Quantum Computing Assignment 1 Solutions

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Key Insight: The given 4×4 matrices can be expressed as the tensor product of two smaller 2×2 matrices. Using the property of tensor products and eigenvalues, we can deduce the eigenvalues of these matrices efficiently. Let A and B be two matrices with eigenvalues λ and μ, respectively. Then, $A \otimes B$ has eigenvalues $\lambda\mu$.

Part (a)

$$\begin{bmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues of $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ are $\frac{7 \pm \sqrt{57}}{2}$, and the eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are -1 and 1. Combining these:

$$\text{Eigenvalues: } \{-\frac{7+\sqrt{57}}{2}, \frac{7+\sqrt{57}}{2}, -\frac{7-\sqrt{57}}{2}, \frac{7-\sqrt{57}}{2}\}$$

Part (b) Applying column and row swaps to the matrix in (b) transforms it into the form of the matrix in (a). Since eigenvalues are invariant under these operations, the eigenvalues of (b) are the same as those in (a).

Part (c)

$$\begin{bmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 12 & 10 & 8 \\ 9 & 6 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

Eigenvalues of $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ are $\frac{7 \pm \sqrt{57}}{2}$. Thus, eigenvalues of this matrix are:

$$\text{Eigenvalues: } \{\frac{53 \pm 7\sqrt{57}}{2}, -2, -2\}$$

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The given operator O is orthogonally diagonalizable with eigenvalues ± 1. Using the spectral decomposition, O can be expressed as $O = \sum_\lambda \lambda P_\lambda$. Here, P_λ represents the projector onto the eigenspace associated with eigenvalue λ.

For $\lambda = \pm 1$:

$$P_{\pm 1} = \frac{I \pm O}{2}$$

This follows by substituting O and I into the spectral decomposition equations $O = P_1 - P_{-1}$ and $I = P_1 + P_{-1}$.

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We aim to show the equivalence of norm-preservation and unitarity.

1. Norm Preservation $\Rightarrow$ Inner Product Preservation: Assuming A preserves norms, i.e., $|Ax| = |x| \forall x$, consider:

$$\langle Ax, Ay \rangle = \langle A^\dagger (Ax), y \rangle = \langle x, y \rangle$$

2. Inner Product Preservation $\Rightarrow$ Norm Preservation: Assuming $\langle Ax, Ay \rangle = \langle x, y \rangle$, set $y = x$:

$$||Ax||^2 = \langle x, x \rangle = ||x||^2$$

3. Inner Product Preservation $\Rightarrow$ Unitarity:

$$\langle A^\dagger Ax, y \rangle = \langle Ax, Ay \rangle = \langle x, y \rangle \implies A^\dagger A = I$$

Thus, the conditions are equivalent.

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Using Qiskit, we plot the states $|0\rangle$, $|1\rangle$, $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$. These states correspond to specific directions on the Bloch sphere.

Directions: - $|0\rangle$: North pole (positive z-axis). - $|1\rangle$: South pole (negative z-axis). - $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$: Equator (positive x-axis). - $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$: Equator (negative x-axis).

Solution to Question 5

To prepare $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$: 1. Apply a Hadamard gate to the first qubit. 2. Use controlled-NOT gates to entangle the first qubit with the second and third. This generalizes to n-qubits by applying H to the first and $n-1$ CNOT gates.