UNIT-2

Group

A group G denoted by {G, •}, is a set under some operations (•) if it satisfies the CAIN properties.

- C - Closure
- A - Associative
- I - Identity
- N - iNverse.

Abelian Group

A group is said to be Abelian if it already a group and Commutative property is also satisfied i.e. $(a \bullet b) = (b \bullet a)$ for all a, b in G.

Group and Abelian Group

Property			Explanation
Abelian Group	Group	A1 - Closure	$a, b \in G$, then $(a \bullet b) \in G$.
		A2 - Associative	$a \bullet (b \bullet c) = (a \bullet b) \bullet c$ for all $a, b, c \in G$.
		A3 - Identity element	$(a \bullet e) = (e \bullet a) = a$ for all $a, e \in G$.
		A4 - Inverse element	$(a \bullet a') = (a' \bullet a) = e$ for all $a, a' \in G$.
	A5 - Commutative		$(a \bullet b) = (b \bullet a)$ for all $a, b \in G$.

Cyclic Group

A group G denoted by {G, •}, is said to be a cyclic group, if it contains at-least one generator element.

Cyclic Group

Question 1: Prove that (G, *) is a cyclic group, where $G = \{1, \omega, \omega^2\}$.

Solution:

Composition Table

*	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$1^1 = 1$
$1^2 = 1*1 = 1$
$1^3 = 1*1*1 = 1$
$1^4 = 1*1*1*1 = 1$

$\omega^1 = \omega$
$\omega^2 = \omega*\omega = \omega^2$
$\omega^3 = \omega^2*\omega = 1$
$\omega^4 = \omega^3*\omega = \omega$

$(\omega^2)^1 = \omega^2$
$(\omega^2)^2 = \omega^4 = \omega^3*\omega = \omega$
$(\omega^2)^3 = \omega^6 = \omega^3*\omega^3 = 1$
$(\omega^2)^4 = \omega^8 = \omega^3*\omega^3*\omega^2 = \omega^2$

Cyclic Group

Question 2: When does group G with operation 'x', is said to be a cyclic group?

Solution:

Let us take an element x

$G = \{ \ldots, x^{-4}, x^{-3}, x^{-2}, x^{-1}, 1, x, x^2, x^3, x^4, \ldots \}$

= Group generated by x

If $G = \langle x \rangle$ for some x, then we call G a cyclic group.

Cyclic Group

Question 3: When does group G with operation '+', is said to be a cyclic group?

Solution:

Let us take an element y

$G = \{ \ldots, -4y, -3y, -2y, -y, 0, y, 2y, 3y, 4y, \ldots \}$

$=$ Group generated by y

If $G = \langle y \rangle$ for some y, then we call G a cyclic group.

Rings

A ring R denoted by {R, +, *}, is a set of elements with two binary operations, called addition and multiplication, such that for all $a, b, c \in R$ the following axioms are obeyed:

- Group (A1-A4), Abelian Group(A5).
- Closure under multiplication (M1): If $a, b \in R$ then $ab \in R$
- Associativity of multiplication (M2): $a(bc) = (ab)c$ for all $a, b, c \in R$
- Distributive laws (M3) :

$$a(b + c) = ab + ac \text{ for all } a, b, c \in R$$

$$(a + b)c = ac + bc \text{ for all } a, b, c \in R$$

Commutative Rings

A ring is said to be commutative, if it satisfies the following additional condition:

Commutativity of multiplication (M4): $ab = ba$ for all $a, b \in R$

Integral Domain

An integral domain is a commutative ring that obeys the following axioms:

Multiplicative identity (M5): There is an element $1 \in R$ such that $a1 = 1a = a$ for all $a \in R$.

No zero divisors (M6): If $a, b \in R$ and $ab = 0$, then either $a = 0$ or $b = 0$.

Fields

A field F , sometimes denoted by {F, +,* }, is a set of elements with two binary operations, called addition and multiplication, such that for all a, b, c $\in$ F the following axioms are obeyed:

(A1-M6): F is an integral domain; that is, F satisfies axioms A1 - A5 and M1 - M6.

(M7) Multiplicative inverse: For each a in F, except 0, there is an element a^{-1} in F such that

$$aa^{-1} = (a^{-1})a = 1$$

Note: $a/b = a(b^{-1})$.

Familiar examples of Fields:

- Rational numbers
- Real numbers

Groups, Rings and Fields

	Group	Abelian Group	Ring	Commutative Ring	Integral Domain	Field
A1 - Closure						
A2 - Associative						
A3 - Identity element						
A4 - Inverse element						
A5 - Commutativity of Addition						
M1 – Closure under multiplication						
M2 – Associativity of multiplication						
M3 – Distributive						
M4 - Commutativity of multiplication						
M5 - Multiplicative Identity						
M6 - No Zero Divisors						
M7 - Multiplicative Inverse						

Finite Fields

- A finite field or Galois field (so-named in honor of Évariste Galois) is a field that contains a finite number of elements.
- As with any field, a finite field is a set on which the operations of multiplication, addition, subtraction and division are defined and satisfy certain basic rules.
- The most common examples of finite fields are given by the integers (mod p) when p is a prime number.

Application areas:

- Mathematics and computer science - Number theory, Algebraic geometry, Galois theory, Finite geometry, Cryptography and Coding theory.

Prime Numbers

- Prime Numbers: Has exactly two divisors.
- If 'N' is prime, then the divisors are 1 and N.
- All numbers have prime factors.

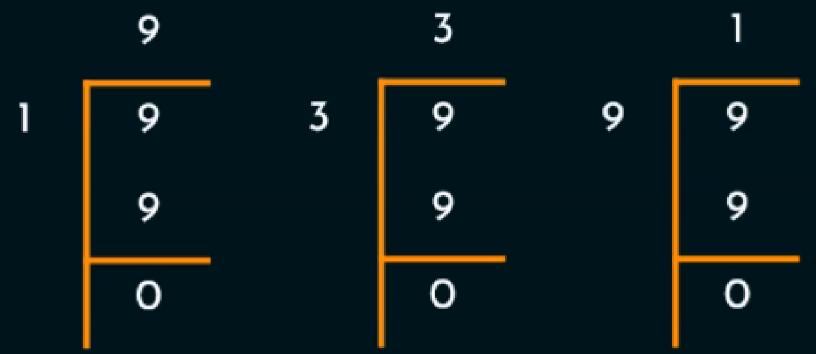
Numbers	10	11	100	37	308	14688
Prime Factorization	$2^1 \times 5^1$	$1^1 \times 11^1$	$2^2 \times 5^2$	$1^1 \times 37^1$	$2^2 \times 7^1 \times 11^1$	$2^5 \times 3^3 \times 17^1$
Prime Numbers	2, 5	1, 11	2, 5	1, 37	2, 7, 11	2, 3, 17

Prime Numbers

A prime number is a number greater than 1 with only two factors - itself and one. It cannot be divided further by any other numbers without leaving a remainder.

Prime Numbers – Example

- ★ 2 is a prime number.
- ★ 3 is a prime number.
- ★ 5 is a prime number.
- ★ 7 is a prime number.
- ★ 9 is not a prime number.
- ★ 9 is a composite number.
- ★ 33 is a composite number.



Divisors of 9: 1, 3 and 9

Facts about primes

- ★ Only even prime : 2
- ★ Smallest prime number : 2
- ★ Is 1 a prime number? No.
- ★ Except for 2 and 5, all prime numbers end in the digit 1, 3, 7 or 9.

Modular Arithmetic

- ★ System of arithmetic for integers.
- ★ Wrap around after reaching a certain value called modulus.
- ★ Central mathematical concept in cryptography.

Congruence

★ In cryptography, congruence(≡) instead of equality(=).

Examples:

$15 \equiv 3 \pmod{12}$

$23 \equiv 11 \pmod{12}$

$33 \equiv 3 \pmod{10}$

$$
\begin{array}{r|l} & 1 \\ \hline 12 & 15 \\ & 12 \\ \hline & 3 \end{array}
\qquad
\begin{array}{r|l} & 1 \\ \hline 12 & 23 \\ & 12 \\ \hline & 11 \end{array}
\qquad
\begin{array}{r|l} & 3 \\ \hline 10 & 33 \\ & 30 \\ \hline & 3 \end{array}
$$

$10 \equiv -2 \pmod{12}$

$$
\begin{array}{r|l} & 0 \\ \hline 12 & 10 \\ & 0 \\ \hline & 10 \end{array}
$$

Valid or Invalid

- ★ $38 \equiv 2 \pmod{12}$
- ★ $38 \equiv 14 \pmod{12}$

$5 \equiv 0 \pmod{5}$

$10 \equiv 2 \pmod{6}$

$13 \equiv 3 \pmod{13}$

$2 \equiv -3 \pmod{5}$

Properties of Modular Arithmetic

1. $[(a \bmod n) + (b \bmod n)] \bmod n = (a + b) \bmod n$
2. $[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$
3. $[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$

Properties of Modular Arithmetic

Property	Expression
Commutative Laws	$(a + b) \bmod n = (b + a) \bmod n$ $(a \times b) \bmod n = (b \times a) \bmod n$
Associative Laws	$[(a + b) + c] \bmod n = [a + (b + c)] \bmod n$ $[(a \times b) \times c] \bmod n = [a \times (b \times c)] \bmod n$
Distributive Laws	$[a \times (b + c)] \bmod n = [(a \times b) + (a \times c)] \bmod n$
Identities	$(0 + a) \bmod n = a \bmod n$ $(1 \times a) \bmod n = a \bmod n$
Additive Inverse	For each $a \in Z_n$, there exists a '$-a$' such that $a + (-a) \equiv 0 \bmod n$

Modular Exponentiation

- It is a type of exponentiation performed over a modulus.
- $a^b \bmod m$ or $a^b \pmod{m}$.

Examples:

$2^{33} \bmod 30$

Example 1

Solve 23^3 mod 30.

23^3 mod 30 $= -7^3$ mod 30 || 23 mod 30 can be 23 or -7.

$= -7^3$ mod 30

$= -7^2 \times -7$ mod 30

$= 49 \times -7$ mod 30

$= -133$ mod 30

$= -13$ mod 30

$= 17$ mod 30

23^3 mod 30 $= 17$

Example 2

Solve 31^{500} mod 30.

$$31^{500} \bmod 30 = 1^{500} \bmod 30$$
$$= 1 \bmod 30$$
$$= 1$$
$$31^{500} \bmod 30 = 1$$

$11^7 \bmod 13 = (-2)^7 \bmod 13$

Solve 242^{329} mod 243.

242^{329} mod 243 = -1^{329} mod 243

= -1^{329} mod 243 || -1^{328} x -1^{1}

= -1 mod 243

= 242

242^{329} mod 243 = 242

Understanding GCD – Example 2

	25	150
Divisors	1, 5, 25	1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, 150
Common Divisors	1, 5, 25	
Greatest Common Divisor (GCD)		

Euclid's Algorithm for finding GCD

Find the GCD(12, 33).

Q	A	B	R

Q	A	B	R
2	33	12	9
1	12	9	3
3	9	3	0
X	$\boxed{3}$	0	X

Euclid's Algorithm for finding GCD

Find the GCD(750, 900).

Q	A	B	R

Euclid's Algorithm for finding GCD

Find the GCD(750, 900).

Q	A	B	R
1	900	750	150
5	750	150	0
X	$\boxed{150}$	0	X

Euclid's Algorithm for finding GCD

Find the GCD(252, 105).

Q	A	B	R
	252	105	

Euclid's Algorithm for finding GCD

Prerequisite: a > b

```
Euclid_GCD (a, b):
        if b = 0 then
                return a;
        else
                return Euclid_GCD (b, a mod b);
```

Euclid's Algorithm – Example 1

Example 1: Find the GCD (50, 12).

Solution:

Here a=50, b=12

GCD (a, b) = GCD (b, a mod b)

GCD (50, 12) = GCD (12, 50 mod 12) = GCD(12, 2)

GCD (12, 2) = GCD (2, 12 mod 2) = GCD(2, 0) = 2

GCD (50, 12) = 2

Euclid's Algorithm – Example 2

Example 2: Find the GCD (83, 19).

Solution:

Here a=83, b=19

GCD (a, b)	= GCD (b, a mod b)	
GCD (83, 19)	= GCD (19, 83 mod 19)	= GCD(19, 7)
GCD (19, 7)	= GCD (7, 19 mod 7)	= GCD(7, 5)
GCD (7, 5)	= GCD (5, 7 mod 5)	= GCD(5, 2)
GCD (5, 2)	= GCD (2, 5 mod 2)	= GCD(2, 1)
GCD (2, 1)	= GCD (1, 2 mod 1)	= GCD(1, 0) = 1

Relatively Prime Numbers

Two numbers are said to be relatively prime, if they have no prime factors in common, and their only common factor is 1.

- ❖ If GCD(a, b) = 1 then 'a' and 'b' are relatively prime numbers.

Relatively Prime Numbers

Question 1: Are 4 and 13 relatively prime?

Solution:

	4	13
Divisors	1, 2, 4	1, 13
Common Divisors	1	
Greatest Common Divisor (GCD)	1	

GCD(4, 13) = 1

Yes, 4 and 13 are relatively prime numbers.

Relatively Prime Numbers

Question 2: Are 15 and 21 relatively prime?

Multiplicative Inverse

$5 \times 5^{-1} = 1$

$5 \times \frac{1}{5} = 1$

$A \times \frac{1}{A} = 1$

$A \times A^{-1} = 1$

Multiplicative Inverse

Under mod n

$A \times A^{-1} \equiv 1 \bmod n$

$3 \times ? \equiv 1 \bmod 5$

$3 \times 2 \equiv 1 \bmod 5$

$2 \times ? \equiv 1 \bmod 11$

$2 \times 6 \equiv 1 \bmod 11$

$4 \times ? \equiv 1 \bmod 5$

$4 \times 4 \equiv 1 \bmod 5$

Multiplicative Inverse using EEA

Q	A	B	R	T_1	T_2	T

Points to Ponder

$A > B$

$$\begin{array}{r|l} & Q \\ \hline B & A \\ \hline & R \end{array}$$

$T_1 = 0$ and $T_2 = 1$

$T = T_1 - T_2 \times Q$

$\boxed{T_1 \text{ is the M.I.}}$

Multiplicative Inverse using EEA

Example 1: What is the multiplicative inverse of 3 mod 5.

Q	A	B	R	T_1	T_2	T
1	5	3	2	0	1	-1
1	3	2	1	1	-1	2
2	2	1	0	-1	2	-5
X	1	0	X	$\boxed{2}$	-5	X

$\therefore$ 2 is the M.I of 3 mod 5.

Multiplicative Inverse using EEA

Example 3: Find the M.I of 11 mod 26.

Q	A	B	R	T_1	T_2	T
2	26	11	4	0	1	-2
2	11	4	3	1	-2	5
1	4	3	1	-2	5	-7
3	3	1	0	5	-7	26
X	1	0	X	$\boxed{-7}$	26	X

The Chinese Remainder Theorem

The Chinese Remainder Theorem (CRT) is used to solve a set of different congruent equations with one variable but different moduli which are relatively prime as shown below:

$X \equiv a_1 \pmod{m_1}$

$X \equiv a_2 \pmod{m_2}$

. . .

$X \equiv a_n \pmod{m_n}$

CRT states that the above equations have a unique solution of the moduli are relatively prime.

$$\boxed{X = (a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} + \ldots + a_n M_n M_n^{-1}) \bmod M}$$

The Chinese Remainder Theorem

Example 1: Solve the following equations using CRT

$X \equiv 2 \pmod 3$

$X \equiv 3 \pmod 5$

$X \equiv 2 \pmod 7$

$X \equiv a_1 \pmod{m_1}$	$X \equiv 2 \pmod 3$
$X \equiv a_2 \pmod{m_2}$	$X \equiv 3 \pmod 5$
$X \equiv a_3 \pmod{m_3}$	$X \equiv 2 \pmod 7$

Solution:

$$X = (a_1M_1\,M_1^{-1} + a_2M_2M_2^{-1} + a_3M_3M_3^{-1}) \bmod M$$

$$X = (a_1M_1\,M_1^{-1} + a_2M_2M_2^{-1} + a_3M_3M_3^{-1}) \bmod M$$

Given		To Find		
$a_1 = 2$	$m_1 = 3$	M_1	M_1^{-1}	M
$a_2 = 3$	$m_2 = 5$	M_2	M_2^{-1}	
$a_3 = 2$	$m_3 = 7$	M_3	M_3^{-1}	

Given		To Find		
$a_1 = 2$	$m_1 = 3$	M_1	M_1^{-1}	$M=105$
$a_2 = 3$	$m_2 = 5$	M_2	M_2^{-1}	
$a_3 = 2$	$m_3 = 7$	M_3	M_3^{-1}	

$M = m_1 \times m_2 \times m_3$

$$M_1 = \frac{M}{m_1}$$
$$M_1 = \frac{105}{3}$$
$$M_1 = 35$$

$$M_2 = \frac{M}{m_2}$$
$$M_2 = \frac{105}{5}$$
$$M_2 = 21$$

$$M_3 = \frac{M}{m_3}$$
$$M_3 = \frac{105}{7}$$

$$M_1 \times M_1^{-1} = 1 \bmod m_1$$
$$35 \times M_1^{-1} = 1 \bmod 3$$
$$35 \times 2 = 1 \bmod 3$$
$$M_1^{-1} = 2$$

$$M_2 \times M_2^{-1} = 1 \bmod m_2$$
$$21 \times M_2^{-1} = 1 \bmod 5$$
$$21 \times 1 = 1 \bmod 5$$
$$M_2^{-1} = 1$$

$$M_3 \times M_3^{-1} = 1 \bmod m_3$$
$$15 \times M_3^{-1} = 1 \bmod 7$$
$$15 \times 1 = 1 \bmod 7$$
$$M_3^{-1} = 1$$

The Chinese Remainder Theorem

Example 1: Solve the following equations using CRT

$X \equiv 2 \pmod{3}$

$X \equiv 3 \pmod{5}$

$X \equiv 2 \pmod{7}$

$a_1 = 2$	$m_1 = 3$	$M_1 = 35$	$M_1^{-1} = 2$	
$a_2 = 3$	$m_2 = 5$	$M_2 = 21$	$M_2^{-1} = 1$	$M=105$
$a_3 = 2$	$m_3 = 7$	$M_3 = 15$	$M_3^{-1} = 1$	

Solution:

$X = (a_1M_1\, M_1^{-1} + a_2M_2M_2^{-1} + a_3M_3M_3^{-1}) \bmod M$

$= (2x35x2 + 3x21x1 + 2x15x1) \bmod 105$

$= 233 \bmod 105$

$X = 23$

Euler's Totient Function

- Denoted as $\Phi(n)$.
- $\Phi(n)$ = Number of positive integers less than 'n' that are relatively prime to n.

Euler's Totient Function

Example 1: Find $\Phi(5)$.

Solution:

Here n=5.

Numbers less than 5 are 1, 2, 3 and 4.

GCD	Relatively Prime?
GCD (1, 5) = 1	✓
GCD (2, 5) = 1	✓
GCD (3, 5) = 1	✓
GCD (4, 5) = 1	✓

Euler's Totient Function

Example 2: Find $\Phi(11)$.

Solution:

Here n=11.

Numbers less than 11 are 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

GCD	Relatively Prime?
GCD (1, 11) = 1	✓
GCD (2, 11) = 1	✓
GCD (3, 11) = 1	✓
GCD (4, 11) = 1	✓
GCD (5, 11) = 1	✓

GCD	Relatively Prime?
GCD (6, 11) = 1	✓
GCD (7, 11) = 1	✓
GCD (8, 11) = 1	✓
GCD (9, 11) = 1	✓
GCD (10, 11) = 1	✓

Euler's Totient Function

Example 3: Find $\Phi(8)$.

Solution:

Here n=8.

Numbers less than 8 are 1, 2, 3, 4, 5, 6, and 7.

GCD	Relatively Prime?
GCD (1, 8) = 1	✓
GCD (2, 8) = 2	✗
GCD (3, 8) = 1	✓
GCD (4, 8) = 4	✗

GCD	Relatively Prime?
GCD (5, 8) = 1	✓
GCD (6, 8) = 2	✗
GCD (7, 8) = 1	✓

Euler's Totient Function

	Criteria of 'n'	Formula
$\Phi(n)$	'n' is prime.	$\Phi(n) = (n\text{-}1)$
	$n = p \times q$. 'p' and 'q' are primes.	$\Phi(n) = (p\text{-}1) \times (q\text{-}1)$
	$n = a \times b$. Either 'a' or 'b' is composite. Both 'a' and 'b' are composite.	$\Phi(n) = n \times \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots$ where $p_1, p_2, \dots$ are distinct primes.

Euler's Totient Function

Example 2: Find Φ(31).

Solution:

Here n=31.

'n' is a prime number.

$\Phi(n) = (n-1)$

$\Phi(31) = (31-1)$

$\Phi(31) = 30$

So, there are 30 numbers that are lesser than 31 and relatively prime to 31.

Example 3: Find $\Phi(35)$.

Solution:

Here n=35.

'n' is a product of two prime numbers 5 and 7.

Let us assign p=5 and q=7.

$$\Phi(n) = (p-1) \times (q-1)$$

$$\Phi(35) = (5-1) \times (7-1)$$

$$\Phi(35) = 4 \times 6$$

$$\Phi(35) = 24$$

So, there are 24 numbers that are lesser than 35 and relatively prime to 35.

Example 4: Find Φ(1000).

Solution:

Here $n = 1000 = 2^3 \times 5^3$.

Distinct prime factors are 2 and 5.

$$\Phi(n) = n \times \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots$$

$$\Phi(1000) = 1000 \times \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)$$

$$\Phi(1000) = 1000 \times \left(\frac{1}{2}\right)\left(\frac{4}{5}\right)$$

$$\Phi(1000) = 400$$

Example 5: Find $\Phi(7000)$.

Solution:

Here $n = 7000 = 2^3 \times 5^3 \times 7^1$

Distinct prime factors are 2, 5 and 7.

$$\Phi(n) \quad = n \times \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)\ldots.$$

$$\Phi(7000) = 7000 \times \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)$$

$$\Phi(7000) = 7000 \times \left(\frac{1}{2}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)$$

$$\Phi(7000) = 2400$$

Euler's Theorem

For every positive integer 'a' & 'n', which are said to be relatively prime, then $a^{\Phi(n)} \equiv 1 \bmod n$.

Example 1: Prove Euler's theorem hold true for a=3 and n=10.

Solution:

Given: a=3 and n=10.

$a^{\Phi(n)} \equiv 1 \pmod{n}$

$3^{\Phi(10)} \equiv 1 \pmod{10}$

$\Phi(10) = 4$

$3^4 \equiv 1 \pmod{10}$

$81 \equiv 1 \pmod{10}$

Therefore, Euler's theorem holds true for a=3 and n=10.

Example 2: Does Euler's theorem hold true for a=2 and n=10?

Solution:

Given: a=2 and n=10.

$a^{\Phi(n)} \equiv 1 \pmod{n}$

$2^{\Phi(10)} \equiv 1 \pmod{10}$

$\Phi(10) = 4$

$2^4 \equiv 1 \pmod{10}$

$16 \equiv 1 \pmod{10}$

Therefore, Euler's theorem does not hold for a=2 and n=10.

Example 3: Does Euler's theorem hold true for a=10 and n=11?

Solution:

Given: a=10 and n=11.

$$a^{\Phi(n)} \equiv 1 \pmod{n}$$

$$10^{\Phi(11)} \equiv 1 \pmod{11}$$

$$\Phi(11) = 10$$

$$10^{10} \equiv 1 \pmod{11}$$

$$-1^{10} \equiv 1 \pmod{11}$$

$$1 \equiv 1 \pmod{11}$$

Fermat's Little Theorem

If 'p' is a prime number and 'a' is a positive integer not divisible by 'p' then $a^{p-1} \equiv 1 \pmod{p}$

Example 1: Does Fermat's theorem hold true for p=5 and a=2?

Solution:

Given: p=5 and a=2.

$a^{p-1} \equiv 1 \pmod{p}$

$2^{5-1} \equiv 1 \pmod{5}$

$2^4 \equiv 1 \pmod{5}$

$16 \equiv 1 \pmod{5}$

Therefore, Fermat's theorem holds true for p=5 and a=2.

Example 3: Prove Fermat's theorem does not hold for p=6 and a=2.

Solution:

$a^{p-1} \equiv 1 \pmod{p}$

$2^{6-1} \equiv 1 \pmod{6}$

$2^{5} \equiv 1 \pmod{6}$

$32 \equiv 1 \pmod{6}$

$32 \equiv 1 \pmod{6}$

Therefore, Fermat's theorem does not hold true for p=6 and a=2.

Question: Does Fermat's theorem hold true for the prime number 11 with the integer 5?

Fermat's Primality Test

Is 'p' prime?

Test:

$a^p - a \longrightarrow$ 'p' is prime if this is a multiple of 'p' for all $1 \leq a < p$.

Example

Question 1: Is 5 prime?

Solution:

$a^p - a \longrightarrow$ 'p' is prime if this is a multiple of 'p' for all $1 \leq a < p$.

$1^5 - 1 \quad = 1 - 1 \quad = 0$

$2^5 - 2 \quad = 32 - 2 \quad = 30$

$3^5 - 3 \quad = 243 - 3 \quad = 240$

$4^5 - 4 \quad = 1024 - 4 \quad = 1020$

$\therefore$ 5 is prime

Example

Question 2: Is 3753 prime?

Solution:

$a^p - a \longrightarrow$ 'p' is prime if this is a multiple of 'p' for all $1 \leq a < p$

$1^{3753} - 1$

$2^{3753} - 2$

$3^{3753} - 3$

$4^{3753} - 4$

...

Question: Is 561 prime?

Miller–Rabin Primality Test

- Miller-Rabin primality test or Rabin-Miller primality test.
- Probabilistic primality test.
- Similar to Fermat primality test and the Solovay-Strassen primality test.
- Checks whether a specific property, which is known to hold for prime values, holds for the number under testing.

Miller–Rabin Primality Test

Algorithm

Step 1: Find $n-1 = 2^k \times m$

Step 2: Choose 'a' such that $1<a<n-1$

Step 3: Compute $b_0 = a^m \pmod{n}, \ldots, b_i = b^2_{i-1} \pmod{n}$

$+1 \longrightarrow$ Composite

$-1 \longrightarrow$ Probably Prime

Example

Question: Is 561 prime?

Solution:

Given n = 561.

Step 1:

$n-1 = 2^k \times m$

$560 = 2^4 \times 35$

So $k = 4$, and $m = 35$

$\frac{560}{2^1} = 280$ | $\frac{560}{2^2} = 140$ | $\frac{560}{2^3} = 70$ | $\frac{560}{2^4} = 35$ | $\frac{560}{2^5} = 17.5$

Step 2:

Choosing $a = 2$; $1<2<560$

Step 3:

Compute $b_0 = a^m \pmod{n}$

$b_0 = a^m \pmod{n}$

Compute $b_0 = a^m \pmod{n}$

$b_0 = a^m \pmod{n}$

$b_0 = 2^{35} \pmod{561} = 263$

Is $b_0 = \pm 1 \pmod{561}$? NO

So calculate b_1

$b_1 = b_0^2 \pmod{n}$

$b_1 = 263^2 \pmod{561}$

$b_1 = 166$

Is $b_1 = \pm 1 \pmod{561}$? NO

$b_2 = b_1^2 \pmod{n}$

$b_2 = 166^2 \pmod{561}$

$b_2 = 67$

Is $b_2 = \pm 1 \pmod{561}$? NO

$b_3 = b_2^2 \pmod{n}$

$b_3 = 67^2 \pmod{561}$

$b_3 = 1 \longrightarrow$ Composite