6) (a) $y^T P y = y^T A^T A y$

$= (Ay)^T A y$

$=$ dot product of vector Ay with itself

≥ 0

$z^T Q z = z^T A A^T z$

$= (A^T z)^T A^T z$

$=$ dot product of vector $A^T z$ with itself

≥ 0

Let y be eigenvector of P with eigenvalue λ,

$Py = \lambda y$

$\underbrace{y^T P y}_{\text{zero or positive}} = \lambda \|y\|^2$

$\Rightarrow \lambda$ should be positive or zero

Similarly

$Qz = \lambda z$

$\underbrace{z^T Q z}_{\text{zero / positive}} = \lambda \|z\|^2$

$\Rightarrow \lambda$ should be zero/positive

(b)

$$Pu = \lambda u$$

$$A^TAu = \lambda u$$

premultiply A

$$\underbrace{AA^T}Au = \lambda Au$$

$$QAu = \lambda Au$$

$\Rightarrow$ Au is eigenvector of Q with eigenvalue λ

$$Qv = \mu v$$

$$AA^Tv = \mu v$$

$$\underbrace{A^TA}A^Tv = \mu A^Tv$$

$$PA^Tv = \mu A^Tv$$

$\Rightarrow$ A^Tv is eigenvector of P with eigenvalue μ.

$$\left.\begin{array}{l} A : m \times n \\ P = A^TA : n \times n \\ Q = AA^T : m \times m \end{array}\right\} \Rightarrow \begin{array}{l} u: n\text{-dimensional vector} \\ v: m\text{-dimensional vector} \end{array}$$

(c) $$A u_i = A \frac{A^T v_i}{\|A^T v_i\|_2}$$

$$= \frac{Q v_i}{\|A^T v_i\|_2} \qquad (\because AA^T = Q)$$

$$= \frac{\lambda_i v_i}{\|A^T v_i\|_2} \qquad (\because Q v_i = \lambda_i v_i)$$

$$= \gamma_i v_i \qquad \text{where } \gamma_i = \frac{\lambda_i}{\|A^T v_i\|_2}$$

(d) $U = [v_1 | v_2 | \dots | v_m]$

$V = [u_1 | u_2 | \dots | u_m]$

$A = U \Gamma V^T$

premultiply by U^T and postmultiply by V

$U^T A V = \Gamma$ (since U and V are orthogonal

$\because u_i^T u_j = \begin{cases} 0, & i \neq j \\ 1 & i = j \end{cases}$

similarly for v) $\frac{(A^T v_i)^T A^T v_i}{\|A^T v_i\|_2^2} = 1$

Now since $\Gamma = U^T A V$,

$$\Gamma_{i,j} = v_i^T A u_j$$

$$= v_i^T A \left(\frac{A^T v_j}{\|A^T v_j\|_2} \right) \quad \left(\because u_i = \frac{A^T v_i}{\|A^T v_i\|_2} \right)$$

$$= \frac{v_i^T Q v_j}{\|A^T v_j\|_2} \quad (\because Q = AA^T)$$

$$= \frac{\lambda_j v_i^T v_j}{\|A^T v_j\|_2} \quad (\because Q v_j = \lambda_j v_j)$$

$$= \gamma_j \underbrace{v_i^T v_j} \quad \left(\because \gamma_i = \frac{\lambda_i}{\|A^T v_i\|_2} \right) \quad \text{positive}$$

$$= \begin{cases} \gamma_i & , \text{if } i = j \\ 0 & , \text{if } i \neq j \end{cases} \quad (\text{also } \gamma_i \geq 0)$$

$\Rightarrow$ $A = U \Gamma V^T$ with Γ being diagonal matrix with non-negative values $\gamma_1, \gamma_2 \dots \gamma_m$