UNIT - 1

Graphs & Graph Models

* A graph 'G' is an ordered pair of two sets 'V' & 'E'
$G=(V,E)$, where 'V' is a finite non empty set of vertices & 'E' is a set of ordered pair of elements taken from the set 'V'. (E → set of edges)

* $V(G)$ & $E(G)$ are the vertex set & Edge set of graph 'G'.

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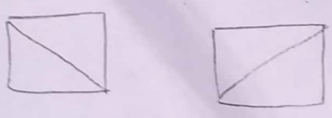
* Vertices are sometimes called points or nodes. & Edges are sometimes called lines.

* Equal graph:- Two graphs 'G' & 'H' are said to be equal if $V(G) = V(H)$ & $E(G) = E(H)$. in which case we write $G = H$.

Ex:-



'G' 'H'

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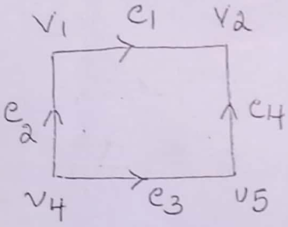
* Directed graph:- Directed graph is a graph in which all the edges must have directions.

Ex:-



In the above graph for the edge 'e_1', 'v_1' is the initial vertex & 'v_2' is the terminal vertex.

Vertex Set $V = \{v_1, v_2, v_4, v_5\}$

Edge Set $E = \{v_1v_2, v_4v_1, v_4v_5, v_5v_2\}$

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① Let $S = \{2, 3, 5, 8, 13, 21\}$ of six specific Fibonacci numbers. Draw the graph 'G' whose vertex set is 'S' & such that $ij \in E(G)$ for $i, j \in S$ if $i+j \in S$ or $|i-j| \in S$.

Soln There are some pairs of distinct integers belonging to 'S' whose sum (or) difference also belongs to 'S'. namely $\{2,3\}, \{2,5\}, \{3,5\}, \{3,8\}, \{5,8\}, \{5,13\}, \{8,13\}, \{8,21\}$ & $\{13,21\}$.

so $V(G) = \{2,3,5,8,13,21\}$

& $E(G) = \{\{2,3\}, \{2,5\}, \{3,5\}, \{3,8\}, \{5,8\}, \{5,13\}, \{8,13\}, \{8,21\}, \{13,21\}\}$.

G :

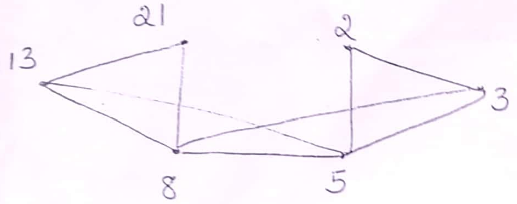


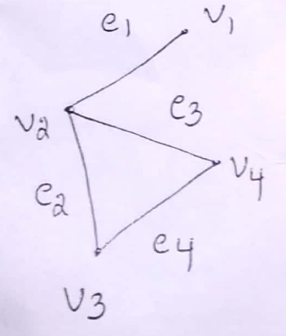
Fig. ①

Note :- If uv is an edge of 'G', then 'u' & 'v' are adjacent in 'G'.

* Order & Size of the Graph :- The order of the graph 'G' is the number of vertices present in the graph 'G'.

& Size of the graph 'G' is the number of edges present in the graph 'G'.

Ex:-



$O(G) = 4 = n$ (number of vertices)

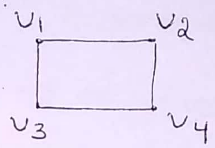
$S(G) = 4 = m$ (number of edges)

* **Trivial graph** :- A graph 'G' is said to be trivial graph if the graph consists of exactly one vertex.

Ex :-

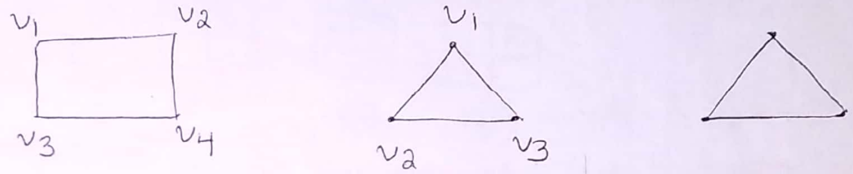
* **Non-trivial graph** :- A graph 'G' is said to be nontrivial graph if the order is atleast 2.

Ex :-



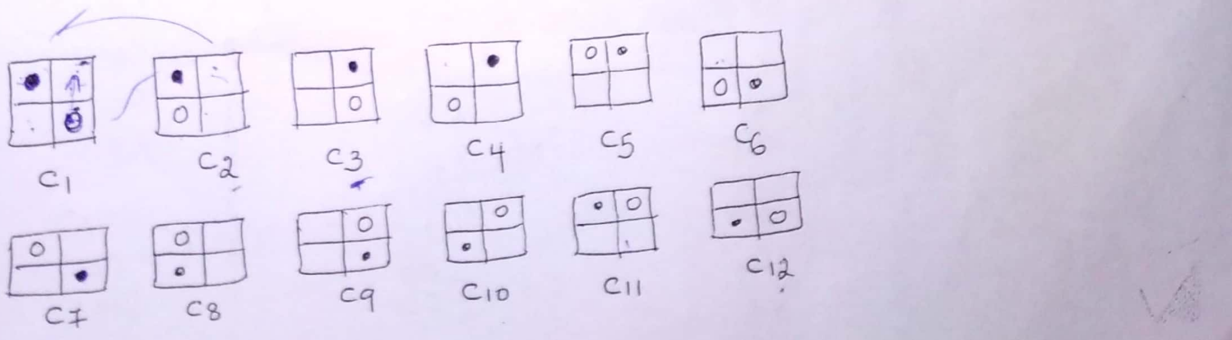
* **Labelled & unlabelled graph** :- A graph is said to be labelled graph if we assign names to vertices otherwise it is unlabelled graph

Ex :-



Labelled graph   Labelled graph   unlabelled graph

(2) Consider the twelve configurations $C_1, C_2, \dots C_{12}$ in below Fig. For every two configurations C_i & C_j, where $1 \le i, j \le 12$, $i \ne j$, it may be possible to obtain C_j from C_i by (first shifting one of the coins in C_i horizontally or vertically & then interchanging the two coins. Model this by a graph 'F' such that $V(F) = \{C_1, C_2, \dots C_{12}\}$ & C_iC_j is an edge of 'F' if C_i & C_j can be transformed in to each other by this 2-step process.

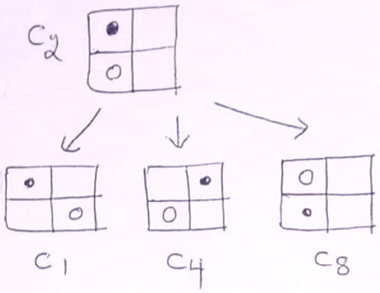


Soln Assume that we have two coins, one silver & one gold, placed on two of the four squares of a 2×2 checker board. There are twelve such configurations as shown in fig. above where shaded coin is the gold coin.

'C_j' can be obtained from 'C_i' by performing 2 steps.

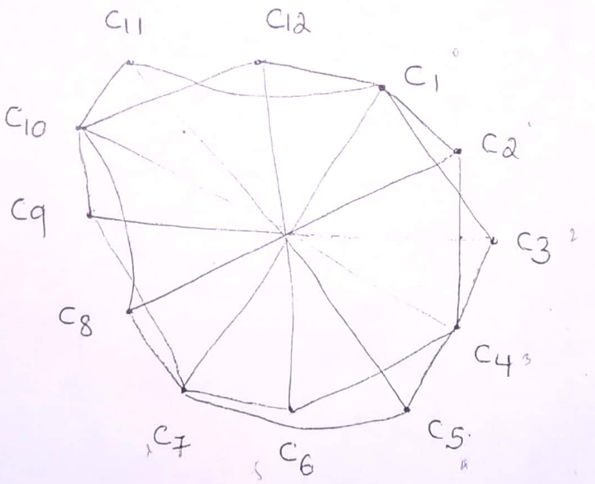
① Moving one of the coins in 'C_i' horizontally or vertically to an unoccupied square.

② Interchanging the two coins in C_i.



Transformations of the configuration C_2.

'F' becomes



$C_1 \left\langle \begin{matrix} C_7 \\ C_{12} \\ C_2 \\ C_3 \end{matrix} \right.$ $C_2 \left\langle \begin{matrix} C_1 \\ C_4 \\ C_8 \end{matrix} \right.$ $C_7 \left\langle \begin{matrix} C_1 \\ C_6 \\ C_8 \\ C_9 \\ C_5 \end{matrix} \right.$ $C_{10} \left\langle \begin{matrix} C_9 \\ C_{11} \\ C_4 \\ C_8 \end{matrix} \right.$

$C_3 \left\langle \begin{matrix} C_1 \\ C_4 \\ C_9 \end{matrix} \right.$ $C_5 \left\langle \begin{matrix} C_4 \\ C_7 \\ C_{11} \end{matrix} \right.$ $C_8 \left\langle \begin{matrix} C_7 \\ C_2 \\ C_{10} \end{matrix} \right.$ $C_{11} \left\langle \begin{matrix} C_7 \\ C_{10} \\ C_5 \end{matrix} \right.$

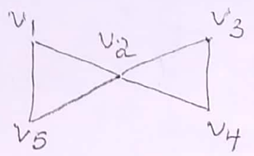
$C_4 \left\langle \begin{matrix} C_{10} \\ C_3 \\ C_5 \\ C_2 \\ C_1 \end{matrix} \right.$ $C_6 \left\langle \begin{matrix} C_{12} \\ C_4 \\ C_7 \end{matrix} \right.$ $C_9 \left\langle \begin{matrix} C_7 \\ C_{10} \end{matrix} \right.$ $C_{12} \left\langle \begin{matrix} C_1 \\ C_6 \\ C_{10} \end{matrix} \right.$

Connected graph :- A graph is said to be connected graph if their exists a path between every pair of vertices.

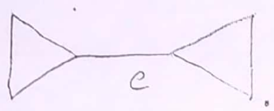
A graph which is not connected graph is called dis connected graph.

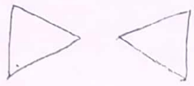
Ex:-

Removal of any edge, graph will disconnect.



Removal of any edge, graph is connected.





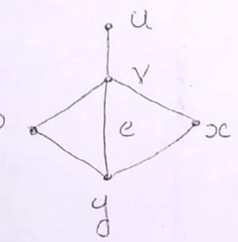
Removal of an edge 'e' will graph is disconnect.

Subgraph :- A graph 'H' is called a subgraph of a graph 'G', written $H \subseteq G$, if $V(H) \subseteq V(G)$ & $E(H) \subseteq E(G)$.

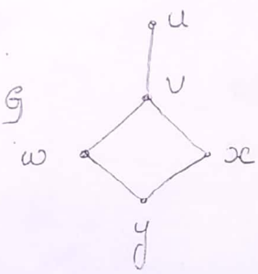
We also say that 'G' contains 'H' as a subgraph. If $H \subseteq G$ & either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then 'H' is a proper subgraph of 'G'.

Ex:-

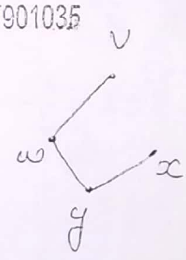
G':



$G' - e = G$



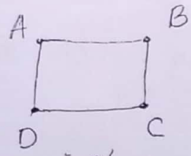
H =



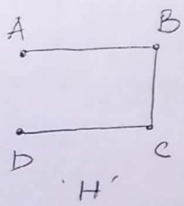
'H' is a proper subgraph of 'G'.

Spanning Subgraph :- ~~Let~~ 'G' be a graph & 'H' is a subgraph of graph 'G', if the graph 'G' then 'H' is called spanning subgraph of graph 'G', vertex set 'H' contains all vertices of 'G'

Ex:-

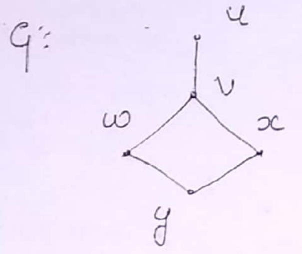


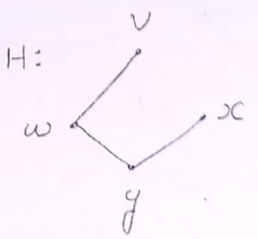
'G'



'H'

Induced Subgraph :- A subgraph 'F' of a graph 'G' is called an induced subgraph of 'G' if whenever u & v are vertices of 'F' & uv is an edge of 'G', then uv is an edge of 'F' as well.





Not induced subgraph

Induced subgraph

Walk :- A walk is the finite alternating sequences of vertices & edges beginning and ending with vertices only.

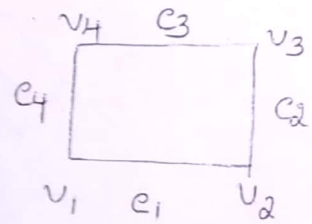
Length of walk is the number of edges present in the walk.

An open walk is a walk in which beginning & terminal vertices are different.

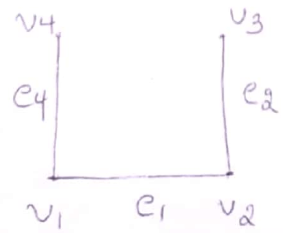
A closed walk is a walk in which beginning & terminal vertices coincides

walk

- Open → Trail → Path
- Closed → Circuit → Cycle



(a)



(b)

$v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_1$ — Closed walk

$v_4 e_4 v_1\ e_1\ v_2\ e_2\ v_3$ — Open walk

Trail :- It is a open walk in which no edges are repeat.

Circuit :- It is a closed walk in which no edges are repeat.

path :- path is an open walk in which no vertices & no edges are repeated.

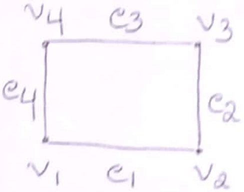
cycle :- cycle is an closed walk in which no vertices & no edges are repeated.

Euler Circuit :- A circuit is said to be Euler circuit if it contains all edges of a graph.

Euler Trail :- A trail is said to be Euler trail if it contains all edges of a graph.

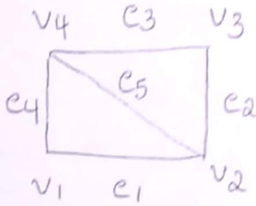
Euler Graph :- A connected graph that contains Euler circuit is called Euler graph.

In a Euler graph every vertices is of even degree



$v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_4\, v_1$

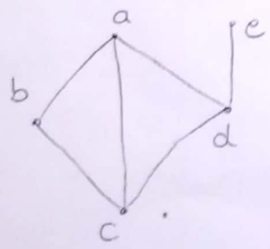
Euler graph



$v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_4\, v_1$ (Not a Euler graph)

does not involve all edges.

(3) Find any four spanning subgraph of the graph given below. and also find any three induced subgraphs for the graph below.



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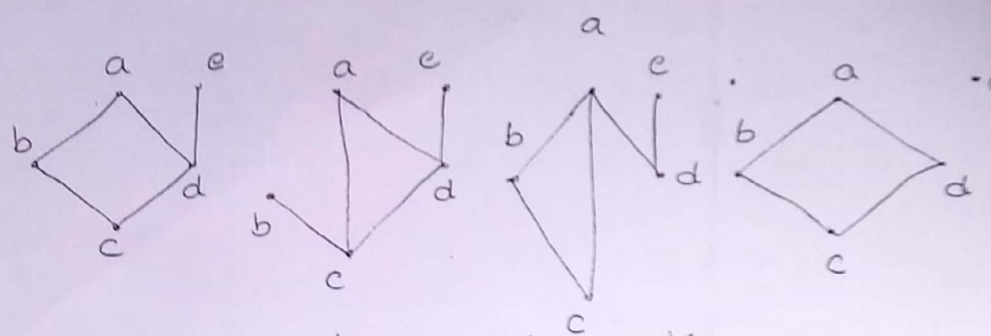
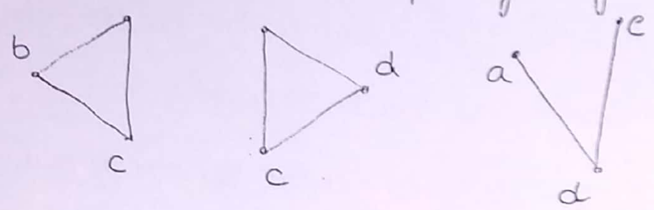
Soln



4 Spanning subgraphs.

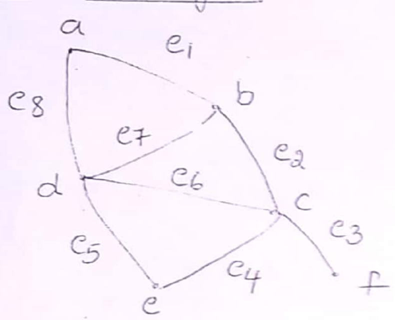


3 Induced subgraphs.

June/July 2008

(4) In the undirected graph a] Find circuit of length 6.

b] A cycle of maximum length.



Soln a] A circuit is a closed walk in which no edges are repeated.

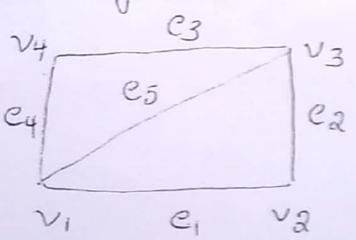
$a\ e_1\ b\ e_7\ d\ e_6\ c\ e_4\ e\ e_5\ d\ e_8\ a$

b] A cycle is a closed walk in which no vertices & no edges are repeated.

$a\ e_1\ b\ e_2\ c\ e_4\ e\ e_5\ d\ e_8\ a$

Hamilton cycle :- Let 'G' be a graph then a cycle in a graph 'G' is said to be Hamilton cycle if it includes all vertices of a graph 'G'

Ex:-



$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$

*** Dec 2010 (5)

<u>Write a note on Konigsberg - bridge problem</u>

<u>Soln</u> In the 18^{th} century city named Konigsberg in East Prussia [Europe], there flowed a river named pregel river which divided the city into 4 parts. Two of these parts were the banks of the river & two were islands. These parts were connected with each other through seven bridges.

The citizens of the city seemed to have posed the following problem, By starting at any of the 4 Land areas, can we return to that area after crossing each of the 7 bridges exactly once?

This problem known as 'Konigsberg bridge problem'.

In 1736, Euler analyzed the problem & gave the solution.

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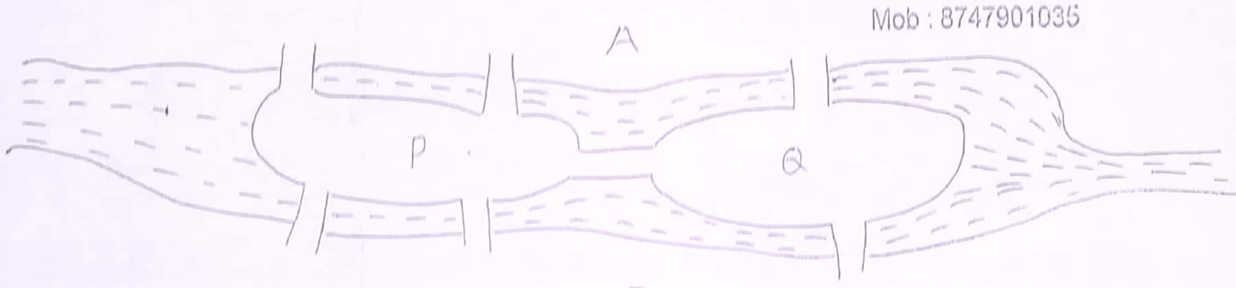
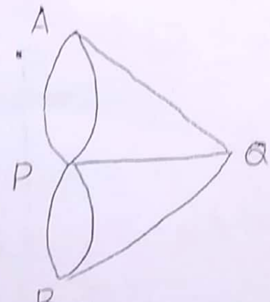


A, B, P & Q are the Land areas. where A & B are the banks of the river, P & Q are the Islands.

Construct a graph by treating 4 Land areas as 4 vertices & seven bridges as seven edges.



$$\left.\begin{array}{l} deg(A) = 3 \\ deg(B) = 3 \\ deg(Q) = 3 \\ deg(P) = 5 \end{array}\right\} \text{Which are not even}$$

Hence Euler graph (Euler circuit) is not possible as the degrees are of odd degree. Hence Euler walk not possible.

June/July 2015

5 For the following graph, determine,

i) A walk from 'b' to 'd' that is not a trail.

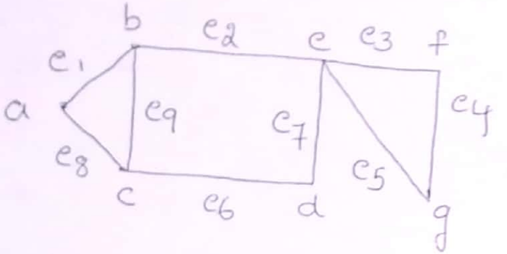
ii) A b-d trail that is not a path.

iii) A path from b-d.

iv) A closed walk from b to b that is not a circuit.

v) A circuit from b to b that is not a cycle.

vi) A cycle from b to b.



Soln

i. Trail is an open walk in which no edges are repeated.

$b\, e_1\, a\, e_8\, c\, e_9\, b\, e_1\, a\, e_8\, c\, e_6\, d$ (Not a trail)

ii. path is an open walk in which no vertices & no edges are repeated

$b\, e_2\, e\, e_3\, f\, e_4\, g\, e_5\, e\, e_7\, d$ (Trail not a path)

iii. $b\, e_9\, c\, e_6\, d$

iv. circuit is a closed walk in which no edges are repeated

$b\, e_9\, c\, e_6\, d\, e_7\, e\, e_2\, b\, e_1\, a\, e_8\, c\, e_9\, b$ (Not a circuit)

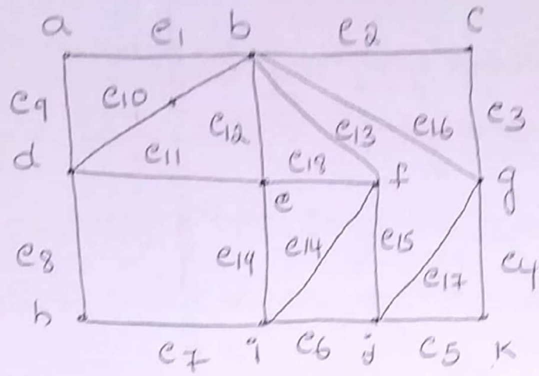
v. cycle is a closed walk in which no vertices & no edges are repeated.

$b\, e_9\, c\, e_6\, d\, e_7\, e\, e_5\, g\, e_4\, f\, e_3\, e\, e_2\, b$ (Not a cycle)

vi. $b\, e_9\, c\, e_6\, d\, e_7\, e\, e_2\, b$

May | June 2010

(6) Find an Euler circuit for the graph shown in Fig.



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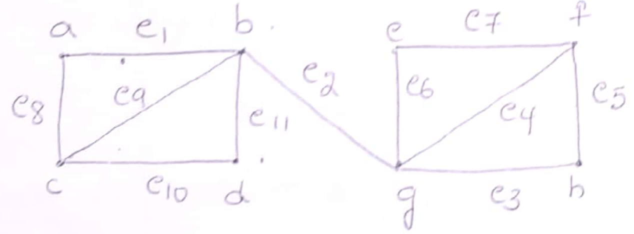
Soln

Euler circuit is a circuit if it contains all edges of a graph.

d e_9 a e_1 b e_2 c e_3 g e_4 k e_5 j e_6 i e_7 h e_8 d e_{10} b e_{16} g e_{17} j e_{15} f e_{18} e e_{12} b e_{13} f e_{14} i e_{19} e e_{11} d

May | June 2010

(7) Let $G = (V, E)$ be the undirected graph in the Fig. How many paths are there in 'G' from 'a' to 'h'? How many of these paths have a length 5?



Soln

path is a open walk in which no vertices & no edges are repeated.

(1) a e_1 b e_2 g e_3 h

(2) a e_1 b e_2 g e_4 f e_5 h

~~(3) a e_1 b e_2 g e_6 e e_7 f e_5 h~~

(4) a e_8 c e_9 b e_2 g e_3 h

~~(5) a e_8 c e_9 b e_2 g e_4 f e_5 h~~

(6) a e_8 c e_9 b e_2 g e_6 e e_7 f e_5 h

~~(7) a e_8 c e_{10} d e_{11} b e_2 g e_3 h~~

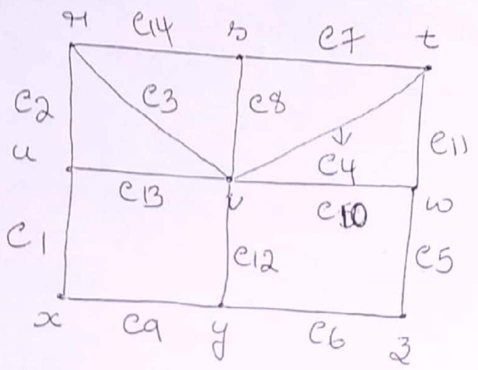
(8) a e_8 c e_{10} d e_{11} b e_2 g e_4 f e_5 h

(9) a e_8 c e_{10} d e_{11} b e_2 g e_6 e e_7 f e_5 h

Totally there are 9 paths are there in 'G' from 'a' to 'h'.
& there are 3 paths have a length 5.

(8) **Text book problem:** For the graph G as shown in fig. Give an example of each of the following (or) Explain why no such example exists

G:



a. An x-y walk of length 6

b. A v-w trail that is not a v-w path

c. An r-z path of length 2.

d. An x-z path of length 3

e. A circuit of length 10.

f. A cycle of length 8.

Soln

a. $x\ e_1\ u\ e_2\ r\ e_3\ v\ e_4\ w\ e_5\ z\ e_6\ y$.

b. $v\ e_4\ t\ e_7\ s\ e_8\ v\ e_4\ w$ (No edges are repeated) trail but not a path

c. path is a open walk in which no vertices & no edges are repeated. In the above graph r-z path of length 2 not exists.

d. x-z path of length 3 not exists

e. $x\ e_1\ u\ e_2\ r\ e_3\ v\ e_4\ t\ e_7\ e_8\ v\ e_{10}\ w\ e_5\ z\ e_6\ y\ e_9\ x$

f. $x\ e_1\ u\ e_2\ r\ e_3\ v\ e_4\ t\ e_{11}\ w\ e_5\ z\ e_6\ y\ e_9\ x$

(7)

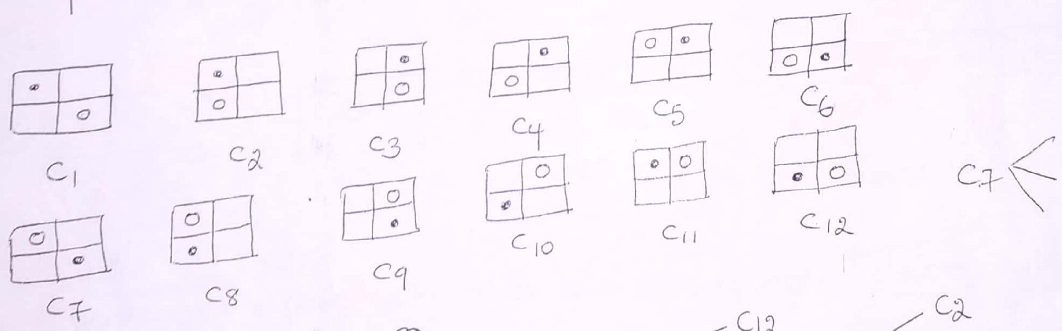
(9). Textbook problem: A graph 'G' of order 12 has vertex set $V(G) = \{c_1, c_2, \ldots c_{12}\}$ for the twelve configurations as shown in Fig. A "move" on this checkboard corresponds to moving a single coin to an unoccupied square, where

a] The gold coin can only be moved horizontally or diagonally.

b] The silver coin can only be moved vertically or diagonally.

Two vertices c_i & c_j $(i \neq j)$ are adjacent if it is possible to move c_i to c_j by a single move.

1. what vertices are adjacent to c_1 in 'G' ?
2. what vertices are adjacent to c_2 in 'G' ?
3. Draw the subgraph of 'G' induced by $\{c_2, c_6, c_9, c_{11}\}$
4. Give an example of a $c_1 - c_7$ path in 'G'.



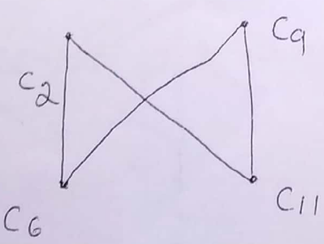
Soln

(1) $c_1 < \begin{matrix} c_{11} \\ c_3 \end{matrix}$

(2) $c_2 < \begin{matrix} c_4 \\ c_6 \\ c_{11} \end{matrix}$ $\quad$ $c_3 < \begin{matrix} c_{12} \\ c_5 \\ c_1 \end{matrix}$ $\quad$ $c_4 < \begin{matrix} c_2 \\ c_5 \end{matrix}$

$c_5 < \begin{matrix} c_4 \\ c_3 \\ c_8 \end{matrix}$ $\quad$ $c_6 < \begin{matrix} c_2 \\ c_9 \\ c_7 \end{matrix}$ $\quad$ $c_9 < \begin{matrix} c_6 \\ c_{11} \\ c_{10} \end{matrix}$ $\quad$ $c_{11} < \begin{matrix} c_9 \\ c_2 \\ c_1 \end{matrix}$

(3)



(4) ~~$c_1\ e_1\ c_{11}$~~

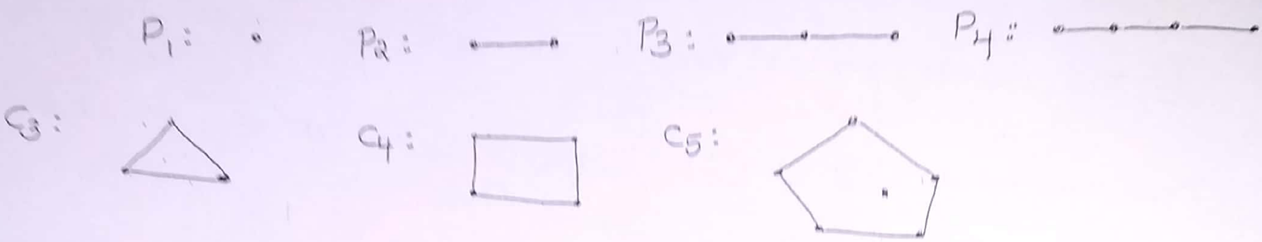
~~$c_1\ e_2\ e_3$~~ c_3 ~~$c_5\ e_4\ c_4$~~ $e_5\ e_2$

~~e_6~~

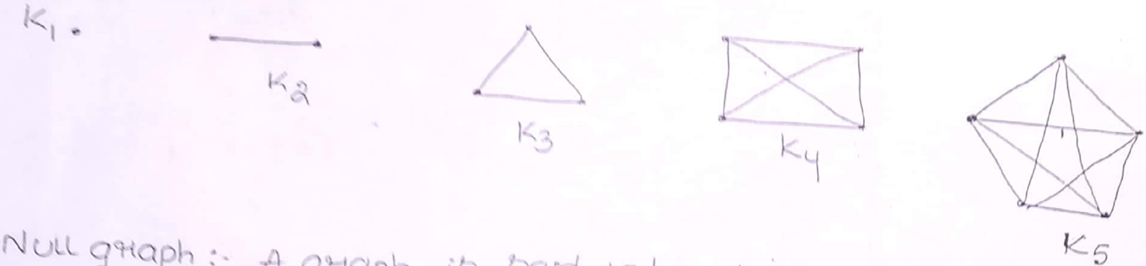
$c_1\ e_1\ c_3\ e_2\ c_5\ e_3\ c_4\ e_4\ c_2\ e_5\ c_6\ e_6\ c_7$.

Common classes of Graphs.

* A graph that is a path of order 'n' is denoted by 'P_n'.
* & A graph that is a cycle of order $n \geq 3$ is denoted by 'C_n'.

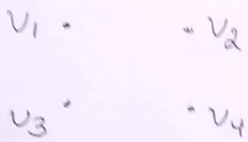


Complete graph:- A graph is said to be complete graph, if their exists an edge b/w every pair of vertices. A complete graph of 'n' vertices is denoted by 'K_n'. The number of pairs of vertices in K_n is $\binom{n}{2}$ & so the size of K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$



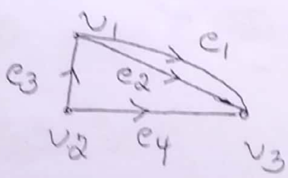
Null graph:- A graph is said to be Null graph in which no edges are present.

Ex:-



Multiple edges:- Two or more edges are said to be multiple edges if they have same initial point & terminal point.

Ex:-



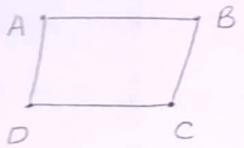
'e_1 & 'e_2' are multiple edges

Bipartite & complete Bipartite graphs

A graph 'G = (V, E)' is said to be Bipartite graph, if the vertex set 'V' can be partitioned in to two non empty subsets 'V1' & 'V2' such that every edge in 'G' joins vertex in 'V1' to a vertex in 'V2'.

A Bipartite graph is said to be complete Bipartite, if there exists an edge b/w every pair of vertices in 'V1' to 'V2'. where 'V1' & 'V2' are the partitions of the vertex set 'V'

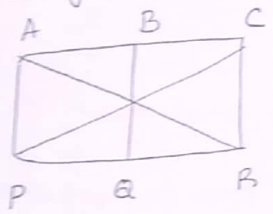
Ex:-

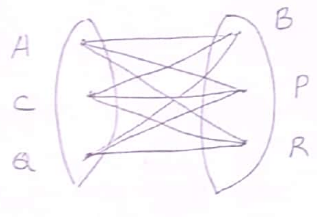


Bipartite graph & complete Bipartite graph.

(10) Verify whether the following graphs are Bipartite graph

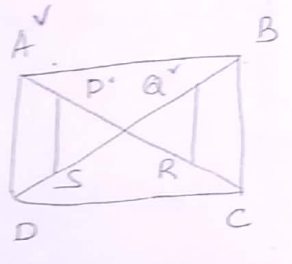
a.

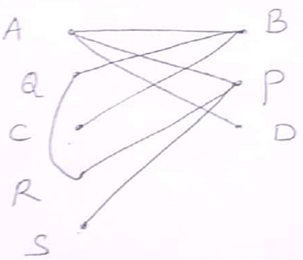




Bipartite graph & complete Bipartite graph

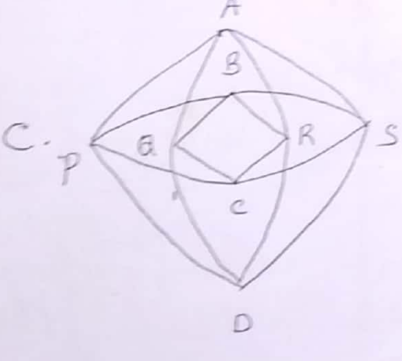
b.

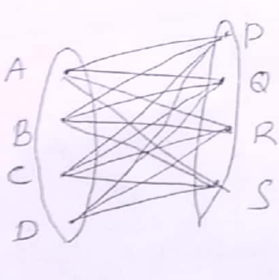




This is not a Bipartite graph because QR edge is there.

c.

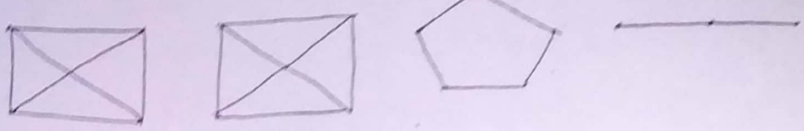




Bipartite & complete Bipartite graph

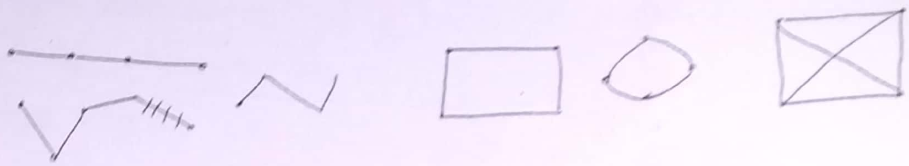
(11) i. Draw the graph for $2K_4 \cup C_5 \cup P_3$

Soln



ii $3P_4 \cup 2C_4 \cup K_4$

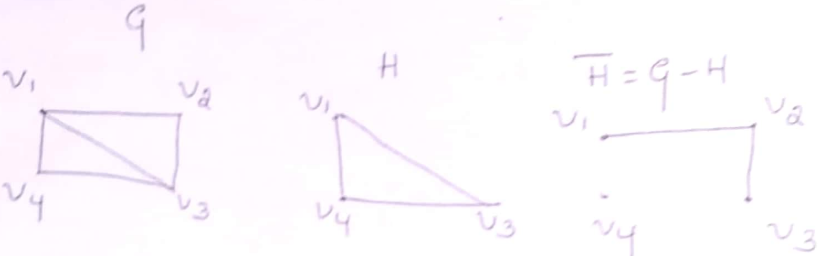
Soln



Complement of a subgraph :-

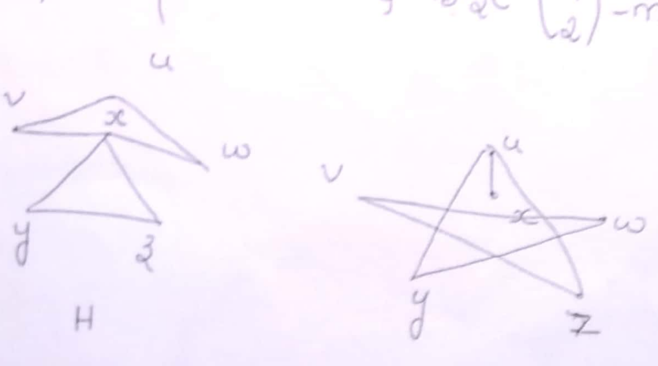
If 'G' is a graph & 'H' is a subgraph of graph 'G', then $\overline{H}$ is called complement of 'H'. which is obtained by deleting all edges of 'H' in 'G'.

Ex:-



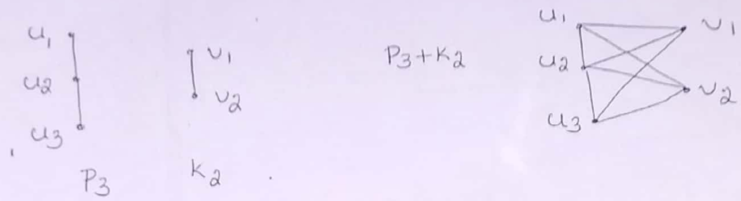
Complement of a graph ($\overline{G}$) :- The complement $\overline{G}$ of a graph G is that graph whose vertex set is $V(G)$ & such that for each pair u,v of vertices of G, uv is an edge of $\overline{G}$ if & only if uv is not an edge of 'G'. (order 'n' & size $\binom{n}{2} - m$)

Ex:-

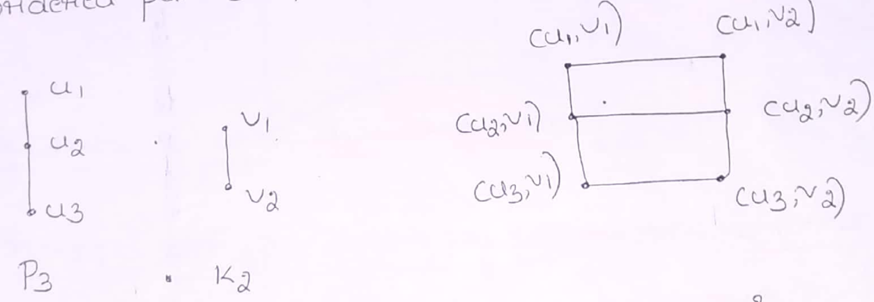


Empty graph :- The graph $\overline{K_n}$ then has 'n' vertices & no edges it is called empty graph of order 'n'.

* The join $G+H$ consists of $G \cup H$ & all edges joining a vertex of G & a vertex of H. the join of P_3 & K_2 is

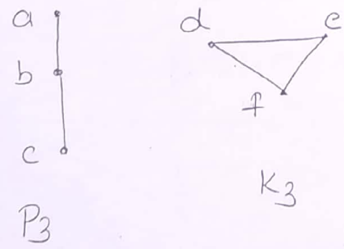


* For two graphs G & H, the cartesian product $G \times H$ has vertex set $V(G \times H) = V(G) \times V(H)$. that is, every vertex of $G \times H$ is an ordered pair (u,v) where $u \in V(G)$ & $v \in V(H)$.

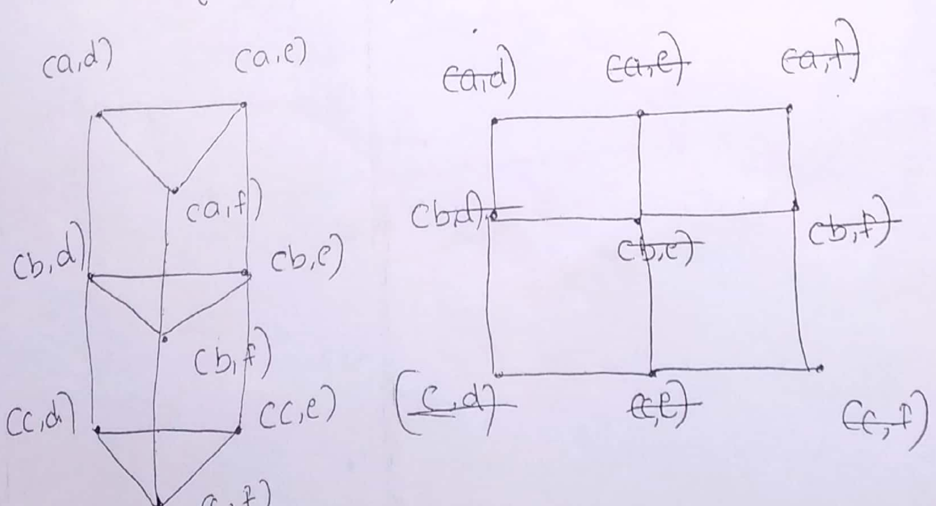


$P_3 \times K_2 = \{(u_1,v_1)\ (u_1,v_2)\ (u_2,v_1)\ (u_2,v_2)\ (u_3,v_1)\ (u_3,v_2)\}$

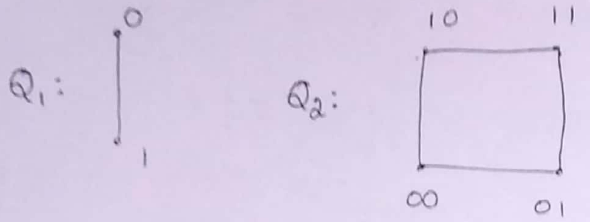
* The cartesian product of P_3 & K_3 is given by



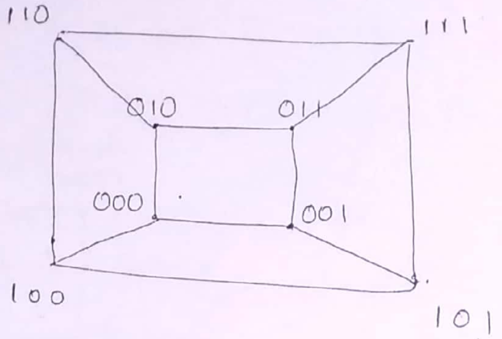
$P_3 \times K_3 = \{(a,d)\ (a,e)\ (a,f)\ (b,d)\ (b,e)\ (b,f)\ (c,d)\ (c,e)\ (c,f)\}$



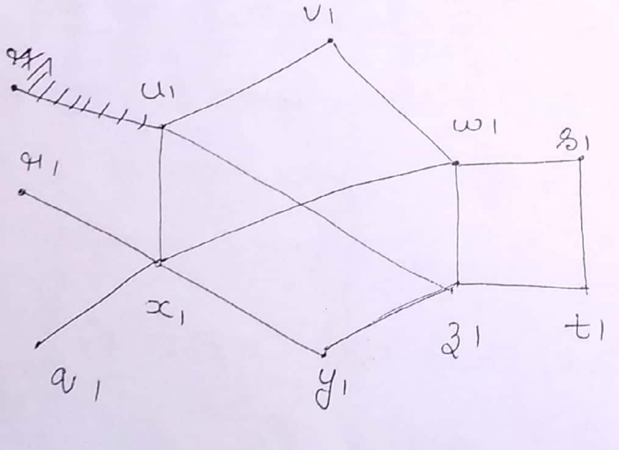
(12) Obtain cartesian product of Q_1 & Q_2.



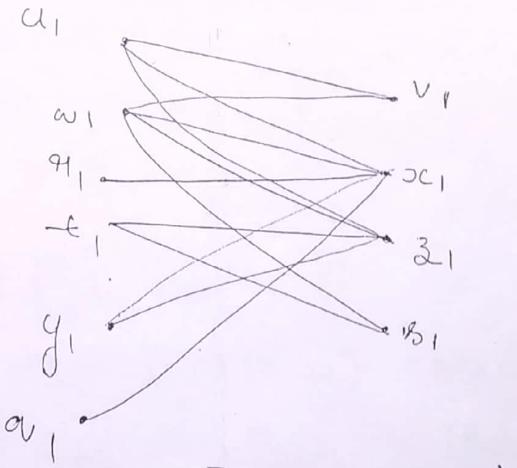
Q_3: $Q_1 \times Q_2 = \{(0,10)\ (0,11)\ (0,00)\ (0,01),\ (1,10)\ (1,11)\ (1,00)\ (1,01)\}$



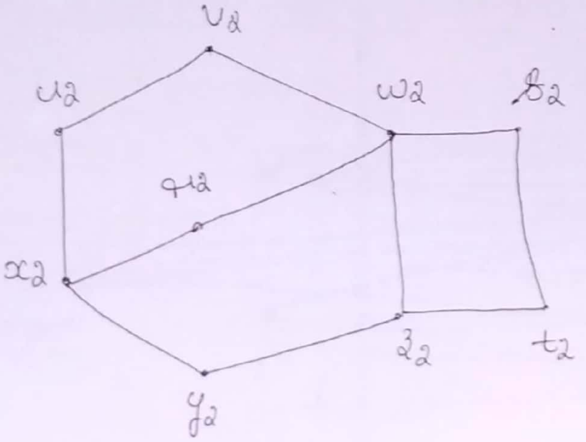
(13) Determine whether the graphs G_1 & G_2 are bipartite. If a graph is bipartite, then redraw it indicating the partite sets; if not, then given an explanation as to why the graph is not bipartite.



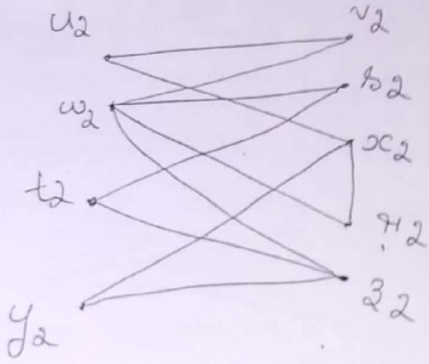
G_1



Bipartite graph



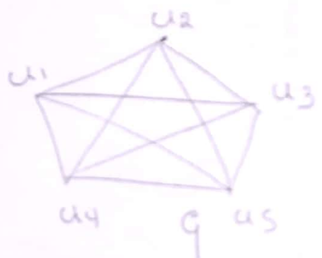
G_2

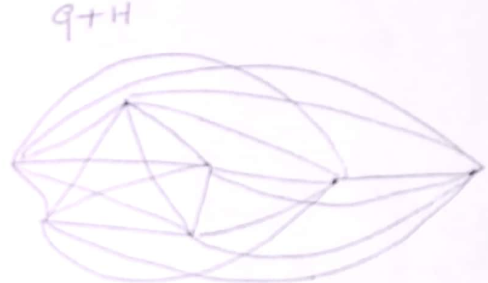


G_2 is not a Bipartite graph because in the second set r_2 to x_2 edge is there.

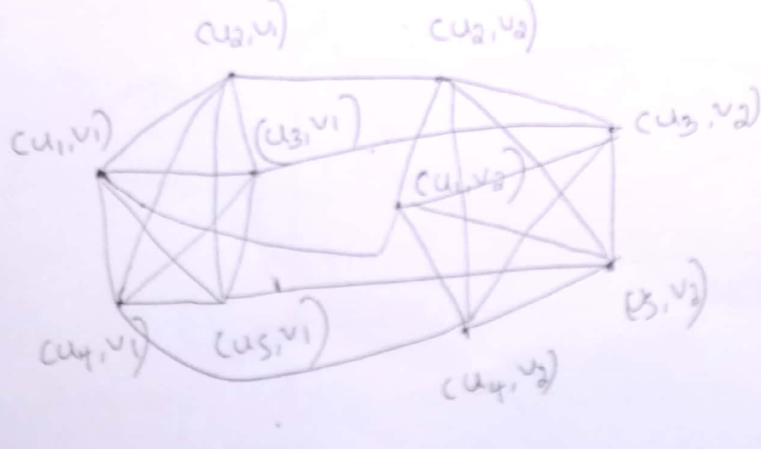
(14) For the following pairs G, H of graphs, draw $G+H$ & $G \times H$

a. $G = K_5$ & $H = K_2$





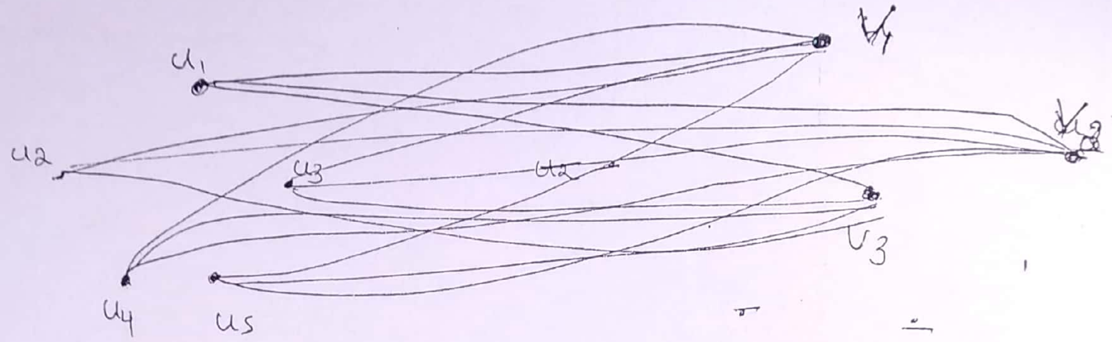
$G \times H = \{ (u_1, v_1)\ (u_1, v_2)\ (u_2, v_1)\ (u_2, v_2)\ (u_3, v_1)\ (u_3, v_2)\ (u_4, v_1)\ (u_4, v_2)$
$(u_5, v_1)\ (u_5, v_2) \}$



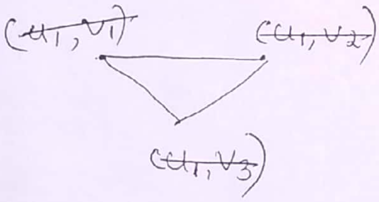
b.

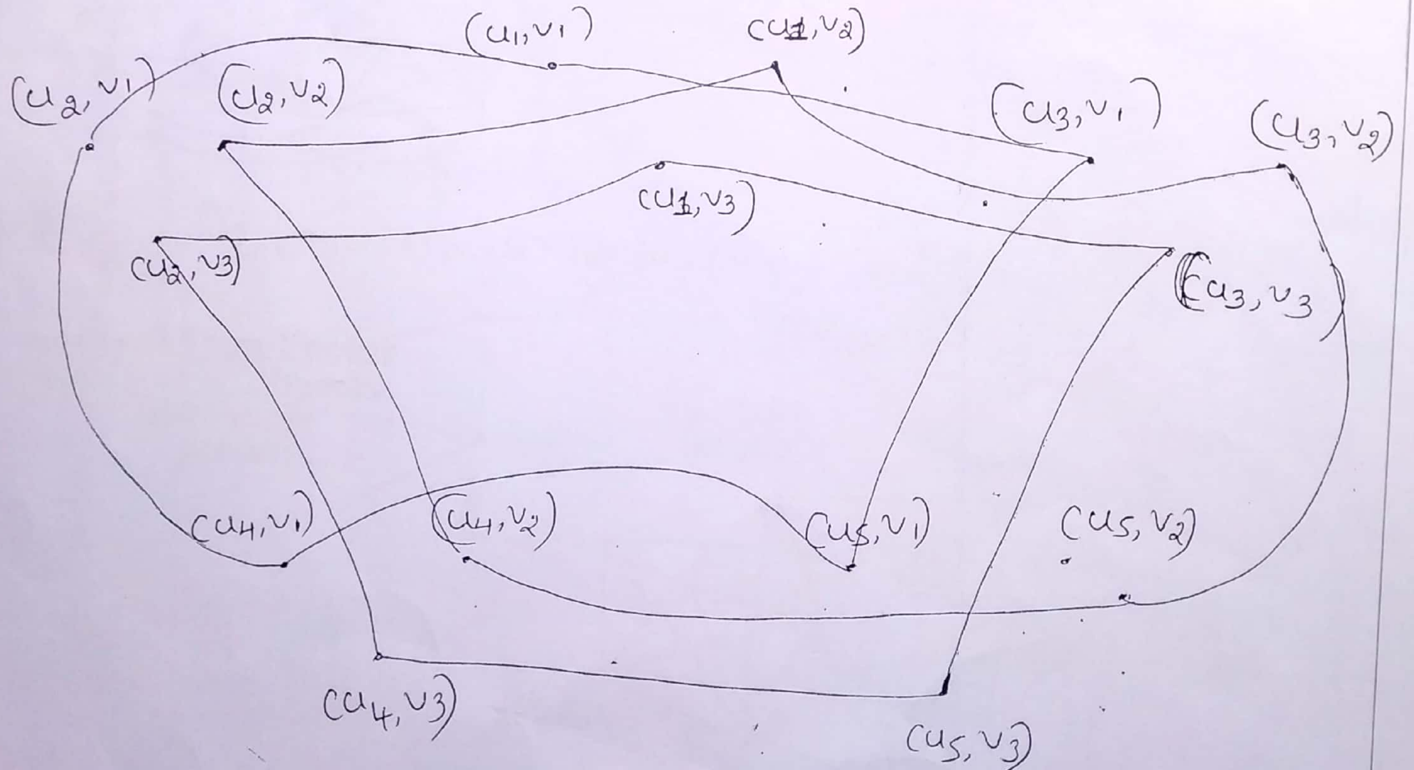
$G = \overline{K_5}$ & $H = \overline{K_3}$

G+H



$G \times H = \{ (u_1, v_1)\ (u_1, v_2)\ (u_1, v_3)\ (u_2, v_1)\ (u_2, v_2)\ (u_2, v_3)$
$(u_3, v_1)\ (u_3, v_2)\ (u_3, v_3)\ (u_4, v_1)\ (u_4, v_2)\ (u_4, v_3)$
$(u_5, v_1)\ (u_5, v_2)\ (u_5, v_3) \}$





(15) A certain graph 'G' has order 14 & size 27. The degree of each vertex of 'G' is 3, 4 or 5. There are 6 vertices of degree 4. How many vertices of 'G' have degree 3 & how many have degree 5?

Soln $\quad n = 14 \quad m = 27$

Let 'K' be the number of vertices of degree 3

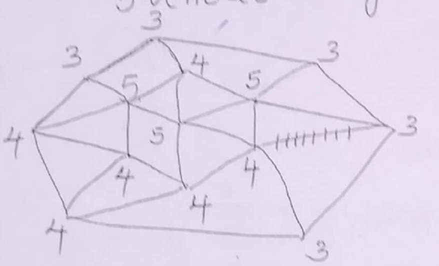
$$K \cdot 3 + 6 \times 4 + (8-K) \cdot 5 = 2 \times 27$$

$$3K + 24 + 40 - 5K = 2 \times 27$$

$$\boxed{K = 5}$$

$8-K$
$8-5 = 3$ $\quad \therefore$ three vertices of degree 5 &

5 vertices of degree 3.



(16) If 'G' is an undirected graph with 'n' vertices & m edges if 'δ' is the minimum & 'Δ' is the maximum of the degree of vertices show that $\delta \le \frac{2m}{n} \le \Delta$

Soln $\quad$ Let $d_1, d_2, \ldots d_n$ be the degree of vertices.

then by handshaking property $d_1 + d_2 + d_3 + \ldots + d_n = 2m \rightarrow (1)$

Since $\delta = \min(d_1, d_2, d_3 \ldots d_n)$

$$\therefore d_1 \ge \delta, d_2 \ge \delta, \ldots d_n \ge \delta$$

$$\therefore d_1 + d_2 + d_3 + \ldots + d_n \ge n\delta \quad \rightarrow (2)$$

Similarly $\quad \Delta = \max(d_1, d_2, d_3, \ldots d_n)$

$$\therefore \Delta \ge d_1, \Delta \ge d_2, \ldots \Delta \ge d_n$$

$$\therefore d_1 + d_2 + d_3 + \ldots + d_n \le n\Delta \rightarrow (3)$$

From (1), (2) & (3) $2m \geq n\delta$ & $2m \leq n\Delta$

$$n\delta \leq 2m \leq n\Delta$$

$$\boxed{\delta \leq \frac{2m}{n} \leq \Delta}$$

(17) Show that in a graph 'G', the number of odd degree vertices is even.

Soln Let 'p' be a graph with 'n' number of vertices.

Let $v_1, v_2, v_3, \dots v_k$ be the odd degree vertices

Let $v_{k+1}, v_{k+2}, v_{k+3}, \dots v_n$ be the even degree vertices

$$\sum_{i=1}^{n} d(v_i) = \underbrace{\sum_{i=1}^{k} d(v_i)}_{\text{odd}} + \underbrace{\sum_{i=k+1}^{n} d(v_i)}_{\text{even}}$$

By Handshaking property

$$\sum_{i=1}^{n} d(v_i) = \text{even edges}$$

$$\Rightarrow \text{Even} = \underbrace{\sum_{i=1}^{k} d(v_i)}_{\text{Odd}} + \underbrace{\sum_{i=k+1}^{n} d(v_i)}_{\text{Even}}.$$

$$\underbrace{\sum_{i=1}^{k} d(v_i)}_{\text{odd}} = \text{Even} - \sum_{i=k+1}^{n} d(v_i)$$

$$= \text{Even} - \text{Even} = \text{Even}$$

$\therefore$ Number of odd degree vertices is even

If 'G' is a bipartite graph, show that 'G' has no cycle of odd length

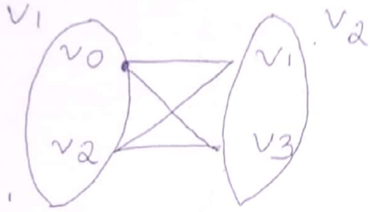
Proof:- Since 'G' is bipartite, we can partition its vertex set 'V' into two disjoint sets (bipartites) V_1 & V_2 so that each edge of 'G' joins a vertex in V_1 & a vertex in V_2.

* Let $v_0 v_1 v_2, \dots v_m v_0$ be a cycle in 'G' & assume that

v_0 is in 'V_1', v_1 is in 'V_2'

v_2 is in 'V_1', v_3 is in 'V_2' & so on.

Thus the vertices in the cycle belong to V_1, V_2 alternately.



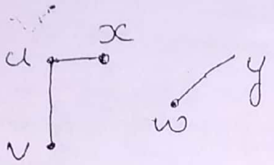
* Since the terminal vertex of the cycle is 'v_0' & it is in V_1, the number of edges that belong to the cycle cannot be '3' or 5 or 7 or any odd number.

Thus 'G' has no cycle of odd length.

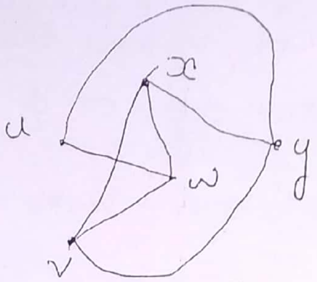
If G is a disconnected graph, then $\overline{G}$ is connected.

Proof:- Since 'G' is disconnected, 'G' contains two or more components. Let 'u' & 'v' be two vertices of $\overline{G}$. We show that 'u' & 'v' are connected in $\overline{G}$. ~~We show that 'u' & 'v' are~~ If 'u' & ~~connected~~ 'v' belong to different components of 'G', then u & v are not adjacent in 'G' & so 'u' & 'v' are adjacent in $\overline{G}$. Hence $\overline{G}$ contains a u-v path of length '1'.

Suppose next that u & v belong to the same component of $\bar{G}$. Let 'w' be a vertex of G that belongs to a different component of $\bar{G}$. Then $uw, vw \notin E(G)$, implying that $uw, vw \in E(\bar{G})$ & so u, w, v is a $u-v$ path in $\bar{G}$.

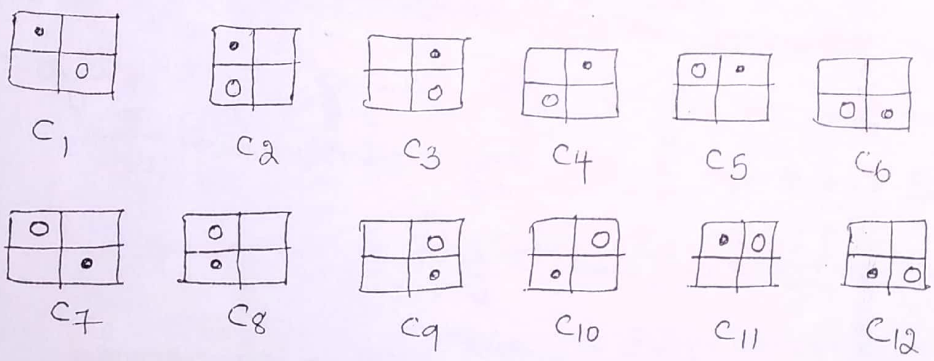


G

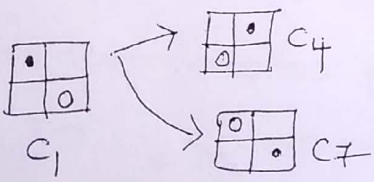


$\bar{G}$

(18) Consider the twelve configurations C_i, $1 \le i \le 12$, in Fig. Draw the digraph 'D', where $V(D) = \{C_1, C_2, \dots C_{12}\}$ & where (C_i, C_j) is a directed edge of 'D' if it is possible to (a) C_j by rotating the configuration C_i either $90°$ or $180°$ clockwise about the midpoint of the checkerboard.



Soln



$C_2 \to C_5, C_9$

$C_4 \to C_7, C_{10}$

$C_6 \to C_8, C_{11}$

$C_3 \to C_6, C_8$

$C_5 \to C_9, C_{12}$

$C_7 \to C_{10}, C_1$

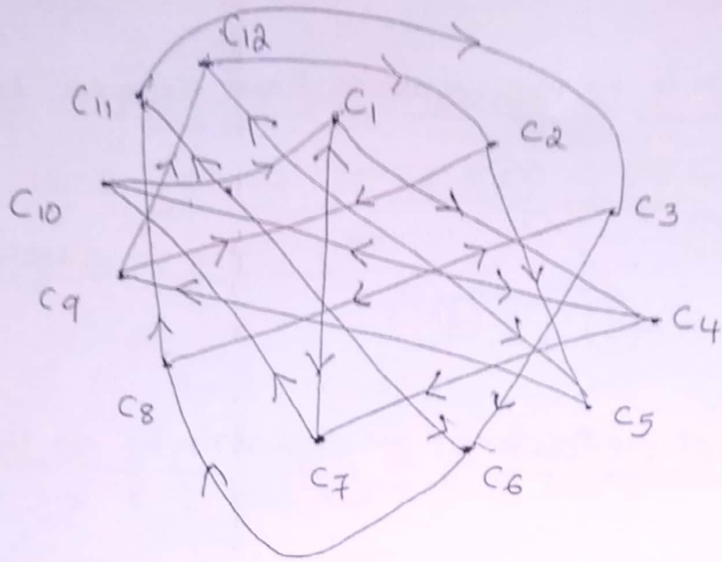
$C_8 \to C_{11}, C_3$

$C_9 \to C_{12}, C_2$

$C_{10} \to C_1, C_4$

$C_{11} \to C_3, C_6$

$C_{12} \to C_2, C_5$



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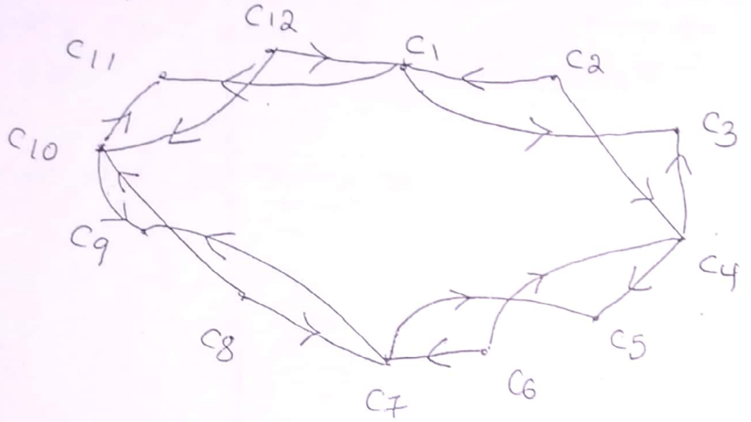
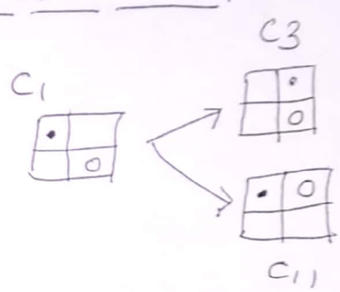
NMIT, Bengaluru 560 064
Mob : 8747901035

(b). if C_j can be obtained by moving one of the coins in C_i to the right or up.

Soln





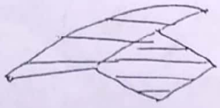
(19) give an example of the following or explain why no such example exists.

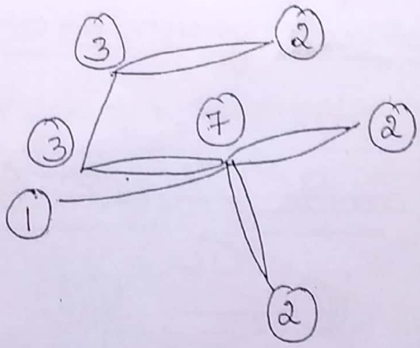
a] A graph of order 7 whose vertices have degree 1, 1, 1, 2, 2, 3, 3

3+3+1+1+1 = 9 odd

Sum of all odd degrees is even but here it is odd

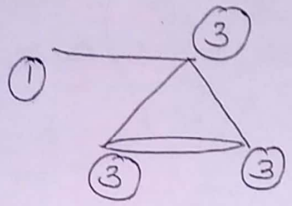
b] 1, 2, 2, 2, 3, 3, 7





1+3+3+7 = even

c] A graph of order 4 whose vertices have degree 1, 3, 3, 3



$1+3+3+3=$ even

(20) Give an example of the following or explain why no such example exists.

a] A graph that has no odd vertices.

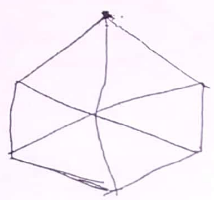
Soln

K_2

K_2 has even vertices.

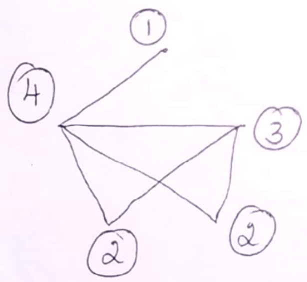
b] A non complete graph, all of whose vertices have degree 3

Soln



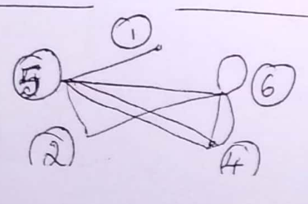
c] A graph 'G' of order 5 or more with the property that $\deg u \neq \deg v$ for every pair u, v of adjacent vertices of G.

Soln



d] A non complete graph 'H' of order '5' or more with the property that $\deg u \neq \deg v$ for every pair u, v of non adjacent vertices of 'H'.

Soln



(21) The degree of each vertex of a certain graph of order 12 & size 31 is either 4 or 6. How many vertices of degree 4 are there?

Soln

$n = 12 \quad m = 31$

Let 'K' be the number of vertices of degree 4

$K \times 4 + (12 - K) \times 6 = 2 \times 31$

$4K + 72 - 6K = 62$

$\cancel{K = 62 - 36}$

$\cancel{K = 26}$

$72 - 62 = 2K$

$10 = 2K$

$\boxed{K = 5}$

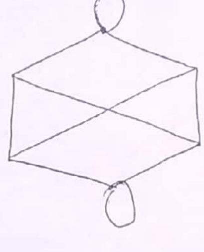
5 vertices of degree 4 & $(12 - K)$

$(12 - 5)$

7 vertices of degree 6.

(22) Give an example of a graph G of order 6 & size 10 such that $\delta(G) = 3$ & $\Delta(G) = 4$

Soln



$\delta(G) = 3$

$\Delta(G) = 4$

(23) The degree of every vertex of a graph 'G' of order 25 & size 62 is 3, 4, 5 or 6. There are two vertices of degree 4 & 11 vertices of degree 6. How many vertices of 'G' have degree 5?

Soln $n=25 \quad m=62$

Let 'K' be the number of vertices of degree 5

$$2\times4+11\times6+K\times5+(12-K)\times3=2\times62$$

$$\cancel{8+66+5K+39-3K=124}$$

$$15\cancel{2-2K=124} \qquad \cancel{2K+113=124}$$

$$\cancel{K=152-124} \qquad \cancel{2K=11}$$

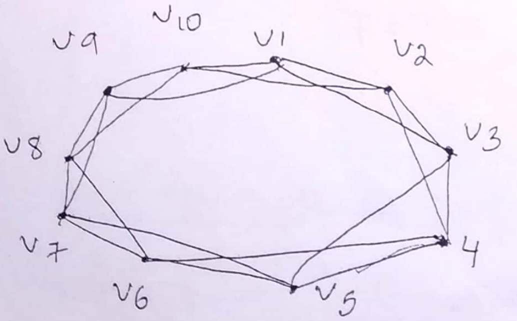
$$\cancel{K=}$$

$$8+66+5K+36-3K=124$$
$$2K+110=124$$
$$2K=14$$
$$\boxed{K=7}$$

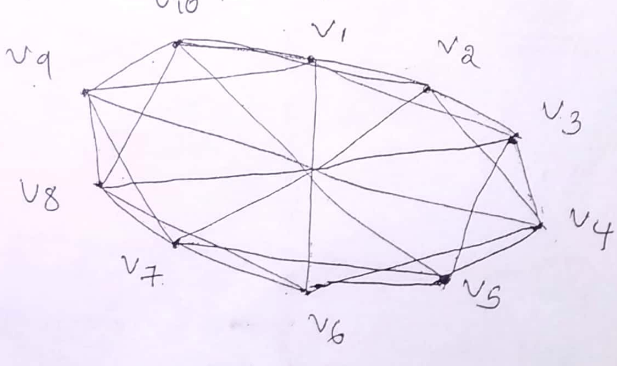
7 vertices of degree 5 & 5 vertices of degree 3.

Harary graphs:- $H_{r,n}$, A graph is said to be Harary graphs only if it is $H_{r,n}$ where 'r' is the r-regular graph & 'n' be the total number of vertices.

Ex:- A 4-regular graph & a 5-regular graph both of order 10.



$H_{4,10}$



$H_{5,10}$

Note:- Let 'r' & 'n' be integers with $0\le r\le n-1$. There exists an r-regular graph of order 'n' if & only if atleast one of 'r' & 'n' is even.

(15)

(24) For the graph 'G' of fig. Find a 5-regular graph 'H' of minimum order containing 'G' as an induced subgraph.

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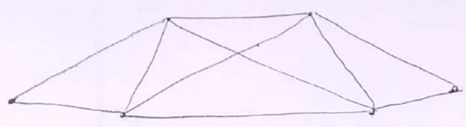
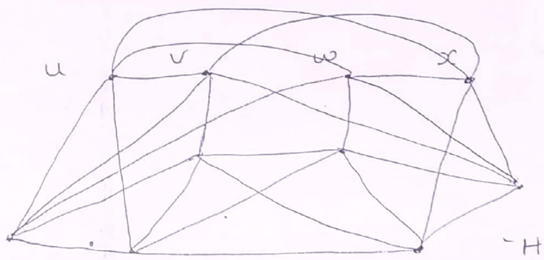


'G'

Soln

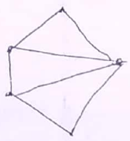


'H' containing 'G' as an induced subgraph

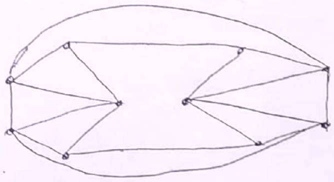
5 regular graph

(25) A 4-regular graph 'H' containing 'G' as an induced subgraph

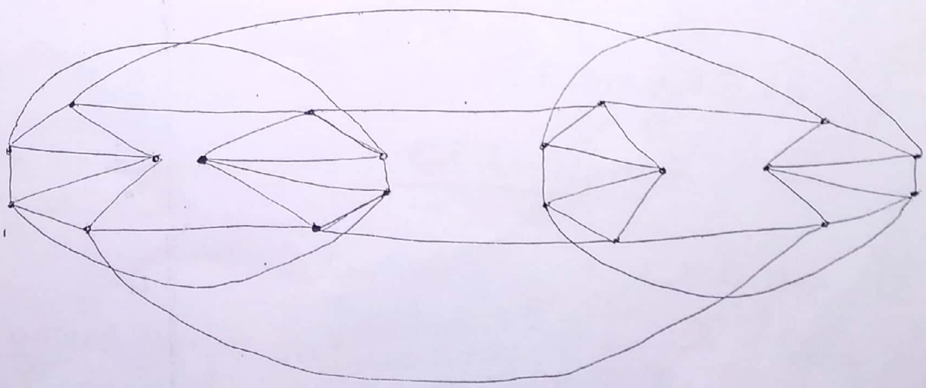
G :



G1 :



H :



Degree sequences

(26) Which of the following sequences are graphical?

Havel-Hakimi Theorem

a] S_1: 3, 3, 2, 2, 1, 1

Decreasing order

Soln Deleting 3 from S_1 & subtracting 1 from next 3 terms

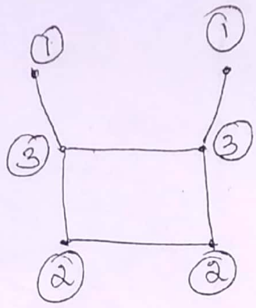
$\cancel{3}$, 3, 2, 2, 1, 1

$\cancel{2}$, 1, 1, 1, 1

Delete 2

0, 0, 1, 1

1, 1, 0, 0.



b] S_2: 6, 5, 5, 4, 3, 3, 3, 2, 2

Soln $\cancel{6}$, 5, 5, 4, 3, 3, 3, 2, 2

Deleting 6 from S_2 & subtract 1 from next 6 terms

$\cancel{4}$, 4, 3, 2, 2, 2, 2, 2

3, 2, 1, 1, 2, 2, 2

$\cancel{3}$, 2, 2, 2, 2, 1, 1

1, 1, 1, 2, 1, 1

$\cancel{2}$, 1, 1, 1, 1, 1

0, 0, 1, 1, 1

1, 1, 1, 0, 0

Graph doesnot exists

c] S_3: 7, 6, 4, 4, 3, 3, 3

Soln $\cancel{7}$, 6, 4, 4, 3, 3, 3

there is no 7th term to delete so graph does not exist

d] S_4: 3, 3, 3, 1

Soln $\cancel{3}$, 3, 3, 1

$\cancel{2}$, 2, 0

1, −1

No degrees having negative number so graph does not exists.

(27) Decide whether the sequence S1: 5, 4, 3, 3, 2, 2, 2, 1, 1, 1 is Graphical.

Soln

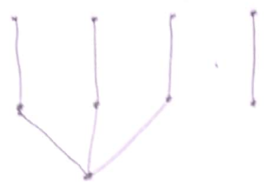
$\cancel{5}$, 4, 3, 3, 2, 2, 2, 1, 1, 1

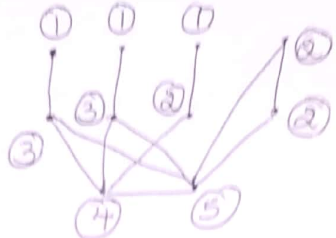
Delete 5 & subtract 1 from 5 terms

3, 2, 2, 1, 1, 2, 1, 1, 1

$\cancel{3}$, 2, 2, 2, 1, 1, 1, 1, 1

1, 1, 1, 1, 1, 1, 1, 1, 1





S2: 7, 7, 4, 3, 3, 3, 2, 1 is graphical

Soln

$\cancel{7}$, 7, 4, 3, 3, 3, 2, 1

$\cancel{6}$, 3, 2, 2, 2, 1, 0

2, 1, 1, 1, 0, -1

Graph doesnot exist as one degree having negative value.

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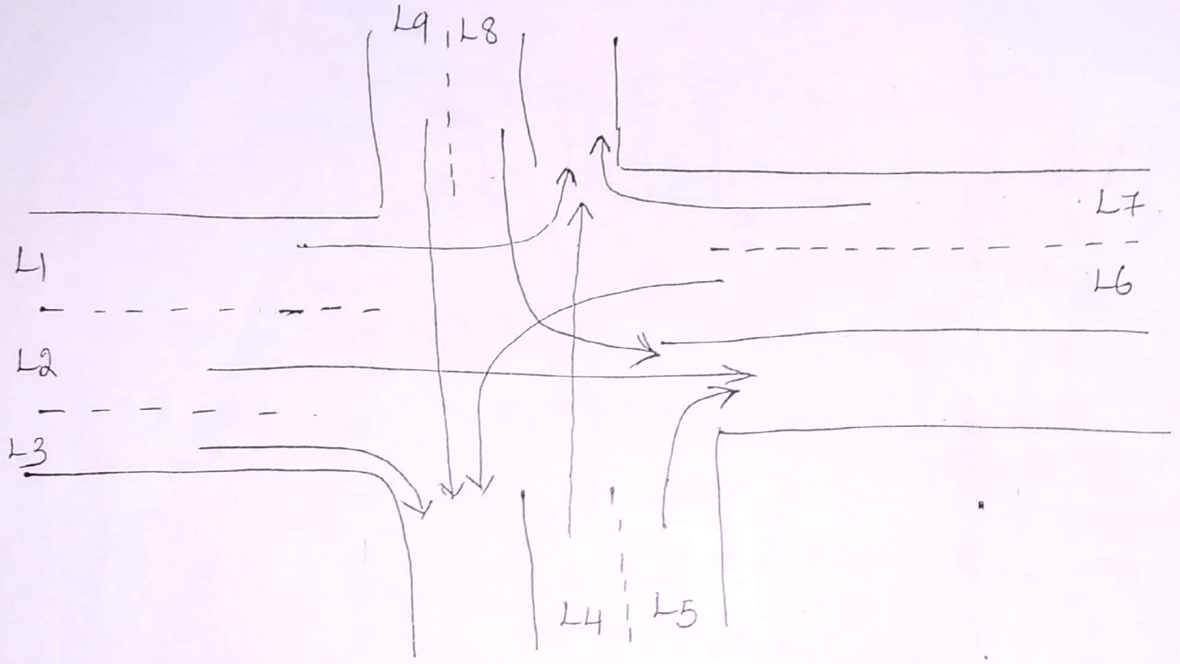
28) Following figure illustrates the traffic lanes at the intersection of two streets. When a vehicle approaches this intersection, it could be in one of the nine lanes : L1, L2, ...L9. Draw a graph 'G' that models this situation, where $V(G) = \{L1, L2, ... L9\}$ & where two vertices are joined by an edge if vehicles in these two lanes cannot safely enter this intersection at the same time.



Soln