Ex: The 10 editors have decided on the 7 committees

$C_1 = \{1, 2, 3\}$

$C_2 = \{1, 3, 4, 5\}$

$C_3 = \{2, 5, 6, 7\}$

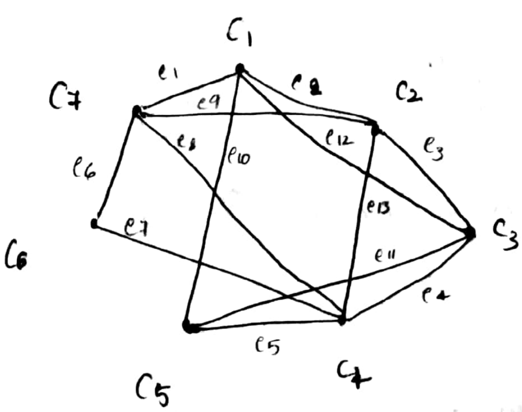
$C_4 = \{4, 7, 8, 9\}$

$C_5 = \{2, 6, 7\}$

$C_6 = \{8, 9, 10\}$

$C_7 = \{1, 3, 9, 10\}$

They have set aside 3 time periods for the seven committees to meet on those fridays when all 10 editors are present. Some pairs of committee cannot meet during the same period because one or two of the editors are on both committees. Model the situation using graphs. Write edge set and vertex set.



1 : C_1 C_2 C_7

2 : C_1 C_3 C_5

3 : C_1 C_2 C_7

4 : C_2 C_4

5 : C_2 C_3

6 : C_3 C_5

7 : C_3 C_4 C_5

8 : C_4 C_6

9 : C_4 C_6 C_7

10 : C_6, C_7

$V(G) = \{C_1\ C_2\ C_3\ C_4\ C_5\ C_6\ C_7\}$ $(C_6\ C_5\ C_2)$ meeting can be connected at once

$E(G) = \{e_1\ e_2\ \dots\ e_{13}\}$

* **Trivial graph**

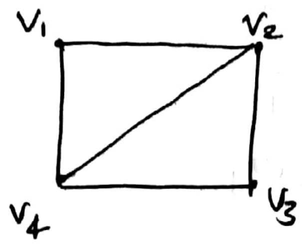
A graph with exactly 1 vertex is called a trivial graph.

G_1 :

* **Non trivial graph**

A graph with more than 1 vertex is called non trivial graph

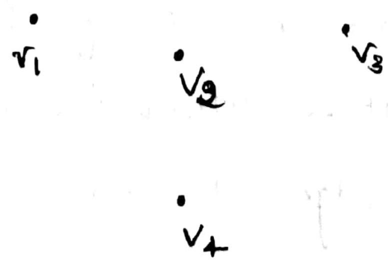
Ex: G_2 :



* **NULL graph**

A graph which does not contain any edges is called a NULL graph

Ex: G_3 :



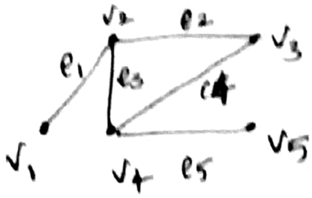
NOTE: * In a NULL graph each vertex is isolated vertex

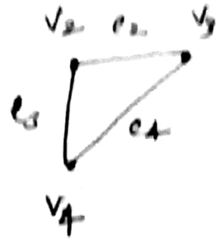
* Degree of each vertex is 0 in a NULL graph.

* **Sub-graphs**

Let $G(V,E)$ be a graph. Then $G_1(V_1,E_1)$ is called a subgraph of G if $V_1(G_1)$ is a subset of $V(G)$ and $E_1(G_1) \subseteq E(G)$ where each edge in

E_1 is incident with vertices in V_1

Ex: G : 

G_1 : 

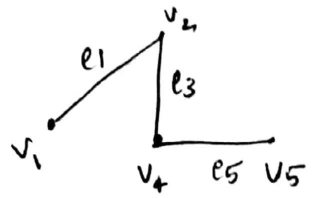
$V_1(G_1) = \{v_2, v_3, v_4\}$

$\therefore V_1(G_1) \subseteq V(G)$

$E_1(G_1) = \{e_2\ e_3\ e_4\}$

$\therefore E_1(G_1) \subseteq E(G)$

$\therefore G_1$ is a subgraph of G ie $G_1 \subseteq G$

G_2 : 

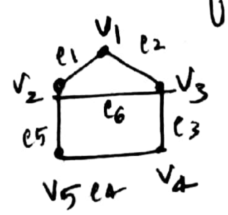
$V_2(G_2) \subseteq V(G)$

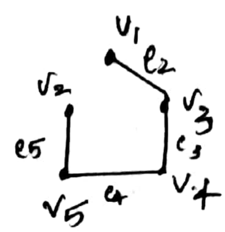
$E_2(G_2) \subseteq E(G)$

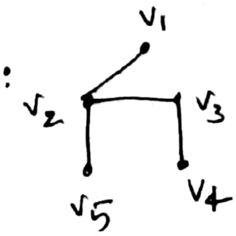
$\therefore G_2 \subseteq G$

Spanning subgraphs

A subgraph G_1 of graph G is called a spanning subgraph of G if G_1 contains all the vertices of G.

G : 

G_1 : 

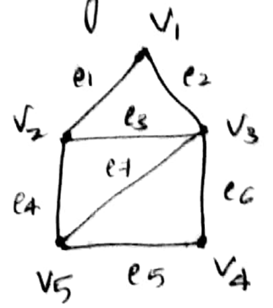
G_2 : 

G_1 & G_2 are called spanning subgraphs of the graph G

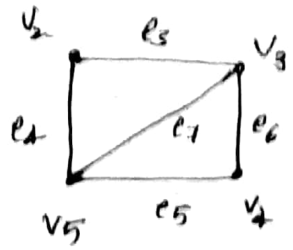
* Induced subgraph

Let $G(V,E)$ be a graph. The subgraph F of the graph G is called induced ~~by F~~ subgraph ~~by F~~ of G where the subgraph contains all the edges incidenting on the vertices of G

G:

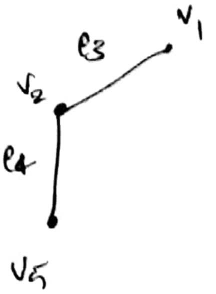


F:



F is an induced subgraph of G

F_1 :



F_1 is an induced subgraph of G

Ex:

1) Suppose we have 2 coins, 1 silver and 1 gold placed on 2 of the 4 squares of a 2×2 checkerboard. An arrangement C_i can be transformed into C_j such that if C_j can be obtained from C_i by performing exactly one of the following 2 steps

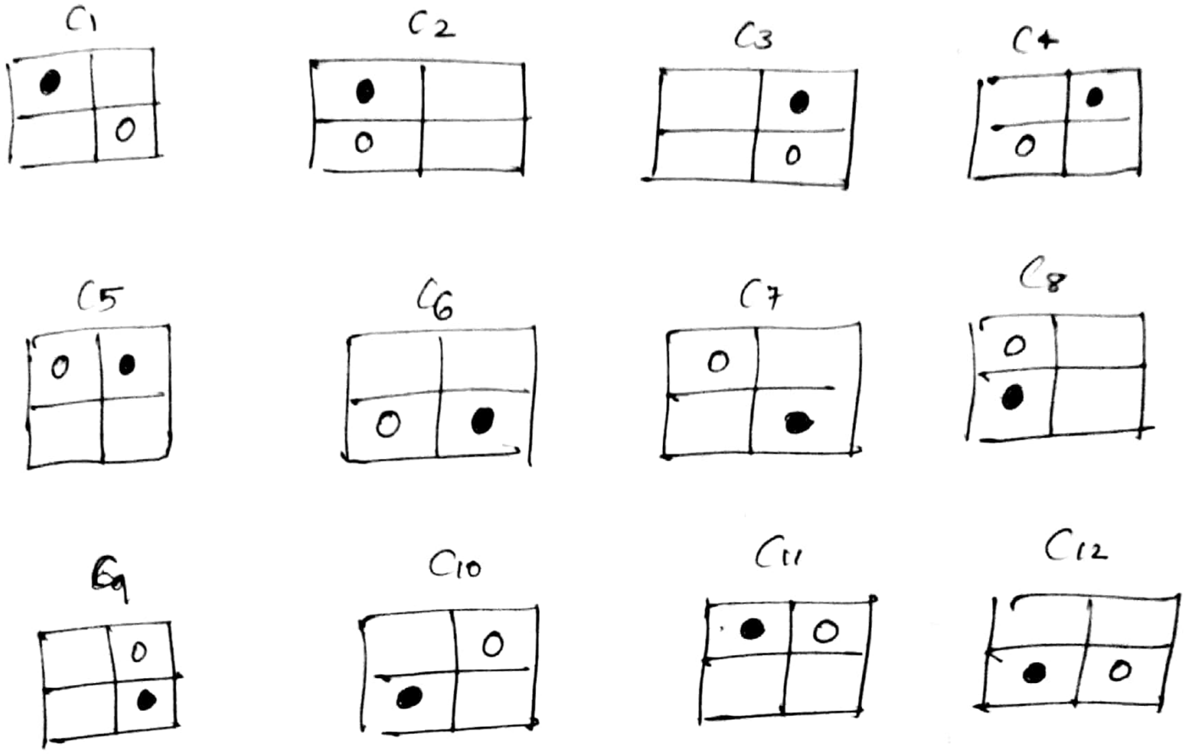
(1) moving one of the coins in C_i horizontally or vertically to an unoccupied square.

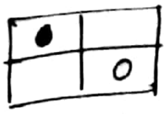
(2) Interchanging the 2 coins in C_i.

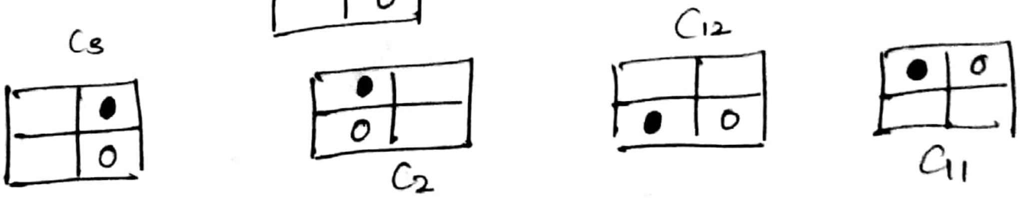
Represent the transforms in a graph.

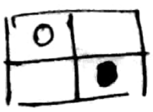
Soln [Arranging 2 coins in 4 squares.

$^4P_2 = \frac{4!}{2!} = 12$ arrangements / possibilities.]



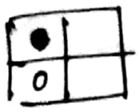
Consider C_1 

(1) 

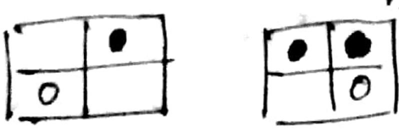
(2) 

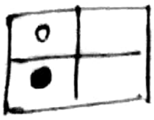
C_7

$\therefore C_1 \rightarrow C_2\ C_3\ C_7\ C_{11}\ C_{12}$

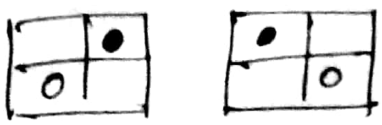
Consider C_2 

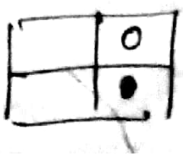
Consider C_3 

(1) 

(2) 

C_8

(1) 



Illy do all

$C_2 \rightarrow C_1 \quad C_4 \quad C_8$

$C_3 \rightarrow C_1 \quad C_4 \quad C_9$

$C_4 \rightarrow C_2 \quad C_3 \quad C_5 \quad C_6 \quad C_{10}$

$C_5 \rightarrow C_4 \quad C_7 \quad C_{11}$

$C_6 \rightarrow C_4 \quad C_7 \quad C_{12}$

$C_7 \rightarrow C_1 \quad C_5 \quad C_6 \quad C_8 \quad C_9$

$C_8 \rightarrow C_2 \quad C_7 \quad C_{10}$

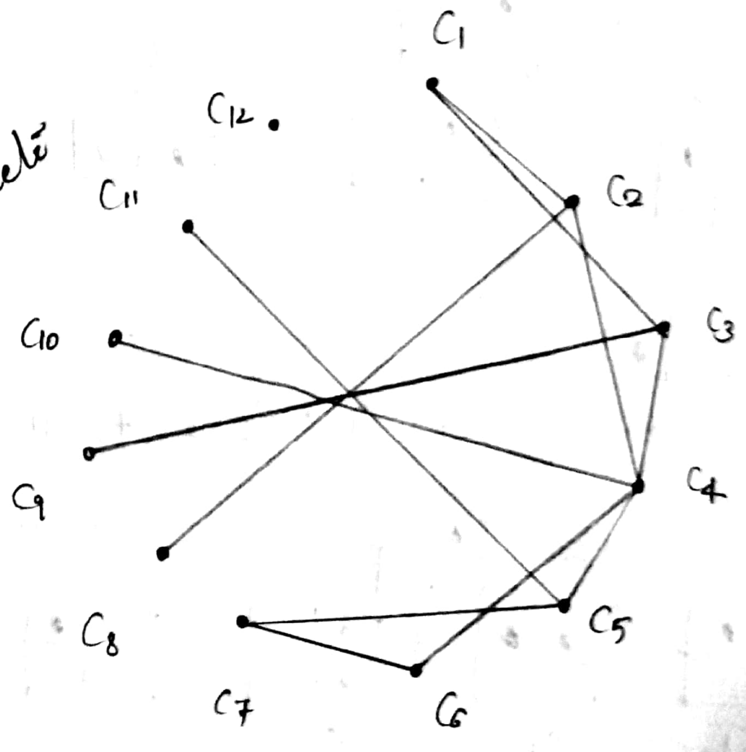
$C_9 \rightarrow C_3 \quad C_7 \quad C_{10}$

$C_{10} \rightarrow C_4 \quad C_8 \quad C_9 \quad C_{11} \quad C_{12}$

$C_{11} \rightarrow C_1 \quad C_5 \quad C_{10}$

$C_{12} \rightarrow C_1 \quad C_6 \quad C_{10}$

incomplete



Word graph :

Let W_1 be a word. The word W_1 can be transformed into W_2 by following the 2 steps

(1) Interchange 2 letters of W_1

(2) Replacing a letter in W_1 by another letter

The word W_1 can be adjacent to W_2.

A graph obtained by following the above 2 steps is called a word graph

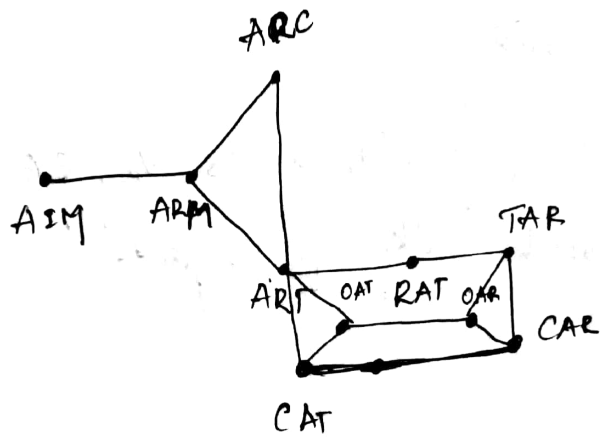
Ex:

1) Given a collectⁿ of 3-letter ~~is~~ English word say ACT, AIM, ARC, ARM, ART, CAR, CAT, OAR, OAT, RAT, TAR.

Draw the word graph

Solⁿ ACT → ART, CAT

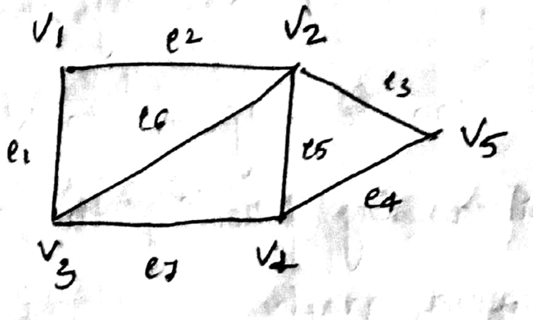
ART → ACT, RAT, ARM, ARC



Walk: Let G be a graph. A walk in G is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices.

Ex:

1) G :



A walk = $\{ v_1, e_2, v_2, e_6, v_3, e_7, v_4 \}$

NOTE

(1) Length of the walk is no. of edges in the walk.

Ex: A walk $\{ v_3, e_1, v_1, e_2, v_2, e_6, v_3, e_7, v_4, e_4, v_5 \}$ is of length 5.

Closed walk

A walk is closed if it begin and end at the same vertex.

Ex: A walk $\{ v_4, e_4, v_5, e_3, v_2, e_6, v_3, e_7, v_4 \}$ is a closed walk of length 4

Trivial walk

A walk of length zero is called trivial walk.

Trial:

A trial is an open walk in which no edge is repeated

A walk $\{v_1, e_1, v_3, e_6, v_2, e_5, v_4, e_4, v_5\}$ is a trial of length 4

A walk $\{v_3, \underline{e_6}, v_2, e_5, v_4, e_7, v_3, \underline{e_6}, v_2, e_2, v_1\}$ is not a trial

NOTE:

* A trial is a walk but a walk need not be a trial

Path:

A path is an open walk in which not vertex is repeated

A walk $\{v_1, e_1, v_3, e_7, v_4, e_5, v_2, e_3, v_5\}$ is a path of length 4

Circuit:

A closed trial is called a circuit.

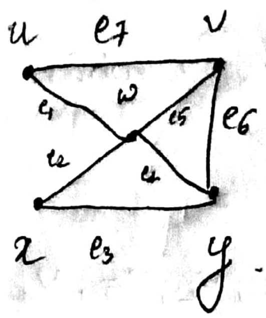
Cycle:

A closed path is called a cycle

(1) Theorem:

If a graph G contains a $u-v$ walk of length l then G contains a $u-v$ path of length l

Proof :



Let P_1 be a smallest walk from u to v

covering all the vertices then

$P = \{u, e_1, \omega, e_2\ x, e_3, y, e_6, v\}$ is a

smallest walk covering all the vertices of G

of length 4

Let $P_1 : \{u, u_1, u_2, u_3, \dots u_l\}$ be the

smallest walk covering all vertices of G

and having length say k

Suppose that there is a path of length $l > k$

Then $P_2 : \{u, e_1, \omega, e_2, x, e_3, y, e_4, \omega, e_5, v\}$

is a walk of length 5

i.e $P_2 = \{u, u_1, u_2, \dots u_l\}$ be a walk

of length k then definitely some

vertex are repeated.

$\therefore$ walk P_2 cannot be a path.

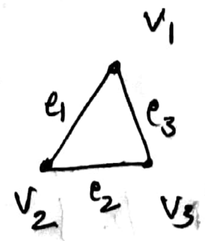
Thus P_1 is a path of length atmost k

Complete graph :

A simple graph in which there exists an edge between every pair of vertices is called a complete graph.

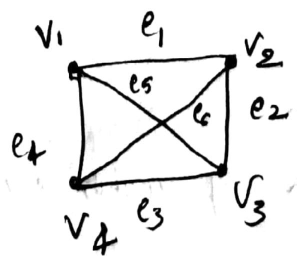
A complete graph of n vertices is denoted by K_n.

Ex: G_1:



is a complete graph of 3 vertices (K_3)

G_2 :



is a complete graph of 4 vertices (K_4).

Bipartite graph: Let G be a graph

If the vertex set $V(G)$ can be partitioned into 2 subsets V_1 and V_2 such that each edge of G has 1 end in V_1 and other end in V_2 then graph G is called a Bipartite graph.

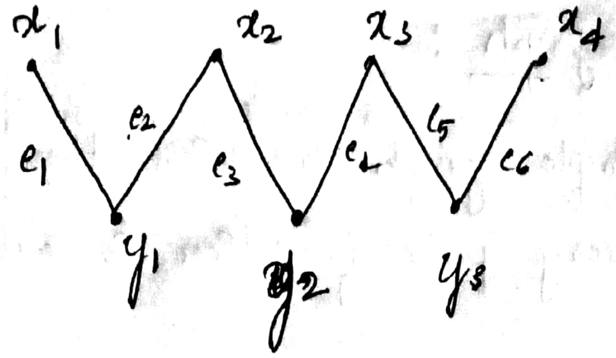
In a Bipartite graph the vertex sets V_1 and V_2 satisfies the following properties

(i) $V_1(G) \cup V_2(G) = V(G)$

(ii) $V_1(G) \cap V_2(G) = \phi$

Ex:

G:



$V_1(G) = \{x_1, x_2, x_3, x_4\}$

$V_2(G) = \{y_1, y_2, y_3\}$

$V_1(G) \cup V_2(G) = \{x_1\ x_2\ x_3\ x_4\ y_1, y_2\} = V(G)$

<u>Complete bipartite graph</u>

A bipartite graph G with each vertex of V_1 is joined with every vertex of V_2 then G is called a complete bipartite graph

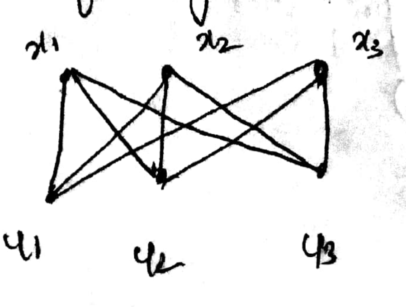
If $|V_1(G)| = m$

$|V_2(G)| = n$ then a complete bipartite has

(1) No. of vertices $= m+n$

(2) No. of edges $= m \times n$

Ex: G



$V_1(G) = \{x_1, x_2, x_3\}$

$V_2(G) = \{y_1, y_2, y_3\}$

$|V_1(G)| = 3$ $|V_2(G)| = 3$

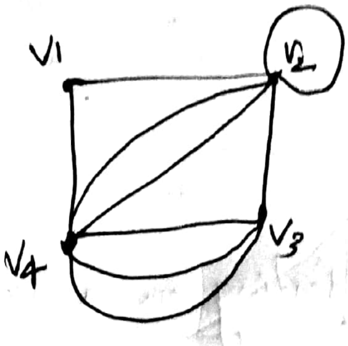
* No. of vertices = $3+3 = 6$

* No. of edges : $3 \times 3 = 9$

Multigraph

A graph G with self loop and parallel edges is called a multigraph

Ex:



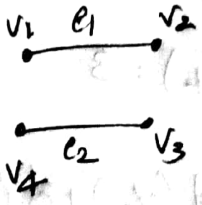
Regular graph

A graph in which all vertices are of equal degree is called a regular graph.

G_1 :

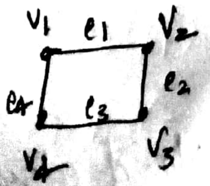
0-regular graph

G_2 :

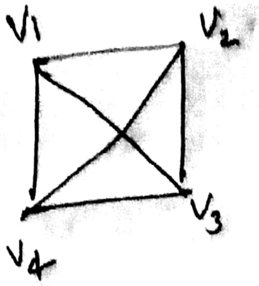


1-regular graph

G_3 :



2-regular graph



3-regular graph

r-regular graph

If the degree of all the vertices of graph G is r then the graph is called r-regular graph.

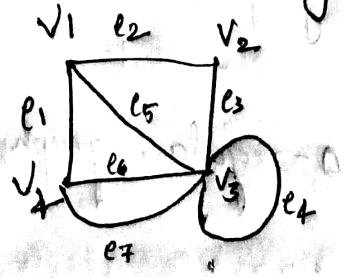
Degree of a vertex

The no. of edges incidenting on a vertex v_i with self loop counted twice is called degree of v_i

It is denoted by $deg(v_i)$ or $d(v_i)$

Ex:

G:



$d(v_1) = 3$

$d(v_2) = 2$

$d(v_3) = 6$

$d(v_4) = 3$

$$\sum_{i=1}^{4} d(v_i) = d(v_1) + d(v_2) + d(v_3) + d(v_4)$$

$$= 3+2+6+3$$

$$= 14 = 2 \times 7 = 2 \times \text{No. of edges}$$

Result 1

(1) If G is a graph of size m then $\Sigma d(v_i) = 2m$

Proof: Each edge contribute 2 to the sum of degrees of vertices.

Since the graph has m edges, therefore contributes exactly $2m$ to the sum of degrees of vertices.

$\therefore \Sigma d(v_i) = 2m$ $\quad$ m: no. of edges.

(Handshake property)

Result 2

(2) Every graph has even no. of odd degree vertices

Proof: ~~Each~~ Let G have v_j no. of even degree vertices and v_k no. of odd degree vertices

$\Sigma d(v_i) = \Sigma d(v_j) + \Sigma d(v_k)$

even = even + $\Sigma d(v_k)$

$\Sigma d(v_k)$ = even - even = even

(Sum of even no. of odd values/no.s is even)

$\therefore$ No. of odd degree vertices is even.