UNIVERSITÄT ZU LÜBECK
INSTITUTE FOR ELECTRICAL
ENGINEERING IN MEDICINE

RO4001 – Model Predictive Control

Exercise Sheet 2

Fall semester 2020/21 Nov. 3, 2020

The exercises may be solved individually or in small groups. You are *not* allowed to use a calculator or computer unless this is explicitly stated.

Exercise 1 (stability and observability 1)

Consider the discrete-time LTI system with the following state space representation:

$$x_{k+1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \alpha & 0 \\ 0 & \frac{1}{2} & -\frac{5}{4} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{3} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ -2 \\ 4 \\ 0 \end{bmatrix} u_k \ , \tag{1a}$$

$$y_k = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} & 0 \end{bmatrix} x_k \ . \tag{1b}$$

a) Let $\alpha = 0$.

   i) Is the system stable?

   ii) Which states of the system belong to the controllable subsystem? (*Hint:* Find an intuitive explanation, without computing the controllability matrix.)

   iii) Which states of the system belong to the observable subsystem? (*Hint:* Find an intuitive explanation, without computing the observability matrix.)

b) The reachable subspace is defined as the set of states the system can reach from the origin. Use your knowledge of controllability to compute the reachable subspace as a function of the parameter α.

c) Now let $\alpha = \frac{1}{2}$. Is it possible to design a stabilizing controller for system (1)?

Exercise 2 (stability and observability 2)

Consider a single-input single-output DT LTI system

$$x_{k+1} = Ax_k + Bu_k \ , \tag{2a}$$

$$y_k = Cx_k + Du_k \ , \tag{2b}$$

with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}$, and $y_k \in \mathbb{R}$ for all $k \in \mathbb{Z}_{0+}$.

a) Suppose the system shall be controlled via linear output feedback. Which of the following statements is correct?

- ◯ The system can be stabilized using feedback control if the controllable subspace contains the unobservable subspace.
- ◯ The system can be stabilized using feedback control if all states which are not in the controllable subspace are stable.
- ◯ The controlled closed-loop system is asymptotically stable if all its eigenvalues lie inside the closed unit circle.
- ◯ A system that is not fully controllable can never be stabilized using feedback control.

b) Now let the system matrix of (2) be diagonal:

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} .$$

Mark the correct answer(s): The system is controllable if and only if

- ◯ all elements of B are non-zero.
- ◯ all eigenvalues of A are non-zero, and all elements of B are non-zero.
- ◯ all eigenvalues of A are distinct, and all elements of B are non-zero.

Exercise 3 (discretization of a CT LTI state-space model)

Note: Use MATLAB (or a similar tool) to solve this exercise.

Consider the following CT LTI system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 2.7 \\ -3.1 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 & 2.1 \\ 1.1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} , \tag{3a}$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} . \tag{3b}$$

a) Discretize the system with a sample time $t_\mathrm{s} = 1\,\mathrm{s}$ using

   i) Euler's method and

   ii) an exact discretization. (*Hint:* Use the Matlab function `expm` to compute matrix exponentials and implement a simple loop for numerical integration.)

b) Use the Matlab function `ss` to define the continuous-time system, then use `c2d` to obtain a time discretization.

c) Compare the outputs of the continuous and the discretized model in a dynamic simulation, starting from the same initial state and applying the same inputs. (*Hint:* Use the Matlab function `lsim` to run simulations.)

Exercise 4 (controller and observer design 1)

Consider the following DT LTI system:

$$x_{k+1} = \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \beta \end{bmatrix}}_{\triangleq A(\beta)} x_k + Bu_k \ , \tag{4a}$$

$$y_k = \underbrace{\begin{bmatrix} 1 & \gamma \end{bmatrix}}_{\triangleq C(\gamma)} x_k \ , \tag{4b}$$

where $\beta, \gamma \in \mathbb{R}$ are two parameters.

a) The first part of the exercise is concerned with the analysis of system (4).

 i) Find the intervals for the parameter β such that the system is stable and asymptotically stable.

 ii) Let $\beta = \frac{2}{3}$. One can choose between three actuators for the system:

$$B^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \qquad B^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \qquad B^{(3)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ .$$

 For which one(s) is the resulting system controllable?

 iii) Let $\beta = \frac{2}{3}$. For which values of the parameter γ is the system observable?

 iv) Let β, γ, B be such that the system is asymptotically stable and observable, but *not* controllable. Is it possible to design a combination of a state observer and a state feedback controller such that

$$\lim_{k\to\infty} y_k = 0 \qquad \text{for any } \ x_0 \in \mathbb{R}^2 \ \ ?$$

 Justify your answer.

 v) Let $\beta = \frac{2}{3}$ and $B = B^{(2)}$. Is it possible to design a state feedback controller $u_k = Kx_k$ such that all poles of the closed-loop system have absolute values no larger than $\frac{1}{2}$? If yes, identify for which $K \in \mathbb{R}^{1\times 2}$ this condition holds.

b) Now we want to design a state observer and a state feedback controller for system (4). The state observer is a Luenberger observer, which has the following structure:

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + L\big(y_k - \hat{y}_k\big) \ , \\ \hat{y}_k &= C\hat{x}_k \ . \end{aligned}$$

Here $L \in \mathbb{R}^{1\times 2}$ denotes the observer gain, $\hat{x}_k$ is the state estimate and $\hat{y}_k$ is the output estimate. The resulting feedback controller has the structure

$$u_k = K\hat{x}_k \ ,$$

where $K \in \mathbb{R}^{2\times 1}$ denotes the feedback gain.

 i) The closed-loop system including the state observer and the feedback controller can be described via the augmented linear system

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = A_{\text{aug}} \begin{bmatrix} x_k \\ e_k \end{bmatrix} \ ,$$

 where $e_k \triangleq x_k - \hat{x}_k$ denotes the estimation error. Derive the state transition matrix $A_{\text{aug}} \in \mathbb{R}^{4\times 4}$ of the augmented system.

ii) Based on the result in b.i), compute the eigenvalues of the matrix A_{aug} in terms of the eigenvalues of $A + BK$ and $A - LC$. (*Hint:* You may use the fact that

$$\det\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \det(X) \cdot \det(Z)$$

for arbitrary square matrices X and Z.)

iii) If the controller gain K is selected such that the closed-loop system with state feedback is stable and if the observer gain L is selected such that the dynamics of the state estimation error are stable, is the closed-loop system including the state observer and the estimated state feedback stable in general? Justify your answer.

Exercise 5 (controller and observer design 2)

Note: You may use MATLAB (or a similar tool) to solve this exercise.

Consider the following DT system with linear dynamics

$$x_{k+1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\triangleq A(\alpha)} x_k + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\triangleq B} u_k \ , \tag{5a}$$

$$y_k = \underbrace{1 \cdot \tanh(x_{1,k}) + 2 \cdot \tanh(x_{2,k}) + 3 \cdot \tanh(x_{3,k})}_{\triangleq h(x_k)} \ , \tag{5b}$$

where $\alpha \in \mathbb{R}$ is a parameter and $x_k = [x_{1,k} \ \ x_{2,k} \ \ x_{3,k}]^{\mathrm{T}}$.

a) In the first part, the goal is to analyze system (5).

i) For which values of α is the system controllable? For which values of α is it stabilizable?

ii) Linearize the output mapping $h(x_k)$ around the $x_k = [0 \ \ 0 \ \ 0]^{\mathrm{T}}$ and $u = 0$, i.e., determine C and D in the linearized output mapping

$$y_k = Cx_k + Du_k \ .$$

(*Hint:* $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.)

For the remainder of this question you can use

$$y_k = \underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{\triangleq C} x_k + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\triangleq D} u_k \ .$$

iii) For which values of α is the linearized system observable? For which values of α is it detectable?

b) For the remainder of this exercise, let $\alpha = 1$. Consider the following linear state feedback controller

$$u_k = -\underbrace{\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}}_{\triangleq K} x_k \ . \tag{6}$$

i) Use the Matlab function `eig` to examine the stability of the closed loop system.

ii) Consider the Lyapunov function candidate $V : \mathbb{R}^3 \to \mathbb{R}$ defined as

$$V = x^{\mathrm{T}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}}_{\triangleq P} x \ .$$

Verify that V is indeed a Lyapunov function for the closed loop system.

c) A state estimator has been designed for the linear system, using a stationary Kalman gain L_∞. Instead of the actual state x_k, the state feedback controller (6) now uses the state estimate $\hat{x}_k$.

i) Determine the state transition matrix $A_{\text{aug}} \in \mathbb{R}^{6\times 6}$ for the augmented closed-loop system as a function of A, B, C, D, K, and L_∞:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = A_{\text{aug}} \begin{bmatrix} x_k \\ e_k \end{bmatrix} \tag{7}$$

ii) Does there exist a Lyapunov function $V_{\text{aug}} : \mathbb{R}^6 \to \mathbb{R}$ for the augmented closed-loop system (7)? Justify your answer.

Exercise 6 (stabilizability and detectability)

Consider the following DT LTI system:

$$x_{k+1} = \begin{bmatrix} -0.4 & -1.1 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 0.9 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_k \ , \tag{8a}$$

$$y_k = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x_k \ . \tag{8b}$$

Which of the following statements are true or false? Justify your answers.

a) The system (8) is stable.
- ◯ Yes.
- ◯ No.

b) The system is controllable and stabilizable.
- ◯ Yes.
- ◯ No.

c) The system is stabilizable, but not controllable.
- ◯ Yes.
- ◯ No.

d) The system is observable, but not detectable.
- ◯ Yes.
- ◯ No.

e) The system is detectable, but it is not observable.
- ◯ Yes.
- ◯ No.