5·5. TWO-DIMENSIONAL RANDOM VARIABLES

So far we have defined only one random variable on a sample space. It is also possible to define more than one random variable on the same sample space. For example, we may be interested in recording the height and weight of every person in a certain educational institution. To describe such experiments mathematically we introduce the study of two random variables.

> **Definition.** *Let X and Y be two random variables defined on the same sample space S, then the function (X, Y) that assigns a point in R^2 (= R × R), is called a two-dimensional random variable.*

Let (X, Y) be a two-dimensional random variable, defined on the sample space S and $\omega \in S$. The value of (X, Y) at ω is given by the pair of real numbers $\{X(\omega), Y(\omega)\}$. The notation $\{X \le a, Y \le b\}$ denotes the event of all elements $\omega \in S$, such that $X(\omega) \le a$ and $Y(\omega) \le b$. The probability of the event $\{X \le a, Y \le b\}$ will be denoted by $P(X \le a, Y \le b)$.

Let $A = \{a < X \le b\}$, $B = \{c < Y \le d\}$ be two events. Then the event

$$\{a < X \le b, c < Y \le d\} = \{a < X \le b\} \cap \{c < Y \le d\} = A \cap B$$

$$\therefore \quad P\{a < X \le b, c < Y \le d\} = P(A \cap B)$$

Remarks 1. A two-dimensional random variable is said to be discrete if it takes at most a countable number of points in R^2.

2. Two random variables X and Y are said to be jointly distributed if they are defined on the same probability space. The sample points consist of 2-tuples. If the joint probability function is denoted by $P_{XY}(x, y)$ then the probability of a certain event E is given by:

$$P_{XY}(x, y) = P[(X, Y) \in E] \qquad \dots(5·13)$$

5·5·1. Two-dimensional or Joint Probability Mass Function. Let X and Y be random variables on a sample space S with respective image sets $X(S) = \{x_1, x_2, \dots, x_n\}$ and $Y(S) = \{y_1, y_2, \dots, y_m\}$. We make the product set

$$X(S) \times Y(S) = \{x_1, x_2, \dots, x_n\} \times \{y_1, y_2, \dots, y_m\}$$

into a probability space by defining the probability of the ordered pair (x_i, y_j) to be $P(X = x_i, y = y_j)$ which we write $p(x_i, y_j)$. The function p on $X(S) \times Y(S)$ defined by: $p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$ is called the *joint probability function* of X and Y and is usually represented in the from of the following table:

X \ Y	y_1	y_2	y_3	...	y_j	...	y_m	*Total*
x_1	p_{11}	p_{12}	p_{13}	...	p_{1j}	...	p_{1m}	$p_{1\cdot}$
x_2	p_{21}	p_{22}	p_{23}	...	p_{2j}	...	p_{2m}	$p_{2\cdot}$
x_3	p_{31}	p_{32}	p_{33}	...	p_{3j}	...	p_{3m}	$p_{3\cdot}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_i	p_{i1}	p_{i2}	p_{i3}	...	p_{ij}	...	p_{im}	$p_{i\cdot}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_n	p_{n1}	p_{n2}	p_{n3}	...	p_{nj}	...	p_{nm}	$p_{n\cdot}$
Total	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	...	$p_{\cdot j}$	...	$p_{\cdot m}$	1

Definition. *If (X, Y) is a two-dimensional discrete random variable, then the joint discrete function of X, Y, also called the joint probability mass function of X, Y, denoted by $p_{X,Y}$ is defined as:*

$$p_{XY}(x_i, y_j) = P(X = x_i, Y = y_j) \text{ for a value } (x_i, y_j) \text{ of } (X, Y)$$

and

$$p_{XY}(x_i, y_j) = 0, \text{ otherwise.} \qquad \dots (5\cdot14)$$

Remark. It may be noted that $\sum\sum p_{XY}(x_i, y_j) = 1$, where the summation is taken over all possible values of (X, Y).

Marginal Probability Function. Let (X, Y) be a discrete two-dimensional *r.v.* which takes up countable number of values (x_i, y_j). Then the probability distribution of X is determined as follows :

$$\begin{aligned} p_X(x_i) &= P(X = x_i) \\ &= P(X = x_i \cap Y = y_1) + P(X = x_i \cap Y = y_2) + \dots \dots + P(X = x_i \cap Y = y_m) \\ &= p_{i1} + p_{i2} + \dots + p_{ij} + \dots + p_{im} = \sum_{j=1}^{m} p_{ij} = \sum_{j=1}^{m} p(x_i, y_j) = p_{i\cdot} \end{aligned} \qquad \dots (5\cdot14a)$$

and is known as *marginal probability mass function or discrete marginal density function* X.

Also $\quad \sum_{i=1}^{n} p_{i\cdot} = p_{1\cdot} + p_{2\cdot} + \dots + p_{n\cdot} = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij} = 1$

Similarly, we can prove that

$$p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = \sum_{i=1}^{n} p(x_i, y_j) = p_{\cdot j} \qquad \dots (5\cdot14\,b)$$

which is the *marginal probability mass function of* Y.

Conditional Probability Function

Definition. *Let (X, Y) be a discrete two-dimensional random variable. Then the conditional discrete density function or the conditional probability mass function of X, given Y = y, denoted by $f_{X|Y}(x|y)$, is defined as :*

$$p_{X|Y}(x|y) = \frac{P(X = x, Y = y)}{P(Y = y)}, \text{ provided } P(Y = y) \neq 0 \qquad \dots (5\cdot14\,c)$$

Since for a fixed y,

$$\sum_i \frac{P(X = x_i, Y = y)}{P(Y = y)} = \frac{1}{P(Y = y)} \sum_i P(X = x_i, Y = y) = \frac{1}{P(Y = y)} P(Y = y) = 1,$$

it follows that the conditional discrete density function $f_{X|Y}(x|y)$ is a discrete density function, when considered as a function of the values of X.

The conditional probability mass function $p_{Y|X}(y|x)$ is similarly defined , *i.e.*,

$$p_{Y|X}(y|x) = \frac{P(X = x, Y = y)}{P(X = x)}.$$

A necessary and sufficient condition for the discrete random variables X and Y to be *independent* is :

$$P(X = x_i, Y = y_j) = P(X = x_i)\,P(Y = y_j) \text{ for all values } (x_i, y_j) \text{ of } (X, Y) \qquad \dots (5\cdot14\,d)$$

5·5·2. Two-dimensional Distribution Function

Definition. *The distribution function of the two-dimensional random variable (X, Y) is a real valued function F defined for all real x and y by the relation :*

$$F_{XY}(x, y) = P(X \le x, Y \le y). \quad \ldots (5\cdot 15)$$

Properties of Joint Distribution Function

1. (i) For the real numbers a_1, b_1, a_2 and b_2,

$$P(a_1 < X \le b_1, a_2 < Y \le b_2) = F_{XY}(b_1, b_2) + F_{XY}(a_1, a_2) - F_{XY}(a_1, b_2) - F_{XY}(b_1, a_2)$$

[For proof see Example 5·46]

(ii) Let $a_1 < a_2, b_1 < b_2$, then $(X \le a_1, Y \le a_2) + (a_1 < X \le b_1, Y \le a_2) = (X \le b_1, Y \le a_2)$ and the events on the L.H.S. are mutually exclusive.

$\therefore$ $F(a_1, a_2) + P(a_1 < X \le b_1, Y \le a_2) = F(b_1, a_2) \Rightarrow F(b_1, a_2) - F(a_1, a_2) = P(a_1 < X \le b_1, Y \le a_2)$

$\therefore$ $F(b_1, a_2) \ge F(a_1, a_2)$ [since $P(a_1 < X \le b_1, Y \le a_2) \ge 0$]

Similarly it follows that : $F(a_1, b_2) - F(a_1, a_2) = P(X \le a_1, a_2 \le Y \le b_2)$

$\therefore$ $F(a_1, b_2) \ge F(a_1, a_2)$, which shows that $F(x, y)$ is monotonic non-decreasing function.

2. $F(-\infty, y) = 0 = F(x, -\infty), \quad F(-\infty, +\infty) = 1.$

3. If the density function $f(x, y)$ is continuous at (x, y), $\dfrac{\partial^2 F}{\partial x \, \partial y} = f(x, y)$

5·5·3. Marginal Distribution Functions. From the knowledge of joint distribution function $F_{XY}(x, y)$, it is possible to obtain the individual distribution functions, $F_X(x)$ and $F_Y(y)$ which are termed as marginal distribution function of X and Y respectively with respect to the joint distribution function $F_{XY}(x, y)$.

$$F_X(x) = P(X \le x) = P(X \le x, Y < \infty) = \lim_{y \to \infty} F_{XY}(x, y) = F_{XY}(x, \infty)$$

Similarly, $$F_Y(y) = P(Y \le y) = P(X < \infty, Y \le y) = \lim_{x \to \infty} F_{XY}(x, y) = F_{XY}(\infty, y) \quad \ldots (5\cdot 16)$$

$\ldots (5\cdot 16\,a)$

$F_X(x)$ is termed as the marginal distribution function of X corresponding to the joint distribution function $F_{XY}(x, y)$ and similarly $F_Y(y)$ is called marginal distribution function of the random variable Y corresponding to the joint distribution function $F_{XY}(x, y)$.

In the case of jointly discrete random variables, the marginal distribution functions are given as :

$$F_X(x) = \sum_y P(X \le x, Y = y), \text{ and } F_Y(y) = \sum_x P(X = x, Y \le y)$$

Similarly in the case of jointly continuous random variable, the marginal distribution functions are given as :

$$F_X(x) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y) \, dy \right\} dx, \qquad F_Y(y) = \int_{-\infty}^{y} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y) \, dx \right\} dy$$

5·5·4. Joint Density Function, Marginal Density Function. From the joint distribution function $F_{XY}(x, y)$ of two-dimensional continuous random variable, we get the joint probability density function by differentiation as follows :

$$f_{XY}(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \lim_{\delta x \to 0, \delta y \to 0} \frac{P(x \le X \le x + \delta x, y \le Y \le y + \delta y)}{\delta x \, \delta y} \quad \ldots (5\cdot 17)$$

Or it may be expressed in the following way also :

"The probability that the point (x, y) will lie in the infinitesimal rectangular region, of area $dx\,dy$ is given by

$$P\left(x-\tfrac{1}{2}dx \le X \le x+\tfrac{1}{2}dx,\ y-\tfrac{1}{2}dy \le Y \le y+\tfrac{1}{2}dy\right) = dF_{XY}(x, y) \quad \ldots (5\cdot17\ a)$$

and is denoted by $f_{XY}(x, y)\,dx\,dy$, where the function $f_{XY}(x, y)$ is called the joint probability density function of X and Y.

The marginal probability function of X and Y are given respectively as follows :

$$f_X(x) = \begin{cases} \sum\limits_y p_{XY}(x, y), & \text{(for discrete variables)} \\ \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy, & \text{(for continuous variables)} \end{cases} \quad \ldots (5\cdot17\ b)$$

and

$$f_Y(y) = \begin{cases} \sum\limits_x p_{XY}(x, y), & \text{(for discrete variables)} \\ \int_{-\infty}^{\infty} f_{XY}(x, y)\,dx, & \text{(for continuous variables)} \end{cases} \quad \ldots (5\cdot17\ c)$$

The marginal density functions of X and Y can be obtained in the following manner also :

$$f_X(x) = \frac{dF_X(x)}{dx} = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy$$

and

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dx \quad \ldots (5\cdot17\ d)$$

Important Remark. If we know the joint *p.d.f.* (*p.m.f.*) $f_{XY}(x, y)$ of two random variables X and Y, we can obtain the individual distributions of X and Y in the form of their marginal *p.d.f.'s* (*p.m.f.'s*) $f_X(x)$ and $f_Y(y)$ by using (5·17 *b*) and (5·17 *c*). However, the converse is not true, *i.e.*, *from the marginal distributions of two jointly distributed random variables, we cannot determine the joint distributions of these two random variables.*

To verify this, it will suffice to show that two different joint *p.m.f.'s* (*p.d.f.'s*) have the same marginal distribution for X and the same marginal distribution for Y. We give below two joint discrete probability distributions which have the same marginal distributions.

JOINT DISTRIBUTIONS HAVING SAME MARGINALS

Probability Distribution I

Y \ X	0	1	$f_Y(y)$
0	0·28	0·37	0·65
1	0·22	0·13	0·35
$f_X(x)$	0·50	0·50	1·00

Probability Distribution II

Y \ X	0	1	$f_Y(y)$
0	0·35	0·30	0·65
1	0·15	0·20	0·35
$f_X(x)$	0·50	0·50	1·00

As an illustration for continuous *r.v.'s*, let (X, Y) be continuous *r.v.* with joint *p.d.f.* :

$$f_{XY}(x, y) = x + y\ ;\ 0 \le (x, y) \le 1 \quad \ldots (5\cdot17\ e)$$

The marginal $p.d.f.$'s of X and Y are given by :

$$f_X(x) = \int_0^1 f(x, y)\, dy = \int_0^1 (x + y)\, dy = \left| xy + \frac{y^2}{2} \right|_0^1$$

$$\left.\begin{aligned} \Rightarrow \quad & f_X(x) = x + \tfrac{1}{2} && ; 0 \le x \le 1 \\ \text{Similarly} \quad & f_Y(y) = \int_0^1 f(x, y)\, dx = y + \tfrac{1}{2} ; && 0 \le y \le 1 \end{aligned}\right\} \quad \text{... (5·17 } f)$$

Consider another continuous joint $p.d.f.$:

$$g(x, y) = \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) ; 0 \le (x, y) \le 1 \quad \text{... (5·17 } g)$$

Then marginal $p.d.f.$'s of X and Y are given by :

$$g_1(x) = \int_0^1 g(x, y)\, dy = \left(x + \frac{1}{2}\right) \int_0^1 \left(y + \frac{1}{2}\right) dy = \left(x + \frac{1}{2}\right) \left| \frac{y^2}{2} + \frac{1}{2} y \right|_0^1$$

$$\left.\begin{aligned} \Rightarrow \quad & g_1(x) = x + \tfrac{1}{2} ; \; 0 \le x \le 1 \\ \text{Similarly} \quad & g_2(y) = y + \tfrac{1}{2} ; \; 0 \le y \le 1 \end{aligned}\right\} \quad \text{... (5·17 } h)$$

(5·17 f) and (5·17 h) imply that the two joint $p.d.f.$'s in (5·17 e) and (5·17 g) have the same marginal $p.d.f.$'s (5·17 f) or (5·17 h).

5·5·5. The Conditional Distribution Function and Conditional Probability Density Function. For two-diamensional random variable (X, Y), the joint distribution function $F_{XY}(x, y)$ for any real numbers x and y is given by :

$$F_{XY}(x, y) = P(X \le x, Y \le y)$$

Now let A be the event $(Y \le y)$ such that the event A is said to occur when Y assumes values upto and inclusive of y. Using conditional probabilities we may now write

$$F_{XY}(x, y) = \int_{-\infty}^{x} P(A \mid X = x)\, dF_X(x) \quad \text{... (5·18)}$$

The *conditional distribution function* $F_{Y|X}(y \mid x)$ denotes the distribution function of Y when X has already assumed the particular value x. Hence

$$F_{Y|X}(y \mid x) = P(Y \le y \mid X = x) = P(A \mid X = x)$$

Using this expression, the joint distribution function $F_{XY}(x, y)$ may be expressed in terms of the conditional distribution function as follows :

$$F_{XY}(x, y) = \int_{-\infty}^{x} F_{Y|X}(y \mid x)\, dF_X(x) \quad \text{... (5·18 } a)$$

$$\text{Similarly} \quad F_{XY}(x, y) = \int_{-\infty}^{y} F_{X|Y}(x \mid y)\, dF_Y(y) \quad \text{... (5·18 } b)$$

The *conditional probability density function* of Y given X for two random variables X and Y which are jointly continuously distributed is defined as follows, for two real numbers x and y :

$$f_{Y|X}(y \mid x) = \frac{\partial}{\partial y} F_{Y|X}(y \mid x) \quad \text{... (5·19)}$$

Remarks 1. $f_X(x) > 0$, then $f_{Y|X}(y|x) = \dfrac{f_{XY}(x, y)}{f_X(x)}$

Proof. We have

$$f_{XY}(x, y) = \int_{-\infty}^{x} f_{Y|X}(y|x)\ dF_X(x) = \int_{-\infty}^{x} f_{Y|X}(y|x) f_X(x)\,dx$$

Differentiating w.r. to x, $\qquad \dfrac{\partial}{\partial x} F_{XY}(x, y) = F_{Y|X}(y \mid x) f_X(x)$

Differentiating w.r. to y, we get $\qquad \dfrac{\partial}{\partial y}\left\{\dfrac{\partial}{\partial x} F_{XY}(x, y)\right\} = f_{Y|X}(y \mid x) f_X(x)$

$$f_{XY}(x, y) = f_{Y|X}(y|x) f_X(x)$$

$$\Rightarrow \qquad f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

2. If $f_Y(y) > 0$, then $\qquad f_{X|Y}(x|y) = \dfrac{f_{XY}(x, y)}{f_Y(y)}$

3. In terms of the differentials, we have

$$P(x < X \le x + dx \mid y < Y \le y + dy) = \frac{P(x < X \le x + dx,\ y < Y \le y + dy)}{P(y < Y \le y + dy)}$$

$$= \frac{f_{XY}(x, y)\,dx\,dy}{f_Y(y)\,dy} = f_{X|Y}(x \mid y)\,dx$$

Hence $f_{X|Y}(x|y)$ may be interpreted as the conditional density function of X on the assumption $Y = y$.

5·5·6. Stochastic Independence. Let us consider two random variables X and Y (of discrete or continuous type) with joint *p.d.f.* (*p.m.f.*) $f_{XY}(x, y)$ and marginal *p.d.f.'s* (*p.m.f.'s*) $f_X(x)$ and $g_Y(y)$ respectively. Then by the compound probability theorem

$$f_{XY}(x,y) = f_X(x)\, g_{Y|X}(y|x)$$

where $g_{Y|X}(y|x)$ is the conditional *p.d.f.* of Y for given value of X = x.

If we assume that $g(y|x)$ does not depend on x, then by the definition of marginal *p.d.f.'s*, we get for continuous *r.v.'s* :

$$g(y) = \int_{-\infty}^{\infty} f(x, y)\,dx = \int_{-\infty}^{\infty} f_X(x)\, g(y \mid x)\,dx$$

$$= g(y \mid x)\int_{-\infty}^{\infty} f_X(x)\,dx \qquad [\text{since } g(y \mid x) \text{ does not depend on } x]$$

$$= g(y \mid x) \qquad [\because f(.) \text{ is } p.d.f. \text{ of X}]$$

Hence

$$g(y) = g(y \mid x) \text{ and } f_{XY}(x, y) = f_X(x)\, g_Y(y) \qquad \ldots(*)$$

provided $g(y \mid x)$ does not depend on x. This motivates the following definition of independent random variables.

Independent Random Variables. *Two r.v.'s X and Y with joint p.d.f. (p.m.f.) $f_{XY}(x,y)$ and marginal p.d.f.'s (p.m.f's) $f_X(x)$ and $g_Y(y)$ respectively are said to be stochastically independent if and only if* $\qquad f_{X,Y}(x, y) = f_X(x)\, g_Y(y) \qquad \ldots(5\cdot20)$

Remarks 1. In terms of the distribution function, we have the following definition :

Two jointly distributed random variables X and Y are stochastically independent if and only if their joint distribution function $F_{X,Y}(\cdot,\cdot)$ is the product of their marginal distribution function $F_X(\cdot)$ and $G_Y(\cdot)$, *i.e.*, if for real (x, y)

$$F_{X,Y}(x, y) = F_X(x)\, G_Y(y) \quad \ldots (5\cdot 20\,a)$$

2. The variables which are not stochastically independent are said to be stochastically dependent.

Theorem 5·2. *Two random variables X and Y with joint p.d.f. f(x, y) are stochastically independent if and only if $f_{X,Y}(x, y)$ can be expressed as the product of a non-negative function of x alone and a non-negative function of y alone, i.e., if*

$$f_{XY}(x, y) = h_X(x) . k_Y(y), \text{ where } h(.) \geq 0 \text{ and } k(.) \geq 0. \quad \ldots (5\cdot 20\,b)$$

Proof. If X and Y are independent, then by *def.*, $f_{X,Y}(x, y) = f_X(x) . g_Y(y)$,

where $f(x)$ and $g(y)$ are marginal *p.d.f.'s* of X and Y respectively. Thus condition (5·20*b*) is satisfied.

Conversely if (5·20*b*) holds, then we have to prove that X and Y are independent. For continuous random variables X and Y, the marginal *p.d.f.'s* are given by :

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{-\infty}^{\infty} h(x)\, k(y)\, dy = h(x) \int_{-\infty}^{\infty} k(y)\, dy = c_1 h(x), \text{ say} \ldots (*)$$

and
$$g_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_{-\infty}^{\infty} h(x)\, k(y)\, d\,x = k(y) \int_{-\infty}^{\infty} h(x)\, dx = c_2 k(y), \text{ say} \ldots (**)$$

where c_1 and c_2 are constants independent of x and y. Moreover,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)\, k(y)\, dx\, dy = 1$$

$$\Rightarrow \quad \left(\int_{-\infty}^{\infty} h(x)\, dx\right)\left(\int_{-\infty}^{\infty} k(y)\, dy\right) = 1 \quad \Rightarrow c_2 c_1 = 1 \quad [\text{From } (**) \text{ and } (*)] \quad \ldots (***)$$

Finally, we get

$$\begin{aligned} f_{XY}(x, y) &= h_X(x)\, k_Y(y) = c_1 c_2\, h_X(x)\, k_Y(y) && [\text{Using } (***)] \\ &= \{c_1 h_X(x)\}\, \{c_2 k_Y(y)\} \\ &= f_X(x)\, g_Y(y) && [\text{From } (*) \text{ and } (**)] \end{aligned}$$

$\Rightarrow$ X and Y are stochastically independent.

Theorem 5·3. *If the random variables X and Y are stochastically independent, then for all possible selections of the corresponding pairs of real numbers (a_1, b_1), (a_2, b_2) where $a_i \leq b_i$ for all i = 1, 2 and where the values $\pm\infty$ are allowed, the events $(a_1 < X \leq b_1)$ and $(a_2 < Y \leq b_2)$ are independent, i.e.,*

$$P\,\{(a_1 < X \leq b_1) \cap (a_2 < Y \leq b_2)\} = P(a_1 < X \leq b_1)\; P\,(a_2 < Y \leq b_2)$$

Proof. Since X and Y are stochastically independent, in the usual notations :

$$f_{X,Y}(x, y) = f_X(x)\, g_Y(y) \quad \ldots (*)$$

In case of continuous *r.v.*'s, we have

$$\begin{aligned} P\,[(a_1 < X \leq b_1) \cap (a_2 < Y \leq b_2)] &= \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x, y)\, dx\, dy \\ &= \left(\int_{a_1}^{b_1} f_X(x)\, dx\right)\left(\int_{a_2}^{b_2} g_Y(y)\, dy\right) && [\text{From } (*)] \end{aligned}$$

$= P(a_1 < X \le b_1)\, P(a_2 < Y \le b_2)$, as desired.

$\Rightarrow$ The events : $(a_1 < X \le b_1)$ and $(a_2 < Y \le b_2)$ are indepndent.

Remark. In case of discrete r.v.'s theorems 5·2 and 5·3 can be proved on replacing integration by summation over the given range of the variables.

5·5·7. Generalisation to n-Dimensional Random Variable. The concept of two-dimensional random variables and their joint and marginal distributions is § 5·5 to § 5·5·6 can be easily generalised to the case of n-dimensional random variable.

Joint and Marginal Probability Mass Function.

Let $(X_1, X_2, \ldots, X_n)$ be a discrete n-dimensional r.v., assuming discrete values, in some region, say, R^n of the n-dimensional space. Then the joint *p.m.f.* of $(X_1, X_2, \ldots, X_n)$ is defined as :

$$p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P\,[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n]$$

$$= P\left[\bigcap_{i=1}^{n} (X_i = x_i)\right] \qquad \ldots(5·21)$$

where,

(i) $p(x_1, x_2, \ldots, x_n) \;\ge\; 0,\ \forall\,(x_1, x_2, \ldots, x_n) \in R^n$, and

(ii) $\sum_{x_1, x_2, \ldots, x_n} p(x_1, x_2, \ldots, x_n) = 1$

The marginal *p.m.f.* of any r.v., say, X_i, is obtained on summing $p(x_1, x_2, \ldots, x_n)$, over the values of all other variables except X_i. Thus,

$$P_{X_i}(x_i) = \sum_{\left(\substack{x_1, x_2, \ldots, x_n \\ \text{except } x_i}\right)} p(x_1, x_2, \ldots, x_n) \qquad \ldots(5·21\ a)$$

In particular, if $p(x_1, x_2, x_3)$ is the joint *p.m.f.* of three r.v.'s X_1, X_2 and X_3, then the marginal *p.m.f.* of, say, X_1 is given by :

$$P_{X_1}(x) = \sum_{x_2, x_3} p(x_1, x_2, x_3), \qquad \ldots(5·21\ b)$$

and so on.

As, in the case of two random variables, *the r.v.'s $X_1, X_2, \ldots, X_n$ are independent if and only if their joint p.m.f. is equal to the product of their marginal p.m.f.'s, i.e., iff :*

$$P_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = p_{X_1}(x_1) \cdot p_{X_2}(x_2) \ldots p_{X_n}(x_n) \qquad \ldots(5·21\ c)$$

Joint and marginal Probability Density Function.

Let $(X_1, X_2, \ldots, X_n)$ be n-dimensional continuous r.v. assuming all the values in some region, say, R_1^n of the n-dimensional space. Then the joint *p.d.f.* of $(X_1, X_2, \ldots, X_n)$ is given by :

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \lim_{dx_1 \to 0,\, dx_2 \to 0, \ldots,\, dx_n \to 0} \frac{P\left[\bigcap_{i=1}^{n} (x_i < X_i < x_i + dx_i)\right]}{dx_1 \cdot dx_2 \ldots dx_n} \qquad \ldots(5·21d)$$

where :

(i) $f(x_1, x_2, \ldots, x_n) \ge 0,\ \forall\,(x_1, x_2, \ldots, x_n) \in R_1^n$, and

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n = 1$

The marginal *p.d.f.* of any variable, say, X_i, is obtained on integrating the joint *p.d.f.* over the range of all the variables except X_i. Thus,

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_1 \ldots dx_{i-1}\, dx_{i+1} \ldots dx_n \quad \ldots (5{\cdot}21\, e)$$

In particular, for three *r.v.'s* X_1, X_2, X_3 with joint *p.d.f.* $f(x_1, x_2, x_3)$, the marginal *p.d.f.* of, say, X_2 is given by :

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, dx_1\, dx_3 \quad \ldots (5{\cdot}21\, f)$$

and so on.

The necessary and sufficient condition for the independence of r.v.'s $X_1, X_2, \ldots, X_n$ is that their joint p.d.f. is the product of their marginal p.d.f.'s i.e.,

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) \quad \ldots (5{\cdot}21\, g)$$

Example 5·30. *In the random placement of three balls in three cells, describe the possible outcomes of the experiment. Let X_i denote the number of balls in cell i; $i = 1, 2, 3$; and N, the number of cells occupied. Obtain the joint distribution of : (a) (X_1, N) and (b) (X_1, X_2).*

Solution. (*a*) Let the three balls be denoted by a, b and c. Then the possible outcomes of placing the three balls in three cells are as follows :

1. {a | b | c}	10. {ac | b | –}	19. {b | ca | –}
2. {a | c | b}	11. {ac | – | b}	20. {b | – | ca}
3. {b | a | c}	12. {– | ac | b}	21. {– | b | ca}
4. {b | c | a}	13. {bc | a | –}	22. {c | ab | –}
5. {c | a | b}	14. {bc | – | a}	23. {c | – | ab}
6. {c | b | a}	15. {– | bc | a}	24. {– | c | ab}
7. {ab | c | –}	16. {a | bc | –}	25. {abc | – | –}
8. {ab | – | c}	17. {a | – | bc}	26. {– | abc | –}
9. {– | ab | c}	18. {– | a | bc}	27. {– | – | abc}

Each of these arrangements represents a sample event, *i.e.,* a sample point. The sample space contains 27 points.

Let N denote the number of occupied cells. The favourable cases for $N = 1$ are at numbers 25, 26 and 27, *i.e.,* 3; for $N = 2$ are at numbers 7 to 24, *i.e.,* 18; and for $N = 3$ are at numbers 1 to 6, *i.e.,* 6. Accordingly, the probability distribution of N is :

$$P(N = 1) = \frac{3}{27}, \quad P(N = 2) = \frac{18}{27}, \quad P(N = 3) = \frac{6}{27}.$$

Let X_1 denote the number of balls placed in the first cell. Then from the above table of sample points, we get

$$P(X_1 = 0) = \frac{8}{27}, \quad P(X_1 = 1) = \frac{12}{27}, \quad P(X_1 = 2) = \frac{6}{27} \quad \text{and } P(X_1 = 3) = \frac{1}{27}.$$

The joint distribution of N and X_1 can be obtained as follows :

$P(N=1, X_1=0) = \frac{2}{27}$, $P(N=1, X_1=1)=0$, $P(N=1, X_1=2)=0$,

$P(N=1, X_1=3) = \frac{1}{27}$, $P(N=2, X_1=0)=\frac{6}{27}$, $P(N=2, X_1=1)=\frac{6}{27}$,

$P(N=2, X_1=2) = \frac{6}{27}$, $P(N=2, X_1=3)=0$, $P(N=3, X_1=0)=0$,

$P(N=3, X_1=1) = \frac{6}{27}$, and $P(N=3, X_1=2)=0$, $P(N=3, X_1=3)=0$.

JOINT DISTRIBUTION OF N AND X_1

X_1 \ N	1	2	3	Distribution of X_1
0	$\frac{2}{27}$	$\frac{6}{27}$	0	$\frac{8}{27}$
1	0	$\frac{6}{27}$	$\frac{6}{27}$	$\frac{12}{27}$
2	0	$\frac{6}{27}$	0	$\frac{6}{27}$
3	$\frac{1}{27}$	0	0	$\frac{1}{27}$
Distribution of N	$\frac{3}{27}$	$\frac{18}{27}$	$\frac{6}{27}$	1

(ii) Proceeding on the same lines, the joint distribution of X_1 and X_2 can be obtained as shown in the adjoining table :

JOINT DISTRIBUTION OF X_1 and X_2

X_1 \ X_2	0	1	2	3	Distribution of X_1
0	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{3}{27}$	$\frac{1}{27}$	$\frac{8}{27}$
1	$\frac{3}{27}$	$\frac{6}{27}$	$\frac{3}{27}$	0	$\frac{12}{27}$
2	$\frac{3}{27}$	$\frac{3}{27}$	0	0	$\frac{6}{27}$
3	$\frac{1}{27}$	0	0	0	$\frac{1}{27}$
Distribution of X_2	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$	1

Example 5·31. *A random observation on a bivariate population (X, Y) can yield one of the following pairs of values with probabilities noted against them :*

For each observation pair	Probability
(1, 1) ; (2, 1) ; (3, 3) ; (4, 3)	$\frac{1}{20}$
(3, 1) ; (4, 1) ; (1, 2) ; (2, 2) ; (3, 2) ; (4, 2) ; (1, 3) ; (2, 3)	$\frac{1}{10}$

Find the probability that Y = 2 given that X = 4. Also find the probability that Y = 2. Examine if the two events X = 4 and Y = 2 are independent.

Solution. $P(Y=2) = P[(1, 2) \cup (2, 2) \cup (3, 2) \cup (4, 2)] = \frac{4}{10} = \frac{2}{5}$.

$$P(X=4) = P[(4, 1) \cup (4, 2) \cup (4, 3)] = \frac{1}{10} + \frac{1}{10} + \frac{1}{20} = \frac{1}{4}$$

$$P(X=4, Y=2) = P[(4, 2)] = \frac{1}{10}$$

$$P(Y=2 \mid X=4) = \frac{P(X=4 \cap Y=2)}{P(X=4)} = \frac{1/10}{1/4} = \frac{2}{5}$$

Now $\quad P(X=4) \cdot P(Y=2) = \frac{1}{4} \cdot \frac{2}{5} = \frac{1}{10} \quad = P(X=4 \cap Y=2)$

Hence the events $X=4$ and $Y=2$ are independent.

Problem 5·32. *The joint probability distribution of two random variables X and Y is given by : $P(X=0, Y=1) = \frac{1}{3}$, $P(X=1, Y=-1) = \frac{1}{3}$, and $P(X=1, Y=1) = \frac{1}{3}$.*

Find (i) Marginal distributions of X and Y, and (ii) the conditional probability distribution of X given Y = 1.

Solution. $P(X=-1)$

$= \sum_y P(X=-1, Y=y)$

$= P(X=-1, Y=-1)$

$+ P(X=-1, Y=0)$

$+ P(X=-1, Y=1) = 0$

Similarly $P(X=0) = \frac{1}{3}$

and $\quad P(X=1) = \frac{2}{3}$

Y \ X	-1	0	1	Marginal Y
-1	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	0
1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
Marginal (X)	0	$\frac{1}{3}$	$\frac{2}{3}$	1

Thus

Marginal distribution of X is :

Values of X, x :	-1	0	1
$P(X=x)$:	0	$\frac{1}{3}$	$\frac{2}{3}$

Marginal distribution of Y is :

Values of Y, y :	-1	0	1
$P(Y=y)$:	$\frac{1}{3}$	0	$\frac{2}{3}$

(*ii*) The conditional probability distribution of X given Y is :

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}. \text{ Now}$$

$$P(X=-1 \mid Y=1) = \frac{P(X=-1, Y=1)}{P(Y=1)} = 0, \; P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}$$

Thus the conditional distribution of X given $Y=1$ is :

Values of $X=x$	-1	0	1
$P(X=x \mid Y=1)$	0	$\frac{1}{2}$	$\frac{1}{2}$

Example 5·33. *For the adjoining bivariate probability distribution of X and Y, find :*

(i) $P(X \le 1, Y = 2)$,

(ii) $P(X \le 1)$,

(iii) $P(Y \le 3)$, *and*

(iv) $P(X < 3, Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution. The marginal distributions are given below :

X \ Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

(i) $P(X \le 1, Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2) = 0 + \frac{1}{16} = \frac{1}{16}$

(ii) $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

(iii) $P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$

(iv) $P(X < 3, Y \le 4) = P(X = 0, Y \le 4) + P(X = 1, Y \le 4) + P(X = 2, Y \le 4)$

$$= \left(\frac{1}{32} + \frac{2}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}\right) = \frac{9}{16}.$$

Example 5·34. *For the joint probability distribution of two random variables X and Y given below :*

X \ Y	1	2	3	4	Total
1	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{10}{36}$
2	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{9}{36}$
3	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{8}{36}$
4	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{9}{36}$
Total	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	1

Find (i) the marginal distributions of X and Y, and (ii) conditional distribution of X given the value of Y = 1 and that of Y given the value of X = 2.

Solution. The marginal distribution of X is defined as :

$$P(X = x) = \sum_y P(X = x, Y = y)$$

$\therefore \quad P(X=1) = \sum_y P(X=1, Y=y)$

$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$

$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}.$

Similarly $P(X=2) = \sum_y P(X=2, Y=y) = \frac{9}{36}$; $P(X=3) = \sum_y P(X=3, Y=y) = \frac{8}{36}$

and $\quad P(X=4) = \sum_y P(X=4, Y=y) = \frac{9}{36}.$

Similarly, we can obtain the marginal distribution of Y.

MARGINAL DISTRIBUTION OF X

Values of X, x	1	2	3	4
$P(X=x)$	$\frac{10}{36}$	$\frac{9}{36}$	$\frac{8}{36}$	$\frac{9}{36}$

MARGINAL DISTRIBUTION OF Y

Values of Y, y	1	2	3	4
$P(Y=y)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$

(*ii*) The conditional probability function of X given Y is defined as follows :

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}. \text{ Therefore}$$

$$\therefore \quad P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{4/36}{11/36} = \frac{4}{11}$$

$$P(X=2 \mid Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

$$P(X=3 \mid Y=1) = \frac{P(X=3, Y=1)}{P(Y=1)} = \frac{5/36}{11/36} = \frac{5}{11}$$

$$P(X=4 \mid Y=1) = \frac{P(X=4, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

Hence the conditional distribution of X given $Y=1$ is :

x :	1	2	3	4
$P(X=x \mid Y=1)$:	$\frac{4}{11}$	$\frac{1}{11}$	$\frac{5}{11}$	$\frac{1}{11}$

Similarly, we can obtain the conditional distribution of Y for $X=2$ as given below :

y :	1	2	3	4
$P(Y=y \mid X=2)$:	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{9}$

Example 5·35. *A two-dimensional r.v. (X, Y) have a bivariate distribution given by :*

$$P(X=x, Y=y) = \frac{x^2+y}{32}, \text{ for } x = 0, 1, 2, 3 \text{ and } y = 0, 1.$$

Find the marginal distributions of X and Y.

(b) A two-dimensional r.v. (X, Y) have a joint probability mass function :

$$p(x, y) = \frac{1}{27}(2x+y), \text{ where } x \text{ and } y \text{ can assume only the integer values 0, 1 and 2.}$$

Find the conditional distribution of Y for X = x.

Solution. (a) We have

Y \ X	0	1	2	3	Marginal distribution of Y, $P(Y=y)$
0	0	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{9}{32}$	$\frac{14}{32}$
1	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{18}{32}$
Marginal distribution of X, $P(X=x)$	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{9}{32}$	$\frac{19}{32}$	1

The marginal probability distribution of X is given by :

$P(X=x)=\sum_y P(X=x, Y=y)$ and is tabulated in last row of above table.

The marginal probability distribution of Y is given by :

$P(Y=y)=\sum_x P(X=x, Y=y)$ and is tabulated in last column of above table.

(b) The joint probability function :

$$p_{XY}(x,y)=\frac{1}{27}(2x+y)\ ;\ x=0,1,2;\ y=0,1,2$$

gives the following table of joint probability distribution of X and Y.

JOINT PROBABILITY DISTRIBUTION $p(x,y)$ OF X AND Y

X \ Y	0	1	2	$f_X(x)$
0	0	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{3}{27}$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$\frac{9}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{15}{27}$

For example, $p(0,0)=\frac{1}{27}(0+2\times 0)=0,\quad p(1,0)=\frac{1}{27}(0+2\times 1)=\frac{2}{27}$;

$p(2,0)=\frac{1}{27}(0+2\times 2)=\frac{4}{27}$; and so on.

CONDITIONAL DISTRIBUTION OF Y FOR $X=x$

Y \ X	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$
2	$\frac{4}{15}$	$\frac{5}{15}$	$\frac{6}{15}$

The conditional distribution of Y for $X=x$ is given by :

$P_{Y|X}(Y=y \mid X=x)=\frac{p_{XY}(x,y)}{p_X(x)}$ and is obtained in the adjoining table.

Example 5·36. *Two discrete random variables X and Y have the joint probability density function :*

$$p_{XY}(x,y)=\frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!\,(x-y)!}\ ,\ y=0,1,2,\ldots,x\ ;\ x=0,1,2,\ldots,$$

where λ, p are constants with $\lambda>0$ and $0<p<1$.

Find (i) The marginal probability density functions of X and Y.
(ii) The conditional distribution of Y for a given X and of X for a given Y.

Solution. (*i*) The marginal *p.m.f.* of *X* is given by :

$$p_X(x) = \sum_{y=0}^{x} p(x, y) = \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!} = \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} \frac{x!\, p^y (1-p)^{x-y}}{y!(x-y)!}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} {}^xC_y\, p^y (1-p)^{x-y} = \frac{\lambda^x e^{-\lambda}}{x!} \left[p + (1-p)\right]^x = \frac{\lambda^x e^{-\lambda}}{x!},\ x = 0, 1, 2, \ldots$$

which is the probability function of a Poisson distribution with parameter λ.

$$p_Y(y) = \sum_{x=y}^{\infty} p(x,y) = \sum_{x=y}^{\infty} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!} = \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!}$$

$$[\because y = 0, 1, 2, \ldots, x \Rightarrow x \le y \Rightarrow x \ge y]$$

$$= \frac{(\lambda p)^y e^{-\lambda}}{y!} e^{\lambda(1-p)} = \frac{e^{-\lambda p} (\lambda p)^y}{y!};\ y = 0, 1, 2, \ldots$$

which is the probability function of a Poisson distribution with parameter λp.

(*ii*) The conditional distribution of *Y* for given *X* is :

$$P_{Y|X}(y \mid x) = \frac{p_{XY}(x, y)}{p_X(x)} = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y} x!}{y!(x-y)!\, \lambda^x e^{-\lambda}} = \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y}$$

$$= {}^xC_y\, p^y (1-p)^{x-y},\ x \ge y \text{ i.e., } y = 0, 1, 2, \ldots x.$$

The conditional probability distribution of *X* for given *Y* is :

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)} = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!} \cdot \frac{y!}{e^{-\lambda p} (\lambda p)^y} \qquad [\text{c.f. Part (i)}]$$

$$= \frac{e^{-\lambda q} (\lambda q)^{x-y}}{(x-y)!};\ q = 1 - p,\ x \ge y \text{ i.e., } x = y, y+1, y+2, \ldots$$

Example 5·37. *If X and Y are two random variables having joint density function :*

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y)\,;\ 0 \le x < 2,\ 2 \le y < 4 \\ 0\,,\ \text{otherwise} \end{cases}$$

Find (i) $P(X < 1 \cap Y < 3)$, (ii) $P(X + Y < 3)$, and (iii) $P(X < 1 \mid Y < 3)$.

Solution. We have

$$(i)\ P(X < 1 \cap Y < 3) = \int_{-\infty}^{1} \int_{-\infty}^{3} f(x, y)\, dx\, dy = \int_{0}^{1} \int_{2}^{3} \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{3}{8}$$

$$(ii)\ P(X + Y < 3) = \int_{0}^{1} \int_{2}^{3-x} \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{5}{24}$$

$$(iii)\ P(X < 1 \mid Y < 3) = \frac{P(X < 1 \cap Y < 3)}{P(Y < 3)} = \frac{3/8}{5/8} = \frac{3}{5}$$

$$\left[\text{From part (i) and } P(Y < 3) = \int_{0}^{2} \int_{2}^{3} \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{5}{8}\right]$$

Example 5·38. *Suppose that two-dimensional continuous random variable (X, Y) has joint p.d.f. given by :*

$$f(x, y) = \begin{cases} 6x^2y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

(i) *Verify that* $\int_0^1 \int_0^1 f(x, y)\, dxdy = 1.$

(ii) *Find* $P(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2)$, $P(X + Y < 1)$, $P(X > Y)$ *and* $P(X < 1 \mid Y < 2)$.

Solution. (i)

$$\int_0^1 \int_0^1 f(x, y)\, dxdy = \int_0^1 \int_0^1 6x^2y\, dxdy = \int_0^1 6x^2 \left|\frac{y^2}{2}\right|_0^1 dx = \int_0^1 3x^2 dx = \left|x^3\right|_0^1 = 1$$

(ii) $$P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right) = \int_0^{3/4} \int_{1/3}^1 6x^2y\, dxdy + \int_0^{3/4} \int_1^2 0.\, dxdy$$

$$= \int_0^{3/4} 6x^2 \left|\frac{y^2}{2}\right|_{1/3}^1 dx = \frac{8}{9} \int_0^{3/4} 3x^2\, dx = \frac{8}{9} \left|x^3\right|_0^{3/4} = \frac{3}{8}.$$

$$P(X + Y < 1) = \int_0^1 \int_0^{1-x} 6x^2y\, dxdy = \int_0^1 6x^2 \left|\frac{y^2}{2}\right|_0^{1-x} dx = \int_0^1 3x^2(1-x)^2 dx = \frac{1}{10}$$

$$P(X > Y) = \int_0^1 \int_0^x 6x^2y\, dxdy = \int_0^1 3x^2 \left|y^2\right|_0^x dx = \int_0^1 3x^4 dx = \frac{3}{5}.$$

$$P(X < 1 \mid Y < 2) = \frac{P(X < 1 \cap Y < 2)}{P(Y < 2)}$$

where $$P(X < 1 \cap Y < 2) = \int_0^1 \int_0^1 6x^2\, y\, dxdy + \int_0^1 \int_1^2 0.\, dxdy = 1$$

and $$P(Y < 2) = \int_0^1 \int_0^2 f(x,y)\, dxdy = \int_0^1 \int_0^1 6x^2\, y\, dxdy + \int_0^1 \int_1^2 0.\, dxdy = 1$$

$\therefore$ $$P(X < 1 \mid Y < 2) = \frac{P(X < 1 \cap Y < 2)}{P(Y < 2)} = 1.$$

Example 5·39. *The joint probability density function of a two-dimensional random variable (X, Y) is given by :*

$$f(x, y) = \begin{cases} 2\,; & 0 < x < 1, 0 < y < x\,; \\ 0, & \text{elsewhere} \end{cases}$$

(i) *Find the marginal density functions of X and Y.*

(ii) *Find the conditional density function of Y given X = x and conditional density function of X given Y = y.*

(iii) *Check for independence of X and Y.*

Solution. Evidently $f(x,y) \geq 0$ and $\int_0^1 \int_0^x 2\, dx\, dy = 2\int_0^1 x\, dx = 1.$

(i) The marginal *p.d.f.*'s of X and Y are given by :

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_0^x 2dy = 2x, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx = \int_y^1 2dx = 2(1-y), 0 < y < 1 \\ 0 \text{, elsewhere} \end{cases}$$

(*ii*) The conditional density function of Y given X, $(0 < x < 1)$ is :

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}, 0 < y < x.$$

The conditional density function of X given Y, $(0 < y < 1)$ is :

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{(1-y)}, y < x < 1$$

(*iii*) Since $f_X(x) f_Y(y) = 2(2x)(1-y) \neq f_{XY}(x, y)$, X and Y are not independent.

Example 5·40. *The joint p.d.f. of two random variables X and Y is given by :*

$$f(x, y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}\ ; 0 \le x < \infty, 0 \le y < \infty$$

Find the marginal distributions of X and Y, and the conditional distribution of Y for X = x.

Solution. Marginal *p.d.f.* of X is given by :

$$f_X(x) = \int_0^{\infty} f(x, y)\, dy = \frac{9}{2(1+x)^4} \int_0^{\infty} \frac{(1+y)+x}{(1+y)^4} dy$$

$$= \frac{9}{2(1+x)^4} \int_0^{\infty} \{(1+y)^{-3} + x(1+y)^{-4}\}\, dy$$

$$= \frac{9}{2(1+x)^4} \left(\left| \frac{-1}{2(1+y)^2} \right|_0^{\infty} + x \left| \frac{-1}{3(1+y)^3} \right|_0^{\infty} \right)$$

$$= \frac{9}{2(1+x)^4} \cdot \left(\frac{1}{2} + \frac{x}{3} \right) = \frac{3}{4} \cdot \frac{3+2x}{(1+x)^4}\ ;\ 0 < x < \infty$$

Since $f(x, y)$ is symmetric in x and y, the marginal *p.d.f.* of Y is given by :

$$f_Y(y) = \int_0^{\infty} f(x, y)\, dx = \frac{3}{4} \cdot \frac{3+2y}{(1+y)^4}\ ;\ 0 < y < \infty$$

The conditional distribution of Y for $X = x$ is given by :

$$f_{XY}(Y = y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \times \frac{4(1+x)^4}{3(3+2x)} = \frac{6(1+x+y)}{(1+y)^4(3+2x)}; 0 < y < \infty$$

Example 5·41. *Joint distribution of X and Y is given by :*

$$f(x,y) = 4xy\, e^{-(x^2+y^2)}; x \ge 0, y \ge 0.$$

Test whether X and Y are independent. For the above joint distribution, find the conditional density of X given Y = y.

Solution. Joint *p.d.f.* of X and Y is : $f_{XY}(x,y) = 4xy\ e^{-(x^2+y^2)}; x \ge 0, y \ge 0.$

Marginal density of X is given by :

$$f_X(x) = \int_0^{\infty} f_{XY}(x, y)\, dy = \int_0^{\infty} 4\,xy\ e^{-(x^2+y^2)}\, dy = 4x\, e^{-x^2} \int_0^{\infty} y\, e^{-y^2}\, dy$$

$$= 4x\, e^{-x^2} \cdot \int_0^{\infty} e^{-t} \cdot \frac{dt}{2} = 2x \cdot e^{-x^2} \left| -e^{-t} \right|_0^{\infty}$$

$$\therefore \quad f_X(x) = 2x\, e^{-x^2}\,;\ x \geq 0$$

Similarly, the marginal *p.d.f.* of Y is given by :

$$f_Y(y) = \int_0^{\infty} f_{XY}(x, y)\, dx = 2y\, e^{-y^2}\,;\ y \geq 0$$

Since $f_{XY}(x, y) = f_X(x) . f_Y(y)$, X and Y are independently distributed. The conditional distribution of X for given Y is given by :

$$f_{X\mid Y}(X = x \mid Y = y) = \frac{f(x, y)}{f_Y(y)} = 2x\, e^{-x^2}\,;\ x \geq 0.$$

Example 5·42. *Let X and Y be jointly distributed with p.d.f. :*

$$f_{XY}(x, y) = \begin{cases} \frac{1}{4}(1 + xy)\,, & \mid x \mid < 1,\ \mid y \mid < 1 \\ 0\,, & \textit{otherwise} \end{cases}$$

Show that X and Y are not independent by X^2 and Y^2 are independent.

Solution. $\quad f_X(x) = \int_{-1}^{1} f(x, y)\, dy = \frac{1}{4}\left| y + \frac{xy^2}{2} \right|_{-1}^{1} = \frac{1}{2}\,, \quad -1 < x < 1\,;$

Similarly, $\quad f_Y(y) = \int_{-1}^{1} f(x, y)\, dx = \frac{1}{2}\,, \quad -1 < y < 1$

Since $f_{X,Y}(x,y) \neq f_X(x)\, f_Y(y)$, X and Y are not independent. However,

$$P(X^2 \leq x) = P(\mid X \mid \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} f_X(x)\, dx = \sqrt{x} \qquad \ldots (*)$$

$$\begin{aligned} P(X^2 \leq x \cap Y^2 \leq y) &= P(\mid X \mid \leq \sqrt{x} \cap \mid Y \mid \leq \sqrt{y}) \\ &= \int_{-\sqrt{x}}^{\sqrt{x}} \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(u, v)\, dv \right] du = \sqrt{x}\,\sqrt{y} \\ &= P(X^2 \leq x) \,.\, P(Y^2 \leq y) \qquad \text{[From (*)]} \end{aligned}$$

Hence, X^2 and Y^2 are independent.

Example 5·43. *A gun is aimed at a certain point (origin of the coordinate system). Because of the random factors, the actual hit point can be any point (X, Y) in a circle of radius R about the origin. Assume that the joint density of X and Y is constant in this circle given by :*

$$f_{XY}(x, y) = \begin{cases} k\,, & \textit{for } x^2 + y^2 \leq R^2 \\ 0\,, & \textit{otherwise} \end{cases}$$

(i) Compute k,

(ii) show that $\quad f_X(x) = \begin{cases} \frac{2}{\pi R}\left\{1 - \left(\frac{x}{R}\right)^2\right\}^{1/2}, & \textit{for } -R \leq x \leq R \\ 0\,, & \textit{otherwise} \end{cases}$

Solution. (*i*) The constant k is computed from the consideration that the total probability is 1, *i.e.*,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx dy = 1 \;\Rightarrow\; \iint\limits_{x^2 + y^2 \leq R^2} k\, dx\, dy = 1 \;\Rightarrow\; 4 \iint\limits_{I} k\, dx\, dy = 1$$

where region I is the first quadrant of the circle $x^2 + y^2 = R^2$.

$$\Rightarrow \quad 4k \int_0^R \left(\int_0^{\sqrt{R^2-x^2}} 1 \, . \, dy \right) dx = 1 \quad \Rightarrow \quad 4k \int_0^R \sqrt{R^2 - x^2}\, dx = 1$$

$$\Rightarrow 4k \left| x\sqrt{R^2-x^2} + \frac{R^2}{2} \sin^{-1}\left(\frac{x}{R}\right) \right|_0^R = 1 \quad \Rightarrow \quad 4k\left(\frac{R^2}{2} \cdot \frac{\pi}{2}\right) = 1 \quad \Rightarrow \quad k = \frac{1}{\pi R^2}$$

$$\therefore \qquad f_{XY}(x, y) = \begin{cases} 1/(\pi R^2) \; ; & x^2 + y^2 \le R^2 \\ 0 & , \textit{otherwise} \end{cases}$$

$$(ii)\; f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \frac{1}{\pi R^2} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1 . \, dy$$

$$\left[\because x^2 + y^2 \le R^2 \quad \Rightarrow \quad -(R^2 - x^2)^{1/2} \le y \le (R^2 - x^2)^{1/2}\right]$$

$$= \frac{2}{\pi R^2} \int_0^{\sqrt{R^2-x^2}} 1 \, dy = \frac{2}{\pi R^2} (R^2 - x^2)^{1/2}$$

$$= \frac{2}{\pi R} \left\{1 - \left(\frac{x}{R}\right)^2\right\}^{1/2}, -R \le x \le R$$

Example 5·44. *Given : $f(x, y) = e^{-(x+y)}\, I_{(0,\infty)}(x) \, . \, I_{(0,\infty)}(y)$. Are X and Y independent? Find (i) $P(X > 1)$, (ii) $P(X < Y \mid X < 2Y)$, (iii) $P(1 < X + Y < 2)$.*

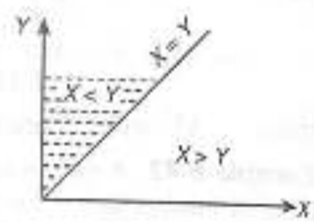
Solution. We are given :

$$f(x, y) = e^{-(x+y)} \; ; 0 \le x < \infty, 0 \le y < \infty$$
$$= (e^{-x})(e^{-y}) = f_X(x) \, . \, f_Y(y) \; ; 0 \le x < \infty, 0 \le y < \infty \qquad \ldots (1)$$

$\Rightarrow$ X and Y are independent and $f_X(x) = e^{-x}$; $x \ge 0$ and $f_Y(y) = e^{-y}$; $y \ge 0$ ...(2)

$$(i) \qquad P(X > 1) = \int_1^{\infty} f_X(x)\, dx$$

$$= \int_1^{\infty} e^{-x}\, dx = \left| \frac{e^{-x}}{-1} \right|_1^{\infty} = \frac{1}{e}$$



$$(ii)\; P(X < Y \mid X < 2Y) = \frac{P(X < Y \cap X < 2Y)}{P(X < 2Y)}$$

$$= \frac{P(X < Y)}{P(X < 2Y)} \qquad \ldots (3)$$

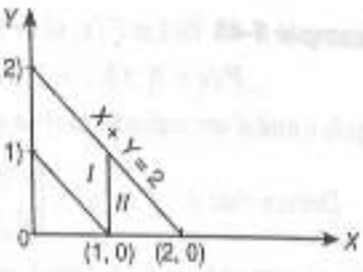
$$P(X < Y) = \int_0^{\infty} \left\{ \int_0^y f(x, y)\, dx \right\} dy = \int_0^{\infty} \left\{ e^{-y} \left| \frac{e^{-x}}{-1} \right|_0^y \right\} dy$$

$$= -\int_0^{\infty} e^{-y}(e^{-y} - 1)\, dy = -\left| \frac{e^{-2y}}{-2} + e^{-y} \right|_0^{\infty} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(X < 2Y) = \int_0^{\infty} \left\{ \int_0^{2y} f(x, y)\, dx \right\} dy = -\int_0^{\infty} e^{-y}(e^{-2y} - 1)\, dy$$

$$= -\left| \frac{e^{-3y}}{-3} + e^{-y} \right|_0^{\infty} = 1 - \frac{1}{3} = \frac{2}{3}.$$

Substituting in (3), $\quad P(X < Y \mid X < 2Y) = \dfrac{1/2}{2/3} = \dfrac{3}{4}.$



(iii) $P(1 < X + Y < 2)$

$$= \iint_{I} f(x, y)\, dx\, dy + \iint_{II} f(x, y) dx\, dy$$

$$= \int_0^1 \left(\int_{1-x}^{2-x} f(x,y)\, dy\right) dx + \int_1^2 \left(\int_0^{2-x} f(x, y)\, dy\right) dx$$

$$= \int_0^1 \left(e^{-x}\int_{1-x}^{2-x} e^{-y}\, dy\right) dx + \int_1^2 \left(e^{-x}\int_0^{2-x} e^{-y}\, dy\right) dx$$

$$= \int_0^1 \frac{e^{-x}}{-1}(e^{x-2} - e^{x-1})\, dx + \int_1^2 \frac{e^{-x}}{-1}(e^{x-2} - 1)\, dx$$

$$= -(e^{-2} - e^{-1}) \int_0^1 1.dx - \int_1^2 (e^{-2} - e^{-x})\, dx$$

$$= -(e^{-2} - e^{-1})\left[\,\left|x\right|_0^1\,\right] - \left[\left|e^{-2}.\,x + e^{-x}\right|_1^2\right] = \frac{2}{e} - \frac{3}{e^2}.$$

Example 5·45. *If the joint distribution function of X and Y is given by :*

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)} & ; x > 0,\ y > 0 \\ 0 & ; \text{elsewhere} \end{cases}$$

(a) Find the marginal densities of X and Y. *(b) Are X and Y independent ?*
(c) Find $P(X \le 1 \cap Y \le 1)$ and $P(X + Y \le 1)$.

Solution. (a) & (b). The joint p.d.f. of the r.v.'s (X, Y) is given by :

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[e^{-y} - e^{-(x+y)}\right] = \begin{cases} e^{-(x+y)}; & x \ge 0,\ y \ge 0 \\ 0; & \text{otherwise} \end{cases} \quad \ldots (i)$$

We have $f_{XY}(x, y) = e^{-x} \cdot e^{-y} = f_X(x) f_Y(y)$ ... (ii)

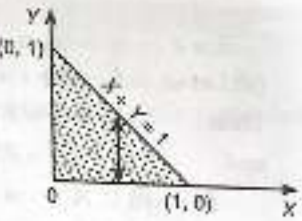
where $f_X(x) = e^{-x};\ x \ge 0;\quad f_Y(y) = e^{-y};\ y \ge 0$ ... (iii)

(ii) ⇒ X and Y are independent, and (iii) gives the marginal p.d.f.'s of X and Y.

(c) $P(X \le 1 \cap Y \le 1) = \int_0^1 \int_0^1 f(x, y)\, dx\, dy$

$$= \left(\int_0^1 e^{-x} dx\right)\left(\int_0^1 e^{-y} dy\right) = (1 - e^{-1})^2$$

$$P(X + Y \le 1) = \iint_{X+Y\le 1} f(x,y) = \int_0^1 \left\{\int_0^{1-x} f(x, y)\, dy\right\} dx$$



$$= \int_0^1 \left[e^{-x}\int_0^{1-x} e^{-y}\, dy\right] dx$$

$$= \int_0^1 e^{-x}\{1 - e^{-(1-x)}\}\, dx = 1 - 2e^{-1}.$$

Example 5·46. (i) Let $F(x, y)$ be the d.f. of X and Y. Show that

$$P(a < X \le b, c < Y \le d) = F(b, d) - F(b, c) - F(a, d) + F(a, c),$$

where a, b, c and d are real constants $a < b$; $c < d$.

Deduce that if : $F(x, y) = \begin{cases} 1, \text{ for } x + 2y \ge 1 \\ 0, \text{ for } x + 2y < 1, \end{cases}$

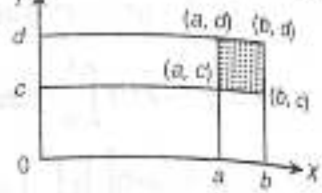
then $F(x, y)$ cannot be joint distribution function of variables X and Y.

(ii) Show that, with usual notations : for all x, y,

$$F_X(x) + F_Y(y) - 1 \quad \le \quad F_{XY}(x, y) \quad \le \quad \sqrt{F_X(x) F_Y(y)}.$$

Solution. (i) Let us define the events :

$A : \{X \le a\}$; $B : \{X \le b\}$; $C = \{Y \le c\}$; $D = \{Y \le d\}$; for $a < b; c < d$



$P(a < X \le b \cap c < Y \le d)$

$= P[(B - A) \cap (D - C)]$ ... (*)

$= P[B \cap (D - C) - A \cap (D - C)]$

(By distributive property of sets)

We know that if $E \subset F \Rightarrow E \cap F = E$, then

$P(F - E) = P(\bar{E} \cap F) = P(F) - P(E \cap F) = P(F) - P(E)$ ...(**)

Obviously $A \subset B \quad \Rightarrow \quad [A \cap (D - C)] \subset [B \cap (D - C)]$

Hence using (**), we get from (*)

$$\begin{aligned} P(a < X \le b \cap c < Y \le d) &= P[B \cap (D - C)] - P[A \cap (D - C)] \\ &= P[(B \cap D) - (B \cap C)] - P[(A \cap D) - (A \cap C)] \\ &= P(B \cap D) - P(B \cap C) - P(A \cap D) + P(A \cap C) \quad \text{...(***)} \end{aligned}$$

[On using (**), since $C \subset D \Rightarrow (B \cap C) \subset (B \cap D)$ and $(A \cap C) \subset (A \cap D)$]

We have : $P(B \cap D) = P[X \le b \cap Y \le d] = F(b, d)$.

Similarly $P(B \cap C) = F(b, c)$; $P(A \cap D) = F(a, d)$ and $P(A \cap C) = F(a, c)$

Substituting in (***), we get

$$P(a < X \le b \cap c < Y \le d) = F(b, d) - F(b, c) - F(a, d) + F(a, c) \quad \text{...(1)}$$

We are given $\quad F(x, y) = \begin{cases} 1, \text{ for } x + 2y \ge 1 \\ 0, \text{ for } x + 2y < 1 \end{cases}$ ...(2)

In (1) let us take $a = 0, b = \frac{1}{2}; c = \frac{1}{4} \cdot d = \frac{3}{4}$ s.t. $a < b$ and $c < d$. Then using (2), we get

$$F(b, d) = 1; \quad F(b, c) = 1; \quad F(a, d) = 1; \quad F(a, c) = 0.$$

Substituting in (1), $P(a < X \le b \cap c < Y \le d) = 1 - 1 - 1 + 0 = -1$; which is not possible since $P(.) \ge 0$.

Hence $F(x, y)$ defined in (2) cannot be the distribution function of r.v.'s X and Y.

(ii) Let us define the events : $A\{X \le x\}$; $B = \{Y \le y\}$

Then : $\quad P(A) = P(X \le x) = F_X(x)$; $\quad P(B) = P(Y \le y) = F_Y(y)$ ...(3)

and $\quad P(A \cap B) = P(X \le x \cap Y \le y) = F_{XY}(x, y)$

$$(A \cap B) \subset A \quad \Rightarrow \quad P(A \cap B) \le P(A) \quad \Rightarrow \quad F_{XY}(x, y) \le F_X(x)$$

$$(A \cap B) \subset B \quad \Rightarrow \quad P(A \cap B) \le P(B) \quad \Rightarrow \quad F_{XY}(x, y) \le F_Y(y)$$