Queueing Theory:

r.v. $N(t)$: # of customers in queueing system at time t

$N_q(t)$: " " in queue in ——— t

$N_s(t)$: " " getting service ——— t.

$$N(t) = N_q(t) + N_s(t)$$

r.v. N : steady state # of customers in queueing system

N_q : " " " " in queue " ———

N_s : " " " " getting service ———

$$N = N_q + N_s$$

$$\Rightarrow \quad E(N) = E(N_q) + E(N_s)$$

$$L = L_q + L_s$$

$L = E(N)$

$L_q = E(N_q)$

$L_s = E(N_s)$

Relationship between time

r.v.

w : total time a customer spends in queueing system

$$w = \underbrace{q}_{\text{time spent in queue}} + \underbrace{s}_{\text{service time}}$$

$$\Rightarrow \quad E(w) = E(q) + E(s)$$

$$W = W_q + W_s$$

$W = E(w)$

$W_q = E(q)$

$W_s = E(s)$

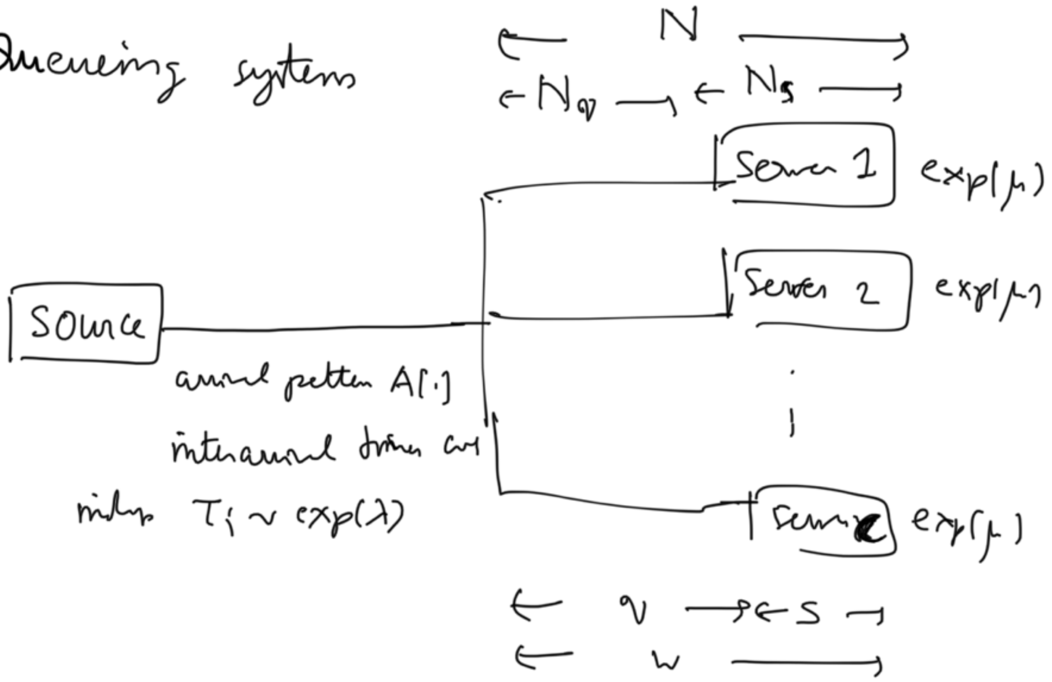
cdf $\quad W(t) = P(w \le t)$

$W_q(t) = P(q \le t)$

$W_q(t) = P(q \le t)$

$$W_s(t) = P(s \le t)$$

Queueing systems



If n servers are busy, then
T time for next service completion
$\sim \exp(n\mu)$

Notation

$$A/B/c/K/m/Z$$

- A: interarrival time distn
- B: service time distn
- c: # of servers
- K: system capacity
- m: # in the population or source
- Z: queue discipline — FIFO or LIFO

A, B:
- GI general iid interarrival time
- G general service time
- M expo. interarrival or service time distn
- U uniform " " ---- ---
- D deterministic " " ---

$N(t)$ # of customers arrival by time t

av. arrival rate of the customer

$$\lambda_a = \lim_{t \to \infty} \frac{N(t)}{t}$$

Basic cost identity:

$$\begin{bmatrix} \text{av. rate at which} \\ \text{the system earns} \end{bmatrix} = \lambda_a \times \begin{bmatrix} \text{av. amount an entering} \\ \text{customer pays} \end{bmatrix}$$

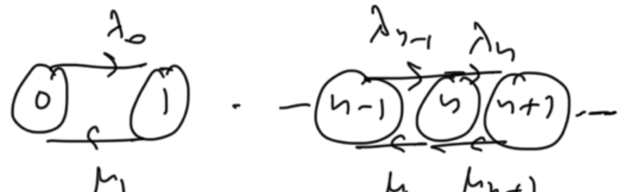
$$L = \lambda_a W$$

$$L_q = \lambda_a W_q \qquad \text{Little's law}$$

$$L_s = \lambda_a W_s$$

— x —

Birth and death process:



In steady state $N = n$

$$P_n = P(N=n)$$

$$S = 1 + C_1 + C_2 + \cdots$$

$$P_0 = \frac{1}{S} \qquad C_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$$

$$P_n = C_n P_0 \quad , n = 1, 2, \ldots$$

M/M/1 queueing system ?

interarrival times are $\exp(\lambda)$

service time is $\exp(\mu)$

# of server

If $N = n$, then
prob. $\frac{}{}$ of an arrival in time interval of length h is

$$e^{-\lambda h}\lambda h = \lambda h\left(1 - \lambda h + \frac{(\lambda h)^2}{2!} - \cdots\right)$$

$$= \lambda h + o(h)$$

$$\therefore \lambda_n = \lambda, \quad n = 0, 1, 2, \ldots$$

$N = n$, service time dist.

$$W_s(t) = P(s \le t) = 1 - e^{-\mu t}$$

If $N = n$,
prob. of Service completion in a small interval of length h

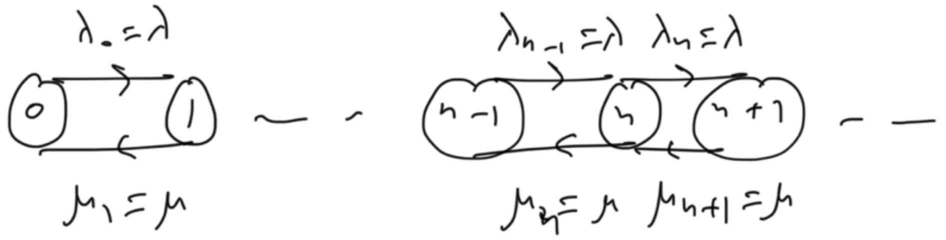
$$= 1 - e^{-\mu h} = 1 - \left(1 - \mu h + \frac{(\mu h)^2}{2!} - \cdots\right)$$

$$= \mu h + o(h)$$

B & D process

$$\therefore \mu_n = \mu, \quad n = 1, 2, \ldots$$

M/M/1



let $\frac{\lambda}{\mu} = \rho$

$P_0 = \frac{1}{S}$; $S = 1 + C_1 + C_2 + \cdots$

$P_0 = 1 - \rho$ $\qquad = 1 + \rho + \rho^2 + \cdots = \frac{1}{1-\rho}$

$$P_n = P(N = n) = C_n P_0 = \rho^n(1-\rho), \quad n = 0, 1, 2, \ldots$$

$$L = E(N) = \frac{\rho}{1-\rho}$$

$s \sim \exp(\mu)$

$$W_s = E(s) = \frac{1}{\mu} = \frac{\rho}{\lambda}$$

$$\sigma_N^2 = \frac{\rho}{(1-\rho)^2}$$

$$\Rightarrow \rho = \lambda W_s$$

$$\lambda_a = \lambda \times 1 = \lambda$$

$$L = \lambda W \Rightarrow W = \frac{L}{\lambda} = \frac{\rho}{} = \frac{W_s}{} = \frac{1}{\mu(1-\rho)}$$

$\lambda \qquad \lambda(1-\rho) \qquad 1-\rho$

$$W_q = W - W_s = \frac{W_s}{1-\rho} - W_s = \frac{\rho}{1-\rho} W_s$$

$$L_q = \lambda W_q = \frac{\lambda \rho}{1-\rho} W_s = \frac{\rho^2}{1-\rho}$$

$$L_s = L - L_q$$

P(server is busy) = 1 − P(server is empty)

$= 1 - P(N=0) = 1-(1-\rho)$

$= \rho$ → server utilization
(fraction of time server is busy)

arrival/service ratio $\quad \rho = \frac{\lambda}{\mu}$

Example Suppose that customers arrive at a Poisson rate of one per every 12 min, and that the service time is expo. at a rate of one service per 8 min. Find the parameters of M/M/1 system.

Sol

$$\lambda = \frac{1}{12} \quad ; \mu = \frac{1}{8}$$

$$\rho = \frac{\lambda}{\mu} = \frac{8}{12} = \frac{2}{3} < 1 \qquad \lambda_a = \lambda = \frac{1}{12}$$

$$L = \frac{\rho}{1-\rho} = \frac{2}{3} \times 3 = 2 \qquad W = \frac{L}{\lambda} = 2 \times 12 = 24 \text{ min}$$

$$L_q = \frac{\rho^2}{1-\rho} = \frac{4}{9} \times 3 = \frac{4}{3} \qquad W_q = W - W_s = 24 - 8 = 16 \text{ min}$$

$$W_s = \frac{1}{\mu} = 8 \text{ min}$$

$$L_s = L - L_q = 2 - \frac{4}{3} = \frac{2}{3}$$

$$P_0 = 1 - \rho = \frac{1}{3}$$

$$P_n = P(N=n) = C_n P_0 = \rho^n (1-\rho) = \left(\frac{2}{3}\right)^n \frac{1}{3} ; n = 0,1,2 \ldots$$

When there is N=0 no ~~customer~~ in queueing system then no queueing time

serve $W_q(0) = P(q=0) = P(N=0) = 1-\rho$

$N=n$, $q = S_1 + S_2 + \cdots + S_n$, $S_n \sim \text{IID } \exp(\mu)$

pdf

$$f_{q|N=n}(t) = \frac{\mu^n}{\Gamma n} e^{-\mu t} t^{n-1} , \quad t>0$$

For $n>0$

$$P(q \le t \mid N=n) = \int_0^t f_{q|N=n}(x)\, dx$$

$$P(0 < q \le t) = E\left(P(0 < q \le t \mid N)\right)$$

$$= \sum_{n=1}^{\infty} P(0 < q \le t \mid N=n) P(N=n)$$

$$= \sum_{n=1}^{\infty} \int_0^t \frac{\mu^n}{(n-1)!} e^{-\mu x} x^{n-1} \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n dx$$

$$= \int_0^t \lambda e^{-\mu x}\left(1-\frac{\lambda}{\mu}\right) \underbrace{\left(\sum_{n=1}^{\infty} \frac{(\lambda x)^{n-1}}{(n-1)!}\right)}_{e^{\lambda x}} dx$$

$$= \frac{\lambda}{\mu} \int_0^t (\mu - \lambda) e^{-(\mu-\lambda)x} dx$$

$$= \frac{\lambda}{\mu}\left(1-e^{-(\mu-\lambda)t}\right) = \rho\left[1-e^{-\mu(1-\rho)t}\right]$$

$$= \rho\left(1-e^{-t/W}\right)$$

For $t>0$

$$W_q(t) = P(q=0) + P(0<q\le t)$$

$$= 1-\rho+\rho\left(1-e^{-t/W}\right) = 1-\rho e^{-t/W}$$

Given $N=n$

$w = S_1+S_2+\cdots+S_{n+1} \sim \text{Gamma}(n+1,\mu)$ $\quad S_r \sim \text{iid } \exp(\mu)$

pdf

$$f_{w|N=n}(t) = \frac{\mu^{n+1}}{\Gamma(n+1)} e^{-\mu t}\, t^n, \quad t>0$$

$\frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1}$

$$W(t) = P(w\le t) = E\left(P(w\le t|N)\right)$$

$$= \sum_{n=0}^{\infty} P(w\le t|N=n)\,P(N=n)$$

$$= \sum_{n=0}^{\infty}\int_0^t \frac{\mu^{n+1}}{n!} e^{-\mu x} x^n \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n dx$$

$$= \int_0^t \mu e^{-\mu x}\left(1-\frac{\lambda}{\mu}\right)\underbrace{\left(\sum_{n=0}^{\infty}\frac{(\lambda x)^n}{n!}\right)}_{e^{\lambda x}} dx$$

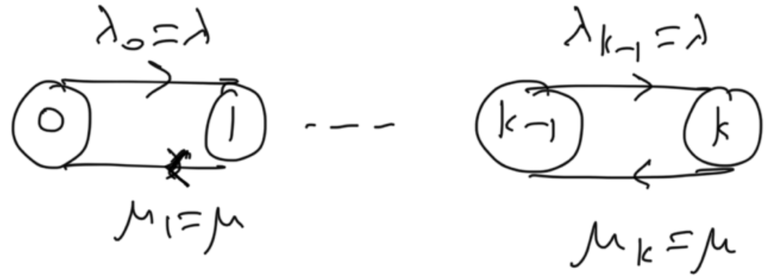
$$= \int_0^t (\mu-\lambda)\, e^{-(\mu-\lambda)x}\, dx$$

$$= 1-e^{-(\mu-\lambda)t} = 1-e^{-\mu(1-\rho)t} = 1-e^{-t/W}$$

$$w \sim \exp\left(\frac{1}{W}\right) \qquad E(w) = W \; ; \; \sigma_w^2 = W^2$$

$$s \sim \exp(\mu) \qquad E(s) = \frac{1}{\mu}$$

<u>M/M/1/K queueing system:</u>



B & D process

$$\lambda_n = \begin{cases} \lambda, & n = 0, 1, \ldots, k-1 \\ 0, & n = k, k+1, \ldots \end{cases}$$

$$\mu_n = \begin{cases} \mu, & n = 1, 2, \ldots, k, \\ 0, & n = k+1, k+2, \ldots \end{cases}$$

$$a = \frac{\lambda}{\mu}$$

✓ $P_n = C_n P_0 = a^n P_0 \; , \; n = 0, 1, 2, \ldots$

$$\sum_{n=0}^{k} P_n = 1 \Rightarrow P_0 + a P_0 + \cdots + a^k P_0 = 1$$

$$\frac{1 - a^{k+1}}{1-a} P_0 = 1 \Rightarrow P_0 = \frac{1-a}{1-a^{k+1}}$$

if $a \neq 1$

if $a = 1 \Rightarrow P_0 = \dfrac{1}{k+1}$

$$P(N=n) = P_n = \begin{cases} \dfrac{(1-a)\, a^n}{1-a^{k+1}} & \text{if } a \neq 1 \end{cases}$$

$$\Big\lfloor \frac{1}{k+1} \qquad \text{if } a=1$$

$a \neq 1$

$$L = \sum_{n=0}^{k} n P_n = \frac{(1-a)a}{1-a^{k+1}} \sum_{n=0}^{k} n a^{n-1}$$

$$\frac{d}{da} a^n$$

$$= \frac{(1-a)a}{1-a^{k+1}} \frac{d}{da} \sum_{n=0}^{k} a^n$$

$$\frac{1-a^{k+1}}{1-a}$$

$$= \frac{(1-a)a}{1-a^{k+1}} \times \left[\frac{-(1-a)(k+1)a^k + (1-a^{k+1})}{(1-a)^2} \right]$$

$$= \frac{a}{1-a} - \frac{(k+1)a^{k+1}}{1-a^{k+1}}$$

if $a=1$

$$L = \sum_{n=0}^{k} n \times \frac{1}{k+1} = \frac{k}{2}$$

$$a = \frac{\lambda}{\mu}$$

$$L = \begin{cases} \dfrac{a}{1-a} - \dfrac{(k+1)a^{k+1}}{1-a^{k+1}} & \text{if } a \neq 1 \\ \dfrac{k}{2} & \text{if } a=1 \end{cases}$$

$$L_s = E(N_s) = E\big(E(N_s \mid N)\big)$$

$$= E(N_s | N=0)\, P(N=0) + E(N_s | N>0)\, P(N>0)$$

$$= 0 \times P_0 + 1 \times (1-P_0) = 1 - P_0$$

$$L_q = L - L_s = L - (1-P_0)$$

$$P_{blocking} = P_k$$

$$\lambda_a = \lambda \times (1 - P_{blocking}) = \lambda \times (1-P_k)$$

$$W = \frac{L}{\lambda_a}, \quad W_q = \frac{W}{\lambda_a} \qquad W_s = \frac{1}{\mu} = \frac{a}{\lambda}$$

$$a = \frac{\lambda}{\mu}$$

true server utilization $= \rho = \lambda_a W_s = \lambda(1-P_k) W_s$

$$= (1-P_k)\, a$$

For $n = 0, 1, \ldots, k-1$

q_n prob. that an arriving customer who enters the system finds n customers already in the system

event A_n : there are n customers in the system

A : an arrival is about to occur

$$P(A|A_n) = \lambda h + o(h)$$

$$P(A) = \sum_{n=0}^{k-1} P(A|A_n)\, P(A_n)$$

$\lambda h + o(h)$

$$= (\lambda h + o(h)) \left(\sum^{k-1} P(A_n) \right)$$

$$n=0$$

$$1-P_k$$

$$= \left(\lambda h + o(h)\right)(1-P_k)$$

$$P(N_a=n) = q_n = P(A_n \mid A) = \frac{P(A \cap A_n)}{P(A)}$$

$$= \frac{P(A \mid A_n)\,P(A_n)}{P(A)}$$

$$= \frac{(\lambda h + o(h))\,P_n}{\left[\lambda h + o(h)\right](1-P_k)} = \frac{P_n}{1-P_k}, \quad n=0,1,\dots,k-1$$

N_a r.v. that counts # of customers in an $M/M/1/k$ system just before a customer arrives to enter the system

$$W(t) = P(w \le t) = E\left(P(w \le t \mid N_a)\right)$$

$$= \sum_{n=0}^{k-1} P(w \le t \mid N_a = n)\,P(N_a=n)$$

$$q_n$$

$$= \sum_{n=0}^{k-1} q_n \int_0^t \frac{\mu^{n+1}}{\Gamma(n+1)} e^{-\mu x} x^n \, dx$$

$$= 1 - \sum_{n=0}^{k-1} q_n \int_t^\infty \frac{\mu^{n+1}}{\Gamma(n+1)} e^{-\mu x} x^{n+1-1} \, dx$$

$$S_n > t \equiv N(t) \le n-1$$

$$N(t) \sim PP(\mu)$$

t

$S_n \sim \text{Gamma}(n$

$$= 1 - \sum_{n=0}^{k-1} q_n \underbrace{\sum_{k=0}^{n} \frac{e^{-\mu t}(\mu t)^k}{k!}}_{Q(n;\mu t)}$$

$$= 1 - \sum_{n=0}^{k-1} q_n \, Q(n;\mu t)$$

$$W_q(t) = P(q \le t)$$

$$= W_q(0) + \sum_{n=1}^{k-1} P(q \le t \mid N_a = n)\, q_n$$

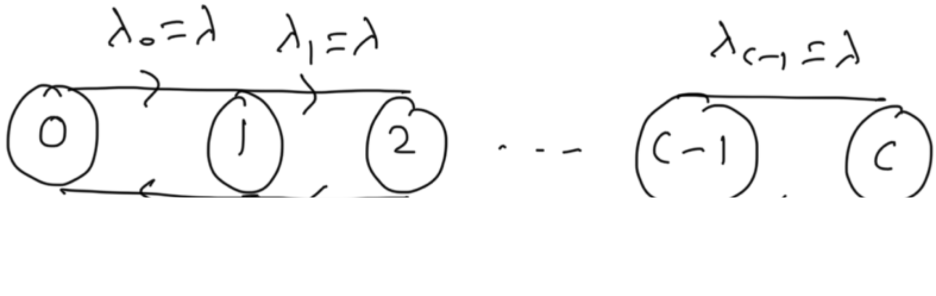
$$= q_0 + \sum_{n=1}^{k-1} q_n \int_0^t \frac{\mu^n}{\Gamma n} e^{-\mu x} x^{n-1}\, dx$$

$$= q_0 + \sum_{n=1}^{k-1} q_n \left[1 - \int_t^{\infty} \frac{\mu^n}{\Gamma n} e^{-\mu x} x^{n-1}\, dx \right]$$

$$= q_0 + (1 - q_0) - \sum_{n=1}^{k-1} q_n \, Q(n-1;\mu t)$$

$$= 1 - \sum_{n=0}^{k-2} q_{n+1} \, Q(n;\mu t)$$

M/M/c/c queuing system:



$\mu_1 = \mu \quad \mu_2 = 2\mu \qquad \qquad \mu_c = c\mu$

B& D process

$$C_n = \frac{a^n}{n!}, \quad n = 1, 2, \ldots c \quad ; \quad a = \frac{\lambda}{\mu}$$

$$W_s = \frac{1}{\mu}$$

$$S = \frac{1}{P_0} = 1 + C_1 + \cdots + C_c$$

$$= 1 + a + \frac{a^2}{2!} + \cdots + \frac{a^c}{c!}$$

$$P_n = C_n P_0 = \frac{a^n/n!}{1 + a + \frac{a^2}{2!} + \cdots + \frac{a^c}{c!}}, \quad n = 0, 1, \ldots, c$$

$$P_{blocking} = P_c$$

$$\lambda_a = \lambda \times (1 - P_c)$$

$$W_q = 0 \quad ; \quad L_q = 0$$

$$L = E(N) = \sum_{n=\cancel{0}\,1}^{c} n P_n = P_0 \sum_{n=1}^{c} n \frac{a^n}{n!} = a P_0 \sum_{n=1}^{c} \frac{a^{n-1}}{(n-1)!}$$

$$= a P_0 \sum_{n=0}^{c-1} \frac{a^n}{n!}$$

$$W(t) = W_s(t) = 1 - e^{-\mu t} = 1 - e^{-t/W_s}$$

M/M/c queueing system:

$$\lambda_0 = \lambda \quad \lambda_1 = \lambda \qquad \qquad \lambda_{c-1} = \lambda \quad \lambda_c = \lambda$$

$$(0) \rightarrow (1) \rightarrow (2) \cdots (c-1) \rightarrow (c) \rightarrow (c+1) \cdots$$

$\mu_1 = \mu \quad \mu_2 = 2\mu \qquad\qquad \mu_c = c\mu \quad \mu_{c+1} = c\mu$

$$\lambda_n = \lambda \ , \ n = 0, 1, 2, \ldots$$

B&D process

$$\mu_n = \begin{cases} n\mu \ , & n = 1, 2, \ldots, c \\ c\mu \ , & n = c+1, c+2, \ldots \end{cases}$$

$$a = \frac{\lambda}{\mu} \qquad \rho = \frac{a}{c}$$

$$C_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$$

$$C_n = \begin{cases} \dfrac{a^n}{n!} & , n = 1, 2, \ldots, c \\ \dfrac{\rho^{n-c} a^c}{c!} & , n = c+1, c+2, \ldots \end{cases}$$

$$S = \frac{1}{P_0} = 1 + C_1 + C_2 + \cdots$$

$$= 1 + a + \frac{a^2}{2!} + \cdots + \frac{a^{c-1}}{(c-1)!} + \frac{a^c}{c!}\left[1 + \rho + \rho^2 + \cdots\right]$$

$$= \sum_{n=0}^{c-1} \frac{a^n}{n!} + \frac{a^c}{c!(1-\rho)}$$

$$P_0 = \frac{1}{S}$$

$$P_n = C_n P_0 = \begin{cases} \dfrac{a^n}{n!} P_0 & , n = 0, 1, \ldots, c \\ \dfrac{\rho^{n-c} a^c}{c!} P_0 & , n = c+1, c+2, \ldots \end{cases}$$

$$C[c, a] = P(N \geq c) = \sum_{n=c}^{\infty} P_n = 1 - \sum_{n=0}^{c-1} P_n$$

Erlang C formula

$$= 1 - P_0 \sum_{n=0}^{c-1} \frac{a^n}{n!} = \frac{a^c}{c!(1-\rho)} P_0$$

$$L_q = \sum_{n=c}^{\infty} (n-c) P_n = P_0 \frac{a^c}{c!} \sum_{n=c}^{\infty} (n-c) \rho^{n-c}$$

$$= P_0 \frac{a^c}{c!} \rho \sum_{k=0}^{\infty} k \rho^{k-1} \quad \to \frac{d}{d\rho} \rho^k$$

$$= P_0 \frac{a^c}{c!} \rho \frac{d}{d\rho} \sum_{k=0}^{\infty} \rho^k \quad \to \frac{1}{1-\rho}$$

$$= P_0 \frac{a^c}{c!} \rho \frac{1}{(1-\rho)^2} = \frac{P_0 a^c \rho}{c! (1-\rho)^2} = C[c,a] \frac{\rho}{1-\rho}$$

$$\lambda_a = \lambda \times 1 = \lambda$$

$$W_q = \frac{L_q}{\lambda} \quad , W_s = \frac{1}{\mu}$$

$$W = W_q + W_s \quad , L = \lambda W$$

$$W_q(0) = P(q=0) = P(N<c) = 1 - P(N \geq c)$$

$$= 1 - C[c,a] \quad \checkmark$$

$q \mid N=n \sim \text{Gamma}(n-c+1, c\mu)$

$$W_q(t) = W_q(0) + P(0 < q \leq t)$$

$$= W_q(0) + \sum_{n=c}^{\infty} P(q \leq t \mid N=n) P_n$$

$$= W_q(0) + \sum_{n=c}^{\infty} \left(\int_0^t \frac{(c\mu)^{n-c+1}}{\Gamma(n-c+1)} e^{-c\mu x} x^{n-c+1-1} dx \right) \frac{P_0 a^n}{c! c^{n-c}}$$

$$= W_q(0) + \frac{P_0 a^c}{(c-1)!} \int_0^t \mu e^{-c\mu x} \underbrace{\left(\sum_{n=c}^{\infty} \frac{(a\mu x)^{n-c}}{(n-c)!} \right)}_{\to\, e^{a\mu x}} dx$$

$$= W_q(0) + \frac{P_0 a^c}{(c-1)!} \left(\underbrace{\int_0^t \mu(c-a)\, e^{-\mu x(c-a)}\, dx}_{1-e^{-\mu t(c-a)}} \right) \times \frac{1}{c-a}$$

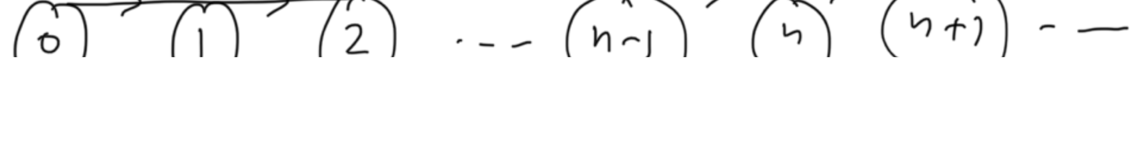
$$= 1 - C[c,a] + \underbrace{\frac{P_0 a^c}{(c-a)(c-1)!}}_{C[c,a]} \left(1 - e^{-\mu t(c-a)}\right)$$

$$= 1 - C[c,a]\, e^{-\mu t(c-a)}$$

$$W(t) = P(w \le t)$$

$$= \begin{cases} 1 - \dfrac{a-c+W_q(0)}{a+1-c} e^{-\mu t} - \dfrac{C[c,a]}{a+1-c} e^{-c\mu t(1-\rho)} & \text{if } a \ne c-1 \\[2ex] 1 - (1 + C[c,a]\mu t) e^{-\mu t} & \text{if } a = c-1 \end{cases}$$

M/M/∞ queueing system:



$\mu_1 = \mu \quad \mu_2 = 2\mu \qquad\qquad \mu_n = n\mu \quad \mu_{n+1} = (n+1)\mu$

$$C_n = \frac{a^n}{n!}, \quad n = 1, 2, \ldots \qquad a = \frac{\lambda}{\mu}$$

$$S = \frac{1}{P_0} = 1 + C_1 + C_2 + \cdots = 1 + a + \frac{a^2}{2!} + \cdots$$

$$= e^a$$

$$P_n = C_n P_0 = \frac{e^{-a} a^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$L = E(N) = a, \quad \sigma_N^2 = a$$

$$L_q = 0, \quad W_q = 0$$

$$L = L_s = a \qquad W = W_s = \frac{1}{\mu}$$

$$W(t) = W_s(t) = 1 - e^{-\mu t}$$

Example: Calls in a telepone system arrive randomly at an exchange at the rate of 140 per hr. If there is very large number of lines available to handle the calls, that last an average of 3 min, what is the av. number of lines in use?

Sol $M/M/\infty$

$$\lambda = 140 \text{ per hr} = \frac{140}{60} \text{ per min}$$

$\mu = \frac{1}{3}$ per min

$$L_s = a = \frac{\lambda}{\mu} = \frac{140}{60} \times 3 = 7$$

av. # of lines in use is 7.

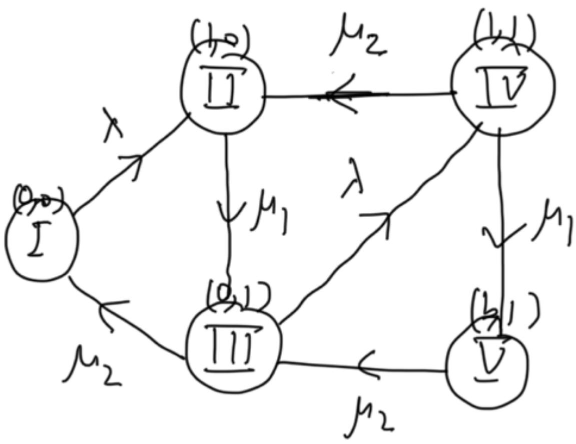
Example Shoeshine shop (Extension)

exp(μ_1) exp(μ_2)

C_1 C_2

$\square \xrightarrow{PP(\lambda)}$

	State	
I	(0,0)	C_1 ✗ C_2 ✗
II	(1,0)	C_1 ✓ C_2 ✗
III	(0,1)	C_1 ✗ C_2 ✓
IV	(1,1)	C_1 ✓ C_2 ✓
V	(b,1)	C_1 ✓b C_2 ✓



$\nu_I = \lambda$, $\nu_{II} = \mu_1$, $\nu_{III} = \lambda + \mu_2$

$\nu_{IV} = \mu_1 + \mu_2$, $\nu_V = \mu_2$ indep $\left\langle \begin{array}{l} Y_2 \sim \exp \mu_2 \\ X \sim \exp(\lambda) \end{array} \right.$

$P_{I,II} = 1$, $P_{II,III} = 1$, $P_{III,I} = P(Y_2 \le X)$

$$P_{III,IV} = \frac{\lambda}{\lambda + \mu_2}$$

$$= \int P(X \ge y)\, \mu_2 e^{-\mu_2 x}\, dx$$

$$= \int_0^\infty e^{-\lambda y} \mu_2 e^{-\mu_2 x}\, dx$$

$$P_{IV, II} = \frac{\mu_2}{\mu_1 + \mu_2} \qquad \boxed{= \frac{\mu_2}{\lambda + \mu_2}}$$

$$P_{IV, V} = \frac{\mu_1}{\mu_1 + \mu_2}, \quad P_{V, III} = 1$$

$$q_{I, II} = \nu_I P_{I, II} = \lambda \times 1$$

$$q_{II, III} = \mu_1, \quad q_{III, I} = \cancel{(\lambda + \mu_2)} \times \frac{\mu_2}{\cancel{\lambda + \mu_2}} = \mu_2$$

$$q_{III, IV} = \lambda, \quad q_{IV, V} = \mu_1, \quad q_{V, III} = \mu_2$$

$$Q = \begin{array}{c} \\ I \\ II \\ III \\ IV \\ V \end{array} \begin{array}{c} \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ 0 & -\mu_1 & \mu_1 & 0 & 0 \\ \mu_2 & 0 & -(\lambda + \mu_2) & \lambda & 0 \\ 0 & \mu_2 & 0 & -(\mu_1 + \mu_2) & \mu_1 \\ 0 & 0 & \mu_2 & 0 & -\mu_2 \end{bmatrix} \end{array}$$

balance equation

$$\tilde{\Pi} Q = 0$$

$$\Pi_I + \Pi_{II} + \Pi_{III} + \Pi_{IV} + \Pi_V = 1$$

proportion of customers entering the system $= \Pi_I + \Pi_{III}$

L = av # of customers in the system

$$= \Pi_{II} + \Pi_{III} + 2(\Pi_{IV} + \Pi_V)$$

$$\lambda_a = \lambda \times (\Pi_{I} + \Pi_{III})$$

$$W = \frac{L}{\lambda_a}$$

—x—