JEE Linear Algebra using Matrix Computation

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To find the equation of the circle, which is the mirror image of the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} = 0$$

in the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3$$

Solution

General equation of circle with centre $\mathbf{c}_1$ and radius r :

$$\|\mathbf{x}-\mathbf{c}_1\|^2 = r^2$$

$$\Rightarrow (\mathbf{x}-\mathbf{c}_1)^T(\mathbf{x}-\mathbf{c}_1) = r^2$$

$$\Rightarrow \mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} = r^2 - \mathbf{c}_1^T\mathbf{c}_1$$

Comparing this equation with

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} = 0$$

We have

$$-2\mathbf{c}_1^T = -\begin{pmatrix} 2 & 0 \end{pmatrix}$$

and

$$r^2 - \mathbf{c}_1^T\mathbf{c}_1 = 0$$

Contd.

$$\Rightarrow \mathbf{c}_1^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$r^2 = \mathbf{c}_1^T \mathbf{c}_1$$

$$\Rightarrow r^2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow r^2 = 1$$

$$\Rightarrow r = 1$$

Contd.

Line in which image to be found:

$$\mathbf{L} : \mathbf{u}^T\mathbf{x} = 3$$

where $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $\mathbf{C}_2$ be the centre of the image circle.
$\mathbf{E}$ and $\mathbf{F}$ be points on the Line $\mathbf{L}$

$$\Rightarrow \mathbf{u}^T\mathbf{E} = 3 \;\; ; \;\; \mathbf{u}^T\mathbf{F} = 3$$

Contd.

$$\Rightarrow \mathbf{u}^T(\mathbf{E}-\mathbf{F}) = 0$$

Since

$$\mathbf{C}_2 - \mathbf{C}_1 \perp \mathbf{E} - \mathbf{F}$$

$$\Rightarrow (\mathbf{C}_2 - \mathbf{C}_1)^T(\mathbf{E}-\mathbf{F}) = 0$$

We know

$$\mathbf{a}^T\mathbf{b} = \mathbf{c}^T\mathbf{b} = 0 \Rightarrow \mathbf{c} = \alpha\mathbf{a}$$

Hence

$$\mathbf{C}_2 - \mathbf{C}_1 = \alpha\mathbf{u}$$

$$\Rightarrow \mathbf{C}_2 = \mathbf{C}_1 + \alpha\mathbf{u}$$

Contd.

Let $\mathbf{A}$ be the intersection point of line $\mathbf{L}$ and $\mathbf{C}_2 - \mathbf{C}_1$

$$\Rightarrow \mathbf{A} = \mathbf{C}_1 + \frac{\alpha \mathbf{u}}{2}$$

Point $\mathbf{A}$ lies on line $\mathbf{L}$

$$\Rightarrow \mathbf{u}^T\mathbf{A} = 3$$
$$\Rightarrow \mathbf{u}^T(\mathbf{C}_1 + \frac{\alpha \mathbf{u}}{2}) = 3$$
$$\Rightarrow \mathbf{u}^T\mathbf{C}_1 + \frac{\alpha \mathbf{u}^T\mathbf{u}}{2} = 3$$
$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\alpha \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{2} = 3$$

Contd.

$$\Rightarrow \alpha = 2$$

We know

$$\mathbf{C}_2 = \mathbf{C}_1 + \alpha \mathbf{u}$$

$$\Rightarrow \mathbf{C}_2 = \mathbf{C}_1 + 2\mathbf{u}$$

$$\Rightarrow \mathbf{C}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{C}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Clearly the radius of the image circle will as same of original circle

Contd.

Hence equation of Image circle:

$$\|\mathsf{x} - \mathsf{c}_2\|^2 = r^2$$

$$\Rightarrow \mathsf{x}^T\mathsf{x} - 2\mathsf{c}_2^T\mathsf{x} = r^2 - \mathsf{c}_2^T\mathsf{c}_1$$

$$\Rightarrow \mathsf{x}^T\mathsf{x} - \begin{pmatrix} 6 & 4 \end{pmatrix}\mathsf{x} + 12 = 0$$

Graphical Verification in Python

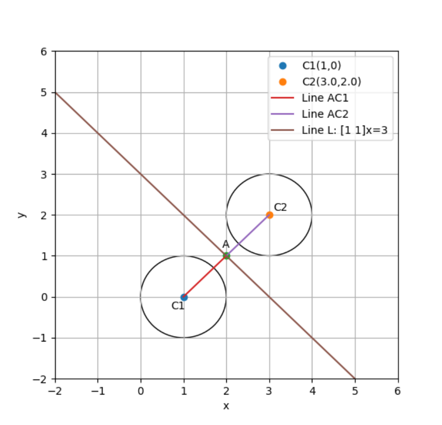

Figure 1: We can see that circles are mirror images in line L