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(1)

$$A = \begin{bmatrix} 1 & 2.01 \\ 1.01 & 2.03 \end{bmatrix} \qquad B = \begin{bmatrix} 1.01 & 1.02 \end{bmatrix}^T$$

$$Ax = B$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

$$|A| = -10^{-4}$$

$$\operatorname{adj}(A) = \begin{bmatrix} 2.03 & -2.01 \\ -1.01 & 1 \end{bmatrix}$$

$$A^{-1} = (-10^{+4}) \begin{bmatrix} 2.03 & -2.01 \\ -1.01 & 1 \end{bmatrix}$$

$$X = A^{-1} B$$

$$= (-10^4) \begin{bmatrix} 2.03 & -2.01 \\ -1.01 & 1 \end{bmatrix} \begin{bmatrix} 1.01 \\ 1.02 \end{bmatrix}$$

$$= (-10^4) \left(10^{-4} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

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$$b+\Delta b = \begin{bmatrix} 1.01 & 1.0201 \end{bmatrix}^T$$

$$A^{-1} = (-10^4)\begin{bmatrix} 2.03 & -2.01 \\ -1.01 & 1 \end{bmatrix}$$

$$X = A^{-1}B$$
$$= (-10^4)\begin{bmatrix} 2.03 & -2.01 \\ -1.01 & 1 \end{bmatrix}\begin{bmatrix} 1.01 \\ 1.0201 \end{bmatrix}$$
$$= (-10^4)\left(10^{-4}\begin{bmatrix} -1.01 \\ 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} +1.01 \\ 0 \end{bmatrix}$$

$$\|A\|_\infty = \max(1+2.01,\ 1.01+2.03)$$
$$= 3.04$$

$$\|A^{-1}\|_\infty = \max\left((2.03+2.01)10^{+4},\ (1.01+1)10^{+4}\right)$$
$$= 4.04\times 10^{+4}$$

Condition Number (relative) $= \cancel{\|A\|\,\|A^{-1}\|}$

$$= \frac{\|\Delta X\|_\infty / \|X\|_\infty}{\|\Delta B\|_\infty / \|B\|_\infty}$$

$$\Delta X = \begin{bmatrix} 2.01 \\ -1 \end{bmatrix} \qquad \Delta B = \begin{bmatrix} 0 \\ 0.0001 \end{bmatrix}$$

Condition number $= \dfrac{2.01/1}{0.0001/1.02}$

$$= 2.0502\times 10^4$$

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(2) (a)

Case (i):

b^2 is close to $4ac$

If this happens then determinant (b^2-4ac) will be wrongly calculated as subtraction is ill conditioned and since this value is in both roots so both x_1, x_2 will be affected

Case (ii):

$b^2 >> 4ac$

if this happens then

$$x_{1,2} = \frac{-b \pm (b+\epsilon)}{2a}$$

$$x_1 = \frac{-b + b + \epsilon}{2a}$$

Since this also has subtraction so x_1 will be affected as subtraction is ill conditioned.

(b) $$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\left(\frac{-b \mp \sqrt{b^2-4ac}}{-b \mp \sqrt{b^2-4ac}}\right)$$

$$x_{1,2} = \frac{b^2 - (b^2-4ac)}{2a\left(-b \mp \sqrt{b^2-4ac}\right)}$$

$$x_{1,2} = \frac{2c}{-b \mp \sqrt{b^2-4ac}}$$

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In above formula,

$$x_2 = \frac{2c}{-b + \sqrt{b^2 - 4ac}}$$

If $b^2 >> 4ac$, then again there will be huge error because of cancellation in subtraction & subtraction is ill-conditioned. So second root will be affected but for first formula x_1 will be affected.

(3) $f(x) = (x-1)^2$, x is real

$f(x) = (x-1) \times (x-1)$

$\tilde{f}(x) = (fl(x) \ominus 1) \otimes (fl(x) \ominus 1)$

$\tilde{f}(x) = [(x(1+\epsilon_1) - 1)(1+\epsilon_2)][(x(1+\epsilon_1) - 1)(1+\epsilon_2)(1+\epsilon_3)]$

where $\epsilon_1, \epsilon_2, \epsilon_3 \leq \epsilon_m$

$\tilde{f}(x) \leq [(x(1+\epsilon_m) - 1)(1+\epsilon_m)][(x(1+\epsilon_m) - 1)(1+\epsilon_m)](1+\epsilon_m)$

$\tilde{f}(x) \leq (x-1)^2 + \epsilon_m(5x^2 - 8x + 3) + k(\epsilon_m^2)$

$\tilde{f}(x) - f(x) \leq \epsilon_m(5x^2 + 3 - 8x) + k(\epsilon_m^2)$

$|\tilde{f}(x) - f(x)| \leq \epsilon_m |5x^2 - 8x + 3| + |k| \epsilon_m^2$

$|\tilde{f}(x) - f(x)| \leq \epsilon_m c + O(\epsilon_m^2)$

where $c = |5x^2 - 8x + 3|$

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(4) (a)

Take column wise interpretation of B as

$$B = \left[B_1 \middle| B_2 \middle| B_3 \text{ --- } \middle| B_p \right]$$

$$C = AB = \left[AB_1 \middle| AB_2 \middle| \text{ --- } \middle| AB_p \right]$$

Say B_x is linearly dependent of some B_i's so after multiplying B with A then AB_x will still be dependent on AB_i's for same values of i's.

So we can say that if B has k linearly dependent vectors then AB ie C will have more than or equal to k linearly dependent vectors or alternatively we can say that if B has r linearly independent vectors then C will have atmost r linearly independent vectors. i.e.

$$\text{rank}(B) \geq \text{rank}(C)$$
$$\text{rank}(B) \leq \min(n, p) = n$$

$$\Rightarrow \boxed{\text{rank}(C) \leq n}$$

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(b) ① A maps non zero to non zero -

We will prove it by contradiction
Let say A maps non zero to zero.

$$AX = 0$$

$$cAX = 0 \quad \text{for any scalar } c$$

$$A(cX) = 0$$

So now A will map 2 vectors X and cX to zero but according to theorem A cannot maps 2 distinct vector to same. Hence contradiction.

② A maps linearly indep. to Linearly Ind.-

We will prove it by contradiction
let say A maps linearly independent (X_i) to Linearly dependent (Y_i) ie

$$\sum_{i=1}^{n} \alpha_i Y_i = 0$$

$$\left(\sum_{i=1}^{n} \alpha_i (AX_i)\right) = 0$$

$$A\left(\sum_{i=1}^{n} \alpha_i X_i\right) = 0$$

Since $A \neq 0$ & X_i are ~~...~~ Linearly independent

$\Rightarrow$ A maps nonzero to a zero vector

~~$\Rightarrow X_i$ are linearly dependent which is contradiction.~~

which is contradiction

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(C) $C_i = AB_i$ where B_i and C_i are i^{th} columns of B and C

Since A is full rank, So from (b) we can say A ~~only~~ maps linearly independent vectors to linearly independent vectors.
and since B is also full rank
ie $\text{rank}(B) = \min(n, p) = n$
$\Rightarrow$ B has n linearly independent vectors
So AB will also have atleast n linearly independent vectors.
$\Rightarrow$ C has atleast n linearly independent vectors

$\Rightarrow \text{rank}(C) \geq n$

But from (a), we have
$\text{rank}(C) \leq n$

So $\boxed{\text{rank}(C) = n}$

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(5) (a) $\|x+y\| \leq \|x\| + \|y\|$

Triangular inequality

take x as $x+y$ and y as $(-y)$

~~$\|x-y+y\|$~~

$$\|(x+y)-y\| \leq \|x+y\| + \|-y\|$$

$$\|x\| \leq \|x+y\| + \|y\|$$

$$\|x+y\| \geq \|x\| - \|y\|$$

(b) Let $I+\epsilon A$ be invertible

$\therefore$ $I+\epsilon A$ is full rank matrix & hence it maps non zero vector x to a non-zero vector.

$\Rightarrow$ $\forall x \neq 0$ $(I+\epsilon A)x \neq 0$

$(I+\epsilon A)x$ is not all zero matrix

$\therefore$ $\|(I+\epsilon A)x\| \neq 0$

$\Rightarrow$ $\|(I+\epsilon A)x\| > 0$

$\|x+\epsilon Ax\| > \big|\|x\| - \epsilon\|Ax\|\big| > 0$

also we know that

$\|Ax\| \leq \|A\| \ \|B\|$

$\|x+\epsilon Ax\| \geq \|x\| - \epsilon\|A\|\,\|x\| > 0$

also x is non zero $\therefore \|x\| \neq 0$

$\Rightarrow \big|1 - \epsilon\|A\|\big| > 0 \Rightarrow \epsilon < \frac{1}{\|A\|} \Rightarrow \delta = \frac{1}{\|A\|}$

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(6) $A = [10^{10} - 0.01, 10^{10} + 0.00, 10^{10} + 0.01]$

Var1 = 2097152.0

Var2 = $6.659344459 \times 10^{-5}$

$\frac{\text{Var1}}{\text{Var2}} > 10^3$

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