$0+4 = 4+0 = 4 \in Z_6$ such that

$\therefore$ $0 \in Z_6$ acts as identity element

$\therefore$ for any $a \in Z_6$, $a+0 = 0+a = a \in Z_6$

$\therefore$ $(Z_6,+)$ satisfies Identity law

(IV) for $0 \in Z_6$, itself is the Inverse element

$0+0=0$

for $1 \in Z_6$, $5 \in Z_6$ is the inverse Element

for $2 \in Z_6$, $4 \in Z_6$ is the inverse element

for $3 \in Z_6$, $3 \in Z_6$ is the inverse element

Such that

$3+3=0$

$\therefore$ $(Z_6,+)$ Satisfies Inverse law

$\therefore$ $(Z_6,+)$ is a Grp

(V) for $2 \in Z_6$, & $5 \in Z_6$

$2+5 = 1 \in Z_6$.

$5+2 = 1 \in Z_6$.

$\therefore$ $2+5 = 5+2$

for ~~a, b ∈~~ any $a, b \in Z_6$, $a+b = b+a$

$\therefore$ $(Z_6,+)$ satisfies Commutative law

Hence $(Z_6,+)$ is an abelian Grp

Show that $(Z_7,\cdot)$ is a group

$\Rightarrow$ **Sub groups** =

let $(G,*)$ be a group & let H be a non empty subset of G.

Then H is a subgroup of G if

(i) $(H, *)$ satisfies closure law.

(ii) $(H, *)$ satisfies inverse law.

Eg: Let E be the set of even integers $(Z, +)$. Is E a subgroup?

Soln: $Z = \{ \ldots -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots \}$

$E = \{ \ldots -4, -2, 0, 2, 4, 6 \ldots \}$

(i) for $-4, 6 \in E$,

$-4+6 = 2 \in E$

for any $a, b \in E$, $a+b \in E$

$\therefore (E, +)$ satisfies closure law

(ii) for $-6 \in E$, $6 \in E$ such that

$-6+6 = 0 \in E$

for $10 \in E$, $\exists$ $-10 \in E$ such that $-10+10 = 0 \in E$

$\therefore (E, +)$ satisfies Inverse law

Hence $(E, +)$ is a subgroup.

RESULT - H is a subgroup of G iff for all $a, b \in H$, $ab \in H$ and $a^{-1} \in H$.

Proof - Suppose H is a subgroup of G

Then, for $a, b \in H$

$ab \in H$

Therefore, $(H, \cdot)$ satisfies closure law.

for $a \in H$, $\exists$ $a^{-1} \in b$ such that $aa^{-1} = a^{-1}a = e \in H$.

$a^{-1} \in H$ is called Inverse element of a.

$(H,\cdot)$ satisfies Inverse law.

Conversely,

(I) for $a,b \in H$, $ab \in H$

$\therefore (H,\cdot)$ satisfies closure law

(II) for $a \in H$, $\exists\, a^{-1} \in H$ such that $aa^{-1} = a^{-1}a = e$

$\therefore (H,\cdot)$ satisfies Inverse law

$\therefore (H,\cdot)$ satisfies Inverse law.

RESULT- H is a subgrp of G iff for all $a,b \in H$, $ab^{-1} \in H$

Proof- Suppose that H is a subgroup of G

(I) for $a \in H$, $a^{-1} \in H$ because H satisfies inverse law such that $aa^{-1} = a^{-1}a = e$

for $b \in H$, $b^{-1} \in H$ because H satisfies Inverse law such that $bb^{-1} = b^{-1}b = e \in H$

$\therefore$ H satisfies Inverse law

(II) for $a, b^{-1} \in H$, $ab^{-1} \in H$ because H satisfies closure law

Conversely,

for $a,b \in H$, $ab \in H$

$(H,\cdot)$ satisfies closure law.

for $a \in H$, $\exists\, a^{-1} \in H$ such that

$aa^{-1} = a^{-1}a = e \in H$

$\therefore (H,\cdot)$ satisfies Inverse Law

Hence $(H,\cdot)$ is a subgroup.

Homomorphism:

Let G_1 & G_2 be two Groups and f be a function from G_1 to G_2 i.e $f: G_1 \to G_2$. f is called homomorphism from G_1 to G_2 if $f(a,b) = f(a) f(b)$ $a,b \in G_1$

Isomorphism -

A function $f: G_1 \to G_2$ is called an isomorphism from $G_1 \to G_2$ if (i) f is homomorphism from G_1 to G_2 &

(ii) $f: G_1 \to G_2$ is a one to one function & onto function.

Eg- Consider the Groups $(R,+)$ & $(R^+, \times)$. Define the f^n $f: R \to R^+$ by $f(x) = e^x$ for $x \in R$.
Is f an isomorphism?

Soln- (i) for $a,b \in R$, $f(a+b) = e^{a+b} = e^a . e^b = f(a).f(b)$

$f(a+b) = f(a) f(b)$

$\therefore f: R \to R^+$ is homomorphic

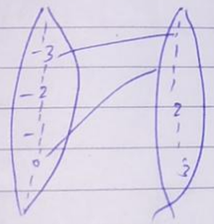
(ii) for $a,b \in R$, $f(a) = e^a$, $f(b) = e^b$

$f(a) \neq f(b) \Rightarrow a \neq b$

$\therefore f: R \to R^+$ is a one to one function

(iii) $Ran(f) = R^+$

$\therefore f: R \to R^+$ is onto function



Every element of R^+ is one or more image of elements of R

$\therefore f: R \to R^+$ is an isomorphism

Eg- Let T be the set of even integers. Show that $f: Z \to T$ defined by $f(a) = 2a$ btw the ~~semi~~groups $(Z^+,+)$ & $(T,+)$ are isomorphic

(i) for $a,b \in Z^+$, $f(a+b) = 2(a+b) = 2a+2b = f(a) + f(b)$

$\therefore f(a+b) = f(a) + f(b)$

$\therefore f: Z^+ \to T$ is homomorphism

(ii) for $a,b \in Z^+$

$f(a) = 2a$, $f(b) = 2b$

$f(a) \neq f(b)$

$\therefore$ No two elements in Z^+ have same image in T

$\therefore f: Z^+ \to T$ is one to one f^n,

(iii) $Ran(f) = T$

$f: Z^+ \to T$ is onto f^n.

$\therefore f: Z^+ \to T$ is isomorphic f^n

Eg- Let $A = Z$ defined by $x * y = x + y$. Is $*$ a binary operation?

Soln- $A = Z = \{ \ldots -2,-1,0,1,2,3 \ldots \}$.

for $-5, 8 \in Z$, $-5 * 8 = -5 + 8 = 3 \in Z$

for $-8, -6 \in Z$, $(-8)*(-6) = -8-6 = -14 \in Z$

$\therefore$ for any $a,b \in Z$ $a*b = a+b \in Z$

$\therefore *$ is a binary operation on Z.

Eg- Let $A = Z^+$ defined by $x*y = x-y$. Is $*$ a binary operation?

$A = Z^+ = \{1, 2, 3, 4, 5 \ldots\}$.

Soln- for $1, 5 \in Z^+$, $1 * 5 = 1 - 5 = -4 \notin Z^+$

for $2, 3 \in Z^+$, $2 * 3 = 2 - 3 = -1 \notin Z^+$

~~for~~ $\therefore$ $*$ is not a binary operation defined on Z^+.

Eg- Let $A = R$ defined by $x * y = x / y$.
Is $*$ a binary operation?

Soln- $A = R = \{\ldots -1, \ldots -2 \ldots -3 \ldots 0 \ldots 1 \ldots 2 \ldots 3\}$.

for $4, 2 \in R$, $4 * 2 = 4/2 = 2 \in R$.
for $1, 0.1 \in R$, $1 * -0.1 = 1/-0.1 = -10 \in R$.

$\therefore$ $*$ is a binary operation defined on R.

Eg- Determine whether binary operation $*$ is commutative & associative on Z^+ where $x * y = x + y + 2$

Soln- $Z^+ = \{1, 2, 3 \ldots\}$.

(I) for $2, 5, 7 \in Z^+$
$(2 * 5) * 7 = (2 + 5 + 2) * 7 = (9 * 7) = 9 + 7 + 2 = 18$
$2 * (5 * 7) = 2 * (5 + 7 + 2) = 2 * (14) = 2 + 14 + 2 = 18$
from ① & ②
$(2 * 5) * 7 = 2 * (5 * 7)$
$\therefore$ for $a, b, c \in Z^+$
$(a * b) * c = a * (b * c)$
$\therefore$ $*$ is associative

(II) for $8, 9 \in Z^+$
$8 * 9 = 8 + 9 + 2 = 19$
$9 * 8 = 9 + 8 + 2 = 19$

$\therefore$ $8 * 9 = 9 * 8$.
for $a, b \in Z^+$
$a * b = b * a$
$\therefore$ $*$ is commutative

⇒ Semigroups

Let S be a set & $*$ be any operation on S.
$(S, *)$ is called a semigroup if it satisfies Associative law.

⇒ Commutative Semi Group:

A Semigroup $(S, *)$ is called a commutative Semi grp if it satisfies commutative law.

⇒ Monoid. A Semigroup $(S, *)$ is called a monoid if it satisfies Identity law.

Eg- Show that $(Z, +)$ is a commutative grp.

soln - $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$.
for $3, 4, 5 \in Z$

Eg - Show $(Z^+, *)$ is a monoid

Eg-

Group Codes

Eg- The word $C = 1010110$ is transmitted through a channel. If $e = 0101101$ is the error pattern, find the word r recieved.
If $P = 0.05$ is the probability that a signal is incorrectly recieved. Find the probability with r is recieved.

Soln- (i) $C = 10\ 10\ 1\ 10$
$e = 0101101$
$\therefore r = C + e$

Arithmetic module 2 addition

$$
\begin{array}{llllllll}
C = & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
e = & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
\hline
r = & 1 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

r differs from C — 2nd, 4th, 5th, 7th

(ii) Prob with which r is recieved with $p = 0.05$ is given by,

$P = p^4 (1-p)^{7-4}$ — no of places corrupted; bit size

$P = p^4 (1-p)^{7-4}$
$= 0.05^4 (1 - 0.05)^3$
$= 0.000005.$

Eg- The word $C = 1010110$ is sent through a binary channel. If $P = 0.02$ is the probability of incorrect receipt of a signal. Find the probability that C is recieved as $r = 1011111$. Determine the error pattern.

Soln-
$C = 10\ 10\ 110$
$r = 10\ 11111$
changes happened in 2 places —

(i) Error probability $= p^2 \cancel{(p-1)} (1-p)^{7-2}$
$= p^2 (1-p)^5$
$= 0.00036.$

(ii) $r = C + e$
$1 = 1 + 0$
$0 = 0 + 0$
$1 = 1 + 0$
$1 = 0 + 1$
$1 = 1 + 0$
$1 = 1 + 0$
$1 = 0 + 1$
$\therefore \underline{e = 0001001}$ is the error pattern

⇒ Encoding ~~functions~~

Let $E: Z_2^m \to Z_2^{m+1}$ be an encoding function defined as follow

$$
W_{m+1} = \begin{cases} 0 & \text{if } w \text{ contains even no of 1's} \\ 1 & \text{if } w \text{ contains odd no of 1's} \end{cases}
$$

Eg- Find the code words assigned by
(i) $E: Z^3 \to Z^4$, for 000, 001, 011, 100, 110, 101, 111, 010
(ii) $E: Z^4 \to Z^5$ for 0000, 0001, 0101, 1111, 1010, 1100, 1101, 1001

Soln - (i) $E: Z^3 \to Z^4$

$E(000) = 0000$ $E(110) = 1100$
$E(001) = 0001$ $E(101) = 1010$
$E(011) = 0110$ $E(111) = 1111$
$E(100) = 1001$ $E(010) = 0101$

(ii) $E: Z^4 \to Z^5$

$E(0000) = 00000$ $E(1010) = 10100$
$E(0001) = 00011$ $E(1100) = 11000$
$E(0101) = 01010$ $E(1101) = 11011$
$E(1111) = 11110$ $E(1001) = 10010$

$(m, 3m)$ Encoding fn →

The encoding fn is $Z^m \to Z^{3m}$

Eg - Find the code word assigned by the encoding fn $E: Z^m \to Z^{3m}$ for 000, 011, 010, 100, 001, 101, 110, 111

Soln - $E(000) = 000000000$ $E(001) = 001\,001\,001$
$E(011) = 011011011$ $E(101) = 101\,101\,101$
$E(010) = 010\,010\,010$ $E(110) = 110\,110\,110$
$E(100) = 100100100$ $E(111) = 111\,111\,111$

⇒ $(3m, m)$ Decoding fn

Let $D: Z^{3m} \to Z^m$ be a decoding fn defined by
$D(\lambda) = S_1, S_2 \ldots S_m$ where

$$S_i = \begin{cases} 1 & \text{if } S a_i\, a^S_{i+m}\ a^S_{i+m} \text{ has majority 1's} \\ 0 & \text{if } S a_i\, a^S_{i+m}\ S a_{i+m} \text{ has majority 0's} \end{cases}$$

Eg - Find the decoded word assigned by $D: Z^6 \to Z^2$ fn

Soln - $D(\lambda) = S_1 S_2$

$D(111111) = 11$ $D(010011) = 01$
$D(101010) = 10$ $D(110110) = 11$
$D(010101) = 01$ $D(010111) = 01$
$D(100100) = 00$ $D(000111) = 01$

$D(000\,000\,000) = 000$ $D(001001001) = 001$
$D(011\,011\,011) = 011$ $D(101101101) = 101$
$D(010010010\,0) = 010$ $D(110110110) = 110$
$D(100100100) = 100$ $D(111111111) = 111$

⇒ Hamming distance -

Let x & y be words in B^m where $x = x_1 x_2 x_3 \ldots x_n$ & $y = y_1 y_2 y_3 \ldots y_n$ & $x_1 \neq y_1$.
The hamming distance b/w x & y is the weight of $x+y$.

It is denoted by $d(x, y) = H(x, y)$
$= \text{weight}(x+y)$

Q = Find the dist b/w x & y when
(i) $x = 110110$, $y = 000101$ (ii) $x = 001100$, $y = 010110$

Soln - (i)
$$\begin{array}{r} 110\,110 \\ 000\,101 \\ \hline 110011 \end{array}$$

* weight → max no of 1's
* mod 2 addition

$d(x,y) = \text{weight}(x+y) = \text{weight}(110011)$
$= |110011| = 4$

(ii)
$$\begin{array}{r} 001100 \\ 010110 \\ \hline 011010 \end{array}$$

$d(x,y) = \text{weight}(x+y) = \text{weight}(011010)$
$= \underline{\underline{3}}$

⇒ Minimum Distance

The min distance of an encoding fn $E : Z^m \to Z^n$ is the min dist btw all the distant pairs of the code words. It is denoted by

min dist = $\min (d(E(x), E(y)) \mid x, y \in B^m))$

Eg- Find the min dist of $E : Z^2 \to Z^5$ encoding fn
$E(00) = 00000$, $E(10) = 00111$, $E(01) = 01110$,
$E(11) = 11111$

Soln-
$d(E(00), E(10)) = d(00000 + 00111) = d(00111) = 3$
$d(E(00), E(01)) = d(00000 + 01110) = d(01110) = 3$
$d(E(00), E(11)) = d(00000 + 11111) = d(11111) = 5$
$d(E(10), E(01)) = d(00000 + 01110) = d(10100) = 2$
$d(E(10), E(11)) = d(00000 + 11111) = d(11000) = 2$
$d(E(01), E(11)) = d(00000 + 11111) = d(10001) = 2$.

Min dist $= \min \{2, 3, 3, 5, 2, 2\}$
$= 2$

Eg- let $E : Z^3 \to Z^6$ defined by
$E(000) = 000111$, $E(001) = 001001$, $E(010) = 010010$
$E(011) = 011100$, $E(100) = 100100$, $E(101) = 101010$
$E(110) = 110001$, $E(111) = 111000$

find min dist

Parity check Matrix

$$G = \begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - & - & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & - & - & a_{mn} \\ 1 & 0 & 0 & - & - & 0 \\ 0 & 1 & 0 & - & - & 0 \\ 0 & 0 & 1 & - & - & 0 \\ \vdots \\ 0 & 0 & 0 & - & - & 1 \end{bmatrix}$$

An encoding fn of G is ~~defined~~ denoted as $E : Z^m \to Z^n$ such that $E(b) = x$ where $b = b_1, b_2 \ldots b_m$ & $x = x_1 x_2 \ldots x_n$

→ To determine $x_1 x_2 \ldots x_n$ we use following Relation

$x_1 = b_1 a_{11} + b_2 a_{21} + b_3 a_{31} + \ldots + b_m a_{m1}$
$x_2 = b_1 a_{12} + b_2 a_{22} + b_3 a_{32} + \ldots + b_m a_{m2}$
$x_3 = b_1 a_{13} + b_2 a_{23} + b_3 a_{33} + \ldots + b_m a_{m3}$
$\vdots$
$x_n = b_1 a_{1n} + b_2 a_{2n} + b_3 a_{3n} + \ldots + b_m a_{mn}$

Eg- An encoding fn $E : Z^2 \to Z^5$ is given by general matrix

$$G = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine all code words of Z^2

Soln- $Z^2 = \{00, 10, 01, 11\}$

i) $e(00) = 00x_1 x_2 x_3$
ii) $e(10) = 10x_1 x_2 x_3$
iii) $e(01) = 01x_1 x_2 x_3$
iv) $e(11) = 11x_1 x_2 x_3$

To find $x_1 x_2 x_3$ of (i).
$x_1 = b_1 a_{11} + b_2 a_{21} = 0\times1 + 0\times0 = 0$
$x_2 = b_1 a_{12} + b_2 a_{22} = 0\times1 + 0\times1 = 0$
$x_3 = b_1 a_{13} + b_2 a_{23} = 0\times0 + 0\times1 = 0$
$\therefore e(00) = 00000$

To find $x_1 x_2 x_3$ of (ii)
$x_1 = 1\times1 + 0\times0 = 1$
$x_2 = 1\times0 + 0\times1 = 1$
$x_3 = 1\times0 + 0\times1 = 0$
$\therefore e(10) = 10110$

To find x_1, x_2, x_3 of (iii)
$x_1 = 0\times1 + 1\times0 = 0$
$x_2 = 0\times1 + 1\times1 = 1$
$x_3 = 0\times0 + 1\times1 = 1$
$\therefore e(01) = 01011$

To find $x_1 x_2 x_3$ of (iv)
$x_1 = 1\times1 + 1\times0 = 1$, $x_2 = 1$, $x_3 = 1$
$\therefore e(11) = 11111$

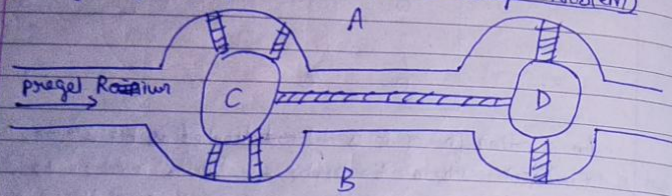
31/10/19

Garry chartend
Introduction to G.T

UNIT-5

GRAPH THEORY

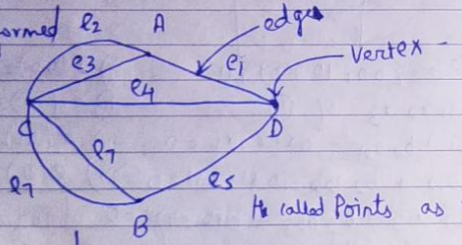
⇒ Konigsberg Bridge Problem (Seven Bridge Problem)



A&B ⟶ Land Area
C&D ⟶ Small island / land area
Total 4 land areas

Problem :- Start from any pt / land area
Cover all bridges exactly once
Reach to the same pt

• Euler transformed this probl into a fig ⟶



↓

He called Graph.

→ A Graph G is associated with only 2 Components Vertex and Edges

$V(G) = \{A, B, C, D\}$
$V(G) = \{v_1, v_2, v_3, v_4 \ldots v_n\}$.
$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$.
$E(G) = \{e_1, e_2, e_3, e_4 \ldots e_n\}$.

$G = G(V, E) : G(4, 7)$:

If it is an euler graph walk is possible
If " " not an " " " " not possible.

→ Graph Theory was first Introduced by Euler in 1736 through his research paper on Konigsberg bridge Problem

→ Konigsberg Bridge Problem (Seven Bridge Problem)

Two islands C&D formed by the Pregel River is connected to each other and the river bands A & B with Seven bridges as Shown in the fig.

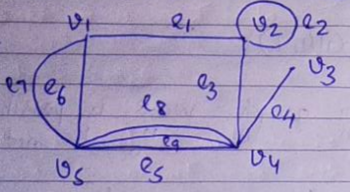
The problem was to Start at any of the four land areas of the city A, B, C or D. Walk over each of the Seven bridges exactly once & reach the Starting point. Many ppl tried to Solve this problem but were unsuccessful. Euler replaced each of land areas by a point and each bridge by a line. This produced a diagram as Shown above.
Through this diagram, euler gave the concept of a Graph.

→ Graph Consists of objects called Vertices & Edges

It is denoted by $G = G(V, E)$ where $V(G) = \{v_1, \ldots v_n\}$ and $E(G) = \{e_1, e_2, \ldots e_n\}$.

A Graph G is also called as <u>Linear Graph</u>.

Eg- $G(5,8)$:



→ **Vertices:** Vertices are called Points, nodes or dots.
A Vertex set of a Graph G is denoted by $V(G)$. For the graph $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

→ **Edges:** A line joining two vertices is called an Edge. An edge can be a straigh line or an arc or a curved line or a circle (Self loop)
The Edge set is denoted by $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

→ **Self-loop-** An edge having the same vertex as both its end points/vertices is called a <u>self-loop</u> or <u>loop</u>.
In the graph, edge e_2 is a self loop.

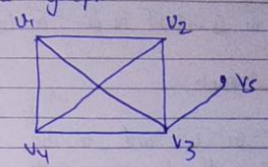
→ **Parallel edges -**

Given a pair of vertices may have more than one edges. Such edges are called Parallel edges. In this graph, e_6 & e_7, e_5, e_8 & e_9 are called Parallel Edges.

⇒ **Simple Graph**

A Graph without a self loop and parallel edges is called a simple graph.

Eg- G_1:



G_1 is a <u>Simple Graph</u>.

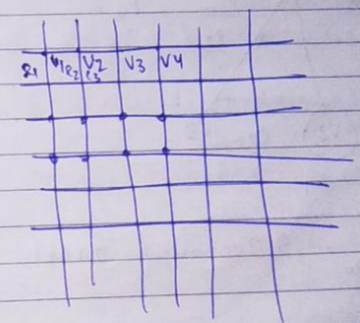
⇒ **Finite Graphs**

A Graph G having finite no of vertices & edges is called a finite graph
G_1 is a finite graph $G(5,7)$

⇒ **Infinite Graphs**

A Graph G having infinite no of Vertices & edges is called an infinite graph.

Eg-



⇒ ~~Components of a Graph~~

⇒ ~~Connected Graph~~

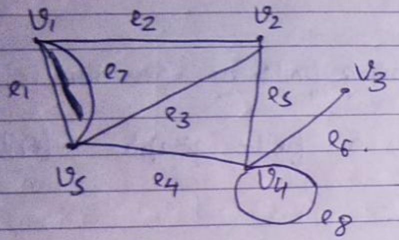
~~A Graph G is said to be a connected graph~~

⇒ Connected Graph

A Graph G is said to be a connected graph if there is atleast one path b/w every pair of vertices. Otherwise G is called Disconnected Graph.

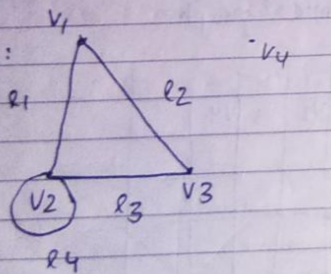
Eg-

$G_1(5,6)$:

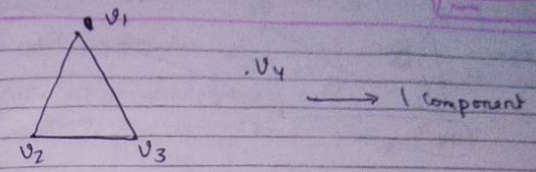


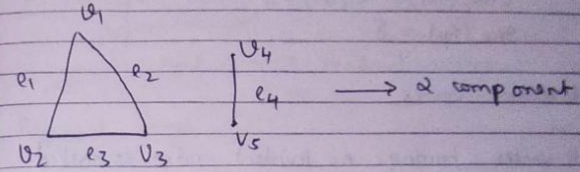
G_1 is called a Connected Graph.

$G_2(4,4)$:



G_2 is a Disconnected Graph



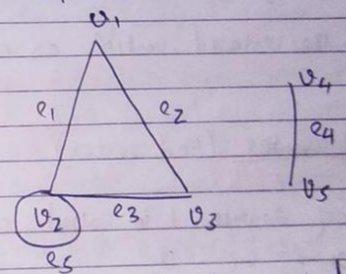


⇒ Components of a graph

A Disconnected Graph have 2 or more connected sub graphs. Each of these connected subgraphs is called a component.

Eg-

$G_2(5,5)$:



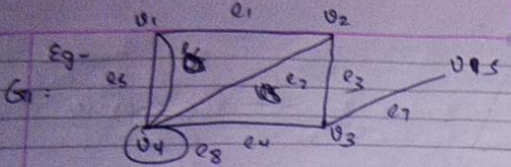
G_2 is 2 component disconnected graph

⇒ Order & Size of a Graph -

The no of vertices in Graph G is called its order & the no of edges is called its size

Eg -

G_1 :



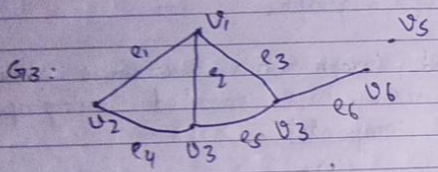
Order $(G_1) = O(G_1) = 5$

Size $(G_1) = 8$

⇒ Isolated Vertex

A vertex having no incident edge is called an isolated vertex

G_3 :

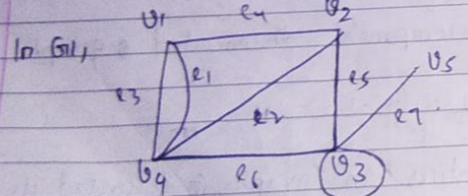


v_5 is the isolated vertice as no edge is incident on it.

⇒ Pendant Vertex (End vertex)

A vertex of degree 1 is called a Pendant vertex.

In G_1,



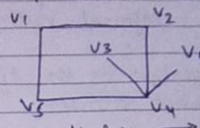
degree of v_3 — 5

degree of v_2 - 3

v_5 = 1.

An edge contributes 2 to the vertex.

v_5 → pendant vertex.

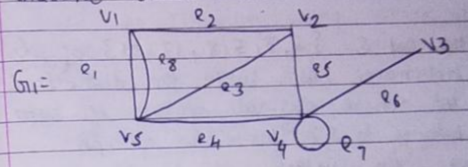


v_3 & v_6 → Pendant vertex

⇒ Adjacent Edges

Two non parallel edges are said to adjacent if they incident on a common vertex

G_1 =



e_2 & e_3 are adjacent edges because they incident on same vertex v_2

e_3 & e_5 " " "

e_1 & e_8 are not " " as they are parallel

e_1 & e_5 are not " " — not incident on same vertex.

e_7 & e_4 — are not adjacent edges

Self loop is not adjacent

⇒ Adjacent Vertices -

Two vertices are said to be adjacent if they are the end vertices of the same edge.

Eg:-

① In G_1,
v_1 & v_2 are the end vertices of the edge e_2
$\therefore$ v_1 & v_2 are the adjacent vertices.

(ii) v_2 & v_5 are the end vertices of the edge e_3
$\therefore$ v_2 & v_5 are the adjacent vertices

(iii) v_3 & v_5 are not the end vertices of the same edge
$\therefore$ v_3 & v_5 are not adjacent vertices.

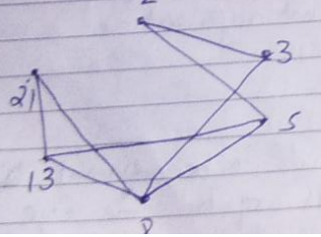
⇒ Subgraphs

Eg- Consider the set $S = \{2, 3, 5, 8, 13, 21\}$ of 6 specific Fibbonacci nos. Write the distinctive pair of int whose absolute value of sum or diff belongs to S? Write the graph representing the system.

$V(G) = \{2, 3, 5, 8, 13, 21\}$

$E = \{\{2,3\}, \{2,5\}, \{3,5\}, \{3,8\}, \{5,13\}, \{8,13\}, \{13,21\}, \{5,8\}, \{8,21\}\}$.

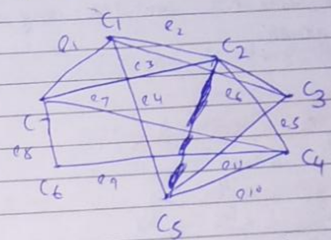
G:



Eg- The 10 editors have decided on the seven committees
$C_1 = \{1, 2, 3\}$
$C_2 = \{1, 3, 4, 5\}$
$C_3 = \{2, 5, 6, 7\}$
$C_4 = \{4, 7, 8, 9\}$
$C_5 = \{2, 6, 7\}$
$C_6 = \{8, 9, 10\}$
$C_7 = \{1, 3, 9, 10\}$.

They have set aside 3 time periods for the seven committees to meet on these fridays when all 10 editors are present. Some pairs of committee cannot meet during the same period because one or two of the editors are on both the committees. Model the situation using Graph. Write Edge set & Vertex set

Soln-



1 - C_1, C_2, C_7.
$C_1 - C_2$
$C_1 - C_7$
$C_2 - C_7$
2 - C_1, C_3, C_5.
$C_1 - C_3$
$C_3 - C_5$
$C_1 - C_5$
3 - C_1, C_2, C_7
$C_1 - C_2$
$C_2 - C_7$
$C_1 - C_7$
4 - C_2, C_4.
5 - C_2, C_3
6 - C_3, C_6.
7 - C_4, C_5
8 - C_4, C_6
9 - C_4, C_6, C_7.
10 - C_6, C_7

Those who are not connected meeting cannot be conducted at same time — C_5, & C_6

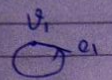
$V(G) = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7\}$

$E(G) = \{e_1, e_2 - - - e_{13}\}$

⇒ Trivial Graph

A Graph with exactly one vertex is called a trivial graph.

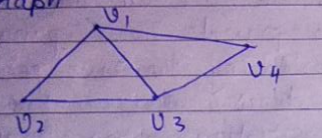
G_1:



⇒ Non-Trivial Graph

A Graph with more than one vertex is called a non-trivial graph

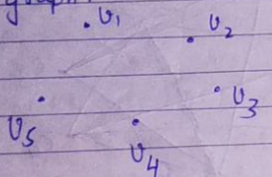
Eg- G_2:



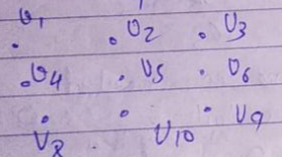
⇒ Null Graph -

A Graph which does not contains any edges is called a null graph.

Eg- G_3.



G_4:



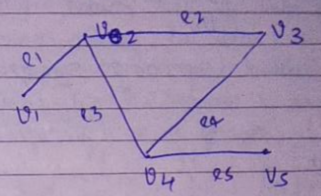
Note- (I) In a null graph, each vertex is an isolated vertex

(II) In a null graph, degree of each vertex is 0.

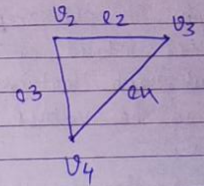
⇒ Subgraph:

let $G(V,E)$ be a graph then $G_1 = (V_1, E_1)$ is called a subgraph of G if $V_1(G_1) \subseteq V(G)$ and $E_1(G_1) \subseteq E(G)$, where each edge in E_1 is incident with vertices in V_1.

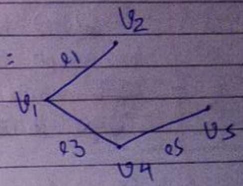
Eg- G:



G_1:



G_2:



$V_1(G_1) = \{v_2, v_3, v_4\}$ G_2 is a subgraph of G

$\therefore V_1(G_1) \subseteq V(G)$

$E_1(G_1) \subseteq E(G)$

$E_1(G_1) = \{e_2, e_3, e_4\}$.

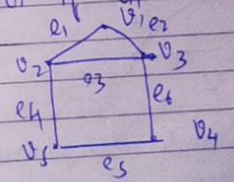
Hence G_1 is a subgraph of G.

$G_1 \subseteq G_2$

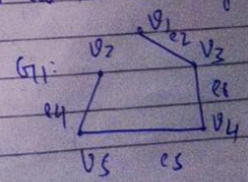
⇒ Spanning Subgraph

Edges and Subset of G. A Subgraph G_1 of a graph G is called a spanning subgraph of G if G_1 contains all the vertices of G.

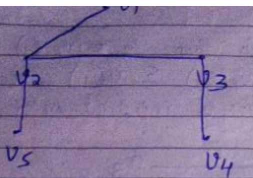
Eg- G:



G_1:



G_2:

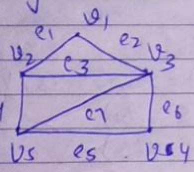


G_1 & G_2 are called spanning subgraphs of graph G.

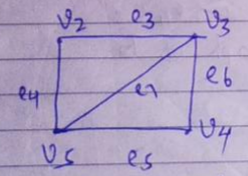
⇒ <u>Induced Subgraph</u> -

Let $G(V,E)$ be a graph. The subgraph F of the graph G is called induced ~~by E~~ subgraph of G. ~~by E~~ where the subgraph contains all the edges incidenting on the vertices of G

Eg - G:

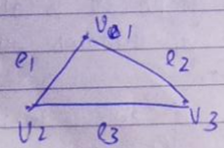


F:

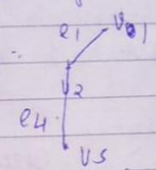


F is an induced subgraph of G.

F_1:



F_2:

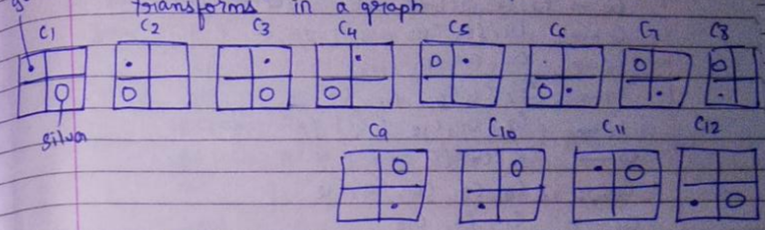


F_1 is an induced subgraph of G & F_2.

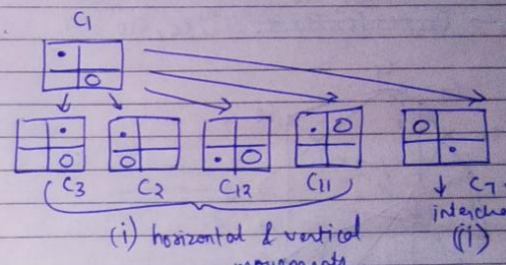
Eg - Suppose we have 2 coins. One silver and one gold placed on two of the four squares of a 2x2 checker board. An Arrangement C_i can be transformed into C_j such that if C_j can be obtained from C_i by performing exactly one of the following steps -

(i) moving one of the coins in C_i horizontally or vertically to an unoccupied square.

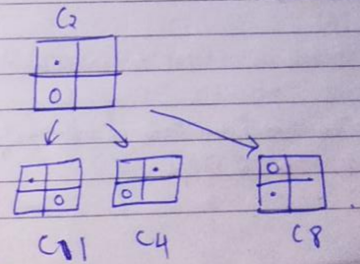
(ii) Interchanging the two coins in C_i. Represent the transforms in a graph



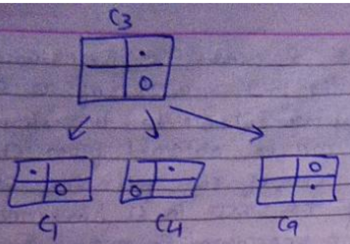
Soln -



$C_1 \to C_2, C_3, C_7, C_{11}, C_{12}$

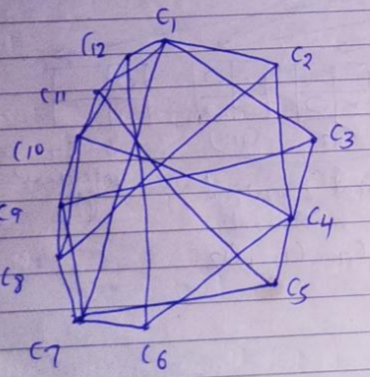


$C_2 \to C_1, C_4, C_8$.



$C_3 \to C_1, C_4, C_9$
$C_4 \to C_2, C_3, C_5, C_6, C_{10}$
$C_5 \to C_4, C_7, C_{11}$
$C_6 \to C_4, C_7, C_{12}$.
$C_7 \to C_1, C_5, C_6, C_8, C_9$
$C_8 \to C_2, C_7, C_{10}$
$C_9 \to C_3, C_7, C_{10}$
$C_{10} \to C_4, C_8, C_9, C_{11}, C_{12}$
$C_{11} \to C_1, C_5, C_{10}$
$C_{12} \to \cancel{C_1, C_5, C_6, C_8, C_9}\ C_1, C_6, C_{10}$

$C_1 \to C_2, C_3, C_7, C_{11}, C_{12}$
$C_2 \to C_1, C_4, C_8$.



⇒ **Word Graph**.
Let w_1 be a word. The word w_1 can be transferred into w_2 by following the 2 steps

(i) Interchange 2 letters of w_1
(ii) Replacing a letter in w_1 by another letter. The word w_1 can be adjacent to w_2

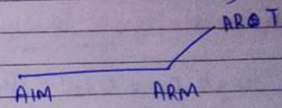
A graph obtained by following the above 2 steps is called a word graph.

Eg- Given a collection of 3 letters english words say ACT, AIM, ARC, ARM, ART, CAR, CAT, OAR, OAT, RAT, TAR. Draw the word graph.

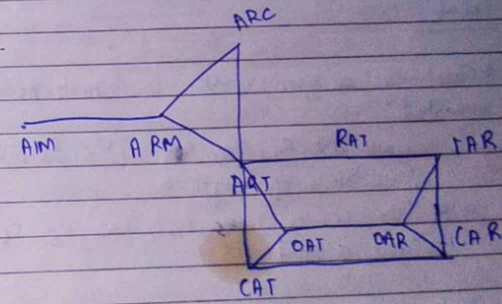
Soln ACT → ART (step 2)
ACT → CAT (step 1)

ART → RAT (1)
ART → ARM, ARC (step 2) } ART → RAT, ARM, AR.

Graph :

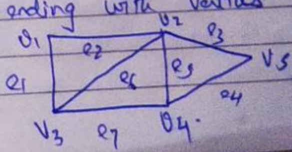


Graph-



⇒ Walk: Let G be a Graph. A walk in G is defined as a finite ordinating sequence of vertices & edges, beginning & ending with vertices

Eg- G:



length = 3.

A walk $\{v_1, e_2, v_2, e_6, v_3, e_7, v_4\}$.

→ length of a walk is no of edges in the walk.

→ A walk $\{v_3, e_1, v, e_2, v_2, e_6, v_3, e_7, v_4, e_4, v_5\}$
length = 5

⇒ Closed walk

A walk is closed if it begins & end at the same vertex.
Eg.
A walk $\{v_4, e_4, v_5, e_3, v_2, e_6, v_3, e_7, v_4\}$
is a closed walk. of length 4.

⇒ Trivial walk

A walk of length 0 is called Trivial walk.

⇒ Trial

A Trial is an open walk in which no edge is repeated
Eg- A walk $\{v_1, e_1, v_3, e_6, v_2, e_5, v_4, e_4, v_5\}$.
is a trial of length 4.
A walk $\{v_3, e_6, v_2, e_5, v_4, e_7, v_3, e_6, v_2, e_2, v_1\}$
is not a trial

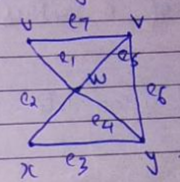
Note- 1. A Trial is a walk but a walk need not be a trial

⇒ Path:
A path is an open walk in which no vertex is repeated
A walk $\{v_1, e_1, v_3, e_7, v_4, e_5, v_2, e_3, v_5\}$
is an ~~open walk~~ path of length 4
~~A walk~~

⇒ Circuit
A closed Trial is called a circuit.
⇒ Cycle
A closed path is called a cycle

⇒ Theorem- If a graph G contains a u-v walk of length l, then G contains a u-v path of length l

G:



Let P1 be the smallest walk from u to v covering all the vertices, then path P1 (length = 4)
path P1 : $\{u, e_1, w, e_2, x, e_3, y, e_6, v\}$ is a smallest walk covering all the vertices of G
Let P_1: $\{u, v_1, v_2, v_3 \_\_\_\_ u_n\}$ be the smallest walk covering all vertices of G & having length K.
Suppose that there is a path of length l greater than K, l > K
P_2: $\{u, e_1, w, e_2, x, e_3, y, e_4, w, e_5, v\}$
(is of length 5).

let the path p2 be

$P_2 = \{ v, e_1, w \text{ - - - - - - } \}$ is a walk of length 5.

i.e $P_2 : \{ v, v, v_2 \text{ - - } v_3 \text{ - - } v_n \}$ be a walk of length k

Then definitly some vertices are repeated

Therefore the walk P_2 cannot be a path.

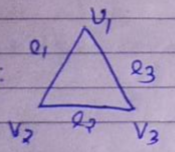
Thus, P_1 is the path of length at most 1

=> Complete Graph-

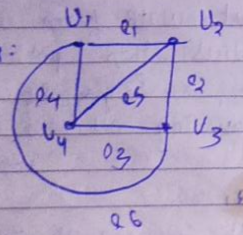
A simple Graph in which there exists an edge btw everywhere of vertices is called a complete Graph.

A complete Graph of n vertices is denoted by K_n

Eg-

G1: 

is a complete Graph of 3 vertices (K_3)

G2: 

is a complete Graph of 4 vertices (K_4)

=> Bipartite Graph-

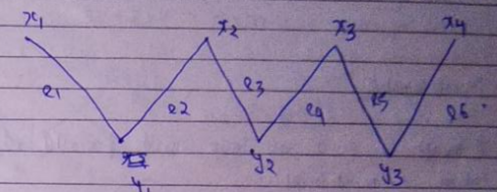
Let G be a Graph. If the vertex set $V(G)$ can be partitioned into 2 subsets V_1 & V_2 such that each edge of G has one ~~edge~~ end in V_1 & other end in V_2. Then the graph G is called a bipartite Graph.

In a bipartite Graph, the vertex sets V_1 & V_2 satisfies the foll properties:-

i) $V_1(G) \cup V_2(G) = V(G)$

ii) $V_1(G) \cap V_2(G) = \phi$

Eg:-

G: 

$V(G) = \{ x_1, y_1, x_2, y_2, x_3, y_3, x_4 \}$

$V_1(G) = \{ x_1, x_2, x_3, x_4 \}$

$V_2(G) = \{ y_1, y_2, y_3 \}$

=> Complete Bipartite Graph-

A Bipartite Graph G with each vertex V_1 is joined with every vertex of V_2. Then G is called a complete bipartite Graph

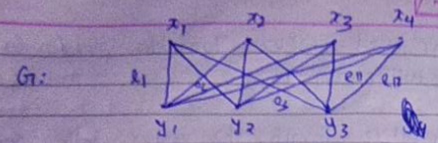
If cardinality

If $|V_1(G)| = m$, $|V_2(G)| = n$ then a complete bipartite Graph has

(i) No of vertices $= m+n$

(ii) no of edges $= m \times n$

Eg-



$v_1(G) = \{x_1, x_2, x_3, x_4\} \Rightarrow n(v_1(G)) = 4$

$v_2(G) = \{y_1, y_2, y_3\} \Rightarrow n(v_2(G)) = 3$

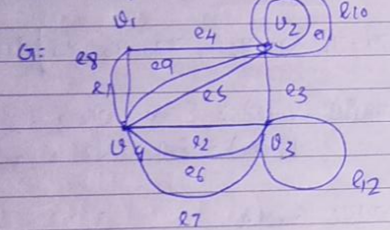
No of vertices $= m+n = 4+3 = 7$

No of edges $= m \times n = 4 \times 3 = 12$

⇒ **Multi Graph**

A Graph G with self loop and parallel edges is called a multi Graph.

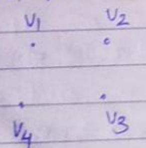
Eg-



⇒ **Regular Graph-**

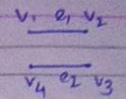
A Graph in which all vertices are of equal degree is called a regular Graph

Eg- G_1:



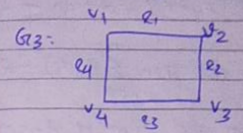
0-regular graph
↓
degree 0

Eg - G_2:



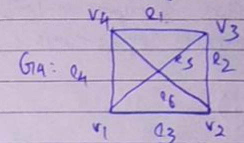
1 - regular

G_3:



2-regular

G_4:



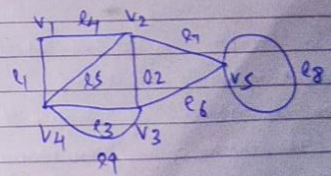
3-regular

~~Note: All regular Graph~~
r regular Graph
If the degree of all vertices of a Graph G is r then the graph is called r regular Graph.

⇒ **Degree of a vertex**.

The no of edges incidenting on a vertex v_i with self loop counted twice is called Degree of v_i. It is denoted by $deg(v_i)$ or $d(v_i)$

Eg-



$deg(v_1) = 2$

$deg(v_2) = 4$

$deg(v_3) = 4$
$deg(v_4) = 4$
$deg(v_5) = 4$

$\sum_{i=1}^{5} d(v_i) = 2+4+4+4+4 = 18 = 2\times 9 = 2 \times$ no of edges.

RESULT If G is a Graph of size m, then $\sum_{i=1}^{n} d(v_i) = 2m$

Proof- Each edge contributes ~~to~~ two the sum of degrees of vertices.
Since the graph has m edges therefore contributes exactly 2m to the sum of degrees of vertices.
Therefore $\sum d(v_i) = 2\times m$

(Hand Shake Property)

RESULT Every Graph has an even no of odd degree vertices

Proof- let G has V_j no of even degree vertices and V_k no of odd degree vertices

$$\underset{\text{even}}{\sum d(v_i)} = \underset{\text{Even}}{\sum d(v_j)} + \sum d(v_k)$$

$\therefore \sum d(v_k) = \text{Even} - \text{Even} = \text{even}$

$\therefore$ no of odd degree vertices is even.

Eg- A certain Graph has order 14 & size 27. The deg of each vertex of G is 3, 4 or 5
There are 6 vertices of degree 4. How many vertices of G have degree 3 & how many have degree 5

Soln- $|V| = n = 14$, $|E| = 27$

let x be the no of vertices of degree 3

$3x + 6\times 4 + (8-x)\times 5 = 2\times|E|$
$3x + 24 + 40 - 5x = 2\times 27$
$-2x + 64 = 54$
$-2x = -10$
$\boxed{x=5}$

Vertices of degree 3 = 5
Vertices of degree 5 = 8 - 5 = 3

Eg- If G is a connected Graph with $|E| = 17$ and $deg(v) \geq 3$ for all $v \in V(G)$. What is the max value of |V|?

Soln - In a connected graph of G, sum of the degree of the vertices $= \sum d(v_i) = 2\times|E|$
$= 2\times 17 = 34$

Since $d(v_i) \geq 3$
$\therefore$ Max no of vertices of Graph $G \leq \left\lfloor \frac{34}{3} \right\rfloor$

$\therefore$ Max no of vertices $|V| = 11$

$\Rightarrow$ Graphical

Let S be a finite sequence of non neg integers. If S forms a degree sequence of some graph then S is called Graphical.

Q= Which of the following sequence are graphical?
Soln- (i) S_1: 3, 3, 2, 2, 1, 1
Is S_1 graphical?
(ii) S_2: 6, 5, 5, 4, 3, 3, 3, 2, 2
(iii) S_3: 7, 6, 4, 4, 3, 3, 3

(iv) $S_4 : 3,3,3,1$

Soln- Procedure for a degree sequence is graphical for a simple Graph

(i) Find no of vertices
(ii) Find no of edges
(iii) Find no of odd degree vertices
(iv) Highest degree of vertex can be $<n-1$

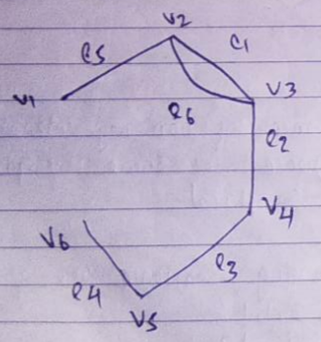
(i) $S_1 : 3,3,2,2,1,1$

(i) No of vertices $= 6$
(ii) No of edges $= \Sigma d(v_i) = 2|E|$
$= 3+3+2+1+1+2$
$= 2\times|E| \Rightarrow |E| = 6$

(iii) No of odd degree vertices $= 4$ (even)
$(1,1,3,3)$

(iv) Highest degree of vertex $< 6-1 = 5$
$3 < 5$ (True)

$\therefore 3,3,2,2,1,1$ is Graphical



Q= Use Havel - Hakimi Theorem, to determine whether the sequence $S: 5,4,3,3,2,2,2,1,1,1$ is Graphical

Soln- $|v| = 10$, $\Sigma d = 2\times|E| \Rightarrow |E| = 12$
No of odd degree vertices $= 6$ (Even)
Highest degree $< n-1 = 10-1 = 9$ (True)
$5 < 9$ (True)
$S : 5,4,3,3,2,2,2,1,1,1$ (non-increasing)

(i) Delete first element 5
$S_1 : 4,3,3,2,2,2,1,1,1$

(ii) Subtracting 1 from each term till $d_1+1 = 4+1 = 5$
$S_1' : 3,2,2,1,1,1,1,1,1$

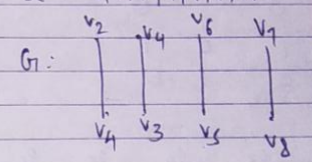
(iii) Arranging in non inc order,

$S_1' : 3,2,2,2,1,1,1,1,1$

(iv) Delete 3
$S_2 : 2,2,2,1,1,1,1,1$
Subtracting from each term till $d_1+1 = 2+1 = 3$

$S_2' : 1,1,1,1,1,1,1,1$

G :



⇒ Adjacency Matrix of a Graph G

(nxn) Let G be a graph of order n & size m. The adjacency matrix of G is an $n\times n$ matrix whose elements are

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{if } v_i v_j \text{ otherwise} \end{cases}$$

⇒ Incidence matrix of Graph G

Let G be a matrix of order m & size n. The incidence matrix of G is an m×n matrix where

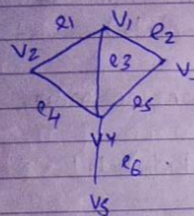
n×m

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

⇒ Eg. Adjacency matrix & Incidence matrix of Graph G given below

Eg-

G:



Soln - (i) Adjacency matrix

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(ii) Incidence matrix

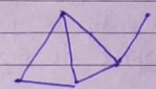
$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

⇒ Isomorphism

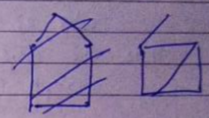
Isomorphic Graphs - Two graphs G_1 & G_2 are said to be isomorphic to each other if there is one to one correspondence btw their vertices & their edges such that there incident relationship is preserved. It is denoted by $G_1 \simeq G_2$.

Eg- Show that the Graphs given below are isomorphic to each other

G_1:



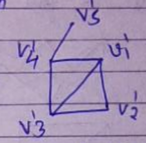
G_2:



Soln- Label any one of the Graph

G_1:

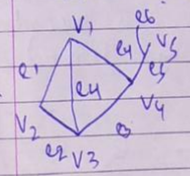
G_2:



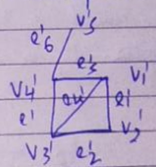
(i) The vertex v_1 in G_1 corresponds to v_1' in G_2.
(ii) The vertex v_2 in G_1 corresponds to v_2' in G_2
(iii) The vertex v_3 in G_1 corresponds to v_3' in G_2
(iv) The vertex v_4 in G_1 corresponds to v_4' in G_2
(v) The vertex v_5 in G_1 corresponds to v_5' in G_2

∴ There is one to one correspondence btw $V_1(G_1)$ & $V_2(G_2)$

(ii) G_1:



G_2:



The edge e_1 in G_1 corresponds to e_1' in G_2
'' '' e_2 '' '' '' to e_2' in G_2
'' '' e_3 '' '' '' e_3' '' ''
'' '' e_4 '' '' '' e_4' '' ''
'' '' e_5 '' '' '' e_5' '' ''

(13) Existential Generalization
$P(c) \Rightarrow \exists x\, P(x)$

(14) Resolution

$p \vee q\ ,\ \neg p \vee r \Rightarrow q \vee r$.

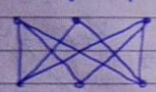
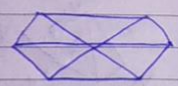
(15)
$\neg(\exists x\, P(x)) \equiv (\forall x)\, \neg P(x)$
$\neg(\forall x\, P(x)) \equiv (\exists x)\, \neg P(x)$

$\therefore$ There is one to one correspondence btw $E(G_1)$ & $E(G_2)$

(ii) The edge e_1 is incident on v_1 & v_2 in G_1
$\therefore$ edge e_1' is incident on v_1' & v_2' on G_2

Similarly
edge e_j is incident on v_j & v_3 on G_1
$\therefore$ edge e_j' is incident on v_j' & v_3' on G_2 and so on
Thus the edge incidence property is preserved.
$\therefore$ Graph G_1 is isomorphic to G_2.

Eg. Show that the foll graphs are isomorphic to each other.

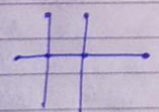
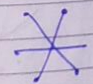
(i) G_1:  G_2: 

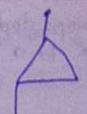
(ii) G_1:  G_2: 

$\Rightarrow$ Trees :-

A Graph G is called a Tree if G is connected & contains no cycles. It is denoted by $T(G,E)$

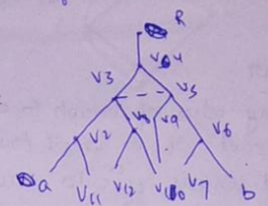
Eg.

T_1:  T_2:  T_3: 

T_4: 

T_4 is not a tree but it is a graph

Theorem: A Graph G is a tree if and only if every 2 vertices of G are connected by a unique path

T: 

Note- In a tree,
If there are n vertices edges are n-1
& if there are n edges (n+1) — vertices
(i) In cycle, more than one path exist

Let G be a tree. We have to prove that there is a unique path in G.
Suppose that there are two paths btw the vertices A & B
$P_1 = \{a, v_1, v_2, v_3, v_4, v_5, v_6 \ldots v_n, b\}$
$P_2 = \{a, v_2, v_3, v_5, v_6 \ldots v_n, b\}$.
$P_3 = \{a, v_2, v_3, v_5, v_4, v_3, v_5, v_6 \ldots b\}$
in which some vertices are repeated. ~~Thus~~ Then the graph has a cycle.
Therefore, G cannot be a Tree
Thus, a connected Graph G is a tree if every two vertices has a unique path.
Conversely, suppose that every 2 distinct vertices has a unique path then we have to prove that G is a Tree
Assume that G has two paths btw a pair of vertices then

In one of the paths, some vertices are repeated which causes a cycle. Then G cannot be a Tree
∴ our assumption is wrong

(13) Thus a tree has a unique path btw every pair of vertices.

(14) Theorem A tree with n vertices has (n-1) number of edges (Refer Text book — Done using mathematical Induction)

(15) => Euler Graph

Euler Circuit or Euler tour

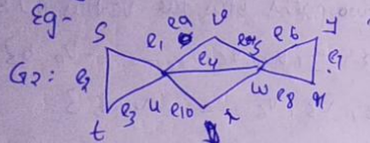
A closed walk through every edges of Graph G exactly once is called an Euler circuit or an euler tour.

Euler Graph — A graph that consists of an euler circuit is called an euler graph.

Euler Trial — An open walk covering all the edges exactly once is called an Euler trial.

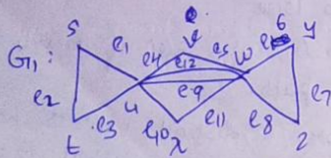
Eg – — Possible only if 2 vertices are odd degree, rest all even degree.

G_2:



An open walk {u, e_1, s, e_2, t, e_3, u, e_4, v, e_5, w, e_6, y, e_7, z, e_8, w, e_9, u, e_{10}, x, e_{11}, w} is called an euler trial.

G_1:



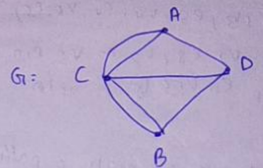
A closed walk ~~t, s~~ {u, e_1, s, e_2, t, e_3, u, e_4, v, e_5, w, e_6, y, e_7, z, e_8, w, e_9, u, e_{10}, x, e_{11}, w, e_{12}, u}

Graph G_1 is called Euler Circuit

-> Since the Graph G_2 contains euler circuit
∴ G_2 is an euler Graph

=> Solution of Konigsberg Bridge Problem

G:

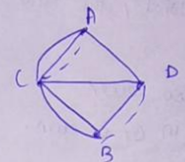


$\deg(A) = \deg(B) = \deg(D) = 3$

$\deg(C) = 5$

G is not a Euler Graph because it does not contain an euler circuit

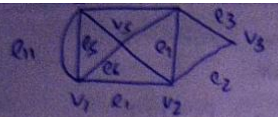
Therefore it is not possible covering all the edges of G exactly once



Note: — (I) An Euler Graph is always a connected Graph

(II) In an Euler Graph, the degree of every vertex is Even

=> Theorem — A non-trivial connected Graph G is an Euler Graph (Eulerian) if & only if every vertex of G has even degree



A closed walk { v_1 e_1 v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_1 e_6, v_7, e_7, v_7 e_8, v_8 e_9 v_5 e_{10} v_6 e_{11} e_{11} v_1 } is an euler circuit

If G is an euler circuit ~~that~~ there exists a path from any v_i to v_j

$\therefore$ G is a connected Graph

Let v_1 be the starting vertex of the euler circuit. In tracing an euler circuit everytime the walk meet the vertex v_i & exit v_i

This entry & exit contribute 2 to the degree of vertex

Since euler circuit is a closed trial, the trial terminates at the starting vertex v_1

Thus, the degree of v_1 is also even.

Hence degree of every vertex in G is even.

Conversely,

Let G be a connected Graph with every vertex of even degree. Construct an euler circuit starting from v_1 & going through all the edges of G such that no edge is traced more than once. Since every vertex is of even degree, we can enter & exit every vertex & stop tracing at v_1 where v_1 is also of even degree. Thus, G is an euler Graph

⇒ Hamiltonian Graph

Hamiltonian Path:-

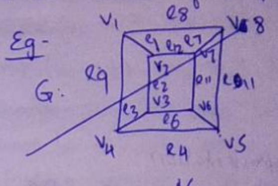
Let G be a connected Graph, a Hamiltonian Path is an open walk in G that contains all the vertices of G exactly once

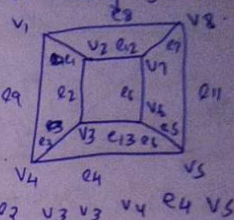
Hamiltonian Cycle

A Hamiltonian Cycle is a closed walk that contains all the vertices of Graph G exactly once

Eg-

G:





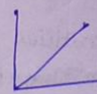
A closed walk { v_1 e_1 v_2 e_2 v_3 v_3 v_4 e_4 v_5 e_5 v_6 e_6 v_7 e_7 v_8 e_8 v_1 }.

is Hamiltonian cycle.

$\therefore$ G is a hamiltonian Graph.

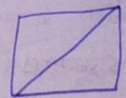
Eg - Determine Hamiltonian Path & hamiltonian Graph

(i) G_1:



(ii) G_2:

~~Sol~~ (iii)



⇒ Planar Graph

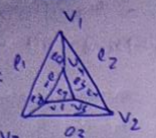
A Graph G is said to be a planar Graph. is G can be draw on a plane such that no two of its edges intersect.

⇒ Non Planar Graph

A Graph that cannot be drawn on a plane without edge intersection is called a non planar Graph.

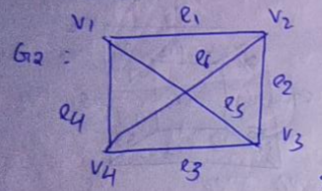
Eg-

G_1:



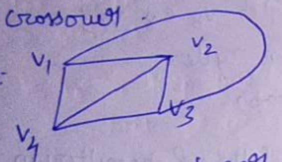
G_1 is a planar Graph because there is no Edge crossover.

G_2 :



G_2 is a non planar ~~Graph~~ representation because of Edge crossover.

G_3:



G_3 is a planar Graph.

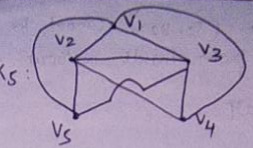
⇒ Kuratoskis Properties on Planarity:

Kuratoskis is a polishian mathematician gave a Unique property on a graph G to say that one of its geometric representation is non planar

Property 1 - A complete Graph of 5 vertices (K_5) is non planar

Proof:

K_5:



Let the five vertices of a complete Graph G be v_1, v_2, v_3, v_4 & v_5.

A complete Graph is a simple Graph where every vertex is joined with every other vertex ~~of~~ by an edge.

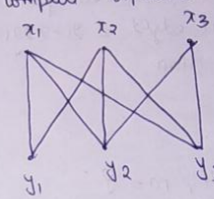
Construct a complete Graph of 5 vertices. First connect ~~v_1 & v_2~~ $(v_1\ v_2)$, $(v_1\ v_3)$, $(v_1\ v_4)$, $(v_1\ v_5)$. Then connect $(v_2\ v_3)$ $(v_2\ v_4)$ $(v_2\ v_5)$, then, (v_3, v_4) ~~An edge~~ The last edge connecting the vertices (v_3, v_5) cannot be drawn either from inside or from outside without edge cross over. Thus the graph G cannot be embedded in a plane.

Therefore, K_5 is non planar

Property 2 - A complete bipartite Graph K_{33} is non planar

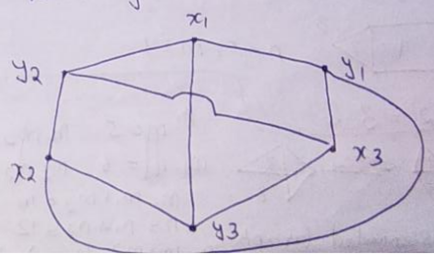
Proof:

K_{33} :



is a complete bipartite Graph.

Redrawing the Graph,



An edge btw the vertices (x_3, y_2) cannot be drawn either from outside or from inside without edge crossover

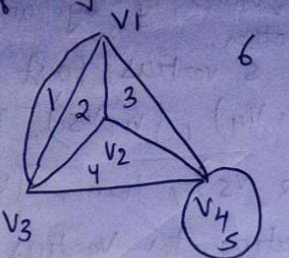
$\therefore K_{33}$ is non planar

Region

A planar representation of a Graph divides the plane into Regions. A Region is categorized by a set of edges forming its boundary.

Eg-

G:



G is a planar Graph dividing the plane into 6 regions

① Eulers Formula on Regions

A connected Graph G with n-vertices & m-edges has $m-n+2$ regions

⑪ Let G be a loop free connected planar Graph with n vertices & m edges & r-regions, where $n \geq 3$, $m > 2$ then

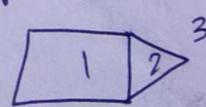
① $3r \leq 2m$

⑪ $m \leq 3m-6$.

Eg1- In the Graph, $n=4$, $m=8$.

No of regions, $r = m-n+2 = 8-4+2 = 4+2 = 6$.

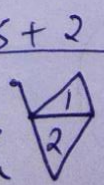
Eg2- G1:

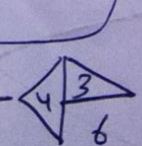


$n=5$, $m=6$

$r = 6-5+2 = 3$.

Eg3- G2:





Disconnected Graph

$n_1 = 5 \quad n_2 = 5$

$m_1 = 6 \quad m_2 = 6$.

$m = m_1 + m_2 = 10$

$n = n_1 + n_2 = 12$

$r = (m_1+m_2) - (n_1+n_2) + 2(2) = 6$

Boolean Algebra

A finite lattice is called a Boolean Algebra if it is isomorphic with B_n where n is a positive integer

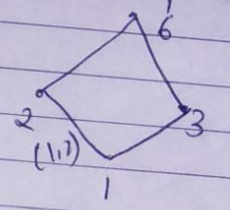
Eg! - Show that lattice (I) D_6 (II) D_{20} (III) D_{30} is a Boolean Algebra

Soln- (I) $D_6 = \{1, 2, 3, 6\}$

$6 \longrightarrow$ Dividend

Hasse Diagram of D_6 is

D_6 :



$|D_6| = 4 = 2^2$

$B_2 = \{00, 10, 01, 11\}$.

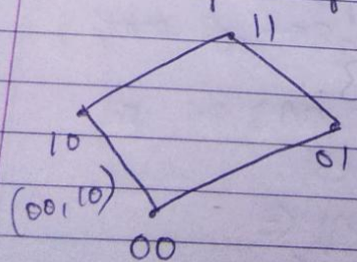
~~Bf~~

$n = 2$

$\therefore D_6$ is a boolean Algebra

Hasse Diagram of B_2

B_2:



$f(1) = 00$, $f(2) = 10$, $f(3) = 01$, $f(6) = 11$.

f from D_6 to B_2 is one to one mapping.

for $(1,2) \in D_6$

$1 < 2$

$\Rightarrow f(1) < f(2)$

$\Rightarrow 00 < 10 \in B_2$

~~for any $a, b \in B_6$~~

~~$f(a) < f(b)$~~

$01 < 11$

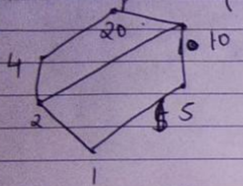
$10 < 11$

$\therefore f : D_6 \rightarrow B_2$ is Isomorphic

Thus, $f : D_6 \rightarrow B_2$ is Boolean Algebra.

(ii) $D_{20} = \{1, 2, 4, 5, 10, 20\}$.

Hasse Diagram of D_{20}



$|D_{20}| = 6 \neq 2^n$

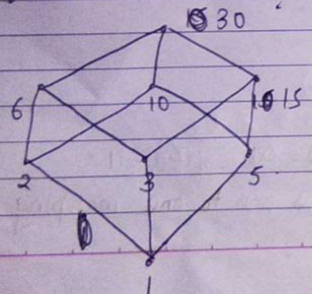
not possible to find value of n

$\therefore$ Not possible to find B_n.

$\therefore B_{20}$ is not a Boolean Algebra.

(iii) $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

D_{30}

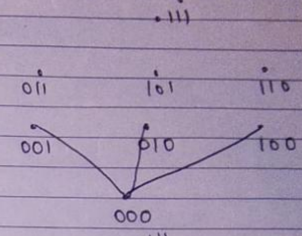


$|D_{30}| = 8 = 2^3$

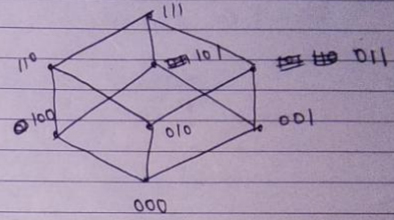
$n = 3$

$B_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

Hasse Diagram of B_3 :-



$B_3 =$



$f(1) = 000$, $f(2) = 100$, $f(3) = 010$, $f(5) = 001$,

$f(6) = 110$, $f(10) = 101$, $f(15) = 011$

$f(30) = 111$.

f from $D_{30} \rightarrow B_3$ is one to one mapping.

for $1, 2 \in D_{30}$

$1 < 2$

$\Rightarrow f(1) < f(2)$

$\Rightarrow 000 < 010 \in B_3$

$f : D_{30} \rightarrow B_3$ is Isomorphic

Thus $D_{30} \rightarrow B_3$ is Boolean Algebra