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2015CS50284

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(1) $(A+\delta A)(q_1+\delta q_1) = (\lambda_1+\delta\lambda_1)(q_1+\delta q_1)$

~~Accept~~ where λ_1 is eigen value corresponding to eigen vector q_1

$Aq_1 + A\delta q_1 + (\delta A)q_1 = \lambda_1 q_1 + \lambda_1 \delta q_1 + \delta\lambda_1 q_1$

$$A\delta q_1 + (\delta A)q_1 = \lambda_1(\delta q_1) + \underline{(\delta\lambda_1)q_1} \quad (1)$$

Now, we have

$$\|q_1+\delta q_1\|_2 = 1$$

$$(q_1+\delta q_1)^T(q_1+\delta q_1) = 1$$

$\Rightarrow \quad q_1^T\delta q_1 = 0 \quad$ and $\quad (\delta q_1)^T q_1 = 0$

Also $\delta q_{11} = 0$

Multiply q_1^T on the left of equation (1)

$$q_1^T A(\delta q_1) + q_1^T(\delta A)q_1 = \cancel{q_1^T(\lambda_1(\delta q_1))}^{\,0} + q_1^T((\delta\lambda_1)q_1)$$

$\delta q_1 = \begin{bmatrix}\delta q_{11}\\ \delta q_{21}\end{bmatrix} \qquad q_1 = \begin{bmatrix}1\\0\end{bmatrix} \qquad A = \begin{bmatrix}a_{11} & 0\\ 0 & a_{12}\end{bmatrix}$

$$\delta A = \begin{bmatrix}\delta a_{11} & \delta a_{12}\\ \delta a_{21} & \delta a_{22}\end{bmatrix}$$

$q_1^T\delta q_1 = 0 \Rightarrow \delta q_{11} = 0; \quad q_1^T\delta A q_1 = \delta a_{11}$

$q_1^T A\delta q_1 = a_{11}\delta q_{11} = 0; \quad q_1^T(\delta\lambda_1 q_1) = \delta\lambda_1$

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$$\delta\lambda_1 = \delta a_{11} \quad \text{--- (2)}$$

$$A q_1 = \lambda_1 q_1$$

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = a_{11} \quad \text{--- (3)}$$

Also, $\delta q_1 = \begin{bmatrix} 0 \\ \delta q_{21} \end{bmatrix}$

Substituting $\delta\lambda_1, \lambda_1, A, \delta A, q_1$ in (1)

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \delta q_{21} \end{bmatrix} + \begin{bmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\lambda_1 \begin{bmatrix} 0 \\ \delta q_{21} \end{bmatrix} + \delta a_{11} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \delta q_{21} = \begin{bmatrix} 0 \\ \dfrac{-\delta a_{21}}{a_{22} - a_{11}} \end{bmatrix}$$

$$q_1 + \delta q_1 = \begin{bmatrix} 1 \\ \dfrac{-\delta a_{21}}{a_{22} - a_{11}} \end{bmatrix}$$

Since a_{22} & a_{11} are eigen values of A, so if they are very close to each other than $a_{22} - a_{11}$ will be very small which makes $\dfrac{-\delta a_{21}}{a_{22}-a_{11}}$ very large -ve value, hence eigen vector is ill conditioned

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(2) (a) Algo for householder QR :

for $k = 1$ to n :

$$x = A_{k:m,k}$$

$$V_k = \text{sign}(x_1) \|x\|_2 e_1 + x$$

$$V_k = \frac{v}{\|x_k\|_2}$$

$$A_{k:m,k:m} = A_{k:m,k:m} - 2 V_k V_k^* A_{k:m,k:m}$$

In this algo, we have to do (m-k) iterations over the rows But for hessenberg matrices in a column element is zero after some point. So we can eliminate iterations for later indices after that one.

So we can change last update to

$$A_{k,k:m} = A_{k,k:m} - 2 V_k V_k^* A_{k,k:m}$$

Now new update takes $O(m)$ time
Also there are m columns
So total of $O(m^2)$ time.

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(3) (a) By power iteration we get eigen vector having largest eigen value 'λ' as we multiply by A at every iteration. But if we multiply by A^{-1} then we will get eigen vector corresponding to eigen value $1/\lambda$ ie smallest one. Also eigen vectors of A & A^{-1} are same ie $\boxed{Av = \lambda v \iff \frac{1}{\lambda} v = A^{-1} v}$

Algo:-

$v^{(0)}$ = Some vector with $\|v^{(0)}\| = 1$

for $k = 1, 2, 3 ----$

$$w = A^{-1} v^{(k-1)}$$
$$v^{(k)} = \frac{w}{\|w\|}$$

return $v^{(k)}$

(b) We can use Simultaneous power iteration to get k^{th} smallest eigen value

get k orthonormal vectors $v_1, ---, v_k$

$$V = [v_1 | v_2 | --- | v_k]$$

Use simultaneous power iteration on V

$$A^{-1}V = [A^{-1}v_1 | ---- | A^{-1}v_k]$$

then do QR decomposition of $A^{-1}V$ then pass orthonormal Q to next iteration

Algo:

$$V^{(0)} = [v_1 | \dots | v_k]$$ orthonormal vectors

for $k = 1, 2, \dots$

$$W = A^{-1} V^{(k-1)} = [A^{-1}v_1 | A^{-1}v_2 | \dots | A^{-1}v_k]$$

$$V^{(k)} = QR$$ (QR decomposition of V)

$$V^{k} = Q$$

return last col. of $V^{(k)}$

④ Let λ_1 be eigen value of A_{11} & v_1 be corresponding eigen vector of A_{11} ie $A_{11}v_1 = \lambda_1 v$
Now $W = Q^T A Q = Q^{-1} A Q$

$$W = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

consider vector $v' = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$

$$Wv' = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} v_1 \\ 0 \end{bmatrix}$$

$$\Rightarrow Wv' = \begin{bmatrix} \lambda_1 v_1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \lambda_1 v'$$

$\Rightarrow$ v' is eigen vector of W with λ_1 as eigen value

$\Rightarrow$ eigen values of A_{11} is same as eigen values of W. and hence same as A as Q is orthogonal and A & W are similar matrices.

Let λ_2 be eigen value of A_{22} & v_2 be corresponding eigen vector of A_{22} ie
$A_{22} v_2 = \lambda_2 v_2$

$$W^T = \begin{bmatrix} A_{11}^T & 0 \\ A_{12}^T & A_{22}^T \end{bmatrix}$$

$\cancel{A_{22} v_2 = \lambda_2 v_2}$

~~Also~~

λ_2 will also be eigen value of A_{22}^T and ~~last~~ lets say y_2 will be corresponding eigen vector of A_{22}^T

ie

$$A_{22}^T y_2 = \lambda_2 y_2$$

$$y' = \begin{bmatrix} 0 \\ y_2 \end{bmatrix}$$

$$w^T y' = \begin{bmatrix} A_{11}^T & 0 \\ A_{12}^T & A_{22}^T \end{bmatrix} \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ A_{22}^T y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \lambda_2 y_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = \lambda_2 y'$$

$\Rightarrow$ λ_2 is also eigen value of W^T and hence W and hence same ~~as~~ as A as Q is orthogonal & A & W^T are similar matrices.

(5) (a) $p(x) = x^3 - x^2 - x + 1$

$p'(x) = 3x^2 - 2x - 1$

$$x_{k+1} = x_k - \frac{p(x_k)}{p'(x_k)}$$

4 iterations :-

$\boxed{x_0 = 2}$

$x_1 = 1.57$, $x_2 = 1.34$, $x_3 = 1.167$, $x_4 = 1.087$

$\boxed{x_0 = -2}$

$x_1 = -1.4$, $x_2 = -1.1$, $x_3 = -1.009$, $x_4 = -1.00007$

(b) We know that convergence is linear if $p'(x_{root}) = 0$ ~~there linear convergence~~ and if $p'(x_{root}) \neq 0$ then quadratic convergence

$p'(1) = 0$ hence linear convergence

$p'(-1) = 4 \neq 0$ hence quadratic convergence.

constant term of linear convergence $= g'(x_{root})$
ie

$$g'(x) = \frac{P(x) \; P''(x)}{[P(x)]^2}$$

$$g'(x) = \frac{(x-1)\,(x-1)\,(x+1)\,(6x-2)}{(3x+1)^2\,(x-1)^2}$$

$$\lim_{x \to 1} g'(x) = \frac{1}{2}$$

(c) Let $f(x) = (x - x_{root})^2\, h(x)$

$$f'(x) = 2(x - x_{root})\, h(x) + (x - x_{root})^2\, h'(x)$$

$$f''(x) = 2(x - x_{root})\, h'(x) + 2\, h(x) + h'(x)\, 2(x - x_{root}) + (x - x_{root})^2\, h''(x)$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = \frac{f(x)\, f''(x)}{(f'(x))^2}$$

But $f'(x_{root}) = 0$
$\Rightarrow$
linear convergence

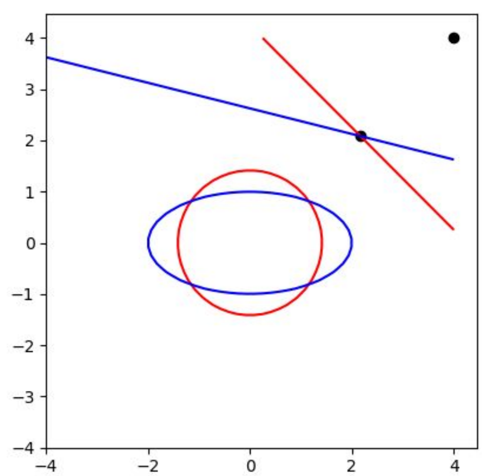
To improve this, we modify

$$g(x) = x - 2\frac{f(x)}{f'(x)} \Rightarrow g'(x_{root}) = 0$$

$e_{k+1} = g'(x)_{root}\, e_k + O(|e_k|^2) = O(|e_k|^2) \Rightarrow$ quadratic convergence.

(6) The next newton iterate lies at the intersection of the 2 linearized functions returned by conic1.linearize(x) & conic2.linearize(x)

Conic1 : $x^2 + y^2 - 2 = 0$ (red)
conic2 : $x^2 - 4y^2 - 4 = 0$ (blue)



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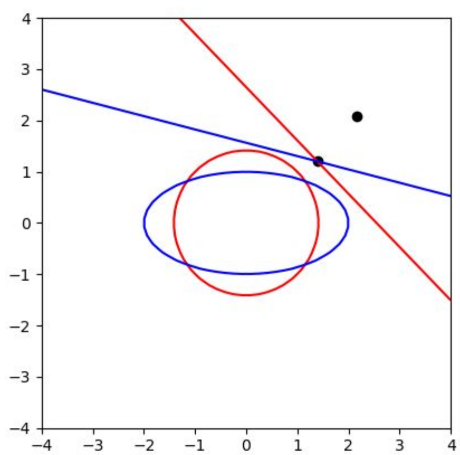

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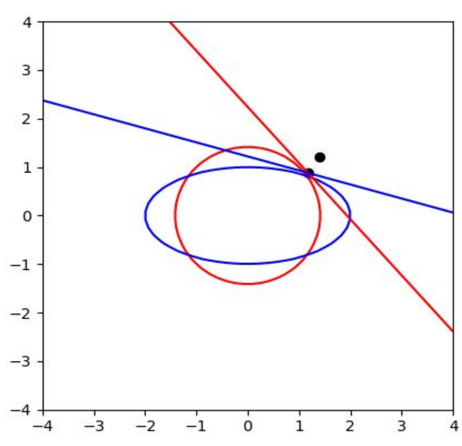

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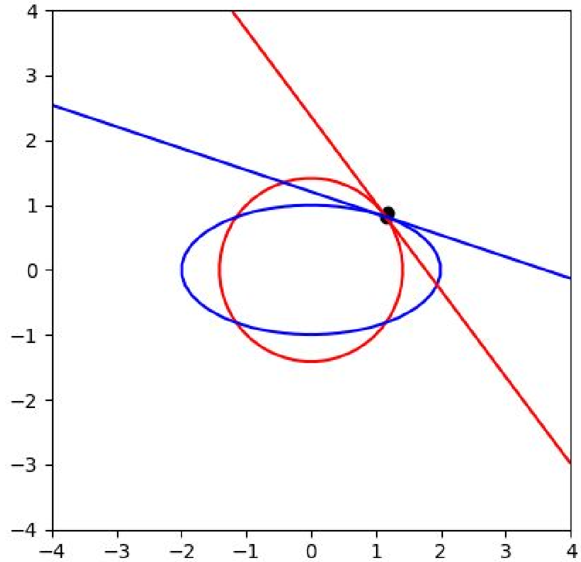
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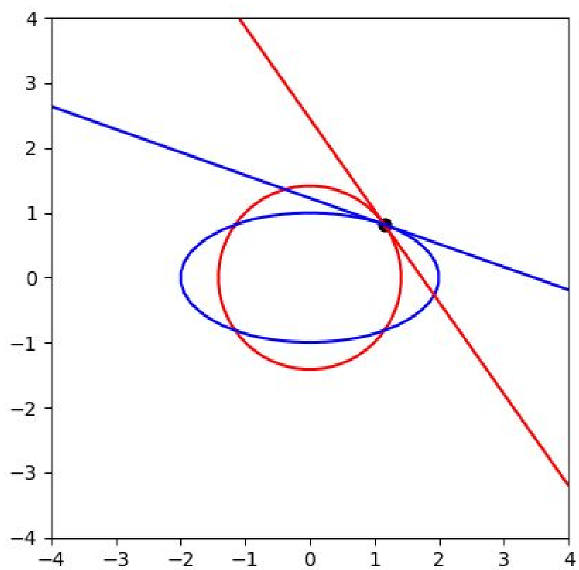
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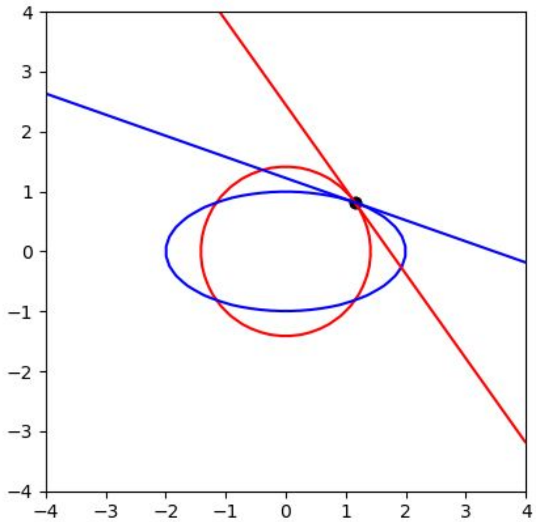


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