Elliptic Curve Cryptography

DISSERTATION

INTEGRATED MASTERS OF SCIENCE
In
APPLIED MATHEMATICS

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Declaration

I hereby certify that the work which is being presented in the thesis entitled **Elliptic Curve Cryptography** in the partial fulfillment of the requirement for the award of the degree of Integrated Master of Science in Applied Mathematics and submitted to the Department of Mathematics, Indian Institute of Technology Roorkee, is an authentic record of my own work carried out during a period from January 2022 to April 2022 under the supervision of **Dr. R.K. Panday**, Associate Professor, Mathematics Department, Indian Institute of Technology Roorkee. The matter presented in this report has not been submitted by me for the award of any other degree of this or any other institute.

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This is certified that the above statement made by the candidate is correct to the best of my knowledge.

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Abstract

This text discuss about Cryptography using Elliptic Curves. It has many practical applications in end-to-end encryption, data and password storing, cryptocurrencies. The motive behind this dissertation is to understand in depth how cryptography is applied in modern technologies such as blockchain which is driving force behind financial revolution of 21^{st} century. This dissertation is divided into two chapters. The first chapter deals with the prerequisite from abstract algebra and number theory that is required to study elliptic curve cryptography. The second chaptes deals with the introduction and algorithms of elliptic curve cryptography.

Acknowledgments

I would like to thank my supervisor, Ram Krishna Panday, Department of Mathematics, Indian Institute of Technology Roorkee for his guidance through each stage of the process.

I would finally like to thank Prof Premananda Bera, Head of Department, Department of Mathematics, Indian Institute of Technology for giving me the permission to carry out this work at IIT Roorkee

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Contents

Chapter 1

Pre-requisite

This chapter covers topics of Abstract Algebra and Number Theory that are required for cryptography and elliptic curves.
This definition of Group, Ring, Field are as follows

1.1 Group

Definition 1. *The set G is equipped with single operation $*$ such the the* 4 *below properties are satisfied is called a Group.*
(1) *Closure:* $\forall x, y \in G, x * y \in G$
(2) *Additive identity:* $\exists 0 \in G$, *such that* $\forall x \in G$, $0 * x = x * 0 = x$
(3) *Associative Property:* $\forall x, y, z \in G$, $(x * y) * z = x * (y * z)$
(4) *Inverse:* $\forall x \in S, \exists y \in G$ *such that*$x * y = 0$ *where* y *is known as inverse of* x *and is denoted by* x^{-1}

1.1.1 Abelian Group

Definition 2. *The set G is equipped with single operation $*$ such the the* 5 *below properties are satisfied is called a Abelian Group.*
(1) *Closure:* $\forall x, y \in G, x * y \in G$
(2) *Additive identity:* $\exists 0 \in G$, *such that* $\forall x \in G$, $0 * x = x * 0 = x$
(3) *Associative Property:* $\forall x, y, z \in G$, $(x * y) * z = x * (y * z)$
(4) *Commutative Property:* $\forall x, y \in G$, $x * y = y * x$
(5) *Inverse:* $\forall x \in G, \exists y \in G$ *such that* $x * y = 0$ *where* y *is known as inverse of* x *and is denoted by* x^{-1}

So, a abelian group G is a group with $\forall x, y \in G, x * y = y * x$

1.2 Ring

Definition 3. *A ring is a set R with two operations $+$ and $*$ which satisfy the below properties*
(1) *It is abelian group under* $+$
(2) *Closure under* $*$*:* $x, y \in R \Rightarrow x * y \in R$
(3) *Associative under* $*$*:* $x, y, z \in R \Rightarrow (x * y) * z = x * (y * z)$
(4) *Distributive property* $x, y, z \in R$

$$x * (y + z) = x * y + x * z$$

$$(x + y) * z = x * z + y * z$$

1.3 Field

Definition 4. *A Field is a set F with two operations $+$ and $*$ with following properties*
(1)*Commutative group under* $+$
(2)*Commutative group under* $*$
(3)*Distributive property* $x, y, z \in F$

$$x * (y + z) = x * y + x * z$$

$$(x + y) * z = x * z + y * z$$

1.4 Fermat's little theorem

Theorem 1. *Let p be any prime number. For any number a such that $p \nmid a$. Then $a^{p-1} \equiv 1 \pmod p$*

Proof: Assume p is a prime number and $p \nmid a$
Every integer is congruent to one of $0, 1, 2, \cdots, p-1 \pmod p$
Only focus on non zero congruence classes, because $0 \pmod p$ contains all the multiples of p (and $p \nmid a$). Focus on $0, 1, 2, \cdots, p-1$
Multiply all of these by a:

$$a, 2a, \cdots, (p-1)a$$

Show that this is a rearrangement of $0, 1, 2, \cdots, p-1$
Case 1: None of these are congruent to 0.

Suppose $r.a \equiv 0 \pmod p$
Then $p \nmid r.a$, but this is impossible since $p \nmid a$ and $r < p$
Case 2: These are distinct, no two are congruent to each other.
Pick two values $r.a$,$s.a$

$$0 < r < p$$
$$0 < s < p$$

Let's show that $r.a \not\equiv s.a \pmod p$
So look at $r.a - s.a = (r - s).a$. As $p \nmid a$, so can $p \mid (r - s)$?

$$0 < r < p$$
$$-p < -s < 0$$

Adding these inequalities gives you:

$$-p < r - s < p$$

So, $p \nmid (r - s)$ which means $a, 2a, \cdots, (p-1)a$ is a rearrangement of $1, 2, \cdots, (p-1)$.

$$a, 2a, \cdots, (p-1)a \equiv 1, 2, \cdots, (p-1) \pmod p$$

$$(p-1)!a^{p-1} \equiv (p-1)! \pmod p$$
$$a^{p-1} \equiv 1 \pmod p$$

Chapter 2

Elliptic Curves and Cryptography

2.1 Introduction to Cryptography

The written word is the most important invention in human history. But as long as human has the ability to share information, they have also had the need to conceal that information as well. This need lead to invention of cryptography.
The word cryptography comes from greek which means "hidden writing". According to Wikipedia, **Cryptography, or cryptology is the practice and study of techniques for secure communication in the presence of adversarial behavior.**
Some of the application of Cryptography includes:

- End-to-end Encryption for e-mail, messaging apps, GSM phones.
- Storing Data: Biggest consumer application of cryptography includes Kindle, iPod which stores books and songs in encrypted format to protect copyright.
- Storing Password: Storing passwords in plane text is not secure. If an attacker has access to the system they can read the password. If the password is converted into hash using one way mapping function and stored. Every time a user logs in, the password will be converted into hash and compared with the stored password.

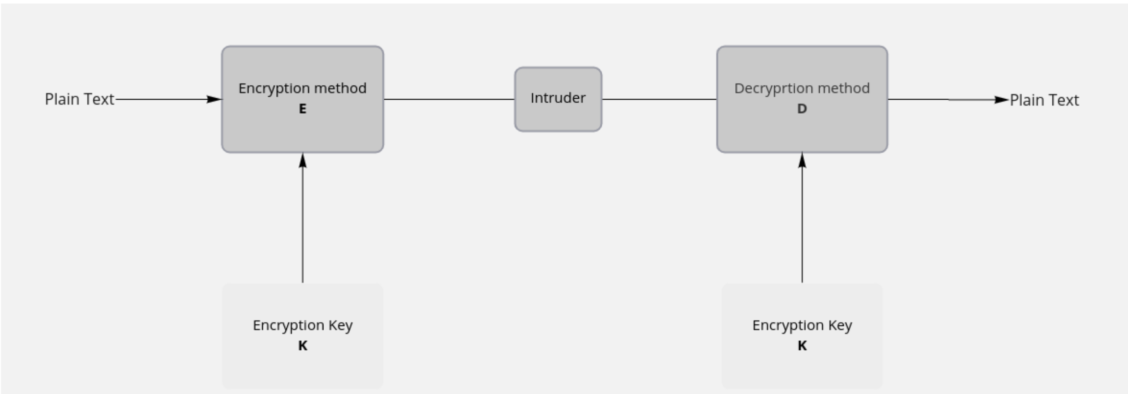
There are mainly two type of cryptography: Symmetric key cryptography and Asymmetric key cryptography.

2.1.1 Symmetric cryptography

Let Alice want to share a message m with Bob. They do so by using a common key and knowledge of some algorithm to encrypt and decrypt message. Alice encrypts the message using the key to produce the cipher text. Now Bob can use key with cipher text to decrypt message.

In symmetric cryptography a common key is used by the sender and receiver.

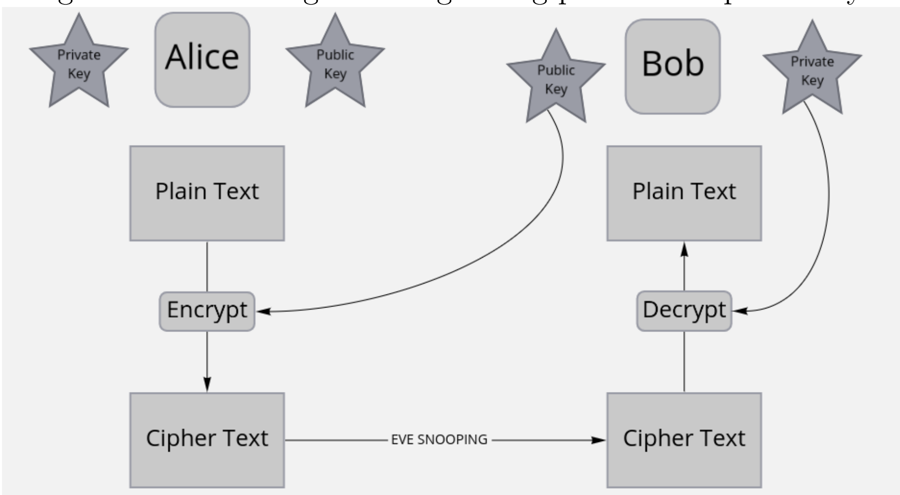
Figure 2.1: A picture of the universe!



2.1.2 Asymmetric cryptography

Asymmetric cryptography works by using private and public key pairs. Each user has a private,public key pair. Public key can be shared freely across the network and is used to verify the owner of a message. Private keys is not transmitted across the network. Public are used to encrypt the message and private key is used to decrypt the message. The major advantage of asymmetric cryptography is that there is no need of a shared key.

Figure 2.2: A message exchange using private and public keys



2.2 Elliptic Curves

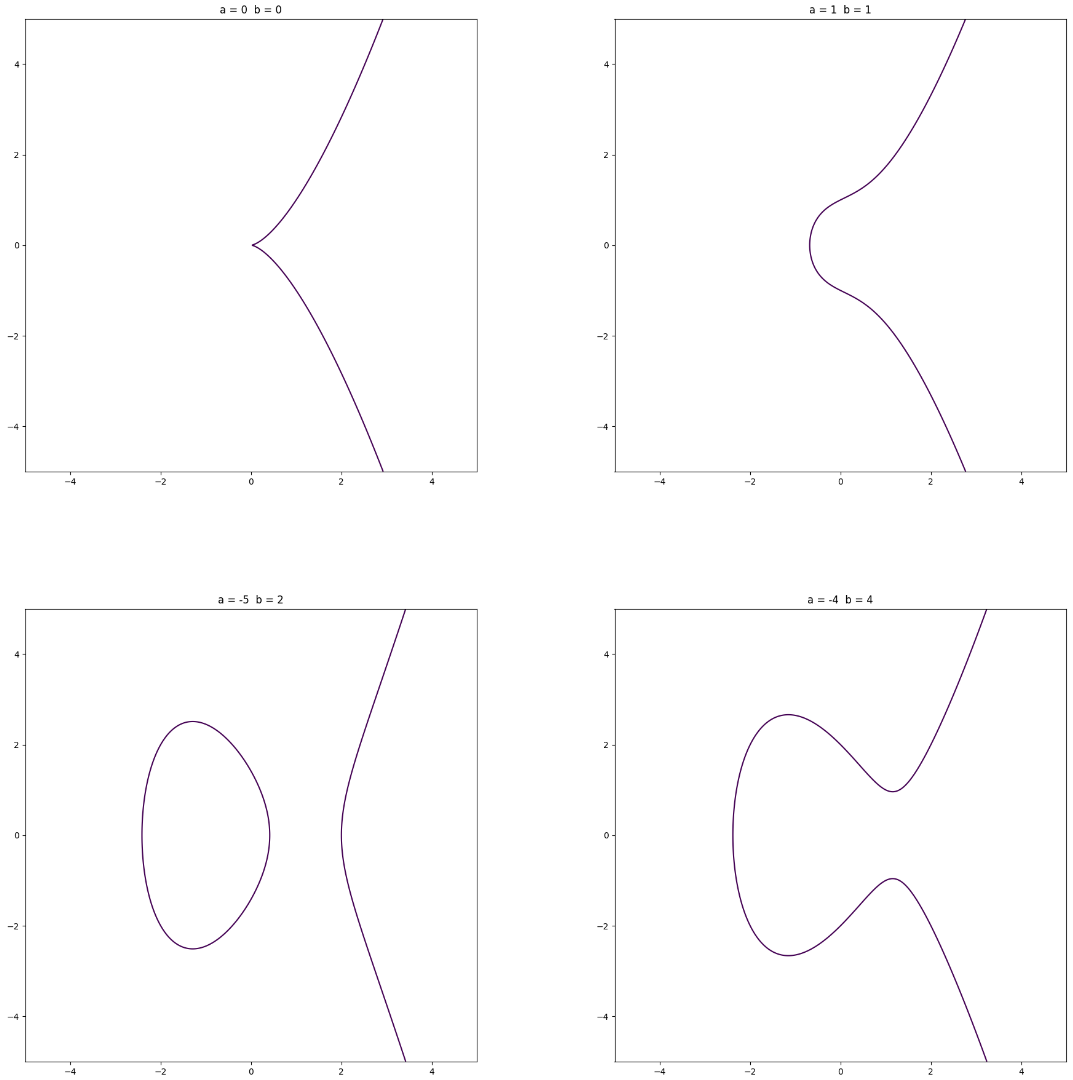
Equation of type $y^2 = x^3 + ax + b$ are called Weierstrass equations. It is named after Karl Weierstrass $(1815 - 1897)$ who studied them in 19^{th} century.

Definition 5. *Elliptic curves are solution sets of Weierstrass equations*

$$E : y^2 = x^3 + ax + b...(1)$$

with $\{ \mathscr{O} \}$ *where* $\Delta_E = 4a^3 + 27b^2 \neq 0$. $\Delta_E \neq 0$ *guarantees that the equation* $x^3 + ax + b$ *has no repeated roots i.e.* $x^3 + ax + b = (x - e_1)(x - e_2)(x - e_3)$ *where* e_1, e_2, e_3 *are distinct.* $\mathscr{O}$ *is defined as the point at infinity which lies on every vertical line.*

Figure 2.3: Example of Elliptic Curves

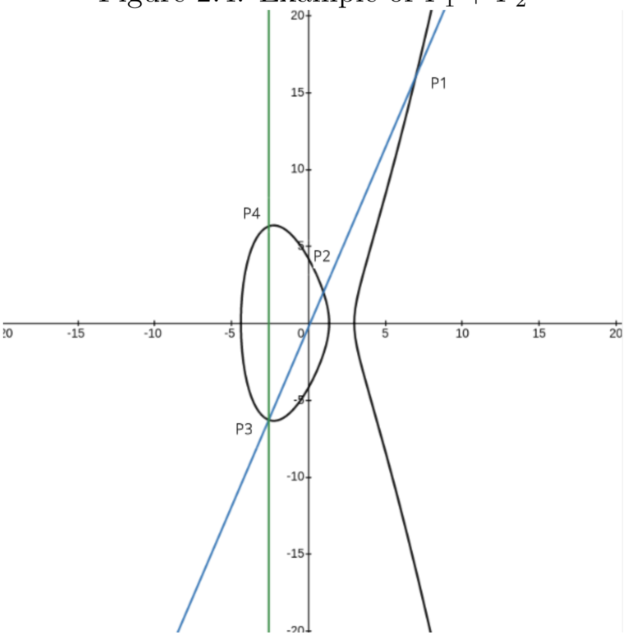


If (x,y) satisfies eq(1), then (x,-y) is also a solution of equation (1). So, elliptic curves are symmetric about x-axis.
The definition of addition "+" operator is a not the usual definition one might expect

$$(a,b) + (c,d) \neq (a+c, b+d)$$

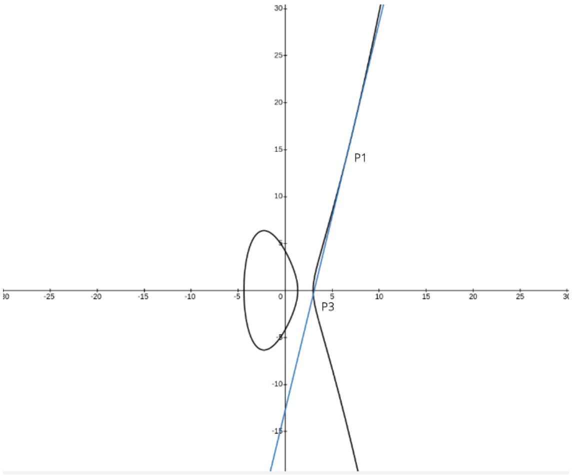
Two points P_1 and P_2 on elliptic curve. If we make a line L that passes through P_1 and P_2, it will intersect the curve at point $P_3 = (x_3, y_3)$. The reflection of P_3 from x-axis i.e. $(x_3, -y_3)$ is called the sum of points P_1 and P_2

Figure 2.4: Example of $P_1 + P_2$



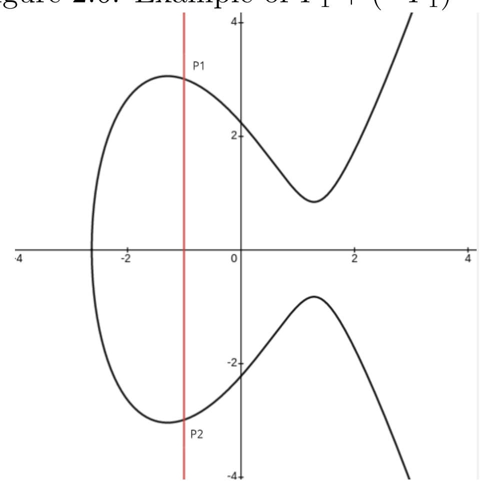
So, what is $P_1 + P_1$? This is the limiting case where $P_2 \to P_1$ and the Line L becomes the tangent to E at P_1. This line will intersect E at P_3. The reflection of P_3 about x-axis is $P_1 + P_1$. Let's look at the case when two

Figure 2.5: Example of $P_1 + P_1$



points on the curve when $P_1 = (x, y)$ and $P_2 = (x, -y)$ are added. In this case line L is $x = a$. L will not intersect the curve at third point. In this case we define $P_1 + P_2 = \mathscr{O}$. We define $\mathscr{O}$ as the point in infinity that lies on every vertical line. If P = (x,y) then -P is defined as (x,-y). So, $P + (-P) = \mathscr{O}$...(2)

Figure 2.6: Example of $P_1 + (-P_1) = \mathscr{O}$



Theorem 2. *Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on elliptic curve $E : Y^2 = X^3 + AX + B$. Then the following are true:*

1. *If $P = \mathcal{O}$, then $P + Q = Q$*
2. *If $Q = \mathcal{O}$, then $P + Q = P$*
3. *If $P = -Q$ then $P + Q = \mathcal{O}$*
4. *If $P \neq Q$ then $\lambda = (y_2 - y_1)/(x_2 - x_1)$ and if $P = Q$ then $\lambda = (3x_1^2 + A)/(2y_1)$. In both cases, $P_1 + P_2 = (\lambda^2 - x_1 - x_2, \lambda(x_1 - x_3) - y_1)$*

Proof. $(1), (2), (3)$ are true as discussed above.
(4) If $P \neq Q$ then λ is the slope of the line passing through P and Q. If $P = Q$ then λ is the slope of the tangent at P. Suppose Line $y = \lambda x + c$ intersects the curve at (x_3, y_3) in addition to (x_1, y_1) and (x_2, y_2).

$$(\lambda x + c)^2 = x^3 + Ax + B$$

$$x^3 - \lambda^2 x^2 + (A - 2c\lambda)x + (B - c^2) = (x - x_1)(x - x_2)(x - x_3)$$

We get $x_3 = \lambda^2 - x_1 - x_3$ by comparing the coefficients of x^2. $y_3 = \lambda x + c = y_1 - \lambda(x_1 - x_2)$. The ordinate of $P + Q$ is $-y_3 = \lambda(x_1 - x_3) - y_1$ □

Theorem 3. *Let E be Elliptic curve. Then E forms abelian group under addition. The following statements are true:*

1. *$P_1 + \mathcal{O} = \mathcal{O} + P_1 = P_1$ for all $P_1 \in E$*
2. *$P_1 + (-P_1) = \mathcal{O}$ for all $P_1 \in E$*
3. *$(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3)$ for all $P_1, P_2, P_3 \in E$*
4. *$P_1 + P_2 = P_2 + P_1$ for all $P_1, P_2 \in E$*

Proof. (1) Claim: $P_1 + \mathcal{O} = P_1$
If a line is drawn through P and $\mathcal{O}$ it will intersect E at -P. Reflection of -P from x-axis is again P. So, $P_1 + \mathcal{O} = P_1$ Similarly, $\mathcal{O} + P_1 = P_1$
(2) Explained above in equation (2)
(3) Associative property is non-trivial to prove using geometry. It can be verified by using Theorem 2 by calculation using substitution.
(4) is true as line passing through P_1 and P_2 is same as the line passing through P_2 and P_1. □

2.3 Elliptic curves over finite fields

Definition 6. *Elliptic curve over a finite field F_p is defined as equation of the form*

$$E : y^2 = x^3 + ax + b$$

where $a, b \in F_p$ and $\Delta_E = 4a^3 + 27b^2 \neq 0$.

Example: Let's suppose $a = 0$, $b = 1$ and $p = 11$.
$F_{11} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.\{a, b\} \in F_{11}$ and $4 * 0^2 + 27 * 1^3 = 27 \neq 0$.
Hence, $E : y^2 = x^3 + 1 \pmod{11}$ is an elliptic curve over finite field.
$E(F_{11}) = \{\mathscr{O}, (10, 0), (0, 10), (0, 1), (9, 2), (9, 9), (2, 3), (2, 8), (6, 8), (8, 4), (8, 7), (3, 5), (3, 6)\}$
with $|E(F_{11})| = 12$

Theorem 4. *Equation of elliptic curve over finite field F_p is $E : Y^2 = X^3 + AX + B$. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on E. If Theorem 2 is applied to points P and Q then the resulting point also lies on E.*

Proof. Theorem 2 is derived by substututing the equation of line to elliptic curve. So the solution automatically satisfies the elliptic curve. Similarly Theorem 4 is also true.

□

Example: Consider the above example. Let $P = (0, 1)$ and $Q = (10, 0)$.
$\lambda = \frac{1-0}{0-10} = \frac{1}{-10} = 1 \pmod{11}$,$x_3 = 1 - 0 - 10 = -9 = 2 \pmod{11}$
$y_3 = 1(0 - (2)) - 1 = -3 = 8$ and $(2, 8) \in E$

Theorem 5. *The elliptic curve over finite field along with addition property forms finite group.*

2.4 Elliptic curve discrete logarithm problem

Definition 7. *Let $(G,.)$ be a group and g be the generator of the group and $h \in G$ such that*

$$g^x = h$$

x is called discrete logarithm of h base x and is denoted by $x = log_g h$.

Elliptic curve group operation is addition as described above.So, in case of elliptic curve discrete logarithm problem is described as follows: Let $P, Q \in E$ such that $nP = Q$. n is called elliptic discrete logarithm of Q base P and is denoted by $n = \log_P Q$.

2.5 Elliptic Diffie–Hellman key exchange

Bibliography