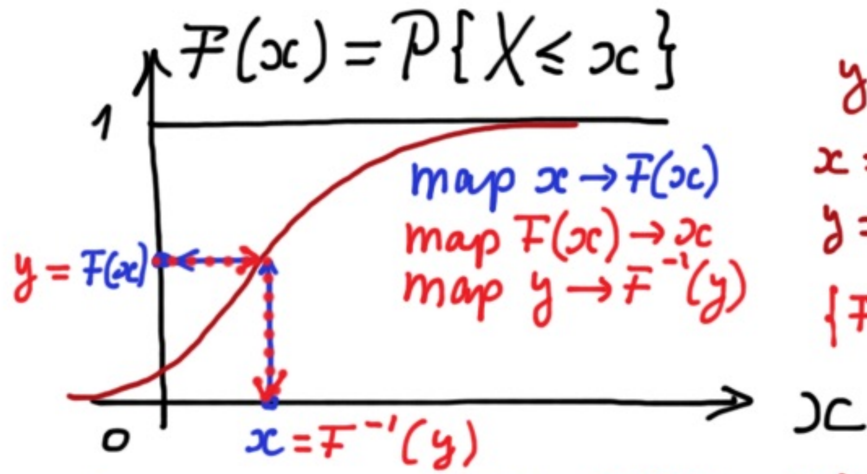
Inverse distribution function and random number simulation.

Let X be a random variable with continuous distribution function $F(x)=P\{X\le x\}$.

$$\boxed{F(F^{-1}(\cdot))=F^{-1}(F(\cdot))=\cdot}$$



$$y = F(x)$$
$$x = F^{-1}(y) = F^{-1}(F(x))$$
$$y = F(x) = F(F^{-1}(y))$$
$$\{F(X)\le x\} \equiv \{F^{-1}(F(X)) \le F^{-1}(x)\}$$
$$\equiv \{X \le F^{-1}(x)\}$$

Random variable $F(X)$: $P\{F(X)\le x\} = P\{X\le F^{-1}(x)\}$
$$\doteq F(F^{-1}(x)) = x$$

What distribution has $F(x)=P\{X\le x\}=x$? $\boxed{\text{Unif}(0,1)}$

Simulation.

We need to simulate random variable $X \sim F(x)$, where $F(x) = P\{X \le x\}$ is an arbitrary distribution.

Step 1. Simulate $U \sim Unif(0,1)$.

Step 2. Set $X = F^{-1}(U)$.

definition U

Then $P\{X \le x\} = P\{F^{-1}(U) \le x\} = P\{U \le F(x)\} = F(x)$

$\boxed{\text{Need } F^{-1}(x)!}$

$\{F(F^{-1}(U)) \le F(x)\} = \{U \le F(x)\}$

Example.

Exponential distribution $Exp(\lambda)$.

$$f(x;\lambda) = \begin{cases} e^{-\lambda x}, & x \in [0,\infty), \\ 0, & x \in (-\infty, 0]. \end{cases} \quad F(x) = 1 - e^{-\lambda x}.$$

$$\boxed{\begin{aligned} F^{-1}(y) &= -\frac{\ln(1-y)}{\lambda} \\ X_E &= -\frac{\ln(1-U)}{\lambda} \end{aligned}}$$

Solve for x: $e^{-\lambda x} = 1 - F(x)$

$$\ln e^{-\lambda x} = -\lambda x = \ln(1 - F(x)); \; x = -\frac{\ln(1-F(x))}{\lambda}$$