Date / /

Page

① (a) Let $x \in S_k$

ie

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$Ax = c_1 A v_1 + c_2 A v_2 + \cdots + c_k A v_k$$

$$Ax = c_1 \sigma_1 u_1 + c_2 \sigma_2 u_2 + \cdots + c_k \sigma_k u_k$$

$$Ax = UZ$$

where

$$Z = \begin{bmatrix} c_1 \sigma_1 \\ \vdots \\ c_k \sigma_k \end{bmatrix}$$

and

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix}$$

and U is orthogonal matrix ——— ①

$$\|Ax\|_2 \quad \|UZ\|_2 \quad \text{———} ②$$

from ① & ②

$$\|Ax\|_2 = \|Z\|_2 = \sqrt{\sum_{i=1}^{k} c_i^2 \sigma_i^2}$$

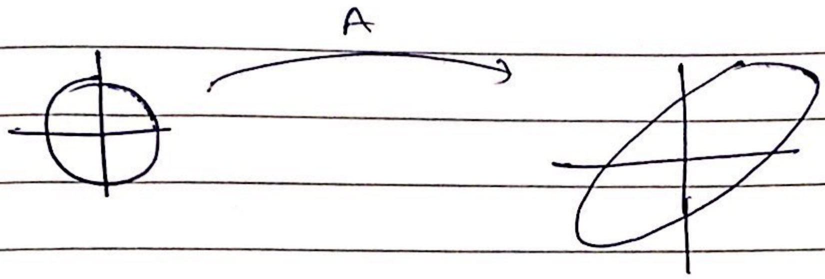
$$x = VC$$

where

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \quad \& \quad C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

Date / /
Page

1(b) for 2-D case



Now, for k dimensions every dimension gets streched by σ_i in final space

~~So~~ We have total of n dimension ie n σ's. ~~A~~nd each dimension out of k is streched by σ_i. So there will be atleast one vector in input space which will be streched by σ_k or less than σ_k as we have to choose k σ's out of n.

Date / /
Page

(2) (a) Lets suppose $\{a_1, ---\} \cup \{b_1 -$
is not linear independent
then

$m_1 a_1 + m_2 a_2 + ---- + m_n a_n +$
$p_1 b_1 + ------ + p_n b_n = 0$

$m_1 a_1 + ---- + m_n a_n =$
$-p_1 b_1 - p_2 b_2 - ------ - p_n b_n$

$m_1 a_1 + ----- + m_n a_n =$
$q_1 b_1 + ------ + q_n b_n = c$ (say)
where $q_i = -p_i$
now we can represent vector c as linear combination of both $\{a_1 ---\}$ & $\{b_1, ---\}$ ie c lies in both S & T but they are complementary so ~~should not be~~
Our supposition must be false as that lead to contradiction.

Hence $\{a_1 ---\} \cup \{b_1 ---\}$ is linearly independent set.

Lets suppose this set do not span R^n
ie we have atleast one vector say v s.t.

$v \neq m_1 a_1 + --- + m_n a_n + p_1 b_1 + ---$
$+ p_n b_n$

$\|x\| = \|Vc\|$

Since V is orthogonal

$\|x\| = \|c\| = \sqrt{\sum_{i=1}^{k} c_i^2}$

$$\frac{\|Ax\|}{\|x\|} = \frac{\sqrt{\sum_{i=1}^{k} c_i^2 \sigma_i^2}}{\sqrt{\sum_{i=1}^{k} c_i^2}}$$

Since

$\sigma_1 \geq \cdots\cdots \geq \sigma_k$

$$\Rightarrow \frac{\|Ax\|}{\|x\|} \geq \frac{\sqrt{\sum_{i=1}^{k} c_i^2 \sigma_k^2}}{\sqrt{\sum_{i=1}^{k} c_i^2}}$$

$$\geq \frac{\sigma_k \sqrt{\sum_{i=1}^{k} c_i^2}}{\sqrt{\sum_{i=1}^{k} c_i^2}} = \sigma_k$$

$$\inf_{x \in S_k} \frac{\|Ax\|}{\|x\|} = \sigma_k$$

Date / /
Page

ie V neither lies in S or T but since S & T are complementary so they must include every vector so by contradiction we get $\{a_1 \_\_\} \cup \{b_1 \_\_\}$ spans R^m.

(b) Let

$$B_1 = \{a_1 \_\_\} \quad \& \quad B_2 = \{b_1 \_\_\}$$

$$B = [B_1 \ \ B_2]$$

$$PB = [PB_1 \ \ PB_2]$$

Since B_2 is null space of P ie $PB_2 = 0$ and B_1 is range (P)

so $PB_1 = B_1$

$$PB = [PB_1 \ \ 0] = [B_1 \ \ 0]$$

columns of B are linearly independent ie B is full rank so B is invertible

$$\Rightarrow \boxed{P = [B_1 \ \ 0]\, B^{-1}}$$

Scanned with CamScanner

Date / /
Page

3(a)

Pv is multiple of x

$\Rightarrow Pv = \lambda x$

$(v - Pv)$ is C-orthogonal to x

ie

$f(x, v - Pv) = 0$

$\Rightarrow x^T C (v - Pv) = 0$

$x^T C (v - \lambda x) = 0$

$\Rightarrow \lambda = \dfrac{x^T C v}{x^T C x}$

$Pv = \lambda x$

$Pv = \dfrac{x^T C v}{x^T C x} x$

Since dim. of x^T is $1 \times n$ & dim of C is $n \times n$ & dim of v is $n \times 1$

hence

$x^T C v$ is scalar.

$\Rightarrow (x^T C v) x = x (x^T C v)$

$Pv = \dfrac{x x^T C v}{x^T C x}$

$\Rightarrow \boxed{P = \dfrac{x x^T C}{x^T C x}}$

(3) (b) $A = \hat{X}\hat{R}$

where R is $n \times n$ upper triangular matrix & X is $m \times n$ matrix

In gram Schmidt we have,

$v_j = a_j - (q_i^T a_j)\, q_i - \cdots$

we can replace inner product $\langle q_i, a_j \rangle$ to modified inner product $f(q_i, a_j)$

$f(x_i, a_j) = x_i^T C a_j$

$v_j = a_j - (x_i^T C a_j)\, x_i - \cdots$

Date / /
Page

(4) $$F^T F = \left(I - \frac{2vv^T}{v^T v}\right)^T \left(I - \frac{2vv^T}{v^T v}\right)$$

$$= \left(I - \frac{2vv^T}{v^T v}\right)\left(I - \frac{2vv^T}{v^T v}\right)$$

$$= I - \frac{2vv^T}{v^T v} - \frac{2vv^T}{v^T v} + \frac{4vv^T}{v^T v}\frac{vv^T}{v^T v}$$

$$= I - \frac{4vv^T}{v^T v} + \frac{4v(v^T v)v^T}{(v^T v)(v^T v)}$$

$$= I - \frac{4vv^T}{v^T v} + \frac{4vv^T}{v^T v}$$

$$= I$$

$$v^T v = (x - x')^T (x - x')$$
$$= x^T x - x^T x' - x'^T x + x'^T x'$$

Since $\|x\| = \|x'\|$ & $x^T x = \|x\|$

$\Rightarrow$ $x^T x = x'^T x'$

Also, $x^T x'$ is single value which is same as $x'^T x$

$$v^T v = 2x^T x - 2x^T x' = 2x^T v$$

$$Fx = \left(I - \frac{2vv^T}{v^T v}\right) \cancel{x} = \left(I - \frac{2vv^T}{2x^T v}\right) x$$

$$Fx = x - \frac{2vv^T x}{2x^T v} = x - v = x'$$ (as $x^T v$ & $v^T x$ is a value)

(5) $b_i^T b_j = a_i^T a_j$ — (1)

$\Rightarrow B^T B = A^T A$ — (2)

where $A = [a_1 | a_2 | \cdots | a_n]$ & $B = [b_1 | \cdots | b_n]$

So if we have to prove (1) it is sufficient to prove (2)

Take SVD of A as

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma^T U^T U \Sigma V^T$$

$$A^T A = V \Sigma^T \Sigma V^T \quad — (3)$$

Take $B = \Sigma V^T$

$$B^T B = V \Sigma^T \Sigma V^T \quad — (4)$$

from (3) & (4)

$$A^T A = B^T B$$

~~$\|A\| = \|U \Sigma V^T\|$~~
~~as U is orthogonal~~
~~$\|A\| = \|\Sigma V^T\|$~~
~~Also~~
~~$\|B\| = \|\Sigma V^T\|$~~
~~$\Rightarrow \|A\| = \|B\|$~~

diagonal element of $A^T A$ is $a_i^T a_i$

Since $A^T A = B^T B$

$\Rightarrow$

$$a_i^T a_i = b_i^T b_i$$

$$\|a_i\|_2 = \|b_i\|_2$$

Scanned with CamScanner

(6) (b) <u>Compress</u> -

firstly, I computed SVD of matrix C

$$C = U\Sigma V^T$$

$U\Sigma = A$ & $B = V^T$

then I truncated A & B to get rank - r approximation of matrix C

$$A \longrightarrow A'$$
$$B \longrightarrow B'$$

then $C' = A'B'$

(c) relative error is computed as follows

$$\frac{\sqrt{\sigma_r^2 + \sigma_{r+1}^2 + \cdots + \sigma_n^2}}{\sqrt{\sigma_1^2 + \cdots + \sigma_n^2}}$$

as error is caused by the singular values that are ~~rem~~ removed while compressing the image.