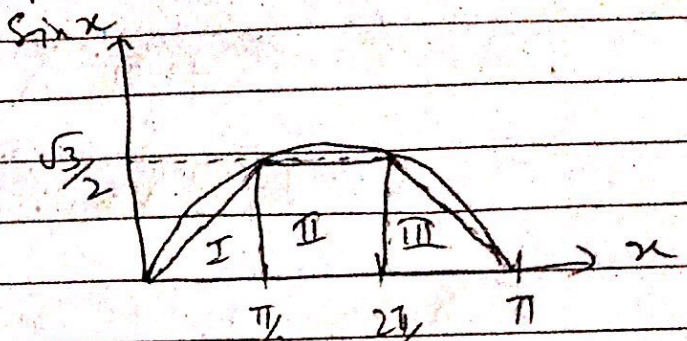
RANKA
DATE / /
PAGE

1(b) $I = \int_0^{\pi} \sin x \, dx$

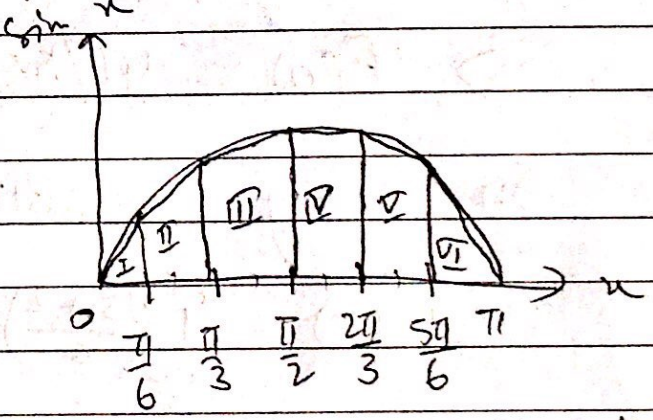
4 equally spaced -



I = Area = Area I + Area II + Area III

$= \frac{\pi}{\sqrt{3}}$

7 equally spaced -



I = Area

I = Area I + Area II + Area III + Area IV + Area V + Area VI

$= \left(\frac{\sqrt{3}+2}{6}\right)\pi$

RANKA
DATE / /
PAGE

Richardson extrapolation-

$$F(h) = a_0 + a_1 h^2 + O(h^4) \quad \text{---} (1)$$

Now if we take $h = \pi/3$ then $F(h) =$ Area

$$F(h/2) = F(\pi/6) = \text{Area 2}$$

$$F(h/2) = a_0 + a_1 \frac{h^2}{4} \quad \text{---} (2)$$

replacing $a_1 h^2$ from (1) in (2) we get

$$a_0 = \frac{4F(h/2) - F(h)}{3}$$

Also

$$F(0) = a_0$$

$$\Rightarrow F(0) = \frac{4F(h/2) - F(h)}{3}$$

on solving by putting values we get

$$F(0) = \frac{4\left(\frac{\sqrt{3}+2}{6}\right)\pi - \left(\frac{\pi}{\sqrt{3}}\right)}{3}$$

order of accuracy $= O(h^4)$

1(b) No, it is not possible to use Richardson extrapolation as Q_4 will have accuracy of order $O(h^5)$ & Q_7 will have accuracy of $O(h^9)$ so it will be difficult to find a $I(h)$ of the form

$$I(h) = a_0 + a_1 h^p + O(h^q)$$

as both Q_4 & Q_7 has different order of accuracies.

(2) Using taylor series we can write

$$f(x+h) = f(x) + f'(x)h + f''(x)h^2 + O(h^3)$$

&

$$f(x+2h) = f(x) + 2f'(x)h + 4f''(x)h^2 + O(h^3)$$

for $h \to 0$

$$4f(x+h) - f(x+2h) =$$

$$3f(x) + 2h f'(x) + O(h^3)$$

$$f'(x) = \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h} + O(h^2)$$

——— (1)

$$f'(0) = \frac{4f(h) - f(2h) - 3f(0)}{2h} + O(h^2)$$

As we can see in ① that $f'(x)$ has accuracy of $O(h^2)$ i.e. second order accurate.

3(a) Proof by contradiction.

i.e.

intersection of all half spaces that contain C is not equal to C itself.

If intersection is not C then there will be 2 cases

① $\exists\ x' \in C$ but x does not belong to intersection

It is not possible as every halfspace must include C so they have to contain x.

② $\exists\ x' \notin C$ but still belongs to intersection.

~~Suppose~~ we construct a halfspace H with $a^T x = b$ S.t.

$a^T x' < b$ & $a^T c > b\ \forall\ c \in C$

Since halfspace including x' does not include convex set C so while creating a space after intersection this set won't be included & hence x' won't be included.

This is contradiction so our assumption must be wrong.

3(b) To prove

f is concave

$\Leftrightarrow$

$\alpha_1 f(x_1) + (1-\alpha) f(x_2)$

$\leq f(\alpha x_1 + (1-\alpha) x_2)$

for any $x_1, x_2 \in C$

Consider 2 balls at pts x_1, x_2 of radii r_1, r_2 having largest value inside C

If $x_1, x_2 \in C$ then

$x_3 = \alpha_1 x_1 + (1-\alpha) x_2 \in C$

Consider this as center of a new ball with radius $r_3 = \alpha_1 r_1 + (1-\alpha) r_2$

Now if we show that this new ball lies inside C then we are done. Any pt of new ball is $X = x_3 + r_3 d$. by substituting x_3 & r_3 from above we get $X = \alpha(x_1 + r_1 d) + (1-\alpha)(x_2 + r_2 d)$

Now this also lies inside C as $x_1 + r_1 d \in C$ & $x_2 + r_2 d \in C$ & X is linear combo of these pts $\Rightarrow$ all pts of new ball lies inside C.

RANKA
DATE / /
PAGE

(4) f is convex if

$$f(y) \geq f(x) + \nabla f^T(x)(y-x) \quad \text{---(1)}$$

f is psd if $h^T \nabla^2 f(x) h \geq 0 \quad \forall x$

(i) f is psd

ie

$h^T \nabla^2 f(x) h \geq 0 \quad \forall x$

$\Rightarrow$

$$h^T \nabla^2 f(x+th) h \geq 0 \quad \text{---(2)}$$

Using taylor series $\frac{1}{2} h^T \nabla^2 f(x+th) h$

$$f(x+h) = f(x) + \nabla f^T(x) h + \frac{1}{2} h^T \nabla^2 f(x+th) h \quad \text{---(3)}$$

Using (2) & (3)

$$f(x+h) - f(x) - \nabla f^T(x) h \geq 0$$

$$f(x+h) \geq f(x) + (x+h-x)^T \nabla f(x)$$

Hence f is convex

(ii) $f(x+h) = f(x) + h\nabla f^T(x) + \frac{1}{2} h^T \nabla^2 f(x) h + O(h^3)$

Also f is convex

$$f(x+h) \geq f(x) + \nabla f^T(x)(y-x)$$

$$\frac{1}{2} h^T \nabla^2 f(x) h + O(h^3) \geq 0$$

Let $h = \sigma \hat{u}$ where $\hat{u}$ a unit vector

$$\frac{\sigma^2}{2} \hat{u}^T \nabla^2 f(x) \hat{u} + \sigma^3 O(\hat{u}^3) \geq 0$$

$$\frac{1}{2} \hat{u}^T \nabla^2 f(x) \hat{u} + \sigma O(\hat{u}^3) \geq 0$$

as $h \to 0 \Rightarrow \sigma \to 0$

$$\lim_{\sigma \to 0} \frac{1}{2} \hat{u}^T \nabla^2 f(x) \hat{u} + \sigma O(\hat{u}^3) \geq 0$$

$$\frac{1}{2} \hat{u}^T \nabla^2 f(x) \hat{u} \geq 0$$

$$\hat{u}^T \nabla^2 f(x) \hat{u} \geq 0$$

Hence f is psd.

6 (b)

$$f(y) = \frac{1}{2} k_{vel} \sum_{i=1}^{12} (y_i - y_{i-1})^2 + k_{obs} \sum_{i=1}^{12} f_{obs}(x_i, y_i)$$

$$\frac{\partial f(y)}{\partial y_i} = \frac{1}{2} k_{vel} \left[\frac{\partial}{\partial y_i} (y_i - y_{i-1})^2 + \frac{\partial}{\partial y_i} (y_{i+1} - y_i)^2 \right]$$

$$+ k_{obs} \frac{\partial}{\partial y_i} f_{obs}(x_i, y_i)$$

$$\frac{\partial f(y)}{\partial y_i} = \frac{1}{2} k_{vel} \left\{ 2(y_i - y_{i-1}) - 2(y_{i+1} - y_i) \right\}$$

$$+ k_{obs} g_{obs}(x_i, y_i)_2$$

$$= k_{vel} (-y_{i-1} + 2y_i - y_{i+1}) + k_{obs} g_{obs}(x_i, y_i)_2$$

(c)

As seen from the graph, algo 2 with hessian converges faster.