Group 8

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Bayesian Decision Theory

1) $\hat{y} = \underset{y' \in R}{\arg\min} \int_R \Delta(y', y)\, P(y|x)\, dy \quad -(1)$

$\Delta(y', y) = (y' - y)^2 \quad -(2)$

From 1 & 2

$\hat{y} = \underset{y' \in R}{\arg\min} \int_R (y'-y)^2\, p(y|x)\, dy$

We obtain min if we set $y' = y$, therefore @ $y = y'$

$= \int_R y'^2 \int p(y=y'|x)\, dy - 2y' \int y\, p(y=y'|x)\, dy$
$+ \int y^2 p(y=y'|x)\, dy$

$= y'^2 - 2y' E(y/x) + E((y/x)^2)$

$[\, Var(x) = E(x^2) - (E(x))^2$

$= y'^2 - 2y' E(y/x) + (E(y/x))^2 + Var(y/x)$

$= (y' - E(y/x))^2 + Var(y/x)$

$\hat{y} = (y' - E(y=y'/x))^2 + Var((y-y')/x)$

To obtain a min $\hat{y}$, first term $(y' - E(y=y'/x))^2$ is minimized when $y' = E(y=y'/x)$ i.e if we choose y' to be expectation (mean) of posterior distrubution $P(y/x)$

Minimal loss is the second term $Var((y-y')/x)$ which is the co-variance of the above posterior distrubution

Bayesian Decision Theory

2) $$\hat{y} = \arg\min_{y' \in \mathcal{Y}} \sum_{y \in \mathcal{Y}} \Delta(y', y)\, P(y|x) \quad -(1)$$

$$\Delta(y', y) = 1 - \delta(y', y) \quad -(2)$$

$$\hat{y} = \arg\min_{y' \in \mathcal{Y}} \sum (1 - \delta(y', y))\, P(y|x)$$

$$= \arg\min_{y' \in \mathcal{Y}} \left[\sum_{y \in \mathcal{Y}} P(y|x) - \sum_{y \in \mathcal{Y}} \delta(y', y)\, P(y|x) \right] \quad -(2a)$$

$$\sum P(y|x) = 1 \quad -(3)$$

$$\sum_{y \in \mathcal{Y}} \delta(y', y)\, P(y|x) = P(y = y'|x) \quad \left[\text{Since } \delta(y', y) = 1 \text{ when } y = y' \right] \quad -(4)$$

from (3) & (4) in 2a

$$\boxed{\hat{y} = \arg\min_{y' \in \mathcal{Y}} \left[1 - P(y = y'|x) \right]}$$

To minimise $\hat{y}$ it is clear that we should maximise $P(y = y'|x)$ i.e y' must be chosen value which is max value of posterior distribution $P(y|x)$

Decision Regions (5 points)

Given $a \le b$

$\sqrt{a} \le \sqrt{b}$ (taking root on both sides)

$\sqrt{a}\sqrt{a} \le \sqrt{a}\sqrt{b}$ ($\times$ by $\sqrt{a}$)

$a \le \sqrt{ab}$

$a \le (ab)^{1/2}$ – (1)

$$p(\text{mistake}) = \int_{R_1} p(x, C_2)\,dx + \int_{R_2} p(x, C_1)\,dx \quad -(1b)$$

To prove – $p(\text{mistake}) \le \int (p(x, C_1)\, p(x, C_2))^{1/2}\,dx$

If error is made in R_1, we get $p(C_1|x) \ge p(C_2|x)$

$\therefore$ from (1) $p(C_2|x) \le \{p(C_1|x)\, p(C_2|x)\}^{1/2}$ – (2)

we know $p(x, C_2) = p(C_2|x)\, p(x)$

$\int_{R_1} p(x, C_2)\,dx = \int_{R_1} p(C_2|x)\, p(x)\,dx$ – (3)

from (2) & (3) $\int_{R_1} p(x, C_2)\,dx \le \int_{R_1} (p(C_1|x)\, p(C_2|x))^{1/2} p(x)\,dx$

$$\le \int_{R_1} \left(\frac{p(C_1, x)\, p(C_2, x)}{p(x)^2}\right)^{1/2} p(x)\,dx$$

$$\le \int_{R_1} (p(x, C_1)\, p(x, C_2))^{1/2}\,dx \quad -(4)$$

Similarly for error made in R_2

we get $\int_{R_2} p(x, C_1)\,dx \le \int_{R_2} (p(x, C_1)\, p(x, C_2))^{1/2}\,dx$ –(5)

from (1b), (4), & (5)

$$p(\text{mistake}) \le \int (p(x, C_1)\, p(x, C_2))^{1/2}\,dx$$

Task 3

① $p(x,y|z) = p(x|z)\,p(y|xz)$

$$p(x,y|z) = \frac{p(xyz)}{p(z)} = \frac{\cancel{p(z)}\,p(x|z)\,p(y|xz)}{\cancel{p(z)}} =$$

$$= p(x|z)\,p(y|xz)$$

$$p(x|y,z) = \frac{p(y|x,z)\,p(x|z)}{p(y|z)}$$

$$p(x|y,z) = \frac{p(xyz)}{p(yz)} = \frac{\cancel{p(z)}\,p(x|z)\,p(y|xz)}{p(y|z)\,\cancel{p(z)}} =$$

$$= \frac{p(x|z)\,p(y|xz)}{p(y|z)}.$$

Monty Hall problem

Player picking door 1 - $P1$

Host picking door 3 = $H3$

Probability of car behind door 1,2,3 = $C1 = C2 = C3 = 1/3$

Assuming the player picks Door1 initially.

If Player picks Door 1, the probability of host choosing door3 is

$P(H3|C1, P1) = 1/2$ [could be H2 or H3, since car is behind Door1]

$P(H3|C2, P1) = 1$ [Host has no choice but open door3, else he will open the door with car]

$P(H3|C3, P1) = 0$ [Host will open a door with no car behind]

So if player chooses Door1 (P1), Host opens door3 (H3), the probability of winning by switching after Host opens door is

$$P(C2|H3, P1) = \frac{P(H3, C2, P1)\, P(C2|P1)}{P(H3|P1)}$$

$$= \frac{P(H3|C2, P1)\, P(C2, P1)}{P(H3|C1, P1)\, P(C1|P1) + P(H3|C2, P1)\, P(C2|P1) + P(H3|C3, P1)\, P(C3|P1)}$$

$$= \frac{P(H3|C2, P1)}{P(H3|C1, P1) + P(H3|C2, P1) + P(H3|C3, P1)}$$

$$= \frac{1}{1/2 + 1 + 0} = \frac{2}{3}$$

∴ probability of winning after host opens the door & player switching is 2/3. So player has more chances of winning.