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SEQUENTIAL DECISION PROBLEMS: A MODEL TO EXPLOIT EXISTING FORECASTERS*†

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A sequential decision problem is partitioned into two parts: a stochastic model describing the transition probability density function of the state variable, and a separate framework of decision choices and payoffs. If a particular sequential decision problem is a recurring one, then there may often exist human forecasters who generate quantitative forecasts at each decision stage. In those cases where construction of a mathematical model for predictive purposes is difficult, we may consider using the forecasts and forecast revisions provided by the existing forecasting mechanism as the state variable. The paper considers a specific class of problems in which improved forecasts of some unknown quantity are available before each decision stage. A small number of actual forecasters are studied through analysis of historical data to see whether the data-generating process for forecast changes is quasi-Markovian. The data are generally, although not entirely, consistent with the hypothesis that ratios of successive forecasts are independent variates; their distribution appears to be conditionally Lognormal. Some possible reasons for these results are explored. In cases where the hypothesis holds, a dynamic programming approach to the sequential decision problem may be used to provide optimal decision rules. The usefulness of the approach is illustrated with a numerical example involving crop planning, and the example is extended to explore the effects of using the methodology when the required assumptions do not hold.

Introduction

A sequential decision problem has been described as

> a problem which involves the making of two or more decisions at different points in time, and which has the property that the later decision(s) may be influenced not only by the previous decisions, but also by some stochastic parameters whose values will actually have been observed before later decisions are made. [5, p. 159]

The key difference between sequential and nonsequential decision problems is that future decisions in sequential problems may be based partially on information known in the future but unknown at present.

Motivating Examples

There are numerous decision situations which fit the basic pattern outlined above, in which subsequent related decisions are made in light of additional information. Initial production scheduling decisions, followed by successive revisions in schedule, constitute a sequential decision problem. The marketing of new products, including research, product development, test marketing, regional introduction, and national

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introduction, illustrates a sequential decision problem in the marketing area. Successive plant expansions, related by economies of scale and affected by patterns of future demand, can be characterized by a sequential decision model.

The example used to investigate the applicability of the approach presented in this paper is a seasonal production supply problem in which the decision-maker cannot fully control his output. Specifically, consider a crop planning problem faced by a food processor, in which the following holds:

(1) Control over crop supply is imperfect, due to uncontrollable weather variations.
(2) Several related decisions affecting crop supply must be made at different stages in the acreage contracting, planting, and harvesting cycle, with more weather information available at later stages.
(3) The same complete set of decisions is faced each year, under essentially unvarying environmental conditions.

Mathematical Formulation of Finite-Horizon Sequential Decision Problem

Each relevant variable in the system is classified as a state variable, uncontrollable by the decision-maker, or a decision variable, controlled by the decision-maker. Assume there are N decision points in time or stages. Let Y_n represent the decision variable chosen at stage n, and let X_n represent the state variable taking on values obtaining at stage n. If future states X_{n+i} are determined with certainty, then the system is deterministic and the problem becomes nonsequential according to the definition given above. Now consider the more general problem in which future states are not known with certainty; the system is then stochastic. Let

$$h_n(X_{n+1} \mid X_1, Y_1, X_2, Y_2, \cdots, X_n, Y_n), \qquad n = 1, 2, \cdots, N-1$$

represent the probability density function for the state of the system at stage $n+1$, given information about all states and decisions up to and including stage n. Together, the state and decision variables and the transition probability density functions fully describe the probabilistic behavior of the system.

In order to use the above system as a model for a decision problem, it is necessary to impose a criterion or a payoff function over the state and decision variables. The payoff function describes the return or payoff resulting from all possible state and decision sequences. Let $R(X_1, Y_1, \cdots, X_N, Y_N)$ represent the total payoff given a complete sequence of states and decisions. Suppose it is possible to restate the total payoff as the sum of N payoff terms; then

$$(1) \qquad R(X_1, Y_1, \cdots, X_N, Y_N) = R_1(X_1, Y_1) + \cdots + R_N(X_N, Y_N).$$

It is also assumed that the expressed goal of the decision-maker is to maximize expected payoff. Finally, the initial state must be given, i.e., X_1 must be specified beforehand.

Markovian or Quasi-Markovian Property

Suppose the probability density function for the state variable depends only on the current state (as defined above), the current decision, and possibly the current stage number, i.e.,

$$(2) \quad h_n(X_{n+1} \mid X_1, Y_1, \cdots, X_n, Y_n) = h_n(X_{n+1} \mid X_n, Y_n), \quad n = 1, 2, \cdots, N-1.$$

Then, following reference [2], the system is called quasi-Markovian. If the probability density function of the state variable were dependent only on the current state, the system would be called Markovian. The quasi-Markovian or Markovian property of a system is important because the solution technique of dynamic programming requires a state space formulation to possess the Markovian (or quasi-Markovian) property. A quasi-Markovian formulation may always be obtained by a crude enlargement of the state variable into a vector-valued variable which includes all preceding variate values, but such an approach leads to a state vector with large dimensionality. Even on a modern electronic computer with large, high-speed memory capacity, dynamic programming computational results are generally limited to problems in which the state vector dimensionality is three or less. Thus a compact state description which meets the quasi-Markovian requirement directly rather than by artificial enlargement is most desirable for computational purposes.

Dynamic Programming Solution to Sequential Decision Problems

The usual approach to solution of sequential decision problems uses the functional equation technique of dynamic programming. Let $f_n(X_n\,; X_1, Y_1, \cdots, X_{n-1}, Y_{n-1})$ represent the payoff associated with optimal management of the system from stage n to the end of the planning horizon (stage N), given the current state is X_n and the past history is $\{X_1, Y_1, \cdots, X_{n-1}, Y_{n-1}\}$.

Assume that the system is quasi-Markovian; that is, assume equation (2) holds. Bellman's Principle of Optimality states that

> an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. [1, p. 83]

Application of the Principle of Optimality results in:

$$(3)\quad f_n(X_n) = \operatorname{Max}_{\{Y_n\}} \Big[R_n(X_n, Y_n)) + \int f_{n+1}(X_{n+1}) h_n(X_{n+1} \mid X_n, Y_n)\, dX_{n+1} \Big], n = 1, 2, \cdots, N-1$$

and

$$(4)\qquad f_N(X_N) = \operatorname{Max}_{\{Y_N\}} [R_N(X_N, Y_N)].$$

The recurrence relations in equation (3) can be solved successively, starting with equation (4), since the transition probability density functions are known. The maximization may be performed either by complete enumeration over a discrete grid approximating allowable state and decision values, by selective search procedures, or by means of calculus if it is applicable. Using the grid approximation approach, at each stage the optimal payoff is calculated for a grid of values approximating all allowable state descriptions, and for each state value considered, the optimal decision choice $Y_n^*(X_n)$ is calculated and recorded as a function of the value of the state variable. Finally, after working backwards from stage N to stage 1, the sequence of optimal decision rules thereby obtained $\{Y_1^*(X_1), Y_2^*(X_2), \cdots, Y_N^*(X_N)\}$ constitutes an optimal policy.

Recurring Sequential Decision Problems: Use of Existing Forecasters

A recurring sequential decision problem is defined as a sequence of identical finite-horizon sequential decision problems in which the following hold:

(1) The last decision of any sequential decision problem is made before the first decision of the succeeding sequential decision problem.
(2) If $X_{j,n}$ represents the state variable at the n^{th} decision stage in the j^{th} sequential decision problem, then the random variables $\{X_{i,m}, X_{j,n}; 1 \leq m, n \leq N\}$ are independent for $i \neq j$.
(3) The criterion or payoff in the j^{th} sequential decision problem is a function only of variables in the j^{th} sequential decision problem.

A sequential decision problem may be partitioned into two parts: a predictive (stochastic) model describing the transition probability density function of the state variable, and a separate framework of decision alternatives and payoffs. This same partitioning may exist in an actual decision problem; e.g., we often see estimates or forecasts being generated and recorded, and subsequently used as one input for a separate decision-making process. If the construction of a mathematical model for predictive purposes is unduly burdensome, we may attempt to use the existing forecast mechanism for predictive purposes. Specifically, the current forecast may be used as the state variable in the dynamic programming framework. Consider a specific class of sequential decision problems in which a revised forecast of the same unknown quantity is available at each decision stage, and in which the current forecast of that unknown is an adequate state description for the problem. If a human (or mechanistic) forecaster exists and if the problem is a recurring one, historical forecast data may be analyzed with the goal of constructing a model of the forecast data-generating process to obtain appropriate transition probabilities. In order to apply the data-generating process to future decisions it must also be assumed that the forecaster's method of forecasting will not change, and that the underlying stochastic process relating the unknown to the forecaster's information sources will not change significantly. That is, we assume the forecaster's performance and environment will not change significantly over the next planning period. Once a data-generating process for the forecast mechanism is obtained, it may be combined with a mathematical model of decision alternatives and payoffs to create a complete sequential decision model.

If the data-generating process for the forecast mechanism exhibits the quasi-Markovian property, then the dynamic programming technique will be computationally feasible and the optimal policy may be obtained. The next section considers empirical grounds for assuming that some forecasters may be approximated by a quasi-Markovian data-generating process.

Empirical Support for Quasi-Markovian Forecast Changes

Conditionally Lognormal Forecast Changes

Before the quasi-Markovian property is explored it is convenient to present results concerning the functional form of the transition probability density function relating successive forecasts. Although the precise functional form of the probability density function is unimportant for purposes of general applicability, most of the historical forecast data analyzed seems to fit (as a first approximation) the following functional form: ratios of successive forecasts are Lognormal variates. Thus as a matter of information this particular hypothesis is explored, although its validity or lack thereof is not crucial to the general method presented in this paper.

The first analysis considers forecasts of regional crop supply made by a manager of a food processing company during the planting, growing, and harvesting periods of three recent years. Forecasts were obtained at the following points in time: February 10, April 10, August 10, and September 10 of each year. An artificial fifth forecast was introduced as of October 15 which was set equal to the actual regional crop supply as measured after the end of the harvest. Since the company forecast does not change from stage one (February) to stage two (April), we examined only the last four company forecasts, denoted by X_2, X_3, X_4, and X_5. Three regions produced sweet corn, and two of these regions also produced snap beans. Company agriculture experts said the two crops were "equally unpredictable." They also emphasized that the three regions could be assumed to be independent of one another, since they were widely separated geographically. In addition, the two crops had different growing cycles, and were susceptible to somewhat different factors which could inhibit their maturing processes. It was therefore assumed that all fifteen forecast series[1] were mutually independent, and in this manner fifteen sets of observations were obtained for the company forecast data-generating process.

Normal probability paper was used to study the hypothesis that ratios of successive forecasts are distributed Lognormally. If a variate is distributed Lognormally, then a cumulative plot of sample observations on Lognormal probability paper would lie on a straight line except for sampling error. However, by definition the natural logarithm of a Lognormal variate is distributed Normally, so we may equivalently plot the cumulative natural logarithms of ratios of successive forecasts on Normal probability paper.

Figures 1 through 3 illustrate plots of natural logarithms of ratios of successive forecasts of regional crop supply. A pattern of distortion exists in the figures which is due to a specific tendency of some of the forecasts not to change from one period to the next. (A lack of new, substantive information between decision stages may explain this pattern.) If the hypothesis of Lognormality is modified to include a probability P_n that the ratio of successive forecasts (X_{n+1}/X_n) is 1.0, and a probability $(1 - P_n)$ that the ratio of successive forecasts is a Lognormally distributed random variable, the quasi-Markovian property still holds. Figures 4, 5 and 6 contain plots of natural logarithms of ratios of successive crop forecasts, conditional on a change in the forecast. The Kolmogorov-Smirnov test [11, pp. 47–52] was used to measure whether the sample data would indicate rejection of the hypothesis of conditional Lognormality. In all three cases, even at the .20 level of significance it was not possible to reject the hypothesis of conditional Lognormality.

Other Forecasts

In order to investigate whether the Lognormal (or conditionally Lognormal) behavior of crop forecast revisions might be found in other types of forecasts, analyses similar to the one above were performed on other types of historical forecast data:

1. Successive forecasts of total U.S. crop supply for processed vegetables made by the U.S. Department of Agriculture and published in monthly *Vegetable Processing* reports distributed by the Crop Reporting Board, Statistical Reporting Service, U.S.D.A.
2. Successive forecasts made by a financial analyst of earnings per share (E/S) for a given year for various corporations in an industry (three separate analysts forecasted E/S for companies in the utility, oil, and drug industries, respectively).
3. Successive forecasts of wholesale demands for cruise-season women's dresses sold by a particular company.

[1] The term "forecast" refers to a particular estimate made at a particular point in time, while "forecast series" refers to a group of successively revised forecasts, all estimating the same unknown. Thus the four successive forecasts X_2, X_3, X_4 and X_5 for snap beans in Region 3 in year 1 constitute one forecast series, and fifteen such series were available for study.

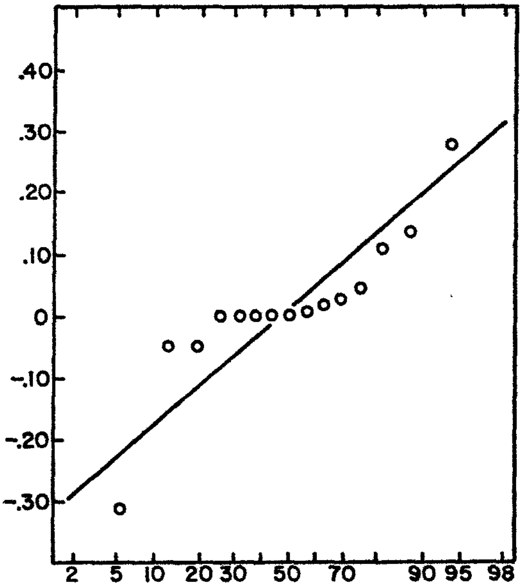

FIGURE 1. Regional Crop Supply Forecasts. Log of X_3/X_2

In each case there were two or more revisions made before the actual outcome occurred; and in all but one case[2] the data was consistent with the Lognormal (or conditionally Lognormal) hypothesis according to the Kolmogorov-Smirnov test.

Dress Sales Forecasts

The results of the analysis for forecasts of dress sales is particularly interesting because, in contrast to any of the others, the forecasts here were produced entirely by a mechanistic process rather than by an individual. This analysis is presented in more detail than the others for this reason.

Three years of historical wholesale demands for cruise-season women's dresses were collected by a group of graduate students at M.I.T. [7]. The wholesale selling season for this product runs from September first to December fifteenth, but most demands occur in October and November. The student group analyzed the data to obtain a graph of average cumulative per cent of total demands versus time. This relationship is illustrated in Figure 7. Then the group used this relationship as a simple forecasting device. At the third week, for example, the graph showed that the average cumulative percent of total demands was 16%. Then if 100 demands had been received for a particular dress style during the first three weeks, they would divide 100 by .16 to obtain an estimated total demand for that dress style of $100/.16 = 625$. The group followed actual demands for the following year and made forecasts at September 25, October 13, and November 16. Figures 8 through 10 contain plots of natural logarithms of ratios of successive sales forecasts for forty-five dress styles (again treating the actual outcome as the last forecast). The hypothesis of Lognormality[3] is consistent with the figures, and the Kolmogorov-Smirnov test again fails to reject the Lognormality hypothesis in each case.

[2] The E/S forecaster for the drug industry.

[3] Since the forecasting procedure for the dress sales is mechanical, there is no specific tendency for successive forecasts not to change.

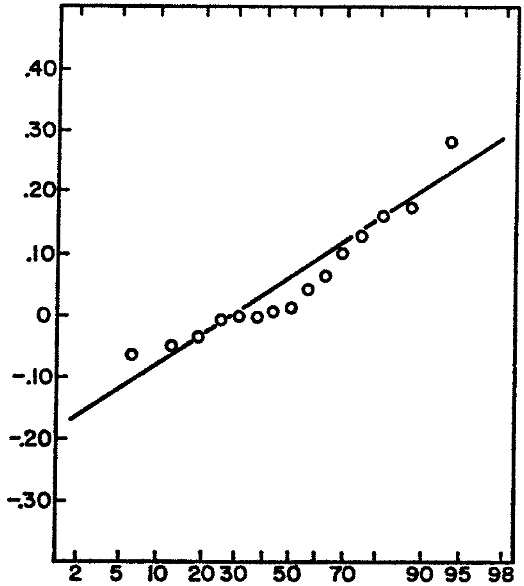

FIGURE 2. Regional Crop Supply Forecasts. Log of X_4/X_3

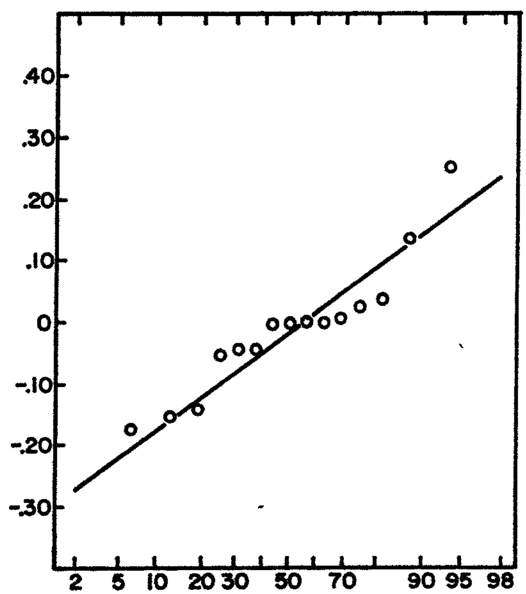

FIGURE 3. Regional Crop Supply Forecasts. Log of X_5/X_4

Empirical Support for Independent Forecast Changes (The Quasi-Markovian Property)

To study the assumption that ratios of successive forecasts are mutually independent, a correlation analysis was performed on the natural logarithms $W_n = \log (X_{n+1}/X_n)$ of the ratios of successive forecasts. The results of the correlation analysis for company crop supply forecasts are contained in Table 1, together with the value of r corresponding to the .05 level of significance. Since none of the correlations are significant at the

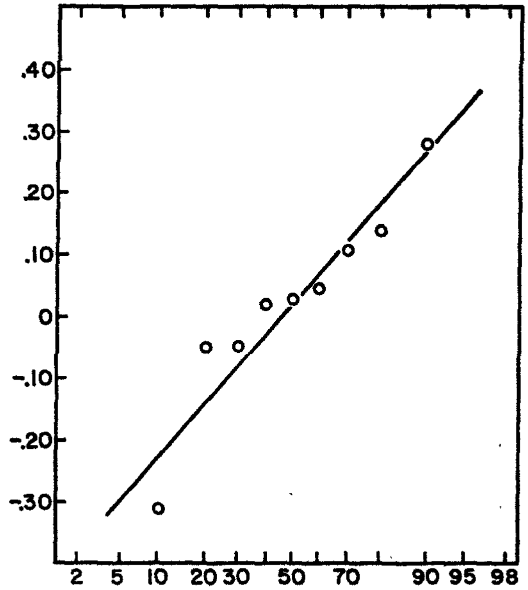

FIGURE 4. Regional Crop Supply Forecasts. Log of X_3/X_2, conditional on a change

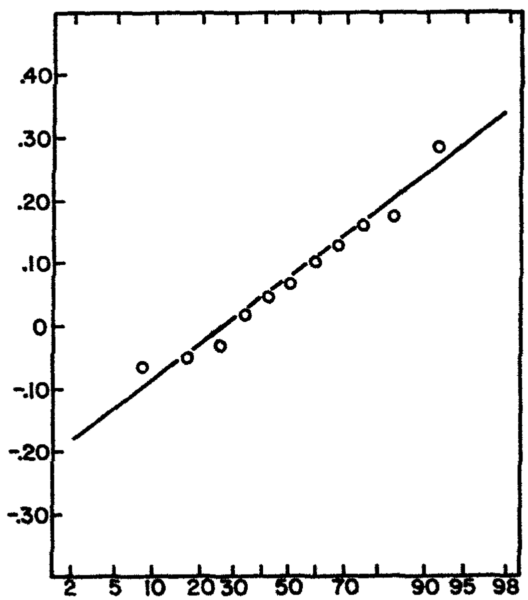

FIGURE 5. Regional Crop Supply Forecasts. Log of X_4/X_3, conditional on a change

.05 level, it seems reasonable to conclude that the company crop forecast changes may be treated as independent; thus company crop forecast changes are quasi-Markovian.

Table 2 contains the intercorrelation matrix for dress sales forecast changes. The hypothesis of independence is also supported by this matrix.

Similar analyses were performed for the other types of forecasts mentioned previously. The results generally, although not entirely,[4] supported the hypothesis of in-

[4] The drug E/S forecasts strongly violated the independence assumption.

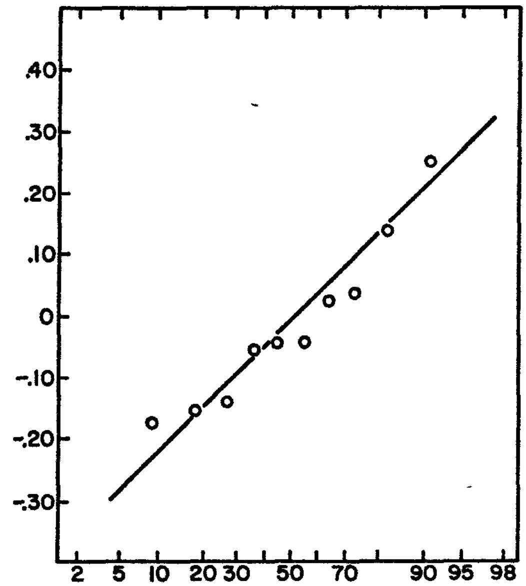

FIGURE 6. Regional Crop Supply Forecasts. Log of X_5/X_4, conditional on a change

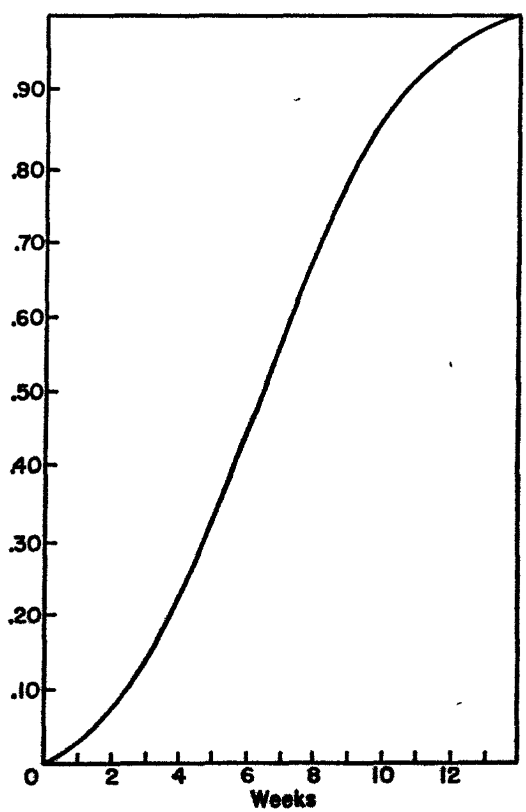

FIGURE 7. Average Cumulative Percent Sales for Dresses

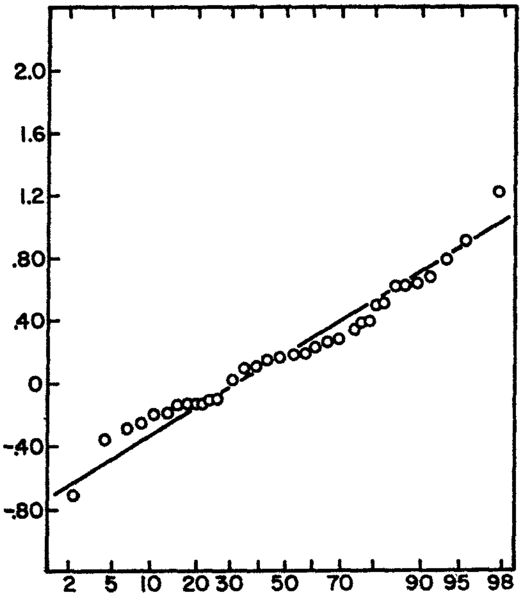

FIGURE 8. Dress Sales Forecasts. Log of X_2/X_1

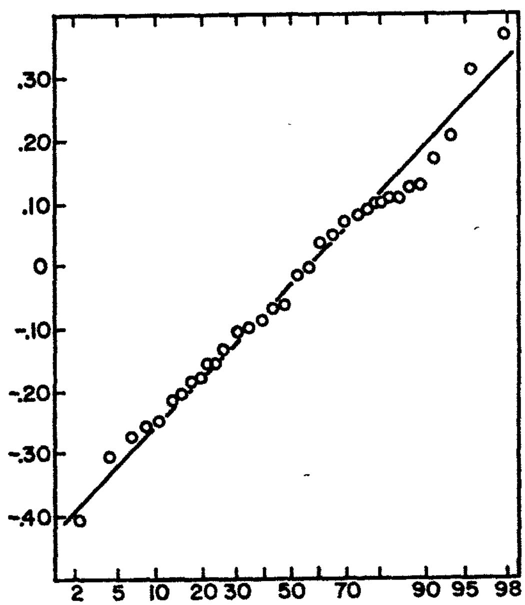

FIGURE 9. Dress Sales Forecasts. Log of X_3/X_2

dependence (i.e., the quasi-Markovian property). In the case of the total U.S. crop supply forecasts made by the U.S.D.A., a particular type of dependence occurred, which could logically be explained by the way in which the U.S.D.A. collected raw data on which their forecasts were based. As long as dependence can be counted on to continue in the future, it is possible to "adjust" biased forecasts so that the sequence of adjusted forecasts contains the quasi-Markovian property. We now illustrate how such an adjustment could be made.

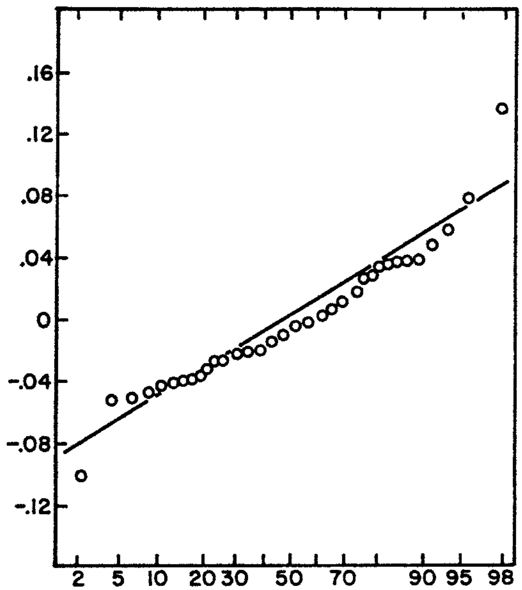

FIGURE 10. Dress Sales Forecasts. Log of X_4/X_3

TABLE 1
Intercorrelation Matrix for Company Crop Forecast Changes

	W_2	W_3
W_3	$-.245$	
W_4	$-.313$	$-.116$
		($r_{.05}$ = .514 for n = 15)

Adjustment to Obtain quasi-Markovian Forecasts

For simplicity of presentation let us consider three forecasts, X_1, X_2, and X_3, with $W_1 = \log (X_2/X_1)$ and $W_2 = \log (X_3/X_2)$. Assume the random variables W_1 and W_2 are correlated with correlation coefficient ρ. Again for simplicity let us assume that the variables W_1 and W_2 have been standardized with means of zero and variances equal to unity. Under these circumstances it is well known[5] that the conditional distribution of W_2 given W_1 is Normal with conditional mean $E(W_2|W_1) = \rho W_1$ and conditional variance $\text{Var}(W_2|W_1) = 1 - \rho^2$. The value of W_1 will now be adjusted to a new value W_1^* (and the second forecast X_2 will be adjusted to a new value X_2^* corresponding to the adjustment of W_1^*) so that the conditional mean of $W_2^* = \log (X_3/X_2^*)$ given W_1^* is zero; then W_2^* and W_1^* will be independent, as sought.

To perform the adjustment let

(5) $$W_1^* = W_1 + \rho W_1$$

and solve for X_2^* in the relationship

(6) $$W_1^* = \log (X_2^*/X_1).$$

[5] *See, for example,* [4, pp. 151–152].

TABLE 2

Intercorrelation Matrix for Dress Sales Forecast Changes

	W_1	W_2
W_2	$-.143$	
W_3	$-.106$	$-.101$
		($r_{.05} = .294$ for $n = 45$)

Raising e to the power of each side of equation (6),

$$(7)\quad e^{W_1^*} = (X_2^*/X_1) = e^{(W_1+\rho W_1)} = (e^{W_1})^{1+\rho} = (X_2/X_1)^{1+\rho} = (X_2/X_1)(X_2/X_1)^{\rho}$$

so that

$$(8)\qquad X_2^* = X_1(X_2/X_1)(X_2/X_1)^{\rho} = X_2(X_2/X_1)^{\rho}.$$

With this adjustment of X_2, namely, multiplying X_2 by $(X_2/X_1)^{\rho}$ to obtain X_2^*, we now consider $W_2^* = \log(X_3/X_2^*)$.

$$\begin{aligned}(9)\quad W_2^* &= \log(X_3) - \log(X_2^*) = \log(X_3) - \log(X_2) + \log(X_2) - \log(X_2^*)\\ &= W_2 + \log(X_2) - \log(X_2) - \rho W_1 \quad \text{(from (8) above)}\\ &= W_2 - \rho W_1 .\end{aligned}$$

Now $E(W_2^* \mid W_1^*) = E(W_2 \mid W_1) - \rho W_1 = \rho W_1 - \rho W_1 = 0.$

Thus by redefining X_2 as X_2^* as indicated in (8) above, the variables W_1^* and W_2^* are no longer correlated, and the sequence of adjusted forecasts $\{X_1, X_2^*, X_3\}$ exhibits the quasi-Markovian property.

In the next section some possible reasons for our findings are considered.

Possible Reasons for Independent, Conditionally Lognormal Forecast Changes

Each of the forecasts considered above is presumably affected in a complex manner by a number of largely unpredictable events or activities. In the cases where human forecasters are active, a vast amount of potentially relevant information becomes available between each forecast stage. In this situation, forecast changes based on a complex "processing" of information may reasonably be assumed to be generated according to the "Theory of Proportional Effect," which states that the change in a variate from one time period to the next is a random proportion of its current value. Let

$$X_{t+1} - X_t = \epsilon_t X_t ,$$

where X_t represents a forecast at time t and ϵ_t is a random variable. Irrespective of the value of the current forecast, the assumption is that the change from one time period to the next is a random percentage of the current forecast. Then

$$\begin{aligned}X_{t+1} &= (1 + \epsilon_t)X_t\\ &= [\textstyle\prod_{i=0}^{t} (1 + \epsilon_i)]X_0 ,\end{aligned}$$

and taking natural logarithms,

$$\ln X_{t+1} = [\textstyle\sum_{i=0}^{t} \ln(1 + \epsilon_i)] + \ln X_0 .$$

Furthermore, if the sequence $\{\epsilon_t\}$ is a sequence of mutually independent random variables with finite variances, then as t approaches infinity, asymptotically the sum of the transformed variates will approach Normality by the Central Limit Theorem. Therefore ln (X_{t+1}/X_0) will tend to be Normally distributed, so by definition (X_{t+1}/X_0) will tend to be Lognormally distributed. If the decision stages and successive forecasts are $t + 1$ time periods apart, then this reasoning provides a rationale for the hypothesis that ratios of successive forecasts are Lognormally distributed. In addition, it was previously mentioned that a possible lack of substantive information between stages could create a tendency of successive forecasts not to change. Such a possible lack of substantive information, when coupled with the Theory of Proportional Effect, provides an explanation for the conditional Lognormal distribution which was encountered in connection with the company crop supply forecasts.

There are two closely related ways to explain the independence of ratios of successive forecasts. Assuming the Theory of Proportional Effect holds, consider three successive forecasts: X_0, X_{t+1}, X_{2t+2}, each $t + 1$ time periods apart.

Let

$$Z_1 = X_{t+1}/X_0,$$

$$Z_2 = X_{2t+2}/X_{t+1}.$$

If the sequence $\{\epsilon_i\}$ which results from a forecaster processing information is a sequence of mutually independent random variables, then Z_1 and Z_2 will also be independent.

A second approach leading to the same result treats independence of ratios of successive forecasts as a desirable attribute of a human forecaster. If ratios of successive forecasts were not independent, then the forecast change from X_0 to X_{t+1} would contain some information relevant to predicting the probable direction and magnitude of the change from X_{t+1} to X_{2t+2}. Recalling that each forecast is a revised forecast of the same unknown quantity, it is clear that the forecast X_{t+1} could be improved upon by adjusting it in the direction in which X_{2t+2} is expected to go.[6] For example, ratios of successive drug company earnings-per-share forecasts were all significantly positively correlated. Thus if the first two forecasts were \$0.80 and \$1.00, the conditional expected value of the third forecast would be higher than the second forecast; a similar tendency would be predicted for the change from the third forecast to the fourth forecast (actual earnings per share). With this knowledge, the second forecast of \$1.00 should be adjusted upward by some amount to reflect the positive dependence of the forecast changes. The drug earnings-per-share forecaster may either be holding back the full impact of his information, or else he may be ignoring a time dependence in the passing events which provide him with information. This last point is important: Even though the flow of information over time may be highly correlated between adjacent time periods, a good forecaster should be aware of such correlations and should discount information to just the right extent so as to avoid either an under-reaction or an over-reaction. Thus good forecasters should generate ratios of successive forecasts which are independent; indeed, the above discussion leads one to *define* a good forecaster partially in terms of the independence characteristic.

It should be noted that the data-generating process described here for forecast revisions is sometimes called "geometric Brownian motion" [10, p. 17] and has frequently been used to represent the stochastic (random-walk) behavior of stock prices; see the bibliography in [6]. In fact, an argument similar in nature to the one just presented has been made in the context of future prices [9].

[6] See the adjustment process described earlier.

Now a simplified example of a crop planning problem will be studied. The example is designed to demonstrate the manner in which existing forecasters may be used to contribute to a computational solution of a sequential decision problem. The example will also illustrate the effects of using the methodology on the drug forecast data, where the assumptions underlying the analysis do not hold.

Example: A Simplified Crop Supply Decision Problem

Consider an agricultural supply problem in which the decision-maker first contracts for a certain amount of crop supply, and then may make incremental additions or deletions (outside purchases or sales) at various points in time during the growing season, including a final adjustment after the harvest to equate desired supply (a known constant) and actual supply. For example, suppose a vegetable processor makes contracts in February, begins growing in April, may make outside purchases or sales commitments in July and in August, and makes a final adjustment (purchase or sale) in October after the harvest is completed in order to obtain a specific amount of product.

Suppose the product can be obtained at minimum cost in February or April, and that any additions or deletions in the later periods carry with them "penalty costs" associated with delays in crop planning; i.e., additions to crop supply after April cost more than the same additions made in February or April, and deletions (sales) beyond April involve similar penalties. It seems reasonable to assume that the penalty costs for crop adjustments increase through time (delay is expensive). For a specific example, suppose the penalty costs of outside purchases or sales are proportional to the square of the amount purchased or sold, and assume that the constant of proportionality increases through time. Then any final discrepancy between actual and desired supply must be removed by a final adjustment with its associated penalty costs.

Given this problem structure, assume that at each stage a revised forecast of current crop supply (including the effects of previously made purchases or sales) is available, and that the data-generating process for forecasts of crop supply is quasi-Markovian. If the decision-maker wishes to meet his supply requirement at minimum penalty cost, his problem may be formulated as the following dynamic programming problem.

Notation:

let r = desired crop supply

X_n = forecast of crop supply made at stage n; $n = 1, 2, \cdots, N$

a_n = action taken by company at stage n to increase $(+)$ or decrease $(-)$ crop supply; $n = 1, 2, \cdots, N$

(Note: a_N is an artificial action which is forced to make up any final discrepancy between actual and desired supply: $a_N = r - X_N$.)

Z_n = ratio of $(n + 1)^{st}$ forecast to n^{th} forecast (adjusted for any action taken at the n^{th} stage); $n = 1, 2, \cdots, N - 1$

$$Z_n = X_{n+1}/(X_n + a_n) \tag{10}$$

$C_n(a_n)$ = penalty cost associated with action a_n taken at stage n; $n = 2, 3, \cdots, N$.

The objective is to minimize expected penalty costs (EPC) associated with obtaining the desired crop supply. In mathematical notation,

$$E.P.C. = \sum_{n=2}^{N} C_n(a_n). \tag{11}$$

Now assume the hypothesis of independent conditional[7] Lognormal forecast changes has been substantiated. Then ratios of successive forecasts $\{Z_n\}$ are independent random variables, each conditionally Lognormally distributed.

Note that even though the costs of crop adjustments are quadratic, the certainty-equivalence property [5, pp. 452–454] does not hold because the total cost function over all stages is not quadratic in the random variables $\{Z_n\}$. Thus it is necessary to use the dynamic programming approach to obtain optimal actions.

Dynamic Programming Solution

Let $f_n(X_n)$ = minimum expected penalty cost associated with observing forecast X_n at stage n, and behaving optimally from that point on. Then since $X_{n+1} = Z_n(X_n + a_n)$, from (10), $f_n(X_n)$ can be written as:

$$(12)\quad f_n(X_n) = \mathrm{Min}_{\{a_n\}}\Big\{C_n(a_n) + P_n f_{n+1}(X_n + a_n) + (1 - P_n)\int_0^\infty f_{n+1}(Z_n \cdot (X_n + a_n)) f_{LN}(Z_n \mid \mu_n, \sigma_n^2)\,dZ_n\Big\},\quad n = 1, 2, \cdots, N-1$$

and

$$(13)\qquad f_N(X_N) = C_N(r - X_N).$$

Equation (12) may be solved recursively, starting with equation (13), if the parameters of the conditional Lognormal density functions for ratios of successive forecasts can be estimated through analysis of historical data.

Using a grid of values for X_n at each stage, the optimal action $a_n^*(X_n)$ which minimizes expected penalty costs can be determined as a function of the state variable X_n. Finally, after working backwards from stage N to stage 1, the sequence of optimal decision rules $\{a_1^*(X_1), a_2^*(X_2), \cdots, a_N^*(X_N)\}$ constitutes an optimal policy. To use such a policy, it is merely necessary at each stage to enter the table $a_n^*(X_n)$ at the current forecast value X_n, and find the optimal action a_n^* to take. The optimal action will either indicate an outside purchase (if $a_n^* > 0$), an outside sale (if $a_n^* < 0$), or no action ($a_n^* = 0$).[8]

Numerical Example

For a specific numerical example, let

$$N = 5 \text{ stages},$$

$$r = 10 \text{ million lbs.},$$

and

$$C_n(a_n) = (.9)^{5-n}\cdot(a_n)^2,\qquad n = 2, 3, 4, 5.$$

Note that as n increases, the coefficient in $C_n(a_n)$ also increases; it becomes more costly to take action as the harvest time approaches. The forecasts of regional crop supply (see Figures 1 through 6) have been used as sources of parameter estimates for the parameters μ_n, σ_n, and P_n. These values are given in Table 3.

A computer program has been written to solve equations (12) and (13) recursively,

[7] That is, a probability P_n that the ratio of successive forecasts Z_n is 1.0, and a probability $(1 - P_n)$ that the ratio is a Lognormal variate with parameters estimated from historical data.

[8] For $n = 1$, a_n indicates the initial target for a processor's own-plant supply.

TABLE 3

Parameter Estimates from Company Crop Supply Forecasts

n	μ_n	σ_n	P_n
1	0.0000	0	1.0
2	0.0207	0.1519	0.40
3	0.0754	0.1020	0.27
4	0.0169	0.1267	0.33

TABLE 4

Expected Penalty Costs for Company Crop Forecasts

X_1	Optimal Policy $f_1(X_1)$	No-Adjustment Policy $f_1(X_1)$
2.0	9.11	59.53
4.0	5.33	33.24
6.0	2.86	14.01
8.0	1.68	4.47
8.4	1.65	3.65
8.8	1.61	3.26
9.2	1.63	3.46
9.6	1.66	3.70
10.0	1.76	4.80
10.4	1.95	5.78
10.8	2.13	7.12
11.2	2.39	9.17
11.6	2.70	11.62
12.0	3.08	13.77
14.0	5.66	31.70
16.0	10.22	50.85
18.0	17.60	70.82

using a grid[9] on X_n, and using the parameter estimates from Table 3. Selected values of $f_1(X_1)$ are presented in Table 4. For comparison, Table 4 also gives selected values of $f_1(X_1)$ for a policy which makes no adjustments until after the harvest. The main output of the program in practice would be a set of tables of optimal actions, $a_n^*(X_n)$.

In order to provide a rough check on the grid approximation as well as to test further the assumptions concerning independence and Lognormality of ratios of successive forecasts, an additional program was written to read in the original set of 15 crop forecast series and to act "optimally" with respect to the actual forecast changes which occurred in the sample of 15 sets of data. From Table 4, the best initial target is 8.8 million lbs., and the expected penalty cost associated with the optimal policy is 1.61. In a sample of 15, we would expect the average penalty cost not to be significantly different from 1.61 if the assumptions and the grid approximation are reasonable. The output of this second program for the 15 company crop supply forecast series is contained in Table 5. The sample average penalty cost, 1.55, is quite close to the expected amount of 1.61, particularly in view of the variation present in Table 5.

As a further check on the methodology, the second computer program was run with no adjustments until after the harvest. From Table 4, the best initial target for the no-adjustment policy is again 8.8 million lbs., and the expected penalty cost associated

[9] The grid runs from 0.2 million lbs. to 20 million lbs., in steps of 0.4 million lbs.

TABLE 5

Actual Penalty Costs, 15 Company Crop Forecast Series

Set	Optimal Policy: $X_1 = 8.8$ Penalty Cost	No-Adjustment Policy: $X_1 = 8.8$ Penalty Cost
1	0.86	2.56
2	1.46	5.76
3	5.98	1.44
4	2.67	4.00
5	2.06	0.64
6	1.59	2.56
7	3.50	5.76
8	0.27	0.64
9	0.56	0.16
10	0.38	1.44
11	0.43	0.16
12	1.04	2.56
13	0.63	1.44
14	1.04	2.56
15	0.79	0.64
Average:	1.55	2.15

TABLE 6

Cost Comparisons, Company Crop Forecasts

	Optimal Policy	No-Adjustment Policy
Expected Costs	1.61	3.26
Sample Average Costs ($n = 15$)	1.55	2.15

with this policy is 3.26. The sample penalty costs for the 15 company crop forecast series under this no-adjustment policy are also given in Table 5. In this case the sample average, 2.15, is lower than the expected cost, 3.26. However, the difference is less than one standard deviation and is therefore not statistically significant.

Value of Optimal Policy

Table 6 below summarizes the cost comparisons for the company crop forecast data. In comparison with the policy of no adjustment until after the harvest,[10] the optimal policy promises an expected reduction in penalty costs of 3.26 − 1.61, or 1.65. The sample evidence, with 15 data points, indicates a realized average saving of 2.15 − 1.55, or 0.60. The magnitudes of these improvements are clearly very much affected by the choice of the penalty cost function $C_n(a_n)$, and actual improvement will be smaller if the penalty for delay is not as large as in our numerical example.

Use of Method where Assumptions are Violated

Often a method may be moderately useful even when it is not strictly applicable. The drug earnings-per-share (E/S) forecasts discussed earlier violated the assumptions of independence and conditional Lognormality. Nevertheless, the same numerical

[10] This policy is in actual use in a number of situations. Food processors feel they do not have enough reliable information to make appropriate adjustments until near or after the harvest.

TABLE 7

Cost Comparisons, Drug Forecasts

	Optimal Policy	No-Adjustment Policy
Expected Costs	1.42	3.11
Sample Average Costs ($n = 24$)	1.92	3.56

example as above was studied using the drug forecasts, in order to determine whether our method could be useful in this case.

Proceeding as before, parameter estimates were generated from the drug forecast data. Then the first computer program was used to derive the optimal policy $a_n^*(X_n)$, *assuming* the properties of independence and conditional Lognormality held. Then the actual drug forecast data (24 sample series)[11] was run through both the "optimal" policy and the "no-adjustment" policy, and the penalty costs were recorded. Table 7 summarizes the cost comparisons.

As expected, sample costs are larger than expected costs in each case, since the assumptions of independence and conditional Lognormality are not met in the case of ratios of successive drug E/S forecasts. Nevertheless, use of the so-called "optimal" policy as compared with the no-adjustment policy offers a reduction in sample costs of 3.56 − 1.92, or 1.64. Thus for this particular example, even though the suggested method is not strictly applicable, the method offers substantial advantages over the policy of no adjustment until the end of the process.

Conclusion

In certain types of recurring sequential decision problems, it may be possible to exploit existing forecast mechanisms by formulating a mathematical model in which the current forecast serves as the state variable. In order for this formulation to lead to computational feasibility, it is necessary that the data-generating process for the forecasting mechanism exhibit the quasi-Markovian property (i.e., no memory). A small number of forecasters have been studied, with results which are generally (although not entirely) consistent with the quasi-Markovian property. Most of these forecasts conform, as a first approximation, to the conditional Lognormal distribution. Some possible reasons for these results are considered. A numerical example illustrates the precise methodology, and indicates that the approach may be moderately useful even when the required assumptions do not hold.

For those cases involving existing, experienced human forecasters, the approach may be described as a kind of "man-mathematics" interaction; the man performs those tasks (forecasting in a complex, poorly structured environment) in which he apparently has a relative advantage, while a mathematical model and an associated optimization technique perform those tasks in which quantitative analysis has a distinct advantage.

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[11] *E/S* was forecasted for 8 drug companies for 3 years.

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