Project 1: Mathematical Calculation of Bullseye Experiment

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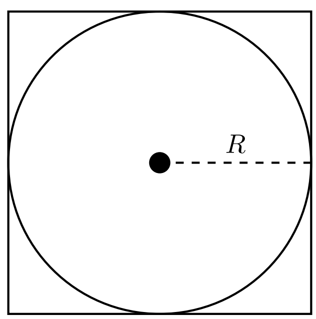
1 Question

This study investigates the precision of dart throwing toward a target modeled after a bullseye. The bullseye consists of a circular target enclosed within a square grid. Each dart thrown randomly lands within this grid, and its proximity to the bullseye determines its score. The task is to estimate, through simulation, the average closeness of darts to the bullseye.



2 Discussions

2.1 Code and observation for the specific case:

The Python script for this problem is given below:

```
import numpy as np

def bullseye(k):
    L=[]
    Points=[]
    while(len(L) < k):
        x = np.random.uniform(-1,1)
        y = np.random.uniform(-1,1)
```

```python
        if x**2 + y**2 <= 1:
            L.append(np.sqrt(x**2 + y**2))
            Points.append((x,y))
    L.sort()
    return L, Points

def bullseye_experiment(darts, trials):
    L = []
    for i in range(trials):
        L.append(min(bullseye(darts)[0]))
    return (np.mean(L))

print(bullseye_experiment(100,100000))
```

The code simulates dart throwing at a bullseye target within a square grid. The `bullseye` function generates random dart positions and calculates their distances from the center. The `bullseye_experiment` function conducts multiple trials, recording the minimum distances in each trial, and computes their mean. Finally, the average minimum distance from the bullseye is printed.

Trial	Average Minimum Distance
1	0.08842389561925132
2	0.08839676669711896
3	0.08810488343346802
4	0.08839672512247582
5	0.08868507061942657

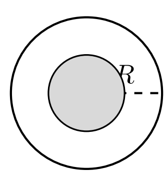
Table 1: Output table.

On average, the minimum distance obtained is approximately 0.088 units.

3 Mathematical Analysis

We are interested in finding the expectation value $E(\text{min})$, which represents the average minimum distance of a dart from the center when 100 darts are thrown inside a circular dartboard of radius R.

1. **Cumulative Density Function (CDF):**



The Cumulative Density Function (CDF) $F(x)$ represents the probability that a dart lands within a circle of radius x. For $0 < x < k$, the CDF is given by:

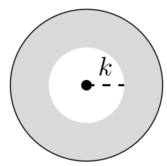
$$F(0 < x < k) = \frac{x^2}{R^2}$$

2. **Probability Density Function (PDF):**

The Probability Density Function (PDF) $P(x = k)$ gives the probability that a dart lands exactly at a distance k from the center. It is the derivative of the CDF with respect to k:

$$P(x = k) = \frac{dF(x)}{dk} = \frac{2k}{R^2}$$

3. **Probability of at least 1 dart near the center:**



The probability of at least one dart landing within a circle of radius k is $1 - F(0 < x < k)$. This represents the cumulative probability from distance k to $R = 1$:

$$F(k < x < 1) = 1 - F(0 < x < k) = 1 - \frac{k^2}{R^2}$$

The probability of at least one dart landing near the center out of n darts is given by:

$$P(\min = k) = \binom{n}{1} \cdot 2k \cdot (1 - k^2)^{n-1}$$

4. **Expectation Value of Minimum Distance k:**

The expectation value $E(\min)$ is calculated by integrating the minimum distance k multiplied by the probability density function $P(\min = k)$ over the range of possible k values:

$$E(\min) = \int_0^1 k \cdot P(\min = k)\, dk$$

Integrating $k \cdot P(\min = k)$ with respect to k, we obtain:

$$E(\min) = \int_0^1 k \cdot \binom{100}{1} \cdot 2k \cdot (1 - k^2)^{99}\, dk$$

Simplifying the integral and applying the Beta function formula:

$$E(\min) = \int_0^1 200k^2(1 - k^2)^{99}\, dk$$

Let $k^2 = t$, After the substitution $2k\,dk = dt$, we have:

$$dk = \frac{dt}{2k}$$

And since $k^2 = t$, we can rewrite dk as:

$$dk = \frac{dt}{2\sqrt{t}}$$

Finally, substituting $k^2\,dk = t^{1/2}\,dt$, we get:

$$E(\min) = \int_0^1 \frac{200}{2} t^{1/2} \cdot (1-t)^{99}\,dt = \frac{200 \cdot B\left(\frac{3}{2}, 100\right)}{2}$$

We know that $B\left(\frac{3}{2}, 100\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(100)}{\Gamma\left(\frac{203}{2}\right)}$.

Using properties of the Gamma function, we have:

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \quad \Gamma(100) = 99!$$

Finally, calculating $\Gamma\left(\frac{203}{2}\right) = \left(\frac{203}{2} - 1\right)\Gamma\left(\frac{203}{2} - 1\right)$, we can obtain the final expression for $E(\min)$.

Substitute the values and simplify:

$$E(\min) = \frac{200 \cdot \frac{\sqrt{\pi}}{2} \cdot 99!}{201 \cdot \Gamma\left(\frac{203}{2} - 1\right)}$$

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$$E(\min) = \frac{200 \cdot 2^{99} \cdot \sqrt{\pi} \cdot 99!}{201 \cdot 199 \cdot 197 \cdots 5 \cdot 3 \cdot \sqrt{\pi}}$$

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After solving this we get the final answer as:

$$E(\min) = \frac{200}{2244}$$

4 Conclusion

From the mathematical calculation, we find that the final value of the expectation $E(\min)$ is approximately 0.08912. On the other hand, from observation of the Python code, we obtained an average minimum distance of approximately 0.088 units.

To quantify the difference between the mathematical calculation and the simulation result, we can calculate the percentage difference as follows:

$$\text{Percentage Difference} = \left| \frac{E_{\text{math}} - E_{\text{sim}}}{E_{\text{math}}} \right| \times 100\%$$

Substituting the values, we get:

$$\text{Percentage Difference} = \left| \frac{0.08912 - 0.088}{0.08912} \right| \times 100\% \approx 1.26\%$$

This indicates that the simulation result deviates from the mathematical calculation by approximately 1.26%.

5 References

References to any external sources or materials used in this project.

1. Sheldon M. Ross, *A First Course in Probability*, Pearson, 2009.

2. Integration calculations performed using Wolfram Alpha,
   Wolfram Beta Function