# Closest Pair of Points

**Input:** $n$ pts $P = \{P_1, P_2, \ldots, P_n\}$

$P_i = (x_i, y_i) \in \mathbb{R}^2$

**Output:** Pair $(i, j)$ $i \neq j$ s.t.

$d(P_i, P_j)$ is minimized

$$d(P_i, P_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

**Assume:** $x_1, \ldots, x_n$ are distinct.
$y_1, \ldots, y_n$

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Let $x^*$ be the median x-value


$|P| = n$

$|Q| = |R| = \frac{n}{2}$

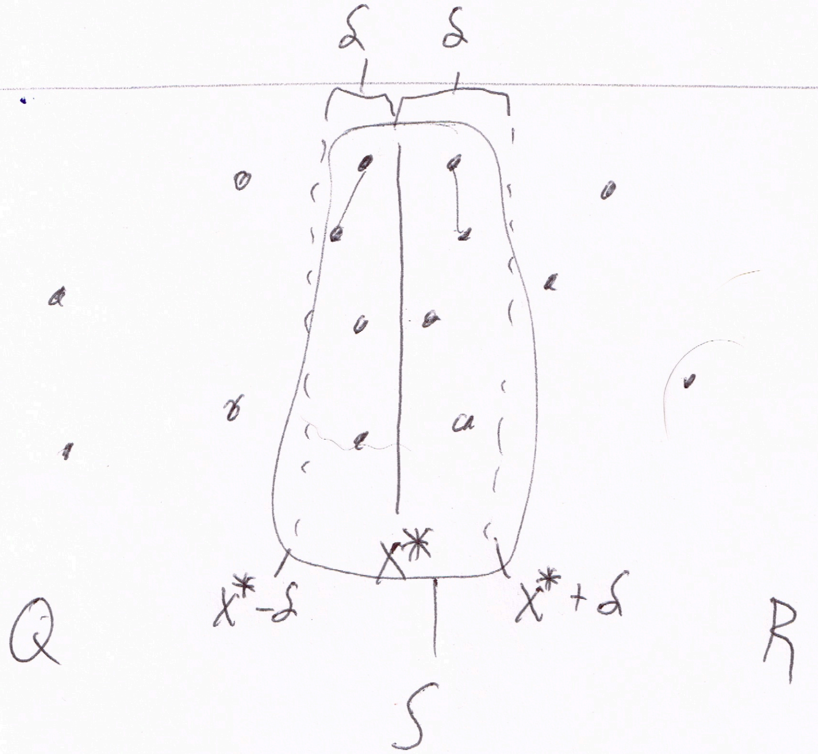
$P_x, Q_x, R_x$ are the sets sorted by x-value

$P_y, Q_y, R_y$ are the sets sorted by y-value

$Q_x, Q_y, R_x, R_y$ can be computed in linear time

Need to find the closest crossing pair in linear time.


$\delta = \min(\delta_1, \delta_2)$




$|S| = O(n)$

$\exists$ a $q \in Q$ and $r \in R$ s.t. $d(q, r) < \delta$
iff $\exists\, s \neq s' \in S$ s.t. $d(s, s') < \delta$

<u>Lemma:</u> if $\exists s, s' \in S$ s.t. $d(s, s') < \delta$

then $s$ and $s'$ are within 15 positions of each other in $S_y$.

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Using the Lemma, we have an $O(n)$ time implementation of closest-in-box.
(i.e. Check each point against the next 15 and take the min distance)

$15n$ is $O(n)$

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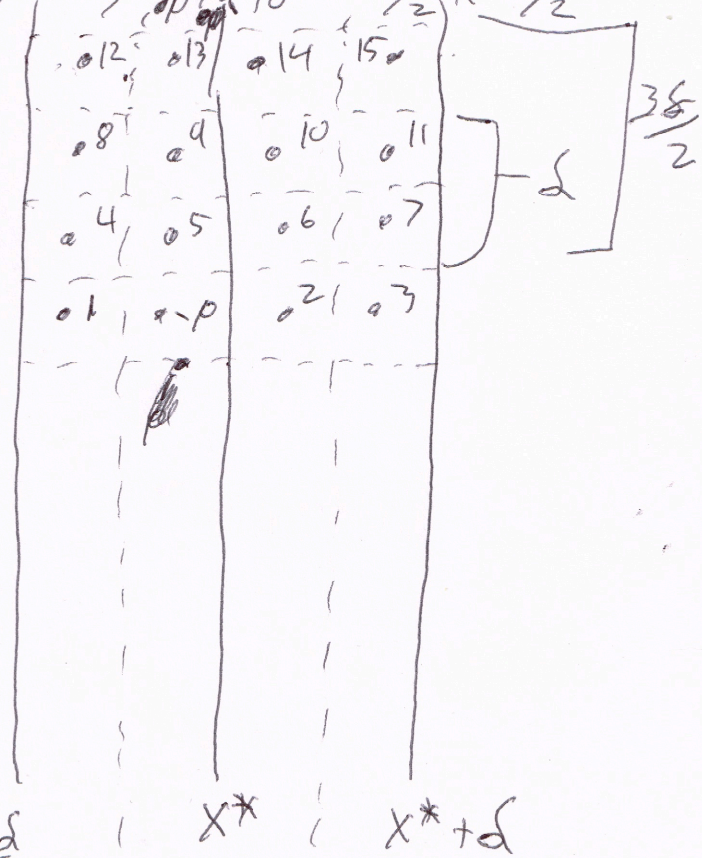
<u>Proof of Lemma</u>

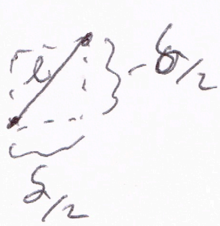
Divide the area for $S$ points into $\frac{\delta}{2} \times \frac{\delta}{2}$ size boxes.




~~p and~~ $d(p, p') > \frac{3\delta}{2} > \delta$

$\therefore d(p, p') < \delta$

<u>main claim:</u> There can only be at most one point in each box of size $\frac{\delta}{2} \times \frac{\delta}{2}$




$\ell = \sqrt{(\delta/2)^2 + (\delta/2)^2}$

$\ell = \frac{\sqrt{2}}{2}\delta < \delta$ ✓