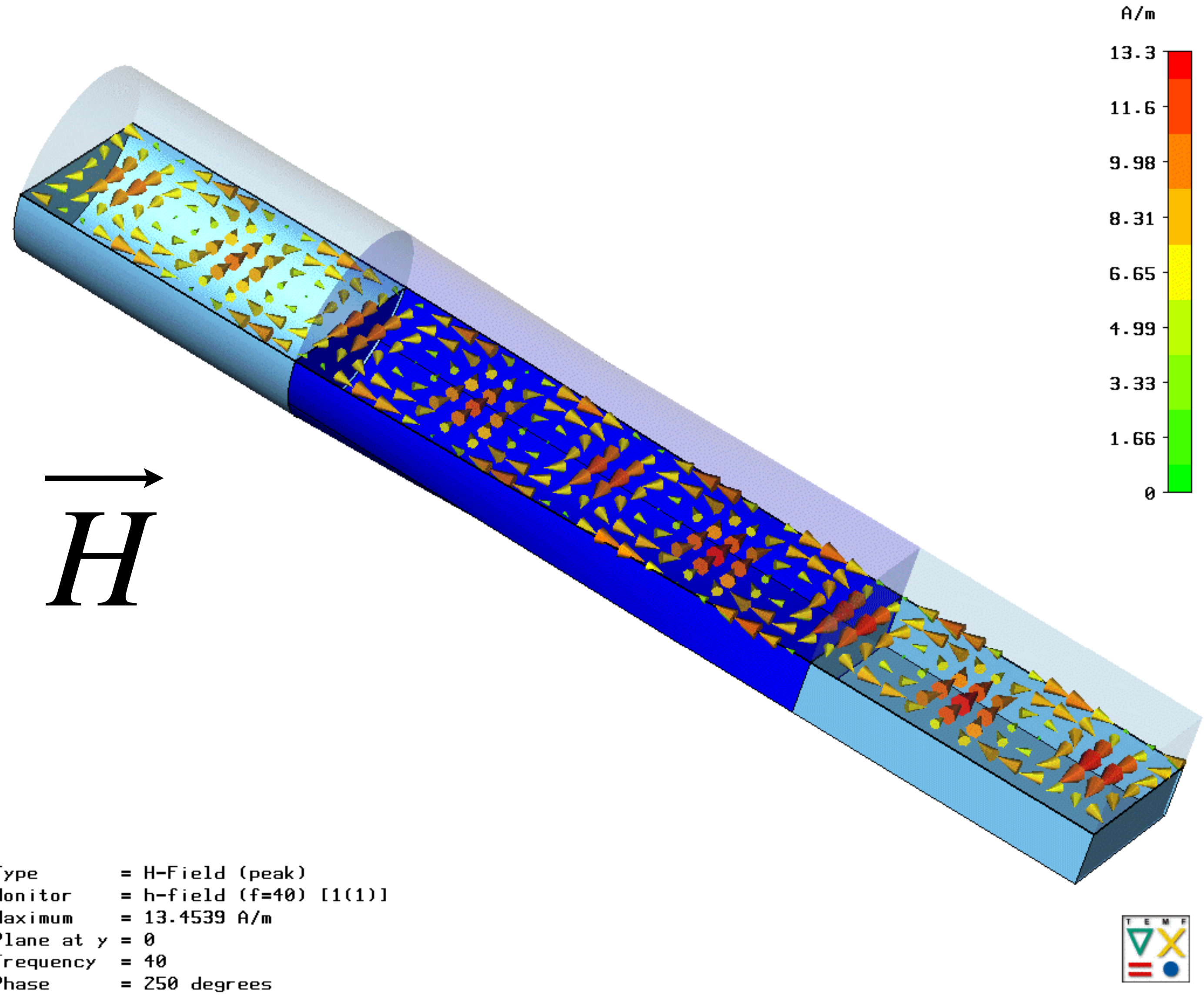
- **Transition from rectangular to circular waveguides:**



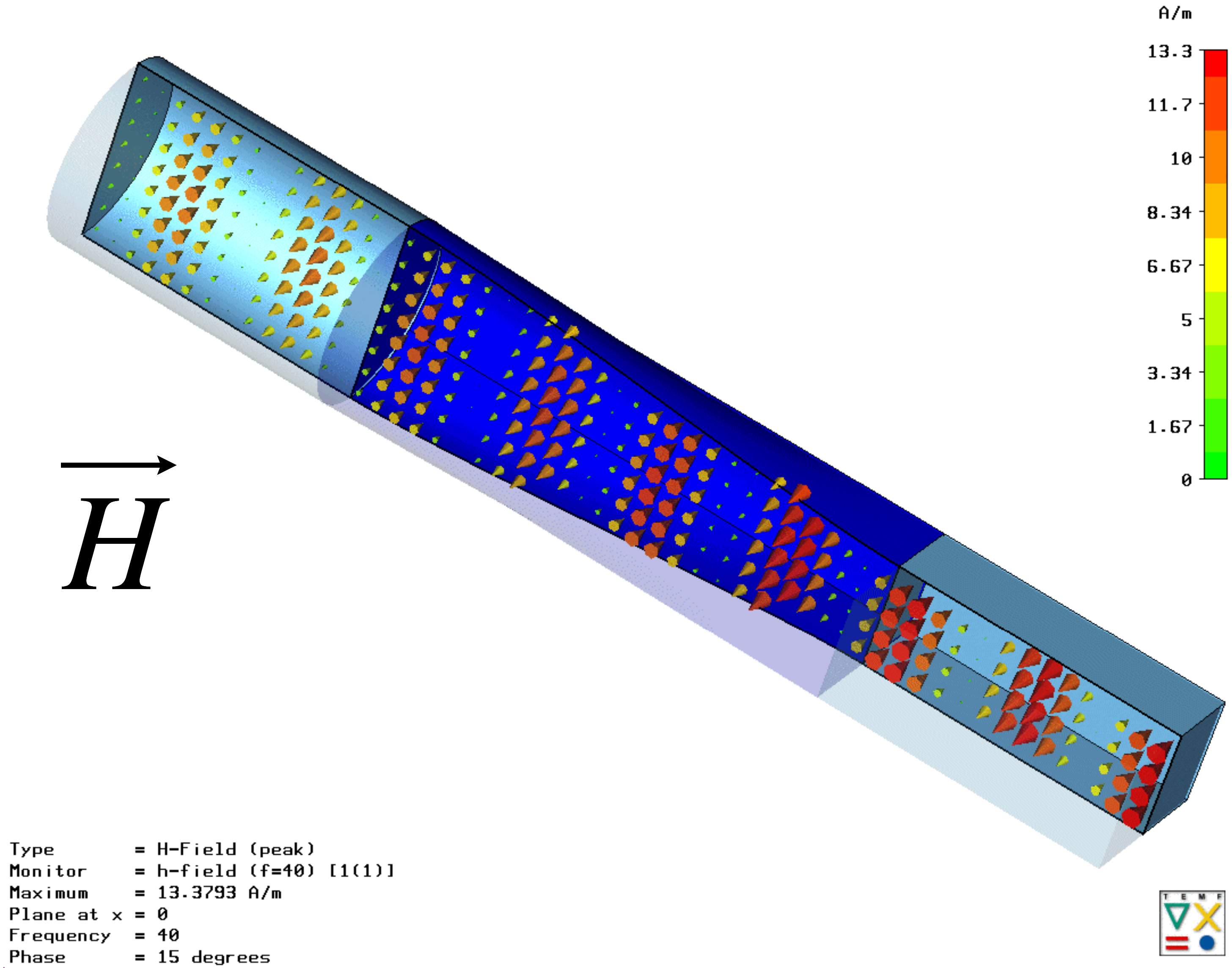
- **Transition from rectangular to circular waveguides:**



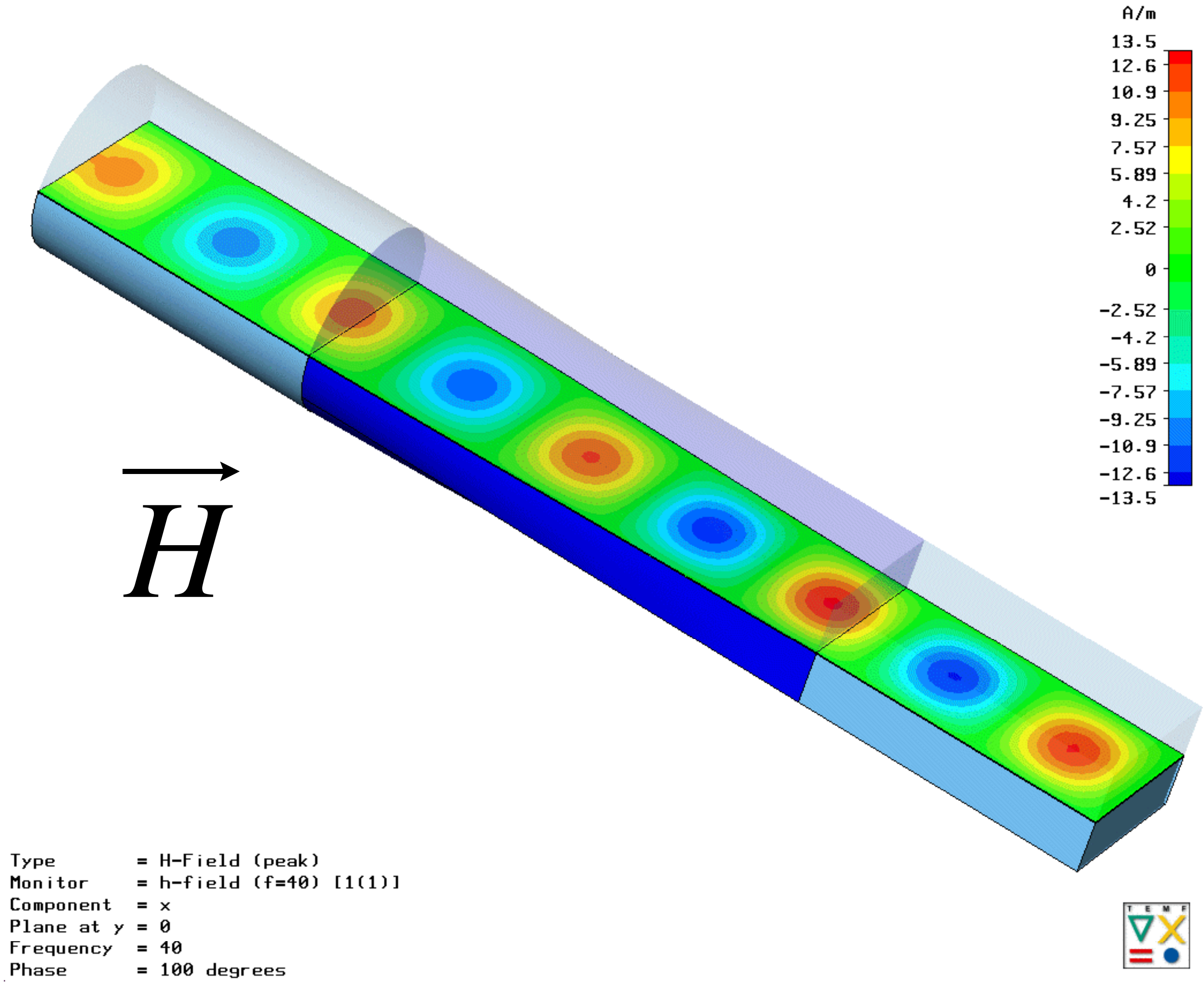
- **Transition from rectangular to circular waveguides:**



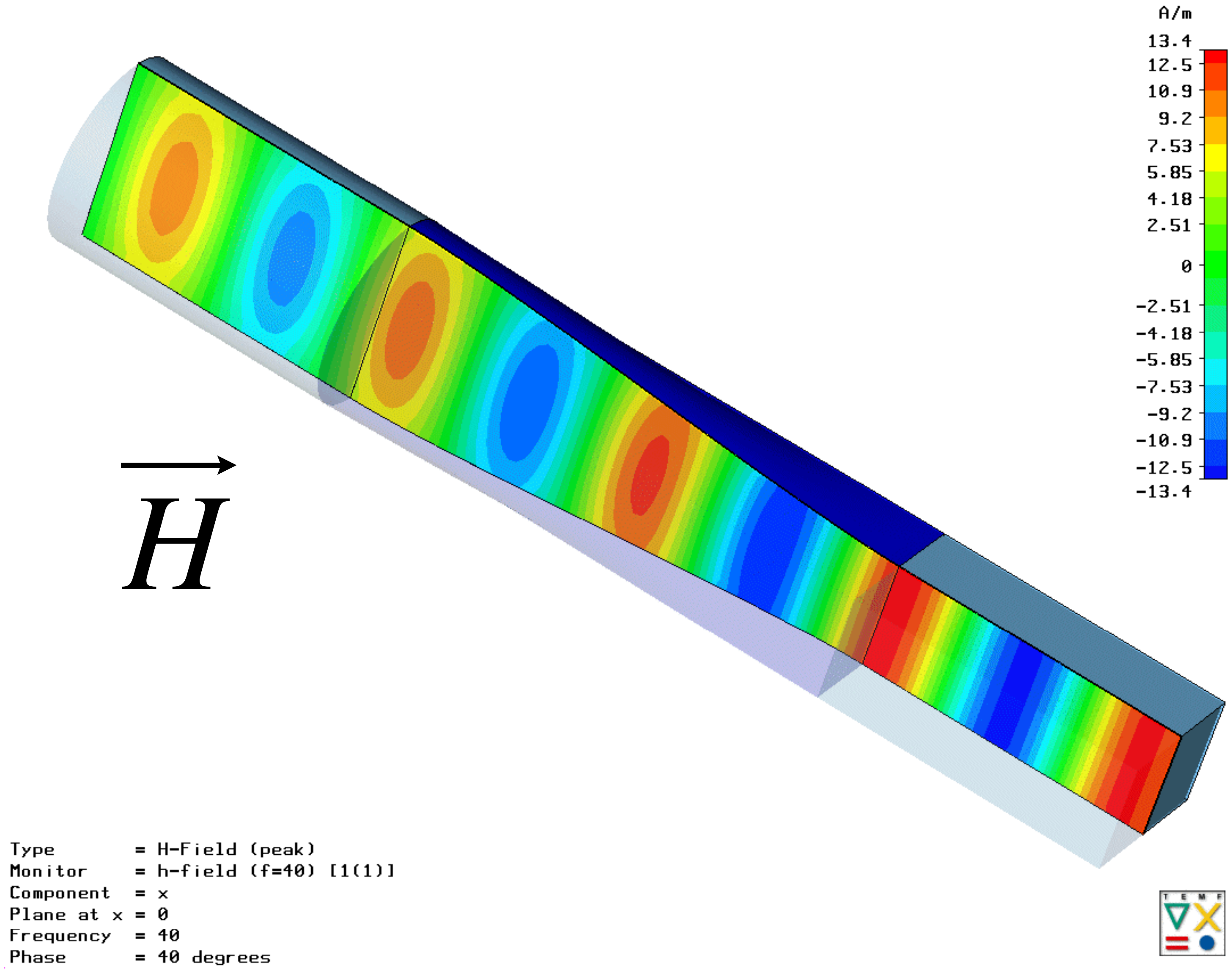
- **Transition from rectangular to circular waveguides:**



- **Transition from rectangular to circular waveguides:**

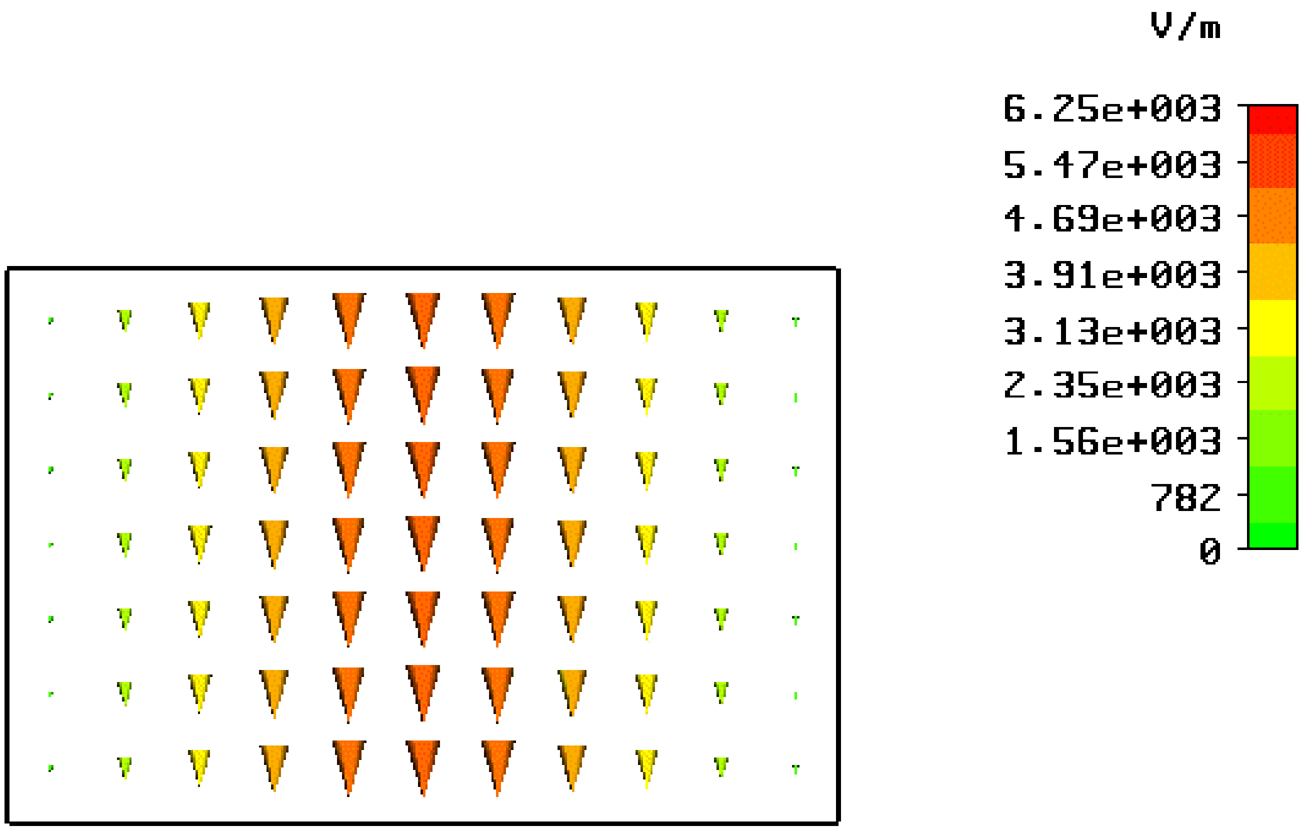


- **Transition from rectangular to circular waveguides:**



- **Transition from rectangular to circular waveguides:**

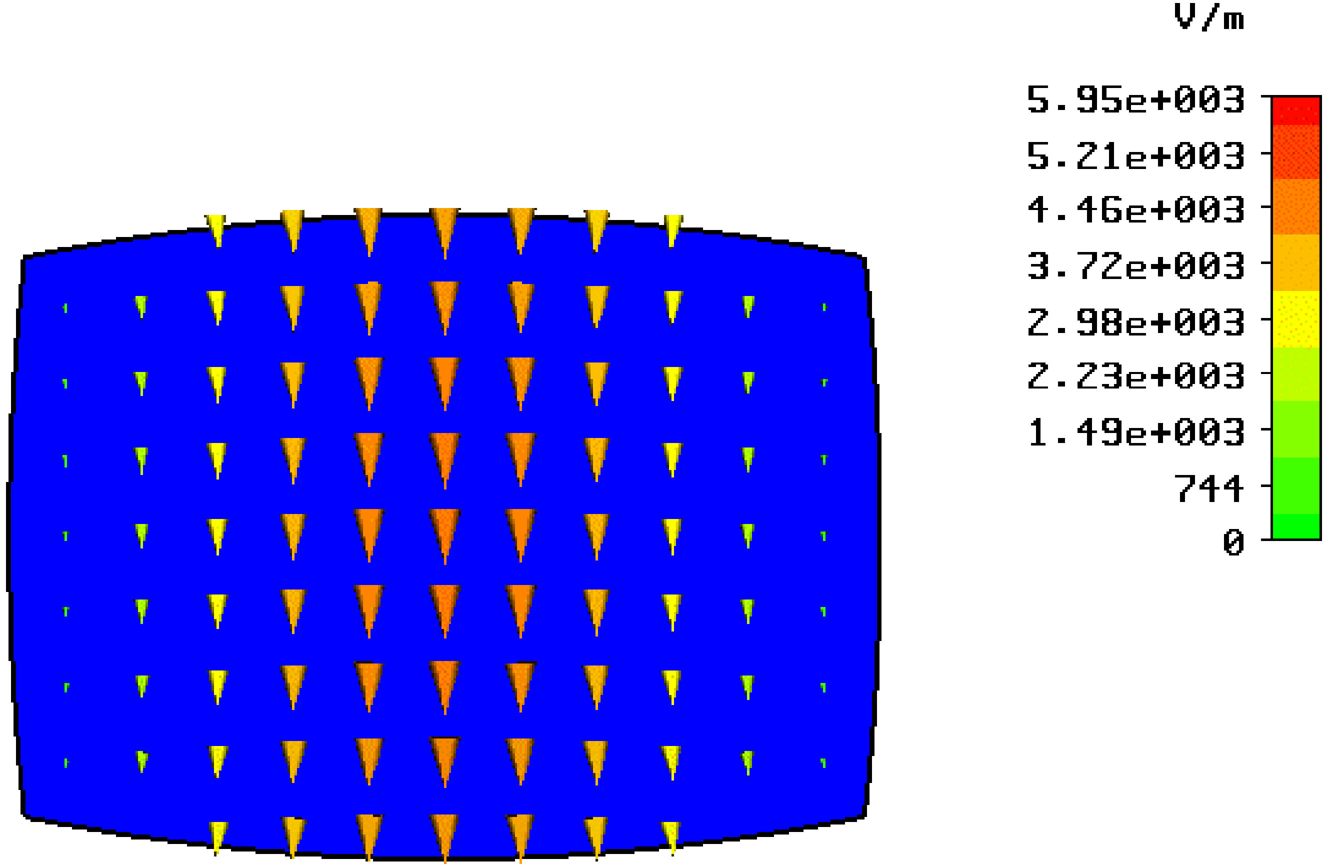
$$\vec{E}$$



Waveguides

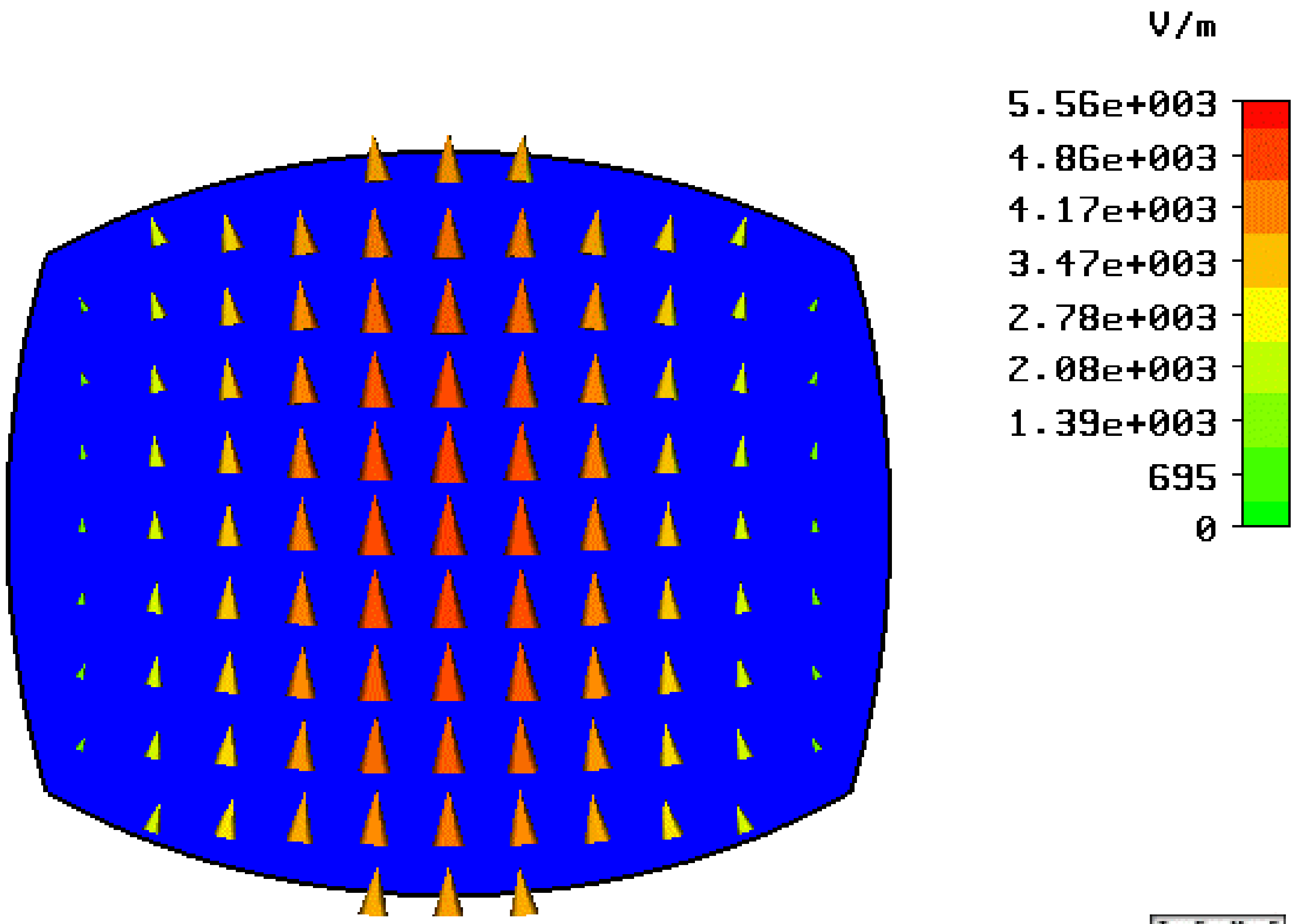
- **Transition from rectangular to circular waveguides:**

$\vec{E}$



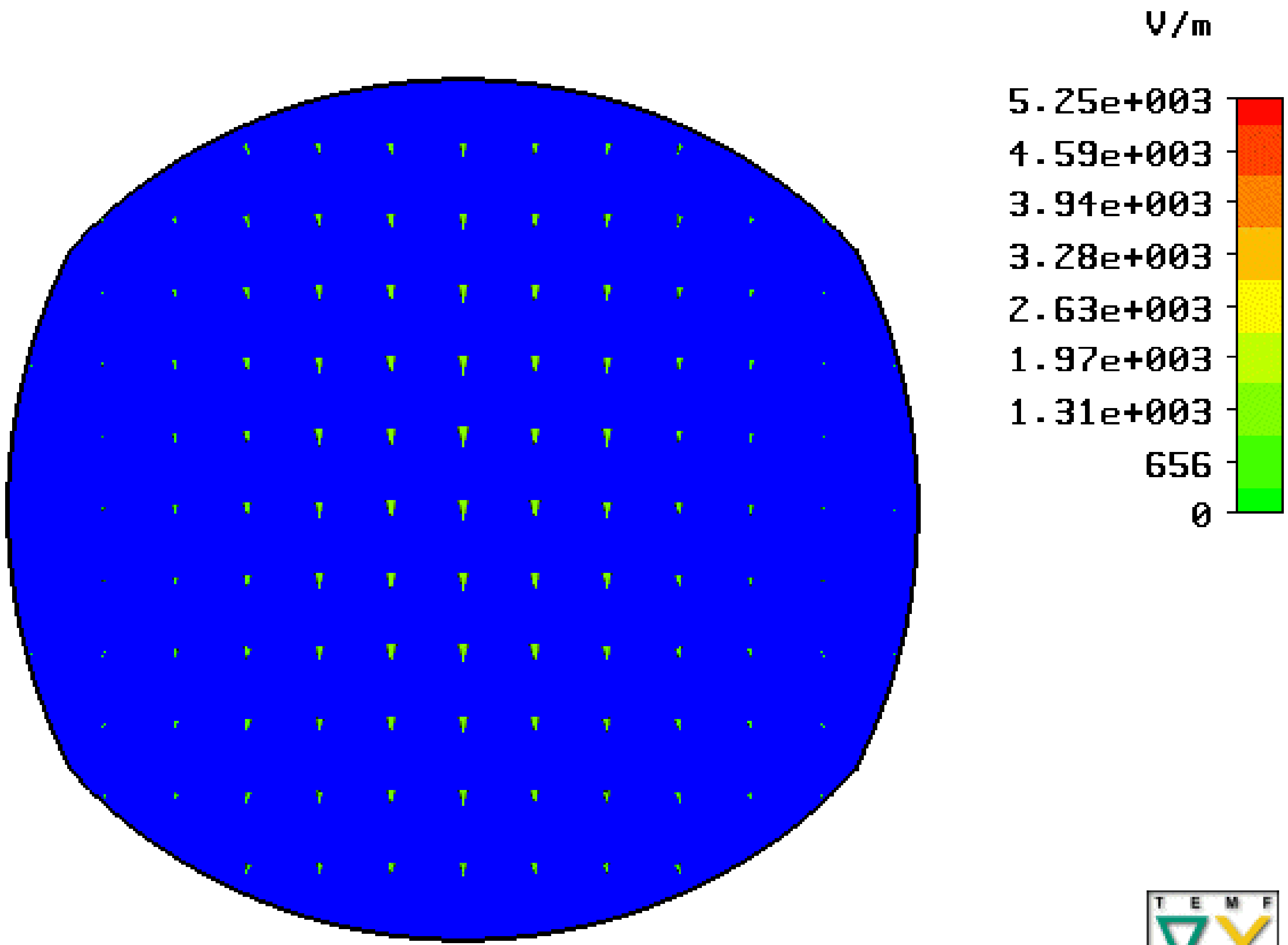
- **Transition from rectangular to circular waveguides:**

$\vec{E}$



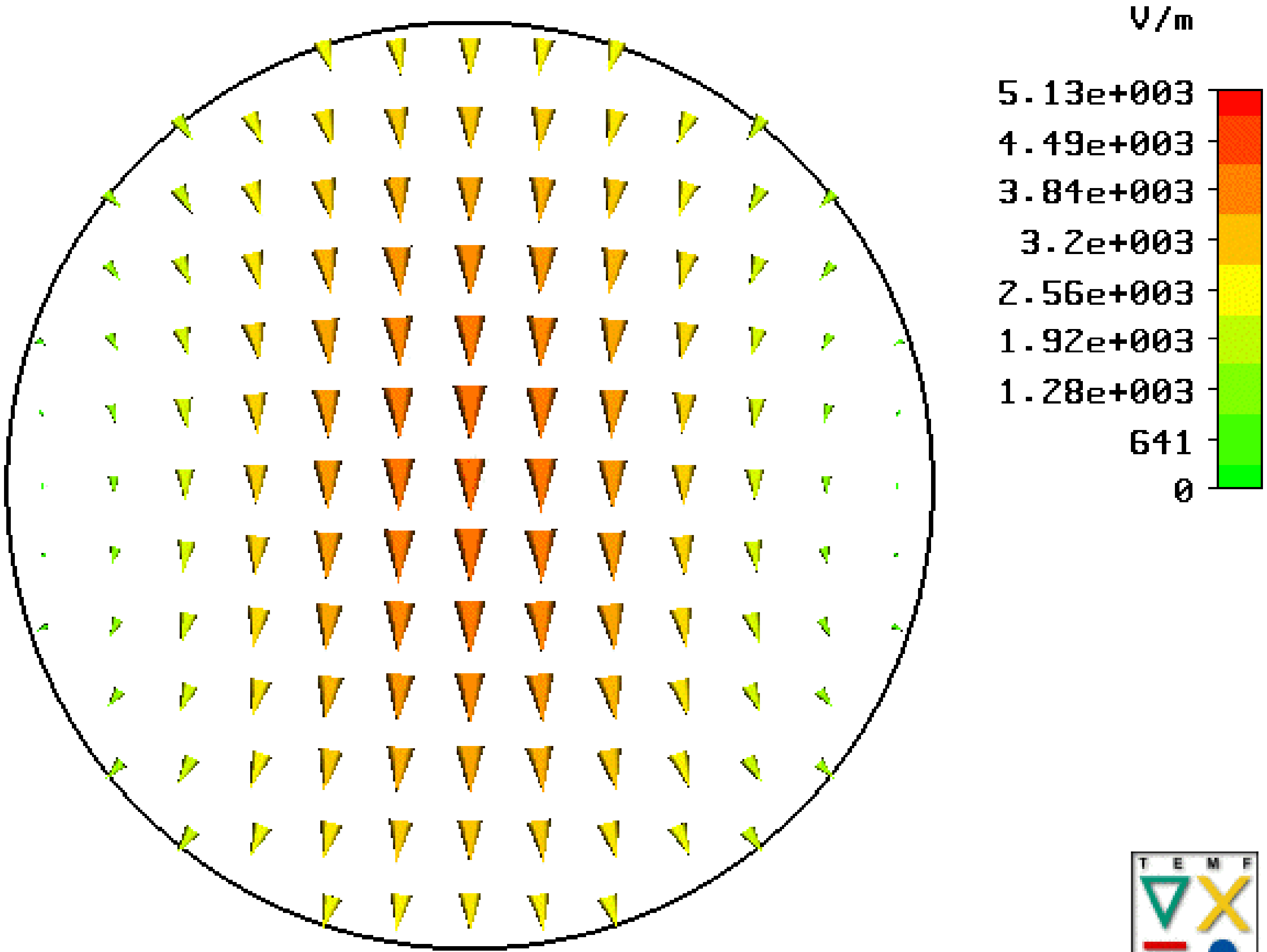
- **Transition from rectangular to circular waveguides:**

$\vec{E}$



- **Transition from rectangular to circular waveguides:**

$\vec{E}$



Coaxial Waveguides

Coaxial Waveguides

- **General solution: combination of Bessel and Neumann functions (field is 0 on the z axis, therefore the Neumann function can also be present in the solution)**
- **General solution:**

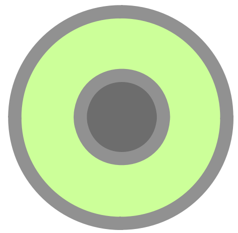
$$\mathrm{Z}_m(Kr) = A\mathrm{J}_m(Kr) + B\mathrm{N}_m(Kr).$$

- **Practical application: modes with cutoff frequency 0**

Coaxial Waveguides

- **TEM waves (cutoff 0 Hz):**

Coaxial WG



$$\omega_c = K c = 0 \Rightarrow K = 0$$

☞ Additionally, no dependency on the angular coordinate

$$\mu = 0$$

☞ Equation after variable separation (see circular waveguide)

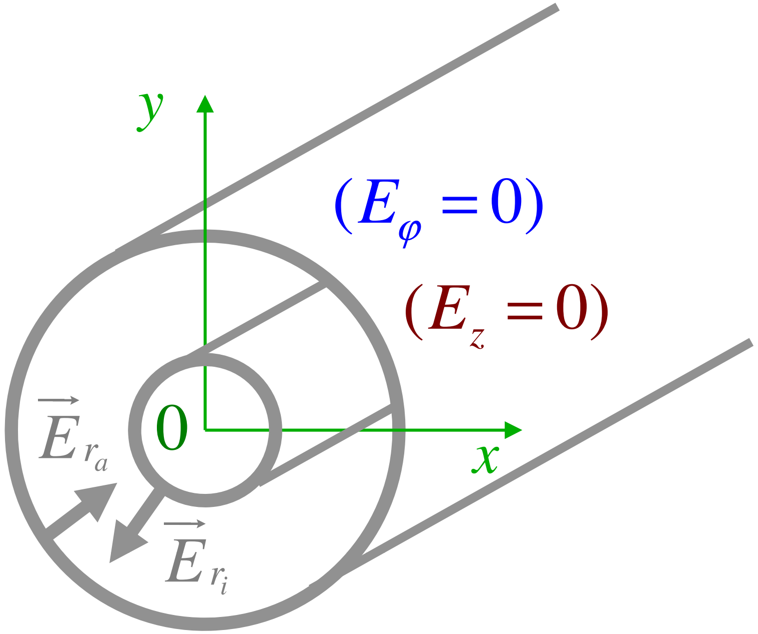
$$\frac{1}{f}\frac{\partial^2 f}{\partial r^2} + \frac{1}{fr}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{1}{g}\frac{\partial^2 g}{\partial \varphi^2} + \frac{1}{h}\frac{\partial^2 h}{\partial z^2} + k^2 = 0$$

$$K^2 = \frac{1}{f}\frac{\partial^2 f}{\partial r^2} + \frac{1}{fr}\frac{\partial f}{\partial r} - \frac{1}{r^2}\mu^2 = 0$$

$$K = 0, \quad \mu = 0 \quad \boxed{\Rightarrow \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = 0}$$

Coaxial Waveguides

- **TEM-solution (No component in propagation direction):**



$$K = 0, \quad \mu = 0 \qquad \text{(Separation variables)}$$

$$A_z = C_1 \ln \frac{r}{C_2} \, e^{-jkz} \qquad \text{(Vector potential)}$$

TEM: ➡ no Bessel-ODE

Field vector components:

$$\left.\begin{aligned} E_r &= \frac{1}{r} C_1 \cdot e^{-jkz} \\ H_\varphi &= \frac{1}{Z}\frac{1}{r} C_1 \cdot e^{-jkz} \end{aligned}\right\} \text{Static solution} \cdot e^{-jkz}$$

$$E_\varphi = E_z = H_r = H_z = 0$$

Continuity:

$$E_\varphi(r = r_i, \varphi, z) = 0$$
$$E_\varphi(r = r_a, \varphi, z) = 0$$
$$E_z(r = r_i, \varphi, z) = 0$$
$$E_z(r = r_a, \varphi, z) = 0$$

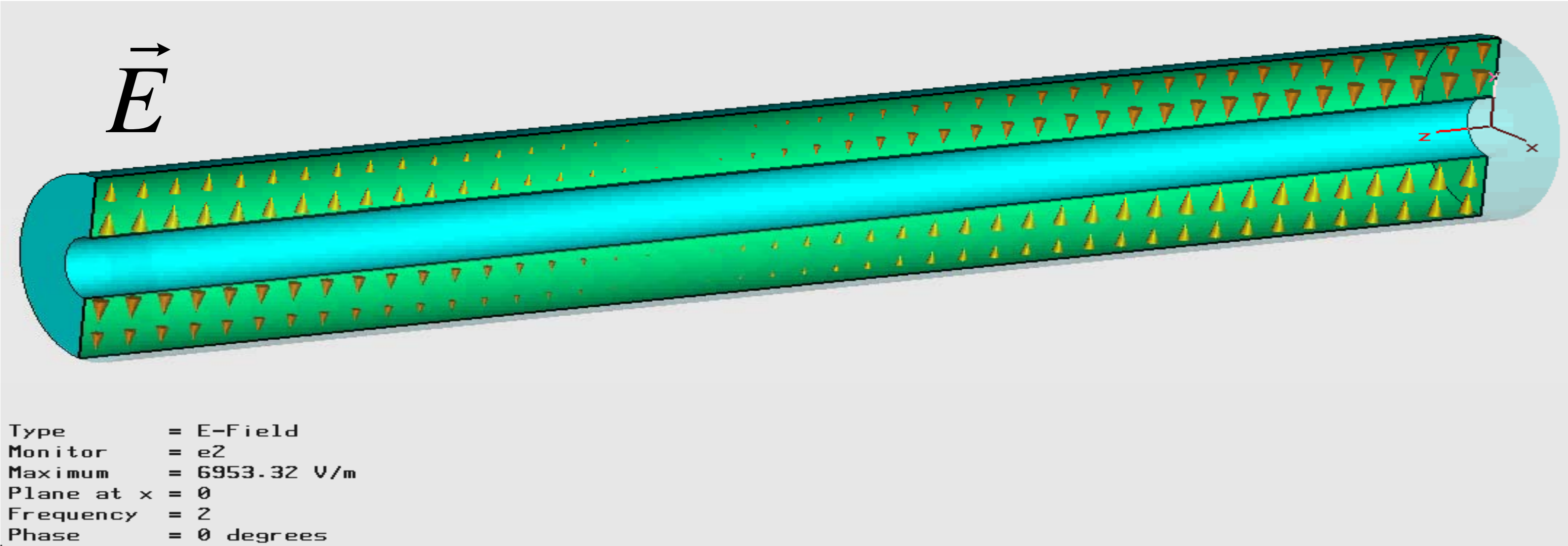
Coaxial Waveguides

- **TEM-Mode:**

$r_i = 1\,mm,\ r_a = 3\,mm$

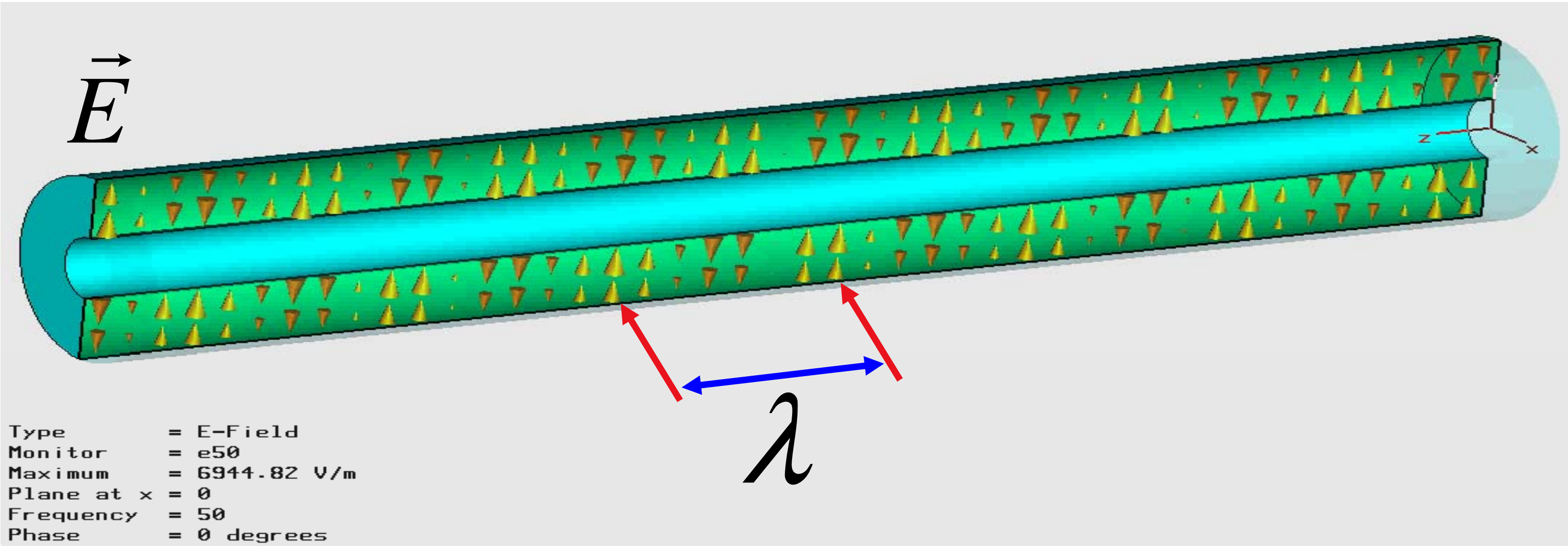
$f = 2\,GHz$

$\lambda = 150\,mm$



$f = 50\,GHz$

$\lambda = 6\,mm$



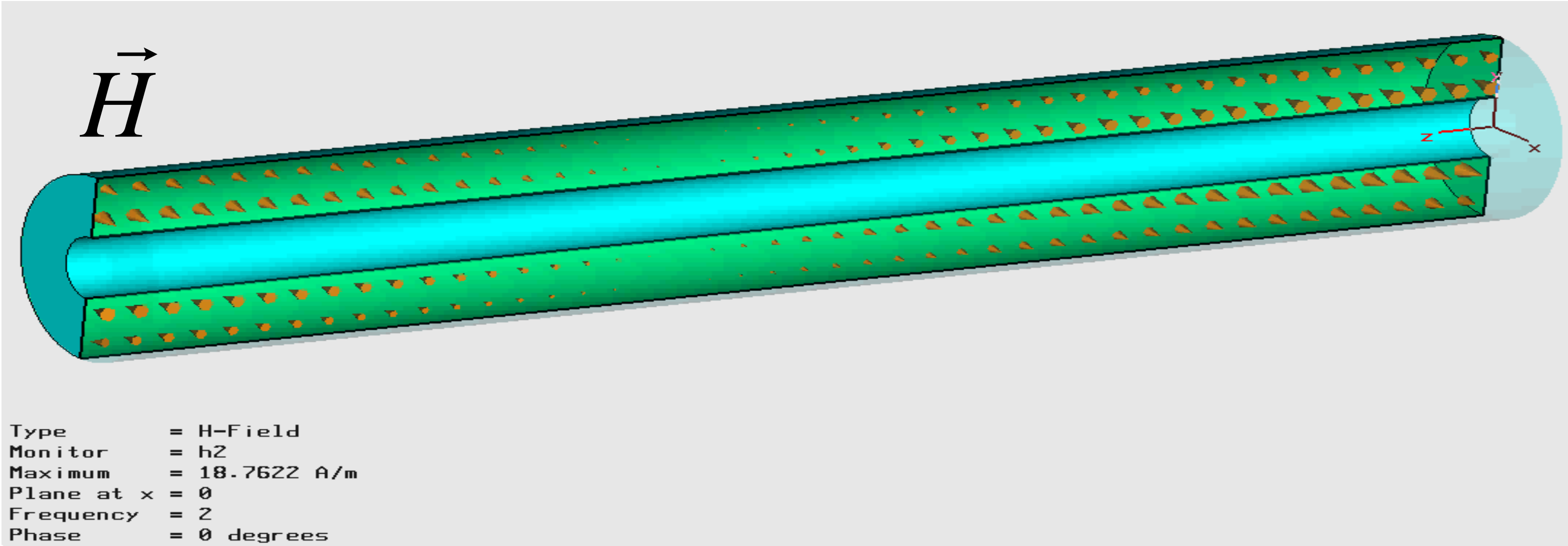
Coaxial Waveguides

- **TEM-Mode:**

$$r_i = 1\,mm,\; r_a = 3\,mm$$

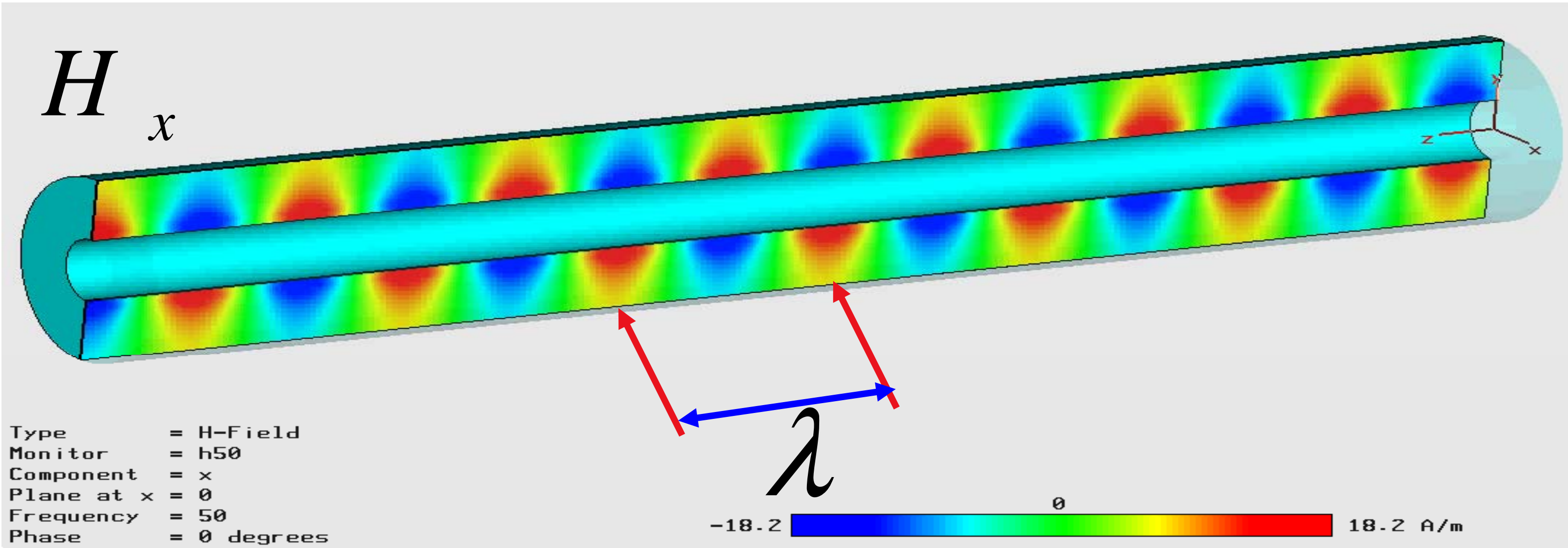
$f = 2\,GHz$

$\lambda = 150\,mm$

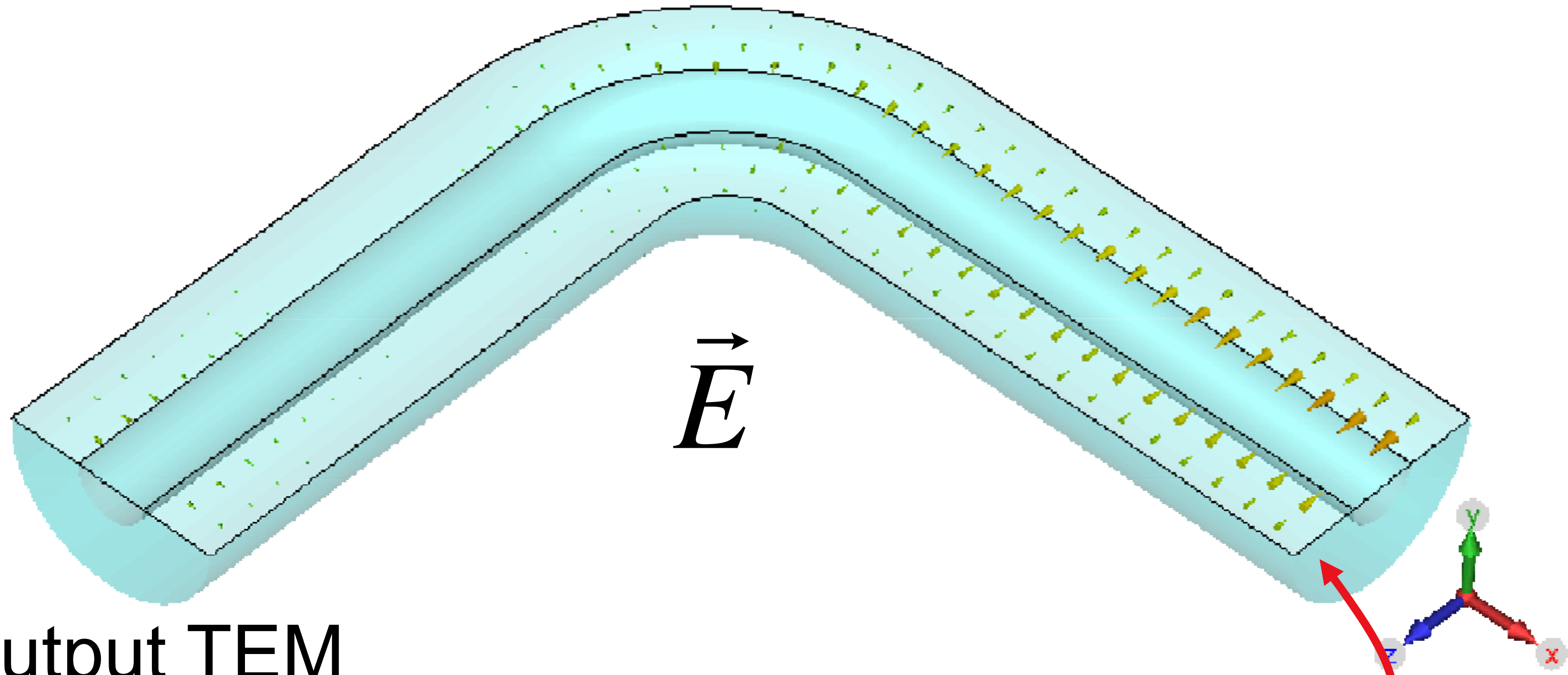


$f = 50\,GHz$

$\lambda = 6\,mm$



$$f = 2\ \text{GHz}$$



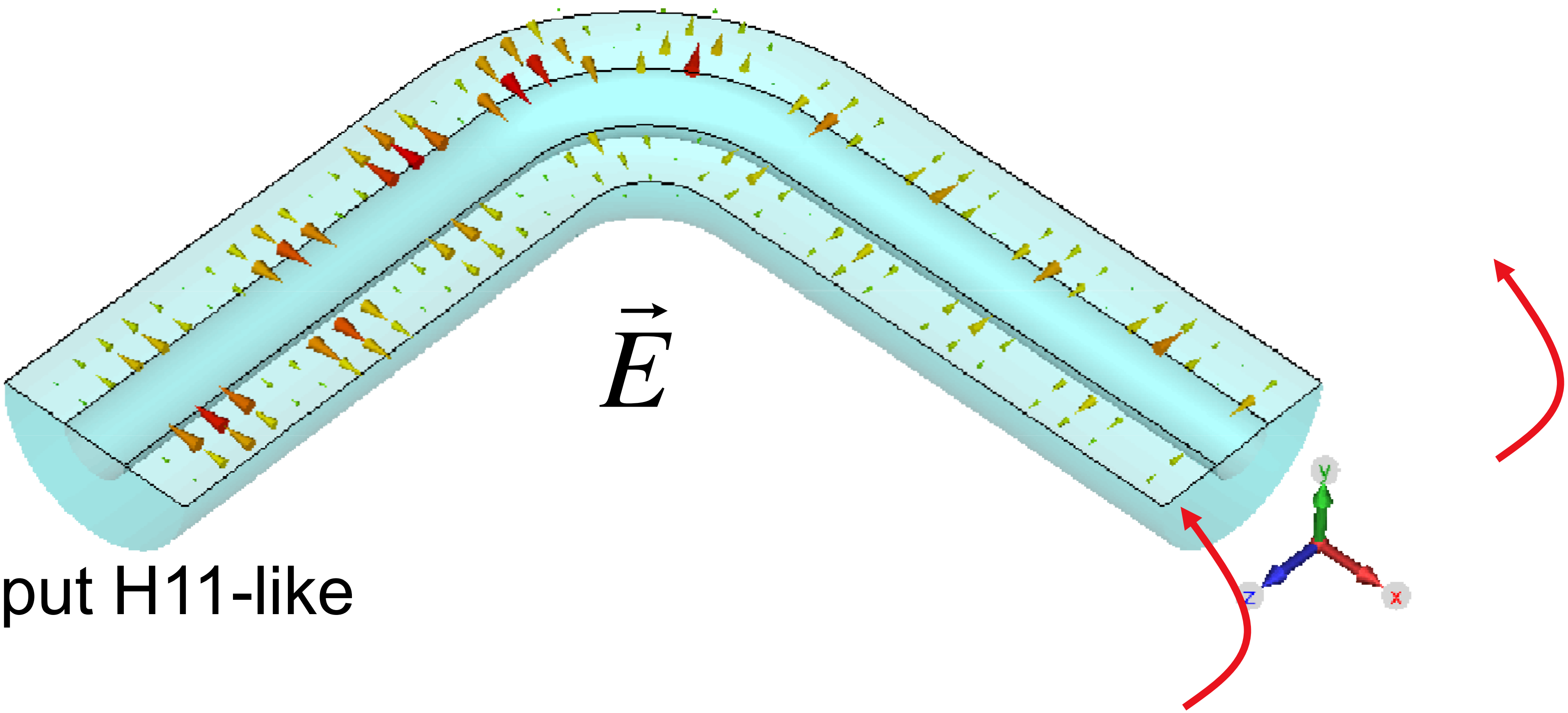
$\vec{E}$

Output TEM

Excitation TEM

Coax Bend

$$f = 40\ \mathrm{GHz}$$



Output H11-like

Excitation TEM

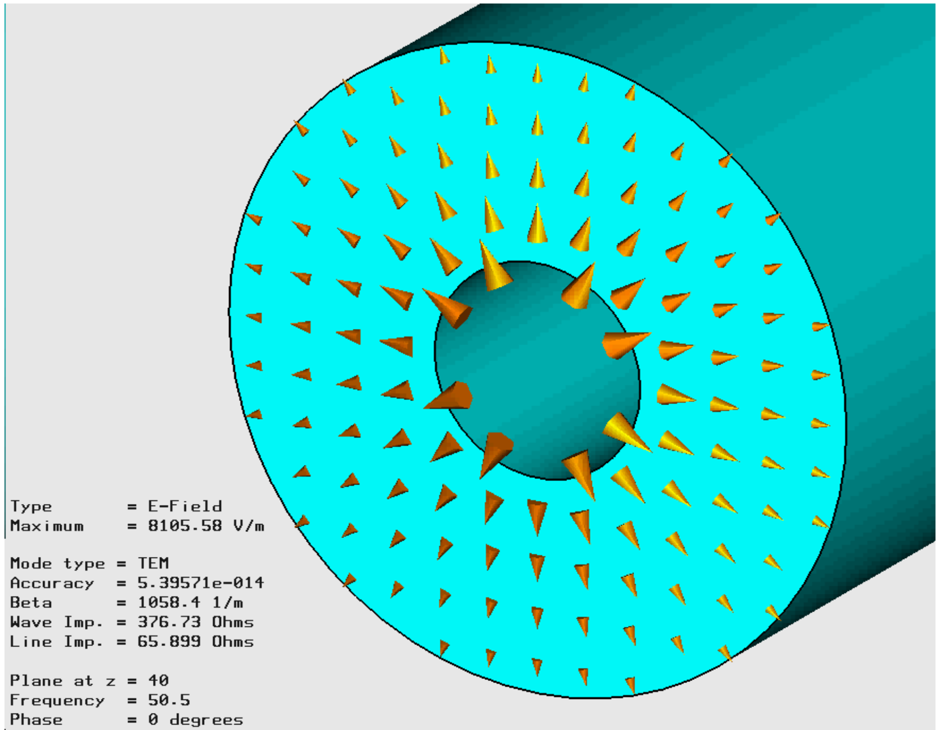
Coaxial Waveguides

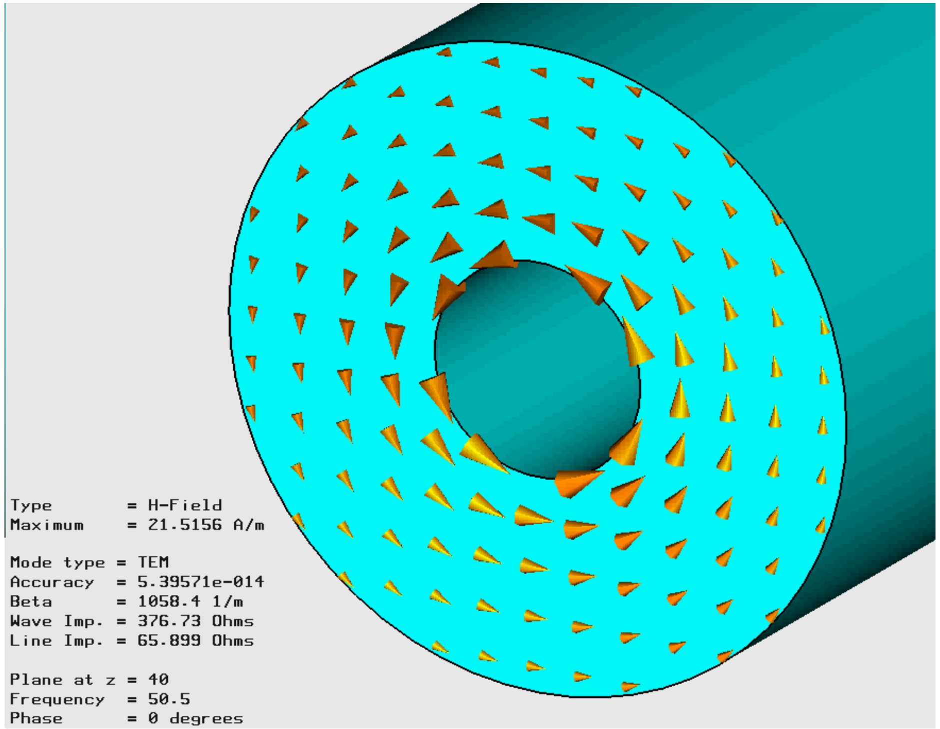
- **TEM-Mode:**

$$E_z = 0 \quad H_z = 0$$

$$f_{\text{cutoff}} = 0\text{Hz}$$

$\vec{E}$ $\vec{H}$

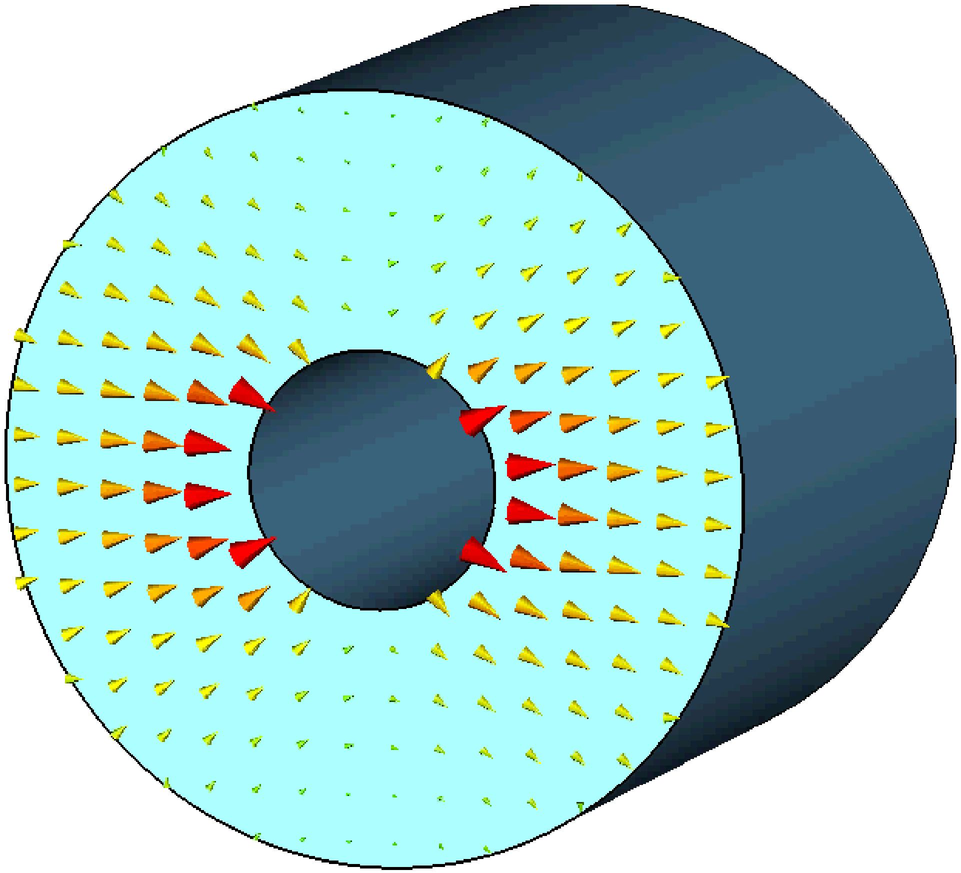




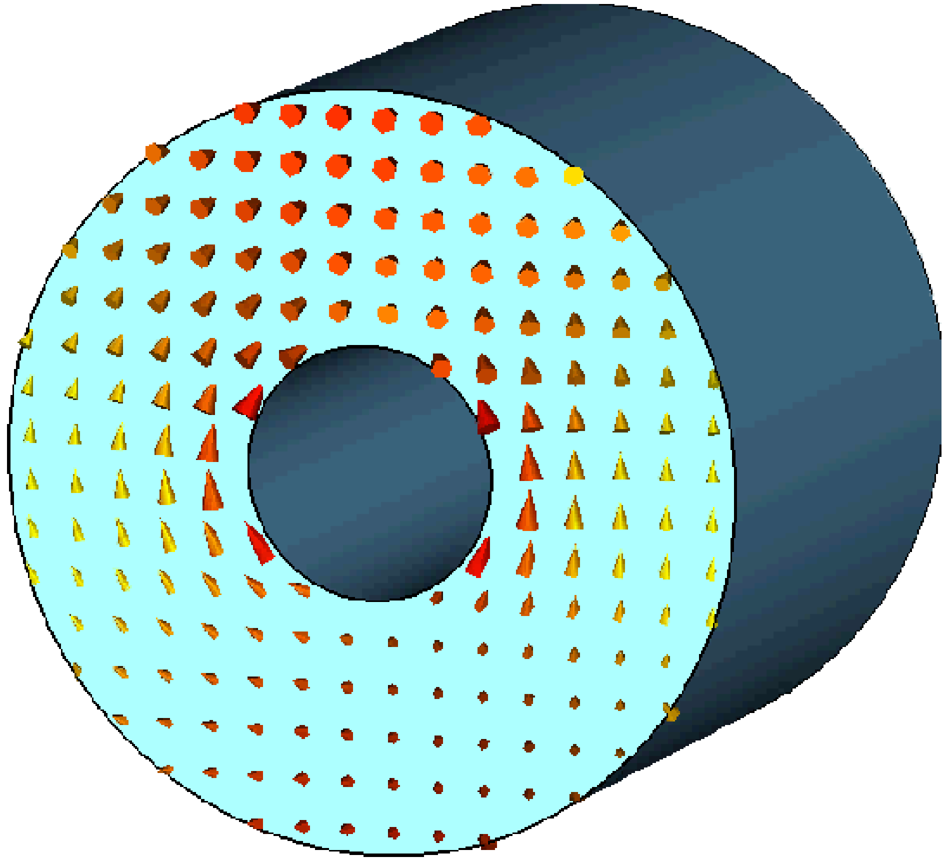
- **Higher Coax-Modes:**

$$r_i = 1mm,\ r_a = 3mm$$

$$f_{cutoff} = 24.5\,\mathrm{GHz}$$



$\vec{E}$ at $\omega t = 0$



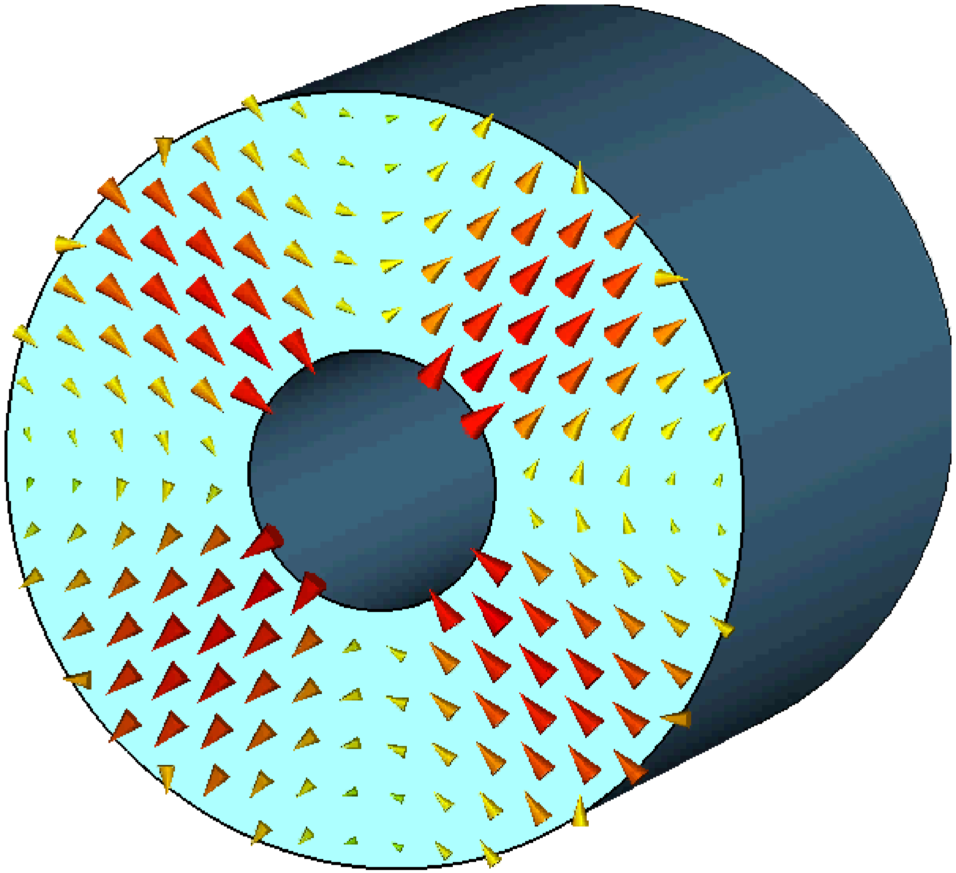
$\vec{H}$ at $\omega t = \pi / 2$

Coaxial Waveguides

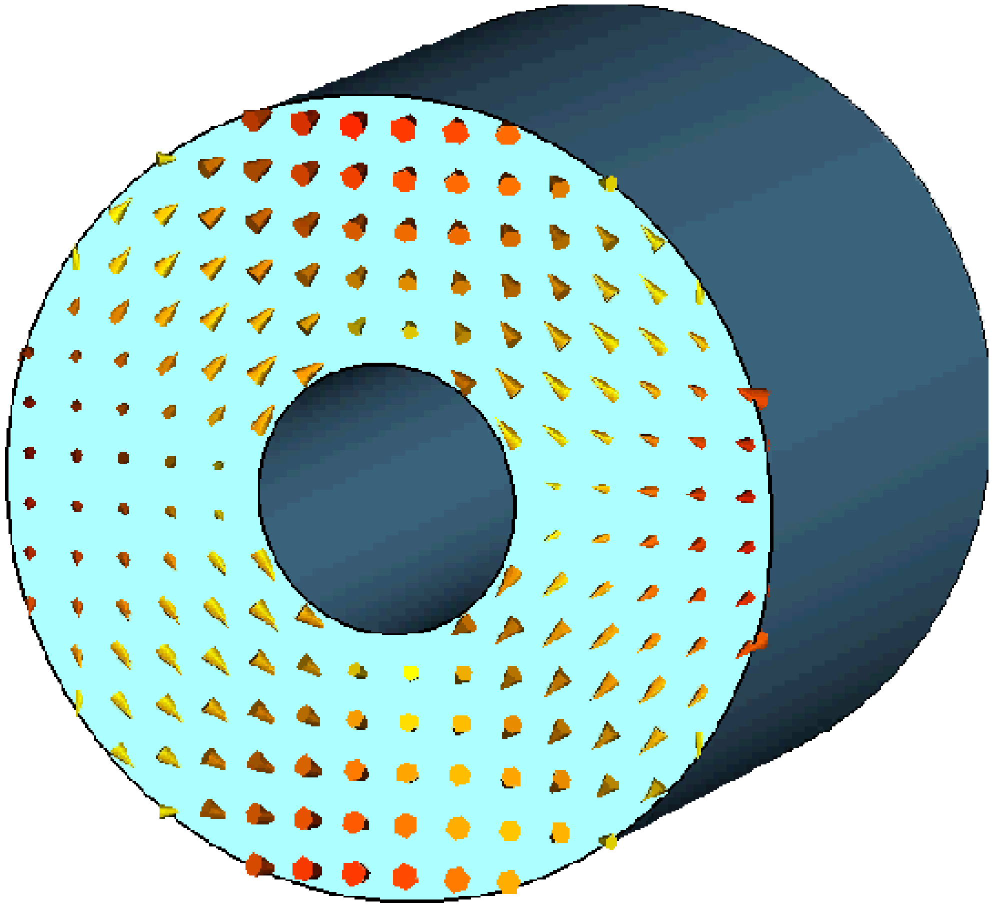
- **Higher Coax-Modes:**

$$r_i = 1mm,\ r_a = 3mm$$

$$f_{cutoff} = 46.3\,GHz$$

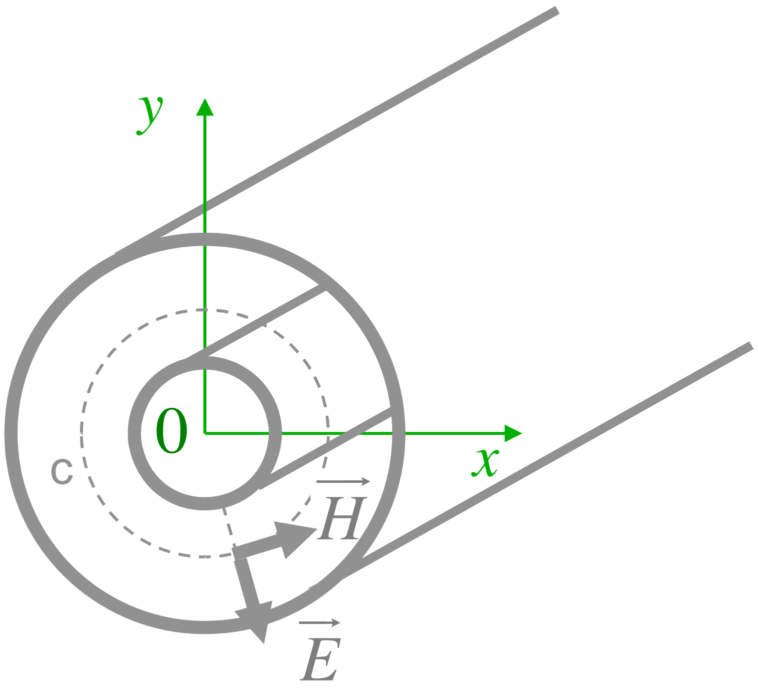


$\vec{E}$ at $\omega t = 0$



$\vec{H}$ at $\omega t = \pi / 2$

- **Definitions of characteristic impedances:**



$$U(z) = \int_{r_i}^{r_a} \vec{E} \cdot \vec{ds} \qquad \text{(voltage)}$$

$$I(z) = \oint_c \vec{H} \cdot \vec{ds} \qquad \text{(current)}$$

Characteristic field impedance:

$$\boxed{Z = \frac{|\vec{E}|}{|\vec{H}|}}$$

Characteristic line impedance:

$$\boxed{Z = \frac{U}{I}}$$

- **Definitions of characteristic impedances:**

Characteristic line impedance:

$$Z = \frac{U}{I}$$

$$U(z) = \int_{r_i}^{r_a} \vec{E} \cdot \vec{ds} \qquad \text{(voltage)}$$

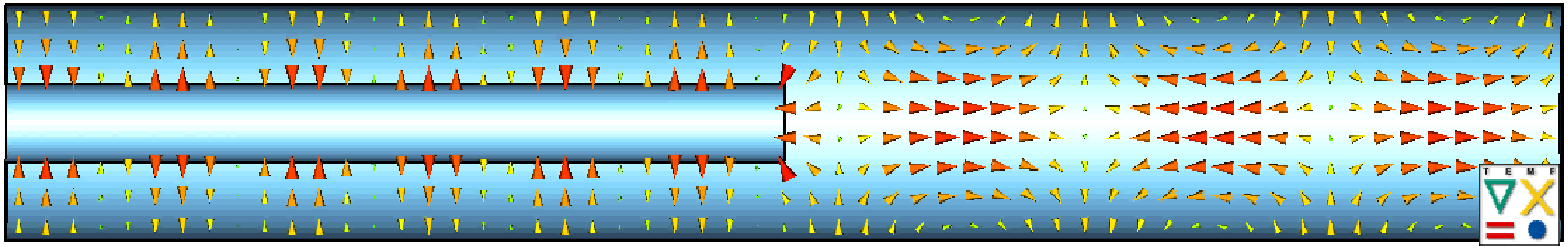
$$I(z) = \oint_c \vec{H} \cdot \vec{ds} \qquad \text{(current)}$$

- **The characteristic line impedance only depends on the dimensions of the cross-section and on the permittivitty / permeability of the dielectric**

- **Transition from coaxial cable to circular waveguides:**

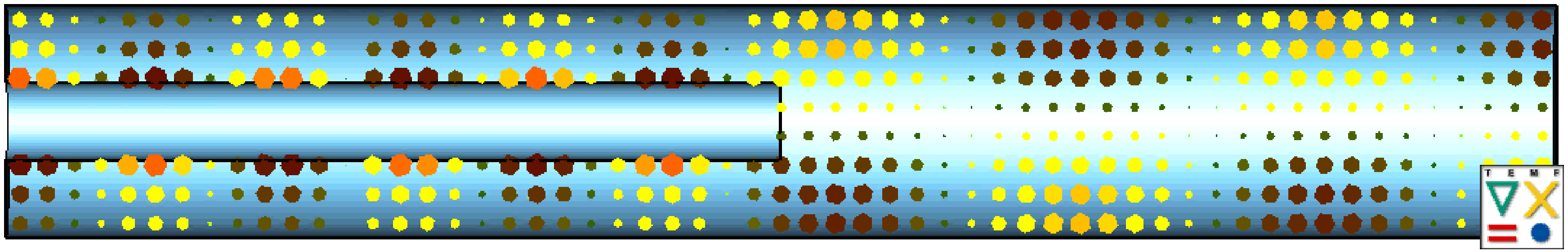
$$(r_i = 1\,\mathrm{mm},\ r_a = 3\,\mathrm{mm})$$

☞ Electric field: $f = 57\ GHz$



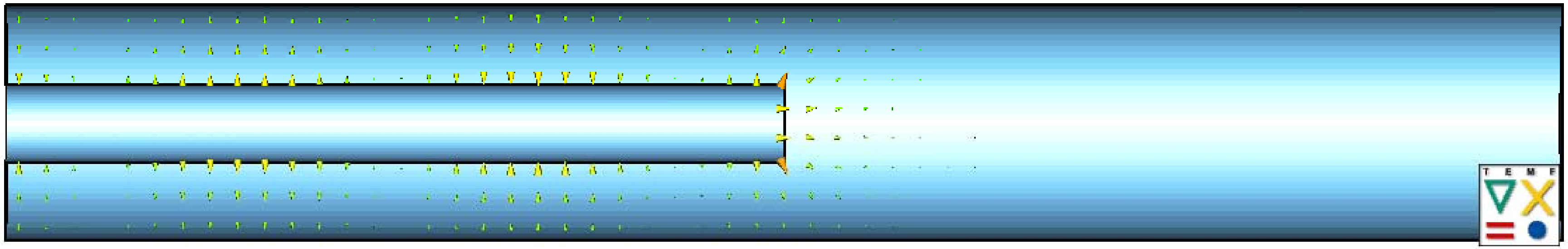
TEM-mode → TM_{01}-Mode

☞ Magnetic field:



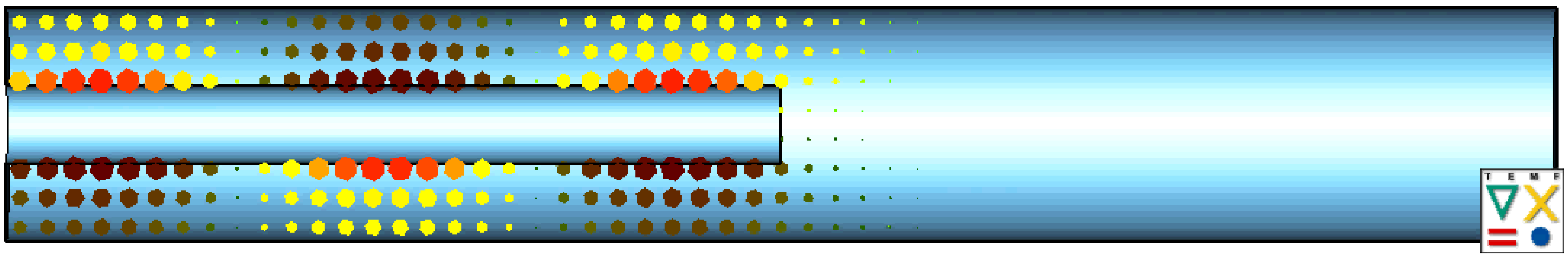
- **Transition from coaxial cable to circular waveguides:**

  - ☞ Electric field: $f = 25\ GHz \quad (f_c = 38.25\ GHz\ !)$

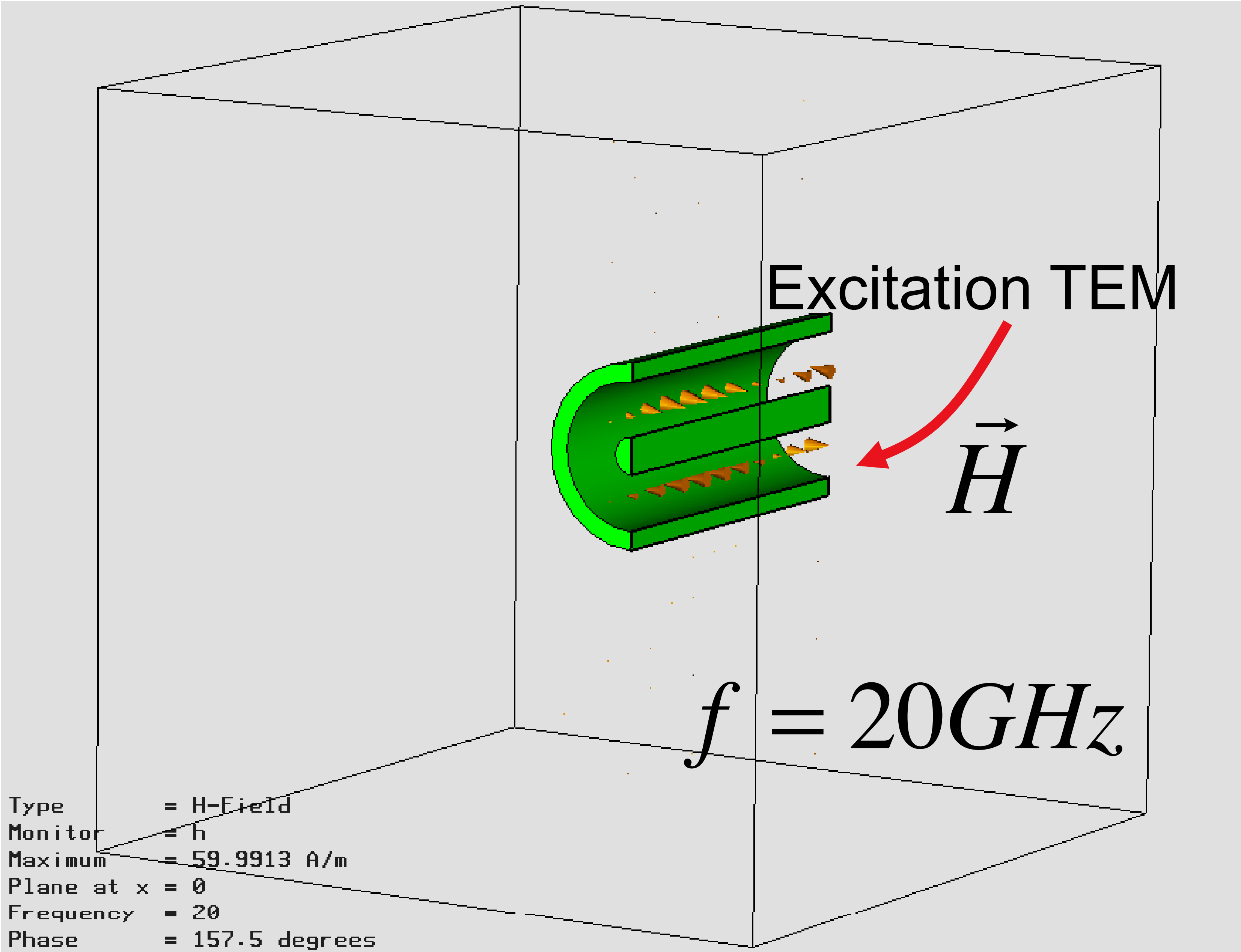


TEM-mode → TM_{01}-Mode below cutoff !

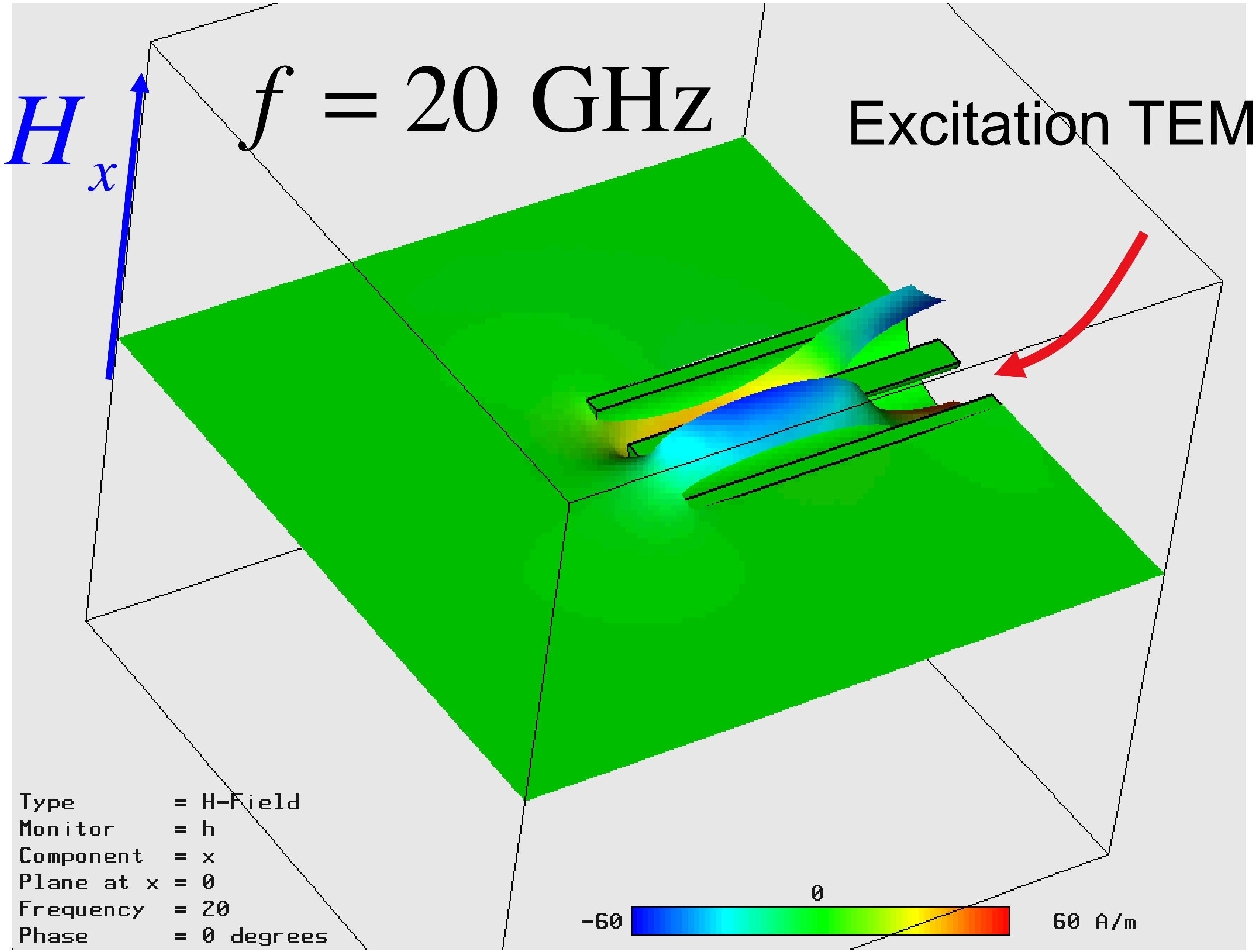
  - ☞ Magnetic field:



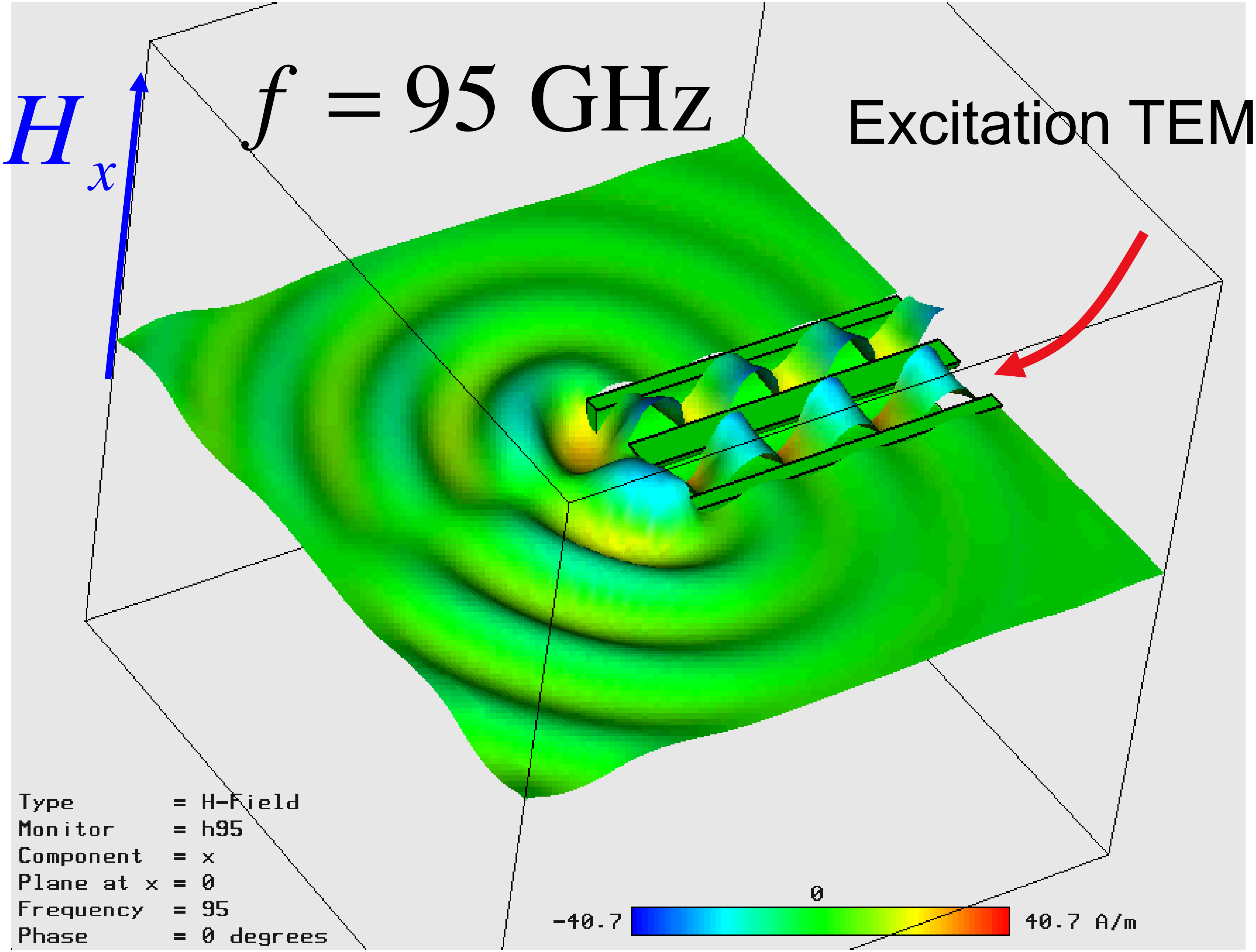
Open Ended Coax



Open Ended Coax



Open Ended Coax



End