MT3732: Multivariate Statistics

Class exercise 1

Principal components of the Equicorrelation matrix

Obtain the principal components of the (3×3) equicorrelation matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}$$

by carrying out an eigenanalysis. For $\rho > 0$, what proportion of the total variation is explained by the first principal component? Show that the remaining variation is spread equally amongst the subsequent PC's.

You should carry out the following steps:

1. Find a cubic equation whose roots are $1 - \lambda$, where λ are the eigenvalues of $\boldsymbol{\Sigma}$.

2. Show that the $(p \times p)$ equicorrelation matrix may be written

$$\boldsymbol{\Sigma} = a\, \boldsymbol{I} + b\mathbf{1}\mathbf{1}^T$$

where $\mathbf{1} = (1, 1, ..., 1)^T$ is the $p-$vector of 1's and deduce that $\mathbf{1}$ is an eigenvector of $\boldsymbol{\Sigma}$ with eigenvalue

$$1 + (p - 1)\, \rho.$$

3. Write down by inspection all the eigenvalues of the $(p \times p)$ matrix $\boldsymbol{M} = \mathbf{1}\mathbf{1}^T$ and verify directly using $|\mathbf{M} - \lambda \boldsymbol{I}| = 0$ for the case $p = 3$.

4. Deduce the remaining eigenvalues and eigenvectors of $\boldsymbol{\Sigma}$, using the result proved at the end of Section 1 of the notes. Verify for the case $p = 3$, that a possible set of eigenvectors of $\boldsymbol{\Sigma}$ is

$$\begin{aligned} \boldsymbol{e}_1 &= \frac{1}{\sqrt{3}}\,(1, 1, 1)^T \\ \boldsymbol{e}_2 &= \frac{1}{\sqrt{2}}\,(1, -1, 0)^T \\ \boldsymbol{e}_3 &= \frac{1}{\sqrt{6}}\,(1, 1, -2)^T \end{aligned}$$

5. Factorize the cubic equation in Step 1 if you have not done so already.