JEE Advanced:2019

G V V Sharma*

Contents

Abstract—**This manual has exercises based on problems in JEE advanced.**

1 2019-1

1.1 Consider the optimization problem

$$\max_z \frac{1}{|z-1|} \tag{1}$$

$$s.t. \quad \left|z-2+\jmath\right| \ge \sqrt{5} \tag{2}$$

If the maximum value is obtained at z_0, find the principal argument of

$$\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2\jmath} \tag{3}$$

Solution: The optimization problem can be reframed as

$$\min_{\mathbf{x}} \|\mathbf{x}-\mathbf{c}_1\|^2 \tag{4}$$

$$s.t. \quad \|\mathbf{x}-\mathbf{c}_2\|^2 \ge 5 \tag{5}$$

where

$$z = \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{6}$$

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7}$$

$$\mathbf{c}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{8}$$

Fig. 1.1.1 explains (4) where z_0 is the set of points comprising of the intersection of the smallest circle Γ : with the largest circle $\Omega : r_2 \ge \sqrt{5}$ with radii r_1 and $r_2 \ge \sqrt{5}$ respectively. The Lagrangian is

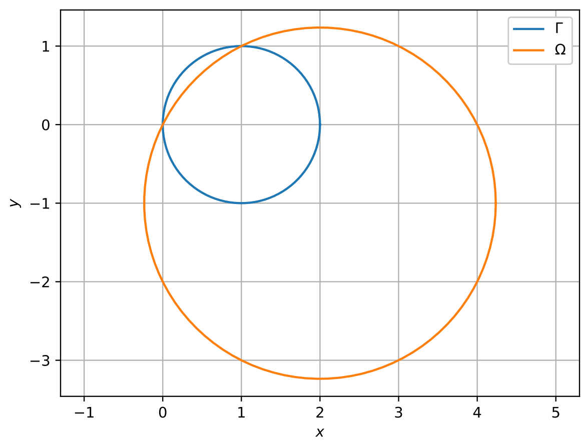

Fig. 1.1.1

$$L(\mathbf{x},\lambda) = \|\mathbf{x}-\mathbf{c}_1\|^2 - \lambda\left\{\|\mathbf{x}-\mathbf{c}_2\|^2 - r_2^2\right\} \tag{9}$$

From the KKT conditions,

$$\frac{\partial L(\mathbf{x},\lambda)}{\partial \mathbf{x}} = 0 \tag{10}$$

$$\implies \mathbf{x}-\mathbf{c}_1 - \lambda(\mathbf{x}-\mathbf{c}_2) = 0 \tag{11}$$

$$\implies \mathbf{x} = \frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1-\lambda} \tag{12}$$

and

$$\frac{\partial L(\mathbf{x},\lambda)}{\partial \lambda} = 0 \tag{13}$$

$$\implies \|\mathbf{x}-\mathbf{c}_2\|^2 - r_2^2 = 0 \tag{14}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

Substituting from (12) in (17),

$$\left\|\frac{\mathbf{c}_1-\lambda\mathbf{c}_2}{1-\lambda}-\mathbf{c}_2\right\|^2-r_2^2=0 \tag{15}$$

$$\implies \lambda = 1 \pm \frac{\|\mathbf{c}_1-\mathbf{c}_2\|}{r_2} \tag{16}$$

$$= 1 \pm \sqrt{\frac{2}{5}} \tag{17}$$

Fig. 1.1.2 plots Γ for

$$\lambda = 1-\sqrt{\frac{2}{5}} \tag{18}$$

Thus, from (12),

$$\mathbf{x}_0 = \frac{\mathbf{c}_1-\lambda\mathbf{c}_2}{1-\lambda} \tag{19}$$

$$\implies z_0 = \frac{1}{1-\lambda}\left(1-2\lambda+\jmath\lambda\right) \tag{20}$$

$$\text{or, } \arg\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2\jmath} = \frac{2-\Re\{z_0\}}{\jmath\left(\Im\{z_0\}+1\right)} \tag{21}$$

$$= \frac{2(1-\lambda)-(1-2\lambda)}{\jmath} = -\jmath \tag{22}$$

Thus, the principal argument is $-\frac{\pi}{2}$.

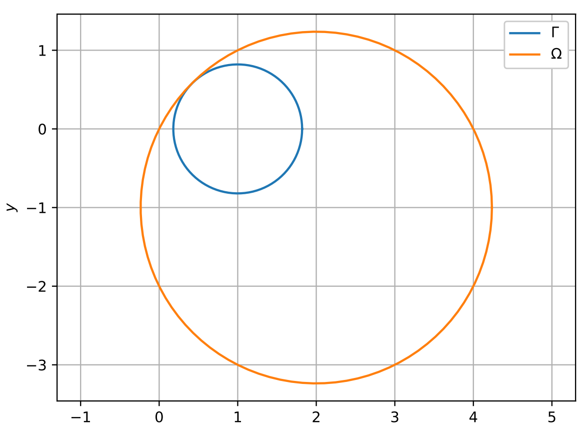

Fig. 1.1.2

1.2 Let

$$\mathbf{M} = \begin{pmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha\mathbf{I}+\beta\mathbf{M}^{-1} \tag{23}$$

where α,β are real functions of θ and $\mathbf{I}$ is the identity matrix. If

$$\alpha^* = \min_\theta \alpha(\theta) \tag{24}$$

$$\beta^* = \min_\theta \beta(\theta), \tag{25}$$

find $\alpha^*+\beta^*$.

Solution: (23) can be expressed as

$$\mathbf{M}^2-\alpha\mathbf{M}-\beta\mathbf{I}=0 \tag{26}$$

which yields the characteristic equation of $\mathbf{M}$ as

$$\lambda^2-\alpha\lambda-\beta=0 \tag{27}$$

Since the sum of the eigenvalues is equal to the trace and the determinant is the product of eigenvalues,

$$\alpha = \sin^4\theta+\cos^4\theta = 1-\frac{\sin^2 2\theta}{2} \tag{28}$$

$$-\beta = \sin^4\theta\cos^4\theta+\left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{29}$$

$$= 2+\frac{\sin^2 2\theta}{4}+\frac{\sin^4 2\theta}{16} \tag{30}$$

$$= \left(\frac{\sin^2 2\theta}{4}+\frac{1}{2}\right)^2+\frac{7}{4} \tag{31}$$

Thus,

$$\alpha^* = \frac{1}{2}, \beta^* = -\frac{37}{16} \tag{32}$$

$$\implies \alpha^*+\beta^* = -\frac{29}{16} \tag{33}$$

1.3 The line

$$\Gamma : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{34}$$

intersects the circle

$$\Omega : \left\|\mathbf{x}-\begin{pmatrix} 3 \\ -2 \end{pmatrix}\right\| = 5 \tag{35}$$

at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. The mid point of PQ is $\mathbf{R}$ such that

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{R} = -\frac{3}{5} \tag{36}$$

Find m.

Solution: Let

$$\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{37}$$

The intersection of (34) and (35) is

$$\|\mathbf{c} + \lambda\mathbf{m} - \mathbf{O}\|^2 = 25 \tag{38}$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^T(\mathbf{c} - \mathbf{O}) + \|\mathbf{c} - \mathbf{O}\|^2 - 25 = 0 \tag{39}$$

Since $\mathbf{P}, \mathbf{Q}$ lie on Γ,

$$\mathbf{P} = \mathbf{c} + \lambda_1\mathbf{m} \tag{40}$$

$$\mathbf{Q} = \mathbf{c} + \lambda_2\mathbf{m} \tag{41}$$

$$\implies \frac{\mathbf{P}+\mathbf{Q}}{2} = \mathbf{c} + \frac{\lambda_1+\lambda_2}{2}\mathbf{m} \tag{42}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\frac{\mathbf{P}+\mathbf{Q}}{2} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{c} + \frac{\lambda_1+\lambda_2}{2}\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{m} \tag{43}$$

$$= \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{c} - \frac{\mathbf{m}^T(\mathbf{c}-\mathbf{O})}{\|\mathbf{m}\|^2} \tag{44}$$

using the sum of roots in (39). From (36) and (37),

$$-\begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}-3 \\ 3\end{pmatrix} = -\frac{3}{5}\left(1+m^2\right) \tag{45}$$

$$\implies m^2 - 5m + 6 = 0 \tag{46}$$

$$\implies m = 2 \text{ or } 3 \tag{47}$$

From (39),

$$\lambda = \frac{-\mathbf{m}^T(\mathbf{c}-\mathbf{O})}{\|\mathbf{m}\|^2} \pm \frac{\sqrt{\left(\mathbf{m}^T(\mathbf{c}-\mathbf{O})\right)^2 - \|\mathbf{c}-\mathbf{O}\|^2 + 25}}{\|\mathbf{m}\|^2} \tag{48}$$

Fig. 1.3 summarizes the solution for $m = 2$.

1.4 Find the area of the region

$$\begin{pmatrix}x \\ y\end{pmatrix} : xy \le 8, 1 \le y \le x^2 \tag{49}$$

Solution: The intersection of $y = 1, y = x^2$ is

$$\mathbf{A} = \begin{pmatrix}1 \\ 1\end{pmatrix} \tag{50}$$

The intersection of $y = 1, xy = 8$ is

$$\mathbf{B} = \begin{pmatrix}8 \\ 1\end{pmatrix} \tag{51}$$

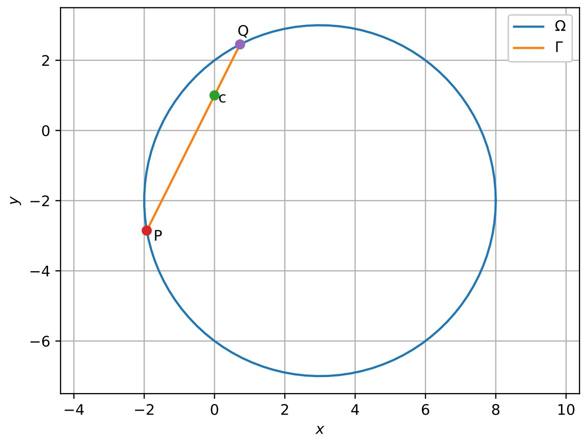

Fig. 1.3

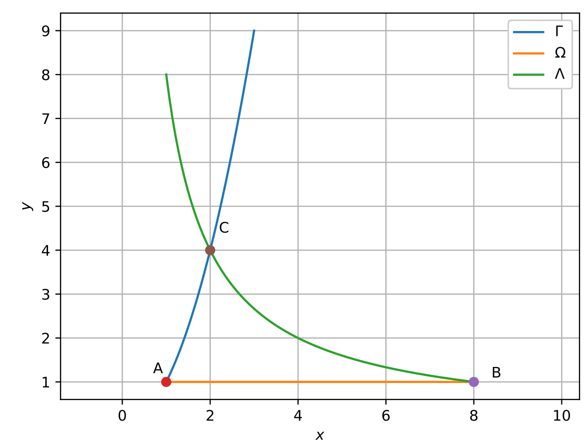

Fig. 1.4

The intersection of $y = x^2, xy = 8$ is

$$\mathbf{C} = \begin{pmatrix}2 \\ 4\end{pmatrix} \tag{52}$$

The desired region is enclosed by the vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ Thus, the area is obtained as

$$\int_1^2 x^2\,dx + \int_2^8 \frac{8}{x}\,dx = \left[\frac{x^3}{3}\right]_1^2 + 8\left[\ln x\right]_2^8 - 7 \tag{53}$$

$$= 16\ln 2 - \frac{14}{3} \tag{54}$$

1.5

1.6

1.7

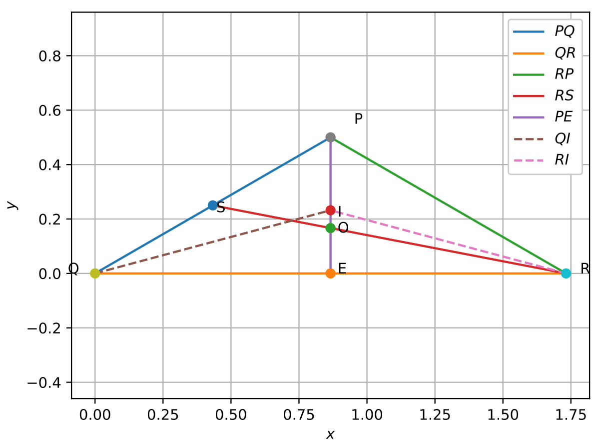

Fig. 1.8

1.8 In $\triangle PQR$, which is not right angled, let

$$PQ = r, QR = p, RP = q \tag{55}$$

The median RS and the altitude PE intersect at $\mathbf{O}$. $p = \sqrt{3}, q = 1$ and the radius of the circumcircle of $\triangle PQR = k = 1$. Find the length of RS, OE, area of $\triangle SOE$, and radius of the incircle of $\triangle PQR$.

Solution: Using the sine formula,

$$\frac{p}{\sin P} = \frac{q}{\sin Q} = 2k \tag{56}$$

$$\implies \sin P = \frac{\sqrt{3}}{2}, \sin Q = \frac{1}{2} \tag{57}$$

If $\angle R \neq \frac{\pi}{2}$, the only possible solution is

$$\angle P = \frac{2\pi}{3}, \angle Q = \frac{\pi}{6}, \angle R = \frac{\pi}{6} \tag{58}$$

$\because \angle Q = \angle R, q = r = 1$. The given information is shown in Fig. 1.8 Using the cosine formula,

$$RS = \sqrt{q^2 + \left(\frac{r}{2}\right)^2 - qr\cos P} \tag{59}$$

$$= \sqrt{1 + \frac{1}{4} + \frac{1}{2}} = \sqrt{\frac{7}{2}} \tag{60}$$

Also, using Baudhayana's theorem,

$$OE = \sqrt{OR^2 - ER^2} \tag{61}$$

$$= \sqrt{\left(\frac{2RS}{3}\right)^2 - \left(\frac{p}{2}\right)^2} \tag{62}$$

$$= \sqrt{\frac{7}{9} - \frac{3}{4}} = \frac{1}{6} \tag{63}$$

Since PE and RS are medians,

$$\text{ar}\left(\triangle SOE\right) = \frac{1}{4}\text{ar}\left(\triangle POR\right), \tag{64}$$

$$\text{ar}\left(\triangle POR\right) = \frac{2}{3}\text{ar}\left(\triangle PER\right), \tag{65}$$

$$\text{ar}\left(\triangle PER\right) = \frac{1}{2}\text{ar}\left(\triangle PQR\right), \tag{66}$$

$$\implies \text{ar}\left(\triangle SOE\right) = \frac{1}{12}\text{ar}\left(\triangle PQR\right) = \frac{\sqrt{3}}{24} \tag{67}$$

I is the incentre in Fig. 1.8. The radius of the incircle is

$$\frac{p}{2\cos\frac{Q}{2}} = \frac{p}{\sqrt{2\left(1 + \cos Q\right)}} \tag{68}$$

$$= \sqrt{\frac{3}{1 + \sqrt{3}}} \tag{69}$$

1.9

1.10

1.11

1.12 Let

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \tag{70}$$

$$L_2 : \quad \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{71}$$

Given that $L_3 \perp L_1, L_3 \perp L_2$, find L_3.

Solution: Let

$$L_3 : \quad \mathbf{x} = \mathbf{c} + \lambda \mathbf{m}_3 \tag{72}$$

Then

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \mathbf{m}_3 = \mathbf{0} \tag{73}$$

Row reducing the coefficient matrix,

$$\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -2 & -2\\ 0 & 1 & 2\end{pmatrix} \tag{74}$$

$$\leftrightarrow \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 2\end{pmatrix} \implies \mathbf{m}_3 = \begin{pmatrix}2\\2\\-1\end{pmatrix} \tag{75}$$

Also, $L_1 \perp L_2$, but $L_1 \cup L_2 = \phi$. The given information can be summarized as

$$L_1: \quad \mathbf{x} = \mathbf{c}_1 + \lambda_1 \mathbf{m}_1 \tag{76}$$
$$L_2: \quad \mathbf{x} = \lambda_2 \mathbf{m}_2 \tag{77}$$
$$L_3: \quad \mathbf{x} = \mathbf{c}_3 + \lambda \mathbf{m}_3 \tag{78}$$

where

$$\mathbf{c}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix}-1\\2\\2\end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix}2\\-1\\2\end{pmatrix} \tag{79}$$

The objective is to find $\mathbf{c}_3$. Since $L_1 \cup L_3 \neq \phi, L_2 \cup L_3 \neq \phi$, from (76)-(78),

$$\mathbf{c}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{c}_3 + \lambda_3 \mathbf{m}_3 \tag{80}$$
$$\lambda_2 \mathbf{m}_2 = \mathbf{c}_3 + \lambda_4 \mathbf{m}_3 \tag{81}$$

Using the fact that $L_1 \perp L_2 \perp L_3$, (80)-(81) can be expressed as

$$\mathbf{m}_1^T \mathbf{c}_1 + \lambda_1 \left\|\mathbf{m}\right\|_1^2 = \mathbf{m}_1^T \mathbf{c}_3 \tag{82}$$
$$\mathbf{m}_2^T \mathbf{c}_1 = \mathbf{m}_2^T \mathbf{c}_3 \tag{83}$$
$$\mathbf{m}_3^T \mathbf{c}_1 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_3 \left\|\mathbf{m}_3\right\|^2 \tag{84}$$
$$0 = \mathbf{m}_1^T \mathbf{c}_3 \tag{85}$$
$$\lambda_2 \left\|\mathbf{m}_2\right\|^2 = \mathbf{m}_2^T \mathbf{c}_3 \tag{86}$$
$$0 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_4 \left\|\mathbf{m}_3\right\|^2 \tag{87}$$

Simplifying the above,

$$\lambda_1 = -\frac{\mathbf{m}_1^T \mathbf{c}_1}{\left\|\mathbf{m}\right\|_1^2} = \frac{1}{9} \tag{88}$$

$$\lambda_2 = \frac{\mathbf{m}_2^T \mathbf{c}_1}{\left\|\mathbf{m}\right\|_2^2} = \frac{2}{9} \tag{89}$$

Substituting in (80) and (81),

$$L_3: \quad \mathbf{x} = \frac{2}{9}\begin{pmatrix}4\\1\\1\end{pmatrix} + \lambda_3 \begin{pmatrix}2\\2\\-1\end{pmatrix} \text{ or} \tag{90}$$

$$L_3: \quad \mathbf{x} = \frac{2}{9}\begin{pmatrix}2\\-1\\2\end{pmatrix} + \lambda_3 \begin{pmatrix}2\\2\\-1\end{pmatrix} \tag{91}$$

The key concept in this question is that orthogonality of L_1 and L_2 doesnot mean that they intersect. They are skew lines.

2 2019-2

2.1 Obtain the 3×3 matrices $\{\mathbf{P}_k\}_{k=1}^6$ from the vectors

$$\mathbf{v}_1 = \begin{pmatrix}0 & 0 & 1\end{pmatrix} \tag{92}$$
$$\mathbf{v}_2 = \begin{pmatrix}0 & 1 & 0\end{pmatrix} \tag{93}$$
$$\mathbf{v}_3 = \begin{pmatrix}1 & 0 & 0\end{pmatrix} \tag{94}$$

and let

$$\mathbf{X} = \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix}2 & 1 & 3\\ 1 & 0 & 2\\ 3 & 2 & 1\end{pmatrix} \mathbf{P}_k^T \tag{95}$$

Verify if

a) $\lambda = 30$ is an eigenvalue of $\mathbf{X}$ and $\mathbf{x} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}^T$ the corresponding eigenvector.

b) $\mathbf{X}$ is symmetric.

c) $\operatorname{tr}(\mathbf{X}) = 18$.

d) $\mathbf{X} - 30\mathbf{I}$ is invertible.

2.2 Let

$$\mathbf{P} = \begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 2\\ 0 & 0 & 3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}2 & x & x\\ 0 & 4 & 0\\ x & x & 6\end{pmatrix} \tag{96}$$

Verify if

a) $PQ = QP$ for some x.

b) $\det R = \det \begin{pmatrix}2 & x & x\\ 0 & 4 & 0\\ x & x & 5\end{pmatrix}$ for all x.

c) for $x = 0$, if $\mathbf{R}\begin{pmatrix}1\\a\\b\end{pmatrix} = 6\begin{pmatrix}1\\a\\b\end{pmatrix}$, then $a + b = 5$. *Use property of eigenvector.*

d) For $x = 1$, there exists a vector $\mathbf{y}$ for whcih $\mathbf{Ry} = \mathbf{0}$. This implies that the null space of $\mathbf{R}$ is nonempty. Also, $\mathbf{R}$ is noninvertible, $\det(R) = 0$ and has a 0 eigenvalue.

2.3

2.4

2.5

2.6

2.7

2.8 Let

$$L_1: \quad \mathbf{r} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{97}$$

$$L_2: \quad \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{98}$$

$$L_3: \quad \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{99}$$

Let $\mathbf{P} \in L_1, \mathbf{Q} \in L_2, \mathbf{R} \in L_3$. Verify if $\mathbf{Q}$ can be

a) $\begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

given that $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are collinear.

Solution: If $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are collinear,

$$\frac{PQ}{QR} = k, \tag{100}$$

$$(k+1)\,\mathbf{Q} = k\mathbf{P} + \mathbf{R}, \tag{101}$$

From (97), (98) and (99),

$$k\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (k+1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (k+1)\,\lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{102}$$

which can be expressed as

$$\begin{pmatrix} k & 0 & 0 \\ 0 & -(k+1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ k+1 \end{pmatrix} \tag{103}$$

Thus,

$$\mathbf{Q} = \begin{pmatrix} 0 \\ \frac{1}{k+1} \\ 1 \end{pmatrix} \tag{104}$$