A tutorial manual for Discrete mathematics

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The goal of this manual is to give you instructions about what is expected from the tutorials and provide you with suggestions what and how to achieve such goal.

1 Tutorials organization

The purpose of the tutorial is to review the theory from the lectures, show students how theoretical results can be applied to practical/numerical problems, and lead students to understanding the theory and applying it to solve related problems independently.

The tutorials *should not* be a full subsidy of the lecture nor a presentation of the advanced parts of the theory (unless required by the lecturer in exceptional cases).

A single class session hence may consist of several parts, e.g.: homework review, a test, theory review, while the main part shall be devoted to problem solving.

Besides learning DM theory, the students should be trained to communicate, i.e.: express themselves, use notation and terminology properly, both orally and in written form. Some tips towards this goal:

- invite students to ask questions;
- encourage students to present their solutions;
- for a given problem solve simple cases or analogous problems first by yourself, and then let the students work on the given one;
- let the students formulate and solve related problems that are easier to solve;
- encourage students to discuss their suggestions, even for incomplete or only specific cases of solutions
- split students to groups whenever appropriate (e.g., by desks or rows), arrange contests, etc.

When repeating the theory, ask the students about the facts from the lectures (definitions, theorems), so that they are guided to prepare also for the tutorials.

Each tutorial follows a specific instance (parallel class) of the lecture and the students should be notified of this fact. We are trying to keep the parallels synchronized, but this will never be perfect. If the students attend a different lecture, they can still join your tutorial, but they should be warned.

1.1 Tutorial preparation

To prepare lessons, we suggest to consult the following resources (besides the instructions given to you directly by the lecturer).

- collection of solved exercises
  `http://matematika.reseneulohy.cz/en/mathematics/combinatorics`,
  Most exercises are publicly available in Czech and/or English, often with hints and/or solutions. Access to some exercises is restricted for they are intended to be used in tests; this includes some problems listed later in this manual. To see them, you need a teacher's account in the collection. Please ask Jirka Fiala for it.
- textbook J. Matoušek, J. Nešetřil: Invitation to Discrete Mathematics,
- DM gitlab `https://gitlab.kam.mff.cuni.cz/koutecky/dm1516/` includes also possible test and homework problems. For access, please ask your lecturer.

Stay in touch with your lecturer, including keeping track of what is currently being presented during the lectures, and build on it.

Schedule tutorials for the whole term so that all compulsory topics are sufficiently covered. Lectures are sometimes delayed and the topics at the end of the term have less time. We should avoid this.

Whenever in doubts in preparation of your lessons (covered topics, difficulty of tests and homework assignments), grading, etc. consult any such issue with your lecturer, in person or by email.

We encourage you to ask the lecturer for help with preparation or leading a lesson. You can receive valuable feedback.

Last but not least, prepare your tutorial carefully and in time, including all solutions and their alternatives.

1.2 Resources for students

Prepare a web/Moodle page with your study materials like assignments, homework, grading results, your office hours, etc, and keep it up to date.

If possible, print the materials you need for the tutorial for each student (class assignments, tests, homework assignments) and also publish them on web for those who use tablets.

2 Credits

From SIS:

> *For credit you need to get 100 points out of at least 150 possible continuously awarded for tests, homework and other activities.*
>
> *The ongoing nature of the inspection does not imply a right to request corrective tests nor alternative homework assignments.*
>
> *In justified cases (long-term illness, stay abroad, etc.) the lecturer may set individual conditions for credit granting.*
>
> *Credit is a condition for taking the exam.*

Therefore the tutorial credit award certifies that a student is able to independently solve problems of sufficient difficulty. This could be achieved in various ways, each having their pros and cons:

2.1 Tests

Tests clearly are one of the most typical ways of validating student's knowledge.Test of appropriate difficulty may stimulate the working atmosphere, while too difficult or tricky tests promptly spoil it.

A complex test for a whole / significant part of a lesson might most appropriately reflect the students' abilities, while it is much more stressful than other methods. Such complex tests also consume time that could be devoted to topics from the currently taught lecture.

Short tests focused on reviewing material from the previous lesson can lead students to ongoing work, but tend to be rather shallow.

Do not expect that students get an idea under pressure and in short time. Short test are suitable to check ability to apply known methods.

Tips for short tests are included in this material below.

2.2 Homework tasks

Other possibility is to pose homework assignments. These can include more difficult problems requiring deeper understanding. They don't stress students as much as tests do. Students can use other resources, which will likely include collaboration with other students. Homework also encourages students to communicate their ideas.

Homework that is too difficult may require an inadequate amount of time. We recommend monitoring the amount of time students typically need for homework. Make a strategy for guiding them in solving problems effectively, e.g., offer a consultation on a partial solution, possible methods, etc. It is also possible to give a hint when the first deadline passes and award less points for a solution afterwards.

Since homework solutions might not have been achieved independently, you might want to complement them with other ways of demonstrating independent work, such as tests, class activities, presentation of homework solutions, discussions, etc.

Homework tips are included in this material below.

For submitting and grading homework, you can use the Postal Owl at `https://kam.mff.cuni.cz/owl/`. Martin Mareš wrote it when his disappointment with Moodle overflowed :) Ask him if you want to use the Owl for your tutorial.

2.3 Class Activities

Passive participation in tutorials should not lead to a credit award.

On the other hand, active participation at problem solving, presentation of ideas and solution may. More talented students might be involved in homework reviews.

2.4 Possible strategies

Design a scoring system according to your preferences, consult them with the lecturer and communicate it to students in the first lesson. You may combine the above methods, for example:

- A short test for 6 points and a homework for 7 points in each tutorial session. This method leads to ongoing work and is perhaps one of the least stressful method.

- Publish five series of homework, where the correctness of the solution for each is worth 20 points and another 10 can be obtained by presenting the solution during tutorials.
- One test in the middle of the term for 50 points, another at the end for 75 points and 25 points for the class activity. In this case, it is necessary to organize alternative tests for those who cannot participate.

The difficulty of getting credit should be comparable across different classes, so this manual also offers sample test and homework assignments. You can alter them while keeping the overall difficulty and demands.

Correct students' work as soon as possible. University regulations mandate correction in two weeks' time, but we strongly recommend faster responses. Continuous assessment can be more time demanding than big tests, but the work is conveniently spread over a longer period. Talented students can be involved in corrections, but their grading must be checked.

All work submitted by the students for grading should be archived (and kept until the end of the following academic year to help resolve potential complaints). It is however important to give feedback to students. So with tests submitted on paper, it is advisable to return the corrected paper to the students, while keeping an electronic (scanned) copy for archival.

The results, or the number of points, should be announced to the students, while paying attention to the principles of personal data protection (GDPR). It is possible either in Moodle or on the website under the chosen nickname, in extreme cases individually in class.

Prepare a plan so that even those who are just a few points short of 100 points can get credit. E.g., those who get at least 80 points can receive the missing points by solving additional tests or homework in the exam period. Inform students about this option at the beginning of the term.

Award credit in time, especially as the credit is necessary for the exam.

2.5 Cooperation vs plagiarism

The line between collaboration and plagiarism is hard to define and, in the case of homework, indeed hard to prove.

You should declare a clear policy on what collaboration is permitted and what is not. A common choice is to allow the students to discuss tasks together and come up with strategies to solve them, but mandate that everyone will write up the solution on their own (using no written notes from the common discussion).

Clear and provable cases of plagiarism should be reported to the faculty disciplinary committee.

3 Topics and lessons

The following list covers the topics. There is no one to one correspondence between topics and lessons. Some lessons may cover more than a single topics, some topics require more lessons.

We ***quote first*** the topic description from SIS.

For the sake of completeness we include also topics that are covered in the course NDMA005 Discrete mathematics of the mathematics study program. However the syllabus shall be approved for that course guarantors.

For tests and homework ***we strongly suggest to design differently phrased problems but focused on the same concept and of similar complexity*** as we expect that the problems assigned to students (in particular these from this manual) will appear soon or later to be publicly available with their solutions.

Moreover, further variants are collected at the git repository.

Disclaimer: The suggested problems were collected from various resources including textbooks, internet and several teachers' materials. Only a fraction of this collection was designed solely by the author of this manual.

3.1 Introduction

At the first lesson inform students how tutorial will be organized, how they can receive credits and present valid study resources. Besides "live" lectures mention in particular:

- collection of solved exercises
  `http://matematika.reseneulohy.cz/cs/matematika/kombinatorika`,
- recorded lectures `https://is.mff.cuni.cz/prednasky/prednaska` (in Czech),
- textbook J. Matoušek, J. Nešetřil: Invitation to Discrete Mathematics.

Mention also how and when they can you reach during your office hours and offer them help if some of the task would seem too difficult.

Notation, motivating problems, the concept of a proof, proof by induction.

- Review notations on sets and operations with sets.
- Review meaning of an implication, how it could be proved.
- Let student solve problems on sets and logic
  `http://matematika.reseneulohy.cz/3344/subset` and so on.
- Review a proof by induction: base case, induction hypothesis, induction step.
- Show by by induction that $4|(6n^2 + 2n)$.
- Let student solve problems on identities with sums, or with products
  `http://matematika.reseneulohy.cz/3312/sums`, on tilings, plane cutting with lines, etc.

Homework assignment tips (each item 2–4 pts; the same in further homework tips):

1. Express $A \cap B$ by applying only the set difference operation several times on the sets A and B.
   Do not use the other set operations, i.e., intersection, complement, union and symmetric difference.

2. Show that every total sum of 3 CZK can be paid in two-crown and five-crown coins.
3. Determine how many chords a convex n-gon has.
   First, derive a formula for this number e.g., by induction, then evaluate it for $n = 1000$.
4. Show that for any positive real numbers $x_1, \ldots, x_n$ it holds: $\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \geq n^2$

Short test tips (each item 6 pts; the same in further short test tips):

1. Simplify $A \setminus (B \setminus (A \setminus (B \setminus (A \setminus (B \setminus A)))))$ to an expression with at most one occurrence of symbols A, B and $\setminus$.
2. By mathematical induction prove the well-known rule for the sum of a geometric series with $q \neq 1$: $1 + q + q^2 + \cdots + q^n = \dfrac{q^{n+1} - 1}{q - 1}$.
3. By mathematical induction prove the following rule for the sum of the series:
   $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n \cdot (n+1)} = \dfrac{n}{n+1}$

3.2 Combinatorics

Relation: mapping (function, permutation), equivalence.

- Review the notion of Cartesian product, binary relation, its representation.
- Review properties of relations: reflexive, symmetric, antisymmetric, transitive.
  Review the composition operation.
- Let student solve problems on properties of relations
  `http://matematika.reseneulohy.cz/3368/verifying-properties-of-relations` and so on.
- Review mappings and their properties: injective, onto, bijective. Recall permutations as mappings.
- Let student solve problems on mappings
  `http://matematika.reseneulohy.cz/3378/definition-of-a-bijection` and so on.
- Review the notion of equivalence, equivalence classes and the theorem that equivalence classes are always nonempty and either identical or disjoint.
- Let student solve problems on equivalences
  `http://matematika.reseneulohy.cz/3394/three-concrete-equivalences` and so on, in particular let them describe all equivalences on five elements so they get familiar with the structure of equivalence on a finite set.

Homework assignment tips:

1. a) Find relations R, S on the same set X such that $R \circ S = S \circ R$.
   b) Find a relation R on a suitable set X such that $\forall n \in \mathbb{N} : R^n \neq R^{n+1}$.
   The expression R^n means the n-fold composition of R, i.e., $R^n = \underbrace{R \circ R \circ \cdots \circ R}_{n\times}$.
2. Let $\sim$ be an equivalence on the set $\{1, \ldots, 20\}$ given by the rule $n \sim m$ if n and m contain the same number of different primes in their prime factorization. Verify that this is an equivalence relation and list its classes.
3. We call a relation R *circular* if it satisfies: $aRb \wedge bRc \Rightarrow cRa$.
   Let R be a circular and reflexive relation on the set $X = \{\eta, \kappa, \lambda, \mu, \sigma\}$ satisfying $\{(\eta, \mu), (\lambda, \sigma), (\mu, \eta)\} \subseteq R$.

a) Decide whether every such relation R is symmetric.
b) Decide whether there exists such a relation R that would be antisymmetric.
c) Decide whether every such relation R is transitive. (a-c 4 pts)
d) *For 2 extra points:* Determine how many relations R with the given properties exist.

4. Let R be a relation on the set $\{1, 2, \ldots, 1000\}$ where $xRy \Leftrightarrow \log_2\left(\frac{x}{y}\right) \in \mathbb{Z}$.
Decide whether the relation R is an equivalence.
If it is an equivalence, determine how many equivalence classes it has.

Short test tips:

1. Let R be the relation on $X = \{1, 2, 3, 4\}$ given by the enumeration of pairs $R = \{(1, 2), (1, 4), (2, 1), (4, 4)\}$.
Determine the relation $R \circ R$ and decide whether it is reflexive.
2. Let R be the relation on $X = \{a, b, c, d\}$ given by the enumeration of pairs:
$R = \{(a, a), (b, b), (c, c), (d, d), (a, c), (b, d), (c, a), (d, c)\}$.
Determine the largest (with respect to inclusion) equivalence of S contained in R, i.e., $S \subseteq R$.
Describe the equivalence S by using equivalence classes.
3. Let R be the relation on $X = \{a, b, c, d\}$ given by the enumeration of pairs:
$R = \{(a, a), (b, b), (c, c), (d, d), (a, c), (b, d), (c, a), (d, c)\}$.
Determine the smallest (with respect to inclusion) equivalence S containing R, i.e., $R \subseteq S$.
Describe the equivalence S by using equivalence classes.

Partial ordering: chains and antichains, large implies tall or wide, Erdös-Szekeres lemma on monotone subsequences.

- Review the notion of partial and linear order.
- Let student solve problems on
`http://matematika.reseneulohy.cz/4148/the-number-of-order-relations`
- Explain the differnce between the largest and a maximal elements.
- Let student solve problems on
`http://matematika.reseneulohy.cz/4151/the-largest-and-maximal-elements`
- Review chains, antichains and the "tall or wide" theorem.
- Let student solve problems on
`http://matematika.reseneulohy.cz/4153/chains-and-antichains-from-hasse-diagram`

Homework assignment tips:

1. Decide which of the following relations (X, R) are partial orders. Justify your decision (i.e., verify the axioms or find a counterexample).
   a) Set of words $X = \{\text{Adam, Eve, Peter, Paul}\}, (xy) \in R \Leftrightarrow x$ has at least as many consonants as y.
   b) $X = \mathbb{N}, (x, y) \in R \Leftrightarrow x = y \vee x^5 \leq y^3$,
   c) $X = \mathbb{R}^+, (x, y) \in R \Leftrightarrow x = y \vee x^5 \leq y^3$,
   d) X is a set of functions of a real variable with domain $[0, 1]$, $(f, g) \in R \Leftrightarrow \forall x \in [0, 1] : f(x) \leq g(x)$

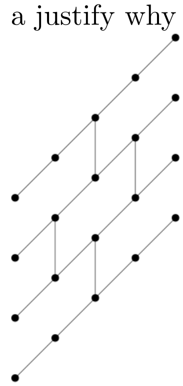
e) X is a set of functions of a real variable with domain $[0,1]$, $(f,g) \in R \Leftrightarrow f = g \vee \forall x \in [0,1] : 2f(x) \leq g(x)$.

2. For a positive integer n, we have a partially ordered set $P_n = (X_n, \preceq)$ where X_n is the set of all pairs of positive integers (i,j) such that $i+j$ is at most n. The relation $\preceq$ is defined by $(x_1, y_1) \preceq (x_2, y_2)$, just when it holds that $x_1 \leq x_2$ and also $y_1 - x_1 \leq y_2 - x_2$.
   a) Verify from the definition that for every positive integer n, P_n is indeed a partially ordered set.
   b) Draw the Hasse diagram of P_6.
   c) Find some longest chain and some longest antichain in P_6. Justify that you cannot find a longer one.
3. Let's say that two historical figures could meet if the periods of their lives are non-empty penetration. So, for example, Jan Amos Comenius could meet both René Descartes and Jonathan Swift, but not Descartes with Swift. Decide which of the following two statements is true:
   a) In any set of 50 personalities, we always find eight of them who could have met, or in this set we find eight such personages, of whom no two meet they couldn't.
   b) There is a set of 50 personalities such that at most seven personalities can be selected from it, what they might have met, and at the same time a maximum of seven such personalities can be selected from this set such that no pair of these seven could meet.

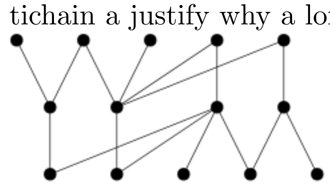
   For 2 extra points: If we have a group of personalities such that any two can meet, there must always be a day when that all these personalities can meet in it at once?

Short test tips:

1. Decide whether the following relation (X, R) is a partial order. Justify your decision (i.e., verify the axioms or find a counterexample).
   X is the set of all English words, $(x,y) \in R \Leftrightarrow$ every letter from x also occurs in y.
2. For the partial order given by the following Hasse diagram, mark some maximal chain a justify why a longer one cannot be found.



3. For the partial order given by the following Hasse diagram, mark some maximal antichain a justify why a longer one cannot be found.



Combinatorial counting: the number of subsets, of subsets of size k, of all mappings, of all injective mappings, of permutations.

- Review the notion of mapping, permutation, number of k-element subsets and their relation to factorials and binomial coefficient.
- Let student solve problems on `http://matematika.reseneulohy.cz/3448/choice-from-a-set-of-objects`
- Continue with *many* word problems.

Homework assignment tips:

1. One hundred senators sit in the Senate, two from each of 50 states. In how many ways can we choose a four-person committee so that it contains no two senators from the same state?
2. Consider an $m \times n$ grid, where m and n indicate the number of horizontal and vertical lines.
   In the following exercises, first derive a general equation and then determine precise numbers for $m = 11$ and $n = 21$, i.e., for a grid with 200 squares.
   a) How many rectangles exist whose sides lie on this grid? (A square is a special case of a rectangle, but a line segment is not.)
   b) *For extra points:*
   How many disjoint pairs of rectangles exist with sides lying on this grid? (Be aware that these are closed rectangles, i.e., including their boundaries.)
3. The town of Gotham has established three committees: the sport committee, the culture committee and the beauty committee. Only councillors or the mayor may sit on committees. The mayor has announced that it is out of the question for him to join the beauty committee, since he himself is a nice enough adornment for the town. Due to disagreements about the form of a statue in the town fountain, the culture and beauty committees may not have any members in common.
   Determine the number of possible ways to populate the committees if the town government contains seven elected councillors in addition to the mayor. Do not forget to include the cases where some committees (or even all committees) contain no members.

Short test tips:

1. Depending on a natural number n, determine the number of ordered triples of sets A, B, C such that
   $B \subseteq A \subseteq \{1, 2, \dots, n\}$ and also $C \subseteq A$.
2. Depending on a natural number n, determine the number of ordered triples of sets A, B, C such that
   $B \subseteq A \subseteq \{1, 2, \dots, n\}$ and also $B \subseteq C \subseteq \{1, 2, \dots, n\}$.
3. How many ten-letter words can we form from the alphabet $A = a, b, c, \dots, h$ in which the letter c appears four times, g twice and the other letters at most once?
4. Determine how many ways there are to arrange (e.g., from left to right):
   Three blue, three yellow and two red crayons.
   All three blue crayons are indistinguishable; similarly for the crayons of the remaining two colors.

5. Determine how many ways there are to arrange (e.g., from left to right):
   Six different crayons of basic colors such that the blue and red crayons must be adjacent. The basic colors are: yellow, red, blue, green, brown and black.
6. We have a rectangular box K with dimensions $7 \times 9 \times 10$ and we wish to cut it into eight smaller rectangular boxes (we also consider a cube to be a rectangular box). We will cut the box using three planes parallel with the walls of the box (although no two of the cutting planes are parallel). We also require that all eight of the new resulting boxes have dimensions that are all (positive) integers. How many possible ways are there to cut the box K, if we require that at least one of the eight smaller boxes contain a cube of dimensions $6 \times 6 \times 6$ (i.e., we want all of its dimensions to be at least 6)?

The binomial theorem. Estimates the factorial function and binomial coefficients.

- Review the notion of factorial, the Stirling's formula, binomial coefficient, and Pascal's triangle.
- Let student solve problems on
  `http://matematika.reseneulohy.cz/3847/factorial-estimate`
- Explain relation between Pascal's triangle and grid walks, in particular between good parenthesizations and walks.
  `http://matematika.reseneulohy.cz/3453/number-of-parenthesizations`

Homework assignment tips:

1. For non-negative integer parameters m, n determine the sum of all binomial coefficients $\binom{k+l}{l}$ which satisfy $k \in \{0, 1, \ldots, m\}$ and $l \in \{0, 1, \ldots, n\}$.
   First solve the exercise in general, and then determine a concrete result for $m = 5$ and $n = 10$.
2. Sort the following expressions by the order of growth (assume that n is a very large number):
   $\binom{2n}{n-1}$, $\binom{2n}{n}$, $\binom{2n}{10}$, $n!$, $n^{\sqrt{n}}$, $(\sqrt{n})^n$, n^{15}, $n^{\log n}$, $(\log n)^n$, $\log(n^n)$, 2^n
3. Show that for each m the product of primes from the interval $[m+1, 2m]$ is at most $\binom{2m}{m}$.

Short test tips:

1. Without using the formula for the binomial coefficient, determine the smallest possible integer constant c for which it holds
   $\binom{123}{59} \leq c \cdot \binom{121}{59}$

Inclusion-exclusion formula. Applications (hatcheck lady).

- Review the Inclusion-exclusion formula in particular explain different ways of formal description. Start with few, eg. three sets.
- Let student solve several word problems like
  `http://matematika.reseneulohy.cz/4158/construction-workers`
- Solve at least one problem on each of:
  - sieve of Eratosthenes,
    `http://matematika.reseneulohy.cz/3486/sieve-of-eratosthenes`

- words without subwords,
  `http://matematika.reseneulohy.cz/3491/words-without-subwords`
- placement of colored stones on chessboard.
  `http://matematika.reseneulohy.cz/3492/stones-on-a-chessboard`

Homework assignment tips:

1. How many ways are there to place 6 red, 6 green and 6 blue stones on a 5×5 chessboard such that some row or column is always covered with stones of the same color?
2. Determine the number of words such that:
   They are composed of letters A, G, H, I, K, L, M, N and after deleting some letters we can not get any of the words LEAGUE, LINE, BOOK or CAMERA.
   These words should contain each of the eight letters listed exactly once.
3. The godfather registration plate of a vehicle is such that in the final four digits it contains at least two identical digits in adjacent positions, such as in 1SJ 7881.
   Determine the share of godfather plates on our roads, eg as a percentage.

Short test tips:

1. Determine the number of words such that they are composed of letters A, B, D, H, O, R, and after deleting some letters we can not get any of the words BAR, ROD, HOD.
   These words should contain each of the six letters listed exactly once.
   Other combinations: A, C, K, M, O, P w/o MOC, MOP, PAK; A, G, H, I, K, L, M, N w/o LIGA, LINKA, KNIHA or KAMIL.
2. Determine the number of positive integers such that they are between 1 and 1,800 (inclusive) and are divisible by 10, 12 or 15.
   Other combinations: [300] w/o multiples of 6, 14 and 15; [600] w/o multiples of 12, 16 or 25.
3. Determine how many there are eight-letter words composed of the letters a, b such that they either contain exactly two letters b or are symmetrical.
   To explain: symmetrical words are read from the back as well as from the front, called palindromes.
4. Determine how many there are six-letter words composed of the letters a, b, c such that they either contain exactly four letters *and* or in which the first three letters match the second three.
   For example, the word *bbcbbc* meets the second of these conditions and should be counted.
5. Determine how many there are four-digit numbers composed of digits from the set $\{1, 2, 3, 4, 5, 6\}$ such that they begin with the digit 1 or the digits in the number are different and gradually grow.
   For example, the number 1,646 meets the first of these conditions and should be counted.
6. Determine how many there are three-digit numbers composed of digits from the set $\{1, 2, 3, 4, 5, 6, 7\}$ such that the sum of the last two digits is 10 or have all digits odd.
   For example, the number 664 meets the first of these conditions and should be counted.
7. How many ways are there to place nine stones on a 5×4 chessboard such that no row or column is completely full? (You may place at most one stone on any given square.) Your answer may include binomial coefficients, whose values you need not calculate.
8. In the integer grid, determine the number of shortest paths between the vertices $(0, 0)$ and $(10, 10)$ that do not pass either of the points $(3, 2)$, $(6, 7)$ and $(2, 6)$.

3.3 Probability

Probability space (at most countable, all subsets are events).

- Review the notion of probability space, event.
- Let student solve several word problems like
  `http://matematika.reseneulohy.cz/3537/three-dice`
  Provide also example(s) when the probability space is when the space is badly designed.
- Let student solve problems on ball selection like
  `http://matematika.reseneulohy.cz/3535/choosing-balls`

Homework assignment tips:

1. A deck contains eight distinct cards, two of each suit. We thoroughly shuffle the deck. What is the probability that we will obtain an ordering in which no two cards of the same suit are adjacent?

Short test tips:

1. We throw two six-sided dice. What is the probability that the maximum of the two numbers thrown will be five?

Independent events, conditional probability.

- Review the notion of (in)dependent events, Bayes law, conditional probability.
- Let student solve problems like
  `http://matematika.reseneulohy.cz/3541/independence-in-rolling-dice`

Homework assignment tips:

1. The success rate for the Discrete Mathematics exam is 90 %. The number of students who pass the exam without even preparing is half as many as those who fail the exam even though they have prepared for it. It turns out that of the students who fail the exam, 70 % have not prepared sufficiently.
   Determine the probability of passing the exam for students who have prepared for it.
   (The numbers in this assignment are made up and are in fact not very realistic.)
2. In the board game Ludo, determine how many squares a piece will advance on average in a single turn if:
   a) The player throws the die only once.
   b) If the player rolls a 6, he rolls again, however in a single turn he may roll at most three times. His piece advances a number of squares equal the sum of the values he has thrown.
   c) If the player rolls a 6, he rolls again and this continues for as long as he rolls sixes. He advances by a number of squares equal to the sum of all the numbers has rolled.

Short test tips:

1. Philip has rolled an 8-sided die three times. The sum of his rolls equals 6. Determine the (conditional) probability that his first roll was a 3.

2. Let π be a random permutation of the numbers $\{1, 2, \ldots, 10\}$. Let A_1 be the event that $\pi(1) > 1$, and A_2 be the event that $\pi(2) > 2$.

   a) Determine the probabilities of the events A_1 and A_2.
   b) Are the events A_1 and A_2 independent?

3. Consider a random permutation π of the set $\{1, \ldots, 6\}$, where all permutations are equally probable. Also consider two events A_1 and A_2, where A_1 occurs when $\pi(3) < \pi(1) < \pi(4)$ and A_2 occurs when $\pi(3) < \pi(2) < \pi(4)$. (Formally, A_i is the set of all permutations π of the set $\{1, \ldots, 6\}$ such that $\pi(3) < \pi(i) < \pi(4)$ for $i \in \{1, 2\}$.)
4. Determine the probabilities of these events $P[A_1]$ and $P[A_2]$.
5. Determine the probability that these two events occur simultaneously, i.e., $P[A_1 \cap A_2]$. Also determine whether the events A_1 and A_2 are independent.

Random variable: distribution function, expectation, examples of calculation, most common distributions.

- Review the notion of random variable, distribution function, expectation.
- Review linearity of expectation.
- Let student solve problems on `http://matematika.reseneulohy.cz/3546/number-of-fixed-points-in-a-random-permuation`
- Give examples of various distributions.

Homework assignment tips:

1. Charles has four 5-crown coins, three 10-crown coins, two 20-crown coins and one 50-crown coin in his pockets. What sum will he obtain on average if:

   a) He chooses one coin at random from his left pocket and one from his right, assuming that his left pocket contains the 5-crown and 20-crown coins and the others are in his right pocket.
   b) He chooses four coins at random.

   (If the analysis of tens of cases seems daunting, please be aware that it is possible to calculate the result much more easily.)
2. A random variable X takes on the values 0,1,...,5 as follows:
   $P[X = 0] = P[X = 1] = P[X = 2] = a$,
   $P[X = 3] = P[X = 4] = P[X = 5] = b$,
   $P[X \geq 2] = 3P[X < 2]$.
   Determine the values of the parameters a, b and also the expected value and variance of X.
3. Imagine that you play the well-known game of Rock-paper-scissors one thousand times, where a victory in a single round wins 1 € and vice versa (i.e., a defeat means a loss of 1 €).
   Find the best upper bound that you can of the probability that after playing all thousand games you will have won at least one hundred euros.
   (Your solution will be evaluated not only on its correctness but also on the quality of the upper bound itself.)

Short test tips:

1. For an independent random variable X and an arbitrary α express $var(\alpha X)$ as a function of $var(X)$ and α.
2. A random trial consists of a throw of two fair dice.
   The random variable X equals the sum of the values thrown, if this sum is divisible by four. Otherwise $X = 0$.
   Determine the expected value of the variable X.
3. A deck consists of 10 cards, of which two are aces. We shuffle the cards.
   The random variable X equals the number of aces at the top of the deck of cards. Determine the expected value of the variable X.
4. We consider a fair k-sided die to be an object whose possible values are the natural numbers 1 through k, each with probability $\frac{1}{k}$. In succession we throw a fair 1-sided die, a 2-sided die, ..., through a 100-sided die (altogether there are 100 throws). Determine the expected value of the sum of these throws.
5. Let n be a positive integer. An increasing triple in a permutation π on $[n]$ is a triple of indices i, j, k, $1 \leq i < j < k \leq n$ such that $\pi(i) < \pi(j) < \pi(k)$. Determine the expected value of the number of increasing triples in a random permutation of $[n]$.

3.4 Graph theory

Path, trail, walk, cycle, subgraph, isomorphism, etc.

- Review the terminology: $K_n, P_n, C_n, K_{m,n}$, difference between path, trail and walk. Explain connected graphs and components.
- Practice concepts of graph containment (subgraphs), graphs that differ by vertex naming (isomorphism).
- Mention that isomophisms preserve $|V|$, $|E|$, degrees, cycles, etc.
- Let student solve problems on
  `http://matematika.reseneulohy.cz/3592/isomorphism-between-two-concrete-graphs`
  in particular let them list all distinct four vertex graphs
  `http://matematika.reseneulohy.cz/3594/graphs-with-four-vertices`
- *For DM for mathematicians only — course NDMA005*
  Review the algorithm on the shortest path

Homework assignment tips:

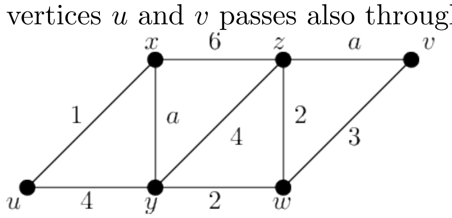
1. Decide whether the following assertions are true. Justify your answers.
   a) A graph $G = (V_G, E_G)$ on the set of vertices $V_G = \{1, 2, \ldots, n\}$ can be isomorphic to at most $n!$ other graphs $H = (V_H, E_H)$.
   b) Two graphs G and H are mutually isomorphic if and only if there exists a bijection between E_G and E_H.
   c) If $f : V_G \to V_H$ is an isomorphism and $u \in V_G$ has a neighbor of degree three, then $f(u)$ also has a neighbor of degree three.
   d) Every bijective mapping of vertices that preserves degrees must be an isomorphism.
   e) Every injective mapping between the vertex sets of two complete graphs of the same size is an isomorphism.
2. Let $2K_2$ be a graph formed from the disjoint union of two graphs K_2.
   a) How many subgraphs does the cycle C_{100} contain that are isomorphic to $2K_2$?

b) How many induced subgraphs does the cycle C_{100} contain that are isomorphic to $2K_2$?

3. Consider the graph G that arises from the cycle C_{1001} on vertices $v_1, \dots, v_{1001}$ when we add all edges (v_i, v_j) for which $i - j \equiv 500 \pmod{1001}$ or $j - i \equiv 500 \pmod{1001}$.

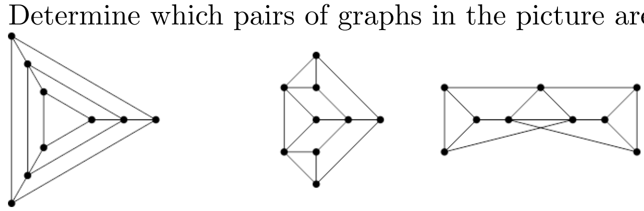
   a) How many triangles (i.e., subgraphs isormophic to C_3) are contained in G?
   b) How many automorphisms does G have, i.e., isomorphisms $f : G \to G$?

4. Mary and her boyfriend are preparing a party, to which they invited four other couples. Pairs that know each other they greet, but — naturally — no one greets with his/her own partner. At the end of the party, Mary asks the others how many people everyone greeted with and receives nine different answers.
   How many people did Mary greet with and how many did her partner?
5. *For DM for mathematicians only — course NDMA005*
   Determine for which values of the parameter a some of the shortest paths between the vertices u and v passes also through the vertices x and w.



Short test tips:

1. Determine which pairs of graphs in the picture are isomorphic. Justify your answers!



2. Prove or refute: A complete bipartite graph cannot be isomorphic to a cycle.
3. We call a graph G self-complementary if it is isomorphic to its complement $\overline{G}$. Find all self-complementary cycles (and prove that no others exist).
4. For any positive integer $n \geq 3$ determine the number of automorphisms of the cycle C_n, i.e., the number of isomorphisms from C_n to C_n.
5. Determine whether the following graphs G and H are isomorphic.

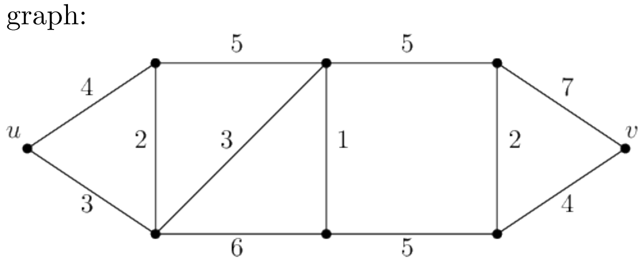
   a) $V_G = \{1, 2, 3, 4, 5, 6\}$
      $E_G = \{(u, v) : 2 \nmid u + v\}$ or $u + v$ is odd
   b) $V_H = \{a, b, c, d, e, f\}$
      $E_H = \{(a, d), (a, e), (a, f), (b, d), (b, e), (b, f), (c, d), (c, e), (c, f)\}$

6. Determine whether the following graphs G and H are isomorphic.

   a) $V_G = \{10, 14, 15, 21, 22, 33\}$
      $E_G = \{(u, v) : gcd(u, v) > 1\}$ or u and v have a non-trivial common divisor
   b) $V_H = \{a, b, c, d, e, f\}$
      $E_H = \{(a, b), (a, e), (a, f), (b, c), (b, e), (c, f), (c, d), (d, e), (d, f)\}$

7. Determine the number of graphs on five vertices (including isomorphic ones) that do not contain a path of length 2 as a subgraph.

8. *For DM for mathematicians only — course NDMA005*
   Find the shortest path from the vertex u to the vertex v in the following edge-weighted graph:



   Use Dijkstra's algorithm and build a table whose columns will contain estimates of distances after each iteration of the algorithm.

Eulerian graphs: characterisation, including directed case; strong and weak connectivity.

- Review again the difference between path, trail and walk; and also connected graphs. Explain directed graphs.
- State characterization of Eulerian graphs, show how induction based proof can be turned to a recursive algorithm.
- Let student solve problems based on an application of the theorem
  `http://matematika.reseneulohy.cz/4175/eulerian-graph-and-union-of-cycles`
  and so on

Homework assignment tips:

1. Consider a graph whose vertices are formed by binary words of length k, and where two words form an edge if and only if when they differ in two positions or in all positions. (E.g., vertex 1110 is adjacent to 0111, 0100, or 0001, but not to 1100.)
   Depending on k decide whether the graph is Eulerian.

Short test tips:

1. Let G be an Eulerian graph whose complement is connected. Depending on the number of vertices of G, determine whether the complement of G is also Eulerian. Justify!

Trees: various characterisations, existence of a leaf.

- Review the notion of tree, leaf, cycle formed by adding a edge.
- Elaborate on different tree definitions
  `http://matematika.reseneulohy.cz/4212/equivalent-definition-of-tree`
- *For DM for mathematicians only — course NDMA005*
  Review the algorithm on the minimum spanning tree

Homework assignment tips:

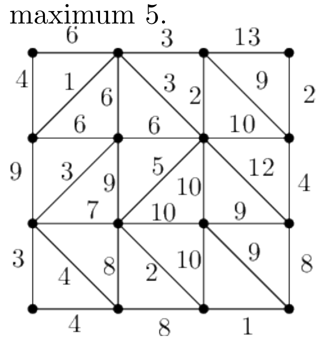
1. Consider the set $\mathcal{G}$ of all graphs G such that: $V_G \subset \mathbb{N}, |E_G| = 303$ and G has 151 vertices of degree 3. (The last condition means that G can have other vertices of other degrees).
   Decide and justify properly whether the following statements hold on the set $\mathcal{G}$:

a) A graph $G \in \mathcal{G}$ is connected if and only if G is a tree.
b) If $G \in \mathcal{G}$ contains a vertex of degree 7, then G is not a tree.
c) If $G \in \mathcal{G}$ is disconnected and does not have isolated vertices, then such G contains a cycle.

2. In the $m \times m$ chessboard, the $2m$ squares are colored blue. We will place a rook on one of the squares. We will move the rook from the blue square again to the blue square, while we will want to move alternately horizontally and vertically. Prove that it is possible to position the rook so that when we move it properly, we will always be be able to make another move.

Short test tips:

1. Let's have a tree on 10 vertices with three leaves. Determine how many vertices of degree two it contains.
2. *For DM for mathematicians only — course NDMA005*
   Find some minimal spanning tree of the weighted graph in the picture. Also mark the intermediate states of the Kruskal's algorithm after processing edges with a weight of maximum 5.



Planar graphs: Euler's formula, maximum number of edges.

- Review the notion of Planar drawing, Jordan theorem, the Euler's formula.
- Let student solve problems on plarity tests
  `http://matematika.reseneulohy.cz/4242/particular-graph`
  and other applications of Euler's formula.
- Review also Platonic solids.

Homework assignment tips:

1. By the *girth* of a graph (which contains a cycle) we mean the length of the shortest cycle contained in that graph. Consider a planar graph G with a planar drawing such that each face, including the outer one, is bounded by a cycle (especially G does not contain any vertices of degrees 0 or 1, and on the other hand it contains at least one cycle).

   a) Prove that if the girth of G is at least 6, then G contains a vertex of degree 2.
   b) Prove that if the girth of G is at least 11, then G contains two adjacent vertices of degree 2.
   c) Find a graph with girth 10 (or with as large girth as possible) that does not contain two adjacent vertices of degree 2.

2. a) In how many different ways can a dodecahedron be inscribed in a cube so that all vertices of the cube are selected vertices of the dodecahedron?
   b) In how many different ways can an octahedron be inscribed in an icosahedron so that all vertices of an octahedron are selected vertices of a icosahedron?

   Justify your answers, e.g., draw a picture.

Short test tips:

1. Decide if there is a cubic (i.e., 3-regular) planar graph satisfying:
   a) All its faces are fourcycles.
   b) It has two pentagonal faces, two octagonal faces and no other.
2. Decide if there is a planar graph on nine vertices with sixteen faces. Justify your decision.
3. Find all connected planar graphs such that they have a planar drawing satisfying that the number of face is by six less than the sum of the number of edges and vertices.
4. Consider a connected planar graph G such that each face, including the outer one, is bounded by a circle of length 5 or 10. Let us further assume that G is cubic, ie all its vertices have degree 3. Finally, suppose that the number of faces bounded by circles of length 5 is exactly 20.

   Specify the number of faces bounded by cycles of length 10. Justify that no other possibility can arise.

   You may verify the existence of such a graph, but you do not have to.
5. A connected planar graph G has a planar drawing in which each face is bounded by a) cycle. We know that one face is bounded by a cycle of length 6, three faces are bounded by a cycle of length 5 and three faces are bounded by a circle of length 3. There are no other faces in this drawing.

   Specify the number of vertices and edges of the graph.
6. Consider the a positive integer $n \geq 3$ and create a graph G by adding two vertices a and b to the cycle C_n. We connect these two vertices with an edge to each of the vertices of C_n, but we do not connect them to each other.
   a) Show that the graph G is planar.
   b) Prove that if we add any edge to G (between two vertices of G that have not been connected yet), then we get a non-planar graph.

Graph coloring: characterisation of bipartite graphs, chromatic number of a d-degenerate graph is at most $d+1$; 5-colorability of planar graphs (using Kempe's chains).

- Review the notion of graphs coloring, relationship to d-degenerate graphs and how these could be treated algorithmically.
- Let student solve problems related to degeneracy like `http://matematika.reseneulohy.cz/4190/four-color-theorem`

Short test tips:

1. Let $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ be two trees with the same set of vertices V. Consider the graph $G = (V, E_1 \cup E_2)$. Prove that G can always be (properly) colored with 4 colors. Also find the example of T_1 and T_2 where three colors are not enough.

3.5 Additional topics

Covering of these topics depends whether they will be covered on the lecture.

Erdős-Szekeres lemma on monotone subsequences. This topic suggested to cover at the end of partial orders.

- Review the theorem statement and provide an example.
- Review arguments of the proof, if students have doubts.

Handshaking lemma, graph score. This topic suggested to cover before/after Eulerian graphs.

- Review the notion of score, Havel-Hakimi theorem and how it could be treated algorithmically.
- Let student solve problems on score
  `http://matematika.reseneulohy.cz/4170/graph-by-score`

Homework assignment tips:

1. In the sequence $(1, 1, 1, 1, \cdot, \cdot, \cdot, 2, 3, \cdot, 5)$ add numbers to the places of the dots so that the resulting sequence is the score of a graph G with required properties.
   If there are several solutions, identify all of them and show that there are no other.

   a) G is a tree.
   b) G is connected and contains at least two vertex disjoint cycles.

Short test tips:

1. A graph G has a score $(1, 1, 2, 3, 3, 3, 5)$. Determine the score of the complement of G. Justify your result!
2. Verify whether the following sequence is the score of a graph, and if so, construct one. $(1, 1, 2, 3, 3, 4, 5, 5, 6, 6, 6)$

Sperner's lemma, application on the HEX game. This topic suggested to cover at the end of the course.

- Review the game rules.
- Review the theorem statement and provide an example.
- Review arguments of the proof, if students have doubts.
- Consider a variant where the goal is to connect three sides of a triangle subdivided into tiles (only at most three tiles may connect at a single point).

Probabilistic proofs (e.g., existence of a 3-paradoxical tournament). This topic suggested to cover at the end of the course.

- Review the theorem statement and provide an example.
- Review arguments of the proof, if students have doubts.