Introduction to Probability
Chapter 7 Continuous Distributions

Nitin Gupta
Department of Mathematics
Indian Institute of Technology Kharagpur,
Kharagpur - 721 302, INDIA.

Outline

1. Exponential distibution
2. Gamma distibution
3. Normal distribution
4. Standard normal distribution

References

1. Probability and statistics in engineering by Hines et al (2003) Wiley.
2. Mathematical Statistics by Richard J. Rossi (2018) Wiley.
3. Probability and Statistics with reliability, queuing and computer science applications by K. S. Trivedi (1982) Prentice Hall of India Pvt. Ltd.

Exponential distribution $X \sim Exp(\lambda)$

- PDF is

$$f(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0,\ \lambda > 0 \end{cases}$$

- CDF is

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

- MGF is $M(t) = \left(1 - \frac{t}{\lambda}\right)^{-1},\ t < \lambda$.
- $E(X) = \frac{1}{\lambda}$; and $Var(X) = \frac{1}{\lambda^2}$.

Relationship between $Exp(\lambda)$ and Poisson Process

Let $N(t)$ counts the number of events occuring in $(0, t]$, i.e., $N(t) \sim PP(\lambda)$, then

$$P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \;\; n = 0, 1, \ldots$$

Let T_1 denote the time of occurance of first event, i.e., time of first arrival of event. Then

$$\begin{aligned} T_1 > t &\equiv N(t) = 0 \\ \Rightarrow \;\; P(T_1 > t) &= P(N(t) = 0) \\ &= e^{-\lambda t}, \;\; t > 0, \;\; \lambda > 0. \end{aligned}$$

Hence $T_1 \sim Exp(\lambda)$

Example

Suppose that on average 30 programme per hour queued to be proceesed in accordance with Poisson process. What is the probability that the server will wait more than 5 minutes before the first programme arrives?

Solution: Let $N(t)$ is the number of programme arriving in $(0, t]$ such that $N(t) \sim PP(\lambda)$

$$\lambda = 30 \text{ per hour} = \frac{30}{60} \text{ per min} = \frac{1}{2} \text{ per min}$$

Let T_1 denote the time for first arrival of programme at server. Here $T_1 \sim exp(\lambda)$. Hence the required probability is

$$\begin{aligned} P(T_1 > 5) &= P(N(5) = 0) \\ &= \frac{e^{-\lambda t}(\lambda t)^0}{0!} \\ &= e^{-\frac{5}{2}} \end{aligned}$$

Memoryless Property of Exponential distribution

- Exponential distribution has memoryless property, i.e., for all $t > 0,\ s > 0$

$$P(X > t + s | X > t) = P(X > s)$$

Gamma distribution $X \sim Gamma(\lambda, r)$

PDF is

$$f(x) = \begin{cases} 0, & x < 0 \\ \frac{\lambda^r}{\Gamma(r)} e^{-\lambda x} x^{r-1}, & x \geq 0,\ \lambda > 0,\ r > 0 \end{cases}$$

here $\Gamma(r) = \int_0^\infty e^{-x} x^{r-1} dx$. Note that $\Gamma(r) = (r-1)\Gamma(r-1)$ and $\Gamma(1) = 1$

MGF is $M(t) = \left(1 - \frac{t}{\lambda}\right)^{-r},\ t < \lambda$. $E(X) = \frac{r}{\lambda}$; and $Var(X) = \frac{r}{\lambda^2}$.

Particular cases:

- If $r = 1$, then $Gamma(\lambda, r) \equiv Exp(\lambda)$
- If $r = \frac{\nu}{2},\ \lambda = \frac{1}{2}$ then $Gamma(\lambda, r) \equiv \chi^2_\nu$

Relationship between $Gamma(\lambda, r)$ and Poisson Process

Let $N(t)$ counts the number of events occuring in $(0, t]$, i.e., $N(t) \sim PP(\lambda)$, then

$$P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \ n = 0, 1, \ldots$$

Let $S_r = \sum_{i=1}^{r} T_i$ denote the time of occurance of r events, i.e., time of r arrivals of event. Then

$$S_r > t \equiv N(t) < r$$

$$\Rightarrow \ P(S_r > t) = P(N(t) \leq r - 1) = e^{-\lambda t} \sum_{i=0}^{r-1} \frac{(\lambda t)^i}{i!}, \ t > 0, \ \lambda > 0.$$

Hence PDF of S_r is $f_{S_r}(t) = -\frac{d}{dt} P(S_r > t) = \frac{\lambda^r}{(r-1)!} e^{-\lambda x} x^{r-1}$.
Hence $S_r \sim Gamma(\lambda, r)$

Example

Suppose that the programmes arrive at a server in according to a Possion process with $\lambda = 10$ programmes per hour. Find the probability that one has to wait more than half hour until the second programme arrive at server?

Solution: $N(t)$ is number of programmes arriving in $(0, t] \sim PP(\lambda)$. S_r denote the time until r arrivals. Here $S_2 \sim \text{Gamma}(\lambda, r)$, $\lambda = 10, \; r = 2$. Required probability is

$$
\begin{aligned}
P\left(S_2 > \frac{1}{2}\right) &= P\left(N\left(\frac{1}{2}\right) \leq 1\right) \\
&= P\left(N\left(\frac{1}{2}\right) = 0\right) + P\left(N\left(\frac{1}{2}\right) = 1\right) \\
&= e^{-10\times\frac{1}{2}} + e^{-10\times\frac{1}{2}}\left(10 \times \frac{1}{2}\right) \\
&= 6e^{-5}
\end{aligned}
$$

Normal distribution $X \sim N(\mu, \sigma^2)$

- PDF is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0.$$

- Probability density function is symmetric around μ, i.e., $f(\mu - x) = f(\mu + x)$.
- MGF is $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.
- $E(X) = \mu$ and $Var(X) = \sigma^2$.

Standard Normal distribution $Z \sim N(0,1)$

- PDF is
$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \ -\infty < x < \infty, .$$
- Probability density function is symmetric around 0.
- CDF
$$\Phi(x) = \int_{-\infty}^{x} \phi(u) du$$
- $\Phi(x) + \Phi(-x) = 1$.
- MGF is $M(t) = e^{\frac{1}{2}t^2}$.
- $E(Z) = 0$ and $Var(Z) = 1$.
- If $Y \sim N(\mu, \sigma^2)$, then $Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$.
- From Standard Normal distribution table we can find the values $\Phi(x)$. For example $\Phi(1.28) = 0.89973$.

Example

A firm manufacture devices which has random life that is normally distributed with mean 700 hours and standard deviation 50 hours. (i) Find the probability that the device survives more than 800 hours. (ii) Find the probablity that the device fails between 675 to 725 hours.

Solution: Let X be the random life of device. $X \sim N(\mu, \sigma^2)$, where $\mu = 700$, $\sigma = 50$. Then $Z = \frac{X-700}{50} \sim N(0,1)$.

$$(i)\ P(X > 800) = P(Z > 2) = 1 - \Phi(2) = 1 - 0.97725 = 0.02275.$$

$$\begin{aligned}(ii)\ P(675 < X < 725) &= P(-0.5 < Z < 0.5) = \Phi(0.5) - \Phi(-0.5)\\ &= 2\Phi(0.5) - 1 = 2 \times 0.69146 - 1 = 0.38292.\end{aligned}$$

Summary

In this chapter we presented some widely used continuous distributions. The exponential, Gamma were presented along with their relation with the Poisson process. Since the normal distribution is widely used in the statisitcal inference. Therefore we introduced the normal distribution.