Spring 2019 MATH 0100 Lecture Notes

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Chapter 1

Introduction and Review

1.1 Course Overview

This course is a continuation of MATH 0090. In MATH 0090 we have learned about differentiation techniques, and a little bit of integration. In this course, we will first learn more about integration techniques, such as the integration by parts, trigonometric substitutions, partial fractions, etc.

The next topic will be Taylor series. The basic idea of Taylor series is to approximate arbitrary functions with functions which are quite familiar to us, polynomials. We will first study basic concepts of general sequences and series, and several tests to determine the convergence behavior of given series. Then we will move onto Taylor series and its applications.

Finally, in the last few weeks, we will learn two minor (but still important) subjects, parametric curves and differential equations.

1.2 Note Structure

Each section, from the next one, will begin with an emphasized *intuitive question,* which is closely related to the content of that section. Then The question will be answered after introducing essential concepts, problem solving strategies and basic examples. We will also study more complicated examples and other related concepts.

⚠ After this signal will be remarks; they could be common mistakes, possible logical errors, or out-of-scope content that might be covered in further courses, but not in this course.

1.3 Review from MATH 0090

Intuitive Question. What have you learned so far about differentiation and integration?

In MATH 0090, we mainly focused on limits, differentiation and derivatives, but we have also studied basic integration, including the concepts of antiderivatives and definite integrals. We also learned about the Fundamental Theorem of Calculus, which says that differentiation and integration are in close relation. Actually they are 'reverse' to each other, so we could integrate many functions using derivatives that we have learned. Here are some stuffs that we know about differentiation and integration so far.

$\boxed{\text{Differentiation Rules}}$

- Sum and Difference Rule: $(f \pm g)' = f' \pm g'$
- Constant Multiplication: $(cf)' = cf'$
- Product Rule: $(fg)' = f'g + fg'$
- Quotient Rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$
- Chain Rule: $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

$\boxed{\text{Derivatives of Common Functions}}$

- Power Rule: $\dfrac{d}{dx}x^n = nx^{n-1}$
- Exponential function: $\dfrac{d}{dx}e^x = e^x$
- Logarithmic function: $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$
- Trigonometric functions:

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$

- Inverse trigonometric functions:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$\boxed{\text{Integration Rules}}$

- Sum and Difference Rule:

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

- Constant Multiplication:

$$\int cf(x)dx = c\int f(x)dx$$

- u-substitution:

$$\int f(g(x))g'(x)dx \overset{u=g(x)}{=} \int f(u)du$$

$\boxed{\text{Antiderivatives of Common Functions}}$

- Power Rule: $\displaystyle\int x^n dx = \frac{1}{n+1}x^{n+1} + C$
- Exponential function: $\displaystyle\int e^x dx = e^x + C$
- Reciprocal function: $\displaystyle\int \frac{1}{x}dx = \ln|x| + C$

  (Here $|x|$ implies the absolute value of x.)
- Trigonometric functions:

$$\begin{aligned}
\int \cos x dx &= \sin x + C \\
\int \sin x dx &= -\cos x + C \\
\int \sec^2 x dx &= \tan x + C \\
\int \sec x \tan x dx &= \sec x + C \\
\int \frac{1}{\sqrt{1-x^2}}dx &= \arcsin x + C \\
\int \frac{1}{1+x^2}dx &= \arctan x + C
\end{aligned}$$

Among these differentiation and integration rules, one that needs special attention is the u-substitution. To illustrate how it works, we take a simple example as follows.

Example. Calculate the integral $\int 2xe^{x^2}dx$.

Letting $u = x^2$, we have $du/dx = 2x$ or $du = 2xdx$. Therefore,

$$\int 2xe^{x^2}dx \overset{u=x^2}{=} \int e^u du = e^u + C = e^{x^2} + C. \quad \diamond$$

As a reverse of the chain rule in derivatives, we can substitute a term with a new variable u to change the given integral into a new one with respect to u. When we choose a term to be substituted by u, it is often useful to take a term 'inside' another function.

Example. Calculate the integral $\int (3x-2)^{10}dx$.

A term $3x-2$ is inside a 10-th power function, so we can try a substitution $u = 3x - 2$ and it actually works. Here $du = 3dx$, so

$$\begin{aligned}\int (3x-2)^{10}dx &= \int u^{10}\left(\frac{1}{3}du\right) = \frac{1}{3}\int u^{10}du \\ &= \frac{1}{3}\left(\frac{1}{11}u^{11}\right) + C = \frac{1}{33}u^{11} + C = \frac{1}{33}(3x-2)^{11} + C. \quad \diamond\end{aligned}$$

Also, if the integrand is a fraction, it is often useful to take a term in the denominator.

Example. Calculate the integral $\int \frac{x^2+2x}{(x^3+3x^2)^2}dx$.

Letting $u = x^3 + 3x^2$, we have $du = (3x^2 + 6x)dx = 3(x^2 + 2x)dx$. Therefore,

$$\begin{aligned}\int \frac{x^2+2x}{(x^3+3x^2)^2}dx &= \int \frac{1}{u^2}\left(\frac{1}{3}du\right) = \frac{1}{3}\int \frac{1}{u^2}du \\ &= \frac{1}{3}\left(-\frac{1}{u}\right) + C = -\frac{1}{3u} + C = -\frac{1}{3(x^3+3x^2)} + C. \quad \diamond\end{aligned}$$

Finally, when we use the u-substitution in definite integrals, we should be careful of bounds since the bounds of u are different from those of x.

Example. Calculate the integral $\int_{e^2}^{e^8} \frac{1}{x\ln x}dx$.

Letting $u = \ln x$, we have $du = \frac{1}{x}dx$. Also, $x = e^2,\ e^8$ correspond to $u = 2,\ 8$, respectively. Therefore,

$$\int_{e^2}^{e^8} \frac{1}{x\ln x}dx = \int_2^8 \frac{1}{u}du = [\ln|u|]_2^8 = \ln 8 - \ln 2 = 3\ln 2 - \ln 2 = 2\ln 2. \quad \diamond$$

Chapter 2

Integration techniques

2.1 Integration by Parts

Intuitive Question. We know that $\int e^x dx = e^x + C$. *Then what would be* $\int xe^x dx$?

Sometimes we want to calculate an integral whose integrand is a product of two functions.

⚠ Of course, the antiderivative of a product is not equal to the product of antiderivatives. For example, $\int 2xdx = x^2 + C$ but $\int (2x)^2 dx \neq x^4 + C$.

Then how can we calculate such integral? To get some intuition, let us go back to the product rule of differentiation,

$$\frac{d}{dx} u(x)v(x) = u'(x)v(x) + u(x)v'(x).$$

Integrating both parts and using the Fundamental Theorem of Calculus we get

$$u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx.$$

Therefore, we have

$$\boxed{\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx}$$

or simply

$$\boxed{\int udv = uv - \int vdu.}$$

How can we use this equation to calculate integrals whose integrand is a product of two function? Actually we cannot use it always, but in some 'good' situations. Explicitly, this 'good' situation implies that one of two functions are integrable. We can let that integrable function as dv (since it is integrable, we can find what is v) and the other as u. Then since we know what are u and v, we can calculate uv and $\int vdu$. If we can integrate $\int udv$ with methods that we already know, then we are done!

Example. Calculate the integral $\int xe^x dx$.

The integrand xe^x is a product of two functions, x and e^x. e^x is integrable with an antiderivative e^x. So we can use the following table.

$$\begin{array}{rl} u = x & dv = e^x dx \\ du = 1dx & v = e^x \end{array}$$

Then we have

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C. \quad \diamondsuit$$

⚠ In some cases, it is important to choose u and dv properly. For example, if we let $u = e^x$ and $dv = xdx$ in the previous example, we get a table

$$\begin{array}{rl} u = e^x & dv = xdx \\ du = e^x dx & v = \frac{1}{2}x^2 \end{array}$$

so

$$\int xe^x dx = \frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x dx.$$

This is not helpful; the integrand becomes more complicated. We need to choose u and dv so that the integrand vdu is 'simpler' than udv.

Applying the same method for several times allows us to calculate the integral with more complicated integrand.

Example. Calculate the integral $\int x^2 e^x dx$.

Intuitively, the derivative of x^2 is $2x$, which is simpler than x^2, while the antiderivative of e^x is still e^x. Therefore, it should be a good idea to apply the integration by parts in this direction. Consider the following table.

$$\begin{array}{rl} u = x^2 & dv = e^x dx \\ du = 2xdx & v = e^x \end{array}$$

Then we have

$$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx.$$

As in the following example, we applied the integration by parts again to get

$$\int 2xe^x dx = 2xe^x - 2e^x + C.$$

Substituting this, we have

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C. \quad \diamondsuit$$

Sometimes we can use the integration by parts even when the integrand does not seem to be a product.

Example. Calculate the integral $\int \ln x dx$.

Although $\ln x$ is just a single term, it can be considered as a product of 1 and $\ln x$. So we can use the following table.

$$\begin{array}{cc} u = \ln x & dv = 1dx \\ du = \frac{1}{x}dx & v = x \end{array}$$

Then we have

$$\int \ln x dx = x \ln x - \int x\frac{1}{x}dx = x \ln x - \int 1dx = x \ln x - x + C. \ \diamondsuit$$

In the following example, the first application of the integration by parts seems to be not helpful. But if we apply it once more and solve the equation algebraically, it eventually gives the value of integral.

Example. Calculate the integral $\int e^x \sin x dx$.

Let $I = \int e^x \sin x dx$ and consider the following table.

$$\begin{array}{cc} u = \sin x & dv = e^x dx \\ du = \cos x dx & v = e^x \end{array}$$

Then we have

$$I = e^x \sin x - \int e^x \cos x dx.$$

To calculate the integral $\int e^x \cos x dx$, we use the integration by parts again. With the following table

$$\begin{array}{cc} u = \cos x & dv = e^x dx \\ du = -\sin x dx & v = e^x \end{array}$$

we have

$$\int e^x \cos x dx = e^x \cos x - \int(-e^x \sin x)dx = e^x \cos x + \int e^x \sin x dx.$$

Substituting this to the first equation, we get

$$I = e^x \sin x - (e^x \cos x + \int e^x \sin x dx) = e^x \sin x - e^x \cos x - I,$$

so

$$2I = e^x \sin x - e^x \cos x.$$

Therefore,

$$I = \int e^x \sin x dx = \frac{1}{2}(e^x \sin x - e^x \cos x). \ \diamondsuit$$

⚠ In the previous example, if we choose the following table in the second integration by parts instead:

$$\begin{array}{ll} u = e^x & dv = \cos x dx \\ du = e^x dx & v = \sin x \end{array}$$

then we get

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx,$$

so

$$I = e^x \sin x - (e^x \sin x - I) = I.$$

This does not give any information about the value of I. Therefore, again, we have to choose u and dv properly.

Actually there is a rule that helps us to choose u and dv in most cases. It is usually called the 'L-I-A-T-E' rule, since the rule implies that it is commonly better to choose u as one that comes first in the following list.

$\boxed{\text{L-I-A-T-E Rule}}$

- **L** - Logarithmic functions
- **I** - Inverse trigonometric functions
- **A** - Algebraic functions (polynomials, square roots of polynomials, etc.)
- **T** - Trigonometric functions
- **E** - Exponential functions

For example, in the very first example $\int xe^x dx$, x is an algebraic function and e^x is an exponential function. Thus, the rule says it is better to take $u = x$ and $dv = e^x dx$. Also, in the example $\int \ln x dx$, 1 is an algebraic function while $\ln x$ is a logarithmic function. Therefore it is natural to take $u = \ln x$ and $dv = 1dx$.

⚠ Of course, this rule does not work always. Also, some integrals can be calculated even if we take u and dv not following the rule, as in the previous example $\int e^x \sin x dx$. In the given solution we took u and dv following the rule ($\sin x$ is a trigonometric function and e^x is an exponential function,) we still be able to apply the integration by parts with $u = e^x$ and $dv = \sin x dx$ at the beginning.

2.2 Trigonometric Integration

Intuitive Question. We know that $\int \sin x dx = -\cos x + C$ *and* $\int \cos x dx = \sin x + C$. *Then what would be* $\int \sin^2 x \cos^3 x dx$*?*

In this section, we deal with integrals of the form $\int \sin^m x \cos^n x$ where m, n are nonnegative integers. Before that, we first review some trigonometric identities that will be used in this section.

Trigonometric Identities

- Definitions:

$$\tan x = \frac{\sin x}{\cos x},\ \csc x = \frac{1}{\sin x},\ \sec x = \frac{1}{\cos x},\ \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

- Pythagorean identities:

$$\sin^2 x + \cos^2 x = 1,\ \tan^2 x + 1 = \sec^2 x$$

- Double-angle formulas:

$$\begin{aligned} \sin 2x &= 2\sin x \cos x, \\ \cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \end{aligned}$$

- Square formulas:

$$\sin^2 x = \frac{1 - \cos 2x}{2},\ \cos^2 x = \frac{1 + \cos 2x}{2}$$

  Note that these formulas are directly obtained from the double-angle formula for $\cos 2x$.

⚠ Indeed, two double-angle formulas are special cases of the following sum formulas:

$$\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta, \\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \end{aligned}$$

These formulas are also called Ptolemy's identities, and they can be proved using some geometric ideas. But the proof, and also the sum formulas themselves, are out of scope for this course.

When one of the exponents m, n is 1 and the other is 0, it is just the integration of $\cos x$ or $\sin x$. If one exponent is still 1 but the other is positive, then we can use the u-substitution to calculate the integral.

Example. Calculate the integral $\int \sin^2 x \cos x dx$.

Letting $u = \sin x$ we have $du = \cos x dx$. Therefore,

$$\int \sin^2 x \cos x dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C. \quad \Diamond$$

We can extend this idea to more general cases, with one of the Pythagorean identities $\sin^2 x + \cos^2 x = 1$.

Example. Calculate the integral $\int \sin^2 x \cos^3 x dx$.

It cannot be done with a single u-substitution, using the Pythagorean identity we can substitute $\cos^2 x$ with $1 - \sin^2 x$. Then we get

$$\int \sin^2 x \cos^3 x dx = \int \sin^2 x(1 - \sin^2 x) \cos x dx.$$

Now we can use the same u-substitution as above.

$$\begin{aligned}
\int \sin^2 x(1 - \sin^2 x) \cos x dx &= \int u^2(1 - u^2) du = \int (u^2 - u^4) du \\
&= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\
&= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C. \quad \Diamond
\end{aligned}$$

We can summarize this process as follows: in the integration $\int \sin^m x \cos^n x dx$ where n is odd, we can use the identity $\cos^2 x = 1 - \sin^2 x$ to 'split' $\cos^n x$, and make the integrand as a product of a single $\cos x$ and a polynomial in $\sin x$. Then we can use the u-substitution $u(x) = \sin x$ to calculate the integral. This process can be applied as well if m is odd; in this case we need to use the identity $\sin^2 x = 1 - \cos^2 x$ and the u-substitution $u(x) = \cos x$.

If both m and n are odd, basically we can choose either method. But in general, it is better to choose a term with smaller exponent for easier calculation.

Example. Calculate the integral $\int \sin^3 x \cos^7 x dx$.

If we try to split $\cos^7 x$, we get

$$\begin{aligned}
\int \sin^3 x \cos^7 x dx &= \int \sin^3 x(\cos^2 x)^3 \cos x dx \\
&= \int \sin^3 x(1 - \sin^2 x)^3 \cos x dx \\
&= \int u^3(1 - u^2)^3 du. \quad (u = \sin x)
\end{aligned}$$

Then we have to expand a cube of $1-u^2$ and integrate a quite complicated polynomial in u. Instead of doing this, we could try the other direction, splitting $\sin^3 x$. Then we have

$$\begin{aligned}\int \sin^3 x\cos^7 x dx &= \int (1-\cos^2 x)\cos^7 x\sin x dx\\ &= \int (1-u^2)u^7(-du) \quad (u=\cos x)\\ &= \int (u^9-u^7)du\\ &= \frac{1}{10}u^{10}-\frac{1}{8}u^8+C=\frac{1}{10}\cos^{10}x-\frac{1}{8}\cos^8 x+C. \quad \diamondsuit\end{aligned}$$

⚠ In the previous example, the answer from splitting $\cos^7 x$, where we did not finish the calculation, would be a polynomial with many terms in $\sin x$. However, it still should be equal to our final answer with only two terms (excluding the integration constant.) Actually, if we use the Pythagorean identity again, we can prove that those two answers are 'mathematically' equivalent.

This is how to calculate the integral $\int \sin^m x\cos^n x dx$ if either m or n is odd. Then what if both m and n are even? Here comes the square formulas from the beginning of this section. When we face the case where both m and n are even, we can use these formulas to calculate the integrand as a function of $\cos 2x$, and see if we can calculate the new integral. We start with the simplest cases.

Example. Calculate the integrals $\int \sin^2 x dx$ *and* $\int \cos^2 x dx$.

Using the identities above, we have

$$\int \sin^2 x dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}\int (1-\cos 2x)dx = \frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)+C$$

and

$$\int \cos^2 x dx = \int \frac{1+\cos 2x}{2}dx = \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C. \quad \diamondsuit$$

Of course, we can do more complicated integrations by applying the identities multiple times.

Example. Calculate the integral $\int \sin^2 x \cos^2 x dx$.

Using the identities above, we have

$$\begin{aligned}\int \sin^2 x \cos^2 x dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx\\ &= \int \frac{1-\cos^2 2x}{4} dx\\ &= \frac{1}{4}\int (1-\cos^2 2x) dx.\end{aligned}$$

Here we use the identity

$$\cos^2 2x = \frac{1+\cos 4x}{2}.$$

(This one can be obtained by substituting $2x$ instead of x in the square formula.) Therefore,

$$\begin{aligned}\int \sin^2 x \cos^2 x dx &= \frac{1}{4}\int \left(1 - \frac{1+\cos 4x}{2}\right) dx\\ &= \frac{1}{4}\int \frac{1-\cos 4x}{2} dx\\ &= \frac{1}{8}\int (1-\cos 4x) dx\\ &= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C. \quad \Diamond\end{aligned}$$

We summarize the whole strategy for integrals of the form $\int \sin^m x \cos^n x dx$ in the following table.

$m \backslash n$	odd	even
odd	Split anything (in general, one with smaller exponent)	Split $\cos^m x$
even	Split $\sin^n x$	Use square formulas to change the integrand into a function of $\cos 2x$, repeat if needed

There are four other elementary trigonometric functions, tangent, secant, cosecant, and cotangent. However, we try to avoid using cosecant and cotangent as possible as we can. (We will see later how to deal with the cases where cosecants or cotangents.) Now naturally, we work on integrals of the form $\displaystyle\int \sec^m x \tan^n x$. First we deal with the simplest cases, antiderivatives of $\sec x$ and $\tan x$. Note that the derivative of $\sec x$ and $\tan x$ is $\sec x \tan x$ and $\sec^2 x$, respectively.

Example. Calculate the integrals $\displaystyle\int \tan x dx$ *and* $\displaystyle\int \sec x dx$.

For the former one we use the u-substitution as follows.

$$\int \tan x dx = \int \frac{\sin x dx}{\cos x} \overset{u=\cos x}{=} \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C.$$

Finding the integral of $\sec x$, however, is not so 'straightforward.' We have to use some kind of trick to calculate it. Considering $\sec x$ as a fraction $^{\sec x}/_1$ and multiplying $\sec x + \tan x$ to both numerator and denominator, we have

$$\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx.$$

Then it turns out that, somewhat magically, the derivative of the denominator $\sec x + \tan x$ is $\sec x \tan x + \sec^2 x$, which is exactly the numerator. Therefore, we can use the u-substitution with $u = \sec x + \tan x$, so that

$$\begin{aligned}\int \sec x dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \int \frac{1}{u} du = \ln|u| + C = \ln|\sec x + \tan x| + C. \quad \Diamond\end{aligned}$$

When the exponents get bigger, it turns out that we can use the similar method - splitting one term using the Pythagorean identity and u-substitution - in most cases.

Example. Calculate the integral $\displaystyle\int \sec^4 x \tan^3 x dx$.

If we split $\sec^4 x$ and use the Pythagorean identity

$$\sec^2 x = \tan^2 x + 1,$$

then we have

$$\begin{aligned}\int \sec^4 x \tan^3 x dx &= \int \sec^2 x \tan^3 x \sec^2 x dx \\ &= \int (\tan^2 x + 1) \tan^3 x \sec^2 x dx \\ (u = \tan x) &= \int (u^2 + 1) u^3 du \\ &= \int (u^5 + u^3) du \\ &= \frac{1}{6} u^6 + \frac{1}{4} u^4 + C = \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C.\end{aligned}$$

We can also split $\tan^3 x$ and use

$$\tan^2 x = \sec^2 x - 1,$$

as follows.

$$\begin{aligned}
\int \sec^4 x \tan^3 x dx &= \int \sec^3 x \tan^2 x \sec x \tan x dx \\
&= \int \sec^3 x(\sec^2 x - 1) \sec x \tan x dx \\
(u = \sec x) &= \int u^3(u^2 - 1) du \\
&= \int (u^5 - u^3) du \\
&= \frac{1}{6}u^6 - \frac{1}{4}u^4 + C = \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C. \quad \diamondsuit
\end{aligned}$$

⚠ As before, we can prove that two answers in the previous example are mathematically equivalent, using the Pythagorean identity again.

Now consider the general integral $\int \sec^m x \tan^n x dx$. From the previous example, with the same logic as the case of $\int \sin^m x \cos^n x$, it turns out that we can split $\sec^m x$ if m is even, and split $\tan^n x$ if n is odd. Therefore, we can make a similar table as above, with a blank in the cell where m is odd and n is even. How can we deal with that case? We briefly see the strategy with a quite simple example.

Example. Calculate the integral $\int \sec x \tan^2 x dx$.

We use the Pythagorean identity

$$\tan^2 x = \sec^2 x - 1$$

to get

$$\begin{aligned}
\int \sec x \tan^2 x dx &= \int \sec x(\sec^2 x - 1) dx \\
&= \int (\sec^3 x - \sec x) dx \\
&= \int \sec^3 x dx - \int \sec x dx.
\end{aligned}$$

Now we calculate $\int \sec^3 x dx$ using a technique from the previous section, integration by parts. We use the following table.

$$\begin{array}{cc}
u = \sec x & dv = \sec^2 x dx \\
du = \sec x \tan x & v = \tan x
\end{array}$$

Therefore,

$$\begin{aligned}
\int \sec^3 x dx &= \sec x \tan x - \int \sec x \tan^2 x dx \\
&= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\
&= \sec x \tan x - \int (\sec^3 x - \sec x) dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx.
\end{aligned}$$

Simplifying the equation gives

$$\int \sec^3 x dx = \frac{1}{2} \left(\sec x \tan x + \int \sec x dx \right),$$

so using the antiderivative of $\sec x$ from above we get

$$\begin{aligned}
\int \sec x \tan^2 x dx &= \int \sec^3 x dx - \int \sec x dx \\
&= \frac{1}{2} \left(\sec x \tan x + \int \sec x dx \right) - \int \sec x dx \\
&= \frac{1}{2} \left(\sec x \tan x - \int \sec x dx \right) \\
&= \frac{1}{2} (\sec x \tan x - \ln|\sec x + \tan x|) + C. \quad \diamondsuit
\end{aligned}$$

⚠ The fact is, indeed, if we change $\tan^n x$ in the integral $\int \sec^m x \tan^n x$ to a polynomial of $\sec x$ using the Pythagorean identity (this is possible since n is even) then the new integrand becomes a sum or difference of several odd powers of $\sec x$. (This is because m is odd.) Also, we can use the integration by parts as well for calculating the antiderivatives of these odd powers of $\sec x$. For example, to calculate $\int \sec^5 x dx$ we use the following table.

$$\begin{array}{cc}
u = \sec^3 x & dv = \sec^2 x dx \\
du = 3 \sec^3 x \tan x & v = \tan x
\end{array}$$

However, we will not go further, since the calculation becomes very complicated as the exponent increases.

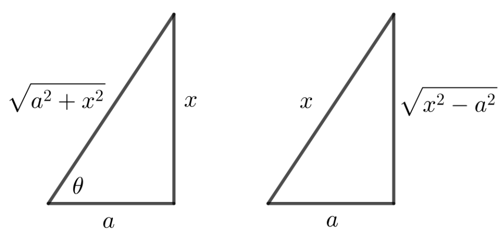
We finish this section with a table showing the general strategy to calculate the integral $\int \sec^m x \tan^n x dx$.

$m \backslash n$	odd	even
odd	Split $\sec^m x$	Use the Pythagorean identity to change the integrand into a sum or difference of odd powers of $\sec x$, and use the integration by parts
even	Split anything (in general, one with smaller exponent)	Split $\sin^n x$

2.3 Trigonometric Substitution

Intuitive Question. We know that $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$ *and* $\int \frac{1}{1+x^2}dx = \arctan x + C$. *Then what would be* $\int \frac{1}{\sqrt{1+x^2}}dx$*?*

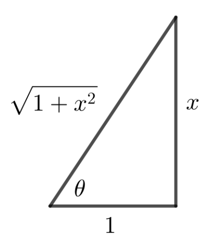
In this section, we study integrals containing $\sqrt{a^2 \pm x^2}$, $\sqrt{x^2 \pm a^2}$, or their powers, where a is a constant. Basically we use a kind of substitution; for example, we use a substitution $x = \tan\theta$ for the intuitive question above. However, it could be hard if we try to memorize every substitution in each cases. Thus, here we approach in a more intuitive way. The key observation is that the above terms usually come from the Pythagorean law. $\sqrt{a^2 + x^2}$, for example, is the length of hypotenuse in a right triangle, where the length of other two sides are a and x, respectively. Similarly, $\sqrt{x^2 - a^2}$ is the length of height (or base, respectively) in a right triangle, where the length of hypotenuse is x and the length of base (or height, respectively) is a. See the following pictures.



Therefore, if the integrand contains $\sqrt{a^2 + x^2}$, then it seems reasonable to set $x = a\tan\theta$, where θ is the angle assigned in the left triangle. Then we can express the integral in terms of θ, where the integrand now becomes a trigonometric function in θ, so we can use the trigonometric integration techniques from the previous section.

Example. Calculate the integral $\int \frac{1}{\sqrt{1+x^2}}dx$.

Using the following triangle, we let $x = \tan\theta$.



Then differentiating both sides gives

$$dx = \sec^2\theta d\theta.$$

Also from the triangle, we have

$$\frac{1}{\sqrt{1+x^2}} = \cos\theta,$$

so we have

$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \cos\theta(\sec^2\theta d\theta) = \int \sec\theta d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C.$$

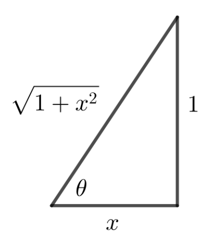
To express the result as a formula of x, we can observe the triangle again and find

$$\sec\theta = \sqrt{1+x^2}, \quad \tan\theta = x.$$

Therefore

$$\int \frac{1}{\sqrt{1+x^2}}dx = \ln\left|\sqrt{1+x^2} + x\right| + C. \quad \Diamond$$

⚠ In the previous example, we could use the following right triangle instead, with the same hypotenuse but with a base x and a height 1. It gives us a substitution $x = \cot\theta$, so



we have to know the derivative of cotangent function. Even if we know that, which gives us $dx = -\csc^2\theta d\theta$, we have $\sin\theta = {}^1/_{\sqrt{1+x^2}}$ so

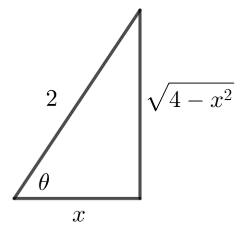
$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \sin\theta(-\csc^2\theta d\theta) = -\int \csc\theta d\theta.$$

Now we have to find the antiderivative of cosecant function; we still can find it, but as mentioned before we try to avoid calculating with these functions, and we can simply avoid it by using the original triangle. In most cases, if one right triangle gives a calculation containing cosecants or cotangents, the 'other version' of it will give a calculation with secants or tangents, which is more familiar to us.

We are now ready to face more complicated examples.

Example. Calculate the integral $\int \frac{x^2}{\sqrt{4-x^2}}dx$.

We use the following triangle.



From the triangle, the proper substitution is $x = 2\cos\theta$, so $dx = -2\sin\theta d\theta$. Also, $\sqrt{4-x^2} = 2\sin\theta$, so we have

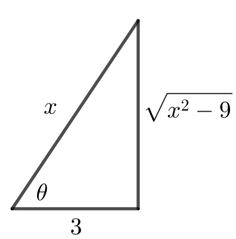
$$\begin{aligned}\int \frac{x^2}{\sqrt{4-x^2}}dx &= \int \frac{(2\cos\theta)^2}{2\sin\theta}(-2\sin\theta d\theta) \\ &= -4\int \cos^2\theta d\theta = -4\int \frac{1+\cos 2\theta}{2}d\theta \\ &= -2\left(\theta + \frac{\sin 2\theta}{2}\right) + C = -2\theta + \sin 2\theta + C \\ &= -2\theta + 2\sin\theta\cos\theta + C.\end{aligned}$$

Note that we used the square formula for $\cos^2 x$ and the double-angle formula for $\sin 2x$ from the previous section. Now using the information from the triangle again we get

$$\begin{aligned}\int \frac{x^2}{\sqrt{4-x^2}}dx &= -2\arccos\left(\frac{x}{2}\right) + 2\left(\frac{\sqrt{4-x^2}}{2}\right)\left(\frac{x}{2}\right) + C \\ &= -2\arccos\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C. \quad \diamondsuit\end{aligned}$$

Example. Calculate the integral $\int \frac{1}{x^2-9}dx$.

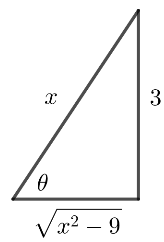
We use the following triangle.



Regarding the triangle, we let $x = 3\sec\theta$ so $dx = 3\sec\theta\tan\theta d\theta$. Also, we have $\sqrt{x^2-9} = 3\tan\theta$ so

$$\begin{aligned}\int \frac{1}{x^2-9}dx &= \int \frac{1}{(3\tan\theta)^2}(3\sec\theta\tan\theta d\theta)\\ &= \int \frac{\sec\theta}{3\tan\theta}d\theta = \frac{1}{3}\int \csc\theta d\theta.\end{aligned}$$

Since we are trying to avoid cosecant, we use the following triangle instead.



Then the substitution becomes $x = 3\csc\theta$, so $dx = -3\csc\theta\cot\theta d\theta$. Also, $\sqrt{x^2-9} = 3\cot\theta$ so

$$\begin{aligned}\int \frac{1}{x^2-9}dx &= \int \frac{1}{(3\cot\theta)^2}(-3\csc\theta\cot\theta d\theta)\\ &= \int \frac{-\csc\theta}{3\cot\theta}d\theta = -\frac{1}{3}\int \sec\theta d\theta\\ &= -\frac{1}{3}\ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

Therefore,

$$\begin{aligned}\int \frac{1}{x^2-9}dx &= -\frac{1}{3}\ln\left|\frac{x}{\sqrt{x^2-9}} + \frac{3}{\sqrt{x^2-9}}\right| + C\\ &= -\frac{1}{3}\ln\left|\frac{x+3}{\sqrt{x^2-9}}\right| + C. \quad \Diamond\end{aligned}$$

⚠ If we try to simplify the answer above using properties of a logarithmic function and a factorization $x^2 - 9 = (x+3)(x-3)$, we could find that it is equal to

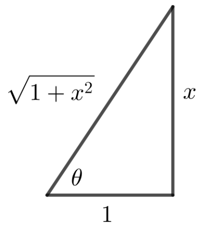
$$-\frac{1}{6}\ln\left|\frac{(x+3)^2}{x^2-9}\right| = -\frac{1}{6}\left|\frac{x+3}{x-3}\right| = -\frac{1}{6}\left(\ln|x+3| - \ln|x-3|\right).$$

Indeed, the fact that the denominator is a polynomial factored as a product of two linear functions allows us to approach the given integral in a totally different way. We will see it in the next section.

One final comment is about the definite integral. Since we are using a substitution, we have to be careful about the bounds.

Example. Calculate the definite integral $\displaystyle\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{3/2}}dx$.

Here we use the same triangle as in the first example.



We also use the same substitutions, so that

$$\int \frac{1}{(1+x^2)^{3/2}}dx = \int (\cos\theta)^3(\sec^2\theta d\theta) = \int \cos\theta d\theta.$$

For the bounds, we see how the triangle looks like for $x = 0$ and $x = \sqrt{3}$. If $x = 0$, it does not seem to be a triangle anymore, but 'approaching' x to 0 it is obvious that $x = 0$ corresponds to $\theta = 0$. For $x = \sqrt{3}$, it is a half of an equilateral triangle, so $\theta = \pi/3$. Therefore,

$$\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{3/2}}dx = \int_0^{\pi/3} \cos\theta d\theta = [\sin\theta]_{\theta=0}^{\theta=\pi/3} = \sin\frac{\pi}{3} - \sin 0 = \frac{\sqrt{3}}{2}. \quad \Diamond$$

⚠ It is also possible to calculate the definite integral after expressing the antiderivative as a function of x. Since

$$\int \frac{1}{(1+x^2)^{3/2}}dx = \int \cos\theta d\theta = \sin\theta + C = \frac{x}{\sqrt{1+x^2}} + C,$$

we have

$$\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{3/2}} = \left[\frac{x}{\sqrt{1+x^2}}\right]_{x=0}^{x=\sqrt{3}} = \frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2}.$$

Two methods are basically the same.

2.4 Partial Fractions

Intuitive Question. We know that $\int \frac{1}{x}dx = \ln|x| + C$ *and* $\int \frac{1}{x^2+1}dx = \arctan x + C$*. Then what would be* $\int \frac{1}{x(x^2+1)}dx$*?*

We have seen several integrals whose integrand is a fraction, especially a fraction of polynomials. (We call such fraction a rational function.) It is often hard to deal with the fraction with complicated denominators, but if the denominator can be factored as a product of 'simple' polynomials, then we can split the whole fraction into a sum or difference of several fractions (they are called partial fractions) and it gives the integral of whole fraction as a sum or difference of the integrals of partial fractions, which can be obtained by known methods. We start with the simplest case, where the denominator is factored as a product of two linear polynomials.

Example. Calculate the integral $\int \frac{2}{(x-1)(x+1)}dx$*.*

First we express the integrand as a sum of two partial fractions, where each fraction has one factor of $(x-1)(x+1)$ as denominator and an undetermined constant as numerator.

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Now we have to find values of A and B satisfying the equality for all values of x. To do this, we calculate the sum in the right hand side using the undetermined constants as follows:

$$\frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1) + B(x-1)}{(x-1)(x+1)} = \frac{(A+B)x + (A-B)}{(x-1)(x+1)}.$$

Comparing this with the original integrand, it seems that A and B should satisfy the following system of equations.

$$\begin{cases} A + B = 0 \\ A - B = 2 \end{cases}$$

We can solve this system by adding and substracting two equations; it gives $A = 1$ and $B = -1$. Now we obtain the partial fraction decomposition of the integrand

$$\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}.$$

Now using the antiderivative of reciprocal functions, we get

$$\int \frac{2}{(x-1)(x+1)}dx = \int \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx = \ln|x-1| - \ln|x+1| + C. \quad \Diamond$$

Sometimes we need to factor the denominator first.

Example. Calculate the integral $\displaystyle\int \frac{3x+11}{x^2-x-6}dx$.

We can factor the denominator as

$$x^2-x-6=(x-3)(x+2),$$

and split the integrand as

$$\frac{3x+11}{x^2-x-6}=\frac{A}{x-3}+\frac{B}{x+2}$$

with undetermined constants A and B. Then the right hand side can be calculated as

$$\frac{A}{x-3}+\frac{B}{x+2}=\frac{A(x+2)+B(x-3)}{(x+2)(x-3)}=\frac{(A+B)x+(2A-3B)}{x^2-x-6},$$

so A and B should satisfy the following system of equations.

$$\begin{cases} A+B=3 \\ 2A-3B=11 \end{cases}$$

Adding 3 times the first equation with the second equation gives $5A=20$ so $A=4$, and it directly gives $B=-1$. Therefore,

$$\int \frac{3x+11}{x^2-x-6}dx=\int\left(\frac{4}{x-3}-\frac{1}{x+2}\right)dx=4\ln|x-3|-\ln|x+2|+C. \quad \diamondsuit$$

In previous examples, we can observe that the partial fraction decomposition is possible since there are only an x-term and a constant term in the original numerator. There is no way to generate x^2-term or terms with higher powers of x in our decomposition process. Therefore, if the degree of the original numerator is greater than or equal to that of the original denominator, we need to use the long division first, and do the partial fraction decomposition with the remainder only. This process is illustrated in the following example.

Example. Calculate the integral $\displaystyle\int \frac{x^3-4x+5}{x^2-x-6}dx$.

Since the degree of numerator (3) is greater than the degree of denominator (2), we use the long division of the numerator by the denominator first.

$$\begin{array}{r|rrrr}
 & & & x & +1 \\ \hline
x^2-x-6 & x^3 & & -4x & +5 \\
 & x^3 & -x^2 & -6x & \\ \hline
 & & x^2 & +2x & +5 \\
 & & x^2 & -x & -6 \\ \hline
 & & & 3x & +11
\end{array}$$

Using this long division, we get

$$\frac{x^3-4x+5}{x^2-x-6} = (x+1) + \frac{3x+11}{x^2-x-6}.$$

Now using the partial fraction decomposition in the previous example,

$$\begin{aligned}\int \frac{x^3-4x+5}{x^2-x-6}dx &= \int \left((x+1) + \frac{3x+11}{x^2-x-6}\right) dx \\ &= \int \left((x+1) + \frac{4}{x-3} - \frac{1}{x+2}\right) dx \\ &= \frac{1}{2}x^2 + x + 4\ln|x-3| - \ln|x+2| + C. \quad \diamondsuit\end{aligned}$$

Based on these basic examples, we now go through some variations. First, if the denominator is factored as a product of more than two linear factors, we can just use a similar method.

Example. Calculate the integral $\int \frac{5x+1}{(x-1)(x+1)(x+2)}dx$.

Since the degree of numerator (1) is strictly smaller than the degree of denominator (3), we do not need a long division. So we directly go to the partial fraction decomposition as

$$\frac{5x+1}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}.$$

The right hand side can be calculated as

$$\begin{aligned}\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} &= \frac{A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)}{(x-1)(x+1)(x+2)} \\ &= \frac{(A+B+C)x^2 + (3A+B)x + (2A-2B-C)}{(x-1)(x+1)(x+2)}.\end{aligned}$$

Therefore, the undetermined constants A, B and C should satisfy the following system of equations.

$$\begin{cases} A+B+C=0 \\ 3A+B=5 \\ 2A-2B-C=1 \end{cases}$$

Adding the first and third equations gives $3A - B = 1$, so with the second equation we can get $6A = 6$ so $A = 1$. Then it is immediate that $B = 2$ and $C = -3$. Therefore,

$$\begin{aligned}\int \frac{5x+1}{(x-1)(x+1)(x+2)}dx &= \int \left(\frac{1}{x-1} + \frac{2}{x+1} - \frac{3}{x+2}\right) dx \\ &= \ln|x-1| + 2\ln|x+1| - 3\ln|x+2| + C. \quad \diamondsuit\end{aligned}$$

⚠ In the previous example, instead of solving the system of equations, we can also just start from identifying the numerators as

$$5x+1 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$

and substituting some special values of x to find A, B and C. Observing the equation above, it is natural to try $x=-2$, $x=-1$ and $x=1$ since they make several terms be zero. Explicitly we have

$$\begin{aligned} x=-2: &\quad -9 = 5(-2)+1 = C(-2-1)(-2+1) = 3C, \\ x=-1: &\quad -4 = 5(-1)+1 = B(-1-1)(-1+2) = -2B, \\ x=1: &\quad 6 = 5(1)+1 = A(1+1)(1+2) = 6A, \end{aligned}$$

so we get $A=1$, $B=2$ and $C=-3$ without solving the system of equation algebraically. This method to find the undetermined constants often allows us to do simpler calculations, especially when the denominator is a product of distinct linear polynomials, but we do not recommend to use this method for all problems since there are several cases where it does not work anymore. Instead, setting up and solving a system of equations work for every case that we deal with in this section.

If the denominator has some multiple factors, it becomes little more complicated. If $(x+a)^2$ is a factor of the denominator for some a, we set two partial fractions corresponding to this; one with $x+a$, and the other with $(x+a)^2$ in the denominator. For other factors in the original denominator, if it is not a multiple factor, we set one partial fraction per a factor as before.

Example. Calculate the integral $\displaystyle\int \frac{3x^2-10x-7}{(x-2)^2(x+3)}dx.$

Note that we do not need a long division since the degree of numerator (2) is strictly smaller than the degree of denominator (3). Since $(x-2)^2$ is the only multiple factor of denominator, we set up the partial fraction decomposition as

$$\frac{3x^2-10x-7}{(x-2)^2(x+3)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+3}.$$

Calculating the right hand side, we have

$$\begin{aligned} \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+3} &= \frac{A(x-2)(x+3) + B(x+3) + C(x-2)^2}{(x-2)^2(x+3)} \\ &= \frac{(A+C)x^2 + (A+B-4C)x + (-6A+3B+4C)}{(x-2)^2(x+3)}. \end{aligned}$$

Therefore, A, B and C should satisfy the following system of equations.

$$\begin{cases} A+C = 3 \\ A+B-4C = -10 \\ -6A+3B+4C = -7 \end{cases}$$

Now subtracting the third equation from 3 times the second equation gives us $9A - 16C = -23$, so with the first equation we get $25C = 50$ then $C = 2$, $A = 1$, and $B = -3$. Therefore,

$$\begin{aligned} \int \frac{3x^2 - 10x - 7}{(x-2)^2(x+3)} dx &= \int \left(\frac{1}{x-2} - \frac{3}{(x-2)^2} + \frac{2}{x+3} \right) dx \\ &= \ln|x-2| + \frac{3}{x-2} + 2\ln|x+3| + C. \quad \Diamond \end{aligned}$$

In general, if $(x+a)^n$ is a factor of the denominator for some a, we set n partial fractions corresponding to this; each one has $x+a$, $(x+a)^2$, $\cdots$, $(x+a)^n$ in the denominator and an undetermined constant in the numerator.

Finally, we study the case with a quadratic factor. To avoid excessive complexity, we only deal with a quadratic factor of the form $x^2 + a$. Note that this quadratic factor should be irreducible, i.e., it should not be factored again as a product of two linear polynomials. If it is really factored, then we should apply the above methods for linear factors. If there is an irreducible quadratic factor, we set only one partial fraction corresponding to the factor, but we put an undetermined linear polynomial (with two undetermined coefficients) in the numerator, not as in cases with linear factors. The rest of process is almost the same. Also, if there is a multiple quadratic factor, then we use a similar method as with a multiple linear factor.

Example. Calculate the integral $\displaystyle\int \frac{x^3 - 2}{(x^2+1)(x-2)} dx$.

Since the degree of numerator (3) is equal to the degree of denominator (3), we first execute a long division as follows. Note that $(x^2+1)(x-2) = x^3 - 2x^2 + x - 2$ in the denominator.

$$\begin{array}{r|rrrr} & & & & 1 \\ \cline{2-5} x^3 - 2x^2 + x - 2 & x^3 & & & -2 \\ & x^3 & -2x^2 & +x & -2 \\ \cline{2-5} & & 2x^2 & -x & \end{array}$$

Using this long division, we get

$$\frac{x^3 - 2}{(x^2+1)(x-2)} = 1 + \frac{2x^2 - x}{(x^2+1)(x-2)}.$$

Now we set up the partial fraction decomposition as follows.

$$\frac{2x^2 - x}{(x^2+1)(x-2)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x-2}$$

Calculating the right hand side, we have

$$\begin{aligned} \frac{2x^2 - x}{(x^2+1)(x-2)} &= \frac{(Ax+B)(x-2) + C(x^2+1)}{(x^2+1)(x-2)} \\ &= \frac{(A+C)x^2 + (-2A+B)x + (-2B+C)}{(x^2+1)(x-2)}. \end{aligned}$$

Therefore, A, B and C should satisfy the following system of equations.

$$\begin{cases} A + C = 2 \\ -2A + B = -1 \\ -2B + C = 0 \end{cases}$$

Adding 2 times the first equation to the second equation, we get $B + 2C = 3$, so with the third equation we get $5C = 6$ and $C = 6/5$, then $B = 3/5$, then $A = 4/5$. Therefore,

$$\begin{aligned} \int \frac{x^3 - 2}{(x^2+1)(x-2)} dx &= \int \left(1 + \frac{1}{5}\left(\frac{4x+3}{x^2+1}\right) + \frac{6}{5}\left(\frac{1}{x-2}\right)\right) dx \\ &= x + \frac{1}{5}\left(2\ln|x^2+1| + 3\arctan x\right) + \frac{6}{5}\ln|x-2| + C. \quad \diamondsuit \end{aligned}$$

We finish this section with a summary of the general process.

Partial fraction process

Step 1) Compare the degree of numerator and denominator. Execute a long division if needed.

Step 2) Set up a partial fraction decomposition with proper denominators corresponding to the factors of the original denominator and numerators with undetermined constants.

Step 3) Calculate the sum of partial fractions and make a system of equation by identifying coefficients, according to the following table.

Factor	#	Denominators	Numerators
Single linear factor $x+a$	1	$x+a$	A
Multiple linear factor $(x+a)^2$	2	$x+a$ $(x+a)^2$	A B
Single quadratic factor x^2+a	1	x^2+a	$Ax+B$
Multiple quadratic factor $(x^2+a)^2$	2	x^2+a $(x^2+a)^2$	$Ax+B$ $Cx+D$

(Here # implies the number of partial fractions corresponding to the given factor.)

Step 4) Solve the equation to find undetermined constants.

Step 5) Find the antiderivative of the original function using obtained partial fraction decomposition and known integration formulas.

2.5 Improper Integrals

Intuitive Question. Consider the graph of a function $y = 1/x$, and the region enclosed by the graph, a line $x = 1$, and the x-axis. How can we find the area of this region? Is it finite or infinite? What if the function is $y = 1/x^2$, or generally $y = 1/x^p$ instead (where p is any number)? Could we find the criteria of the value of p such that the region has a finite area?

In this section, we study some irregular cases in definite integrals. In this section, basically, we deal with two types of improper definite integrals: those with infinite bounds, and those with an undefined point in the interval of integration.

The intuitive question above corresponds to the first type, since we can express the area as a definite integral $\displaystyle\int_1^\infty \frac{1}{x^p}dx$. But we already learned a method to deal with infinity - limits. So to speak, we first set a variable to represent the bound which is originally infinite, calculate the usual definite integral using the variable as the bound instead, and send the variable to the proper infinity to get the desired definite integral, as a limit of a function in the variable. The following example shows how this process works.

Example. Calculate the definite integral $\displaystyle\int_1^\infty \frac{1}{x}dx$.

We use a variable t as the upper bound instead of infinity, and calculate the definite integral:

$$\int_1^t \frac{1}{x}dx = [\ln|x|]_{x=1}^{x=t} = \ln t - \ln 1 = \ln t.$$

Now we can express the original integral as a limit of the above integral in t, as follows.

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty}\ln t = \infty. \quad \Diamond$$

As in the previous example, if the integral (as a limit) does not converge to a finite value, we say that the integral diverges. Otherwise we say that the integral converges, and in this case the value of integral is defined as the limit.

Example. Calculate the definite integral $\displaystyle\int_1^\infty \frac{1}{x^3}dx$.

We again use t as the upper bound, and calculate the definite integral:

$$\int_1^t \frac{1}{x^3}dx = \left[-\frac{1}{2x^2}\right]_{x=1}^{x=t} = \frac{1}{2} - \frac{1}{2t^2}.$$

Then the original integral is

$$\int_1^\infty \frac{1}{x^3}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^3}dx = \lim_{t\to\infty}\left(\frac{1}{2} - \frac{1}{2t^2}\right) = \frac{1}{2}. \quad \Diamond$$

It seems that the convergence behavior of $\displaystyle\int_1^\infty \frac{1}{x^p}dx$ depends on p. Indeed, if $p > 1$, we have

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}dx = \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_{x=1}^{x=t} = \lim_{t\to\infty}\frac{1}{1-p}(t^{1-p}-1).$$

But $1 - p < 0$ since $p > 1$, so t^{1-p} goes to 0 as t goes to infinity. Therefore

$$\int_1^\infty \frac{1}{x^p}dx = -\frac{1}{1-p},$$

which means that the improper integral converges.

On the other hand, if $p < 1$, we still get the same limit, but in this case $1-p > 0$ so t^{1-p} goes to infinity. Therefore

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\frac{1}{1-p}(t^{1-p}-1) = \infty.$$

(The limit is positive infinity, not negative infinity, since $p < 1$.) We can summarize the result, with the first example, as follows. Note that this fact will be revisited in later sections.

$$\boxed{\text{Integration of } \int_1^\infty \frac{1}{x^p}dx}$$

- If $p > 1$, the integral converges to $-{}^1\!/_{(1-p)}$.
- If $p < 1$, the integral diverges.

It is also possible that the lower bound is negative infinity, where we use a similar method.

Example. Calculate the definite integral $\displaystyle\int_{-\infty}^0 \frac{1}{\sqrt{1-x}}dx$.

We use t as the lower bound, so that

$$\begin{aligned}\int_{-\infty}^0 \frac{1}{\sqrt{1-x}}dx &= \lim_{t\to-\infty}\int_t^0 \frac{1}{\sqrt{1-x}}dx\\ &= \lim_{t\to-\infty}\left[-2\sqrt{1-x}\right]_{x=t}^{x=0}\\ &= \lim_{t\to-\infty}(2\sqrt{1-t}-2).\end{aligned}$$

When t goes to negative infinity $1-t$ goes to (positive) infinity, so the limit is infinity. Therefore, the integral diverges. $\Diamond$

Finally, we study a case where both happen; so that the upper bound is positive infinity and the lower bound is negative infinity. In this case, we choose any constant c and divide the integral into two parts, an upper part from c to positive infinity and a lower part from negative infinity to c. Then we use two limits to determine whether each part converges or diverges.

Example. Calculate the definite integral $\int_{-\infty}^{\infty} xe^{-x^2}dx$.

We choose $c = 0$ as the 'splitting' constant. (We could choose other constant as c and it will give the same answer, but $c = 0$ seems to be the most natural here.) Then we split the integral into two parts as follows:

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx.$$

For the upper part, we use t as the upper bound so that

$$\begin{aligned}\int_{0}^{\infty} xe^{-x^2}dx &= \lim_{t\to\infty}\int_{0}^{t} xe^{-x^2}dx \\ &= \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_{x=0}^{x=t} \\ &= \lim_{t\to\infty}\frac{1}{2}\left(1-e^{-t^2}\right) = \frac{1}{2}.\end{aligned}$$

For the lower part, we use s as the lower bound so that

$$\begin{aligned}\int_{-\infty}^{0} xe^{-x^2}dx &= \lim_{s\to-\infty}\int_{s}^{0} xe^{-x^2}dx \\ &= \lim_{s\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_{x=s}^{x=0} \\ &= \lim_{s\to-\infty}\frac{1}{2}\left(e^{-s^2}-1\right) = -\frac{1}{2}.\end{aligned}$$

Note that even though s goes to negative infinity, e^{-s^2} still goes to zero since s^2 goes to positive infinity. Now we add two parts to get

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = -\frac{1}{2} + \frac{1}{2} = 0. \quad \Diamond$$

⚠ We should be careful about two points when we deal with such 'double-infinity' case. First, our two limits to be calculated independently, i.e., we should not calculate two limits at once using the same variable. That is why we used two different variables in the previous example. Second, if any of those two limits diverges, then the whole integral diverges regardless of whether the other limit converges or diverges. We might see some cases where one limit diverges to positive infinity and the another diverges to negative infinity (one easy example is $\int_{-\infty}^{\infty} xdx$) but the whole integral diverges also in this case.

Below is the summary of general formulas for the first type.

$\boxed{\text{Improper integrals with infinite bounds}}$

- Positive infinity in the upper bound:
$$\int_c^\infty f(x)dx = \lim_{t\to\infty}\int_c^t f(x)dx.$$
- Negative infinity in the lower bound:
$$\int_{-\infty}^c f(x)dx = \lim_{t\to-\infty}\int_t^c f(x)dx.$$
- Both bounds are infinite:
$$\begin{aligned}\int_{-\infty}^\infty f(x)dx &= \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx \\ &= \lim_{s\to-\infty}\int_s^c f(x)dx + \lim_{t\to\infty}\int_c^t f(x)dx,\end{aligned}$$
where c is any constant.

We can also approach the second case with limits. When we calculate the definite integral of a function that is not defined at one of the bounds, we first set a variable to represent that bound, calculate the integral with the variable, and send the variable to the original bound. The only difference with the first type is that we sometimes need to use an one-sided limit.

Example. Calculate the definite integral $\int_0^1 \frac{1}{x^2}dx$.

Since the integrand is not defined at the lower bound $x = 0$, we use a variable t as the lower bound. Then we calculate the definite integral,
$$\int_t^1 \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_{x=t}^{x=1} = \frac{1}{t} - 1.$$
Then we use a limit sending t to 0, but in this case we need to use the right handed limit since, intuitively, t represents the lower bound starting somewhere between 0 and 1 and approaching 0 from the right hand side. Therefore, we have
$$\int_0^1 \frac{1}{x^2}dx = \lim_{t\to0^+}\int_t^1 \frac{1}{x^2}dx = \lim_{t\to0^+}\left(\frac{1}{t} - 1\right) = \infty,$$
so the integral diverges. $\diamondsuit$

We can deal similarly with the case where the integrand is not defined at the upper bound. The remaining possibility is an undefined point between two bounds, where we need to split the integral into two integrals divided at the undefined point. In this case, we need to pay a special attention since the integral sometimes seems to be just a normal definite integral.

Example. Calculate the definite integral $\int_0^{\pi} \sec x dx$.

Although the integrand is well defined at both bounds, it is not defined at a point $x = {}^{\pi}/_{2}$ between two bounds. Therefore we first split the integral into two integrals:

$$\int_0^{\pi} \sec x dx = \int_0^{\pi/2} \sec x dx + \int_{\pi/2}^{\pi} \sec x dx.$$

In the first integral, the integrand is not defined at the upper bound so we use a variable t as the upper bound. Here t approaches ${}^{\pi}/_{2}$ from the left hand side, so we use the left handed limit. Therefore we get

$$\begin{aligned}
\int_0^{\pi/2} \sec x dx &= \lim_{t\to\pi/2^-} \int_0^{t} \sec x dx \\
&= \lim_{t\to\pi/2^-} \left[\ln|\sec x + \tan x|\right]_{x=0}^{x=t} \\
&= \lim_{t\to\pi/2^-} (\ln|\sec t + \tan t| - 0) = \infty.
\end{aligned}$$

Here we used the fact that $\sec t = {}^{1}/_{\cos t}$ and $\tan t$ both goes to infinity as t goes to ${}^{\pi}/_{2}$ from the left hand side. Since the first integral diverges, the whole integral diverges regardless of whether the second integral converges or diverges. $\diamondsuit$

We finish this section with the summary of general formulas for the second type. Note that in the third case, as before, we should calculate two integrals independently and the whole integral converges if and only if both integrals converge.

$\boxed{\text{Improper integrals with an undefined point}}$

- Undefined point in the upper bound:

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx.$$

- Undefined point in the lower bound:

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx.$$

- Undefined point between two bounds:

$$\begin{aligned}
\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\
&= \lim_{s\to c^-} \int_a^s f(x)dx + \lim_{t\to c^+} \int_t^b f(x)dx,
\end{aligned}$$

where c is the undefined point.

2.6 Direct Comparison Test for Improper Integrals

Intuitive Question. We can show that $\displaystyle\int_1^\infty \frac{1}{x^3}dx$ *converges to* $1/2$*. What would this imply about an improper integral* $\displaystyle\int_1^\infty \frac{1}{x^3+5}dx$*?*

As the last topic in the integration techniques, we study a powerful method to determine the convergence behavior of certain improper integrals which cannot be directly calculated. Basically, we compare the given integrand with another simple function whose antiderivative can be calculated by known methods. Such process is called the Direct Comparison Test, often abbreviated to DCT.

> *Example. Determine whether the improper integral* $\displaystyle\int_1^\infty \frac{1}{x^3+5}dx$ *converges or diverges.*
>
> It is really hard to calculate the antiderivative of the integrand explicitly. Instead, we use the facts that
> $$0 \le \frac{1}{x^3+5} \le \frac{1}{x^3}$$
> for all $x \ge 1$ and $\displaystyle\int_1^\infty \frac{1}{x^3}dx$ converges. If we draw graphs of two functions, we could see that the graph of $1/(x^3+5)$ is located below the graph of $1/x^3$ if $x \ge 1$. The convergence of $\displaystyle\int_1^\infty \frac{1}{x^3}$ implies that the area of a region enclosed by the graph of $1/x^3$, x-axis and a line $x = 1$ is finite. Therefore the area of the region enclosed by $1/(x^3+5)$, x-axis and $x = 1$ should be also finite, since it is smaller than the area above. Now this implies that the integral $\displaystyle\int_1^\infty \frac{1}{x^3+5}$ converges. $\Diamond$

⚠ In the previous example, although it does not come to the front, one condition that both functions are nonnegative was used importantly. Indeed, $-1/x$ is a function which is also smaller than $1/x^3$ but we can prove that $\displaystyle\int_1^\infty -\frac{1}{x}dx$ diverges.

On the other hand, we can prove that an improper integral diverges by showing that a 'smaller' improper integral diverges, again provided that both integrands are nonnegative.

> *Example. Determine whether the improper integral* $\displaystyle\int_1^\infty \frac{1}{x(1+\sin^2 x)}dx$ *converges or diverges.*
>
> Observing the integrand, we can see that
> $$0 \le \sin^2 x \le 1$$
> for all x. Therefore
> $$0 \le \frac{1}{2x} = \frac{1}{x(1+1)} \le \frac{1}{x(1+\sin^2 x)}$$

for all $x \geq 1$. Now the improper integral $\int_1^\infty \frac{1}{2x}dx$ diverges, so the original integral also diverges.

We can formulate a general theory extending the examples above.

$\boxed{\text{DCT for improper integrals}}$

Fix a constant a. (In above examples, $a = 1$.)

(1) If $0 \leq f(x) \leq g(x)$ for all $x \geq a$ and $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ also converges.

(2) If $0 \leq f(x) \leq g(x)$ for all $x \geq a$ and $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ also diverges.

⚠ It is really important that DCT does not determine every case; lots of students make some mistakes here. For example, using the notations above, if $0 \leq f(x) \leq g(x)$ for all $x \geq a$ and $\int_a^\infty g(x)dx$ diverges, then we cannot conclude anything about $\int_a^\infty f(x)dx$; it could either converge or diverge. Also, the convergence of $\int_a^\infty f(x)dx$ does not guarantee the convergence behavior of $\int_a^\infty g(x)dx$.

When using DCT, a hard part is choosing a function to be compared with the original integrand. Although the basic strategy is to make it simpler, there could be several ways to do that, and some ways might not work well. In the previous example, for instance, we can also compare the original integrand with $1/x$ as

$$0 \leq \frac{1}{x(1+\sin^2 x)} \leq \frac{1}{x},$$

but it is inconclusive since $\int_1^\infty \frac{1}{x}dx$ diverges. Therefore we should try each possible way and see if it works.

Example. Determine whether the improper integral $\int_1^\infty \frac{1}{x+e^x}dx$ *converges or diverges.*

There are two straightforward way to make the integrand simpler, $1/x$ and $1/e^x$. If we compare the integrand with $1/x$, we have

$$0 \leq \frac{1}{x+e^x} \leq \frac{1}{x}$$

for all $x \geq 1$. However, $\displaystyle\int_1^\infty \frac{1}{x} dx$ diverges so it is inconclusive. Instead, we try comparing with ${}^1\!/_{e^x}$; we get

$$0 \leq \frac{1}{x + e^x} \leq \frac{1}{e^x} = e^{-x}$$

for all $x \geq 1$. Then since

$$\int_1^\infty e^{-x} dx = \lim_{t \to \infty} \int_1^t e^{-x} dx = \lim_{t \to \infty} [-e^{-x}]_{x=1}^{x=t} = \lim_{t \to \infty} (e^{-1} - e^{-t}) = e^{-1}$$

converges, we can conclude that the original integral also converges by DCT. $\Diamond$

Chapter 3

Sequences and Series

3.1 Sequences

Intuitive Question. (Zeno's paradox) Assume a dart is shot to a target which is 1 *yard away. To reach the target, the arrow should pass a midpoint, which is* $^1/_2$ *yard away from the starting point. Then the arrow should pass a midpoint between the first midpoint and the target, which is* $^3/_4$ *yard away from the starting point. After that, similarly, it should pass a* $^7/_8$*-yard point, a* $^{15}/_{16}$*-yard point, and so on. Is the dart really possible to reach the target?*

A sequence, as the word itself, is a sequence of numbers. In this section, we are interested in infinite sequences and their limits. The intuitive question gives the idea of infinite sequences. If we write down the distance from the starting point to each midpoint that the dart should pass, it may look like

$$\frac{1}{2},\ \frac{3}{4},\ \frac{7}{8},\ \frac{15}{16},\ \cdots.$$

This is surely an example of infinite sequences. However, we cannot write every term in an infinite sequence, we need to find a way to 'express' the sequence mathematically. The idea is to assign a positive integer to each term as follows,

$$\begin{array}{cccccc} n = & 1 & 2 & 3 & 4 & \cdots \\ & & & & & \\ & \frac{1}{2}, & \frac{3}{4}, & \frac{7}{8}, & \frac{15}{16}, & \cdots \end{array}$$

and express the n-th term as a formula of n. In this example, we can observe that

$$\begin{aligned} \frac{1}{2} &= 1 - \frac{1}{2} = 1 - \frac{1}{2^1} \\ \frac{3}{4} &= 1 - \frac{1}{4} = 1 - \frac{1}{2^2} \\ \frac{7}{8} &= 1 - \frac{1}{8} = 1 - \frac{1}{2^3} \\ \frac{15}{16} &= 1 - \frac{1}{16} = 1 - \frac{1}{2^4} \end{aligned}$$

so the n-th term can be expressed as

$$a_n = 1 - \frac{1}{2^n}.$$

Conversely, if the n-th term formula is given, then we can retrieve the corresponding sequence by substituting positive integers in place of n.

Example. Find the first four terms of a sequence which is defined by an n-th term formula

$$b_n = n^2 - 1.$$

Substituting $n = 1,\ 2,\ 3,\ 4,\ \cdots$, we have

$$\begin{aligned} b_1 &= 1^2 - 1 = 0 \\ b_2 &= 2^2 - 1 = 3 \\ b_3 &= 3^2 - 1 = 8 \\ b_4 &= 4^2 - 1 = 15 \end{aligned}$$

so the sequence begins with

$$0, \quad 3, \quad 8, \quad 15, \quad \cdots. \ \diamondsuit$$

Example. Find the first four terms of a sequence which is defined by an n-th term formula

$$c_n = (-1)^n.$$

Substituting $n = 1,\ 2,\ 3,\ 4,\ \cdots$, we have

$$\begin{aligned} c_1 &= (-1)^1 = -1 \\ c_2 &= (-1)^2 = 1 \\ c_3 &= (-1)^3 = -1 \\ c_4 &= (-1)^4 = 1 \end{aligned}$$

so the sequence begins with

$$-1, \quad 1, \quad -1, \quad 1, \quad \cdots. \ \diamondsuit$$

Now we study limits of infinite sequences. Recall limits of functions that we have learned; if $f(x)$ is a function,

$$\lim_{x\to\infty} f(x) = L$$

implies that $f(x)$ approaches a constant L when x goes to infinity. The same holds for sequences; if a_n is a sequence, i.e., it is a sequence defined by an n-th term formula a_n, then we say that the limit of the sequence is L, or the sequence converges to L, if the sequence approaches L when n goes to infinity and we write as

$$\lim_{L\to\infty} a_n = L.$$

When finding limits, it is usually convenient to use the n-th term formula.

> *Example. Find the limit, if exists, of a sequence which is defined by an n-th term formula*
>
> $$a_n = 1 - \frac{1}{2^n}$$
>
> When n goes to infinity, ${}^1\!/{}_{2^n}$ approaches 0 and so a_n approaches 0. Therefore
>
> $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1 - 0 = 1. \quad \Diamond$$

It is also possible that a sequence does not approach some finite number, but positive or negative infinity. Then we say that the sequence diverges to positive or negative infinity.

> *Example. Find the limit, if exists, of a sequence which is defined by an n-th term formula*
>
> $$b_n = n^2 - 1$$
>
> When n goes to infinity, $n^2 - 1$ approaches positive infinity. Therefore
>
> $$\lim_{n\to\infty} b_n = \lim_{n\to\infty}(n^2 - 1) = \infty,$$
>
> or the sequence diverges to positive infinity. $\Diamond$

Some functions, for example $f(x) = \sin x$, does not converge to a finite number, but also not diverge to any infinity as x goes to infinity. The same situation can happen for sequences; then we say that the limit does not exist. The most standard example is an 'alternating' sequence that we have seen above. Here 'alternating' means that the sign of each term alternates.

> *Example. Find the limit, if exists, of a sequence which is defined by an n-th term formula*
>
> $$c_n = (-1)^n$$
>
> When n goes to infinity, $(-1)^n$ oscillates between -1 and 1. It does not approach one number, and also it does not diverge to any infinity. Therefore the limit does not exist. $\Diamond$

⚠ Although the oscillation in the previous example basically comes from alternating signs caused by $(-1)^n$, it does not imply that the limit of every alternating sequence does not exist. For example, a sequence defined by an n-th term formula

$$d_n = (-1)^n \frac{1}{n}$$

also has alternating signs, but it has a limit 0 since both positive and negative parts eventually approach 0.

Because of a similarly between the limits of functions and sequences, we can apply many methods that we have used to find limits of functions to find limits of sequences.

Example. Find the limit, if exists, of a sequence which is defined by an n-th term formula

$$e_n = \frac{4n^3 + 5n^2 - 2n + 6}{n^3 - n - 2}$$

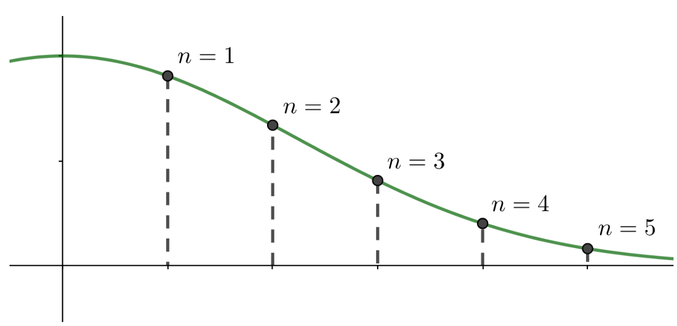
When finding the limit of a function of this form, we found a term with the highest degree and divide every term with it. Here we use the same method; since the highest degree is 3 we divide both numerator and denominator by n^3, so that

$$e_n = \frac{4n^3 + 5n^2 - 2n + 6}{n^3 - n - 2} = \frac{4 + 5/n - 2/n^2 + 6/n^3}{1 - 1/n^2 - 2/n^3}.$$

Now as n goes to infinity, every term except for 4 and 1 goes to 0. Therefore

$$\lim_{n\to\infty} e_n = \frac{4}{1} = 4. \quad \Diamond$$

Generally, for any sequence we can make a corresponding function by simply changing n to x. When we plot the graph of this function, each term in the sequence corresponds to the value of function at $x = 1,\ 2,\ 3,\ \cdots$ as shown in the following picture.



Therefore, if the corresponding function converges to a finite number, then the sequence also converges to the same number. Also, if the function diverges to positive or negative infinity, the sequence also diverges to the same infinity.

Example. Find the limit, if exists, of a sequence which is defined by an n-th term formula

$$f_n = \frac{n^2}{e^n}$$

Recall the limit of the corresponding function

$$\lim_{x\to\infty} \frac{x^2}{e^x}.$$

We have used L'Hospital's Rule twice to find the limit as

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

Since the corresponding function converges to 0, the sequence also converges to 0, i.e.,

$$\lim_{n\to\infty} f_n = 0. \quad \Diamond$$

⚠ The previous argument does not work if the limit of corresponding function does not exist. For example, a function $f(x) = \cos(2\pi x)$ corresponds to a sequence $g_n = \cos(2\pi n)$. The function oscillates between -1 and 1 so the limit of function does not exists. However, when we observe the first few terms of the sequence, they are

$$\begin{aligned} g_1 &= \cos 2\pi = 1, \\ g_2 &= \cos 4\pi = 1, \\ g_3 &= \cos 6\pi = 1, \end{aligned}$$

and so on. Since it is a constant sequence, it obviously converges to 1.

3.2 Series

Intuitive Question. Recall Zeno's paradox from the previous section. From the starting point, the dart travels $1/2$ yard until the first midpoint. Then it travels $1/4$ yard until the second midpoint, and then $1/8$ yard, $1/16$ yard, and so on. Then what can we say about the sum of those distances?

A series is basically an infinite sum of terms in a sequence. So if we are given a sequence $a_1, a_2, a_3, \cdots$, we can think of an infinite sum $a_1 + a_2 + a_3 + \cdots$. We express this infinite sum as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

and we call it an infinite series. Here $i = 1$ means that the sum starts at the first term, and ∞ at the top means that we add infinite terms. Σ is a symbol signifying a series. Since it is a greek letter Sigma, we call this expression a sigma notation.

⚠ Sometimes, with the same sequence, we will think of a sum $a_2 + a_3 + a_4 + \cdots$ which is also infinite. Since it starts at the second term, we can express this sum as

$$\sum_{n=2}^{\infty} a_n = a_2 + a_3 + a_4 + \cdots .$$

To find the value of this infinite sum, we again use the idea of limits. Specifically, we first find a sum of the first N terms, which is called the N-th partial sum and often denoted by s_N, as a formula of N and send N to infinity. If the limit exists as a finite number, we say that the series converges and the sum of series is defined as the number. If the limit diverges to some infinity or it does not exist, we say that the series diverges.

Example. Determine whether the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{n} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

We calculate the first several partial sum as follows.

$$\begin{aligned} s_1 &= a_1 = \frac{1}{2} \\ s_2 &= a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ s_3 &= a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \end{aligned}$$

These partial sums actually make a sequence from the previous section, which can be expressed as

$$s_N = 1 - \frac{1}{2^N}.$$

Since we saw that

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} \left(1 - \frac{1}{2^N}\right) = 1,$$

the given series converges and its sum is 1. ♢

Indeed, the above series is an example of one certain type of series. We can observe that each term can be obtained by multiplying $1/2$ to the previous term. In general, we can think of any series that each term can be obtained by multiplying a constant to the previous term. If we denote the first term as a and the multiplied constant as r, such series will look like

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n.$$

Series of this form is called geometric series. The series in the previous example is a geometric series with $a = r = 1/2$.

The convergence behavior of geometric series basically depends on the value of r. Here we first assume that $a \neq 0$; if $a = 0$ every term becomes 0 so there is nothing to do. If $r = 1$, assuming that $a \neq 0$, it is just a sum of infinite a's so the series diverges. Now we also assume that $r \neq 1$. To examine the criteria for a geometric series to converge, we calculate the N-th partial sum of a general geometric series given above. Since it is the sum of the first N terms, we have

$$s_N = a + ar + ar^2 + \cdots + ar^{N-1}.$$

Now we use a trick; we multiply r to both sides to get

$$rs_N = r(a + ar + ar^2 + \cdots + ar^{N-1}) = ar + ar^2 + ar^3 + \cdots + ar^N,$$

and substract this equation from the previous equation. Since intermediate terms are canceled, we get

$$(1-r)s_N = s_N - rs_N = a - ar^N = a(1-r^N).$$

Therefore,

$$s_N = \frac{a(1-r^N)}{1-r}.$$

Note that we could divide $1-r$ from the both sides since we assumed that $r \neq 1$ above. Now observe that every part of the right hand side does not depend on N, except for r^N. It means that when we send N to infinity, the limit will depend only on the behavior of r^N. If $-1 < r < 1$, then r^N goes to 0 when N goes to infinity, so

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{a(1-r^N)}{1-r} = \frac{a}{1-r}.$$

Therefore, the series converges and its sum is ${}^{a}/_{(1-r)}$. Otherwise, r^N diverges as N goes to infinity or the limit does not exist, the limit of s_N also diverges or does not exist. It follows that the series diverges. We can now summarize the criteria for the convergence behavior of geometric series as follows.

Convergence behavior of geometric series

For a geometric series $a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$ where $a \neq 0$,

- If $-1 < r < 1$, then the series converges and its sum is

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$

- Otherwise, the series diverges.

One more interesting type of series is a telescoping series. In telescoping series, there are cancellations in calculation of partial sums so that we can easily calculate partial sums. The following example is one of the most basic telescoping series.

Example. Determine whether the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Calculating the first several partial sums, we can observe some cancellations as follows.

$$\begin{aligned}
s_1 &= a_1 = \frac{1}{1} - \frac{1}{2} = 1 - \frac{1}{2} \\
s_2 &= a_1 + a_2 = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) = 1 - \frac{1}{3} \\
s_3 &= a_1 + a_2 + a_3 = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) = 1 - \frac{1}{4}
\end{aligned}$$

In general, we have

$$s_N = a_1 + a_2 + \cdots + a_N = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1} \right) = 1 - \frac{1}{N+1}.$$

Since

$$\lim_{N \to \infty} s_N = \lim_{N \to \infty} \left(1 - \frac{1}{N+1} \right) = 1,$$

the series converges and its sum is 1.

⚠ Sometimes the given series does not seem to be a telescoping series. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$$

does not seem to have any cancellation in partial sums. However, since

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)},$$

this series is actually the same one as in the previous example. Thus, in some cases, we might have to modify the given series to make it as a telescoping series.

In general, for any function $f(x)$, we can construct a telescoping series as

$$\sum_{n=1}^{\infty} (f(n) - f(n+1)).$$

In the previous example, for example, it was $f(x) = {}^1\!/\!_x$. But interestingly, if we replace $n+1$ above by $n+2$, $n+3$ or even $n+1000$, we still get a telescoping series. Although we might need to handle more terms, it is still possible to calculate partial sums and find the limit. See the following example.

Example. Determine whether the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{3}{(n+1)(n+4)}$$

First we need to rewrite the term using partial fractions as

$$\frac{3}{(n+1)(n+4)} = \frac{1}{n+1} - \frac{1}{n+4}.$$

Note that if $f(x) = {}^{1}/_{(x+1)}$, then this series can be represented as

$$\sum_{n=1}^{\infty}(f(n) - f(n+3)).$$

There seems to be no cancellation for the first few partial sums, but it eventually happens in the fourth partial sum.

$$\begin{aligned} s_4 &= \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) \\ &= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \end{aligned}$$

In the next partial sum, there are two cancellations;

$$\begin{aligned} s_5 &= \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{9}\right) \\ &= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right). \end{aligned}$$

In general, the first three positive parts and the last three negative parts are 'survived,' so that

$$\begin{aligned} s_N &= \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{N+1} - \frac{1}{N+4}\right) \\ &= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \left(\frac{1}{N+2} + \frac{1}{N+3} + \frac{1}{N+4}\right). \end{aligned}$$

If N goes to infinity, all negative parts go to zero, so

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \left(\frac{1}{N+2} + \frac{1}{N+3} + \frac{1}{N+4}\right) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}.$$

Therefore, the series converges and its sum is ${}^{13}/_{12}$. $\diamondsuit$

Geometric series and telescoping series are two kinds of series that we directly calculate the partial sums to determine the convergence behavior. However, there are other methods that allow us to determine the convergence behavior without calculating the partial sums explicitly. They are called convergence tests, and we will learn what are they and how to use them in next several sections.

Before then, we begin with one basic test as an introduction. It is called the n-th Term Test, because it uses the limit of terms in the series. The main idea is this; each partial sum can be obtained by adding one term to the previous partial sum. Now assume that a series converges, i.e., the partial sums converge to a finite number. Then the difference between adjacent partial sums, which is just a term in the series, should approach zero as the partial sums converge. Mathematically saying, this shows that the terms in a converging series should converge to zero. Conversely, if the terms in a series do not converge to zero then the series should diverge. We can now introduce the n-th Term Test as follows.

$\boxed{n\text{-th Term Test}}$

For a series $\displaystyle\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \neq 0$, i.e., the limit converges to a nonzero number, diverges to any infinity or does not exist, then the series diverges.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^3 + 2n^2 - 5n + 6}{4n^3 - 2n + 7}$$

Although it is impossible to calculate the partial sums explicitly, we know that

$$\lim_{n\to\infty} \frac{n^3 + 2n^2 - 5n + 6}{4n^3 - 2n + 7} = \lim_{n\to\infty} \frac{1 + {}^{2}/_{n} - {}^{5}/_{n^2} + {}^{6}/_{n^3}}{4 - {}^{2}/_{n^2} + {}^{7}/_{n^3}} = \frac{1}{4}$$

which is nonzero. Therefore, by the n-th Term Test, the series diverges. $\diamondsuit$

⚠ It is really important to note that the n-th Term Test only gives the divergence when the limit of terms does not converge to zero. If the limit of terms is zero, then we cannot conclude anything. For example, in both series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

the terms converge to zero but one series converges and the other one diverges. We will see which converges and which diverges in the next section.

3.3 Integral Test

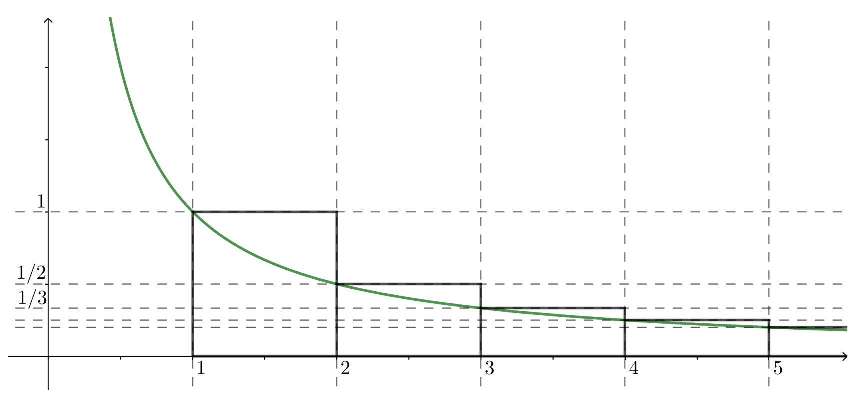
Intuitive Question. We learned that an improper integral $\int_1^\infty \frac{1}{x}dx$ *diverges. What would this imply about the convergence behavior of a series* $\sum_{n=1}^{\infty} \frac{1}{n}$*?*

In some cases, we can study the convergence behavior of series using improper integrals. Before we begin, recall that a definite integral is defined as the area under a curve. We will compare this area with another area, which represents the sum of corresponding series.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

We take a function $f(x) = {}^1\!/\!_x$. Then the improper integral $\int_1^\infty \frac{1}{x}dx$ represents the area under the graph of $f(x)$ from $x = 1$ to infinity. Now we can observe that for each $n = 1,\ 2,\ 3,\ \cdots,\ f(n) = {}^1\!/\!_n$ gives a term in the series $\sum_{n=1}^{\infty} \frac{1}{n}$. Using this fact, we now construct a region whose area represents the sum of this series. To do that, we draw some lines parallel to axes to make rectangles as in the following diagram.



Note that each rectangle has the base of length 1. Also, the leftmost rectangle has the height 1, and generally the n-th rectangle has the height ${}^1\!/\!_n$, so it has the area ${}^1\!/\!_n$. Now merging all of these rectangles, we get a stair-shaped region whose total area is equal to

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}.$$

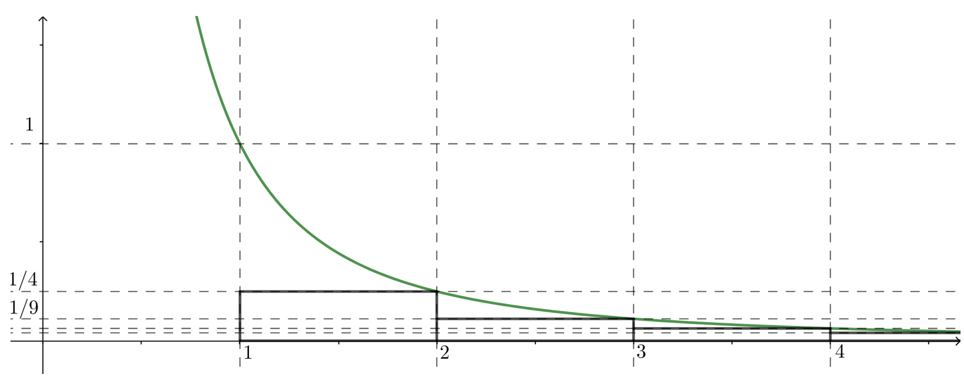
Now geometrically this area fully contains the area under the graph from $x = 1$ to infinity, but we have seen that the improper integral $\displaystyle\int_1^\infty \frac{1}{x}dx$ diverges to infinity, i.e., the area under the graph of $f(x)$ is infinite. Therefore, the sum of series should also be infinite, so we can conclude that the series diverges. $\Diamond$

Since the corresponding improper integral diverges, we made a bigger region so that the area is also infinite. However, if the corresponding improper integral converges, then we need to somehow make a smaller region.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

We take a function $f(x) = {}^1\!/\!_{x^2}$. Then the improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ represents the area under the graph of $f(x)$ from $x = 1$ to infinity. We have seen that this improper integral converges to 1. Now as in the previous example, we draw some lines parallel to axes to make rectangles as in the following diagram.



Again, each rectangle has the base of length 1. However, in this case, the leftmost rectangle has the height $1/4$, and generally the n-th rectangle has the height $1/(n+1)^2$, so it has the area $1/(n+1)^2$. Therefore, the total area of merged stair-shaped region is

$$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Note that the sum starts at $n = 2$, but it does not affect the convergence behavior of the whole series since the only difference is 1. Now this stair-shaped area is fully contained in the area under the graph from $x = 1$ to infinity, which is finite. Therefore, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges and so does the original series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. $\Diamond$

From two previous examples, it seems that the convergence behavior of an improper integral gives the convergence behavior of corresponding series. Therefore, it seems that we can always determine the convergence behavior of a series if the corresponding function is integrable; but unfortunately it is not true. The main idea of comparing two areas only works if the function satisfies certain conditions. Indeed, it should be positive, continuous and decreasing for sufficiently large x. We may check that if any of these three conditions is not satisfied, then we cannot guarantee anything about the relation between the area under the graph and the area of a stair-shaped region corresponding to the series. In two previous examples, both $1/x$ and $1/x^2$ were positive, continuous and decreasing for $x \geq 1$. We can now summarize this process as the Integral Test.

Integral Test

Assume that a function $f(x)$ is positive, continuous and decreasing for sufficiently large x. Then for a series $\displaystyle\sum_{n=1}^{\infty} a_n$, where $a_n = f(n)$ for each n,

- if an improper integral $\displaystyle\int_1^{\infty} f(x)dx$ converges, then the series also converges.
- if an improper integral $\displaystyle\int_1^{\infty} f(x)dx$ diverges, then the series also diverges.

⚠ 'Sufficiently large' in the statement may seem to be vague. Explicitly, our idea works if there is a constant a such that the function is positive, continuous and decreasing for all $x \geq a$. Then we may use the same idea as above to determine the convergence behavior of the 'partial series,' where we drop first few terms. But it is coincident with the convergence behavior of the whole series, since adding the finite number of terms does not affect the convergence behavior.

Example. Assume that p is a constant such that $p > 1$. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

The corresponding function $f(x) = 1/x^p$ is positive, continuous and decreasing for $x \geq 1$. Also, we have seen that the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x^p}dx$ converges in section 2.5. Therefore the series converges by the Integral Test. ♢

The exactly same argument works if $0 < p \leq 1$, where the series diverges then. Also, if $p \leq 0$, then we can observe that the limit of terms is actually nonzero. (Explicitly, it is 1 if $p = 0$ and ∞ if $p < 0$.) Therefore the series also diverges by the n-th Term Test.

Now we can determine the convergence behavior of any series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, which is called a p-series. p-series plays an important role in understanding many other series, so we summarize the results about p-series here.

$\boxed{\text{Convergence behavior of } p\text{-series}}$

A series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

We finish this section with a remark on the decreasing condition. When we check if a given function is decreasing, sometimes it is straightforward since, for example, the function is a fraction and the denominator is increasing while the numerator is fixed as in the case of $f(x) = {}^1\!/\!_x$. However, in some cases it is not obvious. In general, we need to differentiate the function and show that the derivative is negative, which requires some calculation. As in the previous remark, the function should be decreasing just for sufficiently large x, so the derivative also should be negative just for sufficiently large x.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2/4}}$$

The corresponding function is

$$f(x) = \frac{x}{e^{x^2/4}} = xe^{-x^2/4}.$$

It is positive and continuous for $x > 0$. Differentiating $f(x)$ we have

$$f'(x) = e^{-x^2/4} + x\left(-\frac{x}{2}\right)e^{-x^2/4} = \left(1 - \frac{x^2}{2}\right)e^{-x^2/4}.$$

Note that this is positive for small x; for example, $x = 1$ gives

$$f'(1) = \frac{1}{2}e^{-1/4}.$$

However, for sufficiently large x, say $x \geq 2$, we have $f'(x) < 0$ so f is decreasing for $x \geq 2$. Therefore we can apply the Integral Test. Now the improper integral converges since

$$\begin{aligned}
\int_1^{\infty} xe^{-x^2/4}dx &= \lim_{t\to\infty} \int_1^t xe^{-x^2/4}dx \\
&= \lim_{t\to\infty} [-2e^{-x^2/4}]_{x=1}^{x=t} \\
&= \lim_{t\to\infty} 2(e^{-1/4} - e^{-t^2/4}) = 2e^{-1/4},
\end{aligned}$$

so by the Integral Test the given series also converges. $\Diamond$

3.4 Comparison Tests

Intuitive Question. We learned that a series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ *converges. Then what could we say about a series* $\sum_{n=1}^{\infty} \frac{1}{n^3+5}$*?*

Recall DCT (Direct Comparison Test) for improper integrals, in section 2.6. We proved that an improper integral $\int_1^{\infty} \frac{1}{x^3+5}$ converges using the fact that

$$0 \le \frac{1}{x^3+5} \le \frac{1}{x^3}$$

and $\int_1^{\infty} \frac{1}{x^3}$ converges. Now, it is natural to make a similar statement for series. In general, we have the following.

DCT for series

Suppose that $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are series such that $0 \le a_n \le b_n$ for sufficiently large n.

- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.
- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

Note that as in DCT for improper integrals, the nonnegativity condition is required. Also, if $\sum_{n=1}^{\infty} a_n$ converges or $\sum_{n=1}^{\infty} b_n$ diverges, then the test is inconclusive; we cannot conclude anything about the other one.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^3+5}$$

We have

$$0 \le \frac{1}{n^3+5} \le \frac{1}{n^3}$$

for every n. Also, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges since it is a p-series with $p = 3 > 1$. Therefore, the given series converges by DCT. $\Diamond$

Thus, if we are given a series, we can prove the convergence by finding a 'bigger' series which converges, or prove the divergence by finding a 'smaller' series which diverges. However, sometimes it is hard to find a proper series to compare with the given series. For example, if we are given a series $\sum_{n=1}^{\infty} \frac{n+4}{2n^2-1}$, then it is straightforward to see

$$\frac{n+4}{2n^2-1} \geq \frac{n}{2n^2} = \frac{1}{2n},$$

and since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges (it is a half of a p-series of $p = 1$, which diverges) the given series diverges by DCT. However, if the given series is $\sum_{n=1}^{\infty} \frac{n-4}{2n^2+1}$, DCT with the natural comparison

$$\frac{n-4}{2n^2+1} \leq \frac{n}{2n^2} = \frac{1}{2n}$$

is inconclusive. It is still possible to use DCT though; for example, we can prove that

$$\frac{n-4}{2n^2+1} \geq \frac{1}{3n}$$

for sufficiently large n (explicitly for $n \geq 13$) so the given series diverges. However, instead of this, we can use another tool which can be applied directly to much more cases. It is called the Limit Comparison Test, often abbreviated as LCT, since it uses the limit of ratio between terms rather than a direct comparison of them. In LCT, we first calculate the limit of ratio, then the conclusion depends that limit.

LCT for series

Suppose that $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are series such that $0 \leq a_n, b_n$, and

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Then we have the following.

- $0 < L < \infty$: Both series converges or both series diverges. In other words, they have the same convergence behavior.
- $L = 0$: If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.
- $L = \infty$: If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

⚠ Note that we still need the nonnegativity condition; two comparison tests only work for series with nonnegative terms.

In LCT, two special cases where $L = 0, \infty$ is less important since they are basically the same as DCT. For example, $L = 0$ implies that a_n becomes smaller and smaller relative to b_n as n increases, so the convergence of $\sum_{n=1}^{\infty} b_n$ will imply the convergence of $\sum_{n=1}^{\infty} a_n$. Indeed, we do not recommend to use LCT in those cases since the statements for $L = 0$ and $L = \infty$ looks really similar but they are actually different, so it is easy to make mistakes.

However, as mentioned above, LCT is still more useful than DCT in most cases, especially if $0 < L < \infty$. We can find the dominating part in the n-th term (or the dominating parts of numerator and denominator if the n-th term is represented as a fraction) and take a series of dominating part (or the fraction of dominating parts) as the 'simplified' series. Then the original series will have the same convergence behavior with this simplified series, whose convergence behavior could be determined more easily.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^3 - 3n + 3}{5n^4 - 2n^3 + n + 7}$$

We can easily check that every term is nonnegative. Now dominating parts of numerator and denominator are n^3 and $5n^4$, respectively. Since the simplified series $\sum_{n=1}^{\infty} \frac{n^3}{5n^4} = \sum_{n=1}^{\infty} \frac{1}{5n}$ diverges, we could make a guess that the original series also diverges. To be explicit, we take

$$a_n = \frac{n^3 - 3n + 3}{5n^4 - 2n^3 + n + 7}, \quad b_n = \frac{1}{5n}.$$

Then

$$\begin{aligned} L = \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^3 - 3n + 3}{5n^4 - 2n^3 + n + 7}(5n) \\ &= \lim_{n\to\infty} \frac{5n(n^3 - 3n + 3)}{5n^4 - 2n^3 + n + 7} \\ &= \lim_{n\to\infty} \frac{5n^4 - 15n^2 + 15n}{5n^4 - 2n^3 + n + 7} \\ &= \lim_{n\to\infty} \frac{5 - {}^{15}/{n^2} + {}^{15}/{n^3}}{5 - {}^{2}/{n} + {}^{1}/{n^3} + {}^{7}/{n^4}} = \frac{5}{5} = 1. \end{aligned}$$

Since $0 < L < 1$, the original series has the same convergence behavior with the simplified series, so it diverges. $\Diamond$

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n+4}{\sqrt{n^5+7}}$$

Again, we can easily check that every term is nonnegative. Now dominating parts of numerator and denominator are n and $\sqrt{n^5} = n^{5/2}$, respectively, so the simplified series is $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^{5/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges since it is a p-series with $p = {}^3/_2 > 1$. Therefore, letting

$$a_n = \frac{n+4}{\sqrt{n^5+7}}, \quad b_n = \frac{1}{n^{3/2}} = \frac{n}{\sqrt{n^5}}$$

we have

$$\begin{aligned} L = \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(\frac{n+4}{\sqrt{n^5+7}}\right)\left(\frac{\sqrt{n^5}}{n}\right) \\ &= \lim_{n\to\infty} \left(\frac{n+4}{n}\right)\left(\frac{\sqrt{n^5}}{\sqrt{n^5+7}}\right) \\ &= \lim_{n\to\infty} \left(\frac{1+{}^4/_n}{1}\right)\sqrt{\frac{1}{1+{}^7/_{n^5}}} = 1. \end{aligned}$$

Since $0 < L < 1$, the original series has the same convergence behavior with the simplified series, so it converges. $\diamondsuit$

3.5 Ratio and Root Test

Intuitive Question. We know that the series $\sum_{n=1}^{\infty} \frac{1}{e^n}$ *converges, since it is a geometric series with common ratio* $1/e$*. Then would the series* $\sum_{n=1}^{\infty} \frac{n^5}{e^n}$ *also converge?*

We learned how to determine the convergence behavior of geometric series, depending on their common ratio, which is the ratio of two consecutive terms. If that ratio is not steady, then the series is definitely not geometric. However, when the limit of ratio converges to a constant, it could be possible to determine the convergence behavior from the constant. This is the idea of the Ratio Test.

Ratio Test

For a given series $\sum_{n=1}^{\infty} a_n$, assume that the absolute value of the ratio of two consecutive terms converges to a constant, say

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = R.$$

Then we have the following.

- If $R < 1$, then the given series converges.
- If $R > 1$, then the given series diverges.
- If $R = 1$, then the Ratio Test is inconclusive; the series could either converge or diverge.

Basically we can show that the Ratio Test works by comparing with a geometric series. For example, assume that every term is positive and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2.$$

(The positivity assumption is for a technical reason; we will see why we can assume this in the next section.) Then for sufficiently large n, we have

$$\frac{a_{n+1}}{a_n} \geq \frac{3}{2}$$

or equivalently

$$a_{n+1} \geq \frac{3}{2} a_n.$$

Therefore, after some finite number of terms, the series will be bigger than a geometric series with common ratio $3/2 > 1$ which diverges. Therefore, by DCT the original series also diverges. Even if 2 is replaced by any value of R which is bigger than 1, we can use the same method to show that the series diverges; we can take any r such that $1 < r < R$ and replace $3/2$ by r in the argument above. On the other hand, if $R < 1$, then we take any r such that $R < r < 1$ and compare the original series with a geometric series with common ratio r, which converges.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^5}{e^n}$$

Since every term is positive, we can ignore the absolute value. Letting $a_n = n^5/e^n$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} a_{n+1} \cdot \frac{1}{a_n} \\
&= \lim_{n\to\infty} \frac{(n+1)^5}{e^{n+1}} \cdot \frac{e^n}{n^5} \\
&= \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \cdot \frac{e^n}{e^{n+1}} \\
&= 1 \cdot \frac{1}{e} = \frac{1}{e} < 1.
\end{aligned}$$

Therefore, by the Ratio Test, the series converges. $\Diamond$

The Ratio Test is often used to determine the convergence behavior of series containing factorials. As a review, the factorial of a positive integer n, denoted as $n!$, is a product of all positive integers up to n. For example,

$$1! = 1, \quad 2! = 2 \cdot 1 = 2, \quad 3! = 3 \cdot 2 \cdot 1 = 6,$$

and so on. One interesting and important property of the factorial is that the ratio of two consecutive factorials can be expressed simply. Explicitly, we have

$$\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n \cdot (n-1) \cdot \cdots \cdot 1}{n \cdot (n-1) \cdot \cdots \cdot 1} = n + 1.$$

Therefore, if the n-th term of a given series contains some factorials, then the ratio test is often (but not always) useful since the ratio of two consecutive terms can be calculated easily.

⚠ When using the Ratio Test with factorials, be aware of the parentheses. The factorial of $n+1$ is denoted as $(n+1)!$ with the parentheses, and it is definitely different from $n+1!$ which is equal to $n+1$.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

Again, since every term is positive, we can ignore the absolute value. Now letting $a_n = {}^{3^n}/_{n!}$ we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} a_{n+1} \cdot \frac{1}{a_n} \\
&= \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \\
&= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} 3 \cdot \frac{1}{n+1} = 0 < 1.
\end{aligned}$$

Therefore, by the Ratio Test, the series converges. $\Diamond$

Now we study the Root Test, which is also inspired by geometric series. For a geometric series of the form $\sum_{n=1}^{\infty} r^n$, it converges if $|r| < 1$ and diverges if $|r| > 1$. Here r can be thought as the n-th root of r^n; similarly, we could determine the convergence behavior with the n-th root of the n-th term. This is the basic idea of the Root Test.

$\boxed{\text{Root Test}}$

For a given series $\sum_{n=1}^{\infty} a_n$, assume that the absolute value of the n-th root of the absolute value of the n-th term converges to a constant, say

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = R.$$

Then we have the following.

- If $R < 1$, then the given series converges.
- If $R > 1$, then the given series diverges.
- If $R = 1$, then the Root Test is inconclusive; the series could either converge or diverge.

Since both come from geometric series, the Ratio and Root Tests look similar. Indeed, the proofs of them are also similar; to prove the Root Test, we again compare the given series with a certain geometric series. Explicitly, for example, if every term is positive and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 2,$$

then for sufficiently large n we have

$$\sqrt[n]{a_n} \geq \frac{3}{2}$$

or equivalently

$$a_n \geq \left(\frac{3}{2}\right)^n,$$

so the series will eventually be bigger than a geometric series with common ratio $3/2 > 1$ which diverges. Therefore, by DCT the original series also diverges.

The Root Test is especially useful if the n-th term is represented as an n-th power, since then the n-th root and the n-th power will be canceled.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \left(\frac{n^3 - 2n^2 + 3n + 1}{4n^3 - 7n + 5}\right)^n$$

Since every term is positive, we can ignore the absolute value. Now letting

$$a_n = \left(\frac{n^3 - 2n^2 + 3n + 1}{4n^3 - 7n + 5}\right)^n,$$

we have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n^3 - 2n^2 + 3n + 1}{4n^3 - 7n + 5} = \lim_{n\to\infty} \frac{1 - 2/n + 3/n^2 + 1/n^3}{4 - 7/n^2 + 5/n^3} = \frac{1}{4} < 1.$$

Therefore, by the Root Test, the series converges. $\diamondsuit$

However, it is still possible to use the Root Test for other series. For example, suppose that we are given a series

$$\sum_{n=1}^{\infty} \frac{n^5}{e^n}$$

that we have already studied using the Ratio Test. Letting $a_n = n^5/e^n$, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^{5/n}}{e}.$$

Although we cannot directly calculate the limit of $n^{5/n}$, after taking log we have

$$\lim_{n\to\infty} \ln(n^{5/n}) = \lim_{n\to\infty} \frac{5}{n} \ln n = \lim_{n\to\infty} \frac{5 \ln n}{n} = \lim_{n\to\infty} \frac{5/n}{1} = \lim_{n\to\infty} \frac{5}{n} = 0.$$

In the third equality we used L'Hospital's Rule. Now it implies that

$$\lim_{n\to\infty} n^{5/n} = 1,$$

so

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{e} < 1.$$

Therefore we can conclude that the series converges.

⚠ For some series, such as the previous one, we can use both Ratio and Root Test with a simple calculation to determine the convergence behavior. However, in some cases, one works well but the other one has some difficulties. In general, if the n-th term contains some factorials, the Ratio Test works well while the Root Test does not, since it is hard to handle with the n-th root of factorials. On the other hand, if the n-th term contains the n-th power, then the situation is reversed, since the exponent changes as n does.

⚠ As the final remark of this section, the Ratio and Root Test are really useful tests that can be used for a wide range of series, but there are still many series, including relatively simple ones, where both tests are inconclusive. For example, any p-series cannot be dealt with either the Ratio or Root Test, since both the ratio of consecutive terms and the n-th root of the n-th term approach 1 as n approaches infinity. Therefore, it is not a good idea to use the Ratio or Root Test to every series.

3.6 Alternating Series

Intuitive Question. We know that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges, since it is a p-series with* $p = 1$*. Then would the series* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *also diverge?*

In most previous sections, we only deal with series with nonnegative terms. Although the Ratio and Root Test works for arbitrary series, we have assumed that every term is nonnegative so that we could ignore the absolute value. In this section, we will study alternating series, which have alternating signs in their terms, and how to determine its convergence behavior.

In general, a series is called an alternating series if it is of the form

$$\sum_{n=1}^{\infty} (-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

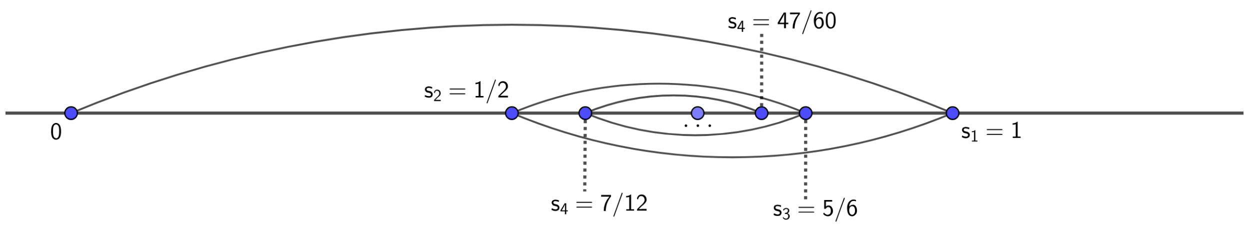
where $b_n \geq 0$, so that the signs of terms are alternating. Note that the starting point could be different. Also, it is possible that the exponent of (-1) is also different, but we can always make the exponent to either n or $n+1$ since $(-1)^2 = 1$. For example,

$$(-1)^{n+2} = (-1)^n(-1)^2 = (-1)^n.$$

Think about the series in the intuitive question above,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Here $a_n = {}^1\!/\!{}_n \geq 0$, and also we can observe that a_n decreases as n increases, and it eventually converges to 0. If we illustrate the partial sums of this series on a horizontal line as follows, it seems that the partial sums oscillate but the amplitude of oscillation decreases, so the series converges.



Indeed, it really converges, and it is also true for all series satisfying the same conditions mentioned above. We can formulate this as the Alternating Series Test, or Leibniz Test.

$\boxed{\text{Alternating Series Test (Leibniz Test)}}$

For a given alternating series

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} b_n$$

where $b_n \geq 0$ for sufficiently large n, assume that the following two conditions are true.

- b_n is decreasing for sufficiently large n.
- $\lim\limits_{n\to\infty} b_n = 0$.

Then the given series converges.

Note that the Alternating Series Test only works for a series of the given form, and it only gives the convergence, not the divergence, of the series.

Example. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+2}{2n^3+3}$$

Here $b_n = {}^{(n+2)}\!/\!{}_{(2n^3+3)}$, which is nonnegative for every n. Now for $f(x) = {}^{(x+2)}\!/\!{}_{(2x^3+3)}$ we have

$$f'(x) = \frac{(2x^3+3) - 6x^2(x+2)}{(2x^2+3)^2} = \frac{-4x^3 - 12x^2 + 3}{(2x^2+3)^2} < 0$$

for every $x \geq 1$. Therefore b_n decreases eventually. Also,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n+2}{2n^3+3} = \lim_{n\to\infty} \frac{1/n^2 + 2/n^3}{2 + 3/n^3} = 0.$$

Since all conditions for the Alternating Series Test are satisfied, we can conclude that the given series converges. $\Diamond$

If we are given an alternating series, or generally an arbitrary series, we can make a 'positive' version of the series, taking the absolute value of each term. Then it is natural to ask about the convergence behavior of this positive version, as well as that of the original series. If we consider two alternating series above

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+2}{2n^3+3},$$

they both converge by the Alternating Series Test. However, for the positive version of them

$$\sum_{n=1}^{\infty}\frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{n+2}{2n^3+3},$$

the former diverges since it is a p-series with $p = 1$, while the latter converges by LCT with a p-series with $p = 2$. These examples show that if a series converges, its positive version can either converge or diverge. Therefore, we can classify convergent series into two cases; depending on the convergence behavior of its positive version.

Absolute Convergence and Conditional Convergence

Suppose that a series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

- If its positive version $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, we say that the original series converges absolutely.
- If its positive version diverges, we say that the original series converges conditionally.

Thus, using those notations, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges conditionally while the other one $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+2}{2n^3+3}$ converges absolutely.

What if the original series diverges? Indeed, we can show that if the original series diverges, then the positive version of the series should diverge. Equivalently, if the positive version of a series converges, then the original series should converge. In fancy words, the absolute convergence always implies the (ordinary) convergence.

Absolute Convergence and Convergence

Consider a series $\sum_{n=1}^{\infty} a_n$. If its positive version $\sum_{n=1}^{\infty} |a_n|$ converges, then the original series also converges.

As a conclusion, a series can converge absolutely, converge conditionally or diverge. If we are given a series, it is natural to ask which convergence behavior it has, and we need to use a proper test to determine its convergence behavior. Here are some general useful strategies; reminds that there always could be some exceptional cases where these strategies do not work properly.

Strategies for Determining Convergence Behavior

(1) It is often useful to think about the limit of the n-th term first. If the limit of the n-th term does not converge to zero, the series diverges by the n-th Term Test.

(2) We can also glance the series to see if it is a geometric series or a p-series. If so, the convergence behavior immediately follows from its common ratio or its p-value.

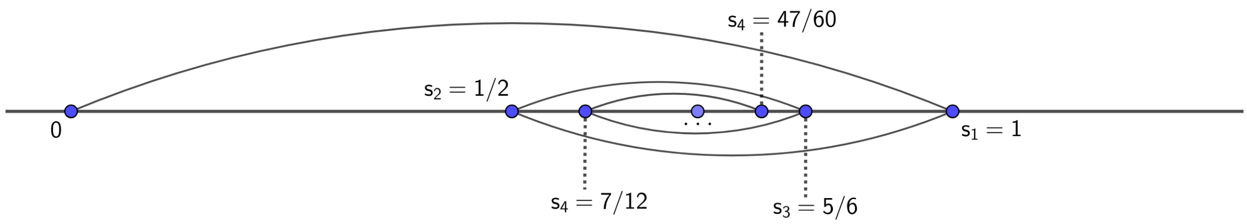
(3) If the series is not exactly a geometric series or a p-series but similar to one of them, try one of the comparison tests. If the n-th term is represented as a fraction, it is often useful to find dominating terms in both numerator and denominator, and use LCT with a series whose term is a fraction of dominating terms. Specifically, if the n-th term is a fraction of polynomials or some roots of polynomials, dominating term is just a term with the highest degree, so we could use LCT with a certain p-series.

(4) Also, if the series is an alternating series, we can try to apply the Alternating Series Test. Note that even if it converges by the Alternating Series Test, we still have to deal with its positive version if we are asked to determine the absolute convergence behavior as well.

(5) If anything does not work, try to use the Ratio and Root Test. In general, if the n-th term contains some factorials or the n-th power of a constant, then the Ratio Test is useful. On the other hand, if the n-th term contains the n-th power of a function of n, then the Root Test is useful.

(6) Finally, see if the corresponding function is integrable (or at least similar to an integrable function) and if so, try to use the Integral Test with possibly the comparison tests.

We finish this section with an interesting topic about alternating series. Recalling the picture showing that a series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges as below, we can observe that the limit of partial sums, which is equal to the sum of the series, should lie between any two consecutive partial sums.



For example, the sum of the series S should be between s_3 and s_4. Therefore the distance between s_3 and S should be less than or equal to s_3 and s_4, or mathematically

$$|S - s_3| \leq |s_4 - s_3|.$$

However, since $s_4 - s_3$ is equal to the fourth term $(-1)^5 {}^1\!/_4 = -{}^1\!/_4$, we have

$$|S - s_3| \leq \left|-\frac{1}{4}\right| = \frac{1}{4}.$$

Similarly, with s_4 and s_5 we will get

$$|S - s_4| \leq \frac{1}{5}.$$

In general, the difference between S and the N-th partial sum s_N is at most the absolute value of the $(N+1)$-th term. Thus, if we use the N-th partial sum as the approximation of the sum of the whole series, then the error should be at most the absolute value of $(N+1)$-th term. This is true in general, so we can formulate this as the error bound of alternating series.

Error Bound of Alternating Series

Suppose that an alternating series

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} b_n$$

satisfies the conditions of the Alternating Series Test, $b_n \geq 0$, b_n is eventually decreasing and $\lim_{n\to\infty} b_n = 0$, so that it converges. If we denote the sum of the series as S, then the following inequality holds for any positive integer N.

$$|S - s_N| \leq b_{N+1}$$

Using the above inequality, we can find how big N should be to approximate the sum as close as we want. If we are given an alternating series satisfying the conditions of the Alternating Series Test and a target error bound, we can calculate its terms to find a term whose absolute value is smaller than or equal to the target bound. Then the partial sum calculated until the previous term will give the desired approximation.

Example. Approximate the sum of the following series with a maximal error 0.05.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+2}{2n^3+3}$$

We have already seen that the series satisfies the conditions in the Alternating Series Test. Now we calculate the absolute value of the first few terms as follows.

$$\begin{aligned}
n=1: &\ \frac{3}{5}\\
n=2: &\ \frac{4}{19}\\
n=3: &\ \frac{5}{57}\\
n=4: &\ \frac{6}{131}\leq\frac{6}{120}=\frac{1}{20}=0.05
\end{aligned}$$

Therefore, if we take the third partial sum

$$\frac{3}{5}-\frac{4}{19}+\frac{5}{57}$$

as the approximation, the error should be at most 0.05. $\Diamond$

Chapter 4

Taylor Series

4.1 Power Series

Intuitive Question. Consider a series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$. *For which values of* x *does this series converge?*

In the previous chapter, we have seen various methods to determine the convergence behavior of given series. Those series were fixed; we could explicitly write the series as an infinite sum or difference of numbers. However, if the series contains a variable, the convergence behavior will depend on the value of such variable. It is then an interesting question to ask the values of variable which make the series converge. If the series is a geometric series, it is easy to determine those values.

Example. Find the values of x *such that the following series converges.*

$$\sum_{n=1}^{\infty} x^n$$

Since it is a geometric series with the common ratio x, we can conclude that the series converges if and only if $|x| < 1$ or equivalently $-1 < x < 1$.

However, it is still possible to formulate a standard method to answer the same question if the n-th term is 'similar' to the n-th power of a variable. Here arises the definition of power series. Explicitly, a power series in x at a constant a is a series of the form

$$\sum_{n=n_0}^{\infty} c_n (x-a)^n,$$

where an integer n_0 is the starting point and c_n is a function of n, i.e., a sequence. The constant a is called a center of this power series.

⚠ When finding the center of a given power series, we sometimes need to be careful. For example, if the series is given as

$$\sum_{n=1}^{\infty} c_n(2x+3)^n,$$

then its center is $-3/2$, not 3 or -3, since this series is the same as

$$\sum_{n=1}^{\infty} c_n 2^n \left(x - \left(-\frac{3}{2}\right)\right)^n.$$

Now suppose that we are given a power series in a variable x, say

$$\sum_{n=0}^{\infty} c_n(x-a)^n.$$

One interesting, and crucial, observation is that this series always converges if $x = a$, since then we have

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 0^0 + c_1 0^1 + c_2 0^2 + \cdots = c_0.$$

Here we used a convention $0^0 = 1$ and $0^n = 0$ for any positive integer n. Indeed, as we will see later, the values of x such that the series converges usually form an interval whose midpoint is the center of the given power series. In such cases, that interval is called the Interval of Convergence (IoC) of the power series. Also, the distance between the midpoint and one endpoint (which is equal to a half of the length of the whole interval) is called the Radius of Convergence (RoC) of the power series. The following example shows a standard method to find IoC and RoC of a given power series; basically we use the Ratio Test.

Example. Find the Radius of Convergence and the Interval of Convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$$

We let

$$a_n = \frac{(-1)^n n}{4^n}(x-3)^n$$

and use the Ratio Test. We have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|a_{n+1}\cdot\frac{1}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x-3)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{(-1)^n n(x-3)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)(x-3)}{4}\cdot\frac{n+1}{n}\right| \\
&= \lim_{n\to\infty}\frac{|x-3|}{4}\cdot\frac{n+1}{n} = \frac{|x-3|}{4}.
\end{aligned}$$

Therefore, the Ratio Test tells us that

- If ${}^{|x-3|}\!/_{4} < 1$, which is equivalent to $|x-3| < 4$ or $-1 < x < 7$, then the series converges.
- If ${}^{|x-3|}\!/_{4} > 1$, which is equivalent to $|x-3| > 4$, then the series diverges.

Now considering an interval from -1 to 7, we can conclude that the series converges inside the interval and diverges outside the interval. It immediately follows that the Radius of Convergence is 4. However, to find the Interval of Convergence we still have to deal with the endpoints where $x = -1, 7$ since the Ratio Test is inconclusive there. If $x = -1$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-1-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-1)^n 4^n = \sum_{n=1}^{\infty} n,$$

which diverges by the n-th Term Test. Similarly, if $x = 7$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(7-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} 4^n = \sum_{n=1}^{\infty} (-1)^n n$$

which also diverges by the same test. As a result, the Interval of Convergence can be represented by an inequality $-1 < x < 7$, or by an open interval $(-1, 7)$. $\Diamond$

Now we generalize this process to an arbitrary power series. Suppose that we have a power series $\displaystyle\sum_{n=1}^{\infty} c_n(x-a)^n$. Letting $a_n = c_n(x-a)^n$ and using the Ratio Test we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| \cdot |x-a| = |x-a| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|.$$

Therefore, if the limit $\displaystyle\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$ exists and it is nonzero and finite, then the series converges if

$$|x-a| < \frac{1}{\displaystyle\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|}$$

and it diverges if

$$|x-a| < \frac{1}{\displaystyle\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|} = R.$$

It directly follows that the Radius of Convergence is R which is defined as above, and the Interval of Convergence has two endpoints $a \pm R$. Since the Ratio Test is inconclusive at the endpoints, the power series could either converge or diverge there. For example, the series diverges at both endpoints in the previous example. However, some power series could converge at both endpoints of its IoC, and it is even possible that it converges at only one endpoint, as in the following example.

Example. Find the Radius of Convergence and the Interval of Convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(x-3)^n$$

We could also use the Ratio Test directly as the previous example, but here we use the general theory above. Since the coefficient is $c_n = {}^{(-1)^n}/_{(4^n n)}$, we can calculate

$$\begin{aligned}\frac{1}{R} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| &= \lim_{n\to\infty}\left|c_{n+1}\cdot\frac{1}{c_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{4^{n+1}(n+1)}\cdot\frac{4^n n}{(-1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-1)}{4}\cdot\frac{n}{n+1}\right| = \frac{1}{4}.\end{aligned}$$

Therefore, the Radius of Convergence is $R = 4$ and the endpoints of the Interval of Convergence are $3 \pm 4 = -1, 7$ as well. Now we see what happens at the endpoints. If $x = -1$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(-1-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(-1)^n 4^n = \sum_{n=1}^{\infty} \frac{1}{n},$$

which still diverges since it is a p-series with $p = 1$. However, if $x = 7$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(7-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}4^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test. Therefore, the Interval of Convergence can be represented by an inequality $-1 < x \leq 7$ or by an half-open interval $(-1, 7]$. $\Diamond$

⚠ We recommend to use the Ratio Test directly, instead of the method in the previous example, since it is easy to make some mistakes when using the ratio ${}^{c_{n+1}}/_{c_n}$. For example, when finding the Radius of Convergence R and ${}^1/_R$ are often confused. Moreover, if the power series contains a coefficient of x which is not equal to 1, we need to pull out the coefficient outside like when we find the center. However, one could omit this process and it leads to a wrong result.

As mentioned, the Interval of Convergence is not always a usual interval. Sometimes it could be so wide that it covers all numbers. On the other hand, it also could be so narrow that it does not cover almost all numbers. However, we should remember that any power series converges at its center.

Example. Find the Radius of Convergence and the Interval of Convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(3x+5)^n}{n!}$$

Letting

$$a_n = \frac{(3x+5)^n}{n!}$$

and using the fact that $(n+1)! = (n+1) \cdot n!$ we have

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(3x+5)^{n+1}}{(n+1)!} \cdot \frac{n!}{(3x+5)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{3x+5}{n+1} \cdot \right| \\
&= |3x+5| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1
\end{aligned}$$

for any values of x, so the Ratio Test says that the given series converges, regardless of x-value. Therefore, the Interval of Convergence can be represented by an inequality $-\infty < x < \infty$ or by an interval $(-\infty, \infty)$. $\diamondsuit$

Example. Find the Radius of Convergence and the Interval of Convergence of the following power series.

$$\sum_{n=1}^{\infty} n!(3x+5)^n$$

First note that the center of this power series is $x = -5/3$, since the power series is equal to

$$\sum_{n=1}^{\infty} n! 3^n \left(x - \left(-\frac{5}{3} \right) \right)^n.$$

Now letting

$$a_n = n!(3x+5)^n$$

and using the fact that $(n+1)! = (n+1) \cdot n!$ we have

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)!(3x+5)^{n+1}}{n!(3x+5)^n} \right| \\
&= \lim_{n\to\infty} |(n+1)(3x+5)| \\
&= |3x+5| \lim_{n\to\infty} (n+1).
\end{aligned}$$

Here it seems that $\lim_{n\to\infty}(n+1)=\infty$ so the limit above is infinity whatever x is. However, we cannot conclude like that if x is the center, i.e., $x=-5/3$, since then the calculation becomes $0\cdot\infty$. Indeed, we have already seen that any power series converges at its center. Therefore, we can conclude that the given series converges if and only if $x=-5/3$. The Interval of Convergence then also can be represented by $x=-5/3$, and the Radius of Convergence is 0. $\Diamond$

These are all the cases that we will deal with in this course. We summarize those cases below to finish this section.

$\boxed{\text{RoC and IoC of a power series}}$

For a given power series

$$\sum_{n=n_0}^{\infty} c_n(x-a)^n,$$

there are three possibilities regarding the RoC and IoC, as follows.

- For
$$R=\frac{1}{\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|},$$
the Interval of Convergence is one of the following
$$(a-R,a+R),\quad [a-R,a+R),\quad (a-R,a+R],\quad [a-R,a+R]$$
depending on its behavior at the endpoints and the Radius of Convergence is R.
- The series converges for all x; the Interval of Convergence is $(-\infty,\infty)$ and the Radius of Convergence is ∞.
- The series diverges for all x except for the center; the Interval of Convergence is $x=a$ and the Radius of Convergence is 0.

4.2 Power Series Representation of Functions

Intuitive Question. Suppose that we are given a function, say $\dfrac{1}{(1-x)^2}$ *or* $\ln(1-x)$*. How could we represent those functions with power series, using the relation between those and* $\dfrac{1}{1-x}$*? For which values of* x *is this possible?*

In the next section we will learn about Taylor Series, which is basically a method to represent arbitrary functions by power series. Before then, we study this idea briefly, with geometric series which is familiar to us.

We take a series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

as a basic example. It is a power series with center 0, and using the ratio test and examining the endpoints it is easy to see that it converges if and only if $|x| < 1$. On the other hand, it is also a geometric series of common ratio x, so if it converges then the sum is equal to $1/(1-x)$. Conversely thinking, we can represent a function $1/(1-x)$ as a power series $\sum_{n=0}^{\infty} x^n$, if (and only if) $|x| < 1$. Starting from this basic example, we can represent several similar functions using power series. See the following examples.

Example. Represent the following functions using power series, and determine the values of x *where the representation is valid, i.e., determine the Interval of Convergence of its power series representation.*

$$\frac{1}{1-x^2}, \quad \frac{1}{1-2x}, \quad \frac{x}{1-x}, \quad \frac{1}{x}, \quad \frac{4x^2}{2-x}$$

For the first function, we can substitute x^2 in place of x in the basic function, so we get

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}.$$

It is valid if and only if $|x^2| < 1$, or equivalently $|x| < 1$. Similarly, substituting $2x$ gives us

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n.$$

In this case, it is valid if and only if $|2x| < 1$, or equivalently $|x| < 1/2$.

For the third function, we can multiply x to both sides in the basic function to get

$$\frac{x}{1-x} = x + x^2 + x^3 + x^4 + \cdots = \sum_{n=1}^{\infty} x^n.$$

Note that it differs from the basic function by only 1, so it is valid if and only if $|x| < 1$ as in the basic function.

The fourth function is interesting because we get a power series centered at 1, not 0. Explicitly, we substitute $1 - x$ in place of x in the basic function, then we have

$$\begin{aligned}\frac{1}{x} = \frac{1}{1-(1-x)} &= 1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots \\ &= \sum_{n=0}^{\infty}(1-x)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n.\end{aligned}$$

It is valid if and only if $|1 - x| < 1$, or equivalently $0 < x < 2$.

Finally, for the last function, we need to rewrite the function as

$$\frac{4x^2}{2-x} = 4x^2 \cdot \frac{1}{2-x} = 4x^2 \cdot \frac{1}{2} \cdot \frac{1}{1 - x/2} = 2x^2 \cdot \frac{1}{1-x/2}.$$

Then substituting $x/2$ in place of x and multiplying $2x^2$ in the basic function we get

$$\begin{aligned}\frac{4x^2}{2-x} &= 2x^2\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots\right) \\ &= 2x^2\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{2}{2^n}x^{n+2} = \sum_{n=0}^{\infty}\frac{1}{2^{n-1}}x^{n+2}.\end{aligned}$$

Using the Ratio Test and examining the series at the endpoints, it turns out that the Integral of Convergence is $|x| < 2$. $\Diamond$

As in the previous example, we can modify the basic example in various ways to generate power series representations of similar functions, while the Interval of Convergence could be changed during modification.

We could also think about differentiation or integration of a given function and its power series representation. Note that when differentiate or integrate a power series, we could just do term by term.

Example. Represent the following functions using power series, and find the Interval of Convergence of the power series representations.

$$\frac{1}{(1-x)^2}, \quad \ln(1-x)$$

Two functions are derivative and antiderivative of our basic function $1/(1-x)$. First, differentiating the power series representation of the basic function we have

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n.$$

Since $n = 0$ gives a zero term, we can 'shift' the index and represent it as an equivalent power series $\sum_{n=0}^{\infty}(n+1)x^n$. Using the Ratio Test and examining at the endpoints, it turns out that the Interval of Convergence is $(-1, 1)$, as same as in the basic example.

On the other hand, integrating the power series representation we get

$$-\ln(1-x) = C + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = C + \sum_{n=1}^{\infty} \frac{1}{n} x^n.$$

Note that we can ignore the absolute value, since the original power series is valid only for $|x| \leq 1$, so that $1 - x \geq 0$. Now substituting $x = 0$ in both sides gives $C = 0$, so indeed we have

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Again using the Ratio Test and examining at the endpoints, the Interval of Convergence is $[-1, 1)$ or $-1 \leq x < 1$. Note that not as in the basic example, the power series converges if $x = -1$. $\Diamond$

We could observe that when differentiating and integrating a power series, the Radius of Convergence remains as 1. It is actually true for any power series.

Differentiation and Integraiton of Power Series

For a given power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

with the Radius of Convergence R, we have

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}(x-a)^n$$

and

$$\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} = C + \sum_{n=1}^{\infty} \frac{c_{n-1}}{n}(x-a)^n.$$

Moreover, both power series also have the Radius of Convergence R.

This allows us to find the Radius of Convergence for the power series representation of some functions using that of their derivatives or antiderivatives. However, as in the previous example, the behavior at the endpoints could be changed, so we still need to examine the endpoints.

Example. Represent the following function using power series, and find the Interval of Convergence of the power series representation.

$$\frac{1}{(1-x)^3}$$

The given function is equal to a half of the derivative of $1/(1-x)^2$, or the second derivative of our basic function $1/(1-x)$. Therefore, using the result above we get

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^n = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2}x^{n-2},$$

or shifting the index

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2}x^n.$$

It follows from the general theory that the Radius of Convergence is 1, as same as the basic function. Since the series diverges for both $x = \pm 1$ (by the n-th Term Test,) the Interval of Convergence is $(-1, 1)$. $\Diamond$

4.3 Taylor and Maclaurin Series

Intuitive Question. Suppose now that we are given an arbitrary function, say $3x^2e^{4x}$. How could we represent those functions with a power series?

In the last section, we learned how to represent a function with a power series, when it is somehow related to a geometric series. However, of course, there are still lots of functions which cannot be related to a geometric series. The idea of Taylor Series is using the derivatives of all orders to formulate the power series representation of any given functions. Although there are some logical complexities to do this, we will assume some conditions so that we could avoid these complexities.

We start with the assumption that we actually have the power series representation already as

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where the power series has the Radius of Convergence R, i.e., the power series representation is valid if (at least) $|x-a| < R$, and we also assume that $f(x)$ is differentiable of any order. Now we see how we can retrieve c_n's from the derivatives of $f(x)$. First, substituting $x = a$ gives $f(a) = c_0$ directly. Next, differentiating both sides gives

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots,$$

and substituting $x = a$ here gives $f'(a) = c_1$. Similarly, differentiating once more induces

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2} = 2c_2 + 6c_3(x-a) + \cdots$$

and substituting $x = a$ gives $f''(a) = 2c_2$ or $f''(a)/2 = c_2$. One more differentiation and substitution gives $f^{(3)}(a) = 6c_3$ or $f^{(3)}(a)/6 = c_3$. Note that 6 in the denominator is indeed 3!. In the same manner, we can get

$$c_4 = \frac{f^{(4)}(a)}{24} = \frac{f^{(4)}(a)}{4!},$$
$$c_5 = \frac{f^{(5)}(a)}{120} = \frac{f^{(5)}(a)}{5!},$$
$$\vdots$$

and in general

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Note that it is valid also for $n = 0$, since the 0-th derivative of f is f itself, $(x-a)^0 = 1$ and $0! = 1$. As a conclusion, we have the following.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

It is called the Taylor Series of $f(x)$ about $x = a$.

Now we go back to an arbitrary function $f(x)$ and a given center a. If $f(x)$ is differentiable of any order, we can write down its Taylor Series about $x = a$. It is important to note that there is no reason that this series converges for any x-value. Explicitly, if we define the N-th Taylor polynomial of $f(x)$ as

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N,$$

then

$$f(x) = P_N(x) + R_N(x)$$

where

$$R_N(x) = \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f^{(N+1)}(a)}{(N+1)!}(x-a)^{N+1} + \frac{f^{(N+2)}(a)}{(N+2)!}(x-a)^{N+2} + \cdots .$$

Now it immediately follows that the power series converges if and only if $\lim_{N\to\infty} R_N(x) = 0$. Therefore, the values of x such that the Taylor Series is valid, i.e., it actually converges to $f(x)$, are the values of x such that the above limit is zero. However, it is quite hard to find those values, we will generally omit the process to find the values of x such that the Taylor Series converges to the original function to concentrate on the Taylor Series itself.

In most cases, we will work on the Taylor Series about $x = 0$. Such Taylor Series are called specifically the Maclaurin Series.

Example. Find the Maclaurin Series, i.e., the Taylor Series about $x = 0$, of

$$f(x) = e^x.$$

It would be the simplest example of the Taylor Series, since the derivatives of $f(x)$ are really simple, even for high orders. Explicitly, we have $f^{(n)}(x) = e^x$ so $f^{(n)}(0) = 1$ for any n, so the Maclaurin Series of $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots . \quad \Diamond$$

Of course, such good situation that all derivatives are the same does not happen in all cases. We often need to differentiate the given function several times, then find some patterns to get a general formula of the n-th derivative evaluated at the given center.

Example. Find the Maclaurin Series, i.e., the Taylor Series about $x = 0$, of

$$f(x) = \sin x.$$

We first calculate the derivatives of small orders and their values at $x = 0$ as follows.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

We can observe that the derivatives $f^{(4)}(0)$ repeat with period 4, so does the values $f^{(n)}(0)$. Then it turns out that the Maclaurin Series of $f(x)$ is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= 0 + 1x + \frac{0}{2}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \quad \diamond \end{aligned}$$

Note that due to some zeros in the values of derivatives evaluated at $x = 0$, we need to use x^{2n+1} instead of x^n to represent that Taylor Series with a sigma notation. Also, with a similar method we can find the Maclaurin Series of $\cos x$,

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Example. Find the Maclaurin Series, i.e., the Taylor Series about $x = 0$, of

$$f(x) = \ln(1 - x).$$

It is actually done in the previous section using the antiderivative of $1/(1-x)$. However, here we approach with the general strategy. Calculating the derivatives of small orders and their values at $x = 0$ we get the following.

$$\begin{aligned} f(x) &= \ln(1-x) & f(0) &= 0 \\ f'(x) &= -\frac{1}{1-x} = -(1-x)^{-1} & f'(0) &= -1 \\ f''(x) &= -(1-x)^{-2} & f''(0) &= -1 \\ f^{(3)}(x) &= -2(1-x)^{-3} & f^{(3)}(0) &= -2 \\ f^{(4)}(x) &= -(3)(2)(1-x)^{-4} & f^{(4)}(0) &= -6 \\ f^{(5)}(x) &= -(4)(3)(2)(1-x)^{-5} & f^{(5)}(0) &= -24 \end{aligned}$$

We can observe that generally

$$f^{(n)}(x) = -(n-1)(n-2)\cdots(2)(1-x)^{-n},$$

so that

$$f^{(n)}(0) = -(n-1)!$$

for any $n \geq 1$. Then it turns out that the Maclaurin Series of $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{-(n-1)!}{n!} x^n = -\sum_{n=1}^{\infty} \frac{x^n}{n},$$

as we calculated in the previous section. $\Diamond$

⚠ Theoretically we need to actually prove that the n-th derivative is indeed represented as the given formula. However, in this course, we will only concentrate on finding a pattern and formulating a general formula.

We could also calculate the Taylor Series about other centers.

Example. Find the Taylor Series about $x = 1$ of

$$f(x) = \frac{1}{x^2} = x^{-2}.$$

Calculating the derivatives of small orders and their values at $x = 1$ we get the following.

$$\begin{aligned} f(x) &= x^{-2} & f(1) &= 1 \\ f'(x) &= -2x^{-3} & f'(1) &= -2 \\ f''(x) &= (3)(2)x^{-4} & f''(1) &= 6 \\ f^{(3)}(x) &= -(4)(3)(2)x^{-5} & f^{(3)}(1) &= -24 \end{aligned}$$

We can observe that generally

$$f^{(n)}(x) = (-1)^n(n+1)(n)\cdots(2)x^{-(n+2)},$$

so that

$$f^{(n)}(1) = (-1)^n(n+1)!$$

for any n. Then it turns out that the Taylor Series of $f(x)$ about $x = 1$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n. \quad \Diamond$$

We are still able to use substitution, differentiation or integration to modify one Taylor Series to get another.

Example. Find the Maclaurin Series, i.e., the Taylor Series about $x = 0$, of

$$f(x) = 3x^2 e^{4x}.$$

We start with

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Substituting $4x$ in place of x, we get

$$e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{4^n}{n!} x^n.$$

Now multiplying $3x^2$ to both sides, we get

$$3x^2 e^{4x} = \sum_{n=0}^{\infty} \frac{3 \cdot 4^n}{n!} x^{n+2}. \quad \Diamond$$

Example. Find the Maclaurin Series, i.e., the Taylor Series about $x = 0$, of

$$f(x) = \arctan x.$$

We start with

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Substituting $-x^2$ in place of x, we get

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Now integrating both sides,

$$\arctan x = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1},$$

and putting in $x = 0$ gives $C = 0$. Therefore

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots . \quad \Diamond$$

Moreover, we can also multiply two Taylor (or Maclaurin) Series to get the Taylor (or Maclaurin) Series of a product of two functions. However, in this case, it is hard to represent it with a sigma notation explicitly; we could only find the first few terms of it.

Example. Find the first four nonzero terms of the Maclaurin Series, i.e., the Taylor Series about $x = 0$, of

$$f(x) = e^x \sin x.$$

We know that the Maclaurin Series of e^x and $\sin x$ are

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

and

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$

When we multiply two functions and two series, we get basically

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right).$$

Now we calculate this product of series term by term, say

$$\begin{aligned}
e^x \sin x &= 1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&\quad + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&\quad + \frac{x^2}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&\quad + \frac{x^3}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&\quad + \frac{x^4}{4!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&\quad + \cdots \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&\quad + \left(x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots\right) \\
&\quad + \left(\frac{x^3}{2} - \frac{x^5}{2 \cdot 3!} + \frac{x^7}{2 \cdot 5!} - \cdots\right) \\
&\quad + \left(\frac{x^4}{3!} - \frac{x^6}{3! \cdot 3!} + \frac{x^8}{3! \cdot 5!} - \cdots\right) \\
&\quad + \left(\frac{x^5}{4!} - \frac{x^7}{4! \cdot 3!} + \frac{x^9}{4! \cdot 5!} - \cdots\right) \\
&\quad + \cdots .
\end{aligned}$$

Now we gather terms with the same degree up to 5, then it becomes

$$\begin{aligned}&x+x^2+\left(\frac{x^3}{2}-\frac{x^3}{3!}\right)+\left(\frac{x^4}{3!}-\frac{x^4}{3!}\right)+\left(\frac{x^5}{4!}-\frac{x^5}{2\cdot 3!}+\frac{x^5}{5!}\right)+\cdots\\=&x+x^2+\frac{x^3}{3}-\frac{x^5}{30}+\cdots.\end{aligned}$$

Note that every term which is not listed above has degree at least 6, so they are actually all terms with degree up to 5. Therefore, the first four nonzero terms are given as

$$x,\quad x^2,\quad \frac{x^3}{3},\quad -\frac{x^5}{30}.\quad \diamondsuit$$

⚠ There are two important issues in solving such problems. First, when we gather terms with the same degree, we need to check that all such terms are included in the calculation. For example, in the previous example, one could miss $x^5/4!$ in degree 5 since it appears quite lately in the expansion. Also, it is always possible that some terms are nontrivially zero due to cancellation, e.g., degree 4 case above, so the nonzero terms could have higher degrees than expected.

4.4 Applications of Taylor Series

Intuitive Question. How can we approximate the value of $e^{0.01}$ very closely, say within the given error bound 10^{-5}?

In this section we study some applications of Taylor Series. The primary purpose of using Taylor Series is to represent an arbitrary function with a power series with a given center. Then, if the variable x is sufficiently close to the center, then terms in the power series will decrease rapidly so that we can approximate the whole power series with only few terms corresponding to small degrees. Explicitly, let $f(x)$ be any function and suppose that we are approximating the value of $f(x_0)$ where x_0 is close to a given center a. Recall the notation

$$f(x)=P_N(x)+R_N(x)$$

where

$$P_N(x)=\sum_{n=0}^{N}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the N-th Taylor polynomial and

$$R_N(x)=\sum_{n=N+1}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the remainder. Since x_0 is close to a, then the absolute value of $(x_0-a)^n$ will become small when n gets bigger, so the Taylor polynomial $P_N(x_0)$ will approximate the value of $f(x_0)$ closely. Also, in most cases, we can make the approximation better by increasing N, i.e., including more terms. Therefore, generally if the 'target' error bound is given, then we can calculate the Taylor polynomial with sufficiently many terms to achieve that error bound. In this section, we will learn how to calculate the error bound explicitly, and how to choose the proper number of terms to be calculated when the target bound is given.

When it turns out that the Taylor Series is alternating, then things become much easier; since then we can apply the error bound of alternating series, covered in section 3.6.

Example. Find the approximated value of $\arctan(0.01)$ *within the maximal error bound* 10^{-10}.

Here $f(x) = \arctan x$, $a = 0$ and $x_0 = 0.01$. Using the Maclaurin Series of $f(x)$ obtained in the previous section, we have

$$\arctan(0.01) = f(0.01) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(0.01)^{2n+1} = (0.01) - \frac{(0.01)^3}{3} + \frac{(0.01)^5}{5} - \cdots .$$

We can observe that it is an alternating series satisfying all conditions in the Alternating Series Test, and the absolute value of the third term is

$$\frac{(0.01)^5}{5} = \frac{10^{-10}}{5} < 10^{-10}.$$

Therefore, using the theory of error bound of alternating series, we can conclude that the Taylor polynomial with two terms

$$(0.01) - \frac{(0.01)^3}{3}$$

approximates the value of $f(0.01)$ within the given error bound 10^{-10}. ♢

However, unfortunately, not all Taylor Series are alternating, so we need to develop a general theory that can be applied to any type of Taylor Series. In general, the error bound of approximation by the N-th Taylor polynomial depends on N and distance to the center as usual, but also the maximal value of the $(N+1)$-th derivative 'near' the center.

Error Bound of General Taylor Series

Let $f(x)$ be any function and suppose that $f(x_0)$ is approximated with the N-th Taylor polynomial $P_N(x_0)$. Then the error, i.e., the absolute value of the remainder $R_N(x_0)$, is bounded as

$$|R_N(x_0)| \leq \frac{M}{(N+1)!}|x_0 - a|^{N+1},$$

where

$$M = \max_{t \text{ is between } x_0 \text{ and } a} |f^{(N+1)}(t)|.$$

We will not deal with the proof in details. It is often hard to calculate the value of M, especially N gets bigger since the $(N+1)$-th derivative will become more complicated. However, it is good to note that in most cases the $(N+1)$-th derivative will be either increasing or decreasing between x_0 and a, so that the maximum occurs at one of the endpoints.

Example. Find the approximation and the maximal error bound of $e^{0.01}$ *when we approximate it with the second Taylor polynomial about* $x = 0$.

Here $f(x) = e^x$, $a = 0$ and $x_0 = 0.01$. Using the Maclaurin Series of e^x, we get

$$e^{0.01} = f(0.01) = \sum_{n=0}^{\infty} \frac{(0.01)^n}{n!} = \left(1 + (0.01) + \frac{(0.01)^2}{2}\right) + \frac{(0.01)^3}{3!} + \cdots .$$

Now setting $N = 2$, $f^{(3)}(x) = e^x$ so we get

$$M = \max_{0 \le t \le 0.01} e^t = e^{0.01}.$$

Therefore the maximal error bound is

$$\frac{M}{3!}(0.01)^3 = \frac{e^{0.01}}{3!}(0.01)^3. \ \diamondsuit$$

Example. Find the approximated value of $\ln(0.99)$ *within the maximal error bound* 10^{-8}.

Here $f(x) = \ln(1-x)$, $a = 0$ and $x_0 = 0.01$. Using the Maclaurin Series of $\ln(1-x)$, we get

$$\ln(0.99) = f(0.01) = -\sum_{n=1}^{\infty} \frac{(0.01)^n}{n} = -0.01 - \frac{(0.01)^2}{2} - \frac{(0.01)^3}{3} - \cdots .$$

Since the series is not alternating, we need to use the general error bound of Taylor Series. Setting $N = 3$, $f^{(4)}(x) = -3!(1-x)^{-4}$ according to an example in the previous section so we get

$$M = \max_{0 \le t \le 0.01} 3!(1-x)^{-4} = 3!(0.99)^{-4} = \frac{3!}{(0.99)^4}.$$

Therefore the error is at most

$$\frac{M}{4!}(0.01)^4 = \frac{3!}{4!}\frac{(0.01)^4}{(0.99)^4} = \frac{1}{4 \cdot 99^4}.$$

Since

$$99^4 > 90^4 = 8100^2 > 8000^2 = 64000000 = 6.4 \cdot 10^7,$$

the error bound is smaller than

$$\frac{1}{4 \cdot 6.4 \cdot 10^7} < \frac{1}{10^8}.$$

Therefore the third Taylor Polynomial

$$-0.01 - \frac{(0.01)^2}{2} - \frac{(0.01)^3}{3}$$

approxmiates the value of $f(0.01)$ within the given error bound 10^{-8}. $\lozenge$

⚠ When approaching such approximation problems, we need to first set the function $f(x)$ and center a, and it is important to set $f(x)$ and a properly so that x_0 is sufficiently close to a, since the error bound mainly corresponds to the powers of $|x_0 - a|$.

Now recall that we can integrate the Taylor Series of a function to find the Taylor Series of its antiderivative. Using this, with the theory of approximation described above, we can estimate some values of definite integrals which cannot be calculated explicitly with known methods.

Example. Find the approximated value of the following definite integral within the maximal error bound 10^{-8}.

$$\int_0^{0.1} e^{-x^2} dx$$

Note that the antiderivative of e^{-x^2} cannot be written as a function of known forms. However, since the Taylor Series of e^{-x^2} is

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots ,$$

we get

$$\begin{aligned}\int_0^{0.1} e^{-x^2} dx &= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \right]_{x=0}^{x=0.1} \\ &= 0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5 \cdot 2!} - \frac{(0.1)^7}{7 \cdot 3!} + \cdots .\end{aligned}$$

This series is alternating, and it satisfies the conditions of the Alternating Series Test. Moreover, the absolute value of the fourth term is smaller than 10^{-8} since

$$\frac{(0.1)^7}{7 \cdot 3!} = \frac{10^{-7}}{42} < \frac{10^{-7}}{10} = 10^{-8}.$$

Therefore, the third partial sum

$$0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5 \cdot 2!}$$

approxmiates the given definite integral within the given error bound 10^{-8}. $\lozenge$

⚠ If the resulting series is not alternating, we need to use the general theory about the error bound of Taylor Series. For example, similarly as the previous example we get

$$\int_0^{0.1} e^{x^2} dx = 0.1 + \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5 \cdot 2!} + \frac{(0.1)^7}{7 \cdot 3!} + \cdots .$$

We can observe that it is equal to a value of function $F(0.1)$, where

$$F(x) = \int_0^x e^{u^2} du.$$

When calculating the error bound with the N-th Taylor polynomial, we need to calculate the $(N+1)$-th derivative of $F(x)$. However, the Fundamental Theorem of Calculus allows us to do that, since $F'(x) = e^{x^2}$ so that the $(N+1)$-th derivative of $F(x)$ is actually the N-th derivative of e^{x^2}.

Other than approximating values of functions or definite integrals, there are two other applications of Taylor Series, calculating certain limits and calculating sums of certain series. We will study those two applications with two examples, one for each.

Example. Calculate the following limit.

$$\lim_{x \to 0} \frac{\frac{1}{1-x} - e^x}{1 - \cos 2x}$$

We can observe that substituting $x = 0$ gives a fraction of the form $^0/_0$. Then we could consider applying L'Hospital's Rule, and some repeated applications of L'Hospital's Rule actually work. However, realizing that the given function is composed of three 'basic' functions, we can deal with this limit using their Maclaurin Series. Explicitly, we have

$$\begin{aligned}
\lim_{x \to 0} \frac{\frac{1}{1-x} - e^x}{1 - \cos 2x} &= \lim_{x \to 0} \frac{(1 + x + x^2 + x^3 + \cdots) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right)}{1 - \left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} - \cdots\right)} \\
&= \lim_{x \to 0} \frac{\frac{x^2}{2} + \frac{2x^3}{3} + \cdots}{2x^2 - \frac{2x^4}{3} + \cdots} \\
&= \lim_{x \to 0} \frac{\frac{1}{2} + \frac{2x}{3} + \cdots}{2 - \frac{2x^2}{3} + \cdots} = \frac{\frac{1}{2}}{2} = \frac{1}{4}.
\end{aligned}$$

Note that we could use the Maclaurin Series since x approaches 0 in the limit; if x approaches some other constant, then we need to use the Taylor Series centered at that constant instead. ◊

Example. Find the sum of the following series.

$$1 - \frac{1}{3! \cdot 10^2} + \frac{1}{5! \cdot 10^4} - \frac{1}{7! \cdot 10^6} + \cdots$$

First, using 0.1 instead of $^1/_{10}$, we can rewrite the given series as

$$1 - \frac{(0.1)^2}{3!} + \frac{(0.1)^4}{5!} - \frac{(0.1)^6}{7!} + \cdots = 10\left((0.1) - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} - \frac{(0.1)^7}{7!} + \cdots\right).$$

Then recalling the Maclaurin Series of $\sin x$, the sum of given series is equal to $10\sin(0.1)$. $\diamondsuit$