5)

To Prove: f is the eigenvector of C with second highest eigenvalue.

Proof:

For minimising the reconstruction error of N vectors projected on e and f directions, following constraint is satisfied: ~~where~~

$$J(e,f) = f^T C f - \lambda_1 (f^T f - 1) - \lambda_2 f^T e$$

(where λ_1 and λ_2 are lagrange multipliers)

$$\frac{dJ}{\partial f} = 0$$

$$Cf - \lambda_1 f - \lambda_2 e = 0 \quad — \quad \star$$

premultiplying e^T

$$e^T C f - \lambda_1 e^T f - \lambda_2 = 0 \quad (e^T e = 1)$$

$$-(1)$$

we know $Ce = \lambda e$

pre and post multiply by e^T

$$e^T C = \lambda e^T$$

$$e^T C f = \lambda e^T f \quad -(2)$$

Since $e \perp f$, $e^T f = 0$

From ②, $e^T C f = \lambda e^T f = 0$ — ③

Putting ③ in ① & using $e^T f = 0$,

$$\lambda_2 = 0$$

Using $\lambda_2 = 0$ in ⊛,

$$Cf = \lambda_1 f$$

Which implies f is eigenvector of C.
Since all non zero eigenvalues of C are distinct and e is eigen~~value~~ vector with highest eigenvalue, f must be eigenvector with second highest eigenvalue.