UNIVERSITÄT ZU LÜBECK
INSTITUTE FOR ELECTRICAL
ENGINEERING IN MEDICINE

**RO4001 – Model Predictive Control**

# Exercise Sheet 4

Fall semester 2020/21 Dec. 1, 2020

The exercises may be solved individually or in small groups. You are *not* allowed to use a calculator or computer unless this is explicitly stated.

## Exercise 1 (quadratic program)

*Note:* Use MATLAB (command `quadprog`) to solve part d) of this exercise.

Consider the following optimization problem

$$\min_{x\in\mathbb{R}^3} \ \frac{1}{2}\left(x_1^2 + x_2^2 + 0.1x_3^2\right) + 0.55x_3 \tag{1a}$$

$$\text{s.t.} \ x_1 + x_2 + x_3 = 1 \tag{1b}$$

$$x_1 \geq 0 \tag{1c}$$

$$x_2 \geq 0 \tag{1d}$$

$$x_3 \geq 0 \tag{1e}$$

a) Show that $x^\star = [0.5 \ \ 0.5 \ \ 0]^\mathrm{T}$ is a local optimum.

b) Is $x^\star$ also the global optimum? Explain why, or why not.

c) Write the problem in the standard form of a quadratic program.

d) Use the MATLAB command `quadprog` (get syntax info via `help quadprog`) to solve problem (1) and compare your solution with $x^\star$ of part a).

## Exercise 2 (Lagrange multipliers)

*Note:* Use MATLAB to solve part b) of this exercise.

Consider the equality constrained quadratic program

$$\min_{x \in \mathbb{R}^n} \ \frac{1}{2} x^{\mathrm{T}} P x + q^{\mathrm{T}} x \tag{2a}$$

$$\text{s.t.} \ \ Ax = b \tag{2b}$$

where $P \in \mathbb{R}^{n \times n}$, $q \in \mathbb{R}^n$ with $P \succ 0$ and $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. There are fewer equality constraints than decision variables, $m < n$, so the optimization problem is well defined.

a) Show that the necessary and sufficient conditions for the optimal solution $x^\star$ of (2) can be expressed as the linear equation system

$$\begin{bmatrix} P & A^{\mathrm{T}} \\ A & 0 \end{bmatrix} \begin{bmatrix} x^\star \\ \nu^\star \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix} \ , \tag{3}$$

where $\nu^\star$ are the optimal Lagrange multipliers. (*Hint:* Use the KKT optimality conditions.)

b) Generate random problem data in Matlab by running

```
M = rand(n,n);
P = M*M';
```

until `rank(P)==n` and by defining

```
q = 10*randn(n,1);
A = randn(m,n);
b = randn(m,1);
```

Verify that (3) gives indeed the correct solution by comparing it with the output of `quadprog` for problem (2).

## Exercise 3 (minimum volume ellipsoid)

*Note:* Use MATLAB (CVX toolbox) to solve this exercise.

The goal of this exercise is to compute a minimum volume ellipsoid

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \ \middle| \ (x - x_{\mathrm{c}})^{\mathrm{T}} P^{-1} (x - x_{\mathrm{c}}) \leq 1 \right\} \ , \tag{4}$$

where the decision variables are the ellipsoid's center $x_{\mathrm{c}} \in \mathbb{R}^n$ and the shape matrix $P \in \mathbb{R}^{n \times n}$, $P \succ 0$ of the ellipsoid. The constraints on $\mathcal{E}$ are that it must contain a given list of points $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$.

a) First, the problem shall be reformulated as a convex semi-definite program (SDP).

   i) Note that the volume of an ellipsoid is proportional to $\det(P)$. Use this fact to state the problem as an SDP and show why it is *non-convex*.

   ii) Because

$$(x - x_{\mathrm{c}})^{\mathrm{T}} P^{-1} (x - x_{\mathrm{c}}) = \left\| (P^{-1/2} (x - x_{\mathrm{c}}) \right\|_2^2 \ ,$$

where $P^{-1} \triangleq P^{-1/2}P^{-1/2}$ and $P^{-1/2} \succ 0$, we can re-write (4) equivalently as:

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \;\middle|\; \|P^{-1/2}(x - x_c)\|_2 \leq 1 \right\} \; .$$

Show that with this expression the problem can be stated as a *convex* SDP. (*Hint:* Use an appropriate change of variables, so that the optimal solution is obtained as $(P^{-1/2})^\star$ and $x_c^\star$.)

b) Use the formulation of a) in CVX to compute a minimum volume ellipsoid in $\mathbb{R}^2$ that contains the following list of points:

$$x_1 = \begin{bmatrix} 7 \\ -1 \end{bmatrix} \;,\quad x_2 = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \;,\quad x_3 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \;,\quad x_4 = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \;,\quad x_5 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \;,\quad x_6 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \;,\quad x_7 = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \;.$$

Plot these points into a MATLAB figure, along with the optimal ellipsoid and its center.

## Exercise 4 (optimal control problem)

*Note:* Use MATLAB (command `fmincon`) to solve this exercise.

Now we want to solve the first optimal control problem in this course. The continuous dynamics of a nonlinear oscillator are given by

$$\ddot{p}(\tau) + p(\tau) + 0.1 \cdot \operatorname{sgn}\big(p(\tau)\big) \cdot \sqrt{\big|p(\tau)\big|}\big) = -u(\tau) \;, \qquad p(0) = 10 \;,\; \dot{p}(0) = 0 \;. \tag{5}$$

Here $\tau \in \mathbb{R}_+$ denotes time and a dot indicates the derivative with respect to time. The functions $p(\tau)$ and $\dot{p}(\tau)$ represent the position and velocity of the oscillator. The oscillator can be controlled via the signal $u(\tau)$. The aim is to bring the oscillator to rest (i.e., velocity $\dot{p} = 0$) at the position $p = 0$ by using the minimum possible control effort.

a) Discretize the oscillator (5) using Euler's method with a sample time $t_s = 0.1$s. Represent the discrete states $x_k \in \mathbb{R}^2$ for $k = 1, 2, \ldots$ as

$$x_k \triangleq \begin{bmatrix} p_k \\ v_k \end{bmatrix} \;,$$

where $p_k$ and $v_k$ are the position and velocity at time step $k$. Also state the initial condition $x_0$ according to (5).

b) The first step is to simulate the dynamics of the oscillator.

i) Write a MATLAB function `x_N=oscisim(U)` that computes the state $x_N$ as a function of the control inputs

$$U \triangleq \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_{N-1} \end{bmatrix}^{\mathrm{T}} \;.$$

Mathematically, this function can be denoted $f_{\text{oscisim}} : \mathbb{R}^N \to \mathbb{R}^2$. (*Hint:* Make sure that `oscisim(U)` returns *only the final state* $x_N$ and not the entire state sequence $\{x_1, x_2, \ldots, x_N\}$!)

ii) Verify that your simulation works by plotting the simulated positions $p_1, p_2 \ldots, p_N$ and velocities $v_1, v_2 \ldots, v_N$ for $N = 100$ and the input $U = 0$. (*Hint:* You should see a slightly unstable oscillatory behavior.)

c) Now we want to solve the optimal control problem

$$\min_{U \in \mathbb{R}^N} \; \|U\|_2^2 \;, \tag{6a}$$

$$\text{s.t.} \;\; f_{\text{oscisim}}(U) = 0 \;. \tag{6b}$$

i) Formulate and solve this problem using `fmincon`. Plot the solution vector $U$ in a MATLAB figure. In another MATLAB figure, plot the positions $p_0, p_1, \ldots, p_N$ and velocities $v_0, v_1, \ldots, v_N$. Verify that the oscillator is finally at rest.

ii) Now add further inequalities to the optimization program that limit the available actuator amplitude,

$$-u_{\max} \leq u_k \leq u_{\max} \qquad \forall\ k = 0, 1, \ldots, N \ .$$

There are two ways to do this in `fmincon`: via inequality constraints or via lower and upper bounds on the decision variables. Experiment with different values of $u_{\max}$, starting with very high ones and gradually making them smaller. If $u_{\max}$ is higher than the maximum amplitude of $U$ in the unconstrained case, the solution will obviously not change at all. Observe what happens to the solution as $u_{\max}$ becomes smaller. What is the minimum value of $u_{\max}$, before the control problem becomes infeasible?