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THE ANALYSIS OF SIMULATION-GENERATED TIME SERIES*

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This study applies spectral analysis to the study of time series data generated by simulated stochastic models. Because these data are autocorrelated, analysis by methods applicable to independent observations is not possible. Mathematical models known as covariance stationary stochastic processes are useful representations of autocorrelated time series. The increased publication of literature describing stochastic processes and spectral analysis, in particular, is making these ideas available to an increasing audience.

Section I presents a rationale for our interest in time series models and spectral analysis. Section II describes the basic notions of covariance stationary processes. It emphasizes the equivalence of these processes in both the time and frequency domains; the compactness of frequency domain analysis seemingly recommends it over correlation analysis. Section III provides a heuristic background for understanding statistical spectral analysis. Simple frequency-domain statistical properties are emphasized and compared with the rather involved sampling properties of estimated correlograms. Several relevant statistical tests are described.

Three simulated experiments are used as examples of how to apply spectral analysis. These are described in Section IV. They are (1) a single-server, first-come-first-served (FCFS) queueing problem with Poisson arrivals and exponentially distributed service times; (2) a similar model with the FCFS assignment rule replaced by one that chooses the job with the shortest service time; and (3) yet another similar model, but with constant service time. In this third example both assignment rules are equivalent.

The state of the queue was observed and recorded at unit intervals for all three examples and forms the time series on which Section V is based; theoretical and statistical correlograms and spectra are compared for example 1; examples 1 and 2 are compared using estimated correlograms and spectra; example 3 serves to show some unusual properties of spectra.

A particular conclusion of this paper is that differences in the statistical properties of queue length that result under various assumptions and operating rules are easily identified using spectral analysis. More generally, however, this estimation procedure provides a tool for (1) comparing simulated time series with real-world data, and (2) for understanding the implications that various alternative assumptions have on the output of simulated stochastic models.

I. Introduction

The past decade has seen computer simulation experiments become a commonly employed tool for quantitative description and analysis. While investigators have considerably enhanced the descriptive power of their simulation models during this period, no concomitant statistical analysis has been presented to adequately analyze the results of these experiments. This study describes a statistical method, spectral analysis, which is unusually well suited for the analysis of these

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results. This technique is a statistical tool commonly employed in the physical sciences to study the time-dependent nature of physical processes and more recently has been used in economics to study the behavior of economic statistics [19], [12], [21, pp. 1–61]. As simulation data are generally autocorrelated, an investigator cannot apply the statistical tools commonly used for studying independent observations. Spectral analysis, however, can be used to study the salient time properties of such processes and to present them in an easily interpretable fashion for descriptive and comparative purposes.

To date, most simulation users have approached statistical data analysis in one of two ways. Either they have computed sample means and variances ignoring the presence of autocorrelation [13], or they have accounted for dependence by dividing the sample record length into intervals that are hopefully longer than the interval of major autocorrelation and have worked with the observations on these supposedly independent intervals [5, pp. 47–61]. In some cases the investigator has performed additional data transformations in order to remove any remaining time dependence [6].

Ignoring autocorrelation is clearly unacceptable, since the reliability of the sample means and variances are thereby overestimated. Our criticism of the method of accounting for dependence is twofold: first, the rather arbitrary methods that are used to create independent observations make us doubt that this purpose is indeed accomplished; second, removing autocorrelation does away with one property of the process with which the investigator should be intimately concerned. Our first criticism seems justified since in most cases reported, the choices of sample record length and sampling interval seem to have neither enough prior nor posterior justification to make them much more than arbitrary. It is true that in some cases the investigator checks for autocorrelation by estimating the correlogram of the process [7, pp. 92–110]; but this technique is not reliable since the estimates of the correlogram are often imprecise and misleading. These estimates are themselves autocorrelated and this dependence is determined by the (unknown) properties of the stochastic process itself [3, pp. 253–256].

Our second criticism concerns the purpose of the simulation experiment. When one studies a stochastic process, he is interested in the average level of activity, deviations from this level, and the length of time these deviations last, once they occur. We can in some sense distinguish two costs that are associated with deviations from the general activity level. The deviation itself includes one cost, and the amount of time the deviation lasts induces the other.

To illustrate the point more fully, consider the following example. A shipment of raw materials is delivered to a manufacturing plant. Circumstances preclude their immediate use and they are therefore stored; the plant incurs the cost of placing them in storage. Now an opportunity cost accrues over the length of time the raw material is in storage. The cost of placing the materials in storage derives from the deviation, whereas the opportunity cost derives from how long the deviation lasts. If storage placement cost is low relative to opportunity cost, then emphasis should be placed on reducing the storage time. If it is relatively high, then one should work on reducing the occurrence of deviations.

Next consider a shop to which people bring appliances for repair. The shop owner is concerned about the waiting-room facilities for these people. Clearly, the more people who wait, the larger must be the waiting room. However, if each person waits only a short time, he will have little concern with his surroundings and the shop owner need not be as attentive to customers' comfort as he would have to be if each one stayed a long time (e.g., providing chairs and magazines). Therefore, his concern is not only with the deviation but also with the length of the deviation from the mean.

Queueing theorists have studied these problems by concentrating on the queue length and waiting-time transient probability distributions, steady-state distributions, the length of busy periods, and first passage time [1, Chapt. 11], [26]. Except for queue length and waiting time, these analytical concepts do not seem to have found their way into simulation experiments.

While the mean activity level and the deviations from it receive explicit or implicit attention, the simulation analyst usually treats autocorrelation as a nuisance and works to eliminate, rather than to understand it. The research described in this paper is an attempt to analyze the statistical properties of simulation output data more fully than has been the case in the past. Our approach emphasizes the linear dependence of a process on its past. The theory of covariance stationary stochastic processes concentrates on this linear dependence rather than on the complete probability law that governs the process [29, Chaps. 1–3]. The first and second-order moments occupy the focus of the theory and, in the case of a multivariate Gaussian process, completely specify the probability law. In the course of our exposition, many of these ideas about covariance stationary processes are developed. Some are used in the examples we offer; others merely acquaint the reader with ideas which will allow him to study data in a broader framework.

Section II describes the basic elements of the theory of covariance or wide-sense stationary processes. We stress the equivalence of the autocorrelation and spectral density functions in the time and frequency domains, respectively, and show that a knowledge of one implies a knowledge of the other. It is our opinion that studying a process in the frequency domain permits more insight into the nature of the process. A frequency domain characterization allows us to study a phenomenon in terms of contributions made by a continuum of processes that have much simpler statistical properties than does the original process. We are essentially studying the phenomenon in parts rather than in sum.

The notion of correlation time is introduced as a measure of autocorrelation. Several examples then show how to interpret the autocorrelation function, the spectral density function, and the correlation time as a statistical description of a stochastic process. The variance of the sample mean is next discussed and is shown to depend asymptotically on three parameters: the population variance, the correlation time of the process, and the length of the sample record.

The spectral decomposition that Section II describes does not lead to an immediate understanding of the problems encountered in estimating the spectrum of a process. The spectrum appears as a Fourier transform and presents estimation problems that are considerably different from those in most commonly encountered statistical problems.[1] Section III describes these problems and the kinds of considerations that must accompany spectral estimation. These are the choice of a statistically consistent estimator, the sharpness of the estimates (resolution), and their statistical stability.

To illustrate how to perform a statistical spectral analysis on simulation output data, we have chosen to simulate a simple single-server queueing model with Poisson arrivals, exponentially distributed or constant service time, and a first-come-first-served (FCFS) or shortest operation (SHOPN) rule for assigning jobs their place in the queue. Section IV describes the details of the simulation. The process we analyze is the state of the queue or simply queue length. In keeping with established practice, queue length includes those waiting and those being served.

Section V contains the substantive analysis of simulation output data for which Sections II, III, and IV have prepared the reader. We begin by estimating and examining the queue-length spectrum for the FCFS exponential service-time case and compare our estimates with known analytical results. Next, we compare this estimated spectrum with that for the queue length using the SHOPN rule. A statistical test of the homogeneity of these curves shows that they are indeed different. This implies that the dynamic behavior of queue length differs significantly with these rules.

A queueing process with constant service time is next observed. Our example shows that the estimated spectrum differs from the others by having peculiar peaks at equidistant frequencies. We explain this by noting that whenever there are jobs in the queue, the constant-service-time assumption implies that jobs will leave the queue at equidistant points in time. That is, the queue length process appears to have a periodic component. The peaks we observe in the spectrum are simply concentrations of variance about the fundamental frequency and its harmonic frequencies. The fundamental frequency is inversely proportional to the constant service time chosen.

II. Stationary Stochastic Processes

This Section briefly describes the theory of stationary stochastic processes and explains the desirability of studying them in the frequency as well as in the time domain. It introduces the relevant nomenclature, discusses the equivalence of certain ensemble and time averages, and establishes the relationship between time domain measures and the spectral density function of the frequency domain. Several examples illustrate the equivalence of time and frequency domain concepts and show the simplicity that the spectral density function offers for the characterization of stochastic processes.

A *stochastic process* $\{X(t), t \varepsilon T\}$ is a family of random variables indexed on a continuous parameter t that takes on all values in the set T. Letting the set T

[1] A comprehensive description of spectral estimation can be found in [4]. Our description relies heavily on [14] and [22, pp. 133–190].

comprise all points on the real line permits us to describe the stochastic process as $\{X(t), -\infty \leqq t \leqq \infty\}$. For our purpose, the parameter t represents time; however, one may also define stochastic processes using space coordinates as well. The observed behavior of a stochastic process during some arbitrary time interval forms a *time series*. This record or *realization* is composed of chronologically ordered observations, recorded either continuously or at discrete time intervals. A time series provides us with the data needed to make statistical inferences about $\{X(t)\}$. The next Section describes the statistical techniques needed to make these inferences.

Just as we are interested in the probability law that governs a random variable, so are we also interested in the probability law that governs a stochastic process. Let $P[X(t_1) \leqq x_1, X(t_2) \leqq x_2, \cdots, X(t_n) \leqq x_n]$ be the n-dimensional joint probability distribution function (p.d.f.) of $\{X(t)\}$ at the points $t_1, t_2, \cdots, t_n$, respectively. If

$$\begin{aligned} P[X(t_1) &\leqq x_1, X(t_2) \leqq x_2, \cdots, X(t_n) \leqq x_n] \\ &= P[X(t_1 + s) \leqq x_1, X(t_2 + s) \leqq x_2, \cdots, X(t_n + s) \leqq x_n] \end{aligned} \tag{2.1}$$

for arbitrary real values of s and for all n, the stochastic process $\{X(t)\}$ is said to be *strictly stationary* [23, p. 70]. This means that the p.d.f. is a function only of the reference times among its arguments and not of historical time; it is invariant under translation along the time axis. Knowledge of the p.d.f. for any $t_1, t_2, \cdots, t_n$ implies a knowledge of how $\{X(t)\}$ behaves during any time interval of length $(t_n - t_1)$.

The analyst seldom knows the multivariate p.d.f. that characterizes $\{X(t)\}$. Furthermore, a knowledge of the *linear dependence* of $\{X(t)\}$ on its past usually provides the most significant amount of information about the process when the p.d.f. is unknown. Therefore, attention is often focused on a class of processes that have less stringent requirements than those of strictly stationary processes and that provide this information on linear dependence. This class requires that the second moments, $E[X(t)X(t + \tau)]$, be finite and be functions only of the reference time τ [23, p. 70]. Stochastic processes with these properties are called *covariance stationary*. Strict stationarity is synonymous with what some writers have called strong stationarity; covariance stationarity coincides with their definition of mean-square, second-order, wide-sense, or weak stationarity.

Let $\{X(t)\}$ be covariance stationary with mean zero and variance σ^2. Its second-order moments then form its *autocovariance function*

$$E[X(t)X(t + \tau)] = \sigma^2\rho(\tau) \qquad 0 \leqq \tau \leqq \infty; \quad -\infty \leqq t \leqq \infty \tag{2.2}$$

where the *autocorrelation function* $\rho(\tau)$ measures the linear dependence of $X(t)$ on its past history. For real valued processes, the function $\rho(t)$ equals unity when τ equals zero and lies inside the interval $(-1, 1)$ for all other values of t.

The second-order moments are relationships between the present and the past. They are formed by

$$E[X(t)X(t + \tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi(t)\xi(t + \tau)\, dP[\xi(t), \xi(t + \tau)], \tag{2.3a}$$

$$= \int_{-\infty}^{\infty} \xi(t)\, dP[\xi(t)] \int_{-\infty}^{\infty} \xi(t+\tau)\, dP[\xi(t+\tau) \mid \xi(t)], \tag{2.3b}$$

and are commonly referred to as probability averages in contrast to the *time* averages,

$$R(\tau) = \lim_{T\to\infty}(1/2T) \int_{-T}^{T} X(t)X(t+\tau)\, d\tau. \tag{2.4}$$

It is known that for covariance stationary processes, time averages converge in probability to their respective probability averages provided that

$$\begin{aligned} \lim_{T\to\infty} \int_0^T E\{[X(t+\tau+\nu)X(t+\tau) - R(\nu)] \\ \cdot[X(t+\nu)X(t) - R(\nu)]\}\, d\tau = 0. \end{aligned} \tag{2.5}$$

The significance of this fact becomes apparent when estimating the autocovariance functions of $\{X(t)\}$ from a single sample record.

Queueing theory customarily focuses on two stochastic processes, *queue length* and *virtual waiting time*. Each is characterized by two probability measures. One is the equilibrium or *steady-state* probability distribution, which tells us the probability that $\{X(t)\}$ assumes some value at an arbitrarily chosen time. The other measure is the non-equilibrium or *transient* probability distribution, which tells us the probability that the process assumes some value at time $t + \tau$, given the value that it has assumed at some prior time t. These probability measures are $dP[\xi(t)]$ and $dP[\xi(t+\tau) \mid \xi(t)]$, respectively, in our notation.

The analytic derivations of these steady state and transient probability distributions have occupied the attention of many queueing theorists. D. G. Kendall has derived the steady state distributions for the n-server general input process with exponential service time and a first-come-first-served discipline [16, pp. 151–185]. Ledermann and Reuter and Bailey have derived the transient distributions for this problem for a single-server queue with Poisson arrivals, and Karlin and McGregor have derived these distributions for n-server queues [18, pp. 321–369], [2, pp. 288–291], and [15, pp. 87–118]. The transient solutions to these rather elementary problems are not in a conveniently usable form and the extension of the method of solution to more sophisticated queueing problems does not appear to be promising.

It is important to note that the queueing problems just described concern the dependence of the random variable $X(t+\tau)$ on $X(t)$, a past observation. This implies that what we are principally studying is the linear dependence of the process on its past; this relationship is described by the autocorrelation function $\rho(\tau)$. While it is seldom possible to determine analytically or empirically the transient probability distribution, it is possible to estimate the autocorrelation function because of the convergence of phase and time averages for covariance stationary processes.

Describing the behavior of queue length by its autocorrelation function is not a new idea. Morse has stressed the relevance of the time-dependent structure of a stochastic process, and has outlined the derivations for the case of Poisson arrivals, exponentially distributed service time, and a first-come-first-served queue discipline [20, pp. 255–261]. As stochastic fluctuations of $\{X(t)\}$ around its mean are the rule rather than the exception, an appropriate understanding of the process requires a knowledge of how long these deviations last and how large their relative magnitudes are. The autocorrelation function and the spectral density function (defined below) provide us with this information The autocovariance function of a covariance stationary process may be represented as the Fourier cosine transform of a real function $F(\lambda)$ of a parameter λ:

$$\sigma^2\rho(\tau) = \sigma^2 \int_0^\infty \cos \lambda\tau \, dF(\lambda) \tag{2.6}$$

where $F(\lambda)$ is known as the *spectral distribution function*. Furthermore, if $F(\lambda)$ is differentiable, an inverse transformation exists such that

$$\sigma^2 f(\lambda) = (2\sigma^2/\pi) \int_0^\infty \rho(\tau) \cos \lambda\tau \, d\tau \tag{2.7}$$

where the *spectral density function* $f(\lambda)$ is the derivative of $F(\lambda)$. Expressions (2.6) and (2.7) constitute the Wiener-Khintchine relationships, which form the basis for modern harmonic analysis [17, pp. 604–615] and [28, pp. 117–258].

The parameter λ is known as angular frequency. An oscillation or fluctuation around the mean of $\{X(t)\}$ may be described by its period (T), the required time for an oscillation to complete itself. Frequency (ν) is the reciprocal $(1/T)$ of the period and is the number of oscillations per unit time. Angular frequency λ, which equals $2\pi\nu$, measures the distance traveled around the unit circle per unit time.

The important contribution of the spectral density function $f(\lambda)$ to the analysis of covariance stationary processes derives from the information it offers about the composition of the variance σ^2. Remember that the quantity $\rho(0)$ is unity. Observe that for τ equal to zero,

$$\sigma^2 = \sigma^2 \int_0^\infty f(\lambda) \, d\lambda \tag{2.8}$$

so that we may think of $f(\lambda)\, d\lambda$ as the contribution made to the variance σ^2 by a small band of angular frequencies around λ (ν cycles per unit time). Essentially $f(\lambda)\, d\lambda$ is a decomposition of the variance into additive contributions made by all positive real frequencies that constitute $\{X(t)\}$. That is, we may think of $\{X(t)\}$ as being made up of a continuum of independent frequency components the sum of whose mean-square variations equals the variance of $\{X(t)\}$. Indeed, Tukey has pointed out how the statistical analysis of time series relates to the classical analysis of variance [27, pp. 191–220].

Note that

$$F(\lambda_0) = \int_0^{\lambda_0} f(\lambda) \, d\lambda \tag{2.9}$$

tells us the relative contribution of angular frequencies from zero to λ_0 to the variance of $\{X(t)\}$. Conversely, the quantity $[1 - F(\lambda_0)]$ tells us the relative contribution that angular frequencies greater than λ_0 make to the variance. If we let

$$T_0 = 2\pi/\lambda_0, \tag{2.10}$$

then we may speak of the quantity $[1 - F(\lambda_0)]$ as the relative importance of oscillations or fluctuations with period T_0 or less. Paraphrasing this technical language somewhat, we may speak of the quantity $[1 - F(\lambda_0)]$ as a measure of the activity that is completed in the process in a time interval T_0 compared to the total activity that takes place in the process. If we decide to observe $\{X(t)\}$ at equidistant points in time, it is important to choose a sampling interval T_0 during which very little change takes place in the process. The choice of an appropriate sampling interval appears again in Section III.

If $\rho(\tau)$ decreases slowly with increasing argument, then the process is highly autocorrelated. When $\rho(\tau)$ falls off slowly, the spectral density decreases rapidly with increasing λ. That is, a highly autocorrelated process is primarily composed of low frequency variations.

While the shapes of these two curves indicate the nature of autocorrelation, it is often desirable to describe time dependence by a single number. To accomplish this, let us consider a hypothetical process that is completely correlated with itself over τ^* time units and completely uncorrelated with itself beyond that time. The corresponding autocorrelation function would be

$$\rho(\tau) = \begin{array}{ll} 1 & 0 \leqq \tau \leqq \tau^* \\ 0 & \tau^* \leqq \tau \leqq \infty. \end{array} \tag{2.11}$$

Expression (2.7) permits us to derive the corresponding spectral density function, which is

$$f(\lambda) = (2/\pi) \int_0^{\tau^*} \cos \lambda\tau \, d\tau, \tag{2.12a}$$

$$f(\lambda) = 2 \sin \lambda\tau^*/(\pi\lambda). \tag{2.12b}$$

The *correlation time* τ^* is then

$$\tau^* = \pi f(0)/2. \tag{2.13}$$

If our actual process had these hypothetical characteristics, the notion of time dependence would indeed be simple. Although this process is not physically realizable, the correlation time τ^* still offers us a convenient, although gross, measure of autocorrelation. Several examples will give the reader a better understanding of the relationships between $\rho(\tau)$, $f(\lambda)$, τ^* and their implications for autocorrelation.

Let $\{X(t)\}$ be a Gaussian process with mean zero and variance σ^2, and autocorrelation function

$$\rho(\tau) = e^{-\alpha\tau} \qquad 0 \leqq \tau \leqq \infty. \tag{2.14}$$

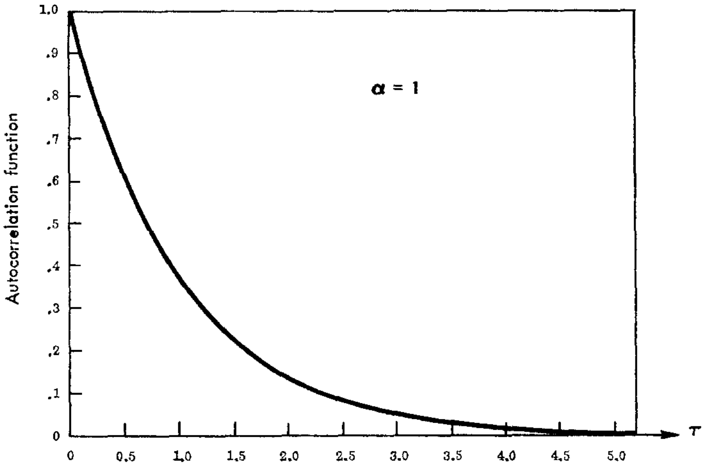

FIG. 1. The autocorrelation function: $\rho(\tau) = e^{-\alpha\tau}$.

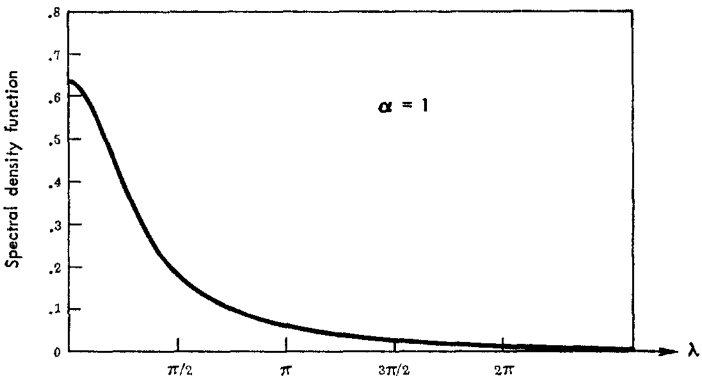

FIG. 2. The spectral density function: $f(\lambda) = 2\alpha/(\pi(\alpha^2 + \lambda^2))$.

The corresponding spectral density function is

$$f(\lambda) = 2\alpha/(\pi(\alpha^2 + \lambda^2)) \qquad 0 \leqq \lambda \leqq \infty. \tag{2.15}$$

Figures 1 and 2 show the shapes of these two curves. Note that the correlation time τ^* is

$$\tau^* = \pi f(0)/2 = 1/\alpha. \tag{2.16}$$

The larger the quantity α, the smaller the correlation time and the smaller is $\rho(\tau)$ for a particular reference time τ. At time τ^* the autocorrelation function is

$$\rho(\tau^*) = 1/e \approx .37. \tag{2.17}$$

Thus, τ^* is the mean time required for the process to go from its mean square deviation from the mean back to 37 percent of this deviation. In two correlation times, the process has gone to about 14 percent of its mean square deviation and, in three correlation times, to about 5 percent.

As a second example, let $\{X(t)\}$ be the sum of N independent Gaussian processes with zero mean and variance

$$\sigma^2 = \sum_{i=1}^{N} \sigma_i^2 \tag{2.18}$$

where the subscript i refers to the i^{th} process. Assume that each has an autocorrelation function of the form given in expression (2.14). Now $\{X(t)\}$ has

$$\rho(\tau) = (1/\sigma^2) \sum_{i=1}^{N} \sigma_i^2 e^{-\alpha_i \tau}, \tag{2.19a}$$

$$f(\lambda) = (2/\pi\sigma^2)[\sum_{i=1}^{N} \sigma_i^2 \alpha_i/(\alpha_i^2 + \lambda^2)]. \tag{2.19b}$$

The corresponding correlation time,

$$\tau^* = 1/\sigma^2 \sum_{i=1}^{N} \sigma_i^2/\alpha_i, \tag{2.20}$$

is the weighted sum of the correlation times of the independent process.

Our two examples have dealt with $\rho(\tau)$'s and $f(\lambda)$'s that are monotonically decreasing. This need not necessarily be the shape of these functions. Suppose that $\{X(t)\}$ has the form

$$X(t) = A(t) \cos (\lambda_0 t + \theta) \tag{2.21}$$

where $A(t)$ is a Markov process with parameter α, θ is uniformly distributed on the interval $(-\pi, \pi)$ and $A(t)$ and θ are independent. The corresponding functions are

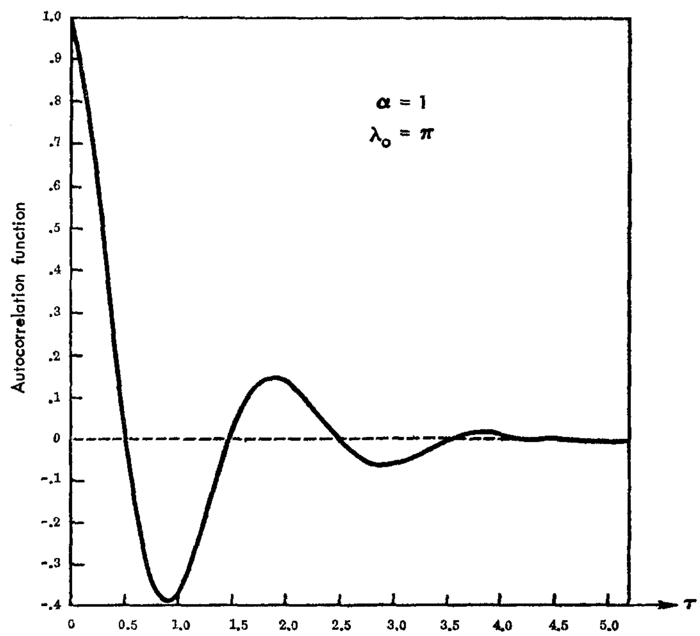

FIG. 3. The autocorrelation function: $\rho(\tau) = e^{-\alpha\tau} \cos (\lambda_0 \tau)$.

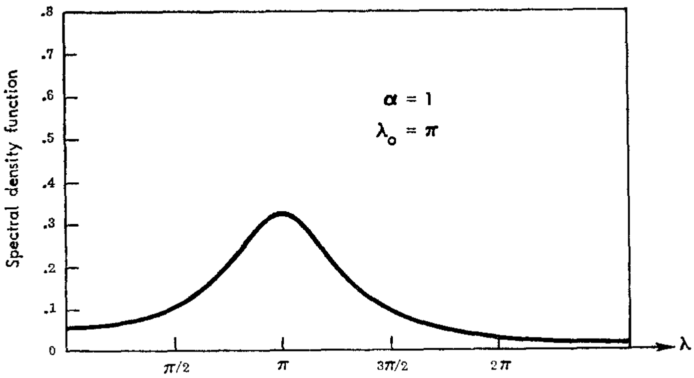

FIG. 4. The spectral density function: $f(\lambda) = (\alpha/\pi)((\alpha^2 + (\lambda - \lambda_0)^2)^{-1} + (\alpha^2 + (\lambda + \lambda_0)^2)^{-1})$

(2.22a) $\rho(\tau) = e^{-\alpha\tau} \cos \lambda_0\tau,$

(2.22b) $f(\lambda) = (\alpha/\pi)[(\alpha^2 + (\lambda + \lambda_0)^2)^{-1} + (\alpha^2 + (\lambda - \lambda_0)^2)^{-1}].$

Figures 3 and 4 show the appearance of these functions. Note that the autocorrelation function has a damped oscillatory appearance, whereas the s.d.f. has a peak in the vicinity of angular frequency λ_0. This concentration of variance around angular frequency λ_0 means that this angular frequency strongly influences the appearance of the process.

The correlation time for this process is

$$\tau^* = \alpha/(\alpha^2 + \lambda_0^2) \tag{2.23}$$

which introduces an interesting fact: the higher the angular frequency λ_0, the smaller the correlation time and therefore the smaller the time dependence.

The correlation time, which we have discussed here, characterizes autocorrelation by a single number. Different processes may have different autocorrelation functions and spectral density functions, and hence each of their correlation times would not seem to imply exactly the same kind of time dependence. This is not always true, however, as we shall show. Assume that $\{X(t)\}$ has mean μ, variance σ^2, and that we have observed the process over the time interval $(0, T)$. That is, we have a time series of length T time units. Our estimate of the mean is

$$m = (1/T) \int_0^T X(t)\, dt \tag{2.24}$$

with variance[2]

$$\begin{aligned} \operatorname{Var}(m) &= E(m - \mu)^2, \\ &= E(m^2) - \mu^2, \\ &= (1/T^2) \int_0^T \int_0^T \{E[X(t)X(s)] - \mu^2\}\, dt\, ds, \end{aligned} \tag{2.25a}$$

[2] The variance of the sample mean for autocorrelated data is discussed in [11]. The purpose in this application is to determine an approximate sample size.

$$= (2\sigma^2/T^2) \int_0^T \int_0^T \rho(t-s)\, dt\, ds,$$

$$= (\sigma^2/T) \int_0^T 2(1 - \tau/T)\rho(\tau)\, d\tau.$$

Now

$$\lim_{T\to\infty} \int_0^T (1 - \tau/T)\rho(\tau)\, d\tau = \pi f(0) \tag{2.25b}$$

so that

$$\lim_{T\to\infty} T \operatorname{Var}(m) = \pi\sigma^2 f(0). \tag{2.26}$$

If we denote the asymptotic variance of m as V and remember that

$$\tau^* = \pi f(0)/2, \tag{2.27}$$

then we have as an asymptotic variance for m

$$V \approx 2\sigma^2\tau^*/T. \tag{2.28}$$

If we were to collect n independent observations on a random variable with mean μ and variance σ^2, then the variance of the sample mean would be

$$V = \sigma^2/n. \tag{2.29}$$

Therefore, we see that if we observe a process with correlation time τ^* for T time units, we have the precision which

$$n = T/(2\tau^*) \tag{2.30}$$

independent observations would give us in estimating the mean. This means that, from the viewpoint of the sample mean, observing the process for $2\tau^*$ time units is equivalent to collecting one independent observation. In this sense, τ^* has the same meaning for all processes.

Correlation analysis and spectral analysis are equivalent ways of studying covariance stationary stochastic processes. A knowledge of the correlation structure implies a knowledge of the spectral structure of the process and vice versa. The choice of which one to use depends on what properties of the $\{X(t)\}$ are relevant for analysis and how these properties are best described. The autocorrelation function stresses dependence along the time axis. The spectral density function conveys this same information in the squared amplitude associated with oscillations at different frequencies. That is, it describes the process in terms of the relative importance of different kinds of oscillations (deviations around equilibrium) which compose the time series. Characterizing a process in terms of independent additive contributions permits greater ease of analysis than does correlation analysis where each value of the autocorrelation function is the weighted summation of the same contributions. Our analyses will nevertheless progress with parallel considerations for both time and frequency analyses so that the reader may draw his own conclusions.

We now apply the basic concepts of covariance stationary processes to an elementary queueing problem. Consider the single-server queueing problem with Poisson distributed arrivals with mean rate μ_1, exponentially distributed service time with mean $1/\mu_2$, a first-come-first-served discipline, and $\mu_1 < \mu_2$. The stochastic process of queue length is characterized in [20, pp. 259–260] by

$$\mu = E[X(t)] = \mu_1/(\mu_2 - \mu_1), \tag{2.31a}$$

$$\sigma^2 = \mu_1 \mu_2/(\mu_2 - \mu_1)^2, \tag{2.31b}$$

$$\rho(\tau) = ((\mu_2 - \mu_1)^3/\pi) \int_0^{2\pi} d\theta(\sin^2 \theta e^{-w\tau}/w^3) \qquad 0 \leqq \tau \leqq \infty, \tag{2.31c}$$

where

$$w \equiv \mu_1 + \mu_2 - 2(\mu_1\mu_2)^{1/2} \cos \theta,$$

and

$$f(\lambda) = (2(\mu_2 - \mu_1)^3/\pi^2) \int_0^{2\pi} \sin^2 \theta \, d\theta/(w^2(w^2 + \lambda^2)) \qquad 0 \leqq \lambda \leqq \infty, \tag{2.31d}$$

$$\tau^* = (\mu_1 + \mu_2)/(\mu_2 - \mu_1)^2. \tag{2.31e}$$

We later use these analytical results for comparative purposes in the Section which analyzes the results of our simulation experiments.

Our discussion has centered on the autocorrelation function and its Fourier cosine transform, the spectral density function. In our simulated examples, we estimate the *correlogram* and the *spectrum* for each sample record. The sample correlogram is a graph of the estimated autocorrelation function and the sample spectrum is a graph of the spectral estimates of

$$g(\lambda) \equiv \sigma^2 f(\lambda), \tag{2.32}$$

$g(\lambda)$ being the spectrum at frequency λ. Since we are interested in variance, it seems wise to estimate $g(\lambda)$ rather than $f(\lambda)$ since it shows us the magnitude of the contribution to the total variance at each frequency. This permits us to compare the variance contributions at each frequency for several spectra.

III. Statistical Spectral Analysis

A simulation experiment permits us to observe $\{X(t)\}$ continuously and without error. For computational convenience, however, it is desirable to sample the process at periodic intervals Δt. We are now considering a sequence $\{X_t\}$ that corresponds to $\{X(t)\}$ at times $\Delta t, 2\Delta t, \cdots, n\Delta t$ where

$$T = n\Delta t. \tag{3.1}$$

The choice of sampling interval becomes quite critical in studying the transient properties of $\{X(t)\}$ for it affects the amount and quality of information that an investigator may derive from a sample record of fixed length. Section II stressed the fact that stochastic fluctuations around the mean are the common behavior of stationary stochastic processes. These fluctuations may be short or long in duration, may be superimposed upon one another, and may have different magnitudes. Scrutinizing the contributions of these oscillations to a process permits the investigator to determine whether or not a time series behaves acceptably. For example, in a queueing problem, a time series may contain high frequency fluctuations with a regular period which would lead to a rejection of the job assignment rule which causes them *if* it were only known that these oscillations existed. Such phenomena may be present without any clear-cut cause, for Slutzky long ago pointed out that periodicities may be induced in a process by a summation of purely random causes [25, pp. 105–146]. Choosing too large a sampling interval precludes any study of rapid oscillations. Unless the sampling interval is short compared to the period of oscillation which is relevant for study, all information about such fluctuations is lost.

Design considerations based on some *a priori* knowledge about the stochastic process play an important role in determining a suitable sampling interval. The investigator has some idea of what kinds of fluctuations are worth studying. For example, if he is interested in studying oscillations that last longer than a day, he must observe the process at intervals of half a day or less so that he may later examine the contribution of the component whose period is one day.

A problem also arises in estimating the spectrum of a continuous stochastic process when the sample record consists of uniformly spaced observations. If a time series is collected at intervals of Δt, then the highest frequency that the resulting spectrum can give us information about is $\pi/\Delta t$; this is known as the Nyquist frequency [4]. However, $\{X(t)\}$ will in general have non-zero contributions from frequencies greater than $\pi/\Delta t$. Since the process is observed at equidistant points, it is impossible to distinguish between contributions from the frequency λ_0, which is less than the Nyquist frequency, and contributions from frequencies $[(2\pi k/\Delta t - \lambda_0), (2\pi k/\Delta t + \lambda_0), k = 1, 2, \cdots \infty]$. The resulting spectrum therefore confounds the contributions of these higher frequencies with those of the frequencies in the interval $(0, \pi/\Delta t)$. This phenomenon is known as *aliasing* and requires appropriate consideration if spectral estimation is to be at all meaningful [4, pp. 31–33] and [14, pp. 143–145].

Assume that an appropriate sampling interval Δt has been chosen. No loss of generality occurs if we define Δt as the unit interval. The number of observations then equals the sample record length T. Before turning to the estimation procedures, it is instructive to investigate the effects of finite record lengths on the spectrum. If $g(\lambda)$ is the spectrum at angular frequency λ, and we have a stochastic sequence $\{X_t\}$ that occurs at unit intervals, then we have

$$g(\lambda) = (1/\pi)\,[2 \textstyle\sum_{\tau=1}^{\infty} R_\tau \cos \lambda\tau + R_0] \qquad 0 \leqq \lambda \leqq \pi, \tag{3.2a}$$

$$R_\tau = \int_0^\pi g(\lambda) \cos \lambda\tau \, d\lambda \qquad \tau = 0, 1, \cdots, \infty. \tag{3.2b}$$

Now consider applying a *weighting function* $k_M(\tau)$ to the autocovariance function such that

$$g^*(\lambda) = (1/\pi)[2 \textstyle\sum_{\tau=1}^{\infty} k_M(\tau) R_\tau \cos \lambda\tau + R_0]. \tag{3.3}$$

Using Expression (3.2b), the function $g^*(\lambda)$ may be rewritten as

$$g^*(\lambda) = (1/\pi)\left\{2 \textstyle\sum_{\tau=1}^{\infty} k_M(\tau)\left[\displaystyle\int_0^\pi g(\omega) \cos \omega\tau \, d\omega\right] \cos \lambda\tau + \displaystyle\int_0^\pi g(\omega) \, d\omega\right\},$$

$$= (1/\pi) \int_0^\pi g(\omega)[2 \textstyle\sum_{\tau=1}^{\infty} k_M(\tau) \cos \omega\tau \cos \lambda\tau + 1] \, d\omega \tag{3.4a}$$

$$\begin{aligned} &= (1/\pi) \int_0^\pi g(\omega) \, [\textstyle\sum_{\tau=1}^{\infty} k_M(\tau) \cos (\omega + \lambda)\tau \\ &\qquad + \textstyle\sum_{\tau=1}^{\infty} k_M(\tau) \cos (\omega - \lambda)\tau + 1] \, d\omega. \end{aligned}$$

Let

$$\mu_M(\omega) = (1/\pi)[2 \textstyle\sum_{\tau=1}^{\infty} k_M(\tau) \cos \omega\tau + 1], \tag{3.5}$$

$$K_M(\omega, \lambda) = [\mu_M(\lambda + \omega) + \mu_M(\omega - \lambda)]/2, \tag{3.6}$$

so that

$$g^*(\lambda) = \int_0^\pi g(\omega) K_M(\omega, \lambda) \, d\omega. \tag{3.7}$$

The spectral function $g^*(\lambda)$ is a weighted average of the true spectrum where the *averaging kernel* $K_M(\omega, \lambda)$ is given by Expression (3.6). Also observe that the function $\mu_M(\omega)$ is the Fourier cosine transform of the weighting function $k_M(\tau)$.

Since the sample record is of finite length T, we can estimate at most $T - 1$ autocovariances. Suppose that we actually know the first M autocovariances with certainty. Our natural instinct would be to replace the upper limit of summation in Expression (3.2a) by M. This corresponds to choosing a weighting function

$$\begin{aligned} k_M(\tau) &= 1 \qquad \tau = 1, 2, \cdots, M \\ &= 0 \qquad \text{elsewhere,} \end{aligned} \tag{3.8}$$

so that we compute a spectral average

$$g^*(\lambda) = (1/\pi)[2 \textstyle\sum_{\tau=1}^{M} R_\tau \cos \lambda\tau + R_0]. \tag{3.9}$$

The corresponding averaging kernel for Expression (3.8) is

$$K_M(\omega, \lambda) = \frac{1}{2\pi}\left\{\frac{\sin [(M + \frac{1}{2})(\omega + \lambda)]}{\sin [(\omega + \lambda)/2]} + \frac{\sin [(M + \frac{1}{2})(\omega - \lambda)]}{\sin [(\omega - \lambda)/2]}\right\} \tag{3.10}$$

which permits us to compute $g^*(\lambda)$—not $g(\lambda)$, the function of interest. If $g(\lambda)$ changes rapidly with λ relative to changes in $K_M(\omega, \lambda)$, then $g^*(\lambda)$ is not very representative of $g(\lambda)$. This presents us with the problem of *resolution*. How does one choose a weighting function $k_M(\tau)$ whose corresponding averaging kernel $K_M(\omega, \lambda)$ concentrates its weight closely around λ so that $g^*(\lambda)$ is highly representative of $g(\lambda)$? The choice of an appropriate weighting function has occupied the attention of many time series analysts for over a decade and the most advantageous choice is still not clearly defined.[3]

[3] See [14] for a comparison of averaging kernels.

In this paper we use the Bartlett weighting function

$$k_M(\tau) = 1 - \tau/M \qquad \tau = 1, 2, \cdots, M - 1 \tag{3.11a}$$

with corresponding averaging kernel

$$K_M(\omega, \lambda) = \frac{1}{2\pi M}\left\{\frac{\sin^2 [M(\omega + \lambda)/2]}{\sin^2 [(\omega + \lambda)/2]} + \frac{\sin^2 [M(\omega - \lambda/2]}{\sin^2 [(\omega - \lambda)/2]}\right\} \tag{3.11b}$$

In retrospect, it would have been more advisable to use the more commonly employed weighting functions suggested by Tukey and Parzen which have certain advantages over Bartlett's suggestion. In the examples in Section IV, the more common weighting functions would, however, offer only meager improvement, the Bartlett weighting function thus being adequate for our purposes.

Our estimates of the autocovariance function are

$$C_\tau = (1/(T - \tau)) \sum_{t=1}^{T-\tau} (X_t - m)(X_{t+\tau} - m) \tag{3.12}$$

where

$$m = (1/T)\sum_{t=1}^{T} X_t, \tag{3.13}$$

and our estimated spectral averages are

$$g^*(\lambda) = (1/\pi)[2 \sum_{\tau=1}^{M} k_M(\tau)C_\tau \cos \lambda\tau + C_0]. \tag{3.14}$$

The focusing power of an averaging kernel is conveniently measured by its *bandwidth*. As used here, bandwidth is defined as the length of a rectangle which has the area and maximum height of the averaging kernel used in the estimation procedure. It essentially measures the degree to which the averaging kernel concentrates its area around the frequency for which a spectral estimate is being derived. For the Bartlett window, bandwidth is

$$\beta(\lambda) = 2\pi/M \tag{3.15}$$

and is measured in radians.

Were we to know the rate of change of the true spectrum, we could pick an appropriate bandwidth to resolve the spectrum. Since this information is lacking, we accomplish resolution by estimating spectra for several values of M to find a narrow enough bandwidth to make $g^*(\lambda)$ a well-resolved estimate of the true spectrum.

The choice of M also determines the approximate number of independent estimates if we assume the underlying sequence to be Gaussian. The Bartlett window permits about $M/3$ independent estimates, which means that estimates are approximately independent if they are at least $\pi/(M/3)$ radians or $3/(2M)$ cycles apart.[4] This independence is a very desirable property of spectral estimation and contrasts with the statistical nature of the correlogram wherein estimates are often so highly correlated with each other that they obscure the true behavior of the time series [3, pp. 254–256], [24, pp. 982–983], and [4, pp. 103–106]. These ap-

[4] Note that in [14] Jenkins uses half the base as his definition of bandwidth. This leads to the same number of independent estimates, however.

proximately independent spectral estimates may be derived for a limited sample size, whereas correlogram estimates are only asymptotically uncorrelated.

If $\{X(t)\}$ is a multivariate Gaussian sequence and the true spectrum varies more slowly than the averaging kernel in the vicinity of λ, then

$$\text{(3.16)} \qquad \operatorname{Var}[\hat{g}^*(\lambda)] \sim (2\pi/T)\hat{g}^2(\lambda) \int_0^\pi K_M^2(\omega, \lambda)\, d\omega$$

and, furthermore, $\hat{g}^*(\lambda)/g(\lambda)$ is approximately distributed as chi square with the number of *equivalent degrees of freedom* given by

$$\text{(3.17)} \qquad \operatorname{EDF}(\lambda) = T\left[\pi \int_0^\pi K_M^2(\omega, \lambda)\, d\omega\right]^{-1}.$$

If $\{X(t)\}$ is Gaussian and the deviations $[g^*(\lambda) - g(\lambda)]$ are not too great, then

$$\text{(3.18)} \qquad \operatorname{Var}\{\log[\hat{g}^*(\lambda)]\} \approx \Psi(M, T)$$

where

$$\text{(3.19)} \qquad \Psi(M, T) = (2\pi/T) \int_0^\pi K_M^2(\omega, \lambda)\, d\omega = 2M/(3T),$$

except[5] at frequencies 0 and π where it is $4M/3T$. For T sufficiently large, we may assume $\log[\hat{g}^*(\lambda)]$ to be Gaussian distributed with mean $\log[g(\lambda)]$ and various $2M/(3T)$. If we are comparing two spectra, we may then regard

$$\{\log[\hat{g}_1^*(\lambda)] - \log[\hat{g}_2^*(\lambda)]\},$$

the statistic for the null hypothesis, as being Gaussian distributed with mean zero and variance $4M/(3T)$.

If $\{X(t)\}$ were not Gaussian, it would be inappropriate to treat spectral estimates as chi square variates for, as Jenkins has noted, the validity of this assumption is highly dependent on the Gaussian nature of $\{X(t)\}$ [14, p. 164]. The assumptions about $\log[\hat{g}^*(\lambda)]$ are, however, fairly insensitive to $\{X(t)\}$ being Gaussian and, consequently, the logarithmic test of the null hypothesis is reasonably valid for non-Gaussian sequences. In many simulation experiments, especially those concerned with queueing problems, the sequence of interest is non-Gaussian. It is therefore advisable to work with $\log[\hat{g}^*(\lambda)]$ when testing the homogeneity of spectra.

The principal purpose of many simulation experiments is to study the mean of a sequence. When the sequence is Gaussian, its sample mean is a Gaussian variate. When the sequence is non-Gaussian, care must be exercised in interpreting the sample mean. Diananda has shown that under rather general conditions the sample mean of a covariance stationary sequence asymptotically converges to normality [9]. The conditions are sufficiently general to include processes usually studied in simulation experiments. For sufficiently large T, we may therefore treat $(m - \mu)/V^{1/2}$ as a standardized Gaussian variate. In Section V we test the validity of this treatment.

[5] We shall generally neglect endpoint considerations in the remainder of the paper.

In general V as well as μ is unknown. If we replace V by an estimate and choose T to be sufficiently large, the effects of the substitution may be regarded as minor. Since our purpose here is to study spectra and not means, we refer the interested reader to a paper by Fishman for additional comments on the study of means in simulation experiments [10].

If the true spectrum changes rapidly compared to the averaging, distortion will be common. To combat this, we may use a procedure known as *prewhitening*. In very simple terms, this amounts to taking weighted differences in place of our original observations. The estimated spectrum of this series is considerably flatter than the spectrum of the original one and, therefore, the spectral averaging takes place over a more uniform spectrum. To obtain the original spectrum, one divides this modified spectrum at each frequency by the frequency domain *transfer function* corresponding to the weighted differencing scheme in the time domain. This builds up the spectrum in the appropriate way making use of the fact that differencing a series in the time domain corresponds to multiplying its frequency function by a transfer function corresponding to the differencing procedure. The net result is a reduction in the distortion in the estimates.

Our analysis in Section V relies on using a sufficient number of lags to accomplish good resolution. Had we prewhitened our data, fewer lags would have been necessary. We did not prewhiten the data in order not to obscure the central purpose of this paper by introducing a sophisticated prewhitening procedure. The reader who is interested in this technique can read the relevant pages in Blackman and Tukey [4, pp. 39–42].

IV. Description of the Model

The model studied is a single-channel queueing system with Poisson arrivals and arbitrary service times. In the experimental runs reported, the service times are either exponentially distributed or constant. The server can be viewed as a machine or as a generalized service facility receiving inputs from some stochastic input stream.

If the server is idle, an arriving job is immediately assigned to it and a sample processing time is generated. If the server is engaged when the job arrives, the job is filed in an input queue, which is a file of jobs waiting to be serviced. In this study two rules for selecting jobs from this queue are used. One rule is the commonly employed FIFO or FCFS (First In, First Out or First Come, First Served) rule, which assigns the highest priority to the job that arrives at the shop earliest. The second rule assigns the highest selection priority to the job in the queue that has the shortest processing time at the time a selection is made; this rule is called the SHOPN, or shortest operation time rule, after the notation of Conway [6, pp. 73–77].

The computer program was written in GASP. It was run on both the IBM 7090 and 7044 computers, because The RAND Corporation changed its computer system during the course of experimentation. The running time of the program varied from 51 seconds to 8 minutes for runs of 4,000 simulated shop-hours, the variation in time being due to the different arrival rates studied. All runs were made at a fixed value of traffic intensity, $\rho = 0.90$.

TABLE 1

Simulation Experiment Statistics

λ	μ	ρ	Priority Rule	Service Distribution	No. of Arrivals	No. of Completions	No. of Jobs That Waited	Time
4.5	5.0	.9	FCFS	exponential	17,978	17,954	16,110	8 min
4.5	5.0	.9	SHOPN	exponential	17,978	17,969	16,125	8 min
.045	.050	.9	—	constant	199	199	196	51 sec

Computation of Model Statistics

The experimental runs were carried out under identical sampling situations. The series of random numbers used were identical for each run; each arrival generated one random variate for the interarrival time between it and the subsequent arrival, and possibly one random variate for the servicing time (in the exponential service time case).

Each run began at 0.00 simulated hours and ran to 4,000.00 simulated hours. Different numbers of jobs were completed in the different runs, because of the randomness of the arrival process and the different priority rules, but the number of jobs was large relative to the differences between runs. Table 1 contains some statistics on the numbers of jobs studied in each of the experimental runs.

Collection of information for use in computing the queueing statistics also began at time 0.00. Our initial feeling was that we should allow some time for the effect of our arbitrary selection of initial condition (zero jobs in queue, the server idle) to die down, but the length of this time, about 114 hours for the FCFS run, was so small relative to our total run length of 4000 hours that we felt it was unnecessary.[6]

Each simulated hour, the number of jobs waiting in the queue plus the job in service (if any) was recorded. At the end of the simulation run this series was input to the spectral analysis program. The simulation program also kept a time-integrated average of the number of jobs in the queue during the run for a check against the computations of the spectral analysis program and as an error check on the simulation program itself.

V. The Analysis of the Simulated Single-Server Queueing Problem

Sections II and III introduced the reader to the elementary concepts of covariance stationary processes and statistical spectral analysis. We now offer several examples which synthesize simulation experiments and spectral estimation. This synthesis provides a comprehensive understanding of simulation output data which form a time series generated by a simulated stochastic process. Our examples are simple, the purpose being to emphasize method rather than problem content.

Consider the simple FCFS single-server queueing model with Poisson arrival rate μ_1 and exponentially distributed service time with mean $1/\mu_2$. For the first simulation experiment we set

[6] This corresponds to three correlation times. See Section II.

(5.1a) $$\mu_1 = 4.5 \text{ arrivals/hour},$$

(5.1b) $$\mu_2 = 5 \text{ jobs serviced/hour},$$

so that $\{X(t)\}$, the queue length process, had[7]

(5.1c) $$\mu = 9,$$

(5.1d) $$\sigma^2 = 90,$$

(5.1e) $$f(0) = 24.19,$$

(5.1f) $$\sigma^2 f(0) = g(0) = 2177.2,$$

and

(5.1g) $$\tau^* = 38 \text{ hours}.$$

We decided to make the sample run approximately 100 times the correlation time so that

(5.2) $$T = 4000 \text{ hours}.$$

The sampling interval was set at

(5.3) $$\Delta t = 1 \text{ hour}.$$

Morse [20, p. 260] has shown that for large frequencies the spectrum is like

(5.4) $$g(\lambda) \approx 2\mu_1/(\pi\lambda^2).$$

Since the sampling interval is one hour, the highest angular frequency which the stochastic sequence $\{X_t\}$ permits is π radians per hour. Therefore the difference between σ^2, the variance of $\{X(t)\}$, and R_0, the variance of $\{X_t\}$, is

(5.5) $$\begin{aligned} \sigma^2 - R_0 &\approx \int_\pi^\infty 2\mu_1/(\pi\lambda^2)\, d\lambda, \\ &\approx 2\mu_1/\pi^2, \\ &\approx .91, \end{aligned}$$

which is less than one per cent of σ^2. This is actually not the only difference between the variances. The two spectra also differ somewhat in the $(0, \pi)$ interval; however, this difference diminishes as the residual variance beyond π radians becomes smaller and, therefore, the two spectra should be quite alike in the relevant interval. The estimated spectrum is, however, subject to possible aliasing and computational roundoff error.

The asymptotic variance of the sample mean m is

(5.6) $$V = 2\sigma^2\tau^*/T = 1.71.$$

One may construct a probability interval such that

(5.7) $$P(\mu - V^{1/2}s_\alpha \leqq m \leqq \mu + V^{1/2}s_\alpha) \approx 1 - \alpha$$

[7] See Section II.

where the quantity s_α is the point on the cumulative Gaussian curve corresponding to probability $\alpha/2$. Choosing the probability $1 - \alpha$ to be .95 results in

$$P(5.65 \leqq m \leqq 12.35) \approx .95. \tag{5.8a}$$

The sample mean for this first experiment was 8.48, well within the set limits.

It is known that the steady-state distribution of queue length for the FCFS discipline is geometric. This means that, were we to sum n independent observations of queue length, the sum would have a negative binomial distribution. That is, the n-fold convolution of a geometric distribution function yields a negative binomial distribution function. To check on Expression (5.8), we summed 34 observations which were 120 simulated hours apart, and derived the probability interval for the negative binomial distribution. A choice of 120 simulated hours was conservative, considering that $3\tau^*$ is equal to 114 hours. This choice of 120 hours led to

$$P(6.06 \leqq m' \leqq 12.44) = .95 \tag{5.8b}$$

where m', the sample average of these independent observations, was 9.26. The similar width in probability intervals lends validity to using our Gaussian assumption for m especially since this former assumption implies the convolution of an infinite number of distribution functions.

Figure 5 shows the actual and estimated autocorrelation functions for the first example. The former was computed by applying Simpson's Rule to Expression (2.32c) in Section II; the latter, by using Expression (3.12) in Section III. Note the smoothness of the estimated curve and its reluctance to monotonically damp out the way the actual function does. Indeed, the estimated function appears to be somewhat periodic. The smoothness results from the high correlation among correlogram estimates. The curve's reluctance to damp out is consistent with the findings of many investigators.

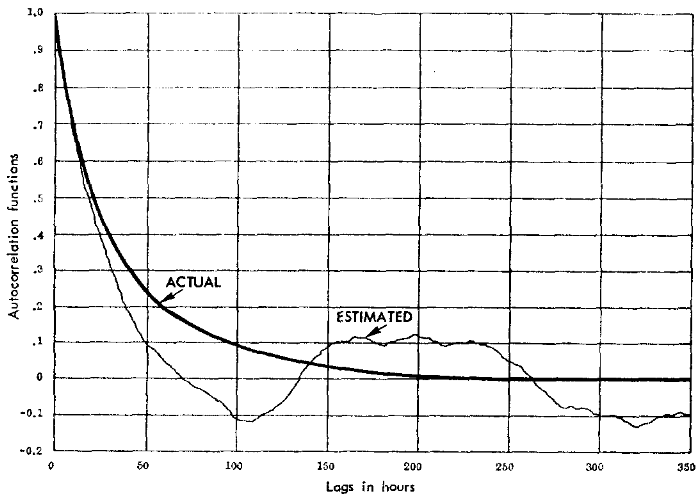

Fig. 5. Actual and estimated FCFS queue length correlograms.

TABLE 2

Estimated Queue-Length Spectra for FCFS Rule

(0–.05 cycles/hour)

Frequency (cycles/hr)	Number of Lags (M)							
	150	200	250	300	350	400	450	500
.0000	843.0	884.0	953.0	1010.0	1030.0	1010.0	985.0	954.0
.0033	842.0	852.0	836.0	824.0	788.0	747.0	715.0	702.0
.0067	617.0	659.0	668.0	656.0	625.0	611.0	605.0	592.0
.0100	325.0	278.0	271.0	256.0	231.0	225.0	220.0	223.0
.0133	233.0	253.0	260.0	272.0	272.0	276.0	273.0	268.0
.0167	143.0	124.0	111.0	111.0	107.0	104.0	102.0	103.0
.0200	92.1	92.4	94.3	98.6	102.0	105.0	110.0	114.0
.0233	79.9	74.7	70.1	68.2	67.7	69.1	68.2	68.3
.0267	58.7	58.5	56.6	55.7	55.6	57.1	58.3	58.4
.0300	51.7	47.4	49.3	50.7	51.7	53.4	55.7	58.7
.0333	48.7	49.2	47.0	46.0	44.7	43.5	41.2	38.1
.0367	42.0	41.5	40.4	39.6	38.2	36.5	35.2	35.3
.0400	29.4	27.0	27.1	26.7	25.5	24.5	25.0	24.6
.0433	27.8	27.2	26.0	25.5	24.1	23.1	21.4	20.6
.0467	23.3	22.8	22.1	21.8	20.5	19.4	18.7	17.9
.0500	22.6	21.0	21.0	21.0	19.9	19.3	18.7	18.4

Spectra were estimated for 150 through 500 lags in 50-lag increments. Table 2 shows the contrasts and similarities among these estimated spectra for low frequencies. These estimates are spaced $\pi/150$ radians or .0033 cycles apart.

Observe that spectral estimates for zero frequency seemingly stabilize at about 1000. No stabilization is evident at .0033 and .0067 cycles, but for M greater than 300 hours, all other estimates appear stable. This means that the averaging kernel's bandwidth corresponding to 300 lags is sufficient to resolve the spectrum almost everywhere. The actual queue-length spectrum was also computed using Expression (2.32) and Simpson's Rule. It varied about 15 per cent over the 300-lag bandwidth of .00167 cycles at zero frequency, which is much less than the averaging kernel varies over this interval. Here the zero estimate seems well resolved with 300 lags.

Figure 6 shows the actual and estimated FCFS queue-length spectra for 300 lags. Although they differ in magnitude, one observes they have the same general shape. The estimates are spaced $\pi/150$ or .0033 cycles apart. If the original sequence were Gaussian, then every other estimate would be independent. Frequencies greater than .1 cycle contribute little to the process relative to frequencies below this point. Had we sampled at 5-hour instead of unit intervals, little information would have been lost. No peaks appear in the midband frequency range. This implies that no regular cycles are to be observed in the process.

To test whether or not $\hat{g}^*(\lambda)$ may be considered to have been drawn from a process with the actual spectrum shown, we assume that

$$\log [\hat{g}^*(\lambda)] \sim N\{\log [g(\lambda)], \Psi(M, T)\}, \tag{5.9}$$

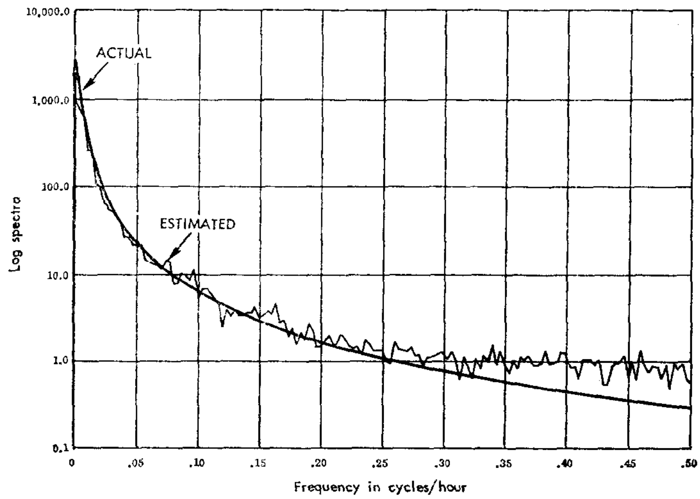

FIG. 6. Actual and estimated FCFS queue length spectra ($M = 300$).

where

$$\Psi(M, T) \approx 2M/(3T) \tag{5.10}$$

This leads to

$$P[e^{-\Phi} \leqq \hat{g}^*(\lambda)/g(\lambda) \leqq e^{\Phi}] = 1 - \alpha \tag{5.11}$$

$$\Phi = s_\alpha(\Psi(M, T))^{1/2}. \tag{5.12}$$

In the midband range,

$$\Psi(300, 4000) \approx .05, \tag{5.13a}$$

so that for $\alpha = .95$,

$$\Phi = 1.96(.05)^{1/2} = .438 \tag{5.13b}$$

and

$$P[.644 \leqq \hat{g}^*(\lambda)/g(\lambda) \leqq 1.553] = .95. \tag{5.13c}$$

Figure 7 shows the function $\hat{g}^*(\lambda)/g(\lambda)$ and its midband probability interval. One observes from Figure 7 that almost all points lie within the band except near the high frequencies. In this interval the ratios usually exceed the upper limit. This is probably due to the folding-back effect caused by aliasing. This effect is of course most noticeable where the estimated spectrum is not much greater than the aliasing, which in this case is the high frequency range.

The width of the probability interval is surprisingly large in Figure 7 at first glance, but a moment's reflection shows that these results are not unexpected. The variance of our estimates in the midband range is

$$\operatorname{Var}[\hat{g}^*(\lambda)] \approx .05g^2(\lambda). \tag{5.14}$$

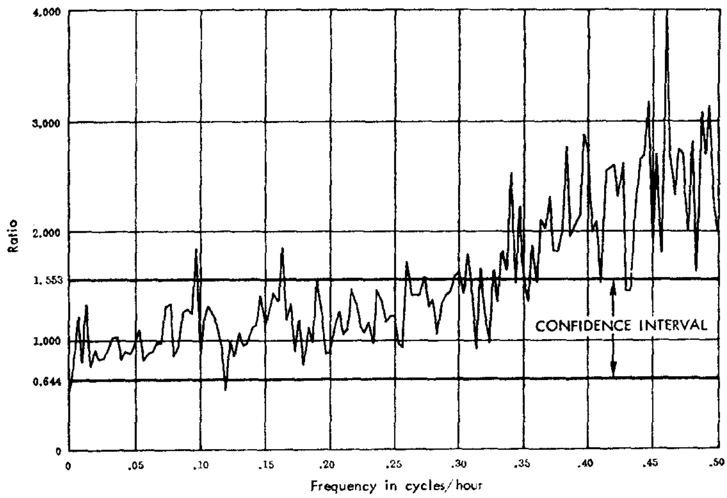

FIG. 7. 95 percent confidence interval for ratio of estimated and actual FCFS queue ength spectra (M = 300).

TABLE 3

Sample Record Lengths (T) for Various Minimum Relative Deviations

$[\Psi(300, T)]^{1/2}$	.05	.10	.15	.20	.25
T (hours)	80,000	20,000	8,889	5,000	3,200

The standard deviation is then greater than 22.4 per cent of the magnitude of the spectrum. This seems relatively large and clearly indicates that, if one considers the estimated spectrum with 300 lags sufficient for analysis, the sample record length T must be substantially longer to gain precision in estimating the spectrum. Table 3 shows T for several relative deviations $[\Psi(300, T)]^{1/2}$ in the midband range. Remember that these sample record lengths will give this approximate precision for midband frequencies. To obtain similar precision at zero frequency would require sample record lengths double these sizes.

While the magnitudes of these sample record lengths appear large, one should remember that, with a correlation time of 38 hours, one can only consider observations that are 76 hours apart as being equivalent to independent observations.[8] Therefore, the 4000-hour record offers only the equivalent of about 53 independent observations. Indeed the .10 relative deviation for M equal to 300 requires 263 equivalent independent observations. Viewed in this way, these large sample record lengths do not appear so enormous.

The numbers in Table 3 may be misleading to the reader unless he remembers that our original concern was with the width of the probability interval. Table 4 shows the logarithmic test's interval widths for various relative deviations.

[8] See Section II.

TABLE 4

Width of Probability Intervals (α = .05) for Various Relative Deviations

$[\Psi(300, T)]^{1/2}$	.05	.10	.15	.20	.25
Lower limit	.91	.82	.75	.68	.61
Upper limit	1.11	1.22	1.34	1.48	1.63
Width	.2	.4	.59	.80	1.02

The choice of interval width of course depends on the purpose of the statistical analysis. This is especially important in this example, where we note that a 25-fold increase in sample record length from 3,200 to 80,000 attenuates the interval from 1.02 to .2. One must judge whether this reduction in interval width is indeed worth the increase in the cost of a longer record.

This first example offers us some worthwhile insights into the nature of the stochastic process of queue length. We note in passing that the variance is primarily concentrated in a frequency band between zero and .1 cycle per hour. This means that the changes in queue length within a ten-hour interval are minor relative to changes which occur over longer time intervals. The absence of concentrations at any other frequency but zero is evidence that no cyclic elements are present in $\{X(t)\}$. The rather rapid sampling fluctuations which result from independence are small compared to the over-all spectral shape, and emphasize the truth of our earlier remarks about the desirability of looking at the estimated spectrum at appropriately chosen points, rather than at the estimated correlogram, for a characterization of the process.

Our prior knowledge has simplified the analysis of the FCFS single-server queueing model with Poisson arrivals and exponentially distributed service time. More often than not, such prior information does not exist for simulation experiments. As a second example, consider this same queueing problem with the job assignment priority rule now being the shortest operation (SHOPN) rule. Under this rule, the next job chosen to receive service is the one with the shortest processing time. Again we chose

(5.15a) $$T = 4000 \text{ hours},$$

(5.15b) $$\Delta t = 1 \text{ hour}.$$

Figure 8 shows the estimated correlograms for both the FCFS and SHOPN rules. The SHOPN correlogram drops off faster than does the FCFS one, but both exhibit the cyclic phenomenon referred to earlier.

Table 5 shows the estimated queue-length spectrum for low frequencies .0033 cycles apart. All spectra seem similar. One must remember that although more lags improve resolution, they reduce stability. For example, the estimated spectrum using 500 lags has only 60 per cent of the equivalent degrees of freedom that estimates with 300 lags have.

Figure 9 compares the estimated FCFS and SHOPN queue-length spectra for 300 lags. Here we are concerned not with detailed resolution but rather with

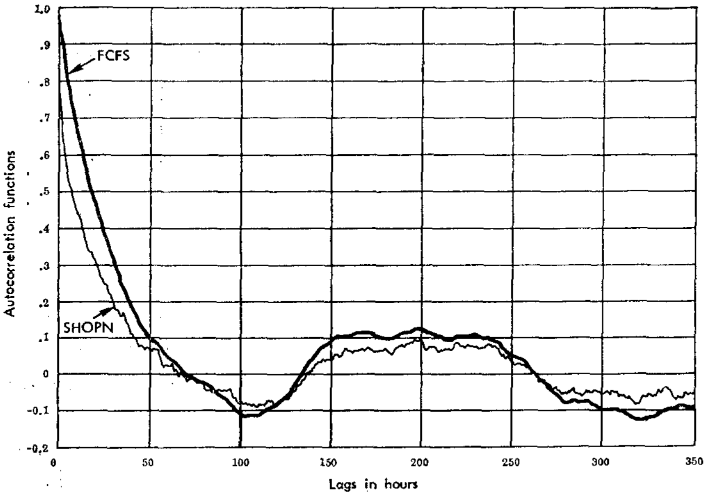

FIG. 8. Estimated queue length correlograms for FCFS and SHOPN rules.

TABLE 5
Estimated Queue-Length Spectra for SHOPN Rule
(0–0.5 cycles/hour)

Frequency (cycles/hr.)	Number of Lags (M)					
	150	200	250	300	350	500
.0000	59.7	60.8	64.9	68.7	69.5	64.3
.0033	61.4	63.1	62.6	62.2	60.1	55.7
.0067	46.1	48.6	48.8	47.5	45.1	43.1
.0100	25.3	22.4	22.1	21.2	19.7	19.4
.0133	18.9	20.2	20.7	21.5	21.7	21.4
.0167	12.4	11.0	10.0	10.1	10.0	9.7
.0200	9.4	9.7	10.0	10.4	10.8	11.6
.0233	7.5	6.9	6.4	6.1	6.0	5.9
.0267	5.7	5.6	5.6	5.5	5.5	5.9
.0300	5.5	5.2	5.4	5.5	5.7	6.6
.0333	5.2	5.2	4.9	4.8	4.8	4.4
.0367	4.8	4.8	4.8	4.9	4.9	5.1
.0400	4.2	4.0	4.0	3.9	3.9	4.0
.0433	3.9	4.0	4.0	3.8	3.7	3.4
.0467	2.7	2.5	2.3	2.3	2.1	1.7
.0500	2.9	2.8	2.9	3.1	3.1	2.9

the appearance of the over-all curves. Visual observation shows that the SHOPN spectrum is clearly lower than the FCFS spectrum and also exhibits a weak contribution from .1 cycle per hour upward.

Our next step is to test whether $\hat{g}_F^*(\lambda)$ is significantly greater than $\hat{g}_S^*(\lambda)$. The subscript F refers to the FCFS rule. Regard the statistic

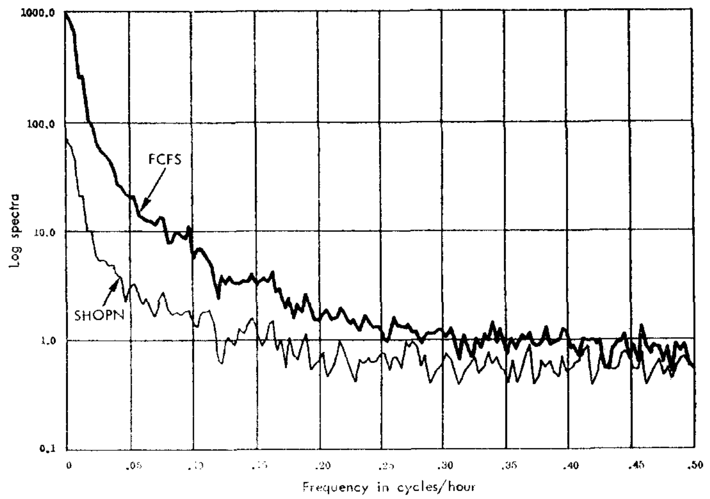

FIG. 9. Estimated queue length spectra for FCFS and SHOPN rules with 300 lags ($M = 300$).

$$\frac{\log[\hat{g}_F^*(\lambda)] - \log[g_S(\lambda)] - \log[\hat{g}_F^*(\lambda)] + \log[g_S(\lambda)]}{(\Psi(M_F, T_F) + \Psi(M_S, T_S))^{1/2}} \tag{5.16}$$

as Gaussian-distributed with mean zero and unit variance; we test the null hypothesis that $g_F(\lambda)$ equals $g_S(\lambda)$ against the alternative hypothesis that $g_F(\lambda)$ is greater than $g_S(\lambda)$. We have a one-tail test with

$$1 - \alpha = .95, \tag{5.17}$$

$$M_F = M_S = 300, \tag{5.18}$$

$$T_F = T_S = 4000. \tag{5.19}$$

Therefore we accept the null hypothesis if the ratio $\hat{g}_F^*(\lambda)/\hat{g}_S^*(\lambda)$ lies in the acceptance region below 1.68.

Figure 10 shows the ratio of 150 equispaced estimates. Note that the estimates lie outside the acceptance region up to about .25 cycles or 4 hours. Therefore, we reject the null hypothesis and accept the alternative one that $g_F(\lambda)$ is significantly greater than $g_S(\lambda)$ in this low frequency band. Beyond .25 cycles the ratio is less than 1.68. Here our acceptance of the null hypothesis should be weighted by the fact that the higher frequencies contribute little to either spectra and that these parts of the spectra are more amenable to computing roundoff error and aliasing.

The above test has ignored the fact that the sampling properties of the spectra are different near and at zero and .5 cycles. Explicit consideration of this fact would not significantly alter our conclusions and, for the purposes of hypothesis testing on the over-all spectra, we shall not bother with these qualifications.

Our treatment of the SHOPN spectrum has relied on its estimates for 300 lags. This results because of the fairly stable estimate at zero frequency. Were we to ignore this consideration, we could have used the estimates for 150 lags.

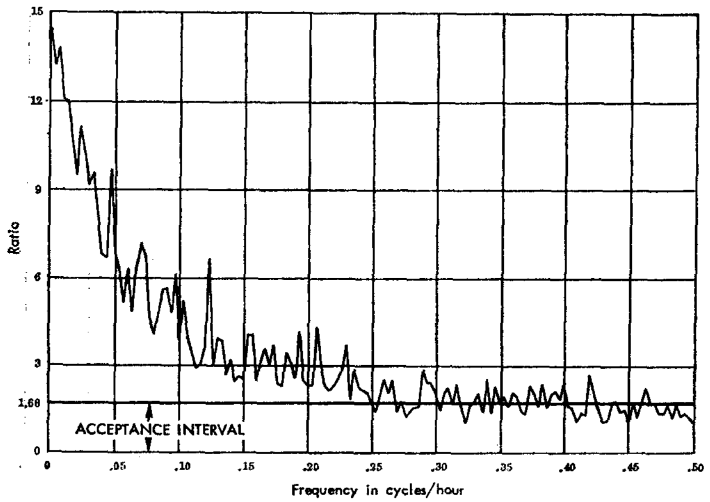

Fig. 10. Testing the ratio of estimated FCFS and SHOPN queue length spectra at the 95 percent level ($M = 300$).

Then

$$\text{Var}\{\log [\hat{g}_S^*(\lambda)]\} \approx .025. \tag{5.20}$$

Using Expression (5.20), the upper bound on the null hypothesis acceptance region becomes 1.57, which is not substantially smaller than 1.68, the present boundary of the acceptance region.

Figure 6 shows that the ratio of estimated spectra decreases with increasing frequency. The ratio about which we are drawing inferences is $\{(\sigma_F^2/\sigma_S^2)[f_F(\lambda)/f_S(\lambda)]\}$. The first term in this product is constant for all λ. If the second term is also invariant with changing λ, then both queue-length processes have the same autocorrelation structure. If this second ratio decreases with increasing λ, as in the case of our estimates, then the FCFS queue-length process is more highly autocorrelated than the SHOPN process. This occurs because the spectral density function of the more highly correlated process is more concentrated at the low-frequency end and therefore decreases faster than the spectral density function of the less autocorrelated process. Therefore, Figure 10 leads us to conclude that the FCFS rule generates a more highly autocorrelated rule than the SHOPN rule does.

This deduction about time dependence will not surprise those readers who have observed the estimated correlograms in Figure 8. The point to remember is that the deduction from the spectra is based on approximately independent estimates, whereas the correlogram estimates are highly correlated and must rely on an asymptotic argument for independence, the rate of which depends on the properties of the time series itself. This is the advantage of deducing information from estimated spectra rather than from estimated correlograms.

The foregoing examples have facilitated our comparing the stochastic properties of two processes. While the value of spectral estimation for comparisons is evident, equally important information can be drawn from the study of individual spectra. Now, it is not always necessary to completely resolve the spectrum (estimate its point magnitude sharply) to study stochastic properties. These estimates are actually spectral averages, and it is noteworthy that the shape of the spectral average curve contains information about the process. That is, besides knowing the relative importance of all frequency contributions, we would also like to identify frequencies which account for any regularity in the process. Our two examples have shown that low frequencies are much more important than high frequencies, but no single frequency induces regularity in the process. Our final example focuses on the question of regularity.

Consider a single-server queueing problem with Poisson arrival rate μ_1 and a constant service time $1/\mu_2$. Let $\{X(t)\}$ be the queue-length process and specify

$$\mu_1 = .045 \text{ arrivals/hour,} \tag{5.21}$$

$$\mu_2 = .050 \text{ jobs serviced/hour,} \tag{5.22}$$

$$T = 4000 \text{ hours,} \tag{5.23}$$

$$\Delta t = 1 \text{ hour.} \tag{5.24}$$

Therefore [8, p. 54]

$$E[X(t)] = (\mu_1/\mu_2)[1 - \mu_1/(2\mu_2)]/(1 - \mu_1/\mu_2) = 4.95. \tag{5.25}$$

The sample mean is

$$m = 5.8 \tag{5.26}$$

and the estimated correlogram is shown in Figure 11. Note the smoothness induced by the correlation among estimates, and the fact that the correlogram has not approached zero in 500 hours. Perhaps more noteworthy is the periodic scalloped effect every 20 hours. The reluctance to damp out or oscillate about zero means that we should not expect well-resolved low-frequency spectral estimates, since the correlogram indicates correlation beyond 500 hours.

Figure 12 displays the estimated spectrum for 500 lags at intervals of .0033 cycles with approximately 24 degrees of freedom in the midband range. As usual, the low-frequency estimates are poorest because we have the least knowledge about long-run movements; however, it is clear that low frequencies do dominate the process, especially over intervals greater than 100 hours.

The peaked phenomena which occur at regular intervals beginning with .05 cycles may seem odd to the reader.[9] Each peak is at least an order of magnitude greater than other estimates in its proximity. These correspond to the scalloping phenomenon in the correlogram but are clearly more pronounced in the spec-

[9] To determine whether these peaks were real or a spurious consequence of the choice of averaging kernel, we estimated the spectrum using the kernel suggested by Parzen [22]. The appearance of the newly estimated spectrum was close to the original, suggesting that the peaks were not a consequence of the choice of averaging kernel.

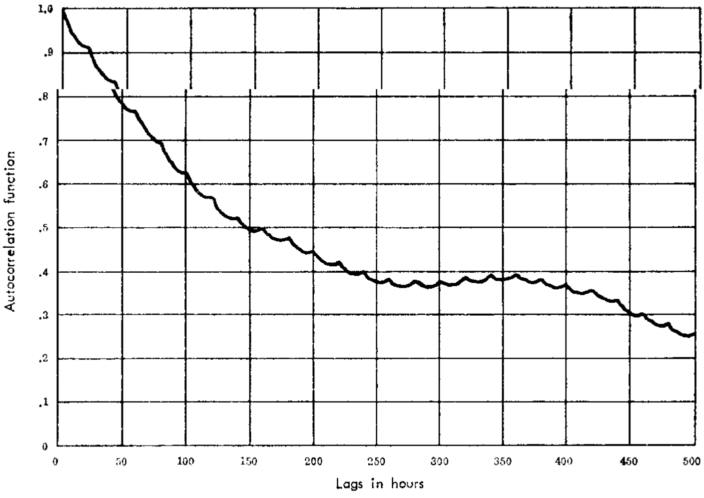

FIG. 11. Estimated queue length correlogram with constant service time.

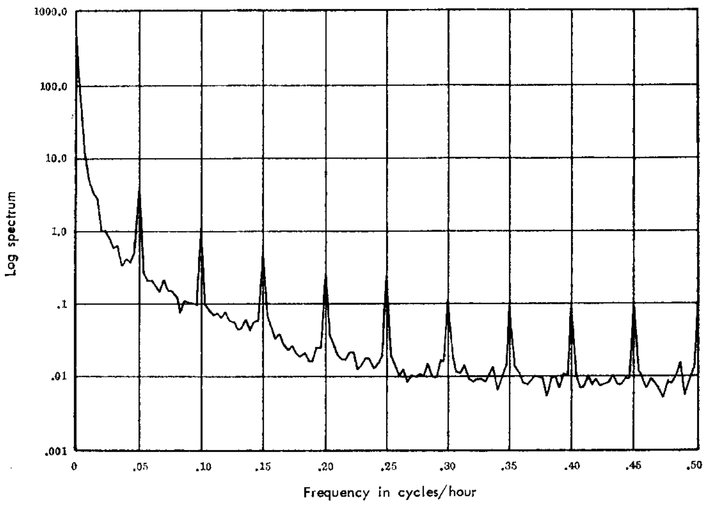

FIG. 12. Estimated queue length spectrum with constant service time.

trum. A little thought explains why these concentrations of variance at equispaced frequencies occur. As long as the queue is not empty, jobs leave the queue at precisely 20-hour intervals. This results from using a constant service time. Therefore, a regularity exists in the decrease of the queue length as long as there are jobs in the queue. The fundamental frequency of this decrease is .05 cycles, which is the reciprocal of the service time.

This regularity in the process is of relatively little importance when compared with the enormous long-run deviations that occur around the mean. Over short intervals, however, these effects are not noticeable and may be altogether unknown to the investigator who has designed the model. It is true that the periodicity is discernible in the correlogram, but its prominence in the spectrum is more conspicuous and it results from independent estimates.

This regularity brings up another consideration regarding the nature of completions. In the first example, arrivals had a Poisson distribution and exponentially distributed service times; therefore completions are Poisson distributed and form a patchwork of periods of activity and blank periods when the queue is empty [20, p. 256]. In the constant-service-time example, the completions process is quite different from that in Example 1. When the queue has x jobs, the completions pattern for at least the next x/μ_2 time units will consist of one completion every $1/\mu_2$ time units. These periods will be mixed with blank periods when the queue is empty, a pattern considerably different from Example 1.

Some confusion may exist in the reader's mind over why these peaks occur in the estimated spectrum at multiples of .05 cycles. Each of these multiples is a harmonic of this fundamental frequency. Were a peak present at the fundamental frequency only, the regularity in the process would be a sinusoidal wave. Queue length, however, is a step function or a square wave. As such, the periodic part of the step function is shaped by the fundamental frequency and all of its harmonics. This accounts for the presence of peaks at the harmonic.

The several examples which this Section has described convey, in our judgment, the kinds of information which may be extracted from simulation output data. We have used but a few of the suggestions which Section III described—our purpose being to exemplify, rather than to be exhaustive. The examples made two very strong points: more elaborate comparisons among data are possible than are currently being performed, and the means for making these comparisons are within the reach of the conscientious investigator.

VI. Conclusions

The paper has pointed out that the sampling properties of the correlogram present unmanageable problems due to the correlation among the estimates. As this correlation depends on the nature of the stochastic process itself, it is hard to draw inferences from the estimated correlogram whose sampling properties only simplify asymptotically. Alternatively, the sampling properties of the estimated spectrum are simple, and it is relatively easy to derive reasonably independent spectral estimates.

One conclusion emerges very sharply. As our investigation has studied the shape of the estimate of a continuous function, we have not only concerned ourselves with stability but have also explicitly considered the resolution of estimates so that, when plotted, they take on the shape of the function we are estimating.

Section III has shown that simple statistical tests are available for *smoothing* our results. After all, hypothesis testing and confidence interval derivations are merely ways of ordering results into smooth forms from which inferences are easily made. This goes a long way toward freeing an investigator from a reliance on point estimates, and leads him to drawing inferences from the shape that a set of point estimates form.

The results of Section V show the success that can be achieved using spectral analysis. The dissimilarity between the actual and estimated FCFS queue-length correlogram vindicated our assertions about their sampling properties. The contrastingly good fit of the associated spectra partly verified our reasons for choosing spectral rather than correlogram analysis. The interval of dissimilarity in this last graph derived perhaps more from an injudicious choice of sampling interval than from any other single source.

Our comparison of estimated spectra for the FCFS and SHOPN rules show how easy it is to do hypothesis testing using spectral analysis. Establishing the significant difference between these rules employed commonly known statistical tests. The inclusion of the single-server queueing problem with constant service time had a dual purpose. First, it showed that the stochastic properties of queue length were quite different from those in our first two examples. Second, it established this result by relying not so much on stability as on resolution. To gain precision for its estimated spectrum, a larger run would be required. However, the different shape of the curve was clearly evident and informed us that a phenomenon was present which might otherwise have gone unnoticed.

This paper has concentrated on simple examples as the easiest way to familiarize the reader with our proposals for studying simulated time series. Several logical extensions of our work suggest themselves when the simulation model under scrutiny is more complicated than our examples. Many simulations consist of a series of simple simulation models in which the output of one is the input to another. In the construction phase of the simulation, it is worthwhile knowing something about these individual simulated output processes. A simple spectral analysis of the output time series generated by each part of the simulation would provide the investigator with a basic understanding of how each part works. In many cases data characterizing the real world, which the simulation supposedly resembles, are available. Comparing the estimated spectra of the real-world data and the simulated data also seems worthwhile since it allows the investigator to judge how well the simulation does resemble the system it was designed to emulate.

The problems we have discussed have been simple in content. Our current research applies and extends the spectral methodology of this paper to more complex simulation environments. More specifically, we are investigating the series-server queueing problem for the purpose of more fully understanding the kinds of queues it generates and the nature of unnecessary idle time to which different job assignment priority rules lead.

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