

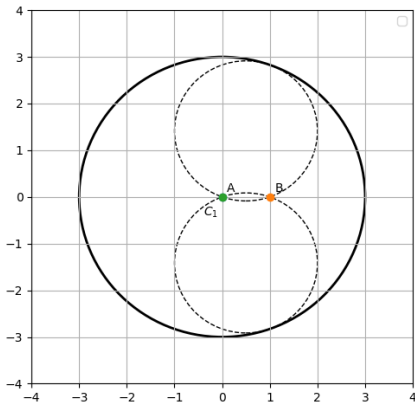
JEE Linear Algebra using Matrix Computation

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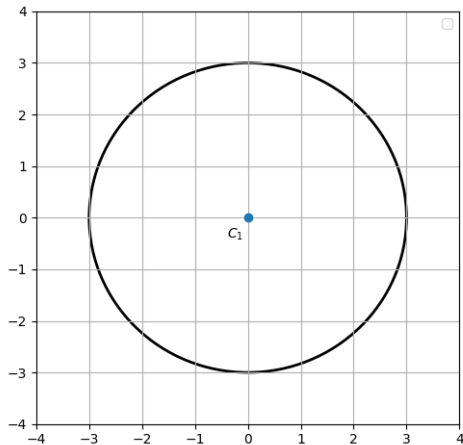
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Problem

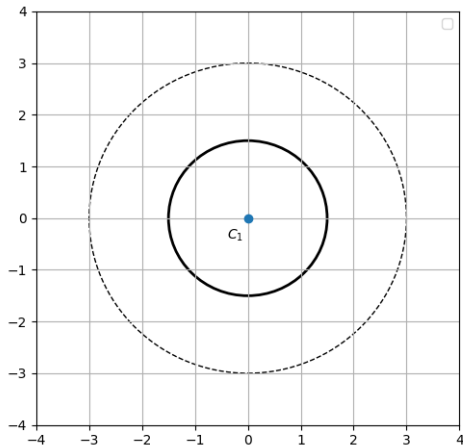
Find the centre of the circle passing through A: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and B: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
and touching the circle $x^2 + y^2 = 9$ (IIT-JEE 2002)



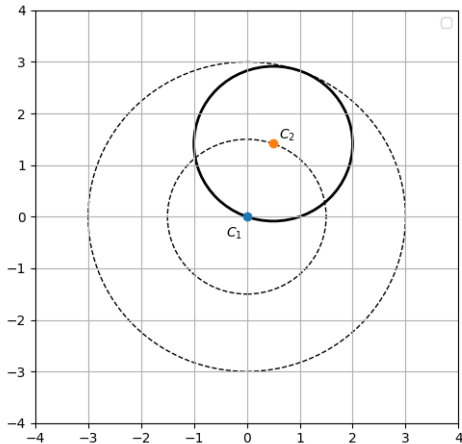
Given Circle: $x^2 + y^2 = 9$



Scaling



Translation

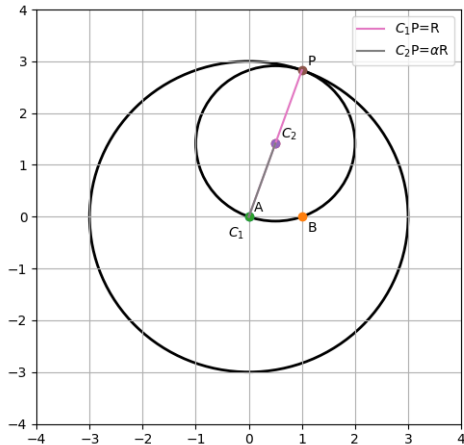


The circle G_2 is obtained from circle G_1 by SCALING and TRANSLATION.
The net result in AFFINE Transformation:

$$T(\mathbf{x}) = \alpha \mathbf{I} \mathbf{x} + (\mathbf{C}_2 - \mathbf{C}_1)$$

Where \mathbf{I} is Identity matrix

Constraints



G_1 has centre $\mathbf{C}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and radius $R = 3$

G_2 has centre \mathbf{C}_2 and radius αR

Constraints:

1. $\|\mathbf{C}_2 - \mathbf{C}_1\| = R(1 - \alpha)$as shown in figure

2. Equation of G_2 : $(\mathbf{x} - \mathbf{C}_2)^T(\mathbf{x} - \mathbf{C}_2) = (\alpha R)^2$ is satisfied by $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\mathbf{C}_2 \mathbf{C}_2^T = R^2(1 - \alpha)^2 \dots \text{from Constraint 1}$$

$$(\mathbf{x} - \mathbf{C}_2)^T(\mathbf{x} - \mathbf{C}_2) = (\alpha R)^2 \dots \text{from Constraint 2}$$

$$\Rightarrow \mathbf{x}^T \mathbf{x} - \mathbf{x}^T \mathbf{C}_2 - \mathbf{C}_2^T \mathbf{x} + \mathbf{C}_2^T \mathbf{C}_2 = (\alpha R)^2$$

$$\Rightarrow 2\mathbf{x}^T \mathbf{C}_2 = \mathbf{x}^T \mathbf{x} + R^2(1 - \alpha^2) - (\alpha R)^2$$

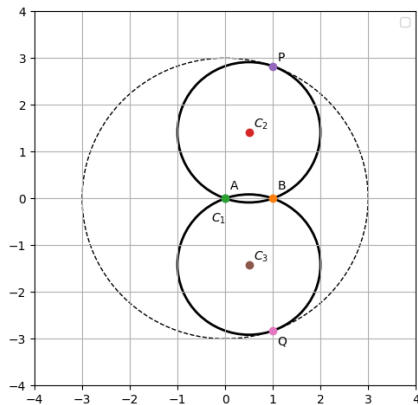
$$\Rightarrow 2\mathbf{x}^T \mathbf{C}_2 = \mathbf{x}^T \mathbf{x} + R^2(1 - \alpha^2) - (\alpha R)^2$$

Above equation is satisfied by $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solving gives us:

$$\alpha = 0.5 \quad ; \quad \mathbf{C}_2 = \begin{pmatrix} 0.5 \\ \sqrt{2} \end{pmatrix} \quad \text{OR} \quad \mathbf{C}_2 = \begin{pmatrix} 0.5 \\ -\sqrt{2} \end{pmatrix}$$

Final Solution



$$C_2 = \begin{pmatrix} 0.5 \\ \sqrt{2} \end{pmatrix} \quad \text{AND} \quad C_3 = \begin{pmatrix} 0.5 \\ -\sqrt{2} \end{pmatrix}$$