03/02/2021

NORMAL DISTRIBUTION

BINOMIAL
POISSON → Discrete $P(X = x_i)$ $i = 1, 2, 3, \ldots n$

$p \to 0$
$n \to \infty$

Continuous

$X \to (-\infty, \infty)$

NORMAL DISTRIBUTION → Binomial distribution

where $n \to \infty$, $p \to 1/2$

prob. of success

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

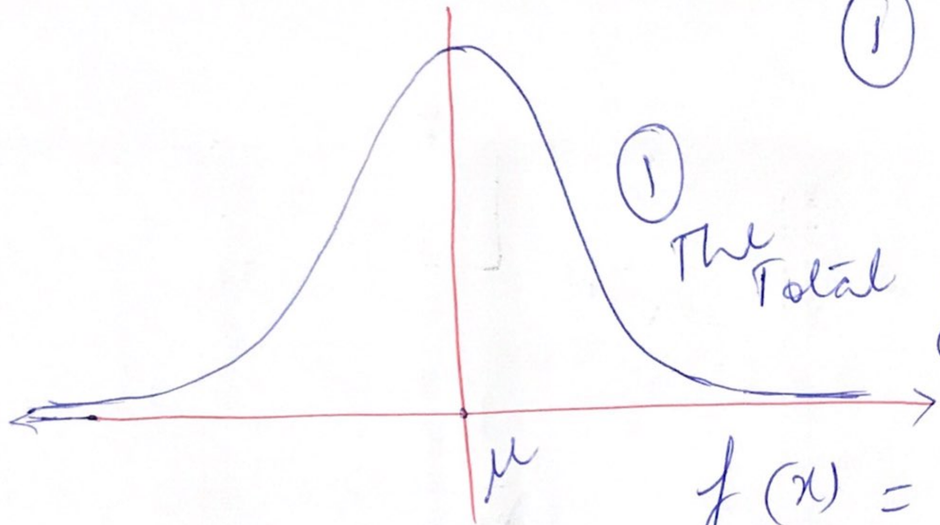
$-\infty < x < \infty$

$$X \sim N(\mu, \sigma^2)$$

$\sigma > 0$

$-\infty < \mu < \infty$

① PROPERTIES of Normal distribution



① The Total area under the probability curve is one.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Consider;

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put

$$\frac{x-\mu}{\sqrt{2}\,\sigma} = t$$

$$\Rightarrow dx = \sqrt{2}\,\sigma\,dt$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\,\sigma\,dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$ Hence Pd

$$\boxed{\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx \quad n>0}$$

$$I = \int_{-\infty}^{\infty} e^{-t^2}\, dt = 2\int_0^{\infty} e^{-t^2}\, dt$$

$$t^2 = z$$

$$2t\, dt = dz$$

$$\Rightarrow dt = \frac{dz}{2\sqrt{z}}$$

$$\Rightarrow dt = \frac{1}{2} dz \left(z^{-1/2}\right)$$

$$\boxed{\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx}$$

$$I = 2\int_{0}^{\infty} e^{-z}\, \frac{1}{2} z^{-1/2}\, dz$$

$$= \boxed{\frac{1}{2} \cdot 2\int_{0}^{\infty} e^{-z} z^{-1/2}\, dz}$$

$$= \int_0^{\infty} e^{-z} z^{-1/2}\, dz = \int_0^{\infty} e^{-z} z^{1/2-1}\, dz = \Gamma\left(\tfrac{1}{2}\right)$$

$$= \sqrt{\pi}$$

(2) Mean of the Normal distribution

$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $\frac{x-\mu}{\sqrt{2}\,\sigma} = t$

$dx = \sqrt{2}\sigma\, dt$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu)\, e^{-t^2} (\sqrt{2}\,\sigma\, dt)$$

$$= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{\infty} \sqrt{2}\sigma\, \underbrace{t e^{-t^2}}_{\text{odd fn}}\, dt + \mu \int_{-\infty}^{\infty} e^{-t^2} dt \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[0 + \mu \int_{-\infty}^{\infty} e^{-t^2} dt \right]$$

$$= \frac{1}{\sqrt{\pi}} \mu \cdot \sqrt{\pi} = \mu$$ = Mean of the Normal distribution.

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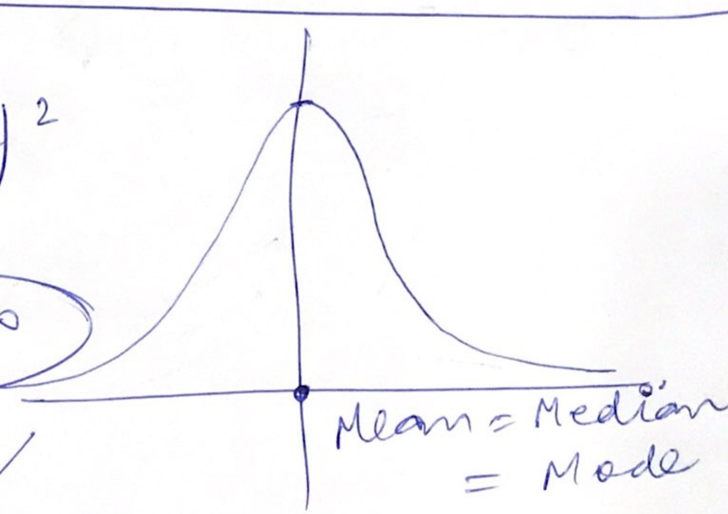
③ MODE

$f_X(x) = \checkmark$

$$f_X(x) \left(= f(x)\right) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$-\infty < x < \infty$



$\boxed{f'(x) = 0}$ ✓ → Stationary points

$\boxed{f''(x) < 0}$ ✓