$$= EX^r + a\frac{1}{\sqrt{2\pi}}\int_0^\infty y^r \sin(2\pi\log y).\frac{1}{y}\exp\left\{-\frac{1}{2}(\log y)^2\right\}dy$$

$$= E(X^r) + \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{rz - z^2/2}\sin(2\pi z)\,dz \qquad [\log y = z \Rightarrow y = e^z]$$

$$= E(X^r) + \frac{a.\,e^{r^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(z-r)^2/2}\sin(2\pi z)\,dz$$

$$= EX^r + \frac{a\,e^{r^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}.\sin(2\pi t)\,dt = E(X^r)$$

[$\because z - r = t \Rightarrow \sin(2\pi z) = \sin(2\pi r + 2\pi t) = \sin 2\pi t$, r being a positive integer.]

the value of the integral being zero, since the integrand is an odd function of t.

$\therefore$ $E(Y^r)$ is independent of 'a' in (**).

Hence, $\{g(y) = g_a(y); -1 \le a \le 1\}$, represents a family of distributions, each different from the other, but having the same moments. This explains that the moments may not determine a distribution uniquely.

6·8. MOMENTS OF BIVARIATE PROBABILITY DISTRIBUTIONS

The mathematical expectation of a function $g(x, y)$ of two-dimensional random variable (X, Y) with p.d.f. $f(x, y)$ is given by :

$$E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)\,f(x, y)\,dx\,dy \qquad \dots(6·43)$$

(If X and Y are continuous variables.)

$$= \sum_i\sum_j x_i\,y_j\,P(X = x_i \cap Y = y_j), \qquad \dots(6·43\,a)$$

(If X and Y are discrete variables.)

provided the expectation exists.

In particular, the rth and sth product moment about origin of the random variables X and Y respectively is defined as :

$$\mu_{rs}' = E(X^r Y^s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^s f(x,y)\,dx\,dy$$

or $$\mu_{rs}' = \sum_i\sum_j x_i^r y_j^s\,P(X = x_i \cap Y = y_j) \qquad \dots(6·44)$$

The joint rth central moment of X and sth central moment of Y is given by :

$$\mu_{rs} = E\left[\{X - E(X)\}^r\{Y - E(Y)\}^s\right]$$

$$= E\left[(X - \mu_X)^r(Y - \mu_Y)^s\right], \quad \{E(X) = \mu_X,\ E(Y) = \mu_Y\} \qquad \dots(6·45)$$

In particular,

$$\mu_{00}' = 1 = \mu_{00}\ ,\quad \mu_{10} = 0 = \mu_{01},\quad \mu_{10}' = E(X)\ ,\quad \mu_{01}' = E(Y)$$

$$\mu_{20} = \sigma_X^2\ ,\quad \mu_{02} = \sigma_Y^2 \ \text{ and } \ \mu_{11} = \text{Cov}(X, Y).$$

6·9. CONDITIONAL EXPECTATION AND CONDITIONAL VARIANCE

Discrete Case. The conditional expectation or mean value of a continuous function $g(X, Y)$ given that $Y = y_j$, is defined by

$$E\{g(X, Y) \mid Y = y_j\} = \sum_{i=1}^{\infty} g(x_i, y_j)\, P(X = x_i \mid Y = y_j) = \frac{\sum_{i=1}^{\infty} g(x_i, y_j)\, P(X = x_i \cap Y = y_j)}{P(Y = y_j)}$$

... (6·46)

i.e., $E\{g(X, Y) \mid Y = y_j\}$ is nothing but the expectation of the function $g(X, y_j)$ of X *w.r.t.* the conditional distribution of X when $y = y_j$. In particular, the conditional expectation of a discrete random variable X given $Y = y_j$ is :

$$E(X \mid Y = y_j) = \sum_{i=1}^{\infty} x_i\, P(X = x_i \mid Y = y_j)$$

... (6·47)

The conditional variance of X given $Y = y_j$ is likewise given by

$$V(X \mid Y = y_j) = E\left[\{X - E(X \mid Y = y_j)\}^2 \mid Y = y_j\right]$$

... (6·47a)

The conditional expectation of $g(X, Y)$ and the conditional variance of Y given $X = x_i$ may also be defined in an exactly similar manner.

Continuous Case. The conditional expectation of $g(X, Y)$ on the hypothesis $Y = y$ is defined by :

$$E\{g(X, Y) \mid Y = y\} = \int_{-\infty}^{\infty} g(x, y)\, f_{X|Y}(x \mid y)\, dx = \frac{\int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx}{f_Y(Y)}$$

... (6·48)

In particular, the conditional mean of X given $Y = y$ is defined by

$$E(X \mid Y = y) = \frac{\int_{-\infty}^{\infty} x f(x, y)\, dx}{f_Y(y)}.$$

...(6·48a)

$$\text{Similarly, } E(Y \mid X = x) = \frac{\int_{-\infty}^{\infty} y f(x, y)\, dy}{f_X(x)}$$

... (6·48a)

The conditional variance of X may be defined as

$$V(X \mid Y = y) = E[\{X - E(X \mid Y = y)\}^2 \mid Y = y].$$

$$\text{Similarly, } V(Y \mid X = x) = E[\{Y - E(Y \mid X = x)\}^2 \mid X = x]$$

... (6·49)

Theorem 6·1. *The expected value of X is equal to the expectation of the conditional expectation of X given Y. Symbolically,* $E(X) = E\{E(X \mid Y)\}$

... (6·50)

Proof. $E\{E(X \mid Y)\} = E\{\sum_i x_i P(X = x_i \mid Y = y_j)\}$

$$= E\left\{\sum_i x_i \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right\} = \sum_j \left[\sum_i \left\{x_i \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right\}\right] P(Y = y_j)$$

$$= \sum_j \sum_i x_i P(X = x_i \cap Y = y_j) = \sum_i [x_i \{\sum_j P(X = x_i \cap Y = y_j)\}] = \sum x_i P(X = x_i) = E(X)$$

Theorem 6·2. *The variance of X can be regarded as consisting of two parts, the expectation of the conditional variance and the variance of the conditional expectation. Symbolically,* $V(X) = E[V(X \mid Y)] + V[E(X \mid Y)]$

...(6·51)

Proof. $E[V(X \mid Y)] + V[E(X \mid Y)]$

$$= E[E(X^2 \mid Y) - \{E(X \mid Y)\}^2] + E[\{E(X \mid Y)\}^2] - [E\{E(X \mid Y)\}]^2$$
$$= E[E(X^2 \mid Y)] - E\{E(X \mid Y)\}^2 + E\{E(X \mid Y)\}^2 - [E\{E(X \mid Y)\}]^2$$
$$= E\{E(X^2 \mid Y)\} - \{E(X)\}^2 = E\{\sum_i x_i^2 P(X = x_i \mid Y = y_j)\} - \{E(X)\}^2$$
$$= E\left\{\sum_i x_i^2 \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right\} - \{E(X)\}^2$$
$$= \sum_j \left[\left\{\sum_i x_i^2 \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right\} P(Y = y_j)\right] - \{E(X)\}^2$$
$$= \sum_i \{x_i^2 \sum_j P(X = x_i \cap Y = y_j)\} - [E(X)]^2 = \sum_i x_i^2 P(X = x_i) - [E(X)]^2$$
$$= E(X^2) - \{E(X)\}^2 = \text{Var}(X)$$

Hence the theorem.

Remark. The proofs of Theorems 6·1 and 6·2 for continuous *r.v.*'s X and Y are left as an exercise to the reader.

Theorem 6·3. *Let A and B be two mutually exclusive events, then*

$$E(X \mid A \cup B) = \frac{P(A)\, E(X \mid A) + P(B)\, E(X \mid B)}{P(A \cup B)} \qquad \ldots (6·52)$$

where by def., $\quad E(X \mid A) = \dfrac{1}{P(A)} \sum\limits_{x_i \in A} x_i\, P(X = x_i).$

Proof. $\quad E(X \mid A \cup B) = \dfrac{1}{P(A \cup B)} \sum\limits_{x_i \in A \cup B} x_i\, P(X = x_i) \qquad \ldots (*)$

Since A and B are mutually exclusive events,

$$\sum_{x_i \in A \cup B} x_i\, P(X = x_i) = \sum_{x_i \in A} x_i\, P(X = x_i) + \sum_{x_i \in B} x_i\, P(X = x_i)$$

$$\therefore \quad E(X \mid A \cup B) = \frac{1}{P(A \cup B)} \{P(A)\, E(X \mid A) + P(B)\, E(X \mid B)\} \qquad \text{[From (*)]}$$

Cor. $\quad E(X) = P(A)\, E(X \mid A) + P(\bar{A})\, E(X \mid \bar{A}) \qquad \ldots (6·53)$

The corollary follows by putting $B = \bar{A}$ in the above Theorem.

Example 6·29. *Two ideal dice are thrown. Let X_1 be the score on the first die and X_2 the score on the second die. Let Y denote the maximum of X_1 and X_2, i.e., $Y = max(X_1, X_2)$. (i) Write down the joint distribution of Y and X_1, and (ii) find the mean and variance of Y and covariance (Y, X_1).*

Solution. Each of the random variables X_1 and X_2 can take six values 1, 2, 3, 4, 5, 6, each with probability $\frac{1}{6}$, *i.e.*,

$$P(X_1 = i) = P(X_2 = i) = \tfrac{1}{6}; \quad i = 1, 2, 3, 4, 5, 6 \qquad \ldots (i)$$

$$Y = \text{Max}(X_1, X_2).$$

Obviously $P(X_1 = i, Y = j) = 0$, if $j < i = 1, 2, \ldots, 6.$

$$P(X_1 = i, Y = i) = P(X_1 = i \cap X_2 \le i) = \sum_{j=1}^{i} P(X_1 = i \cap X_2 = j)$$

$$= \sum_{j=1}^{i} P(X_1 = i)\, P(X_2 = j) \qquad (\because X_1, X_2 \text{ are independent.})$$

$$= \sum_{j=1}^{i} \left(\frac{1}{36}\right) = \frac{i}{36}\,;\; i = 1, 2, \ldots\; 6.$$

$$p(X_1 = i, Y = j) = P(X_1 = i, X_2 = j) = P(X_1 = i)\, P(X_2 = j) = \frac{1}{36}\,;\; j > i = 1, 2, \ldots\; 6.$$

The joint probability table of X_1 and Y is given as follows :

X_1 \ Y	1	2	3	4	5	6	*Marginal Totals*
1	1/36	1/36	1/36	1/36	1/36	1/36	6/36
2	0	2/36	1/36	1/36	1/36	1/36	6/36
3	0	0	3/36	1/36	1/36	1/36	6/36
4	0	0	0	4/36	1/36	1/36	6/36
5	0	0	0	0	5/36	1/36	6/36
6	0	0	0	0	0	6/36	6/36
Marginal Totals	1/36	3/36	5/36	7/36	9/36	11/36	1

Aliter. For an alternate solution, proceed as in Example 6·31.

$$E(Y) = 1.\frac{1}{36} + 2.\frac{3}{36} + 3.\frac{5}{36} + 4.\frac{7}{36} + 5.\frac{9}{36} + 6.\frac{11}{36} = \frac{161}{36}$$

$$E(Y^2) = 1^2.\frac{1}{36} + 2^2.\frac{3}{36} + 3^2.\frac{5}{36} + 4^2.\frac{7}{36} + 5^2.\frac{9}{36} + 6^2.\frac{11}{36} = \frac{791}{36}$$

$$V(Y) = E(Y^2) - \{E(Y)\}^2 = \frac{791}{36} - \left(\frac{161}{36}\right)^2 = \frac{2555}{1296}$$

$$E(X_1) = \frac{6}{36}(1 + 2 + 3 + 4 + 5 + 6) = \frac{126}{36} = \frac{21}{6}$$

$$E(X_1 Y) = 1.\frac{1}{36} + 2.\frac{1}{36} + 3.\frac{1}{36} + 4.\frac{1}{36} + 5.\frac{1}{36} + 6.\frac{1}{36}$$

$$+ \left(4.\frac{2}{36} + 6.\frac{1}{36} + 8.\frac{1}{36} + 10.\frac{1}{36} + 12.\frac{1}{36}\right)$$

$$+ \left(9.\frac{3}{36} + 12.\frac{1}{36} + 15.\frac{1}{36} + 18.\frac{1}{36}\right) + \left(16.\frac{4}{36} + 20.\frac{1}{36} + 24.\frac{1}{36}\right)$$

$$+ \left(25.\frac{5}{36} + 30.\frac{1}{36}\right) + 36.\frac{6}{36}$$

$$= \frac{1}{36}(21 + 44 + 72 + 108 + 155 + 216) = \frac{1}{36} \times 616$$

$$\text{Cov}(X_1, Y) = E(X_1 Y) - E(X_1)\, E(Y) = \frac{616}{36} - \frac{21}{6}.\frac{161}{36} = \frac{3696 - 3381}{216} = \frac{315}{216} = \frac{35}{24}$$

Example 6·30. *Let X and Y be two random variables each taking three values –1, 0, and 1, and having the joint probability distribution*

Y \ X	−1	0	1	Total
−1	0	·1	·1	·2
0	·2	·2	·2	·6
1	0	·1	·1	·2
Total	·2	·4	·4	1·0

(i) Show that X and Y have different expectations.
(ii) Prove that X and Y are uncorrelated.
(iii) Find Var X and Var Y.
(iv) Given that Y = 0, what is the conditional probability distribution of X.
(v) Find V (Y | X = − 1).

Solution. (i) $E(Y) = \sum p_i y_i = -1(\cdot 2) + 0(\cdot 6) + 1(\cdot 2) = 0$

$$E(X) = \sum p_i x_i = -1(\cdot 2) + 0(\cdot 4) + 1(\cdot 4) = \cdot 2$$

$\therefore$ $E(X) \neq E(Y)$

(ii) $E(XY) = \sum p_{ij} x_i y_j = (-1)(-1)(0) + 0(-1)(\cdot 1) + 1(-1)(\cdot 1)$
$\quad + 0(-1)(\cdot 2) + 0(0)(\cdot 2) + 0(1)(\cdot 2)$
$\quad + 1(-1)(0) + 1(0)(\cdot 1) + 1(1)(\cdot 1) = -0\cdot 1 + 0\cdot 1 = 0$

$\therefore$ $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$

$\Rightarrow$ X and Y are uncorrelated (*c.f* Chapter 10)

(iii) $E(Y^2) = (-1)^2(0\cdot 2) + 0(0\cdot 6) + 1^2(0\cdot 2) = 0\cdot 4$

$\therefore$ $V(Y) = E(Y^2) - \{E(Y)\}^2 = 0\cdot 4$

$$E(X^2) = (-1)^2(0\cdot 2) + 0(0\cdot 4) + 1^2(0\cdot 4) = 0\cdot 2 + 0\cdot 4 = 0\cdot 6$$

$$V(X) = E(X^2) - \{E(X)\}^2 = 0\cdot 6 - 0\cdot 04 = 0\cdot 56$$

(iv) $P(X = -1 \mid Y = 0) = \dfrac{P(X = -1 \cap Y = 0)}{P(Y = 0)} = \dfrac{0\cdot 2}{0\cdot 6} = \dfrac{1}{3}$

$$P(X = 0 \mid Y = 0) = \frac{P(X = 0 \cap Y = 0)}{P(Y = 0)} = \frac{0\cdot 2}{0\cdot 6} = \frac{1}{3}$$

$$P(X = 1 \mid Y = 0) = \frac{P(X = 1 \cap Y = 0)}{P(Y = 0)} = \frac{0\cdot 2}{0\cdot 6} = \frac{1}{3}$$

(v) $V(Y \mid X = -1) = E(Y \mid X = -1)^2 - \{E(Y \mid X = -1)\}^2$ ...(*)

$$E(Y \mid X = -1) = \sum_y y\, P(Y = y \mid X = -1) = (-1)0 + 0(\cdot 2) + 1(0) = 0$$

$$E(Y \mid X = -1)^2 = \sum_y y^2\, P(Y = y \mid X = -1) = 1(0) + 0(\cdot 2) + (0) = 0$$

$\therefore$ $V(Y \mid X = -1) = 0.$ [From (*)]

Example 6·31. *Two tetrahedra with sides numbered 1 to 4 are tossed. Let X denote the number on the downturned face of the first tetrahedron and Y denote the larger of the downturned numbers. Investigate the following :*

(a) Joint density function of X, Y and marginals f_X and f_Y,
(b) P {X ≤ 2, Y ≤ 3}, *(c) Cov (X, Y),* *(d) E (Y | X = 2),*

(e) construct joint density different from that in part (a) but possessing same marginals f_X and f_Y.

Solution. The sample space is $S = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ and each of the 16 sample points (outcomes) has probability $p = \frac{1}{16}$, of occurrence.

Let X : Number on the first dice and Y : Larger of the numbers on the two dice. Then the above 16 sample points, in that order, give the following distribution of X and Y.

Sample Point	:	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(2, 1)	(2, 2)	(2,3)	(2, 4)
X	:	1	1	1	1	2	2	2	2
Y	:	1	2	3	4	2	2	3	4
Sample Point	:	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(4, 1)	(4, 2)	(4, 3)	(4, 4)
X	:	3	3	3	3	4	4	4	4
Y	:	3	3	3	4	4	4	4	4

Since each sample point has probability $p = \frac{1}{16}$, the joint density functions of X and Y and the marginal densities f_X and f_Y are given below :

(a)

Y \ X	1	2	3	4	*Total* (f_Y)
1	p	0	0	0	p
2	p	$2p$	0	0	$3p$
3	p	p	$3p$	0	$5p$
4	p	p	p	$4p$	$7p$
Total (f_X)	$4p$	$4p$	$4p$	$4p$	$16p = 1$

(e)

Y \ X	1	2	3	4	*Total* (f_Y)
1	p	0	0	0	p
2	p	$2p$	0	0	$3p$
3	p	$p + \epsilon$	$3p - \epsilon$	0	$5p$
4	p	$p - \epsilon$	$p + \epsilon$	$4p$	$7p$
Total (f_X)	$4p$	$4p$	$4p$	$4p$	1

(b) $P(X \le 2, Y \le 3) = p + p + 2p + p + p = 6p = \frac{3}{8}$. $\quad \left(\because p = \frac{1}{16}\right)$

(c) $\text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{15}{2} - \frac{25}{4} = \frac{5}{4}$ (Try it)

$\text{Var}(Y) = E(Y^2) - \{E(Y)\}^2 = \frac{85}{8} - \left(\frac{25}{8}\right)^2 = \frac{55}{64}$ (Try it)

$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{135}{16} - \frac{5}{2} \times \frac{25}{8} = \frac{5}{8}$ (Try it)

(d) $$E(Y \mid X = 2) = \Sigma y . f(y \mid x = 2) = \Sigma\left\{y \cdot \frac{f(x = 2 \cap y)}{f(x = 2)}\right\}$$

$$= 4\,\Sigma y\, f(2, y) = 4\,(0 + 4p + 3p + 4p) = 44p = \frac{11}{4}$$

(e) Let $0 < \varepsilon < p$. The joint density of X and Y given in *(e)* above is different from that in *(a)*, but has the same marginals as in *(a)*.

Example 6·32. *(a) Given two variates X_1 and X_2 with joint density function $f(x_1, x_2)$, prove that conditional mean of X_2 (given X_1) coincides with (unconditional) mean only if X_1 and X_2 are independent (stochastically).*

(b) Let $f(x_1, x_2) = 21x_1^2 x_2^3$, $0 < x_1 < x_2 < 1$, and zero elsewhere, be the joint p.d.f. of X_1 and X_2. Find the conditional mean and variance of X_1 given $X_2 = x_2$, $0 < x_2 < 1$.

Solution. (*a*) Conditional mean of X_2 given X_1 is given by :

$$E(X_2 \mid X_1 = x_1) = \int_{x_2} x_2 f(x_2 \mid x_1)\, dx_2, \quad \ldots (*)$$

where $f(x_2 \mid x_1)$ is conditional *p.d.f.* of X_2 given $X_1 = x_1$.

The joint *p.d.f.* of X_1 and X_2 is given by :

$$f(x_1, x_2) = f_1(x_1) \cdot f(x_2 \mid x_1) \quad \Rightarrow \quad f(x_2 \mid x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

where $f_1(.)$ is marginal *p.d.f.* of X_1.

Substituting in (*), $\quad E(X_2 \mid X_1 = x_1) = \displaystyle\int_{x_2} \left\{ \frac{x_2 f(x_1, x_2)}{f_1(x_1)} \right\} dx_2, \quad \ldots (**)$

Unconditional mean of X_2 is given by :

$$E(X_2) = \int_{x_2} x_2 f_2(x_2)\, dx_2, \text{ where } f_2(\cdot) \text{ is marginal } p.d.f. \text{ of } X_2. \quad \ldots (***)$$

From (**) and (***), we conclude that the conditional mean of X_2 (given X_1) will coincide with unconditional mean of X_2 only if

$$\frac{f(x_1, x_2)}{f_1(x_1)} = f_2(x_2) \quad \Rightarrow \quad f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$$

i.e., if X_1 and X_2 are (stochastically) independent.

(*b*) $\quad f(x_1, x_2) = \begin{cases} 21\, x_1^2 x_2^3 \,; 0 < x_1 < x_2 < 1 \\ 0, \text{ otherwise} \end{cases}$

Marginal *p.d.f.* of X_2 is given by :

$$f_2(x_2) = \int_0^{x_2} f(x_1, x_2)\, dx_1 = 21\, x_2^3 \int_0^{x_2} x_1^2\, dx_1 = 21\, x_2^3 \left| \frac{x_1^3}{3} \right|_0^{x_2} = 7\, x_2^6 \,; 0 < x_2 < 1$$

$\therefore \quad$ Conditional *p.d.f.* of X_1 (given X_2) is given by

$$f_1(x_1 \mid x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} = 3\, \frac{x_1^2}{x_2^3} \,; 0 < x_1 < x_2 ;\; 0 < x_2 < 1$$

Conditional mean of X_1 is :

$$E(X_1 \mid X_2 = x_2) = \int_0^{x_2} x_1 f_1(x_1 \mid x_2)\, dx_1 = \frac{3}{x_2^3} \int_0^{x_2} x_1^3\, dx_1 = \frac{3}{x_2^3} \left| \frac{x_1^4}{4} \right|_0^{x_2} = \frac{3x_2}{4} \,; 0 < x_2 < 1$$

Now $\quad E(X_1^2 \mid X_2 = x_2) = \displaystyle\int_0^{x_2} x_1^2 f_1(x_1 \mid x_2)\, dx_1 = \frac{3}{x_2^3} \int_0^{x_2} x_1^4\, dx_1 = \frac{3}{x_2^3} \cdot \frac{x_2^5}{5} = \frac{3}{5} x_2^2$

$$\therefore \quad \text{Var}(X_1 \mid X_2 = x_2) = E(X_1^2 \mid X_2 = x_2) - \{E(X_1 \mid X_2 = x_2)\}^2$$

$$= \frac{3}{5} x_2^2 - \frac{9}{16} x_2^2 = \frac{3}{80} x_2^2 \,; 0 < x_2 < 1.$$

Example 6·33. *Two random variables X and Y have the following joint probability density function :*

$$f(x, y) = \begin{cases} 2 - x - y\,;\, 0 \le x \le 1, 0 \le y \le 1 \\ 0, \text{ otherwise} \end{cases}$$

Find (i) *Marginal probability density functions of X and Y ;*

(ii) *Conditional density functions,* (iii) *Var (X) and Var (Y); and*

(iv) *Covariance between X and Y.*

Solution. (i) $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_0^1 (2 - x - y)\, dy = \frac{3}{2} - x$

$\therefore \quad f_X(x) = \begin{cases} \frac{3}{2} - x, 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$

Similarly $\quad f_Y(y) = \begin{cases} \frac{3}{2} - y, 0 < y < 1 \\ 0, \text{otherwise} \end{cases}$

(ii) $\quad f_{X|Y}(x \mid y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{(2 - x - y)}{\left(\frac{3}{2} - y\right)},\ 0 < (x, y) < 1$

$\quad f_{Y|X}(y \mid x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{(2 - x - y)}{\left(\frac{3}{2} - x\right)},\ 0 < (x, y) < 1$

(iii) $\quad E(X) = \int_0^1 x f_X(x)\, dx = \int_0^1 x\left(\frac{3}{2} - x\right) dx = \frac{5}{12}$

$\quad E(Y) = \int_0^1 y f_Y(y)\, dy = \int_0^1 y\left(\frac{3}{2} - y\right) dy = \frac{5}{12}$

$\quad E(X^2) = \int_0^1 x^2\left(\frac{3}{2} - x\right) dx = \left|\frac{3}{6}x^3 - \frac{x^4}{4}\right|_0^1 = \frac{1}{4}$

$\quad V(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$

Similarly $\quad V(Y) = \frac{11}{144}$

(iv) $\quad E(XY) = \int_0^1 \int_0^1 xy\,(2 - x - y)\, dx\, dy = \int_0^1 \left|2\frac{x^2 y}{2} - \frac{x^3 y}{3} - \frac{x^2 y^2}{2}\right|_{x=0}^{x=1} dy$

$\quad = \int_0^1 \left(\frac{2}{3}y - \frac{1}{2}y^2\right) dy = \left|\frac{y^2}{3} - \frac{y^3}{6}\right|_0^1 = \frac{1}{6}$

$\therefore \quad \mathrm{Cov}(X, Y) = E(XY) - E(X)\,E(Y) = \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12} = \frac{-1}{144}.$

Example 6·34. *Let $f(x, y) = 8xy,\ 0 < x < y < 1$; $f(x, y) = 0$ elsewhere. Find* (a) $E(Y \mid X = x)$, (b) $E(XY \mid X = x)$, (c) $Var(Y \mid X = x)$.

Solution. $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = 8x\int_x^1 y\, dy = 4x(1 - x^2),\ 0 < x < 1$

$\quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = 8y\int_0^y x\, dx = 4y^3,\ 0 < y < 1$

$\quad f_{X|Y}(x \mid y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{2x}{y^2}, \qquad f_{Y|X}(y \mid x) = \dfrac{2y}{1 - x^2},\ 0 < x < y < 1.$

(a) $$E(Y \mid X=x) = \int_x^1 y\left(\frac{2y}{1-x^2}\right)dy = \frac{2}{3}\left(\frac{1-x^3}{1-x^2}\right) = \frac{2}{3}\left(\frac{1+x+x^2}{1+x}\right)$$

(b) $$E(XY \mid X=x) = x\,E(Y \mid X=x) = \frac{2}{3} \times \frac{x(1+x+x^2)}{(1+x)}$$

(c) $$E(Y^2 \mid X=x) = \int_x^1 y^2\left(\frac{2y}{1-x^2}\right)dy = \frac{1}{2}\left(\frac{1-x^4}{1-x^2}\right) = \frac{1+x^2}{2}$$

$$\text{Var}(Y \mid X=x) = E(Y^2 \mid X=x) - \{E(Y \mid X=x)\}^2 = \frac{1+x^2}{2} - \frac{4}{9}\cdot\frac{(1+x+x^2)^2}{(1+x)^2}$$

CHAPTER CONCEPTS QUIZ

1. Fill in the blanks :

(i) Expected value of a random variable X exists if —
(ii) If $E(X^r)$ exists then $E(X^s)$ also exists for ....
(iii) If X is a random variable, expectation of $(X - \text{constant})^2$ is minimum when the constant is ...
(iv) $E \mid X - A \mid$ is minimum when A is ...
(v) Var (c) = ..., where c is a constant.
(vi) Var $(X + c)$ = ..., where c is a constant.
(vii) Var $(aX + b)$ = ..., where a and b are constants.
(viii) If X is a $r.v.$ with mean μ and variance σ^2, then $E\left(\frac{X-\mu}{\sigma}\right)$ = ..., Var $\left(\frac{X-\mu}{\sigma}\right)$ = ...
(ix) $\{E(XY)\}^2$ ... $E(X^2) \cdot E(Y^2)$. (Inequality relationship)
(x) $V(aX \pm bY)$ = ..., where a and b are constants.
(xi) If $f(x, y) = 4xy$, for $0 < x < 1, 0 < y < 1$, then
(a) $E(Y \mid x)$ = ... (b) $V(Y \mid x)$ = ...
(xii) $P(X > Y) = 1$ implies that $E(X)$ … $E(Y)$.
(xiii) If X, Y and Z are three random variables, the covariance between X and Y for a given value of Z, *i.e.*, cov $(X, Y \mid Z)$ = ...

2. Mark the correct answer in the following :

(i) For two random variables X and Y, the relation $E(XY) = E(X)\,E(Y)$, holds good
(a) if X and Y are statistically independent,
(b) for all X and Y,
(c) if X and Y are identical.
(ii) If Var $(X) = 1$, then Var $(2X \pm 3)$ is
(a) 5 (b) 13 (c) 4
(iii) $E(X - k)^2$ is minimum when
(a) $k < E(X)$, (b) $k > E(X)$, (c) $k = E(X)$.

3. Comment on the following :

If X and Y are mutually independent variables, then
(i) $E(XY + Y + 1) - E(X + 1)\,E(Y) = 0$
(ii) X and Y are independent if and only if Cov $(X, Y) = 0$
(iii) For every univariable distribution :
(a) $V(cX) = c^2 V(X)$ (b) $E(c/X) = c/E(X)$
(iv) Expected value of a $r.v.$ always exists.
(v) If X and Y are two random variables such that their expectations exist and $P(X \le Y) = 1$, then