UNIVERSITÄT ZU LÜBECK
INSTITUTE FOR ELECTRICAL
ENGINEERING IN MEDICINE

RO4001 – Model Predictive Control

Exercise Sheet 6

Fall semester 2020/21 Jan. 12, 2021

The exercises may be solved individually or in small groups. You are *not* allowed to use a calculator or computer unless this is explicitly stated.

Exercise 1 (polyhedral computations using MPT)

Note: Use MATLAB (MPT toolbox) to solve this exercise.

In this exercise, you will compute various sets for the following DT LTI system:

$$x_{k+1} = \underbrace{\begin{bmatrix} 1.5 & 1.2 \\ 0 & 1 \end{bmatrix}}_{\triangleq A} x_k + \underbrace{\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}}_{\triangleq B} u_k \ , \tag{1a}$$

$$y_k = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\triangleq C} x_k \ . \tag{1b}$$

The system is subject to the following input and state constraints:

$$u_k \in \underbrace{\left\{ u \in \mathbb{R} \mid \|u\|_\infty \leq 1 \right\}}_{\triangleq \mathbb{U}} \ , \qquad x_k \in \underbrace{\left\{ x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 5 \right\}}_{\triangleq \mathbb{X}} \ , \qquad \forall \ k = 0, 1, 2, \dots \ . \tag{2}$$

a) As a first step, you shall install and familiarize yourself with the MPT 3 toolbox for Matlab.

   i) To install the MPT 3 toolbox, download and execute the following Matlab script:

      `https://www.mpt3.org/Main/Installation?action=download&upname=install_mpt3.m`

   ii) Get a quick overview of the capabilities of MPT by running the three basic demos. Each demo can be started with a MATLAB command:

      `mpt_demo_sets1`

      `mpt_demo_sets2`

      `mpt_demo_sets3`

b) For system (1), compute the LQR gain K_∞ for the following weights:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \qquad R = 1 \ .$$

(*Hint:* Use the MATLAB command `dlqr`.)

i) In the remainder of part b), we consider the closed-loop (autonomous) system under LQR control,

$$x_{k+1} = \underbrace{(A - BK_\infty)}_{\triangleq A_\mathrm{a}} x_k \ . \tag{3}$$

Use MPT to compute the pre sets $\mathrm{pre}_1(\mathbb{X})$, $\mathrm{pre}_2(\mathbb{X})$, and $\mathrm{pre}_3(\mathbb{X})$ of (3). Plot them along with $\mathbb{X}$ in different colors into a single figure. (*Hints:* Use the syntax `plot(P,'color','r')` to plot a polytope object `P` in red. For nicer overlappings, plot the sets in the order of $\mathrm{pre}_3(\mathbb{X})$, $\mathrm{pre}_2(\mathbb{X})$, $\mathrm{pre}_1(\mathbb{X})$, $\mathbb{X}$.)

ii) Use MPT to compute the reach sets $\mathrm{reach}_1(\mathbb{X})$, $\mathrm{reach}_2(\mathbb{X})$, and $\mathrm{reach}_3(\mathbb{X})$ of (3). Plot them along with $\mathbb{X}$ in different colors into a single figure. (*Hint:* For nicer overlappings, plot the sets in the order of $\mathbb{X}$, $\mathrm{reach}_1(\mathbb{X})$, $\mathrm{reach}_2(\mathbb{X})$, $\mathrm{reach}_3(\mathbb{X})$.)

iii) Can you use the plots of the pre sets and the reach sets to infer the stability of system (3)?

iv) How could we take the input constraints into account? Modify steps b.ii) and b.iii) accordingly.

c) The next task is to compute and verify a positive invariant set for system (1).

i) Implement the algorithm from the script to compute the maximal positive invariant set $\mathcal{O}_\infty$ as a polytope object.

ii) Obtain the vertices of the minimal V-representation of $\mathcal{O}_\infty$. For each vertex, simulate 10 steps of the closed loop system (3). Create a figure in which you plot $\mathcal{O}_\infty$ along with all state trajectories (in the $x_{1,k}$,$x_{2,k}$-plane) starting from the vertices. Create another figure in which you plot the corresponding control inputs $u_k = -K_\infty x_k$ over time $k = 0, 1, \ldots, 9$.

iii) Why is it sufficient to consider the vertices to verify the invariance of $\mathcal{O}_\infty$?

d) From now on, consider system (1) *without* a fixed control law, i.e., for general $u_k \in \mathbb{U}$.

i) Use MPT to compute and plot the pre sets of $\mathbb{X}$, $\mathrm{pre}_1(\mathbb{X})$, $\mathrm{pre}_2(\mathbb{X})$, and $\mathrm{pre}_3(\mathbb{X})$ for the case of the control system, analogous to b.i).

ii) Use MPT to compute and plot the reach sets $\mathbb{X}$, $\mathrm{reach}_1(\mathbb{X})$, $\mathrm{reach}_2(\mathbb{X})$, and $\mathrm{reach}_3(\mathbb{X})$ for the case of the control system, analogous to b.ii).

e) Compute the maximal control invariant set $\mathcal{C}_\infty$ for the control system (1) by implementing the corresponding algorithm from the script. Compare the result with the maximal positive invariant set $\mathcal{O}_\infty$ of c.ii).

Exercise 2 (MPC with terminal conditions)

Note: Use MATLAB (MPT toolbox) to solve this exercise.

In this exercise, you will design linear quadratic MPCs with different terminal conditions for the following DT LTI system:

$$x_{k+1} = \underbrace{\begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix}}_{\triangleq A} x_k + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\triangleq B} u_k \ , \tag{4a}$$

$$y_k = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\triangleq C} x_k \ . \tag{4b}$$

The system is subject to the following input and state constraints:

$$u_k \in \underbrace{\left\{ u \in \mathbb{R} \mid \|u\|_\infty \leq 1 \right\}}_{\triangleq \mathbb{U}} \ , \qquad x_k \in \underbrace{\left\{ x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 15 \right\}}_{\triangleq \mathbb{X}} \ , \qquad \forall \ k = 0, 1, 2, \ldots \ . \tag{5}$$

Use your code from exercise sheet 5 to design the MPCs and your code from exercise 1 to compute invariant sets. Unless stated otherwise, use the cost weights $Q = I$, $R = 1$ and the prediction horizon $N = 3$.

a) Choose a zero terminal constraint $\mathbb{X}_\mathrm{f} = \{0\}$ and a terminal weight $P = Q$, so that closed-loop stability is guaranteed. (*Hint:* Design an appropriate equality constraint in `quadprog` to enforce the terminal constraint.)

   i) For the initial condition $x_0 = [2 \ \ -1]^\mathrm{T}$, plot the closed-loop trajectory (i.e., $x_{2,k}$ over $x_{1,k}$) for 4 steps, along with the corresponding open-loop predictions of the MPC. Analyze the mismatch between the open loop and the closed loop trajectories.

   ii) Change the prediction horizon to $N = 10$ and do the same analysis, again for the initial condition $x_0 = [2 \ \ -1]^\mathrm{T}$.

   iii) Compute the 3-step controllable set $\mathcal{K}_3$ and the 10-step controllable set $\mathcal{K}_{10}$ of $\mathbb{X}_\mathrm{f}$ for the two MPC controllers. Compare the two sets. Are these sets also stabilizable sets? (*Hint:* To compute the controllable sets, modify your code from exercise 1 for computing pre sets.)

b) Choose as a terminal weight P the infinite horizon cost matrix P_∞ of the LQR and as a terminal set $\mathbb{X}_\mathrm{f}$ the positive invariant set of the system under LQR state feedback $u_k = -K_\infty x_k$. (*Hint:* Use the code from exercise 1 to compute $\mathbb{X}_\mathrm{f}$, then integrate it into your MPC controller of exercise sheet 5).

   i) Pick again $N = 3$. For the initial condition $x_0 = [2 \ \ -1]^\mathrm{T}$, plot the closed-loop trajectory (i.e., $x_{2,k}$ over $x_{1,k}$) for 4 steps, along with the corresponding open-loop predictions of the MPC. Compare your results to task a.i).

   ii) Compute the 3-step controllable set $\mathcal{K}_3$ of $\mathbb{X}_\mathrm{f}$ for this MPC controller. Compare it with the result of task a.iii). Is it also a stabilizable set? (*Hint:* Re-use the code from part a.iii).)