JEE Linear Algebra using Matrix Computation

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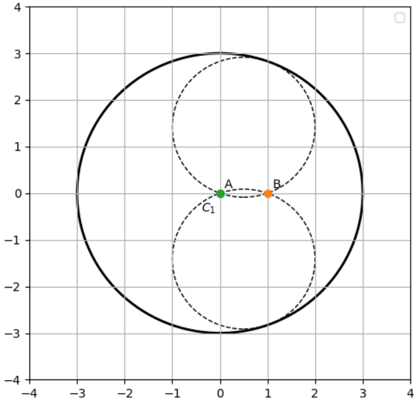
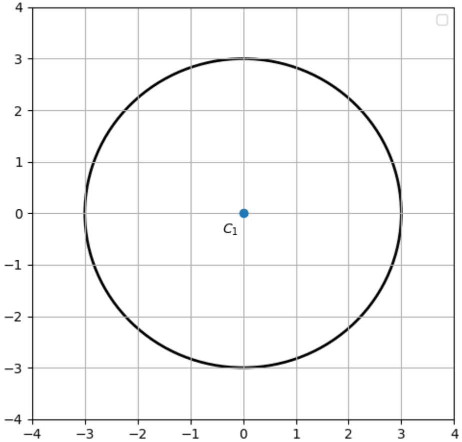
Problem

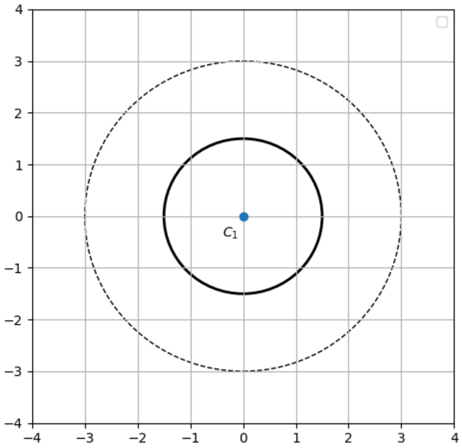
Find the centre of the circle passing through A: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and B: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and touching the circle $x^2 + y^2 = 9$ (IIT-JEE 2002)



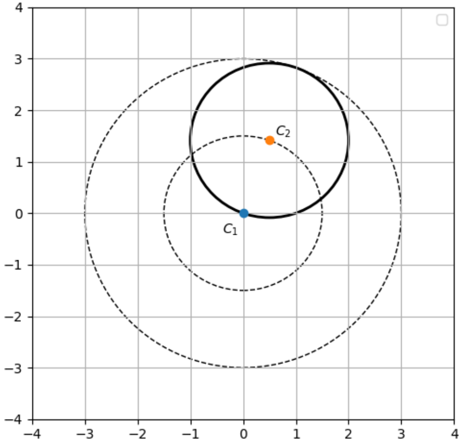
Given Circle: $x^2 + y^2 = 9$



Scaling



Translation



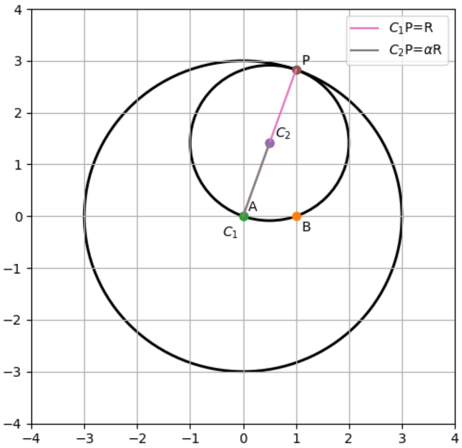
Transformation

The circle G_2 is obtained from circle G_1 by SCALING and TRANSLATION. The net result in AFFINE Tranformation:

$$T(\mathbf{x}) = \alpha \mathbf{I}\mathbf{x} + (\mathbf{C}_2 - \mathbf{C}_1)$$

Where $\mathbf{I}$ is Identity matrix

Constraints



Contd.

G_1 has centre $\mathbf{C}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and radius $R = 3$

G_2 has centre $\mathbf{C}_2$ and radius αR

Constraints:

1. $\|\mathbf{C}_2 - \mathbf{C}_1\| = R(1-\alpha)$........as shown in figure

2. Equation of $G_2 : (\mathbf{x} - \mathbf{C}_2)^T(\mathbf{x} - \mathbf{C}_2) = (\alpha R)^2$

is satisfied by $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Contd.

$C_2 C_2^T = R^2(1-\alpha)^2$....from Constraint 1
$(x - C_2)^T(x - C_2) = (\alpha R)^2$....from Constraint 2

$$\Rightarrow x^T x - x^T C_2 - C_2^T x + C_2^T C_2 = (\alpha R)^2$$

$$\Rightarrow 2x^T C_2 = x^T x + R^2(1-\alpha^2) - (\alpha R)^2$$

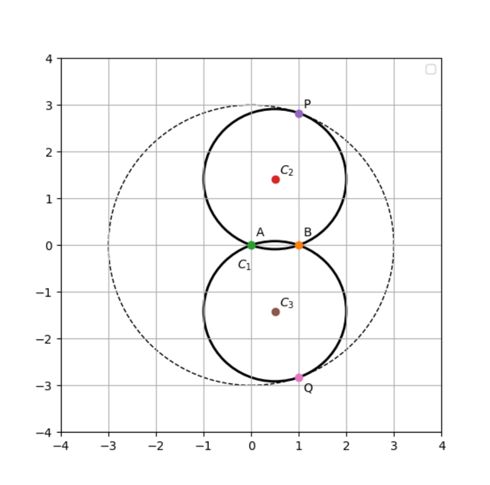
$$\Rightarrow 2x^T C_2 = x^T x + R^2(1-\alpha^2) - (\alpha R)^2$$

Above equation is satisfied by $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solving gives us:

$$\alpha = 0.5 \quad ; \quad C_2 = \begin{pmatrix} 0.5 \\ \sqrt{2} \end{pmatrix} \quad \text{OR} \quad C_2 = \begin{pmatrix} 0.5 \\ -\sqrt{2} \end{pmatrix}$$

Final Solution



$$\mathrm{C}_2 = \begin{pmatrix} 0.5 \\ \sqrt{2} \end{pmatrix} \quad ; \quad \mathrm{C}_3 = \begin{pmatrix} 0.5 \\ -\sqrt{2} \end{pmatrix} \quad ; \quad r = \alpha R = 1.5$$