Homework - 3

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Problem 1

$$\mu_1 = \frac{1}{4}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix}5\\4\end{bmatrix} = \begin{bmatrix}5/4\\1\end{bmatrix} = \begin{bmatrix}1.25\\1\end{bmatrix}$$

$$\mu_2 = \frac{1}{4}\left(\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}3\\3\end{bmatrix} + \begin{bmatrix}3\\0\end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix}10\\8\end{bmatrix} = \begin{bmatrix}2.5\\2\end{bmatrix}$$

Within class scatter matrix:

$$S_1 = \sum_{i=1}^{4}(x_i-\mu_1)(x_i-\mu_1)^T$$

$$= \begin{bmatrix}0.25\\0\end{bmatrix}\begin{bmatrix}0.25 & 0\end{bmatrix} + \begin{bmatrix}-0.25\\1\end{bmatrix}\begin{bmatrix}-0.25 & 1\end{bmatrix} + \begin{bmatrix}0.75\\0\end{bmatrix}\begin{bmatrix}0.75 & 0\end{bmatrix} + \begin{bmatrix}-0.25\\1\end{bmatrix}\begin{bmatrix}-0.25 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0.0625 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0.0625 & 0.25\\0.25 & 1\end{bmatrix} + \begin{bmatrix}0.5625 & 0 & \end{bmatrix} + \begin{bmatrix}0.0625 & 0.25\end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0.25 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 & 0 \\ 0 & 2 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \begin{bmatrix} -0.5 & 1 \end{bmatrix} + \begin{bmatrix} -0.5 \\ 0 \end{bmatrix} \begin{bmatrix} -0.5 & 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \end{bmatrix}$$
$$+ \begin{bmatrix} 0.5 \\ -2 \end{bmatrix} \begin{bmatrix} 0.5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & -0.5 \\ -0.5 & 1 \end{bmatrix} + \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1 \end{bmatrix} +$$
$$\begin{bmatrix} 0.25 & -1 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 6 \end{bmatrix}$$

$$\therefore S_W = S_1 + S_2 = \begin{bmatrix} 0.75 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1.75 & -1 \\ -1 & 8 \end{bmatrix}$$

$$S_W^{-1} = \frac{1}{8 \times 1.75 - 1} \begin{bmatrix} 8 & 1 \\ 1 & 1.75 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 8 & 1 \\ 1 & 1.75 \end{bmatrix}$$

For Fisher's FLD :—

$$w = S_W^{-1} (\mu_1 - \mu_2)$$

$$= \frac{1}{13}\begin{bmatrix} 8 & 1 \\ 1 & 1.75 \end{bmatrix}\begin{bmatrix} -1.25 \\ -1 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -11 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.846 \\ 0.230 \end{bmatrix}$$

(b) Plotting covariance matrices:

$$S_1 = \begin{bmatrix} 0.75 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\lambda_1 = \frac{0.75+2}{2} + \sqrt{\left(\frac{0.75-2}{2}\right)^2 + 0^2}$$

$$= 1.375 + 0.625$$

$$= 2$$

$$\lambda_2 = \frac{0.75+2}{2} - \sqrt{\frac{(0.75-2)^2}{2^2} + 0^2}$$

$$= 1.375 - 0.625$$

$$= 0.75.$$

$$\theta = \frac{\pi}{2} = 90^\circ$$

So, abscissa $= \sqrt{2} = 1.414$

ordinate $= \sqrt{0.75} = 0.86$

$$S_2 = \begin{bmatrix} 1 & -1 \\ -1 & 6 \end{bmatrix}$$

$$\lambda_1 = \frac{1+6}{2} + \sqrt{\left(\frac{1-6}{2}\right)^2 + (-1)^2}$$

$$= 3.5 + \sqrt{(2.5)^2 + 1}$$

$$= 3.5 + 2.69$$

$$= 6.2 \qquad \therefore \sqrt{\lambda_1} = 2.48$$

$$\lambda_2 = \frac{1+6}{2} - \sqrt{\left(\frac{1-6}{2}\right)^2 + (-1)^2}$$

$$= 3.5 - 2.69$$

$$= 0.8 \qquad \therefore \sqrt{\lambda_2} = 0.89$$

$$\theta = \operatorname{atan2}\left(\frac{6.19258 - 1}{-1}\right) = 1.76 = 100.84°.$$

This is sketched in the plot.

(c) This is also sketched in the plot.

(d) To find the projected results:

$$y = w^T x$$

T

1. $= \begin{bmatrix} 0.84 \\ 0.23 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.84 & 0.23 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= 0.84 + 0.23 = 1.07$

2. $0.84 + 0.23 \times 2 = 1.3$

3. $0.84 \times 2 + 0.23 \times 1 = 1.91$

4. $0.84 \times 1 + 0.23 \times 0 = 0.84$

5. $0.84 \times 2 + 0.23 \times 3 = 2.37$

6. 2.14

7. 3.21

8. 2.52

(b) (i) Minimum distance classifier:

$$\mu_1 = \frac{1.07 + 1.3 + 1.91 + 0.84}{4} = 1.07$$

$$\mu_2 = \frac{2.37 + 2.14 + 3.21 + 2.52}{4} = 2.37$$

For $x = (2 \ \ 1.4)'$

$$w^T x = \begin{bmatrix} 0.84 \\ 0.23 \end{bmatrix} \begin{bmatrix} 2 \\ 1.4 \end{bmatrix} = 2.002,$$

which is closer to $\mu_2 (0.368)$ than $\mu_1 (0.932)$.

Thus, we will classify it to group/class2.

(ii) For kNN, we will check the closest projection point to 2.002:

$1 - \lvert(1.07-2)\rvert = 0.93$	$5 - (2.37-2) = 0.37$
$2 - \lvert 1.3-2 \rvert = 0.7$	$6 - (2.14-2) = 0.14$
$3 - (1.91-2) = 0.09$	$7 - (3.21-2) = 0.21$
$4 - (0.84-2) = 1.16$	$8 - (2.52-2) = 0.52$

So, 3 is the closest point.
Thus the test sample belongs to class 1.

(c) & (d): Mahalanobis Distance for contour distance.

$$(\mu^T \Sigma^{-1} \mu)^{1/2}$$

$$= \left[\begin{bmatrix} 1.07 \\ 2.37 \end{bmatrix}^T \frac{1}{13} \times \frac{1}{3} \begin{bmatrix} 8 & 1 \\ 1 & 1.75 \end{bmatrix} \begin{bmatrix} 1.07 \\ 2.37 \end{bmatrix} \right]^{1/2}$$

$$= \left(\frac{1}{39} \times 24.06\right)^{1/2} = (0.6169)^{1/2} = 0.78$$

Problem 2

(a) $\mu_X = \dfrac{1+1+2+1+2+2+3+3}{8} = 1.875$

$\mu_Y = \dfrac{1+2+1+0+3+2+3+0}{8} = 1.5$

$Var_X = 0.696$ [Using calculator]

$Var_Y = 1.428$ [" "]

$Cov_{X,Y} = 0.214$

$\therefore \Sigma = \begin{bmatrix} 0.696 & 0.214 \\ 0.214 & 1.428 \end{bmatrix}$

To find eigenvalues:

$\det(\Sigma - \lambda I) = 0$

$\Rightarrow \det \begin{bmatrix} (0.696-\lambda) & 0.214 \\ 0.214 & (1.428-\lambda) \end{bmatrix} = 0$

$\Rightarrow (0.696-\lambda)(1.428-\lambda) - 0.214^2 = 0$

$\Rightarrow \lambda = 1.48$ and 0.638

Using only the larger one, we get the following eigenvector:

$$e_1 = \begin{bmatrix} 0.26 \\ 0.96 \end{bmatrix}$$

This was found by solving $(\Sigma - \lambda I)x = 0$ for $\lambda = 1.48$.

(b) Error $= \dfrac{\lambda_2}{\lambda_1 + \lambda_2} = \dfrac{0.638}{1.48 + 0.638} = 0.30 = 30\%$

(c) Please check the plot.

(d) Please check the plot.

(e) • $y_1 = e_1' x_1 = \begin{bmatrix} 0.26 & 0.96 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= 0.26 + 0.96$

$= 1.22$

• $y_2 = e_1' x_2 = 2.18$

• $y_3 = 1.48$

• $y_4 = 0.26$

• $y_5 = 3.4$

• $y_6 = 2.44$

• $y_7 = 3.66$

• $y_8 = 0.78$

(f) $y = \begin{bmatrix} 0.26 & 0.96 \end{bmatrix} \begin{bmatrix} 2 \\ 1.4 \end{bmatrix}$

$= 2 \times 0.26 + 1.4 \times 0.96$

$= 1.864.$

- For min^m distance classifier:

$$\mu_1 = \frac{1.22 + 2.18 + 1.48 + 0.26}{4} = 1.285$$

$$\mu_2 = \frac{3.4 + 2.44 + 3.66 + 0.78}{4} = 2.57$$

dist from class 1 = 0.579

" " " 2 = 0.706

So, this should be classified to class 1.

- For KNN distance classifier

dist: (only absolute)

$1 \rightarrow 1.864 - 1.22 = 0.644$	$5 \rightarrow 1.536$
$2 \rightarrow 0.316$	$6 \rightarrow 0.576$
$3 \rightarrow 0.384$	$7 \rightarrow 1.796$
$4 \rightarrow 1.604$	$8 \rightarrow 1.084$

The closest point is $\underline{\underline{2}}$. Thus the test point belongs to class 1.

For (c) & (d) : Mahalanobis distance for contour:

$$\left(\begin{bmatrix} 1.875 \\ 1.5 \end{bmatrix}^T \begin{bmatrix} 0.696 & 0.214 \\ 0.214 & 1.428 \end{bmatrix}^{-1} \begin{bmatrix} 1.875 & 1.5 \end{bmatrix} \right)^{1/2}$$

$$= (5.66)^{1/2} = 2.38.$$

For plotting this correlation matrix,

$$\lambda_1 = \frac{0.696 + 1.428}{2} + \sqrt{\left(\frac{0.696 - 1.428}{2}\right)^2 + (0.214)^2}$$

$$= 1.062 + 0.4239$$

$$= 1.4859.$$

$$\lambda_2 = 1.062 - 0.4239$$

$$= 0.6381.$$

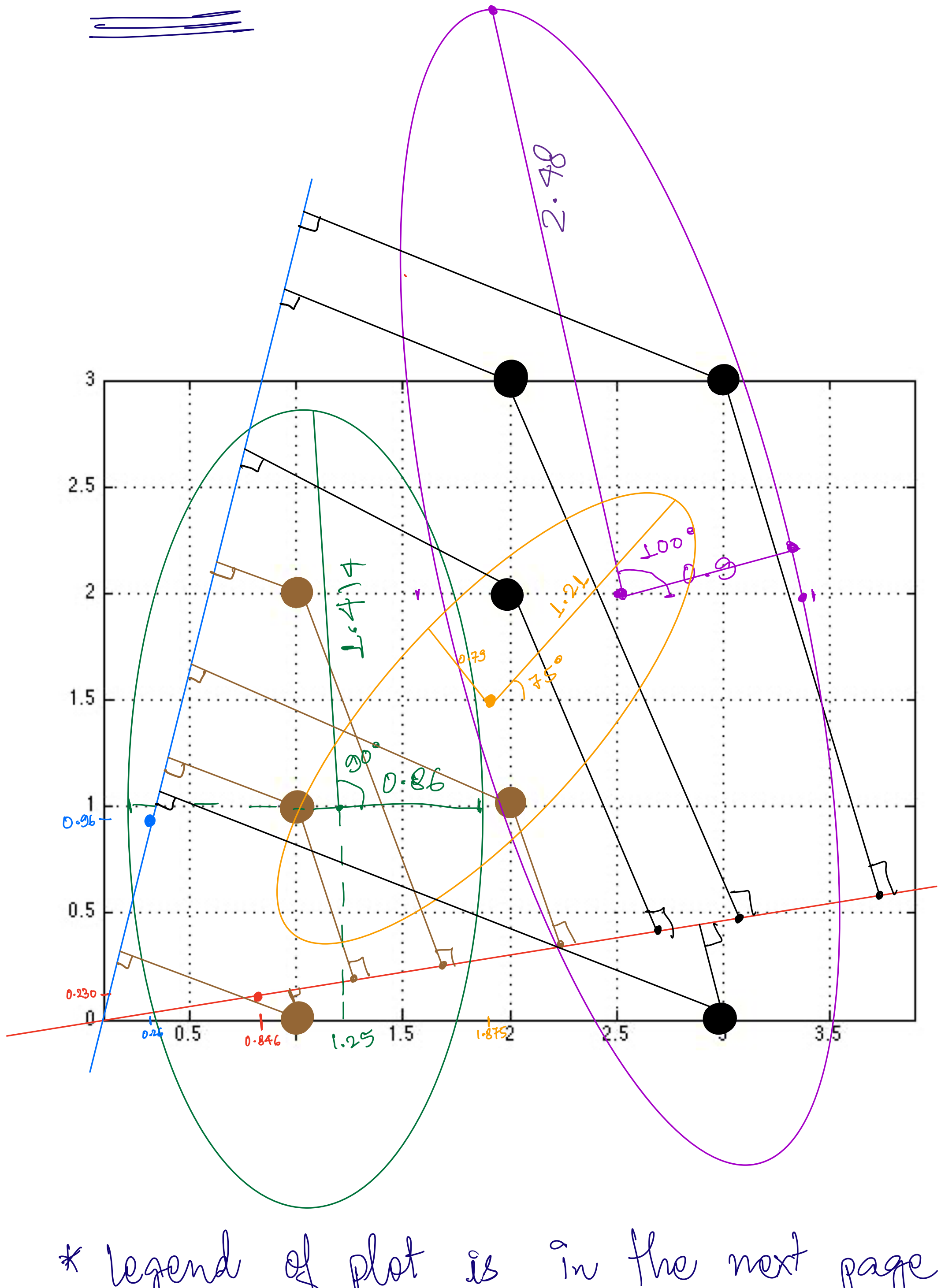
$$\text{abcissa} = \sqrt{\lambda_1} = 1.21$$

$$\text{ordinate} = \sqrt{\lambda_2} = 0.79$$

$$\theta = \operatorname{atan2}(1.4859 - 0.696, 0.214)$$

$$= 1.307 = 74.9^\circ = 75^\circ$$

⇒ Plots



* legend of plot is in the next page.

⇒ <u>Legend</u>

Brown points belong to class 1.

Black points belong to class 2.

Red line shows Fisher's LDA's projection.

Green line shows contour plot for Class 1 in Fisher's LDA.

Purple line shows contour plot for Class 2 in Fisher's LDA

Orange line shows contour plot for PCA.

Blue line is the projection line for PCA.