

1 (a) ① $U = A$, $L = I$

② for $k = 1$ to $m-1$

③ for $j = k+1$ to $\min(m, k+l)$

④
$$l_{jk} = u_{jk} / u_{kk}$$

⑤
$$u_{j, k: \min(m, k+l)} = l_{jk} u_{k, k: \min(m, k+l)}$$

outer loop takes m steps inner loop takes l steps for every outer step updating u takes u steps for every inner step

Hence Algo has $O(lum)$

Outer loop takes m flops

inner loop takes l flops

⑤ Step takes $2u$ flops

time complexity = $2mlu$ flops.

(2) Let Cholesky factorisation of M is

$$M = L_1 L_1^T$$

and say L_1 can be written as

$$L_1 = \begin{bmatrix} L_2 & x \\ y & q \end{bmatrix}$$

where

$$x, y \in \mathbb{R}^m, q \in \mathbb{R}, L_2 \in \mathbb{R}^{m \times m}$$

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix} = \begin{bmatrix} L_2 & x \\ y & q \end{bmatrix} \begin{bmatrix} L_2^T & y^T \\ x^T & q \end{bmatrix}$$

Also since L_1 is lower triangular matrix, so $x = 0$ and L_2 is lower triangular

$$\Rightarrow \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} = \begin{bmatrix} L_2 & 0 \\ y & q \end{bmatrix} \begin{bmatrix} L_2^T & y^T \\ 0 & q \end{bmatrix}$$

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix} = \begin{bmatrix} L_2 L_2^T & L_2 y^T \\ (L_2 y)^T & y y^T + q^2 \end{bmatrix}$$

$$A = L_2 L_2^T$$

Since Cholesky factorisation is unique for every matrix & L_2 is lower triangular

$$\Rightarrow L_2 = L$$

$$\text{Also } b = L_2 y^T, \quad c = y y^T + q^2$$

$$\Rightarrow b = L y^T \quad e$$

We can get y by backsubstitution method which takes less than $O(m^2)$ time.

Once we get y , we can compute the value of c from above equation.

$$L_1 = \begin{bmatrix} L_2 & x \\ y & q \end{bmatrix}$$

$$L_1 = \begin{bmatrix} L & 0 \\ y & q \end{bmatrix}$$

y, q can be computed in $O(m^2)$ time hence L_1 can be computed in $O(m^2)$ time. Cholesky factorisation of M can be computed in $O(m^2)$ time.

(3)(a)

$$x_k \leftarrow \frac{1}{a_{kk}} \left(b_k - \sum_{j \neq k} a_{kj} x_j \right)$$

If we write the above equation in matrix form by forming a matrix ~~with~~ from $\sum_{j \neq k} a_{kj} x_j$ with diagonal elements as zero.

$$x^{(k+1)} = D^{-1} (b - (L+U) x^{(k)})$$

where $A = D + L + U$ strictly
 $\downarrow \quad \downarrow \quad \rightarrow$ upper
 diagonal strictly lower Δ

This method performs Δ Jacobi iteration.

(b) equation p_k updates var. q_k

$$x_{q_k} \leftarrow \frac{b_{p_k} - \sum_{j \neq q_k} a_{p_k j} x_j}{a_{p_k q_k}}$$

P matrix ~~consists~~ is permutations matrix of $\{p_1, \dots, p_m\}$ & similarly Q is permutation ~~matrix~~ of $\{q_1, \dots, q_m\}$

~~Consider the matrix~~

Consider $A' = P A Q^T$

where $A' = D' + L' + U'$ as in (a)

$$Q x^{n+1} = D'^{-1} (P b - (L' + U') Q x^n)$$

$$(5)(a) \quad \|A - I\|_2 = 0.6$$

$$\|A - I\|_2 \geq \frac{\|(A - I)v\|_2}{\|v\|_2} \text{ for any } v$$

$$\|(A - I)v\|_2 \leq 0.6\|v\|_2$$

$$\|Av - v\|_2 \leq 0.6\|v\|_2$$

Since v is any arbitrary vector
So let's take eigen vector ~~of~~ of A
(v_i) as v

$$\|Av_i - v_i\|_2 \leq 0.6\|v_i\|_2$$

$$\|Av_i - v_i\|_2 \leq 0.6\|v_i\|_2$$

$$\|(\lambda - 1)v_i\|_2 \leq 0.6\|v_i\|_2$$

$$|\lambda - 1|(\|v_i\|_2) \leq 0.6\|v_i\|_2$$

Since $\|v_i\| > 0$ as A is SPD

$$|\lambda - 1| \leq 0.6$$

$$\Rightarrow \lambda \in [0.4, 1.6]$$

$$\frac{\|e_n\|_A}{\|e_0\|_A} \leq 2 \left(\frac{\sqrt{K(A)} - 1}{\sqrt{K(A)} + 1} \right)^n$$

Also $K(A) = \frac{\lambda_{\max}}{\lambda_{\min}} \leq \frac{1.6}{0.4} = 4$

$$\frac{\|e_n\|_A}{\|e_0\|_A} \leq 2 \left(\frac{2-1}{2+1} \right)^n = \frac{2}{3^n}$$

(b) $B = A + w w^T$
 $B = A + 49 v_1 v_1^T$

$$B v_1 = A v_1 + 49 v_1 v_1^T v_1$$

$$B v_1 = \lambda_1 v_1 + 49 v_1 \|v_1\|^2$$

$$\lambda_1 = 1 \quad \angle \|v_1\| = 1$$

$$B v_1 = 50 v_1 \quad \text{so} \quad \boxed{\lambda'_1 = 50}$$

Take other eigenvectors of A , v_i $i \neq 1$

$$B v_i = A v_i + 49 v_1 v_1^T v_i$$

Since A is spd, so every eigenvectors are orthogonal to each other i.e. $v_i^T v_j = 0$

$$B v_i = \lambda_i v_i + 0 = \lambda_i v_i$$

i.e. v_i & λ_i are eigenvector & eigenvalue of B

So we have, ~~the~~ same eigenvector of A & B

6 (a)

$$a_{ij} = \begin{cases} \frac{-1}{R_{ij}} & i \neq j \\ \sum_{\substack{\text{connected} \\ \text{to } i}} \frac{1}{R_{ij}} & i = j \end{cases}$$

Since $R_{ij} = R_{ji} \Rightarrow a_{ij} = a_{ji}$ Hence A is Symmetric

1(b)

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 13 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.5 & -0.25 & -0.175 & -0.125 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 10 & 5 & 0 & 0 & 0 \\ 0 & 10 & 7 & 0 & 0 \\ 0 & 0 & 10 & 7 & 0 \\ 0 & 0 & 0 & 10 & 13 \\ 0 & 0 & 0 & 0 & 1.5925 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$