3.7 Exponential and Gamma Distributions

Definition. (Exponential Distribution)

A random variable X is said to have an **exponential distribution** *with parameter* $\lambda > 0$, *if its density is given by*

$$f_X(x) = \lambda e^{-\lambda x} \text{ for all } x \geq 0 \text{ and } \lambda > 0.$$

We write it as $X \sim Expo(\lambda)$.

Definition. (Gamma Distribution)

A random variable X is said to have a **gamma distribution** *if its density is given by*

$$f_X(x) = \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x} \text{ for all } x \geq 0,$$

where $r > 0$ *and* $\lambda > 0$ *are called the parameters of the gamma distribution.*

We write $X \sim gam(\lambda\,;r)$ or $X \sim G(\lambda\,;r)$.

Remark 1. $\Gamma(r)$ is called *gamma function* and is defined as

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx,\ r > 0.$$

It is easy to verify that

$\Gamma(1) = 1, \Gamma(a+1) = a\,\Gamma(a), \Gamma(n) = (n-1)!$, n a positive integer.

Remark 2. *Taking* $r = 1$, *we see that gamma density becomes exponential density* ($\because \Gamma(1) = 1$).

Theorem 3.7.1. *If X has an exponential distribution, then*

$$E[X] = \frac{1}{\lambda},\ var[X] = \frac{1}{\lambda^2} \text{ and } m_X(t) = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda.$$

Proof. We have

$$E[X] = \int_0^\infty x f_X(x)\, dx = \int_0^\infty x \lambda e^{-\lambda x}\, dx,\ \lambda > 0.$$

Integrating by parts, we get

$$E[X] = \lambda \left| x\left(\frac{e^{-\lambda x}}{-\lambda}\right)\right|_0^\infty + \int_0^\infty 1 \cdot e^{-\lambda x}\, dx$$

$$= 0 - \frac{1}{\lambda}\left| e^{-\lambda x}\right|_0^\infty = -\frac{1}{\lambda}(0-1) = \frac{1}{\lambda}. \qquad \ldots(1)$$

Now $$E[X^2] = \int_0^\infty x^2 (\lambda e^{-\lambda x})\, dx$$

$$= \lambda \left| x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right)\right|_0^\infty + 2\int_0^\infty x e^{-\lambda x}\, dx$$

$$= 0 + \frac{2}{\lambda}\int_0^\infty x \lambda e^{-\lambda x}\, dx = \frac{2}{\lambda}\left(\frac{1}{\lambda}\right) = \frac{2}{\lambda^2}, \text{ by (1).}$$

$\therefore$ $$\operatorname{var}[X] = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

The m.g.f. of X is given by

$$m_X(t) = E[e^{tX}] = \int_0^\infty e^{tx}(\lambda e^{-\lambda x})\, dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx,\ \lambda > t$$

$$= -\frac{\lambda}{\lambda - t}\left| e^{-(\lambda - t)x}\right|_0^\infty = \frac{\lambda}{t-\lambda}(0-1)$$

$$= \frac{\lambda}{\lambda - t}, \text{ for } t < \lambda$$

Hence $$m_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}, \frac{t}{\lambda} < 1.$$

Theorem 3.7.6. *If the random variable X has an exponential distribution with parameter* $\lambda > 0$, *then*

$$P[X > a + b \mid X > a] = P[X > b], \text{ for } a > 0, b > 0.$$

Proof. $P[X > a + b \mid X > a] = \dfrac{P[X > a + b \text{ and } X > a]}{P[X > a]} = \dfrac{P[X > a + b]}{P[X > a]}$

$$[\because a + b > a \text{ as } b > 0]$$

$$= \int_{a+b}^{\infty} \lambda e^{-\lambda x} dx \div \int_{a}^{\infty} \lambda e^{-\lambda x} dx = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$

$$= e^{-\lambda b} = \int_{b}^{\infty} \lambda e^{-\lambda x} dx = P[X > b].$$

EXAMPLES

Example 3.7.1. *If* $X \sim Expo(\lambda)$ *with* $P[X \le 1] = P[X > 1]$, *find* $var[X]$.

Solution. We have

$$P[X \le 1] = P[X > 1] = 1 - P[X \le 1]$$

$$\Rightarrow \quad 2P[X \le 1] = 1 \quad \text{or} \quad P[X \le 1] = \frac{1}{2}.$$

$$\therefore \quad \frac{1}{2} = \int_0^1 \lambda e^{-\lambda x} dx = -\left| e^{-\lambda x} \right|_0^1 = -(e^{-\lambda} - 1) = 1 - e^{-\lambda}$$

$$\Rightarrow \quad e^{-\lambda} = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow e^{\lambda} = 2 \Rightarrow \lambda = \log_e 2.$$

Since $X \sim \text{Expo}(\lambda)$, $\text{var}[X] = \dfrac{1}{\lambda^2}$ [Theorem 3.7.1.]

Hence $\text{var}[X] = \dfrac{1}{(\log 2)^2}$.

Example 3.7.2. *If X has exponential distribution with mean 2, find*

$$P[X<1 \mid X<2].$$

Solution. We are given that $\frac{1}{\lambda}=2$ *i.e.*, $\lambda=\frac{1}{2}$.

Now $$P[X<1 \mid X<2]=\frac{P[X<1]}{P[X<2]}=\frac{\int_0^1 \lambda e^{-\lambda x}\,dx}{\int_0^2 \lambda e^{-\lambda x}\,dx}$$

$$=\frac{-\left| e^{-\lambda x}\right|_0^1}{-\left| e^{-\lambda x}\right|_0^2}=\frac{1-e^{-\lambda}}{1-e^{-2\lambda}}$$

$$=\frac{1}{1+e^{-\lambda}}=\frac{1}{1+e^{-1/2}}.$$

Example 3.7.3. *If* $X \sim U(0, 2)$ *and* $Y \sim Expo(\lambda)$, *find the value of* λ *such that*

$$P[X<1]=P[Y<1].$$

Solution. We have

$$f(x)=\frac{1}{2-0}=\frac{1}{2},\ 0<x<2.$$

By the given hypothesis, we obtain

$$\int_0^1 \frac{1}{2}\,dx=\int_0^1 \lambda e^{-\lambda x}\,dx$$

$$\Rightarrow \quad \frac{1}{2}=-\left| e^{-\lambda x}\right|_0^1=-(e^{-\lambda}-1)=1-e^{-\lambda}$$

$$\Rightarrow \quad e^{-\lambda}=\frac{1}{2} \Rightarrow -\lambda=\log\left(\frac{1}{2}\right)=-\log 2.$$

Hence $\lambda=\log 2$.

Example 3.7.5. *Show that $Y = -(1/\lambda)\log F(x)$ is Expo (λ).*

Solution. The m.g.f. of Y is

$$m_Y(t) = E[e^{tY}] = E\exp(tY) = E\exp\left[-\frac{t}{\lambda}\log F(x)\right]$$

$$= E\exp[\log\{F(x)\}^{-t/\lambda}]$$

$$= E[\{F(x)\}^{-t/\lambda}]$$

$$= E[Z^{-t/\lambda}],\ Z = F(x) \sim U(0,1)$$

$$= \int_0^1 z^{-t/\lambda}\,dz$$

$$= \left(1-\frac{t}{\lambda}\right)^{-1}, \text{ which is m.g.f. of } Expo\ (\lambda).$$

[See Theorem 3.7.1.]

Hence Y is $Expo\ (\lambda)$.

EXERCISES

1. A random variable X has a uniform distribution over $(-3, 3)$, find $P[|X| < 2]$.

   [**Hint.** $P[|X| < 2] = \int_{-2}^{2} \frac{1}{6}\,dx = \frac{2}{3}$.]

2. If X is uniformly distributed over $(-a, a)$, determine a if

   $$P[|X| > 2] = \frac{3}{4}.$$ [**Ans.** $a = 8$]

   [**Hint.** $\frac{3}{4} = 1 - P[|X| \le 2] = 1 - P[-2 \le X \le 2] = 1 - \int_{-2}^{2} \frac{1}{2a}\,dx$]

3. For the rectangular distribution :

   $$dF = k\,dx,\ 1 \le x \le 2$$

   show that : A.M. > G.M. > H.M.

   [**Hint.** $f(x) = k,\ 1 \le x \le 2$.]

4. If X is uniformly distributed with mean 2 and variance 3, find $P[X < 2]$. $\left[\textbf{Ans. } \frac{1}{2}\right]$

   [**Hint.** Similar to Example 3.6.1.]

5. If X has the distribution

$$f(x)=\frac{5}{A},\ \frac{-A}{10}\le x<\frac{A}{10};\ f(x)=0 \text{ otherwise},$$

determine the constant A when

$$P[\,|X|\le 2]=2\,P[\,|X|>2].$$ [**Ans.** $A=60$]

[**Hint.** We have $P[X\le 2]=2[1-P[\{\,|X|\le 2\}]$ or $\frac{2}{3}=P[X\le 2]$

or $$\frac{2}{3}=\int_{-A/10}^{2}(5/A)\,dx.\ ]$$

6. If $X\sim$ Expo (λ), find the value of k such that

$$\frac{P[X>k]}{P[X\le k]}=a.$$ $\left[\textbf{Ans. } k=\frac{1}{\lambda}\log(1+a^{-1})\right]$

[**Hint.** $P[X\le k]=\int_0^k \lambda e^{-\lambda x}dx=1-e^{-\lambda k},\ P[X>k]=e^{-\lambda k}$]

7. Find the median of the exponential distribution. [**Ans.** $\log 2/\lambda$]

[**Hint.** If m be the median, then

$$\tfrac{1}{2}=\int_0^m \lambda e^{-\lambda x}dx=1-e^{-\lambda m}.\ ]$$

8. If $X\sim U(-1,3)$ and $Y\sim$ Expo (λ). Find λ such that

$$\sigma_X^2=\sigma_Y^2\ \textit{i.e.},\ \text{var}[X]=\text{var}[Y].$$ [**Ans.** $\lambda=2/\sqrt{3}$]

$\left[\textbf{Hint. } \text{var}[X]=\frac{1}{12}(b-a)^2=\frac{1}{12}(3+1)^2=\frac{4}{3},\ \text{var}[Y]=\frac{1}{\lambda^2}\right]$

9. Show that the mean and variance of the simple gamma distribution

$$f(x)=\frac{e^{-x}x^{r-1}}{\Gamma(r)},\ r>0,\ 0<x<\infty$$

is each r. Further show that $m_X(t)=(1-t)^{-r},\ |t|<1$.

[Take $\lambda=1$ in the gamma density of Sec. 3.7.]

10. If X_1, X_2 are independent random variables having exponential distributions with parameters λ_1, λ_2; respectively, show that $Z=\min(X_1, X_2)$ has exponential distributions with parameter $\lambda_1+\lambda_2$.

[**Hint.** Let G be c.d.f. of Z. Then

$$G(z)=P[Z\le z]=1-P[Z>z]=1-P[X_1>z]\,P[X_2>z]$$

$$=1-\left[\int_z^\infty \lambda_1 e^{-\lambda_1 x}dx\right]\left[\int_z^\infty \lambda_2 e^{-\lambda_2 x}dx\right]$$