CSE 331: Introduction to Algorithm Analysis and Design Summer 2013

HOMEWORK 0 SOLUTIONS

This homework is to refresh your memory on stuff you should have seen in CSE 191 and/or CSE 250. This homework will not count towards your final grade.

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1. (**What is a proof?**) (40 points) Consider the following "proof":

   - Brad Pitt has a beard
   - Every goat has a beard
   - Hence, Brad Pitt is a goat.

   State precisely where the "proof" above fails logically.

   Follow-up question: Can you prove that Brad Pitt is not a goat?

   Proof Idea: The main flaw in the proof above is that the conclusion seems to require the following statement as the second statement

   > Everyone who has a beard is a goat.

   Also there is insufficient information to *prove* that Brad Pitt is not a goat, even though we know from "common knowledge" that Brad Pitt is not a goat. However, in problems that you solve in this course you *cannot* assume anything outside of what is given in the problem statement.

   More formally, we'll use predicate logic to show the flaw in the proof that `Brad Pitt` is a goat. First, we'll construct a few predicates which we can use to express the given statements. Let $b(x)$ represent the statement "x has a beard". Let $g(x)$ represent the statement "x is a goat". □

   Solution: The given proof, in our predicate notation, is:
   $b(\texttt{ Brad Pitt})$
   $\forall x\,(g(x) \rightarrow b(x))$
   Therefore $g(\texttt{ Brad Pitt})$

   In order to draw the conclusion $g(\texttt{ Brad Pitt})$ from $b(\texttt{ Brad Pitt})$, we would need the conditional statement $\forall x\,(b(x) \rightarrow g(x))$. However, we don't know that to be true, and it's not equivalent to any of our statements, so this "proof" fails. We can, however, use a statement that is logically equivalent to the conditional statement we do have, $\forall x\,(g(x) \rightarrow b(x)) \equiv \forall x\,(\neg g(x) \vee b(x))$, to show that we can not disprove that `Brad Pitt` is a goat. Since we know our first statement $b(\texttt{ Brad Pitt})$ is true, the second statement is true regardless of `Brad Pitt`'s "goat-ness": if `Brad Pitt` is not a goat, then $\neg g(\texttt{ Brad Pitt})$ is true, and $\neg g(\texttt{BradPitt}) \vee b(\texttt{BradPitt})$ is true; but if `Brad Pitt` is a goat, then $\neg g(\texttt{ Brad Pitt})$ is false, but $\neg g(\texttt{BradPitt}) \vee b(\texttt{BradPitt})$ is still true. So strictly based on the statements we're given, not counting the faulty conclusion above, we can't determine whether `Brad Pitt` is a goat or not. ■

2. (**Proof by Contradiction**) (45 points) Assume that the following are true:

   - Every `blockbuster movie` has a `hero`.
   - `Jake Sully` and `Neytiri` are dating.
   - The highest grossing `movie` ever is a `blockbuster`.
   - A `hero` in a `movie` never `dies`.
   - The `movie Avatar` has made the most money ever.
   - The `heroine` always dates the `hero`.
   - `Neytiri` is the `heroine` of `Avatar`.

   Prove by contradiction that `Jake Sully` is `alive` at the end of the `movie Avatar`. Please clearly state any assumptions that you needed to make in the proof.

   Proof Idea: We're asked to prove that `Jake Sully` is `alive` at the end of `Avatar` by contradiction, so we'll need to start by assuming the negation of this statement is true, that `Jake Sully` is `dead` at the end of `Avatar`. We will use the other statements that we're given to arrive at a contradiction, a situation where our assumption makes another statement simultaneously true and false. A contradiction means that the assumption must be false, and since the assumption is the negation of what we want to prove, we've shown that "*not* `Jake Sully` is `dead`" is **false**, so "`Jake Sully` is `alive`" is **true**. Two assumptions that we need to make in order to clarify the language of the statements we're given are: `dead` is the negation of `alive`; and by making the most money ever, `Avatar` is the highest grossing movie ever. These assumptions may seem obvious, but in proofs, you can't assume that your reader will always know what you mean unless you state it explicitly. Also, some later statements in your proofs may not be true without some initial assumptions, so if you fail to state your assumptions ahead of time, your proof will be incorrect. (*Not-so-subtle hint: the *Proof Idea* section of your answers to homework questions is a good place to state your assumptions.) □

   Solution: Assume to the contrary that `Jake Sully` is `dead` at the end of `Avatar`. Since the `hero` in a movie never `dies`, then `Jake Sully` is not the `hero` of the movie `Avatar`. However, since `Avatar` made the most money ever (i.e. was highest grossing), it was a `blockbuster`, and since every `blockbuster` has a `hero`, we know that some other character besides `Jake Sully` is the `hero` of `Avatar`. We needed to make this point, because `Avatar` could have been some style of movie that doesn't have a `hero`, so there wouldn't necessarily have to be anyone `alive` at the end (like `Reservoir Dogs`), and our assumption that `Jake Sully` is `dead` doesn't get us anywhere. Also, since `Neytiri` is the `heroine` of `Avatar`, and the `heroine` always dates the `hero`, we can conclude that `Neytiri` is dating the `hero` of `Avatar`. Since we already showed that there is a `hero` in `Avatar`, and that that `hero` is not `Jake Sully`, we can further conclude that `Neytiri` is not dating `Jake Sully`. And here is our contradiction, we have a statement being simultaneously true and false: "`Jake Sully` and `Neytiri` are dating" was given as a true statement, but we just proved "`Jake Sully` and

`Neytiri` are not dating". Therefore, our initial assumption is false, so `Jake Sully` is `alive` at the end of `Avatar`. ■

3. (**Induction**) (15 points) Let $n \geq 1$ be an integer. Given n men and n women, a *perfect matching* is a way to assign every man to a unique woman and vice-versa. For example for $n = 2$, where `Brad Pitt` and `Billy Bob Thornton` are the two men and `Jennifer Aniston` and `Angelina Jolie` are the two women, one possible perfect matching consist of the two pairs (`Brad Pitt`, `Jennifer Aniston`) and (`Billy Bob Thornton`, `Angelina Jolie`).

   Prove that the total number of perfect matchings when you have n men and n women is $n!$. (Recall $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.)

   (*Hint:* Use induction. Also note that the number of perfect matching just depends on n and is *independent* of the identities of the men and women.)

   Proof Idea: We'll prove this by induction on n, the number of people of each gender that we are matching. Our base case will be $n = 1$, where it will be straightforward to show that the number of perfect matchings is $1! = 1$. We will assume that for a set of $n = k - 1$ people of each gender, the total number of possible perfect matchings is $(k-1)!$. Then we will use our assumption for $n = k - 1$ to show that with one more man and one more woman, that our number of possible perfect matchings is $k \times (k-1)! = k!$. □

   Solution: Base case: $n = 1 :$ there is one perfect matching of 1 man and 1 woman: (m, w). (Note that $1! = 1$.)

   As the induction hypothesis, assume that for $n = k - 1$, the number of perfect matchings is $= (k-1)!$.

   Now consider the case $n = k$. Consider a man $m_1 \in M$. There exists k possible matches for m_1 (to $\{w_1, w_2, ..., w_k\}$). Regardless of the choice for m_1, $k-1$ men and $k-1$ women remain to be matched. We know from the inductive hypothesis that there are $(k-1)!$ perfect matchings for this set of remaining people. The total number of matchings, then, is: $k \times (k-1)! = k!$, as desired. ■