Chapter 6

1. Let us assume that the state is (n, m). Male i mates at a rate λ with female j, and therefore it mates at a rate λm. Since there are n males, matings occur at a rate λnm. Therefore,

   $$v_{(n,m)} = \lambda nm$$

   Since any mating is equally likely to result in a female as in a male, we have

   $$P_{(n,m);(n+1,m)} = P_{(n,m)(n,m+1)} = \frac{1}{2}$$

2. Let $N_A(t)$ be the number of organisms in state A and let $N_B(t)$ be the number of organisms in state B. Then clearly $\{N_A(t); N_B(t)\}$ is a continuous Markov chain with

   $$v_{\{n,m\}} = \alpha n + \beta m$$
   $$P_{\{n,m\};\{n-1;m+1\}} = \frac{\alpha n}{\alpha n + \beta m}$$
   $$P_{\{n,m\};\{n+2;m-1\}} = \frac{\beta m}{\alpha n + \beta m}$$

3. This is not a birth and death process since we need more information than just the number working. We also must know which machine is working. We can analyze it by letting the states be

   b : both machines are working
   1 : 1 is working, 2 is down
   2 : 2 is working, 1 is down
   0_1: both are down, 1 is being serviced
   0_2: both are down, 2 is being serviced

   $$v_b = \mu_1 + \mu_2,\ v_1 = \mu_1 + \mu,\ v_2 = \mu_2 + \mu,$$
   $$v_{0_1} = v_{0_2} = \mu$$
   $$P_{b,1} = \frac{\mu_2}{\mu_2 + \mu_1} = 1 - P_{b,2}, \quad P_{1,b} = \frac{\mu}{\mu + \mu_1}$$
   $$= 1 - P_{1,0_2}$$
   $$P_{2,b} = \frac{\mu}{\mu + \mu_2} = 1 - P_{2,0_1}, \quad P_{0_1,1} = P_{0_2,2} = 1$$

4. Let $N(t)$ denote the number of customers in the station at time t. Then $\{N(t)\}$ is a birth and death process with

   $$\lambda_n = \lambda\alpha_n, \quad \mu_n = \mu$$

5. (a) Yes.
   (b) It is a pure birth process.
   (c) If there are i infected individuals then since a contact will involve an infected and an uninfected individual with probability $i(n-i)/\binom{n}{2}$, it follows that the birth rates are $\lambda_i = \lambda i(n-i)/\binom{n}{2}$, $i = 1, \ldots, n$. Hence,

   $$E[\text{time all infected}] = \frac{n(n-1)}{2\lambda}\sum_{i=1}^{n} 1/[i(n-i)]$$

6. Starting with $E[T_0] = \dfrac{1}{\lambda_0} = \dfrac{1}{\lambda}$, employ the identity

   $$E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}E\left[T_{i-1}\right]$$

   to successively compute $E[T_i]$ for $i = 1, 2, 3, 4$.

   (a) $E[T_0] + \cdots + E[T_3]$
   (b) $E[T_2] + E[T_3] + E[T_4]$

7. (a) Yes!
   (b) For $n = (n_1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1})$ let

   $$S_i(n) = (n_1, \ldots, n_{i-1}, n_{i+1} + 1, \ldots, n_{k-1}), \quad i = 1, \ldots, k-2$$
   $$S_{k-1}(n) = (n_1, \ldots, n_i, n_{i+1}, \ldots n_{k-1} - 1),$$
   $$S_0(n) = (n_1 + 1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1})$$

   Then

   $$q_n, S_1(n) = n_i\mu, \qquad i = 1, \ldots, k-1$$
   $$q_n, S_0(n) = \lambda$$

8. The number of failed machines is a birth and death process with

   $$\lambda_0 = 2\lambda \qquad \mu_1 = \mu_2 = \mu$$
   $$\lambda_1 = \lambda \qquad \mu_n = 0, n \neq 1, 2$$

$\lambda_n = 0, n > 1.$

Now substitute into the backward equations.

9. Since the death rate is constant, it follows that as long as the system is nonempty, the number of deaths in any interval of length t will be a Poisson random variable with mean μt. Hence,

$$P_{ij}(t) = e^{-\mu t}(\mu t)^{i-j}/(i-j)!, \quad 0 < j \le i$$

$$P_{i,0}(t) = \sum_{k=i}^{\infty} e^{-\mu t}(\mu t)^k/k!$$

10. Let $I_j(t) = \begin{cases} 0, & \text{if machine } j \text{ is working at time } t \\ 1, & \text{otherwise} \end{cases}$

Also, let the state be $(I_1(t), I_2(t))$.

This is clearly a continuous-time Markov chain with

$$v_{(0,0)} = \lambda_1 + \lambda_2 \;\; \lambda_{(0,0);\,(0,1)} = \lambda_2 \;\; \lambda_{(0,0);\,(1,0)} = \lambda_1$$

$$v_{(0,1)} = \lambda_1 + \mu_2 \;\; \lambda_{(0,1);\,(0,0)} = \mu_2 \;\; \lambda_{(0,1);\,(1,1)} = \lambda_1$$

$$v_{(1,0)} = \mu_1 + \lambda_2 \;\; \lambda_{(1,0);\,(0,0)} = \mu_1 \;\; \lambda_{(1,0);\,(1,1)} = \lambda_2$$

$$v_{(1,1)} = \mu_1 + \mu_2 \;\; \lambda_{(1,1);\,(0,1)} = \mu_1 \;\; \lambda_{(1,1);\,(1,0)} = \lambda_2$$

By the independence assumption, we have

(a) $P_{(i,j)(k,\ell)}(t) = P_{(i,k)}(t)Q_{(j,\ell)}(t)$

where $P_{i,k}(t)$ = probability that the first machine be in state k at time t given that it was at state i at time 0.

$Q_{j,\ell}(t)$ is defined similarly for the second machine. By Example 4(c) we have

$$P_{00}(t) = [\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1]/(\lambda_1 + \mu_1)$$

$$P_{10}(t) = [\mu_1 - \mu_1 e^{-(\mu_1+\lambda_1)t}]/(\lambda_1 + \mu_1)$$

And by the same argument,

$$P_{11}(t) = [\mu_1 e^{-(\mu_1+\lambda_1)t} + \lambda_1]/(\lambda_1 + \mu_1)$$

$$P_{01}(t) = [\lambda_1 - \lambda_1 e^{-(\mu_1+\lambda_1)t}]/(\lambda_1 + \mu_1)$$

Of course, the similar expressions for the second machine are obtained by replacing (λ_1, μ_1) by (λ_2, μ_2). We get $P_{(i,j)(k,\ell)}(t)$ by formula (a). For instance,

$$\begin{aligned} P_{(0,0)(0,0)}(t) &= P_{(0,0)}(t)Q_{(0,0)}(t) \\ &= \frac{\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]}{(\lambda_1 + \mu_1)} \times \frac{\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_2 + \mu_2)} \end{aligned}$$

Let us check the forward and backward equations for the state $\{(0, 0);\ (0, 0)\}$.

Backward equation

We should have

$$P'_{(0,0),(0,0)}(t) = (\lambda_1 + \lambda_2)\left[\frac{\lambda_2}{\lambda_1+\lambda_2}P_{(0,1)(0,0)}(t) + \frac{\lambda_1}{\lambda_1+\lambda_2}P_{(1,0)(0,0)}(t) - P_{(0,0)(0,0)}(t)\right]$$

or

$$P'_{(0,0)(0,0)}(t) = \lambda_2 P_{(0,1)(0,0)}(t) + \lambda_1 P_{(1,0)(0,0)}(t) - (\lambda_1 + \lambda_2)\,P_{(0,0)(0,0)}(t)$$

Let us compute the right-hand side ($r.h.s.$) of this expression:

$r.h.s.$

$$\begin{aligned}
&= \lambda_2 \frac{\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]\left[\mu_2 - \mu_2 e^{-(\mu_2+\lambda_2)t}\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&\quad + \frac{\left[\mu_1 - \mu_1 e^{-(\lambda_1+\mu_1)t}\right]\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&\quad - (\lambda_1 + \lambda_2) \\
&\quad \frac{\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&= \frac{\lambda_2\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&\quad \times \left[\mu_2 - \mu_2 e^{-(\mu_2+\lambda_2)t} - \lambda_2 e^{-(\lambda_2+\mu_2)t} - \mu_2\right] \\
&\quad + \frac{\lambda_1\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&\quad \times \left[\mu_1 - \mu_1 e^{-(\mu_1+\lambda_1)t} - \mu_1 - \lambda_1 e^{-(\lambda_1+\mu_1)t}\right] \\
&= \left[-\lambda_2 e^{-(\lambda_2+\mu_2)t}\right]\left[\frac{\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1}{\lambda_1+\mu_1}\right] \\
&\quad + \left[-\lambda_1 e^{-(\lambda_1+\mu_1)t}\right]\left[\frac{\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2}{\lambda_2+\mu_2}\right] \\
&= Q'_{00}(t)P_{00}(t) + P'_{00}(t)Q_{00}(t) = [P_{00}(t)Q_{00}(t)]' \\
&= \left[P_{(0,0)(0,0)}(t)\right]'
\end{aligned}$$

So, for this state, the backward equation is satisfied.

Forward equation

According to the forward equation, we should now have

$$P'_{(0,\,0)(0,\,0)}(t) = \mu_2 P_{(0,\,0)(0,\,1)}(t) + \mu_1 P_{(0,\,0)(1,\,0)}(t) - (\lambda_1 + \lambda_1)P_{(0,\,0)(0,\,0)}(t)$$

Let us compute the right-hand side:

r.h.s.

$$= \mu_2 \frac{\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]\left[\lambda_2 - \lambda_2 e^{-(\lambda_2+\mu_2)t}\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$+ \mu_1 \frac{\left[\lambda_1 - \lambda_1 e^{-(\lambda_1+\mu_1)t}\right]\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda 2+\mu_2)}$$

$$-(\lambda_1+\lambda_2)\frac{\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]\left[\lambda_2 e^{-(\mu_2+\lambda_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$= \frac{\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]}{(\lambda_1+\mu_1)} \times \frac{\left[\mu_2\lambda_2 - \lambda_2 e^{-(\lambda_2+\mu_2)t} - \lambda_2\left[\lambda_2 e^{-(\mu_2+\lambda_2)t} + \mu_2\right]\right]}{\lambda_2+\mu_2}$$

$$+ \frac{\left[\lambda_2 e^{-(\mu_2+\lambda_2)t} + \mu_2\right]}{(\lambda_2+\mu_2)} \times \frac{\left[\mu_1\left[\lambda_1 - \lambda_1 e^{-(\lambda_1+\mu_1)t}\right] - \lambda_1\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]\right]}{(\lambda_1+\mu_1)}$$

$$= P_{00}(t)\left[-\lambda_2 e^{-(\mu_2+\lambda_2)t}\right] + Q_{00}(t)\left[-\lambda_1 e^{-(\lambda_1+\mu_1)t}\right]$$

$$= P_{00}(t)Q'_{00}(t) + Q_{00}(t)P'00(t) = \left[P_{(0,0)(0,0)}(t)\right]$$

In the same way, we can verify Kolmogorov's equations for all the other states.

11. (b) Follows from the hint upon using the lack of memory property and the fact that ϵ_i, the minimum of $j-(i-1)$ independent exponentials with rate λ, is exponential with rate $(j-i+1)\lambda$.

(c) From (a) and (b)

$$P\{T_1 + \cdots + T_j \le t\} = P\left\{\max_{1\le i\le j} X_i \le t\right\} = (1-e^{-\lambda t})^j$$

(d) With all probabilities conditional on $X(0) = 1$

$$\begin{aligned} P_{1j}(t) &= P\{X(t) = j\} \\ &= P\{X(t) \ge j\} - P\{X(t) \ge j+1\} \\ &= P\{T_1 + \cdots + T_j \le t\} \\ &\quad - P\{T_1 + \cdots + T_{j+1} \le t\} \end{aligned}$$

(e) The sum of independent geometrics, each having parameter $p = e^{-\lambda t}$, is negative binomial with parameters i, p. The result follows since starting with an initial population of i is equivalent to having i independent Yule processes, each starting with a single individual.

12. (a) If the state is the number of individuals at time t, we get a birth and death process with

$$\lambda_n = n\lambda + \theta, \qquad n < N$$
$$\lambda_n = n\lambda, \qquad n \ge N$$
$$\mu_n = n\mu$$

(b) Let P_i be the long-run probability that the system is in state i. Since this is also the proportion of time the system is in state i, we are looking for $\sum_{i=3}^{\infty} P_i$.

We have $\lambda_k P_k = \mu_{k+1}P_{k+1}$.

This yields

$$P_1 = \frac{\theta}{\mu}P_0$$

$$P_2 = \frac{\lambda+\theta}{2\mu}P_1 = \frac{\theta(\lambda+\theta)}{2\mu^2}P_0$$

$$P_3 = \frac{2\lambda+\theta}{2\mu}P_2 = \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{6\mu^3}P_0$$

For $k \ge 4$, we get

$$P_k = \frac{(k-1)\lambda}{k\mu}P_{k-1}$$

which implies

$$P_k = \frac{(k-1)(k-2)\cdots(3)}{(k)(k-1)\cdots(4)}\left[\frac{\lambda}{\mu}\right]^{k-3}$$

$$P_k = \frac{3}{k}\left[\frac{\lambda}{\mu}\right]^{k-3}P_3$$

therefore $\displaystyle\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3 \sum_{k=3}^{\infty}\frac{1}{k}\left[\frac{\lambda}{\mu}\right]^k,$

but $$\sum_{k=1}^{\infty} \frac{1}{k}\left[\frac{\lambda}{\mu}\right]^k = \log\left[\frac{1}{1-\frac{\lambda}{\mu}}\right]$$

$$= \log\left[\frac{\mu}{\mu-\lambda}\right] \text{ if } \frac{\lambda}{\mu} < 1$$

So $$\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3\left[\log\left[\frac{\mu}{\mu-\lambda}\right]\right.$$

$$\left. -\frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]$$

$$\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]$$
$$\frac{\theta(\lambda+\theta)(2\lambda+\theta)}{6\mu^3}P_0$$

Now $\sum_0^{\infty} P_i = 1$ implies

$$P_0 = \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2} + \frac{1}{2\lambda^3}\theta(\lambda+\theta)(2\lambda+\theta)\right.$$
$$\left.\times\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right]^{-1}$$

And finally,

$$\sum_{k=3}^{\infty} P_k = \left[\left[\frac{1}{2\lambda^3}\right]\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right.$$
$$\left.\theta(\lambda+\theta)(2\lambda+\theta)\right] \Big/ \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2}\right.$$
$$+\frac{\theta(\lambda+\theta)(2\lambda+\theta)}{2\lambda^3}$$
$$\left.\times\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right]$$

13. With the number of customers in the shop as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = 3, \quad \mu_1 = \mu_2 = 4$$

Therefore

$$P_1 = \frac{3}{4}P_0, \quad P_2 = \frac{3}{4}, \quad P_1 = \left[\frac{3}{4}\right]^2 P_0$$

And since $\sum_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{16}{37}$$

(a) The average number of customers in the shop is

$$P_1 + 2P_2 = \left[\frac{3}{4} + 2\left[\frac{3}{4}\right]^2\right] P_0$$
$$= \frac{30}{16}\left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{30}{37}$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1-P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{16}\cdot\frac{16}{37} = \frac{28}{37}$$

(c) Now $\mu = 8$, and so

$$P_0 = \left[1 + \frac{3}{8} + \left[\frac{3}{8}\right]^2\right]^{-1} = \frac{64}{97}$$

So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left[\frac{3}{8}\right]^2 \frac{264}{97} = 1 - \frac{9}{97} = \frac{88}{97}$$

The rate of added customers is therefore

$$\lambda\left[\frac{88}{97}\right] - \lambda\left[\frac{28}{37}\right] = 3\left[\frac{88}{97} - \frac{28}{37}\right] = 0.45$$

The business he does would improve by 0.45 customers per hour.

14. Letting the number of cars in the station be the state, we have a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 20, \quad \lambda_i = 0,\ i > 2$$

$$\mu_1 = \mu_2 = 12$$

Hence,

$$P_1 = \frac{5}{3}P_0,\ P_2 = \frac{5}{3}P_1 = \left[\frac{5}{3}\right]^2 P_0$$
$$P_3 = \frac{5}{3}P_2 = \left[\frac{5}{3}\right]^3 P_0$$

and as $\sum_0^3 P_i = 1$, we have

$$P_0 = \left[1 + \frac{5}{3} + \left[\frac{5}{3}\right]^2 + \left[\frac{5}{3}\right]^2\right]^{-1} = \frac{27}{272}$$

(a) The fraction of the attendant's time spent servicing cars is equal to the fraction of time there are cars in the system and is therefore $1 - P_0 = 245/272$.

(b) The fraction of potential customers that are lost is equal to the fraction of customers that arrive when there are three cars in the station and is therefore

$$P_3 = \left[\frac{5}{3}\right]^3 P_0 = 125/272$$

15. With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \ \lambda_i = 0, \quad i \geq 4$$
$$\mu_1 = 2, \ \mu_2 = \mu_3 = 4$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0, \ P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0, \ P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0$$

And therefore,

$$P_0 = \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right]^{-1} = \frac{32}{143}$$

(a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1-P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \times \frac{32}{143} = \frac{116}{143}$$

(b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0 \ P_2 = \frac{3}{4}P_1 = \left[\frac{3}{4}\right]^2 P_0 \ P_3 = \left[\frac{3}{4}\right]^3 P_0$$

and $P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2 + \left[\frac{3}{4}\right]^3\right]^{-1} = \frac{64}{175}$

So that now

$$1 - P_3 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = \frac{148}{175}$$

16. Let the state be

0: an acceptable molecule is attached
1: no molecule attached
2: an unacceptable molecule is attached.

Then this is a birth and death process with balance equations

$$P_{12} = \frac{\mu}{\lambda}P_0$$
$$P_2 = \frac{\lambda(1-\alpha)}{\mu_1}P_1 = \frac{(1-\alpha)}{\alpha}\frac{\mu_2}{\mu_1}P_0$$

Since $\sum_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{\mu_2}{\lambda\alpha} + \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1}\right]^{-1}$$
$$= \frac{\lambda\alpha\mu_1}{\lambda\alpha\mu_1 + \mu_1\mu_2 + \lambda(1-\alpha)\mu_2}$$

P_0 is the percentage of time the site is occupied by an acceptable molecule.

The percentage of time the site is occupied by an unacceptable molecule is

$$P_2 = \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1}P_0 = \frac{\lambda(1-\alpha)\mu_2}{\lambda\alpha\mu_1 + \mu_1 + \lambda(1-\alpha)\mu_2}$$

17. Say the state is 0 if the machine is up, say it is i when it is down due to a type i failure, $i = 1, 2$. The balance equations for the limiting probabilities are as follows.

$$\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$\mu_1 P_1 = \lambda p P_0$$
$$\mu_2 P_2 = \lambda(1-p)P_0$$
$$P_0 + P_1 + P_2 = 1$$

These equations are easily solved to give the results

$$P_0 = (1 + \lambda p/\mu_1 + \lambda(1-p)/\mu_2)^{-1}$$
$$P_1 = \lambda p P_0/\mu_1, \qquad P_2 = \lambda(1-p)P_0/\mu_2$$

18. There are $k+1$ states; state 0 means the machine is working, state i means that it is in repair phase i, $i = 1, \ldots, k$. The balance equations for the limiting probabilities are

$$\lambda P_0 = \mu_k P_k$$
$$\mu_1 P_1 = \lambda P_0$$
$$\mu_i P_i = \mu_{i-1} P_{i-1}, \quad i = 2, \ldots, k$$
$$P_0 + \cdots + P_k = 1$$

To solve, note that

$$\mu_i P_i = \mu_{i-1} P_{i-1} = \mu_{i-2} P_{i-2} = \cdots = \lambda P_0$$

Hence,

$$P_i = (\lambda/\mu_i)P_0$$

and, upon summing,

$$1 = P_0\left[1 + \sum_{i=1}^{k}(\lambda/\mu_i)\right]$$

Therefore,

$$P_0 = \left[1 + \sum_{i=1}^{k}(\lambda/\mu_i)\right]^{-1}, \quad P_i = (\lambda/\mu_i)P_0,$$

$i = 1, \ldots, k$

The answer to part (a) is P_i and to part (b) is P_0.

19. There are 4 states. Let state 0 mean that no machines are down, state 1 that machine 1 is down and 2 is up, state 2 that machine 1 is up and 2 is down, and 3 that both machines are down. The balance equations are as follows:

$$\begin{aligned}
(\lambda_1 + \lambda_2)P_0 &= \mu_1 P_1 + \mu_2 P_2 \\
(\mu_1 + \lambda_2)P_1 &= \lambda_1 P_0 + \mu_1 P_3 \\
(\lambda_1 + \mu_2)P_2 &= \lambda_2 P_0 \\
\mu_1 P_3 &= \mu_2 P_1 + \mu_1 P_2
\end{aligned}$$

$$P_0 + P_1 + P_2 + P_3 = 1$$

These equations are easily solved and the proportion of time machine 2 is down is $P_2 + P_3$.

20. Letting the state be the number of down machines, this is a birth and death process with parameters

$$\lambda_i = \lambda, \quad i = 0, 1$$
$$\mu_i = \mu, \quad i = 1, 2$$

By the results of Example 3g, we have

$$E[\text{time to go from 0 to 2}] = 2/\lambda + \mu/\lambda^2$$

Using the formula at the end of Section 3, we have

$$\begin{aligned}
&Var(\text{time to go from 0 to 2}) \\
&= Var(T_0) + Var(T_1) \\
&= \frac{1}{\lambda^2} + \frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda^3} + \frac{\mu}{\mu+\lambda}(2/\lambda + \mu/\lambda^2)^2
\end{aligned}$$

Using Equation (5.3) for the limiting probabilities of a birth and death process, we have

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2}$$

21. How we have a birth and death process with parameters

$$\lambda_i = \lambda, \quad i = 1, 2$$
$$\mu_i = i\mu, \quad i = 1, 2$$

Therefore,

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2/2}$$

and so the probability that at least one machine is up is higher in this case.

22. The number in the system is a birth and death process with parameters

$$\lambda_n = \lambda/(n+1), \quad n \ge 0$$
$$\mu_n = \mu, \quad n \ge 1$$

From Equation (5.3),

$$1/P_0 = 1 + \sum_{n=1}^{\infty}(\lambda/\mu)^n/n! = e^{\lambda/\mu}$$

and

$$P_n = P_0(\lambda/\mu)^n/n! = e^{-\lambda/\mu}(\lambda/\mu)^n/n!, \quad n \ge 0$$

23. Let the state denote the number of machines that are down. This yields a birth and death process with

$$\lambda_0 = \frac{3}{10}, \ \lambda_1 = \frac{2}{10}, \ \lambda_2 = \frac{1}{10}, \ \lambda_i = 0, \quad i \ge 3$$
$$\mu_1 = \frac{1}{8}, \ \mu_2 = \frac{2}{8}, \ \mu_3 = \frac{2}{8}$$

The balance equations reduce to

$$P_1 = \frac{3/10}{1/8}P_0 = \frac{12}{5}P_0$$

$$P_2 = \frac{2/10}{2/8}P_1 = \frac{4}{5}P_1 = \frac{48}{25}P_0$$

$$P_3 = \frac{1/10}{2/8}P_2 = \frac{4}{10}P_3 = \frac{192}{250}P_0$$

Hence, using $\sum_0^3 P_i = 1$ yields

$$P_0 = \left[1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250}\right]^{-1} = \frac{250}{1522}$$

(a) Average number not in use

$$= P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}$$

(b) Proportion of time both repairmen are busy

$$= P_2 + P_3 = \frac{672}{1522} = \frac{336}{761}$$

24. We will let the state be the number of taxis waiting. Then, we get a birth and death process with $\lambda_n = 1\mu_n = 2$. This is a $M/M/1$, and therefore,

(a) Average number of taxis waiting $= \frac{1}{\mu - \lambda}$

$$= \frac{1}{2-1} = 1$$

(b) The proportion of arriving customers that get taxis is the proportion of arriving customers that find at least one taxi waiting. The rate of arrival of such customers is $2(1 - P_0)$. The proportion of such arrivals is therefore

$$\frac{2(1-P_0)}{2} = 1 - P_0 = 1 - \left[1 - \frac{\lambda}{\mu}\right] = \frac{\lambda}{\mu} = \frac{1}{2}$$

25. If $N_i(t)$ is the number of customers in the ith system $(i = 1, 2)$, then let us take $\{N_1(t), N_2(t)\}$ as the state. The balance equation are with $n \geq 1, m \geq 1$.

(a) $\lambda P_{0,\,0} = \mu_2 P_{0,\,1}$

(b) $P_{n,\,0}(\lambda + \mu_1) = \lambda P_{n-1,\,0} + \mu_2 P_{n,\,1}$

(c) $P_{0,\,m}(\lambda + \mu_2) = \mu_1 P_{1,\,m-1} + \mu_2 P_{0,\,m+1}$

(d) $P_{n,\,m}(\lambda + \mu_1 + \mu_2) = \lambda P_{n-1,\,m} + \mu_1 P_{n+1,\,m-1} + \mu_2 P_{n,\,m+1}$

We will try a solution of the form $C\alpha^n\beta^m = P_{n,\,m}$. From (a), we get

$$\lambda C = \mu_2 C\beta = \beta = \frac{\lambda}{\mu_2}$$

From (b),

$$(\lambda + \mu_1)\, C\alpha^n = \lambda C\alpha^{n-1} + \mu_2 C\alpha^n\beta$$

or

$$(\lambda + \mu_1)\,\alpha = \lambda + \mu_2\alpha\beta = \lambda + \mu_2\alpha\frac{\lambda}{\mu} = \lambda + \lambda\alpha$$

and $\mu_1\alpha = \lambda \Rightarrow \alpha = \frac{\lambda}{\mu_1}$

To get C, we observe that $\sum_{n,\,m} P_{n,\,m} = 1$

but

$$\sum_{n,\,m} P_{n,\,m} = C\sum_n \alpha^n \sum_m \beta^m = C\left[\frac{1}{1-\alpha}\right]\left[\frac{1}{1-\beta}\right]$$

and $C = \left[1 - \frac{\lambda}{\mu_1}\right]\left[1 - \frac{\lambda}{\mu_2}\right]$

Therefore a solution of the form $C\alpha^n\beta^n$ must be given by

$$P_{n,\,m} = \left[1 - \frac{\lambda}{\mu_1}\right]\left[\frac{\lambda}{\mu_1}\right]^n\left[1 - \frac{\lambda}{\mu_2}\right]\left[\frac{\lambda}{\mu_2}\right]^m$$

It is easy to verify that this also satisfies (c) and (d) and is therefore the solution of the balance equations.

26. Since the arrival process is Poisson, it follows that the sequence of future arrivals is independent of the number presently in the system. Hence, by time reversibility the number presently in the system must also be independent of the sequence of past departures (since looking backwards in time departures are seen as arrivals).

27. It is a Poisson process by time reversibility. If $\lambda > \delta\mu$, the departure process will (in the limit) be a Poisson process with rate $\delta\mu$ since the servers will always be busy and thus the time between departures will be independent random variables each with rate $\delta\mu$.

28. Let P^x_{ij}, V^x_i denote the parameters of the $X(t)$ and P^y_{ij}, V^y_i of the $Y(t)$ process; and let the limiting probabilities be P^x_i, P^y_i, respectively. By independence we have that for the Markov chain $\{X(t), Y(t)\}$ its parameters are

$$V_{(i,\,\ell)} = V^x_i + V^y_\ell$$

$$P_{(i,\,\ell),(j,\,\ell)} = \frac{V^x_i}{V^x_i + V^y_\ell}P^x_{ij}$$

$$P_{(i,\,\ell),\,(i,\,k)} = \frac{V^y_\ell}{V^x_i + V^y_\ell}P^y_{\ell k}$$

and

$$\lim_{t\to\infty} P\{(X(t), Y(t)) = (i, j)\} = P^x_i P^y_j$$

Hence, we need show that

$$P^x_i P^y_\ell V^x_i P^x_{ij} = P^x_j P^y_\ell V^x_j P^x_{ji}$$

(That is, rate from $(i,\,\ell)$ to $(j,\,\ell)$ equals the rate from $(j,\,\ell)$ to $(i,\,\ell)$). But this follows from the fact that the rate from i to j in $X(t)$ equals the rate from j to i; that is,

$$P^x_i V^x_i P^x_{ij} = P^x_j V^x_j P^x_{ji}$$

The analysis is similar in looking at pairs $(i,\,\ell)$ and (i, k).