UNIVERSITÄT ZU LÜBECK
INSTITUTE FOR ELECTRICAL
ENGINEERING IN MEDICINE

RO4001 – Model Predictive Control

Exercise Sheet 7

Fall semester 2020/21 Jan. 26, 2021

The exercises may be solved individually or in small groups. You are *not* allowed to use a calculator or computer unless this is explicitly stated.

Exercise 1 (multi-parametric QP)

Note: Use MATLAB (MPT toolbox) to solve part b) of this exercise.

Consider the following multi-parametric quadratic program (mpQP)

$$
\begin{aligned}
f^\star(x_1, x_2) = \min_{z\in\mathbb{R}} \ & \frac{1}{2}z^2 + 2x_1 z + x_2^2 && \text{(1a)}\\
\text{s.t.} \quad & +z \le 1 + x_1 \ , && \text{(1b)}\\
& -z \le 1 - x_2 \ , && \text{(1c)}
\end{aligned}
$$

where $x_1, x_2 \in \mathbb{R}$ are the two parameters. We denote the optimal value function $f^\star(x_1, x_2)$ and the optimal solution $z^\star(x_1, x_2)$.

a) In the first part, the solution of problem (1) will be analyzed analytically.

   i) Write down the Lagrangian and the KKT conditions of the QP.

   ii) Determine the set of parameters x_1, x_2 for which the QP is feasible.

   iii) Based on the complementary slackness condition, and using the primal and dual feasibility conditions, state explicitly the complementary cases than can occur.

   iv) For each complementary case, solve for $z^\star(x_1, x_2)$ and $f^\star(x_1, x_2)$. Also state the critical region of the parameter space x_1, x_2 on which this solution holds.

   v) Draw the critical regions into a the x_1,x_2-plane of the parameter space.

b) In the second part, the parametric solution of problem (1) will be computed with the MPT toolbox.

   i) Define the decision variable z and the paramrters x_1, x_2 as symbolic variables:

```
z = sdpvar(1,1);
x = sdpvar(2,1);
```

ii) Specify the objective function and the constraints of the quadratic program:

```
f = 0.5*z(1)^2 + 2*x(1)*z + x(2)^2;
C = [z(1)<=1+x(1) , -z(1)<=1-x(2) , -5<=x(1)<= 5 , -5<=x(2)<=5];
```

Here the additional constraints $\|x\|_\infty$ are included to keep the solution in a compact set of the parameter space.

iii) Call the MPT routine to solve multi-parametric program:

```
mpQP = Opt(C,f,x,z);
solution = mpQP.solve();
```

Use `help Opt` and `help Opt.solve` to get more information on these two commands.

iv) Visualize the critical regions, the parametric value function $f^\star(x_1, x_2)$ and the parametric solution $z^\star(x_1, x_2)$ via the commands:

```
figure(); solution.xopt.plot();
figure(); solution.xopt.fplot('primal');
figure(); solution.xopt.fplot('obj');
```

Use `help PolyUnion.plot` and `help PolyUnion.fplot` to get more information on these commands.

v) Are the functions $f^\star(x_1, x_2)$ and $z^\star(x_1, x_2)$ piecewise linear / quadratic? Are the continuous and/or continuously differentiable?

Exercise 2 (multi-parametric QP)

Note: Use MATLAB (MPT toolbox) to solve this exercise.

Consider again the DT LTI system from exercise 1 on exercise sheet 6:

$$x_{k+1} = \underbrace{\begin{bmatrix} 1.5 & 1.2 \\ 0 & 1 \end{bmatrix}}_{\triangleq A} x_k + \underbrace{\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}}_{\triangleq B} u_k \ , \tag{2a}$$

$$y_k = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\triangleq C} x_k \ . \tag{2b}$$

The system is subject to the following input and state constraints:

$$u_k \in \underbrace{\left\{u \in \mathbb{R} \mid \|u\|_\infty \leq 1\right\}}_{\triangleq \mathbb{U}} , \qquad x_k \in \underbrace{\left\{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 5\right\}}_{\triangleq \mathbb{X}} , \qquad \forall\ k = 0, 1, 2, \dots \ . \tag{3}$$

In this problem, an explicit MPC shall be designed for this system based on a quadratic cost function with $Q = I$ and $R = 1$.

a) In a first step, the system is modeled as a *LTI system object* called `model`. The process (also for more complicated models) is described here:

`https://www.mpt3.org/UI/Systems`

Also add the upper and lower bounds of the constraints (3) and the state and input penalty to the `model`, as described here:

`https://www.mpt3.org/UI/RegulationProblem`

b) Pick a prediction horizon of $N = 3$ and use MPT to generate an *MPC controller object* `mpc1`:

```
mpc1 = MPCController(model,N);
```

Plot the critical regions of the explicit solution, the state feedback $u^\star_{k|k} = \kappa_{\text{MPC}}(x_k)$ and the optimal cost function $J^\star(x_k)$ by using the following commands:

```
expmpc1 = mpc1.toExplicit();
figure()
title('critical regions');
expmpc1.partition.plot();
figure()
title('control input');
expmpc1.feedback.fplot();
figure()
title('cost function');
expmpc1.cost.fplot();
```

Characterize $\kappa_{\text{MPC}}(x_k)$ and $J^\star(x_k)$ if it is piecewise linear / quadratic, continuous and/or continuously differentiable.

c) Use this *MPC controller object* `mpc1` to simulate the closed-loop system starting from the initial condition $x_0 = [3 \ \ -1]^{\mathrm{T}}$. Plot the resulting trajectory (i.e., $x_{2,k}$ over $x_{1,k}$) for $k = 1, 2, \ldots, 12$ steps, along with the corresponding open-loop predictions of the MPC. (*Hints:* Re-use your code from exercise 2.b.i) on exercise sheet 6 to generate the plot. The sequence of open-loop inputs can be obtained via

`[u, feasible, openloop] = mpc1.evaluate(x); U = openloop.U;`)

d) Compare the domain of $\kappa_{\text{MPC}}(x_k)$ and $J^\star(x_k)$ to the maximum control invariant set of system (2), as computed in exercise 1.e) on exercise sheet 6. How do you explain the difference between the two? What does it mean for the performance of the designed MPC controller?

e) Now generate a new *MPC controller object* `mpc2` by adding a terminal condition compared to `mpc1`. To this end, you have to modify your *LTI system object* as follows:

```
Tset = model.LQRSet;
P = model.LQRPenalty;
model.x.with('terminalSet');
model.x.terminalSet = Tset;
model.x.with('terminalPenalty');
model.x.terminalPenalty = P;
```

Then generate the new *MPC controller object* for $N = 3$:

```
mpc2 = MPCController(model,N);
```

Plot again the critical regions of the explicit solution, the state feedback $u^\star_{k|k} = \kappa_{\text{MPC}}(x_k)$ and the optimal cost function $J^\star(x_k)$.

f) Compare the domain of $\kappa_{\text{MPC}}(x_k)$ and $J^\star(x_k)$ to the maximum control invariant set of system (2), as computed in exercise 1.e) on exercise sheet 6. Does the result make sense? What happens if you increase the horizon length, e.g., to $N = 30$?