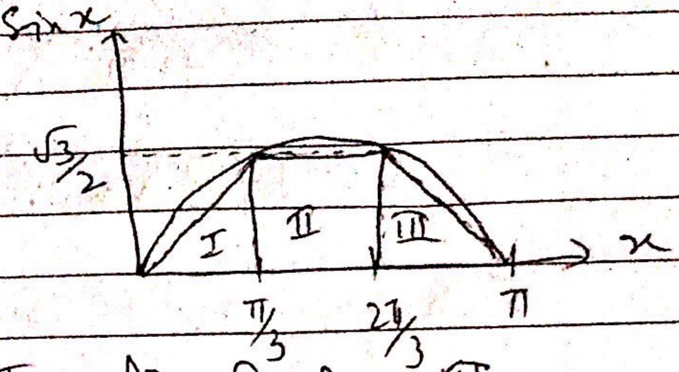
1(a) $I = \int_0^{\pi} \sin x \, dx$

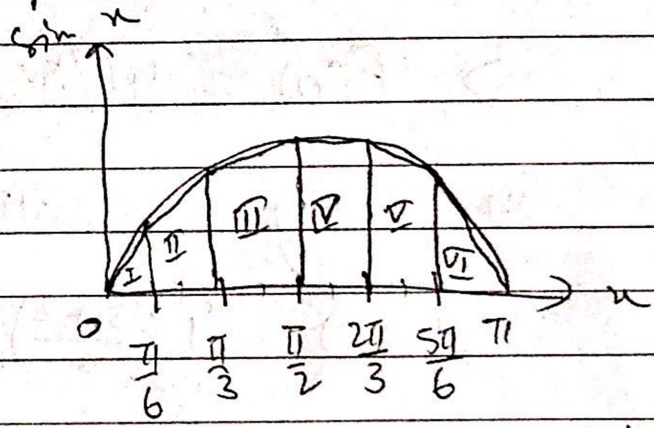
4 equally spaced -



$I = \text{Area} = \text{Area I} + \text{Area II} + \text{Area III}$

$$= \frac{\pi}{\sqrt{3}}$$

7 equally spaced -



$I = \text{Area} =$

$I = \text{Area I} + \text{Area II} + \text{Area III} + \text{Area IV}$
$+ \text{Area V} + \text{Area VI}$

$$= \left(\frac{\sqrt{3}+2}{6}\right)\pi$$

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Richardson extrapolation-

$$F(h) = a_0 + a_1 h^2 + O(h^4) \quad (1)$$

Now if we take $h = \pi/3$ then $F(h) = $ Area

$F(h/2) = F(\pi/6) = $ Area 2

$$F(h/2) = a_0 + a_1 \frac{h^2}{4} \quad (2)$$

replacing $a_1 h^2$ from (1) in (2)
we get

$$a_0 = \frac{4F(h/2) - F(h)}{3}$$

Also

$$F(0) = a_0$$

$$\Rightarrow F(0) = \frac{4F(h/2) - F(h)}{3}$$

on solving by putting values we get

$$F(0) = \frac{4\left(\frac{\sqrt{3}+2}{6}\right)\pi - \left(\frac{\pi}{\sqrt{3}}\right)}{3}$$

order of accuracy $= O(h^4)$

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1(b) No, it is not possible to use Richardson extrapolation as Q_4 will have ~~[illegible]~~ accuracy of order $O(h^5)$ & Q_7 will have accuracy of $O(h^9)$ so it will be difficult to find a $I(h)$ of the form

$$I(h) = a_0 + a_1 h^p + O(h^q)$$

as both Q_4 & Q_7 has different order of accuracies.

(2) Using taylor series we can write

$$f(x+h) = f(x) + f'(x)h + f''(x)h^2 + O(h^3)$$

&

$$f(x+2h) = f(x) + 2f'(x)h + 4f''(x)h^2 + O(h^3)$$

for $h \to 0$

$$4f(x+h) - f(x+2h) = 3f(x) + 2h f'(x) + O(h^3)$$

$$f'(x) = \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h} + O(h^2) \quad (1)$$

$$f'(0) = \frac{4f(h) - f(2h) - 3f(0)}{2h} + O(h^2)$$

As we can see in (11) that $f'(x)$ has accuracy of $O(h^2)$ i.e. second order accurate.

3(a) Proof by contradiction.
i.e.
intersection of all half spaces that contain C is not equal to C itself.

If intersection is not C then there will be 2 cases

(1) $\exists x_i' \in C$ but x does not belong to intersection

It is not possible as every halfspace must include C so they have to contain x.

(2) $\exists x' \notin C$ but still belongs to intersection.

~~Suppose~~ we construct a halfspace H with $a^T x = b$ s.t.
$a^T x' < b$ & $a^T c > b \; \forall c \in C$

Since half space including x' does not include convex set C so while creating a space after intersection this set won't be included & hence ~~x'~~ won't be included.
This is contradiction so our assumption must be wrong.

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3(b) To prove

f is concave

$\Longleftrightarrow$

$$\alpha_1 f(x_1) + (1-\alpha) f(x_2) \leq f(\alpha x_1 + (1-\alpha) x_2)$$

for any $x_1, x_2 \in C$

Consider 2 ~~pts~~ balls at pts x_1, x_2 of radii r_1, r_2 having largest value inside C

If $x_1, x_2 \in C$ then
$$x_3 = \alpha_1 x_1 + (1-\alpha) x_2 \in C$$

Consider this pt as center of ~~of~~ a new ball with radius $r_3 = \alpha r_1 + (1-\alpha) r_2$

~~We show that~~ Now If we show that this new ball lies inside C then we are done. Any pt of ~~the~~ new ball ~~is~~ is $X = x_3 + r_3 d$. by substituting x_3 & r_3 from above we get $X = \alpha(x_1 + r_1 d) + (1-\alpha)(x_2 + r_2 d)$
Now this also lies inside C as $x_1 + r_1 d \in C$ & $x_2 + r_2 d \in C$ & X is linear combo of those pts $\Rightarrow$ all pts of new ball lies inside C.

(4) f is convex if

$$f(y) \geq f(x) + \nabla f^T(x)(y-x) \quad (1)$$

f is psd iff $\left(h^T \nabla^2 f(x) h\right) \geq 0 \quad \forall x$

(i) f is psd ie

$$h^T \nabla^2 f(x) h \geq 0 \quad \forall x$$

$\Rightarrow$

$$h^T \nabla^2 f(x+th) h \geq 0 \quad (2)$$

Using taylor series $\frac{1}{2} h^T \nabla^2 f(x+th) h$

$$f(x+h) = f(x) + \nabla f^T(x) h + \frac{1}{2} h^T \nabla^2 f(x+th) h \quad (3)$$

Using (2) & (3)

$$f(x+h) - f(x) - \nabla f^T(x) h \geq 0$$

$$f(x+h) \geq f(x) + \left((x+h-x)^T \nabla f^T(x)\right)^T$$

Hence f is convex

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(ii) $f(x+h) = f(x) + h\nabla f^T(x) + \frac{1}{2} h^T \nabla^2 f(x) h + O(h^3)$

Also f is convex

$$f(x+h) \geq f(x) + \nabla f^T(x)(y-x)$$

$$\frac{1}{2} h^T \nabla^2 f(x) h + O(h^3) \geq 0$$

Let $h = \sigma \hat{u}$ where $\hat{u}$ is unit vector

$$\frac{\sigma^2}{2} \hat{u}^T \nabla^2 f(x) \hat{u} + \sigma^3 O(\hat{u}^3) \geq 0$$

$$\frac{1}{2} \hat{u}^T \nabla^2 f(x) \hat{u} + \sigma O(\hat{u}^3) \geq 0$$

as $h \to 0 \Rightarrow \sigma \to 0$

$$\lim_{\sigma \to 0} \frac{1}{2} \hat{u}^T \nabla^2 f(x) \hat{u} + \sigma O(\hat{u}^3) \geq 0$$

$$\frac{1}{2} \hat{u}^T \nabla^2 f(x) \hat{u} \geq 0$$

$$\hat{u}^T \nabla^2 f(x) \hat{u} \geq 0$$

Hence f is psd.

6 (b)

$$f(y) = \frac{1}{2} k_{vel} \sum_{i=1}^{12} (y_i - y_{i-1})^2 + k_{obs} \sum_{i=1}^{12} f_{obs}(x_i, y_i)$$

$$\frac{\partial f(y)}{\partial y_i} = \frac{1}{2} k_{vel} \left[\frac{\partial}{\partial y_i} (y_i - y_{i-1})^2 + \frac{\partial}{\partial y_i} (y_{i+1} - y_i)^2 \right] + k_{obs} \frac{\partial}{\partial y_i} f_{obs}(x_i, y_i)$$

$$\frac{\partial f(y)}{\partial y_i} = \frac{1}{2} k_{vel} \left[2(y_i - y_{i-1}) - 2(y_{i+1} - y_i) \right] + k_{obs}\, g_{obs}(x_i, y_i)_2$$

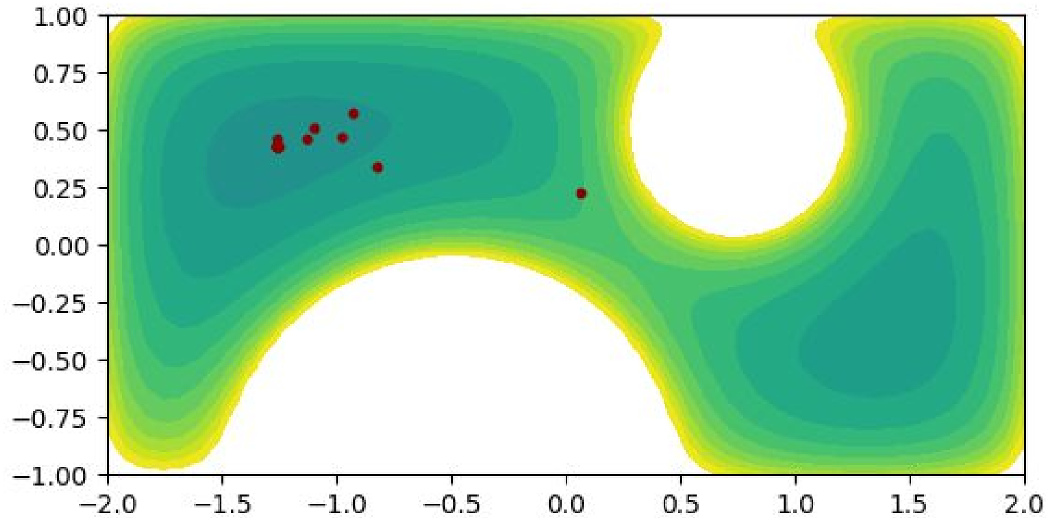
$$= k_{vel} (-y_{i-1} + 2y_i - y_{i+1}) + k_{obs}\, g_{obs}(x_i, y_i)_2$$

(c)

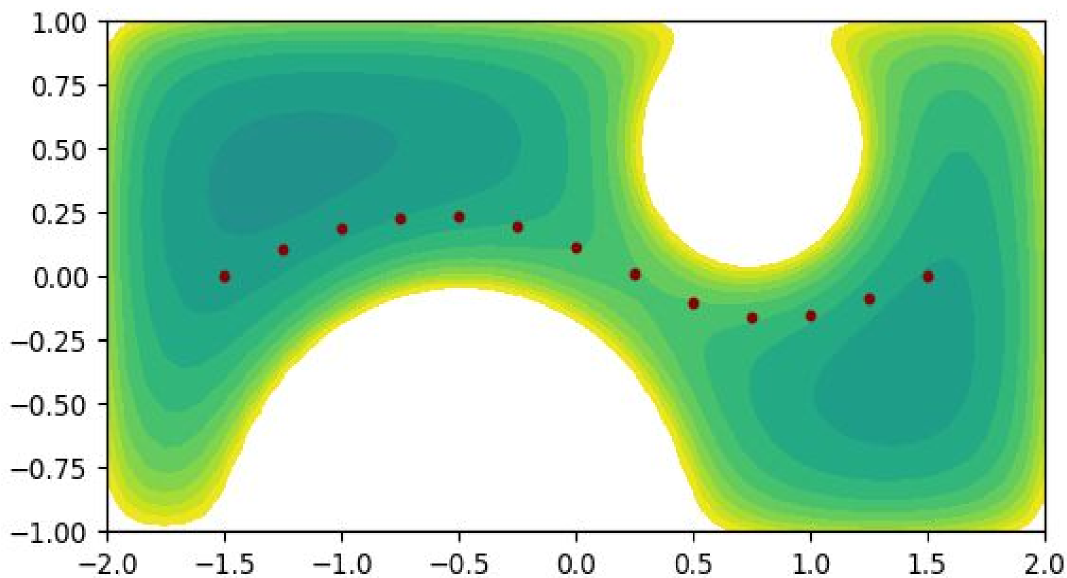
As seen from the graph, algo 2 with hessian converges faster.

6.)

(a)


1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

(b)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0.75
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−0.25
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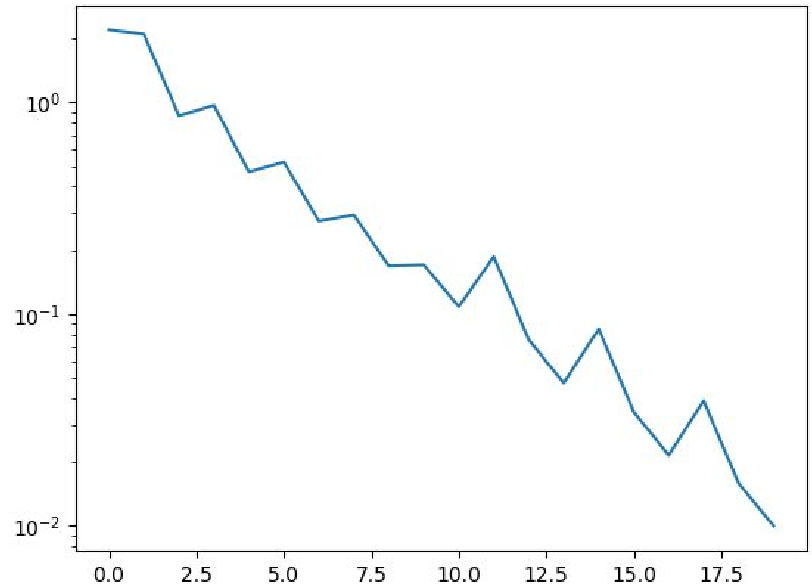
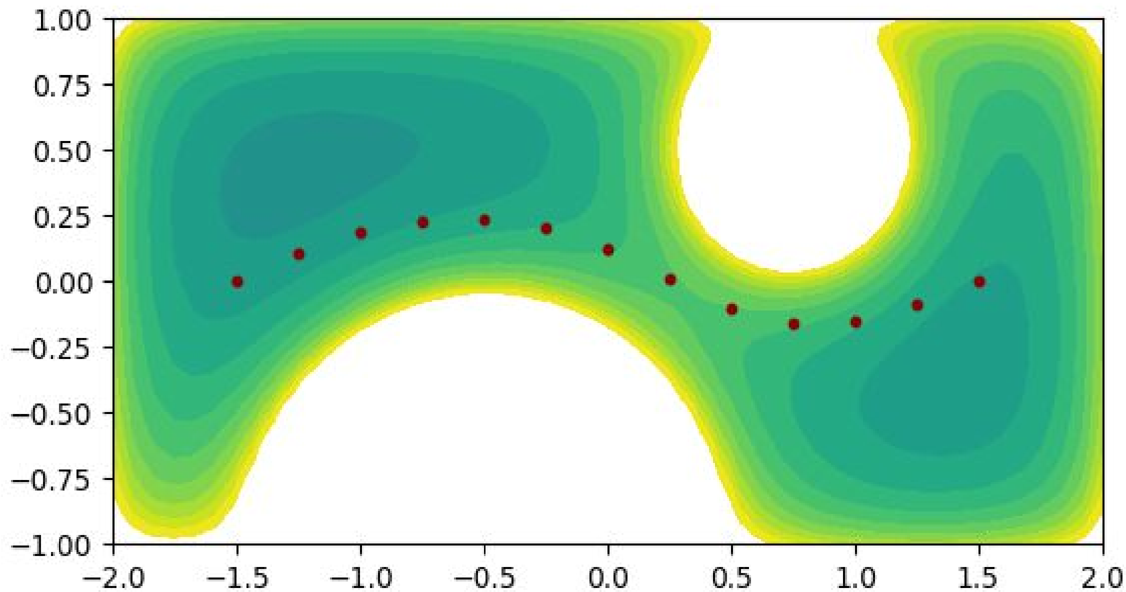
−0.5
0.0
0.5
1.0
1.5
2.0

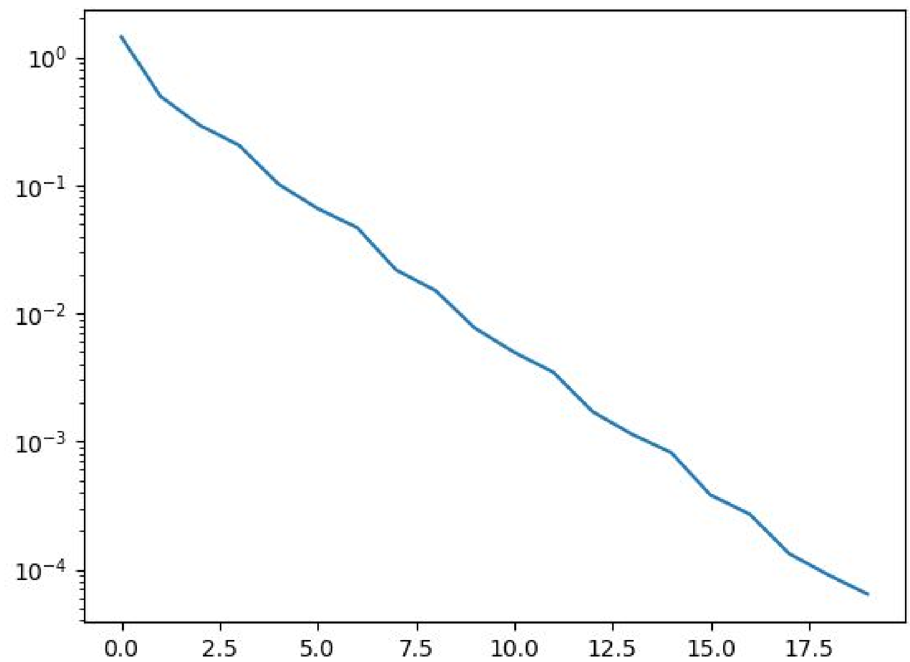


10^0
10^{-1}
10^{-2}
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5

(c)



1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0



10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5