* Variational Model:

The general variational framework used in image processing can be written as an energy function:

$$E[u(x)] = \int_{\Omega}(D(u) + S(u))dx$$

where $D(u)$ represents the data component and $S(u)$ is the smoothing term of the function. We are required to determine $u(x)$ on $\Omega \subset R^2$, that minimizes the energy:

$$u_{min} = \arg\min_{u \in U} E[u(x)]$$

In this case, we consider an image u_0 affected by the Gaussian noise. The general form of energy function is:

$$E[u] = \int_{\Omega}(u-u_0)^2 + \alpha\psi(\|\nabla u\|^2), \alpha > 0 \quad \text{—(1)}$$

where ψ represents the regularizer of smoothing term and α is the regularization parameter.

We are to develop an efficient smoothing component, and thus we consider the following regularizer: $\psi : [0,\infty) \to [0,\infty]$:

$$\psi(s) = \eta\sqrt{\frac{k}{\beta}}\ln\left(s + \sqrt{s^2 + \frac{\gamma}{\beta}}\right) + v.s; \quad k > 0, \eta, \beta, \gamma, v \in (0,1) \quad \text{—(2)}$$

We consider the values of $k, \eta, \beta, \gamma, v$ and α which provide a successful denoising. Now we compute a minimizer for the energy function given by (1), using the previous eq^n:

$$u_{min} = \arg\min_{u \in U} E(u) = \arg\min_{u \in U} \int_{\Omega}(u-u_0)^2 + \alpha\psi(\|\nabla u\|^2)\,dx\,dy$$

The minimization result u_{min} will correspond to the denoised image. The minimization process is performed by solving the following Euler-Lagrange equation [7, 10, 11]:

$$u - u_0 - \alpha\,\text{div}\left(\psi'(\|\nabla u\|^2)\nabla u\right) = 0 \Leftrightarrow \frac{u-u_0}{\alpha} - \text{div}\left(\psi'(\|\nabla u\|^2)\nabla u\right) = 0$$

Thus, we obtain the following PDE:

$$\frac{\partial u}{\partial t} = \text{div}\left(\psi'(\|\nabla u\|^2)\nabla u\right) - \frac{u-u_0}{\alpha} \quad \text{—(3)}$$

where the positive function ψ' is obtained by computing the derivative of the function given by ②,

$$\cancel{\psi^2}\ \psi'(s^2) = \frac{\nu\sqrt{\beta s^2+\gamma}+\eta\sqrt{k}}{\sqrt{\beta s^2+\gamma}}$$

Therefore, ③ becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{\eta\sqrt{\beta\|\nabla u\|^2+\gamma}+\alpha\sqrt{k}}{\sqrt{\beta\|\nabla u\|^2+\gamma}}\cdot\nabla u\right) - \dfrac{u-u_0}{\alpha} \\ u(0,x,y) = u_0 \end{cases}$$

We can demonstrate the above PDE model converges to a unique strong solution, i.e. $u^* = u_{min}$.

Now, we propose a robust discretisation scheme for solving it.

From ③, we have

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\psi'(\|\nabla u\|^2)\nabla u\right) - \frac{u-u_0}{\alpha}$$ which

leads to

$$u(x,y,t+1) \equiv u(x,y,t) + \operatorname{div}\left(\psi'(\|\nabla u\|^2)\nabla u\right) - \frac{u-u_0}{\alpha}$$

We can approximate the above eq^n using the image gradient magnitudes in particular directions,

$$u^{t+1} = u^t + \lambda \sum_{q\in N(p)} \psi'(\|\nabla u_{p,q}(t)\|^2)\nabla u_{p,q}(t) - \frac{u-u_0}{\alpha}$$

where $\lambda \in (0,1)$ and $t = 1, ---, N$.

In the above equation, $N(p)$ represents the 4-neighborhood of the argument pixel, described by it's coordinates $p=(x,y)$. Obviously, it represents a set of image pixels given by their coordinates:

$$N(p) = \{(x-1, y), (x+1, y), (x, y-1), (x, y+1)\}$$

Also, $\nabla u_{p,q}$ is the image gradient magnitude in the direction given by pixel q at iteration t,

$$\nabla u_{p,q}(t) = u(q,t) - u(p,t)$$

The maximum number of iterations N, are empirically chosen.