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# On the bias of intermittent demand estimates

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**Abstract**

Forecasting and inventory control for intermittent demand items has been a major problem in the manufacturing and supply environment. Croston (Operational Research Quarterly 23 (1972) 289), proposed a method according to which intermittent demand estimates can be built from constituent events. Croston's method has been reported to be a robust method but has shown more modest benefits in forecasting accuracy than expected. In this research, one of the causes of this unexpected performance has been identified, as a first step towards improving Croston's method. Certain limitations are identified in Croston's approach and a correction in his derivation of the expected estimate of demand per time period is presented. In addition, a modification to his method that gives approximately unbiased demand per period estimates is introduced. All the conclusions are confirmed by means of an extended simulation experiment where Croston's and Revised Croston's methods are compared. The forecasting accuracy comparison corresponds to a situation of an inventory control system employing a re-order interval or product group review. 

*Keywords:* Forecasting; Inventory; Intermittent demand

## 1. Introduction

Intermittent demand appears at random, with many time periods having no demand. Moreover, when a demand occurs, the request is sometimes for more than a single unit. Intermittent demand creates significant problems in the manufacturing and supply environment as far as forecasting and inventory control are concerned. It is not only the variability of the demand size, but also the variability of the demand pattern that make intermittent demand so difficult to forecast.

In practice, exponential smoothing is often used when dealing with this type of demand. Exponential smoothing places more weight on the most recent data, resulting, in the case of intermittent demand, in a series of estimates that are highest just after a demand occurrence and lowest just before demand occurs again.

Croston [1] proposed a method that builds demand estimates taking into account both demand size and the interval between demand incidences. Despite the theoretical superiority of such an estimation procedure, empirical evidence suggests modest gains in performance when compared with simpler forecasting techniques; some evidence even suggests losses in performance.

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In an effort to identify the causes of the forecast inaccuracy, as a first step towards improving Croston's method, a mistake was found in Croston's mathematical derivation of the expected estimate of demand. That mistake contributes towards the unexpectedly modest benefits of the method when applied in practice. Subsequently, a modification in Croston's method was developed that, theoretically, eliminates the forecast bias. The improvement in forecast accuracy achieved by the Revised Croston's method is shown, by means of simulation, to be statistically significant at the 0.01 significance level.

The paper has been organised as follows: in Section 2, the theoretical background and empirical evidence on Croston's method are presented; in Section 3, a correction in Croston's method is made and a modification in his method that gives approximately unbiased demand per period estimates is introduced; Section 4 describes the simulation study conducted in order to evaluate our findings, and the subsequent results; finally, in Section 5 we state the conclusions, contributions and potential extensions of this paper.

## 2. Theoretical background and research motivation

### 2.1. Exponential smoothing

Croston [1] (as corrected by Rao [2]) proved the inappropriateness of exponential smoothing as a forecasting method when dealing with intermittent demands and he expressed in a quantitative form the bias associated with the use of this method when demand appears at random with many time periods showing no demand at all.

He first assumes deterministic demands of magnitude $\mu$ occurring every $p$ review intervals. Subsequently, the demand $Y_t$ is represented by

$$Y_t = \begin{cases} \mu, & t = np + 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $n = 0, 1, 2, \ldots$ and $p \geqslant 1$.

Conventional exponential smoothing updates estimates every inventory review period whether or not demand occurs during this period. If we are forecasting one period ahead, $Y'_t$, the forecast of demand made in period $t$, is given by

$$Y'_t = Y'_{t-1} + \alpha e_t = \alpha Y_t + (1-\alpha)Y'_{t-1}, \tag{2}$$

where $\alpha$ is the smoothing constant value used, $0 \leqslant \alpha \leqslant 1$, and $e_t$ the forecast error in period $t$.

Under these assumptions if we update our demand estimates only when demand occurs the expected demand estimate per time period is not $\mu/p$, i.e. the population expected value, but rather:

$$E(Y'_t) = \frac{\mu}{p}\frac{p\alpha}{1-(1-\alpha)^p} = \frac{\mu\alpha}{1-\beta^p}, \tag{3}$$

where $\beta = 1 - \alpha$.

Croston then refers to a stochastic model of arrival and size of demand, assuming that demand sizes, $z_t$, are normally distributed, $\mathrm{N}(\mu, \sigma^2)$, and that demand is random and has a Bernoulli probability $1/p$ of occurring in every review period (subsequently, the inter-demand intervals, $p_t$, follow the geometric distribution with a mean $p$). Under these conditions the expected demand per unit time period is

$$E(Y_t) = \frac{\mu}{p}. \tag{4}$$

In this case, Croston showed that when demand estimates are updated every period using exponential smoothing, then the mean demand estimate equals the population expected value, and the variance of the demand estimates equals

$$\mathrm{Var}(Y'_t) = \left[\frac{p-1}{p^2}\mu^2 + \frac{\sigma^2}{p}\right]\frac{\alpha}{2-\alpha}. \tag{5}$$

If we isolate, though, the estimates that are made after a demand occurs, Croston showed that these estimates have the biased expected value

$$E(Y'_t) = \mu(\alpha + \beta/p). \tag{6}$$

The error, expressed as a percentage of the average demand, is shown to be $100\alpha(p-1)$ and reveals an increase in estimation error produced by the Bernoulli arrival of demands as compared with constant inter-arrival intervals.

### 2.2. Croston's method

Croston, assuming the above stochastic model of arrival and size of demand, introduced a new method for characterising the demand per period by modelling the demand in one period from constituent events. According to his method, separate exponential smoothing estimates of the average size of the demand and the average interval between demand incidences are made after demand occurs. If no demand occurs the estimates remain the same. If we let:

$p'_t$ = the exponentially smoothed inter-demand interval, updated only if demand occurs in period $t$ so that $E(p'_t) = E(p_t) = p$, and

$z'_t$ = the exponentially smoothed size of demand, updated only if demand occurs in period $t$ so that $E(z'_t) = E(z_t) = z$,

then following Croston's estimation procedure, the forecast, $Y'_t$ for the next time period is given by:

$$Y'_t = \frac{z'_t}{p'_t}, \tag{7}$$

and, according to Croston, the expected estimate of demand per period in that case would be

$$E(Y'_t) = E\left(\frac{z'_t}{p'_t}\right) = \frac{E(z'_t)}{E(p'_t)} = \frac{\mu}{p} \tag{8}$$

(i.e. the method is unbiased).

Now more accurate estimates can be obtained and a significant advantage of the method is that when demand occurs every period the method is identical to exponential smoothing. Thus, it can be used not only for the intermittent demand items but for the rest of the SKUs (Stock Keeping Units) as well. Johnston and Boylan [3], show how intermittent the demand must be in order to benefit from Croston's method replacing exponential smoothing (inter-demand interval greater than 1.25 inventory review periods).

The variance of the demand estimates per time period is approximated (Stuart and Ord [4], give the approximation of the variance of the ratio of two independent random variables) by

$$\mathrm{Var}\left(\frac{z'_t}{p'_t}\right) \approx \frac{\alpha}{2-\alpha}\left[\frac{(p-1)^2}{p^4}\mu^2 + \frac{\sigma^2}{p^2}\right]. \tag{9}$$

Lead-time replenishment decisions take place only in the time periods following demand occurrence and are based on the equation

$$R_t = z'_t + Km_t, \tag{10}$$

where $m_t$ is the mean absolute deviation of the demand size forecast errors and $K$ is a safety factor.

### 2.3. Empirical evidence presented in the literature

Croston's method is based on assumptions of independence (successive intervals are independent, successive demand sizes are independent and intervals and sizes are mutually independent) and normality of the demand size. Willemain et al. [5], found correlations and distributions in real world data that violated Croston's assumptions. So they conducted a comparative evaluation of exponential smoothing and Croston's method under less idealised conditions using:

(a) Monte Carlo Simulation. Theoretical demand data was generated for different scenarios (lognormal distribution of demand size, cross-correlation between sizes and intervals, autocorrelated sizes and autocorrelated intervals) that violated Croston's assumptions. The comparison with exponential smoothing was mainly based on the mean absolute percentage error (MAPE).

(b) Industrial data, focusing on the MAPE for one step ahead forecasts.

The researchers concluded that Croston's method is robustly superior to exponential smoothing and can provide tangible benefits to manufacturers forecasting intermittent demand. A very important feature of their research, though, was the fact that industrial results showed very modest benefits as compared with the simulation results.

Sani and Kingsman [6], conducted, with the use of simulation, a comparison between alternative forecasting methods evaluating them with respect to the cost and service level resulting from their implementation. The analysis was carried out on real and simulated low demand data. The forecasting methods compared were: a modification of

Croston's method [7] as far as the variance, utilised for the replenishment levels calculation, is concerned; exponential smoothing updating every inventory review period and every nine inventory review periods; an empirical forecasting method developed by one of the dealers who provided some of the real demand data that was used and the 1 year (52 weeks) moving average updating every period. The modification to Croston's method is based on the following calculation of demand variance:

$$\operatorname{Var}(Y_t) \approx \max\left\{\frac{\operatorname{var} z_t'}{p_t'}, \frac{1.1 z_t'}{p_t'}\right\}. \tag{11}$$

The proposed method of calculating the variance was motivated by the excessive replenishment stocking resulting from Eq. (10) and is based on the reasonable assumption [8] that demand follows the negative binomial distribution and subsequently the variance has to be greater than the mean.

The forecasting methods were compared across ten periodic inventory control methods (five empirically developed "simple" rules and five heuristics proposed in the academic literature).

The results showed that the best forecasting methods in terms of cost (ordering, holding and shortage costs are considered) were the empirical forecasting method used and the moving average followed by Croston's method. When the service level was used as the performance criterion then the exponential smoothing updating every review period was the best method followed by the Croston forecast and the moving average. Overall the best forecasting method taking into account both cost and service level was concluded to be the 52 week moving average followed by Croston's method. Croston's method seems to perform very accurately under well-stated assumptions and when its performance is assessed on theoretically generated data. When real data is used, there is some evidence that less sophisticated, in fact, very simple, forecasting methods seem to provide more accurate results and lead to more effective inventory control.

## 3. Expectation of the demand per time period

### 3.1. *Expected estimate of demand – Croston's method*

The empirical evidence suggests that Croston's method's theoretical superiority is not reflected in the forecasting accuracy associated with the use of this method. Subsequently, in an effort to identify the causes of this unexpected forecasting performance, a mistake was found in Croston's mathematical derivation of the expected estimate of demand per time period.

We know (assuming that order sizes and intervals are independent) that

$$E\left(\frac{z_t'}{p_t'}\right) = E(z_t')E\left(\frac{1}{p_t'}\right), \tag{12}$$

but

$$E\left(\frac{1}{p_t'}\right) \neq \frac{1}{E(p_t')}. \tag{13}$$

The expected demand per time period, see Appendix A, for a smoothing constant of unity ($\alpha = 1$), is

$$E\left(\frac{z_t'}{p_t'}\right) = E(z_t')E\left(\frac{1}{p_t'}\right) = \mu\left[-\frac{1}{p-1}\log\left(\frac{1}{p}\right)\right]. \tag{14}$$

So if, for example, the average size of demand when it occurs is $\mu = 6$, and the average inter-demand interval is $p = 3$, the average estimated demand per time period using Croston's method (for $\alpha = 1$) is not $\mu/p = 6/3 = 2$ but it is $6 * 0.549 = 3.295$ (64.75% bias implicitly incorporated in Croston's estimate).

The maximum bias over all possible smoothing parameters is given by

$$\mu\left[-\frac{1}{p-1}\log\left(\frac{1}{p}\right)\right] - \frac{\mu}{p}. \tag{15}$$

This is attained at $\alpha = 1$. For realistic $\alpha$ values, the magnitude of the error is much smaller. This is quantified, using a simulation approach, in Section 4.3 of this paper.

### 3.2. *Corrected estimate of demand – Revised Croston's method*

Since Croston's method fails to produce unbiased demand per period estimates, the authors considered finding a different expectation, that equals $\mu/p$. Such an expectation is the following (see Appendix B):

$$E(Y'_t) = E(z'_t)E\left(\frac{1}{p'_t c^{p'_t - 1}}\right) = \frac{\mu}{p}, \tag{16}$$

where $c$ is a constant. Theoretically, $c$ has to be infinitely large to allow (16) to be true. Nevertheless the result is a good approximation if $c$ is sufficiently large.

Therefore, based on Croston's concept of building demand estimates from constituent events, we propose an alternative updating procedure so that the bias associated with Croston's method is, in theory, eliminated. When demand occurs, the length of the latest inter-demand interval, $p_t$, is recorded and the value of the ratio $1/p_t c^{p_t - 1}$ is calculated. Consequently both demand size and $1/p_t c^{p_t - 1}$ are updated with exponential smoothing, only after demand occurs, and they are combined as follows:

$$Y'_t = z'_t \frac{1}{p'_t c^{p'_t - 1}}, \tag{17}$$

where $c > 100$.

If no demand occurs, the estimates remain exactly the same as in Croston's method.

## 4. Simulation study

### 4.1. Objectives and assumptions

At this stage, a simulation experiment was considered essential to confirm the expected lack of bias from the Revised Croston's method and to quantify the bias of Croston's method for realistic smoothing constants. The bias associated with Croston's method is expected to increase as the $\alpha$ value increases and a certain pattern according to which the bias increases is expected to emerge. Therefore, it was decided, from a theoretical perspective, to use an extended range of $\alpha$ values (up to 1), although most of these values are not utilised in practice.

Moreover, the effect of using the updating procedure developed during this research, Eq. (17), is also assessed and Croston's and Revised Croston's methods are directly compared with respect to the bias resulting from their implementation on the theoretically generated demand data series.

Once more the demand is assumed to occur as a Bernoulli process (probability of demand occurrence, $1/p$) and the demand sizes are assumed to be normally distributed (with mean, $\mu$ and standard deviation, $\sigma$). Although the second assumption is highly doubted by the authors normality is still assumed for the purpose of this simulation experiment as the purpose was to evaluate and compare Croston's method under the original assumptions that he developed his method.

### 4.2. Design of the simulation experiment

Two comparisons between Croston's and Revised Croston's method could be considered, the first referring to the estimates observed at every point in time and the second making the comparison only immediately after an issue has occurred. The former would correspond to the situation of a system employing a re-order interval or product group review and is the one considered for the purpose of this research, whilst the second option would be associated with a re-order level stock replenishment system.

The simulation experiment included 200 factor combinations. Ten different smoothing constant values were used: 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, five different inter-demand intervals: 2, 4, 6, 8, 10, and four different combinations of the means and standard deviations of demand size ($\mu = 1$ and $\sigma = 0$, $\mu = 6$ and $\sigma = 1$, $\mu = 10$, $\sigma = 2$ and $\mu = 20$ and $\sigma = 2$). The standard normal variates were generated by using the Box–Muller method [9]. The simulation model was run for each of the 200 combinations. The programme was written in *Visual Basic* (embedded in the *Excel* version) and each run consisted of 20,000 simulated demand periods of which the first 100 were considered the transient interval. In addition five replications were applied to each of the 200 combinations. The averages of the values obtained are the ones to be used as the final results.

For certain means and standard deviations of demand size, Tables 1 and 2 give the results of 20 simulation runs (factor combinations).

*Column 1*: The smoothing constant value used.

*Column 2*: The population expected demand per time period ($\mu/p$).

*Column 3*: The simulated average demand per unit time period.

*Column 4*: Simulated results for Croston's method average estimate of demand per time period.

*Column 5*: Maximum biased estimates: Eq. (14).

*Column 6*: Simulated results for estimating the demand per time period using the Revised Croston's method, estimation procedure: Eq. (17). The constant value, $c$ is set to 100.

*Column 7*: Error (bias) in Croston's method. Difference between columns 3 and 4.

*Column 8*: Croston's error is expressed as a percentage of the average demand per period (column 3).

### 4.3. Simulation results

The bias implicitly incorporated in Croston's estimates increases, as expected, in all cases (different possible combinations of $\mu$, $\sigma$ and $p$) as the smoothing constant value increases.

For the Revised Croston's method, the demand per time period estimates equal, approximately, for all the different factor combinations, the population expected demand per time period: $\mu/p$. By using the updating procedure given in Eq. (17) as

Table 1
Simulation results for $N$ (6, 1), $p = 2$, $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$

| 1<br>$\alpha$ value | 2<br>$E(Y'_t) = \mu/p$ | 3<br>Simulated $\mu/p$ | 4<br>Croston's $E(Y'_t)$ | 5<br>Max. biased estimates | 6<br>Revised Croston's $E(Y'_t)$ | 7<br>Croston's error | 8<br>% error |
|---|---|---|---|---|---|---|---|
| 0.1 | 3 | 3.00 | 3.07 | 4.15 | 3.00 | 0.07 | 2.51 |
| 0.2 | 3 | 2.99 | 3.15 | 4.15 | 3.02 | 0.15 | 5.18 |
| 0.3 | 3 | 3.02 | 3.26 | 4.15 | 3.02 | 0.23 | 7.85 |
| 0.4 | 3 | 2.98 | 3.31 | 4.15 | 2.97 | 0.32 | 10.93 |
| 0.5 | 3 | 3.01 | 3.44 | 4.15 | 3.04 | 0.43 | 14.33 |
| 0.6 | 3 | 2.99 | 3.52 | 4.15 | 2.97 | 0.52 | 17.38 |
| 0.7 | 3 | 3.00 | 3.64 | 4.15 | 2.97 | 0.63 | 20.99 |
| 0.8 | 3 | 2.97 | 3.75 | 4.15 | 2.97 | 0.77 | 26.10 |
| 0.9 | 3 | 3.00 | 3.93 | 4.15 | 3.01 | 0.92 | 30.79 |
| 1 | 3 | 2.99 | 4.12 | 4.15 | 2.94 | 1.13 | 37.87 |

Table 2
Simulation results for $N$ (10, 2), $p = 4$, $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$

| 1<br>$\alpha$ value | 2<br>$E(Y'_t) = \mu/p$ | 3<br>Simulated $\mu/p$ | 4<br>Croston's $E(Y'_t)$ | 5<br>Max. biased estimates | 6<br>Revised Croston's $E(Y'_t)$ | 7<br>Croston's error | 8<br>% error |
|---|---|---|---|---|---|---|---|
| 0.1 | 2.5 | 2.49 | 2.60 | 4.62 | 2.50 | 0.10 | 4.33 |
| 0.2 | 2.5 | 2.53 | 2.74 | 4.62 | 2.55 | 0.20 | 8.27 |
| 0.3 | 2.5 | 2.47 | 2.81 | 4.62 | 2.58 | 0.34 | 13.76 |
| 0.4 | 2.5 | 2.52 | 3.01 | 4.62 | 2.58 | 0.48 | 19.37 |
| 0.5 | 2.5 | 2.47 | 3.13 | 4.62 | 2.54 | 0.65 | 26.55 |
| 0.6 | 2.5 | 2.49 | 3.30 | 4.62 | 2.59 | 0.80 | 32.14 |
| 0.7 | 2.5 | 2.46 | 3.49 | 4.62 | 2.53 | 1.02 | 41.45 |
| 0.8 | 2.5 | 2.48 | 3.79 | 4.62 | 2.63 | 1.30 | 52.59 |
| 0.9 | 2.5 | 2.51 | 4.17 | 4.62 | 2.72 | 1.66 | 66.11 |
| 1 | 2.5 | 2.47 | 4.69 | 4.62 | 2.67 | 2.21 | 89.30 |

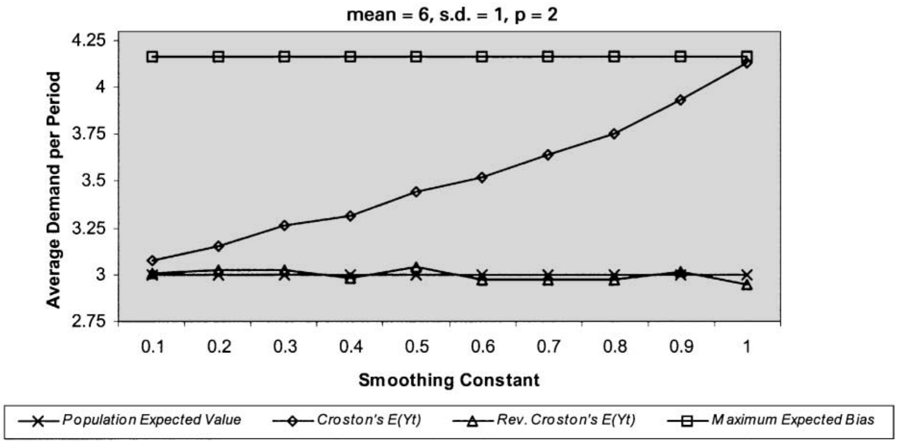


Fig. 1. Mean = 6, s.d. = 1, $p = 2$.

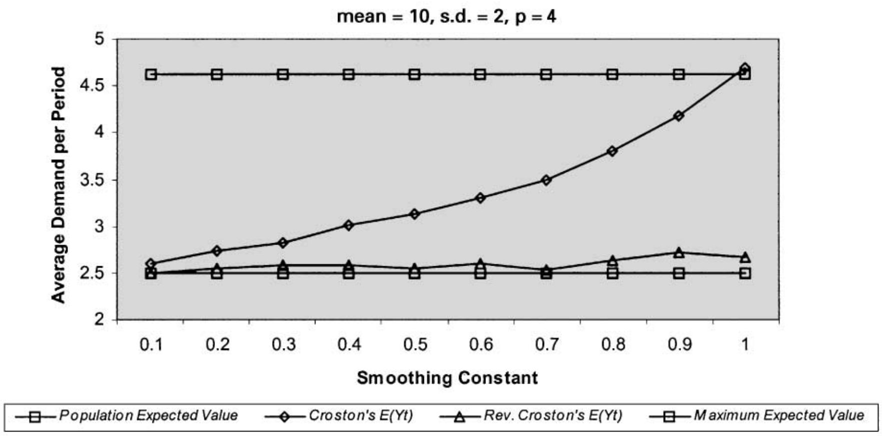


Fig. 2. Mean = 10, s.d. = 2, $p = 4$.

opposed to that utilised by Croston's method we obtain, as indicated in column 6, unbiased forecasts for all the smoothing constant values.

Figs. 1 and 2 show for certain $\alpha$ values the relationship between: Croston's method simulated results for the average estimate of demand per time period, simulated results for the average demand estimate per period when we use the Revised Croston's method, the theoretically expected demand per time period, $\mu/p$, and the maximum possible biased estimates, Eq. (14).

Unfortunately, a particular equation, according to which the bias in Croston's method can be calculated, could not be determined. As the smoothing constant values increase, Croston's method overestimates the actual demand per time period and for $\alpha = 1$ the estimate made by Croston's method can theoretically be quantified by (14) and the bias implicitly incorporated in this estimate by (15). For the rest of the smoothing constant values the bias implicitly incorporated in Croston's estimates can be shown only by means of simulation.

#### *4.3.1. Statistical significance of results*

At this stage, we need to recognise the fact that the Revised Croston's method is exactly unbiased for constant $c$ set at infinity. Hence, for any finite value of $c$ the method is expected to have a very small bias. We also know that for low $\alpha$ values the original Croston's method has a small bias. Therefore, a statistical test is employed to demonstrate whether or not the Revised Croston's method gives the reduction in bias that is anticipated.

The simulated bias associated with the use of both methods was recorded for each of the 200 simulation runs (factor combinations) and the improved forecasting accuracy (i.e. bias reduction) has been tested for statistical significance by using the $t$-test (difference between population means). We cannot assume that the populations of bias are normal or that the population variances are known. Because though the sample size is very large (200 observations) the test statistic, for testing the difference between the bias given by the two methods considered, can be the $t$-test (one sided). The null hypothesis developed was that there is no significant difference between the bias given by the two different methods. For a significance level, $\alpha = 0.01$, the critical region is $t > 2.32$. So if the test statistic is greater than the above specified value we have to reject the null hypothesis and conclude that there is significant difference between the bias associated with the use of the two forecasting techniques.

The test was repeated for the specific factor combinations that included the $\alpha$ values most commonly used in practice (i.e. $\alpha$ values not greater than 0.3). It was decided that this was important in order to assess the practical benefit of employing the Revised Croston's method. In this latter case, the sample size was reduced from 200 to 60 (the $\alpha$ values considered were 0.1, 0.2 and 0.3). The number of observations was large enough to allow the $t$-test to be employed.

The test statistic value was for the above-described cases 9.55 and 5.73, respectively, demonstrating the improvement in forecasting accuracy achieved by employing the Revised Croston's method.

## 5. Conclusions, contribution and extensions

Croston's concept of building future demand estimates from constituent events, has been claimed to be of great value for manufacturers that deal with intermittent demand. Despite the theoretical superiority of such an estimation procedure, empirical evidence suggests modest gains in performance when compared with simpler forecasting techniques; some evidence even suggests losses in performance. As a first step towards improving Croston's method, an effort has been made during this research to identify a cause of this unexpected performance.

Croston's separate estimates of the demand size and the inter-demand interval are correct, but if combined as a ratio fail to produce accurate estimates of demand per time period. Hence, if inventory rules rely on the separate estimates only, then no difficulty arises. If, however, any rules depend on the expected demand per time period, then bias should always be expected when Croston's method is used. During the research that is described in this paper an effort was also made to associate the bias in Croston's method with certain smoothing constant values. Apart from the fact that the bias increases as the $\alpha$ value increases no specific relationship was identified. Nevertheless, we are able now to quantify the maximum bias that can be implicitly incorporated in Croston's estimates. For $\alpha = 1$ the maximum bias is given by (15). For the rest of the possible smoothing constant values simulation is necessary in order to quantify the bias.

Moreover, Croston's method is recommended only for low values of $\alpha$. In all simulation runs, the bias becomes pronounced for $\alpha$ values above 0.15. Therefore, a modification in his method is regarded as essential in order to produce unbiased, demand per time period, estimates. In this paper, a modification in Croston's method is introduced that gives approximately unbiased demand per period estimates. We call the method the Revised Croston's method, and the updating procedure is given in Eq. (17). The improved forecasting accuracy achieved is tested by means of simulation for the case of an inventory system that employs a re-order interval or product group review and subsequently confirmed statistically by employing the $t$-test.

Nevertheless, how this modification performs on real demand data is something that still needs to be assessed. Moreover, the simulation experiment should be extended in order to quantify the effect of the forecasting improvement on inventory control when cost and/or service level are considered. Preliminary results on approximately 17,000 intermittent demand data files (covering demand history of 2 up to 7 years) confirm that improvements in forecasting accuracy translate into cost and/or service benefits. The data files come from the automotive and aerospace industries and they were selected on the basis of having an at least 1.25 review periods inter-demand interval [3]. The simulation experiment conducted on those files, the significance tests performed, the effect of improved forecasting accuracy on inventory control and the final results is a topic left for a later paper.

## Appendix A. Expectation of the inter-demand interval and its reciprocal mathematical derivation

(Assume that $\alpha = 1$ so that $p'_{t+1} = p_t$)

If we denote by $p_t$ the inter demand interval that follows the geometric distribution including the first success (i.e. demand occurring period) and by $1/p_t$ the probability of demand occurrence at period $t$, we then have:

$$E\left(\frac{1}{p_t}\right) = \sum_{x=1}^{\infty} \frac{1}{x}\frac{1}{p}\left(1 - \frac{1}{p}\right)^{x-1}$$

$$= \frac{1}{p}\sum_{x=1}^{\infty} \frac{1}{x}\left(\frac{p-1}{p}\right)^{x-1}$$

for $p > 1$ (i.e. demand does not occur in every single time period)

$$= \frac{1}{p}\sum_{x=1}^{\infty} \frac{1}{x}\frac{((p-1)/p)^x}{((p-1)/p)^1}$$

$$= \frac{1}{p}\frac{1}{(p-1)/p}\sum_{x=1}^{\infty} \frac{1}{x}\left(\frac{p-1}{p}\right)^{x}$$

$$= \frac{1}{p-1}\left[\frac{p-1}{p} + \frac{1}{2}\left(\frac{p-1}{p}\right)^2 + \frac{1}{3}\left(\frac{p-1}{p}\right)^3 + \cdots\right]$$

$$= -\frac{1}{p-1}\log\left(\frac{1}{p}\right).$$

## Appendix B. An expectation that gives unbiased estimates and the associated mathematical derivation

$$E\left(\frac{1}{p_t c^{p_t - 1}}\right)$$

$$= \textstyle\sum_{x=1}^{\infty} \frac{1}{x}\frac{1}{p}\frac{1}{c^{x-1}}\left(1 - \frac{1}{p}\right)^{x-1}$$

$$= \frac{1}{p}\textstyle\sum_{x=1}^{\infty} \frac{1}{x}\left(\frac{1}{c} - \frac{1}{cp}\right)^{x-1}$$

$$= \frac{1}{p}\textstyle\sum_{x=1}^{\infty} \frac{1}{x}\left(\frac{p-1}{cp}\right)^{x-1}$$

$$= \frac{1}{p}\left[1 + \textstyle\sum_{x=2}^{\infty} \frac{1}{x}\left(\frac{p-1}{cp}\right)^{x-1}\right],$$

$$\text{as } c \to \infty,\ E\left(\frac{1}{p_t c^{p_t - 1}}\right) \to \frac{1}{p}.$$

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