Elliptic Curve Cryptography

DISSERTATION

INTEGRATED MASTERS OF SCIENCE
In
APPLIED MATHEMATICS

Submitted by

Harsh Kumar Chourasia

supervised by
Dr. Ram Krishna Pandey

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, ROORKEE
DEPARTMENT OF MATHEMATICS
ROORKEE-247667

Declaration

I hereby certify that the work which is being presented in the thesis entitled **Elliptic Curve Cryptography** in the partial fulfillment of the requirement for the award of the degree of Integrated Master of Science in Applied Mathematics and submitted to the Department of Mathematics, Indian Institute of Technology Roorkee is an authentic record of my work carried out during a period from January 2022 to April 2022 under the supervision of **Dr. R.K. Pandey**, Associate Professor, Mathematics Department, Indian Institute of Technology Roorkee. The matter presented in this report has not been submitted by me for the award of any other degree of this or any other institute.

Harsh Kumar Chourasia
I. M.Sc, Applied Mathematics
Department of Mathematics
IIT Roorkee
Date:
Place:

CERTIFICATE

This is certified that the above statement made by the candidate is correct to the best of my knowledge.

Dr. RK Pandey
Associate Professor
Department of Mathematics
IIT Roorkee
Date:
Place:

Abstract

This text discusses Cryptography using Elliptic Curves. It has many practical applications in end-to-end encryption, data and password storing, and cryptocurrencies. The motive behind this dissertation is to understand in-depth how cryptography is applied in modern technologies such as blockchain, which is the driving force behind the financial revolution of 21st century. This dissertation is divided into two chapters. The first chapter deals with the prerequisite from abstract algebra and number theory that is required to study elliptic curve cryptography. The second chapter deals with the introduction and algorithms of elliptic curve cryptography.

Acknowledgments

I would like to thank my supervisor, Ram Krishna Pandey, Department of Mathematics, Indian Institute of Technology Roorkee, for his guidance through each stage of the process.

I would finally like to thank Prof Premananda Bera, Head of Department, Department of Mathematics, Indian Institute of Technology, for permitting me to carry out this work at IIT Roorkee

Harsh Kumar Chourasia
I. M.Sc, Applied Mathematics
Department of Mathematics
IIT Roorkee
Date:
Place:

Contents

Chapter 1

Fundamental concepts

This chapter covers topics of Abstract Algebra and Number Theory that are required for cryptography and elliptic curves. The definition of Group, Ring, Field are as follows

1.1 Group

Definition 1. *The set G equipped with single operation $*$ such the 4 below properties given below are satisfied is called a Group.*
(1) Closure: $\forall x, y \in G, x * y \in G$
(2) Additive identity: $\exists 0 \in G$, such that $\forall x \in G, 0*x = x*0 = x$
(3) Associative Property: $\forall x, y, z \in G, (x * y) * z = x * (y * z)$
(4) Inverse: $\forall x \in S, \exists y \in G$ such that $x*y = 0$ where y is known as inverse of x and is denoted by x^{-1}

1.1.1 Abelian Group

Definition 2. *The set G is equipped with single operation $*$ such the 5 below properties are satisfied is called a Abelian Group.*
(1) Closure: $\forall x, y \in G, x * y \in G$
(2) Additive identity: $\exists 0 \in G$, such that $\forall x \in G, 0*x = x*0 = x$
(3) Associative Property: $\forall x, y, z \in G, (x * y) * z = x * (y * z)$
(4) Commutative Property: $\forall x, y \in G, x * y = y * x$

(5) *Inverse:* $\forall x \in G, \exists y \in G$ *such that* $x * y = 0$ *where* y *is known as inverse of* x *and is denoted by* x^{-1}

So, a abelian group G is a group with $\forall x, y \in G, x * y = y * x$

1.2 Ring

Definition 3. *Ring is a set* R *with two operations* $+$ *and* $*$ *which satisfy the below properties*
(1) *It is abelian group under* $+$
(2) *Closure under* $*$: $x, y \in R \Rightarrow x * y \in R$
(3) *Associative under* $*$: $x, y, z \in R \Rightarrow (x * y) * z = x * (y * z)$
(4) *Distributive property* $x, y, z \in R$

$$x * (y + z) = x * y + x * z$$

$$(x + y) * z = x * z + y * z$$

1.3 Field

Definition 4. *Field is a set* F *with two operations* $+$ *and* $*$ *with following properties*
(1) *Commutative group under* $+$
(2) *Commutative group under* $*$
(3) *Distributive property* $x, y, z \in F$

$$x * (y + z) = x * y + x * z$$

$$(x + y) * z = x * z + y * z$$

1.4 Fermat's little theorem

Theorem 1. *Let* p *be any prime number. For any number* a *such that* $p \nmid a$. *Then* $a^{p-1} \equiv 1 \pmod{p}$

Proof. Assume p is a prime number and $p \nmid a$

Every integer is congruent to one of $0, 1, 2, \cdots, p-1 \pmod{p}$

Only focus on non zero congruence classes, because $0 \pmod{p}$ contains all the multiples of p (and $p \nmid a$). Focus on $0, 1, \cdots, p-1$

Multiply all of these by a:

$$a, 2a, \cdots, (p-1)a$$

Show that this is a rearrangement of $0, 1, 2, \cdots, p-1$

Case 1: None of these are congruent to 0.

Suppose $r.a \equiv 0 \pmod{p}$

Then $p \nmid r.a$, but this is impossible since $p \nmid a$ and $r < p$

Case 2: These are distinct, no two are congruent to each other.

Pick two values $r.a$,$s.a$

$$0 < r < p$$
$$0 < s < p$$

Let's show that $r.a \not\equiv s.a \pmod{p}$

So look at $r.a - s.a = (r-s).a$. As $p \nmid a$, so can $p \mid (r-s)$?

$$0 < r < p$$
$$-p < -s < 0$$

Adding these inequalities gives you:

$$-p < r - s < p$$

So, $p \nmid (r-s)$ which means $a, 2a, \cdots, (p-1)a$ is a rearrangement of

$1, 2, \cdots, (p-1)$.

$$a, 2a, \cdots, (p-1)a \equiv 1, 2, \cdots, (p-1) \pmod{p}$$
$$(p-1)!a^{p-1} \equiv (p-1)! \pmod{p}$$
$$a^{p-1} \equiv 1 \pmod{p}$$

$\square$

4

Chapter 2

Elliptic Curves and Cryptography

2.1 Introduction to Cryptography

The written word is the most important invention in human history. But as long as humans can share information, they have also needed to hide information. This has led to the invention of cryptography.

The word cryptography comes from Greek which means "hidden writing". According to Wikipedia [2], **Cryptography, or cryptology, is the practice and study of techniques for secure communication in the presence of adversarial behavior.**

Some of the application of Cryptography includes:

- End-to-end Encryption for e-mail, messaging apps, GSM phones.

- Storing Data: Biggest consumer application of cryptography includes Kindle, and iPod, which store books and songs in an encrypted format to protect copyright.

- Storing Password: Storing passwords in plain text is not secure. If an attacker gains access to the system, they can

read the password. If the password is converted into a hash using a one-way mapping function and stored. Every time a user logs in, the password will be converted into the hash and compared with the stored password.

There are main types of cryptography: symmetric key cryptography and asymmetric key cryptography.

2.1.1 Symmetric cryptography

Let Alice want to share a message m with Bob. They do so by using a common key and knowledge of some algorithm to encrypt and decrypt the message. Alice encrypts the message using the key to produce the ciphertext. Now Bob can use the key with the ciphertext to decrypt the message.

In symmetric cryptography, a common key is used by the sender and receiver.

Figure 2.1: A picture of the universe!
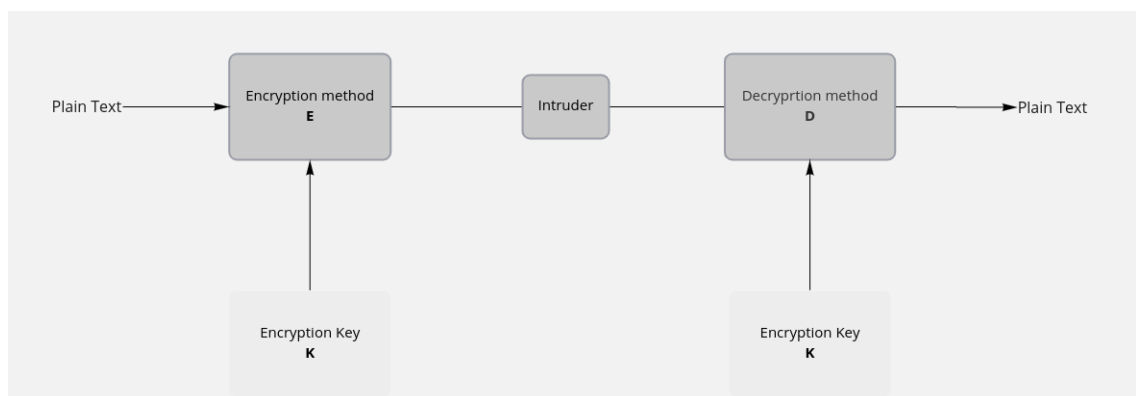
2.1.2 Asymmetric cryptography

Asymmetric cryptography works by using private and public key pairs. Each user has a private, public key pair. The public key can be shared freely across the network and is used to verify the owner of a message. The private key is not transmitted across the network. Public key are used to encrypt the message and the private key is used to decrypt the message. The major advantage of asymmetric cryptography is that there is no need for a shared key.

Figure 2.2: A message exchange using private and public keys
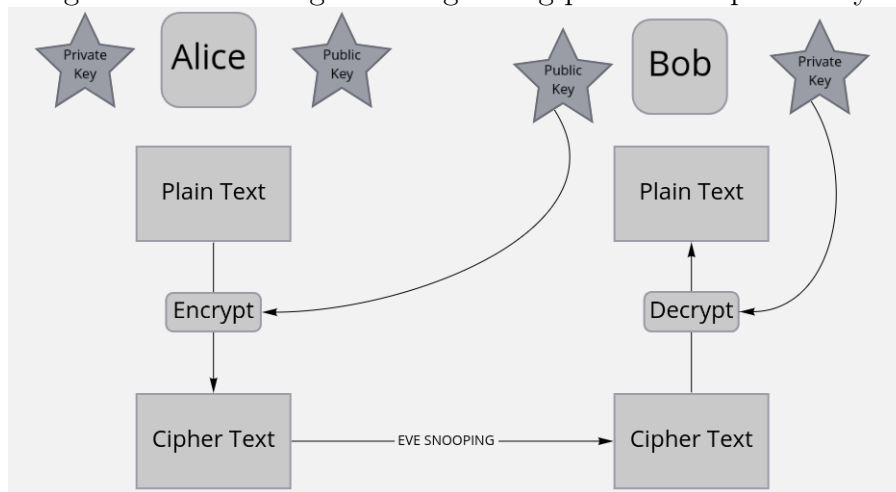
2.2 Elliptic Curves

Equation of type $y^2 = x^3 + ax + b$ are called Weierstrass equations. It is named after Karl Weierstrass $(1815 - 1897)$ who studied them in 19th century.

Definition 5. *Elliptic curves are solution sets of Weierstrass equations*

$$E : y^2 = x^3 + ax + b...(1)$$

with $\{\ \mathscr{O}\ \}$ where $\Delta_E = 4a^3 + 27b^2 \neq 0$. $\Delta_E \neq 0$ guarantees that the equation $x^3 + ax + b$ has no repeated roots i.e. $x^3 + ax + b = (x - e_1)(x - e_2)(x - e_3)$ where e_1, e_2, e_3 are distinct. $\mathscr{O}$ is defined as the point at infinity which lies on every vertical line.

Figure 2.3: Example of Elliptic Curves
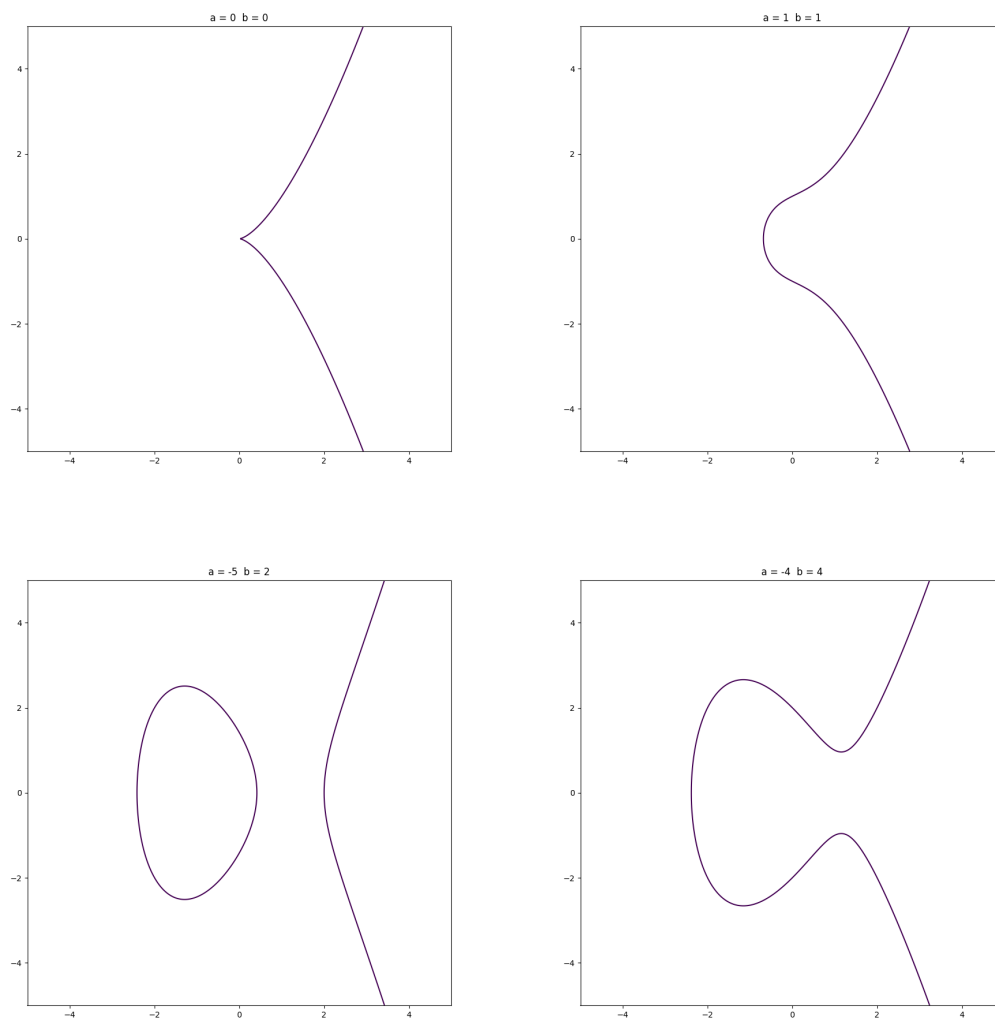
If (x,y) satisfies eq(1), then (x,-y) is also a solution of equation (1). So, elliptic curves are symmetric about x-axis.

The definition of addition "+" operator is a not the usual definition one might expect

$$(a, b) + (c, d) \neq (a + c, b + d)$$

Two points P_1 and P_2 on elliptic curve. If we make a line L that passes through P_1 and P_2, it will intersect the curve at point $P_3 = (x_3, y_3)$. The reflection of P_3 from x-axis i.e. $(x_3, -y_3)$ is called the sum of points P_1 and P_2 So, what is $P_1 + P_1$? This

Figure 2.4: Example of $P_1 + P_2$
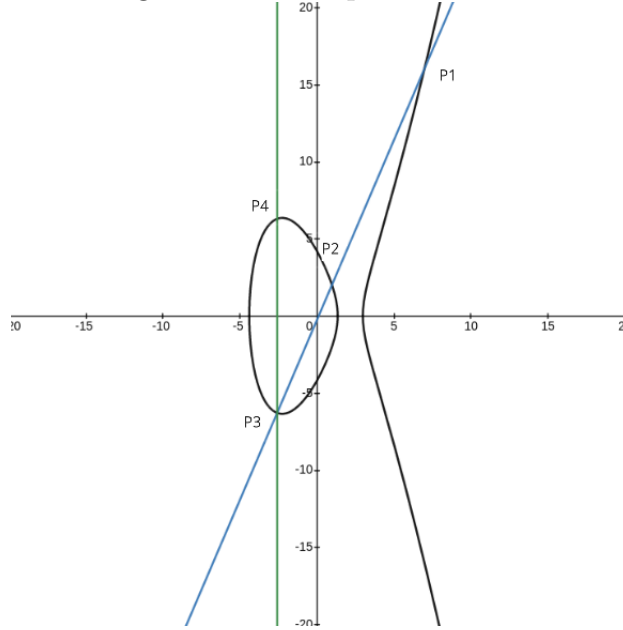
is the limiting case where $P_2 \to P_1$ and the Line L becomes the tangent to E at P_1. This line will intersect E at P_3. The reflection of P_3 about x-axis is $P_1 + P_1$. Let's look at the case when two points on the curve when $P_1 = (x, y)$ and $P_2 = (x, -y)$ are added. In this case line L is $x = a$. L will not intersect the curve at third point. In this case we define $P_1 + P_2 = \mathcal{O}$. We define $\mathcal{O}$ as the point in infinity that lies on every vertical line.

Figure 2.5: Example of $P_1 + P_1$
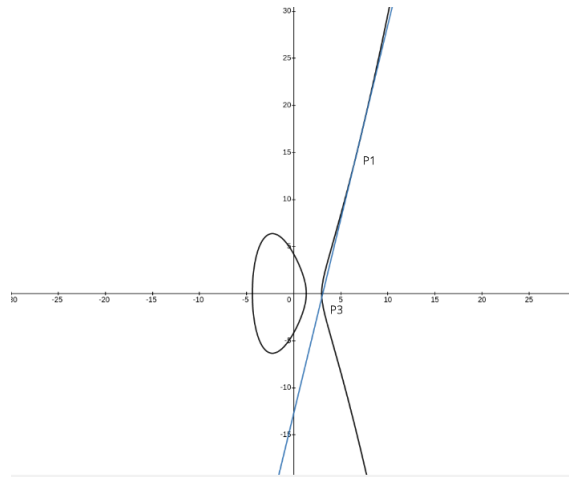
If P $=$ (x,y) then -P is defined as (x,-y). So, $P + (-P) = \mathcal{O}...(2)$

Figure 2.6: Example of $P_1 + (-P_1) = \mathcal{O}$

11
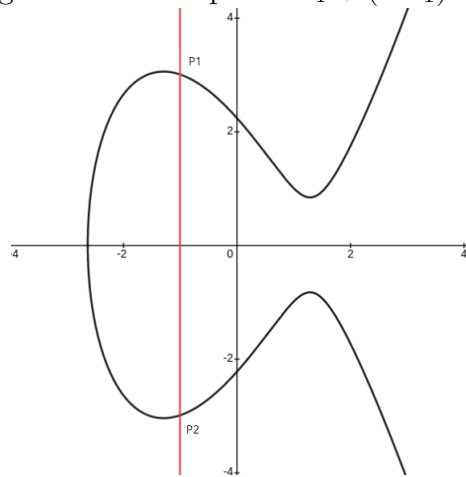
Theorem 2. *Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on elliptic curve $E : Y^2 = X^3 + AX + B$. Then the following are true:*

1. *If $P = \mathcal{O}$, then $P + Q = Q$*

2. *If $Q = \mathcal{O}$, then $P + Q = P$*

3. *If $P = -Q$ then $P + Q = \mathcal{O}$*

4. *If $P \neq Q$ then $\lambda = (y_2 - y_1)/(x_2 - x_1)$ and if $P = Q$ then $\lambda = (3x_1^2 + A)/(2y_1)$. In both cases,*
   $$P_1 + P_2 = (\lambda^2 - x_1 - x_2, \lambda(x_1 - x_3) - y_1)$$

Proof. $(1), (2), (3)$ are true as discussed above.
(4) If $P \neq Q$ then λ is the slope of the line passing through P and Q. If $P = Q$ then λ is the slope of the tangent at P. Suppose Line $y = \lambda x + c$ intersects the curve at (x_3, y_3) in addition to (x_1, y_1) and (x_2, y_2).

$$(\lambda x + c)^2 = x^3 + Ax + B$$

$$x^3 - \lambda^2 x^2 + (A - 2c\lambda)x + (B - c^2) = (x - x_1)(x - x_2)(x - x_3)$$

We get $x_3 = \lambda^2 - x_1 - x_3$ by comparing the coefficients of x^2. $y_3 = \lambda x + c = y_1 - \lambda(x_1 - x_2)$. The ordinate of $P + Q$ is $-y_3 = \lambda(x_1 - x_3) - y_1$ □

Theorem 3. *Let E be Elliptic curve. Then E forms abelian group under addition. The following statements are true:*

1. $P_1 + \mathcal{O} = \mathcal{O} + P_1 = P_1$ *for all $P_1 \in E$*

2. $P_1 + (-P_1) = \mathcal{O}$ *for all $P_1 \in E$*

3. $(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3)$ *for all $P_1, P_2, P_3 \in E$*

4. $P_1 + P_2 = P_2 + P_1$ *for all $P_1, P_2 \in E$*

Proof. (1) Claim: $P_1 + \mathcal{O} = P_1$

If a line is drawn through P and $\mathcal{O}$, it will intersect E at -P. Reflection of -P from x-axis is again P. So, $P_1 + \mathcal{O} = P_1$ Similarly, $\mathcal{O} + P_1 = P_1$

(2) Explained above in equation (2)

(3) Associative property is non-trivial to prove using geometry. It can be verified by using Theorem 2 by calculation using substitution.

(4) is true as line passing through P_1 and P_2 is same as the line passing through P_2 and P_1. □

2.3 Elliptic curves over finite fields

Definition 6. *Elliptic curve over a finite field F_p is defined as equation of the form*

$$E : y^2 = x^3 + ax + b$$

where $a, b \in F_p$ and $\Delta_E = 4a^3 + 27b^2 \neq 0$.

Example: Let's suppose $a = 0$, $b = 1$ and $p = 11$. $F_{11} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.\{a, b\} \in F_{11}$ and $4 * 0^2 + 27 * 1^3 = 27 \neq 0$. Hence, $E : y^2 = x^3 + 1 \pmod{11}$ is an elliptic curve over finite field.
$E(F_{11}) = \{\mathcal{O}, (10, 0), (0, 10), (0, 1), (9, 2), (9, 9), (2, 3), (2, 8), (6, 8), (8, 4), (8, 7), (3, 5), (3, 6)\}$ with $|E(F_{11})| = 12$

Theorem 4. *Equation of elliptic curve over finite field F_p is $E : Y^2 = X^3 + AX + B$. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on E. If Theorem 2 is applied to points P and Q then the resulting point also lies on E.*

Proof. Theorem 2 is derived by substituting the equation of line to elliptic curve. So the solution automatically satisfies the elliptic curve. Similarly Theorem 4 is also true.

□

Example: Consider the above example. Let $P = (0, 1)$ and $Q = (10, 0)$. $\lambda = \frac{1-0}{0-10} = \frac{1}{-10} = 1 \pmod{11}$,$x_3 = 1 - 0 - 10 = -9 = 2 \pmod{11}$

$y_3 = 1(0 - (2)) - 1 = -3 = 8$ and $(2, 8) \in E$

Theorem 5. *The elliptic curve over finite field along with addition property forms finite group.*

2.4   Elliptic curve discrete logarithm problem

Definition 7. *Let $(G,.)$ be a group and g be the generator of the group and $h \in G$ such that*

$$g^x = h$$

x is called discrete logarithm of h base x and is denoted by $x = log_g h$.

Elliptic curve group operation is addition as described above. So, in case of elliptic curve discrete logarithm problem is described as follows: Let $P, Q \in E$ such that $nP = Q$. n is called elliptic discrete logarithm of Q base P and is denoted by $n = \log_P Q$.

It should be noted that if Q is not multiple of P, then n will not be defined. But for all practical purposes, Q is obtained from repeatedly adding P. So n will exist.

We also note that in every finite group every element has finite order. Let order of P be s i.e. $sP = \mathcal{O}$. If n_0 is the smallest number satisfying $Q = sP$. Then for all $n = n_0 + is$ $\forall i \in \mathbb{Z}$, $Q = nP$. It implies $\log_P Q$ is an element of $\mathbb{Z}/s\mathbb{Z}$. We could also set the value to be n_0. But if we define the value to be in $\mathbb{Z}/s\mathbb{Z}$ the the following equation is satisfied.

$$\log_P(Q + R) = \log_P(Q) + \log_P(R)$$

Therefore the function $log_P : E(F_p) \to \mathbb{Z}/s\mathbb{Z}$ defines group homomorphism.

2.4.1   Double and Add Algorithm

The value of n in $Q = nP$ is large for practical application. If nP is calculated by adding P repeatedly n number of times, it will $O(n)$ operation. Suppose n has k binary digits. Then the complexity will be $O(2^k)$. By the use of double and add

algorithm we can reduce the complexity to $O(\log n)$ or $O(k)$.
Take $n = 151$. Binary representation of 151 is 10010111_2.

$151 = 1.2^7 + 0.2^6 + 0.2^5 + 1.2^5 + 0.2^4 + 1.2^3 + 1.2^2 + 1.2^1 + 1.2^0$

or

$$151P = 2^7P + 2^4P + 2^2P + 2^1P + 2^0P$$

Obtain $151P$ using double and add algorithm

1. Start with P

2. Double it to get $2P$ $(P + P)$

3. Add $2P$ to P (result will be $2^1P + 2^0P$)

4. Double $2P$ to get $4P$ $(2P + 2P)$

5. Add $4P$ to the result (result will be $2^2P + 2^1P + 2^0P$)

6. Double $4P$ to get $8P$ $(4P + 4P)$

7. Double $8P$ to get $16P$ $(8P + 8P)$

8. Add $16P$ to the result (result will be $2^4P + 2^2P + 2^1P + 2^0P$)

9. Keep Doubling $16P$ till $128P$ is obtained

10. Add $128P$ to the result (result will be $2^7P + 2^4P + 2^2P + 2^1P + 2^0P$)

We obtained $151P$ using only four addition and seven doubling operation!

Pseudo-code of double and add algorithm

Let $n = n_0 + 2n_1 + 2^2 n_2 + ... + 2^m n_m$ where $n_0, ... n_m \in 0, 1, m = \lfloor \log_2 n \rfloor$

1. $bits = [n_0, n_1, ...n_m]$

2. $result = \mathcal{O}$ // infinity point

3. $temp = P$

4. for bit in bits:

5.   if bit == 1:

6.   result = result + temp   // addition operation

7.   temp = temp + temp   // doubling operation

8. return result

2.4.2 Hasse's Theorem

Hasse's theorem estimates the number of points in an elliptic curve over a finite field. We state Hasse's theorem without proof.

Theorem 6. *Let N is the number of points on an Elliptic Curve E over finite field F_q. Then the following inequality is true*

$$|N - (q + 1)| < 2\sqrt{q}$$

or

$$q + 1 - 2\sqrt{q} < N < q + 1 + 2\sqrt{q}$$

2.4.3 How to solve ECDLP

Suppose P and Q are points on an Elliptic Curve E over finite field F_p such the $Q = nP$. The value of P and Q are known and we wish to find the value of n.

Exhaustive Search Method
Compute the value of $P, 2P, 3P...$ until you find a multiple such that it is equal to Q. Time complexity of this algorithm is $O(p)$.

Collision Search Method
Compute two lists for randomly chosen integers $m_1, m_2, ...m_r$ and $n_1, n_2, ...n_r$ where each integer is between 1 and p.
List 1: $m_1 P, m_2 P, ..., m_r P$
List 2: $n_1 P + Q, n_2 P + Q, ..., n_r P + Q$
As soon as we find $m_u P$ and $m_v P + Q$ such that $m_u P = m_v P + Q$, then $Q = (m_v - m_u)P$ i.e. $n = m_v - m_u$. Time complexity of collision search method is $O(\sqrt{p})$ by the birthday paradox[1].

The value of p is taken very large for practical purpose. For example, Secp256k1 is the name of the elliptic curve used by Bitcoin to implement its public key cryptography. The value of p used by it is $p = 2^{256} - 2^{32} - 2^9 - 2^8 - 2^7 - 2^6 - 2^4 - 1$.

Chapter 3

Elliptic Diffie–Hellman key exchange

In this section, we will use the Elliptic curve to do cryptography. The biggest problem of symmetric cryptography is sharing of private keys. Historically secret key used to be shared physically and then the encrypted message was shared over a public channel. Diffie-Hellman key exchange helps the sender and receiver of the message to securely exchange cryptographic keys over a public channel. Its security does depend on mathematical principles.

Elliptic Diffie-Hellman Key Exchange Algorithm

1. A large prime number p, an Elliptic Curve E over F_p and a point $P \in E(F_p)$ is shared between Ram and Shayam over public channel.

2. Ram chooses a large random number n_R and computes $Q_R = n_R P$

3. Shayam chooses a large random number n_S and computes $Q_S = n_S P$

4. Ram and Shayam shares Q_R and Q_S over the public channel.

5. Ram now compute $n_R Q_S$

6. Shayam now compute $n_S Q_R$

7. The shared secret key is $Q = n_R Q_S = n_R(n_S P) = n_S(n_R P) = n_S Q_R$

Suppose Ghanshyam wants to obtain the secret key. Ghanshyam knows the value of P, Q_R, Q_S as they are available on the public channel. Ghanshyam can solve Elliptic Curve Discrete Logarithm Problem, i.e., the equation $Q_R = n_R P$ or $Q_S = n_S P$ to obtain the value of n_R and n_S respectively. After the value of Q can be obtained using the formula $Q = n_R Q_S$ or $Q = n_S Q_R$.

Example Ram and Shayam want to exchange a secret key using the Diffie-Hellman Algorithm with the following curve, prime and point. $p = 7$ $E : y^2 = X^3 + 1$ $P = (2,3) \in E(F_7)$
Ram and Shayam choose their random secret key as $n_R = 5$ and $n_S = 3$ respectively.
Ram computes $Q_R = n_R P = 5(2,3) = (2,4) \in E(F_7)$
Shayam computes $Q_S = n_S P = 3(2,3) = (6,0) \in E(F_7)$
Ram and Shayam will share the value of Q_R and Q_S with each other.
Ram computes $n_R Q_S = 5(6,0) = (6,0)$
Shayam computes $n_S Q_R = 3(2,4) = (6,0)$
Ram and Shayam have shared the secret key $(6,0)$. They will use x-coordinate as the shared key and ignore y-coordinate.

Chapter 4

Implementation

This chapter deals with the implmentation of ECC using javascript.

```javascript
1  INF_POINT = null;
2  class EllipticCurve {
3    constructor(a, b, p) {
4      this.a = a;
5      this.b = b;
6      this.p = p;
7      this.points = [];
8      this.definePoints();
9    }
10
11   definePoints() {
12     this.points.push(INF_POINT);
13     for (let x = 1; x < this.p; x++) {
14       for (let y = 1; y < this.p; y++) {
15         if (this.equalModP(y ** 2, x ** 3 + this.a * x + this
   .b)) {
16           this.points.push([x, y]);
17         }
18       }
19     }
20   }
21
22   add(p, q) {
23     if (p == null) {
24       return q;
25     }
26     if (q == null) {
27       return p;
28     }
```

```
29
30        const x1 = p[0];
31            const y1 = p[1];
32            const x2 = q[0];
33            const y2 = q[1];
34        let lambda;
35        if (this.equalModP(x1, x2) && this.equalModP(y1, -y2)) {
36          return INF_POINT;
37        }
38        if (this.equalModP(x1, x2) && this.equalModP(y1, y2)) {
39          lambda = this.reduceModP((3 * x1 * x1 + this.a) * this.
      inverseModP(2 * y1));
40        } else {
41          lambda = this.reduceModP((y1 - y2) * this.inverseModP(
      x1 - x2));
42        }
43        const x3 = this.reduceModP(lambda ** 2 - x1 - x2);
44        const y3 = this.reduceModP(lambda * (x1 - x3) - y1);
45        return [x3, y3];
46    }
47
48    doubleAndAddAlgorithm(n, P) {
49      let Q = INF_POINT;
50      while (n != 0) {
51        if (n & 1 !== 0) {
52          Q = this.add(Q, P);
53        }
54        P = this.add(P, P);
55        n >>= 1;
56      }
57      return Q;
58    }
59
60    reduceModP(n) {
61      while (n < 0) {
62        n += this.p;
63      }
64      return n % this.p;
65    }
66
67    equalModP(x, y) {
68      return this.reduceModP(x - y) == 0;
69    }
70
71    inverseModP(x) {
72      for (let y = 1; y < this.p; y++) {
73        if (this.equalModP(x * y, 1)) {
74          return y;
75        }
```

```
76        }
77        return null;
78      }
79 }
80 ec = new EllipticCurve(0, 1, 7);
81 console.log(ec.points);
82 console.log(ec.doubleAndAddAlgorithm(5, [2, 4]));
```

Chapter 5

Conclusion

Bibliography

[1] Jeffrey Hoffstein, Jill Pipher, Joseph H Silverman, and Joseph H Silverman. *An introduction to mathematical cryptography*, volume 1. Springer, 2008.

[2] Wikipedia contributors. Cryptography — Wikipedia, the free encyclopedia, 2022. [Online; accessed 29-April-2022].