Mixed Precision Strategies for Memory-Bound Linear Algebra

LANS Seminar
September 8th 2021

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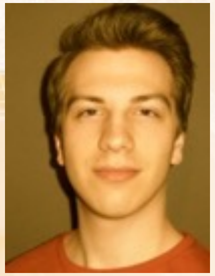
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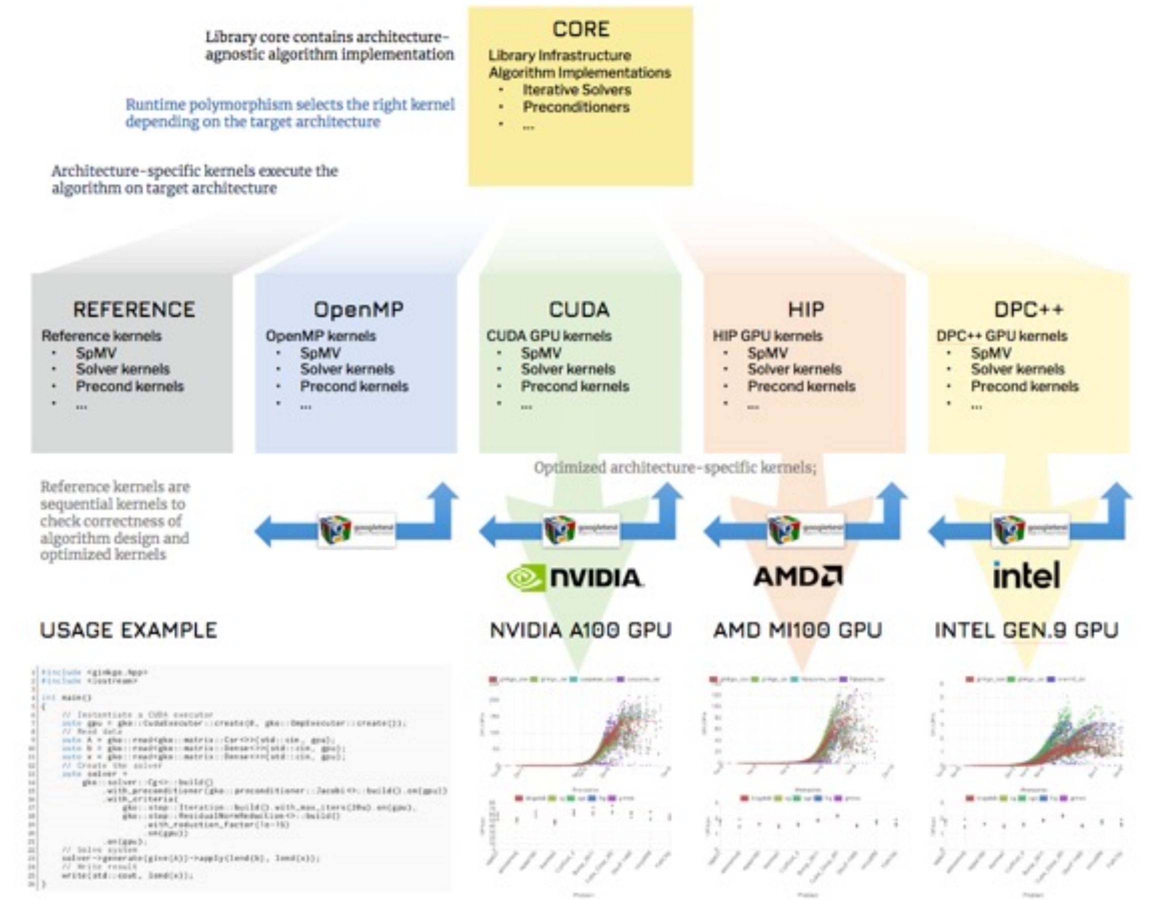
This research was supported by the Exascale Computing Project (17-SC-20-SC), a collaborative effort of the U.S. Department of Energy Office of Science and the National Nuclear Security Administration and the Helmholtz Impuls und VernetzungsfondVH-NG-1241.

Ginkgo

GPU-centric high performance sparse linear algebra. Sustainable and extensible C++ ecosystem with full support for AMD, NVIDIA, Intel GPUs.

- **High performance sparse linear algebra**
  - Linear algebra building blocks: SpMV, SpMM, SpGEAM,...;
  - Linear solvers: BiCG, BiCGSTAB, CG, CGS, FCG, GMRES, IDR;
  - Advanced preconditioning techniques: ParILU, ParILUT, SAI;
  - Batched iterative solvers;
- **Exascale early systems GPU-readiness**
  - Available: Nvidia GPU (CUDA), AMD GPU (HIP), Intel GPU (DPC++), CPU Multithreading (OpenMP);
  - C++, CMake build;
- **Open source, community-driven**
  - Freely available (BSD License), GitHub, and Spack;
  - Part of the **xSDK and E4S software stack**;
  - Can be used from **deal.II** and **MFEM**;
- **Modular precision ecosystem**
  - Decoupling of arithmetic precision and memory precision;
  - Compressed Basis (CB) Krylov methods;
  - Mixed precision algorithms: adaptive precision Jacobi, FSPAI;



https://ginkgo-project.github.io/

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Open source, community-driven

- ***Ginkgo is a child of the ECP effort***:
  - Lessons learnt in other software projects (e.g. MAGMA) are incorporated;
  - Focus on sparse linear algebra;
- **Better Scientific Software** (BSSW.io) proved to be a valuable resource;
- Freely available (BSD License), **GitHub**, and **Spack**;
- Modern programming language (C++14) and build system (**CMake**);
- Continuous Integration and Comprehensive Unit testing;
- From the beginning on, part of the **xSDK and E4S software stacks;**
- **Portability as central design principle;**
- Using the **architecture-native languages** to push the performance to the limit;

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High performance sparse linear algebra

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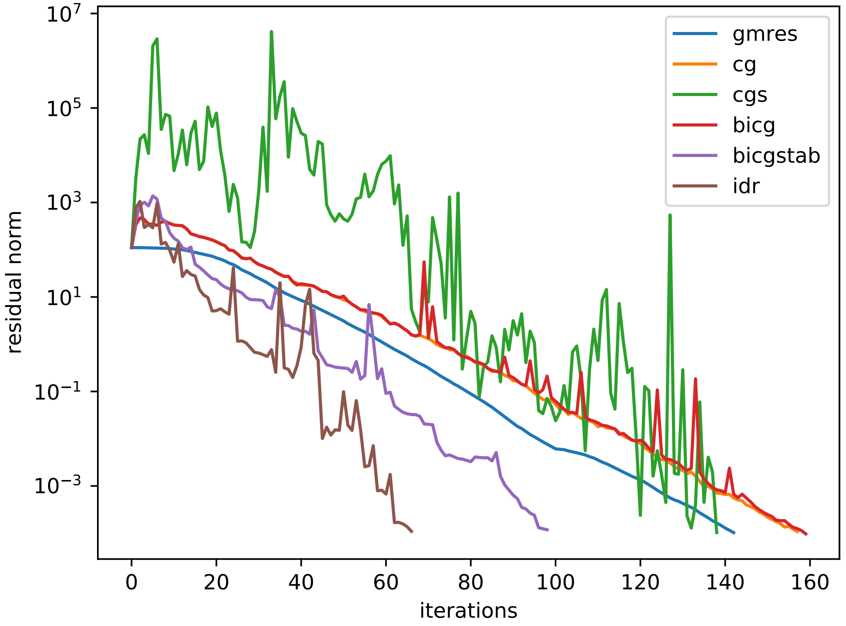
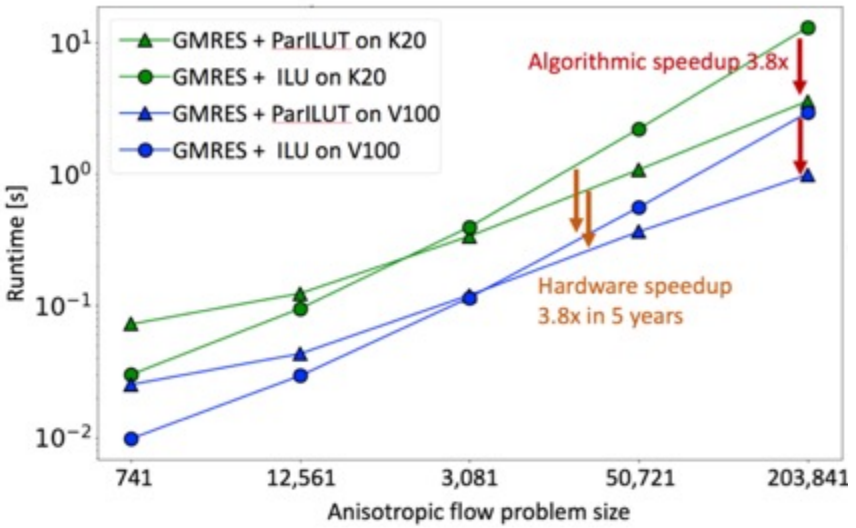
- Linear solvers: BiCG, BiCGSTAB, CG, CGS, FCG, GMRES, IDR;
- Mixed precision algorithms:
  CB-GMRES, adaptive precision Jacobi, FSPAI;
- Advanced preconditioning techniques: ParILU, ParILUT, SAI;
- Batched solver technology;
- *Extensible, sustainable, production-ready;*





GPU-centric high performance sparse linear algebra. Sustainable and extensible C++ ecosystem with full support for AMD, NVIDIA, Intel GPUs.

Library core contains architecture-agnostic algorithm implementation

CORE

Library Infrastructure
Algorithm Implementations

- Iterative Solvers
- Preconditioners
- ...

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Library core contains architecture-agnostic algorithm implementation

CORE

Library Infrastructure
Algorithm Implementations

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- Preconditioners
- ...

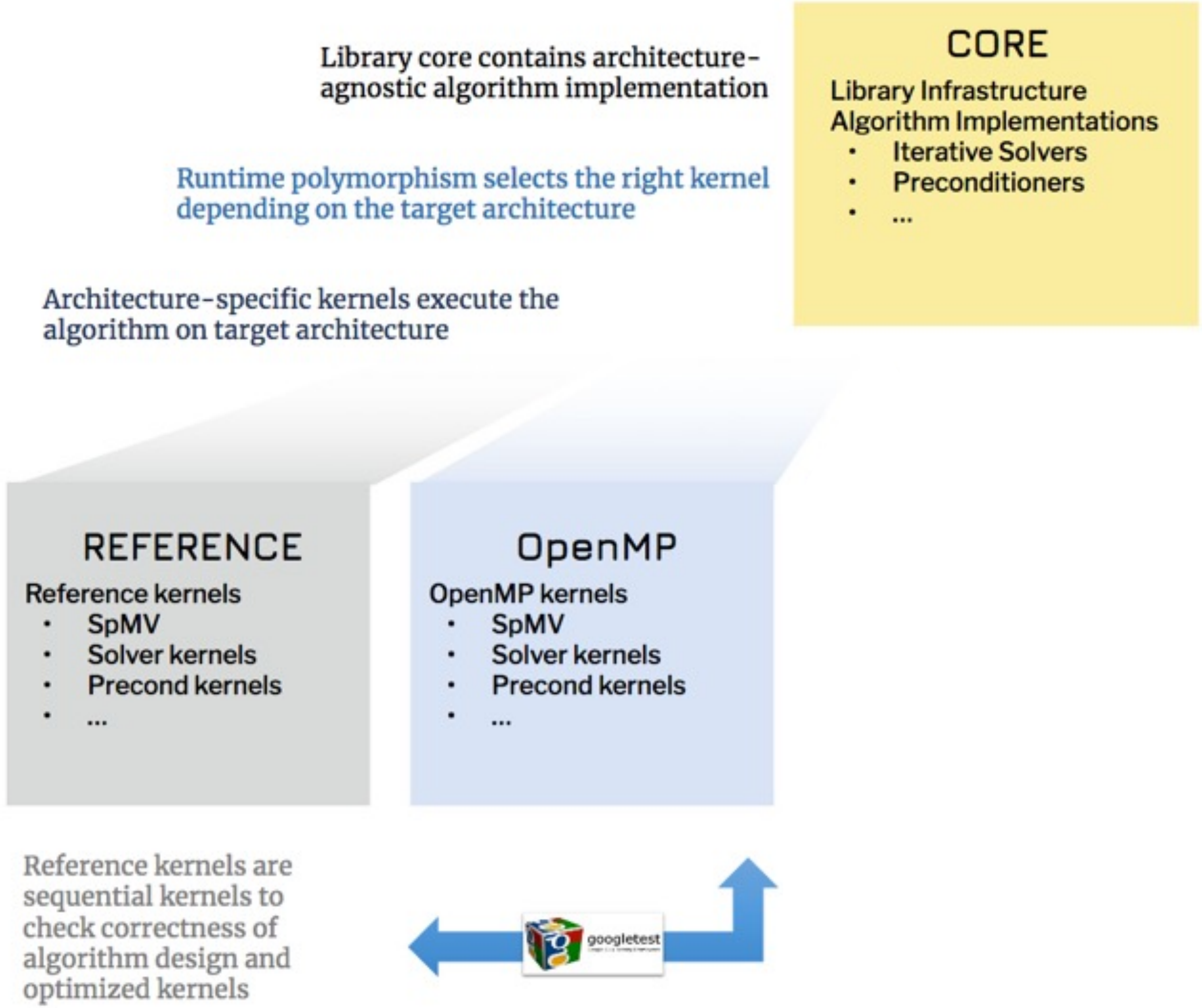
REFERENCE

Reference kernels

- SpMV
- Solver kernels
- Precond kernels
- ...

Reference kernels are sequential kernels to check correctness of algorithm design and optimized kernels

GPU-centric high performance sparse linear algebra. Sustainable and extensible C++ ecosystem with full support for AMD, NVIDIA, Intel GPUs.



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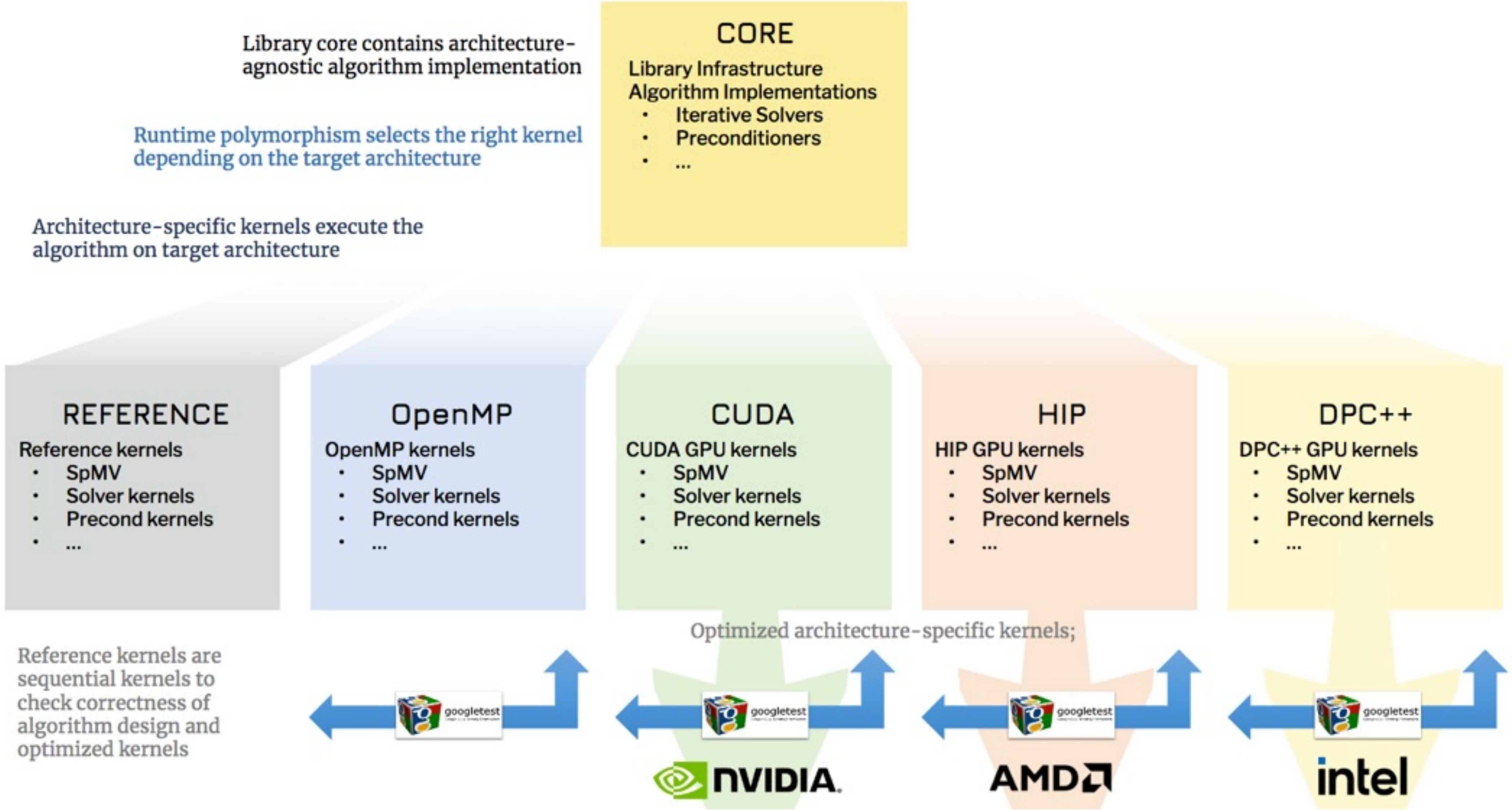
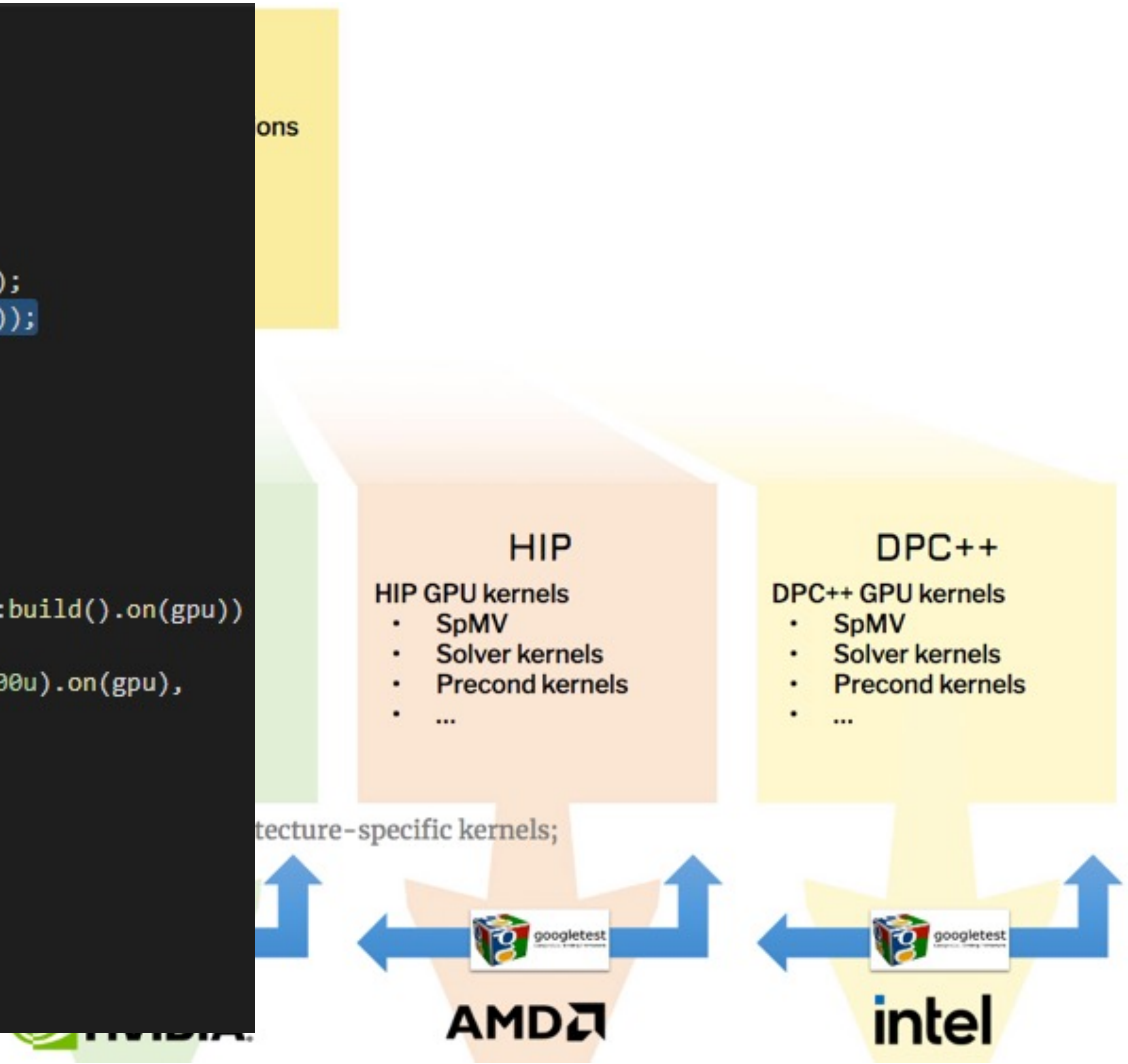


Ginkgo

GPU-centric high performance sparse linear algebra. Sustainable and extensible C++ ecosystem with full support for AMD, NVIDIA, Intel GPUs.

```cpp
#include <ginkgo/ginkgo.hpp>
#include <iostream>

int main()
{
    // Instantiate a GPU executor
    auto gpu =
-       gko::CudaExecutor::create(0, gko::OmpExecutor::create());
+       gko::DpcppExecutor::create(0, gko::OmpExecutor::create());
    // Read data
    auto A = gko::read<gko::matrix::Csr<>>(std::cin, gpu);
    auto b = gko::read<gko::matrix::Dense<>>(std::cin, gpu);
    auto x = gko::read<gko::matrix::Dense<>>(std::cin, gpu);
    // Create the solver
    auto solver =
        gko::solver::Cg<>::build()
            .with_preconditioner(gko::preconditioner::Jacobi<>::build().on(gpu))
            .with_criteria(
                gko::stop::Iteration::build().with_max_iters(1000u).on(gpu),
                gko::stop::ResidualNormReduction<>::build()
                    .with_reduction_factor(1e-15)
                    .on(gpu))
            .on(gpu);
    // Solve system
    solver->generate(give(A))->apply(lend(b), lend(x));
    // Write result
    write(std::cout, lend(x));
}
```



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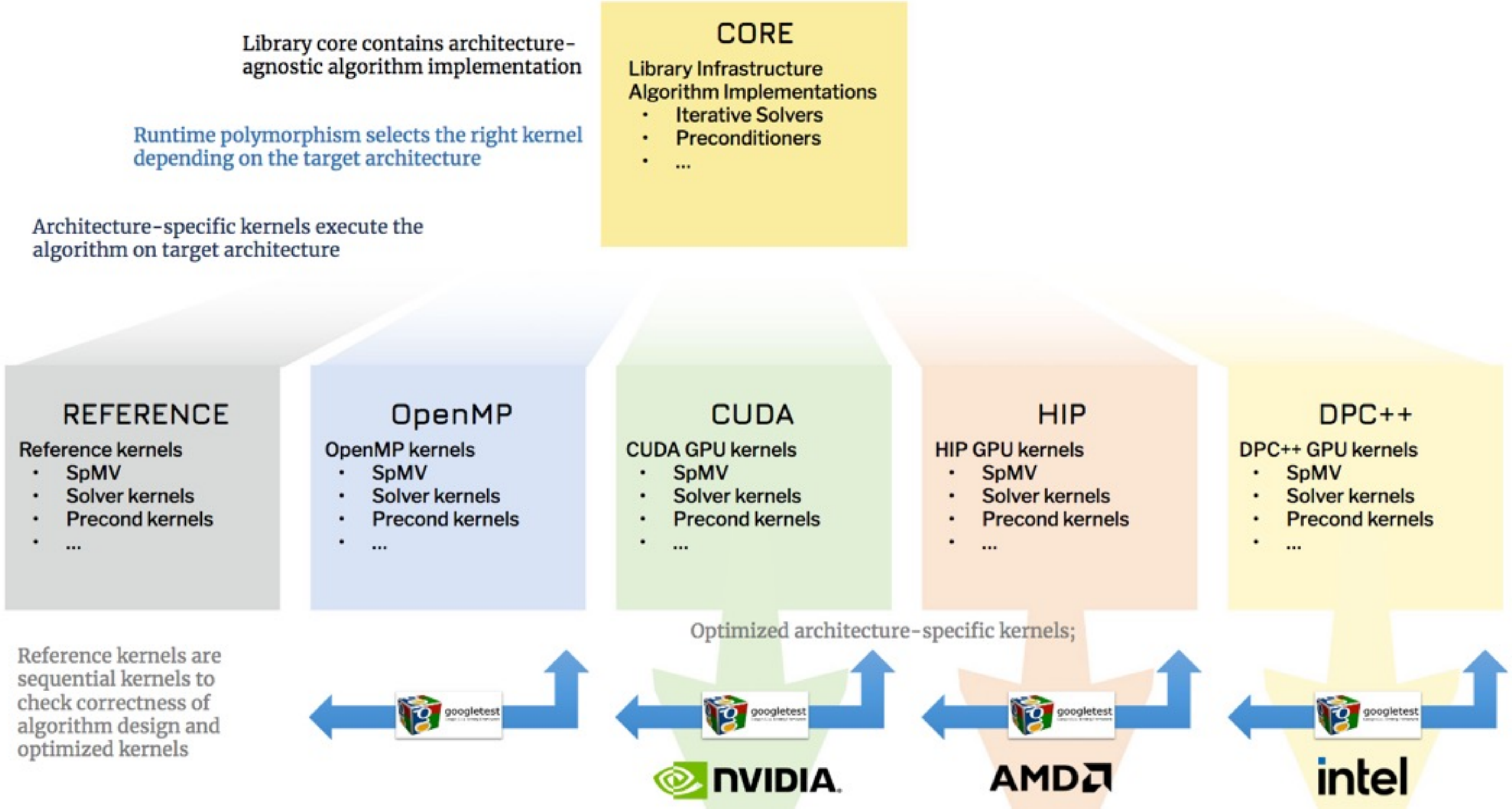

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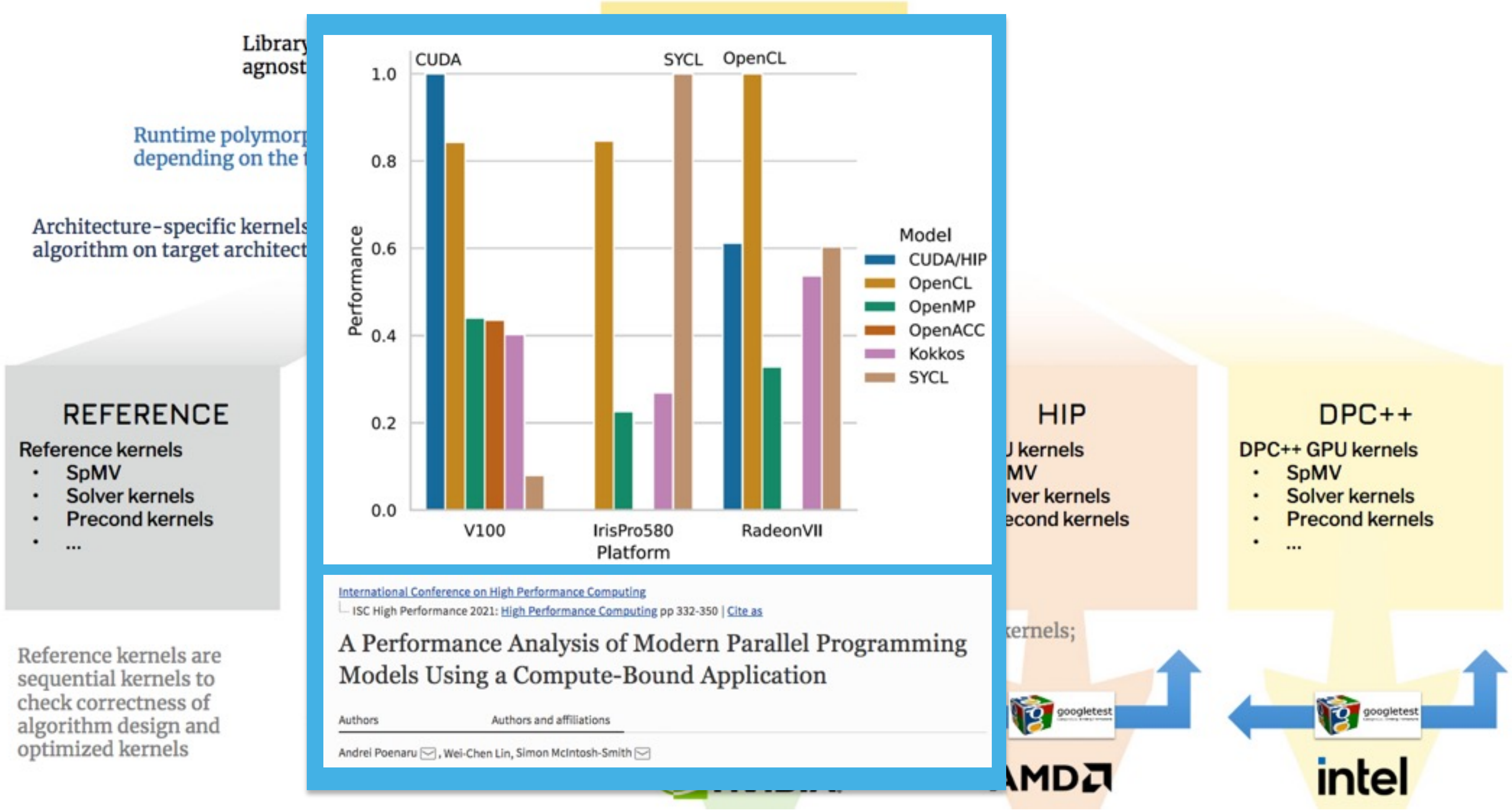

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A Performance Analysis of Modern Parallel Programming Models Using a Compute-Bound Application

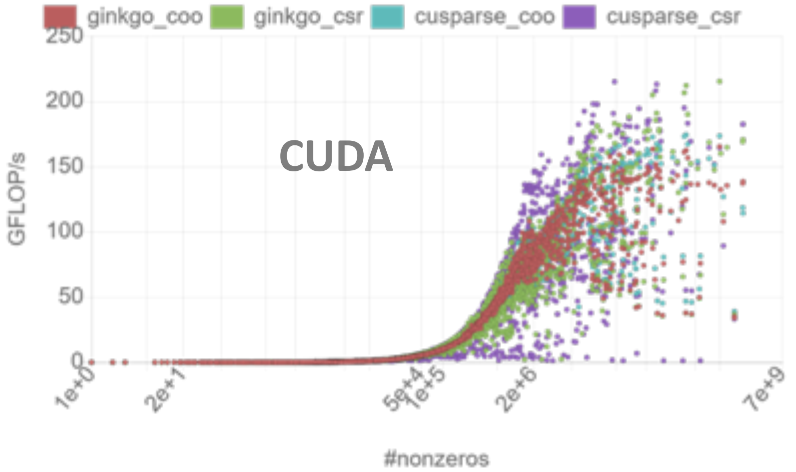
Authors | Authors and affiliations

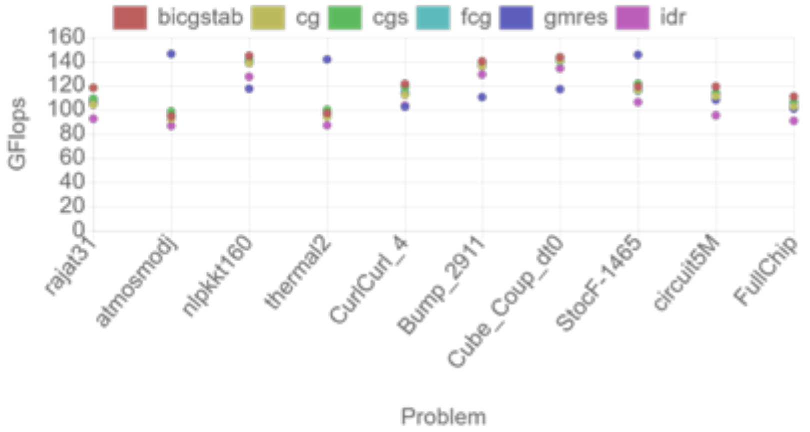
Andrei Poenaru, Wei-Chen Lin, Simon McIntosh-Smith

GPU-centric high performance sparse linear algebra. Sustainable and extensible C++ ecosystem with full support for AMD, NVIDIA, Intel GPUs.

NVIDIA A100 GPU

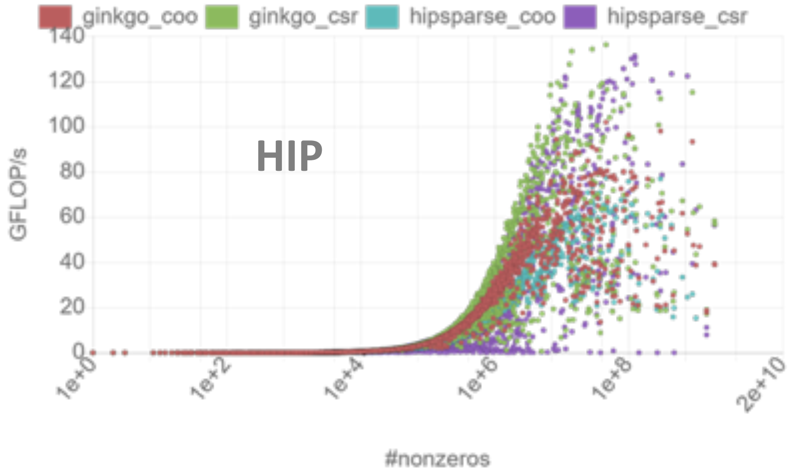
Perlmutter-type, external

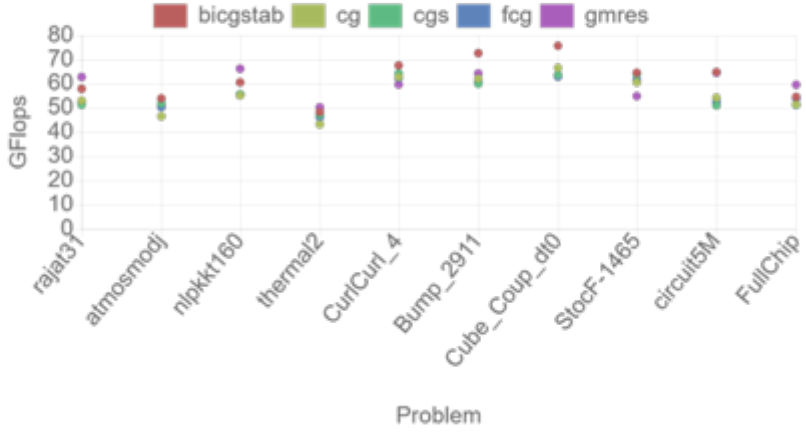




AMD MI100 GPU

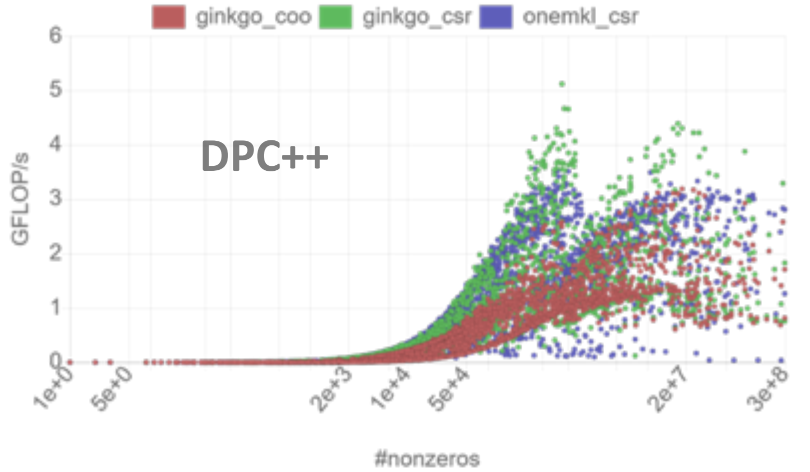
Pre-Frontier system Tulip (approved)

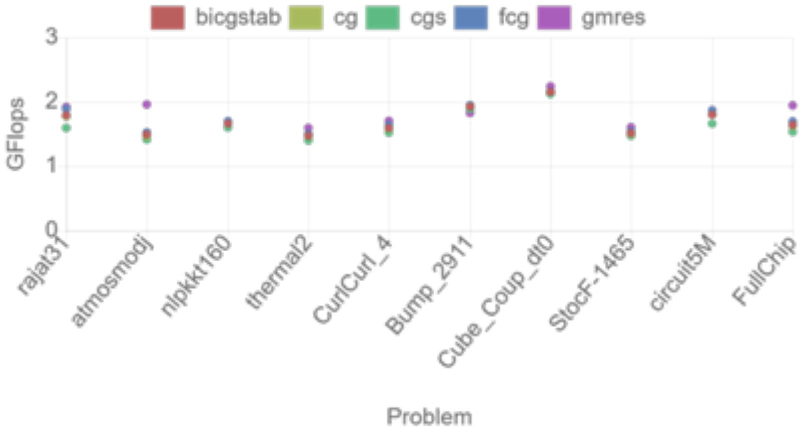




intel Gen9 GPU

integrated GPU





Test problem from Suite Sparse: https://people.engr.tamu.edu/davis/suitesparse.html

GPU-centric high performance sparse linear algebra. Sustainable and extensible C++ ecosystem with full support for AMD, NVIDIA, Intel GPUs.

Linear Operator Design

We express everything as Linear Operator and leverage C++ class inheritance.

Matrix-Vector Product	Preconditioner (for matrix A)	Solver (for system $Ax = b$)
$x := A \cdot b$	$x := M^{-1} \cdot b$	$x := S \cdot b$
	$M^{-1} \approx A^{-1}$	$S \approx A^{-1}$
	$M^{-1} = \Pi(A)$	$S = \Sigma(A)$

All of them can be expressed as

Application of a linear operator* (LinOp) $L : \mathbb{F}^m \rightarrow \mathbb{F}^m$

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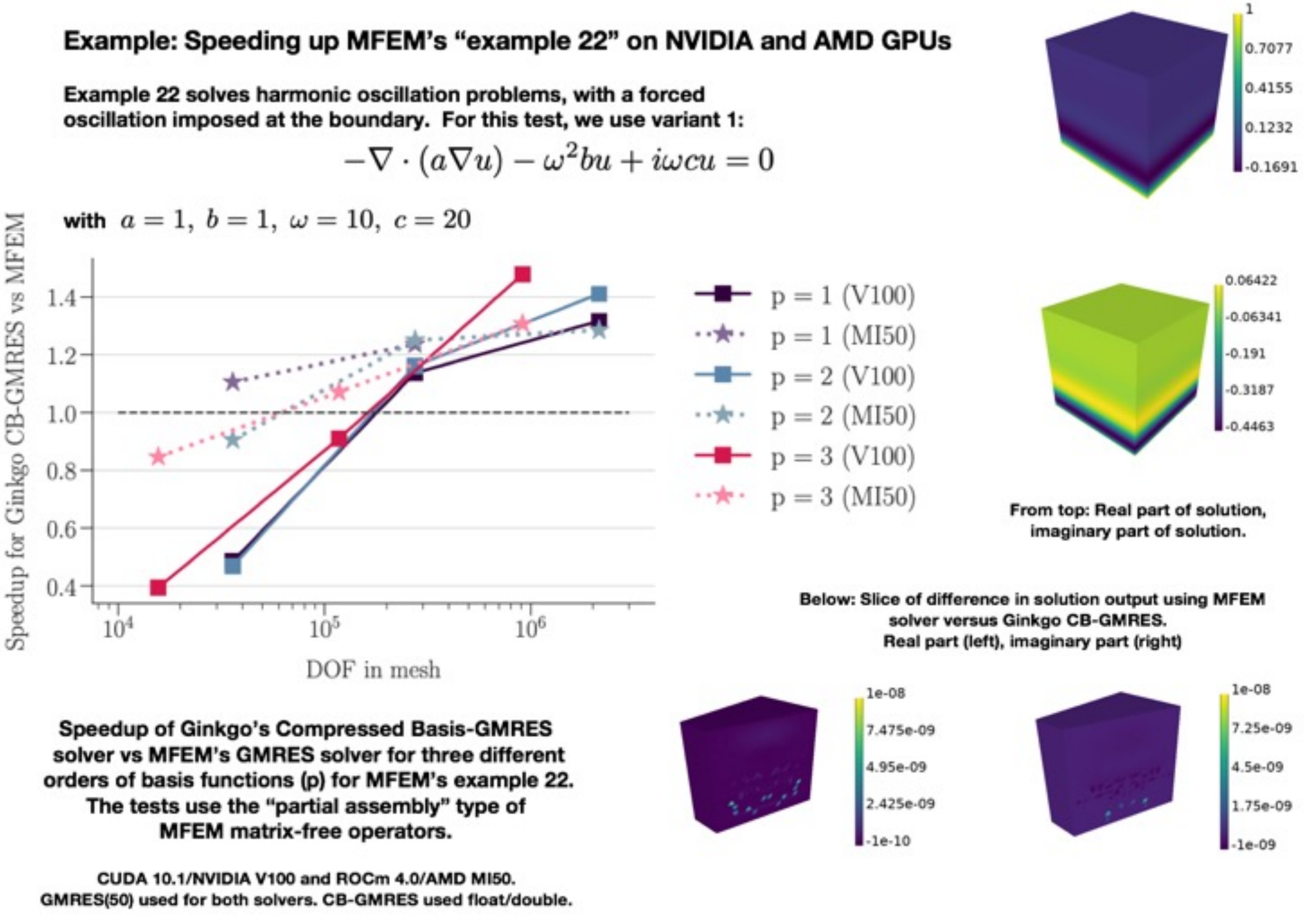

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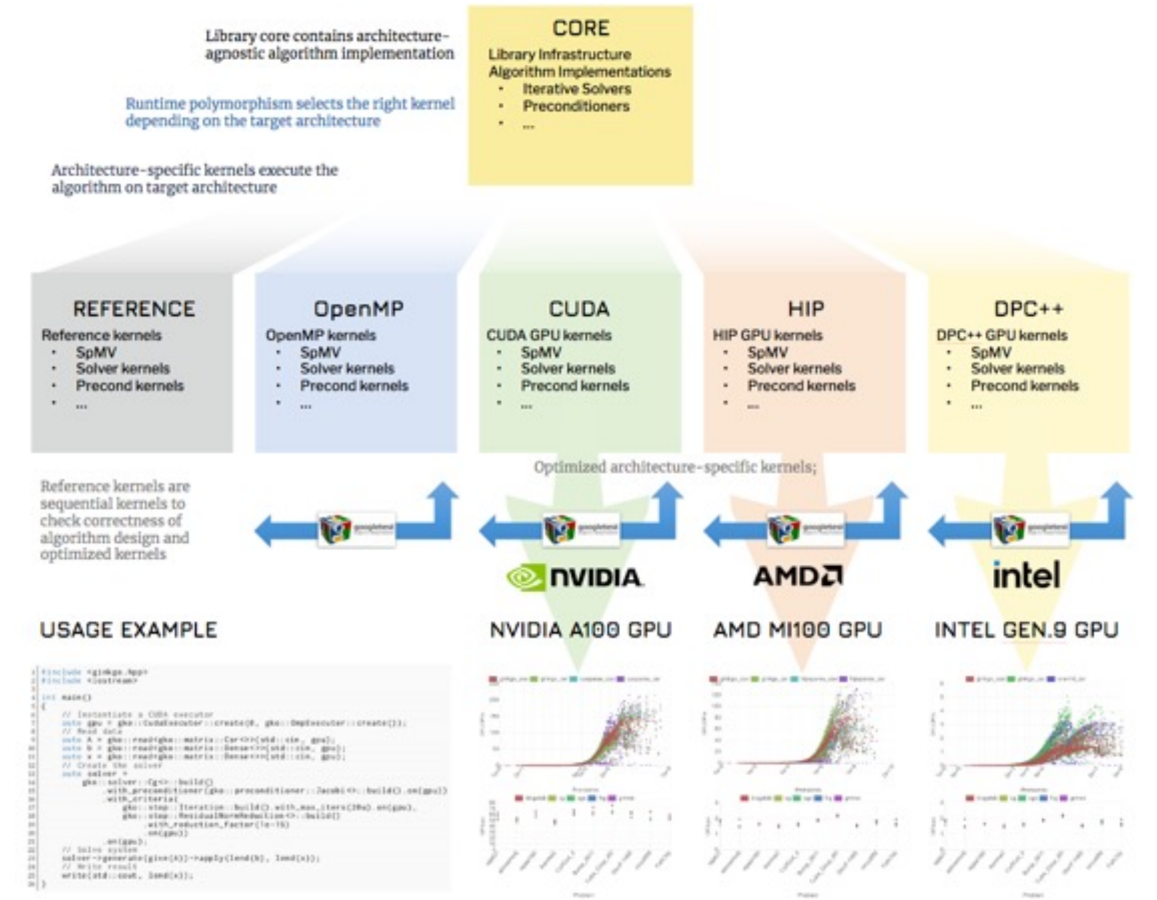
  - C++, CMake build;
- **Open source, community-driven**
  - Freely available (BSD License), GitHub, and Spack;
  - Part of the **xSDK and E4S software stack**;
  - Can be used from **deal.II** and **MFEM**;
  - Working on **SUNDIALS** integration;



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Simplifying research on mixed precision algorithms

- Generally, designing mixed precision algorithms in C/C++ is tedious and messy;
  - Requires writing code with many explicit format conversions;
  - Significant level of data duplication (in different precisions);

```
float A;
double x, y;

y = A*x;
```

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```
float A;
double x, y;

y = A*x;
```

- **Idea: provide a framework, where the compiler handles all format conversions:**
  - Create objects in different precisions and combine them in the code;
  - Automatically-generated temporary copies are used to match the precisions;
  - After completion of the arithmetic, the output is converted back to the requested format;
  - No need to worry about explicit conversions;
  - The framework may not provide good performance;
  - It still allows for easy and quick investigation of numerical effects;
  - If an idea has proven successful, one can replace the on-the-fly conversion with hardware-optimized implementations;

Ginkgo

"Mix & Match Precisions in Ginkgo"

```
using mtx =
        gko::matrix::Csr<float, int>;
using vec =
        gko::matrix::Dense<double>;
…
auto solver = solver_gen->generate(A);
solver->apply(lend(b), lend(x));
```

Mixing precisions in Ginkgo

Ginkgo

1. Create Objects in different precision formats

```
using HighPrecision = double;
using LowPrecision = float;
using hp_mtx = gko::matrix::Ell<HighPrecision, IndexType>;
using lp_mtx = gko::matrix::Ell<LowPrecision, IndexType>;

// read the matrix into HighPrecision and LowPrecision.
auto hp_A = share(gko::read<hp_mtx>(std::ifstream("data/A.mtx"), exec));
auto lp_A = share(gko::read<lp_mtx>(std::ifstream("data/A.mtx"), exec));
// Set the shortcut for each dimension
auto A_dim = hp_A->get_size();
auto b_dim = gko::dim<2>{A_dim[1], 1};
auto x_dim = gko::dim<2>{A_dim[0], b_dim[1]};
auto host_b = hp_vec::create(exec->get_master(), b_dim);
// fill the b vector with some random data
std::ranlux48 rand_engine(32);
auto dist = std::uniform_real_distribution<RealValueType>(0.0, 1.0);
for (int i = 0; i < host_b->get_size()[0]; i++) {
    host_b->at(i, 0) = get_rand_value<HighPrecision>(dist, rand_engine);
}
```

ginkgo -> examples -> mixed-spmv

https://github.com/ginkgo-project/ginkgo/tree/develop/examples/mixed-spmv

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}
```

2. Combine different precisions in computations

```
// Hp * Hp -> Hp
auto hp_sec = timing(exec, hp_A, hp_b, hp_x);
// Lp * Lp -> Lp
auto lp_sec = timing(exec, lp_A, lp_b, lp_x);
// Hp * Lp -> Hp
auto hplp_sec = timing(exec, hp_A, lp_b, hplp_x);
// Lp * Lp -> Hp
auto lplp_sec = timing(exec, lp_A, lp_b, lplp_x);
// Lp * Hp -> Hp
auto lphp_sec = timing(exec, lp_A, hp_b, lphp_x);
```

ginkgo -> examples -> mixed-spmv

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Mixing precisions in Ginkgo

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// Hp * Lp -> Hp
auto hplp_sec = timing(exec, hp_A, lp_b, hplp_x);
// Lp * Lp -> Hp
auto lplp_sec = timing(exec, lp_A, lp_b, lplp_x);
// Lp * Hp -> Hp
auto lphp_sec = timing(exec, lp_A, hp_b, lphp_x);
```

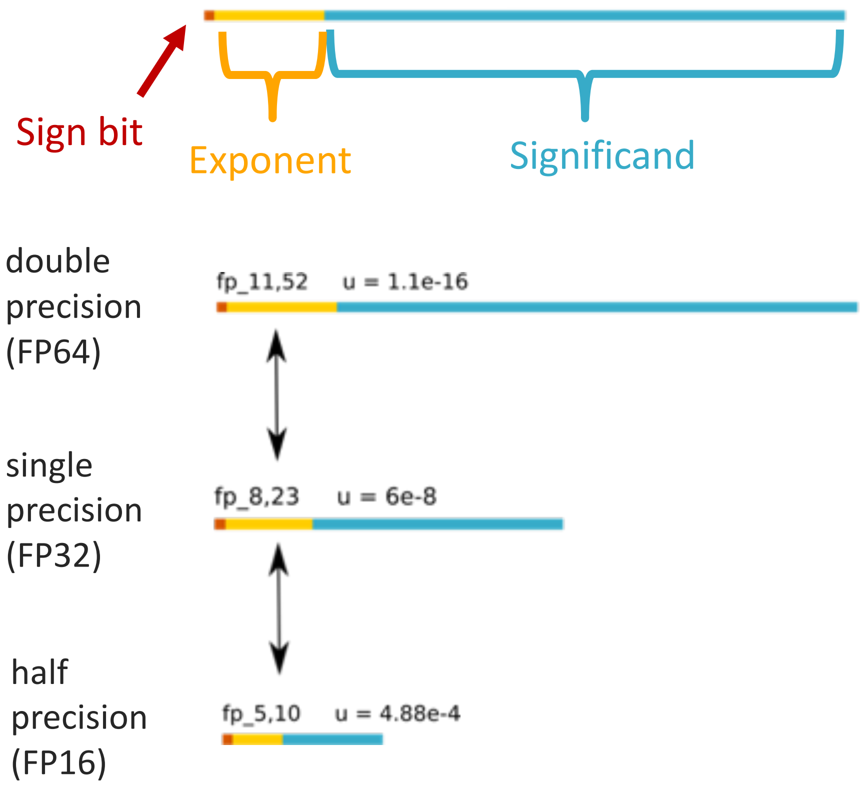
3. Investigate rounding effects

```
High Precision time(s): 9.8980000000e-07
High Precision result norm: 2.5547848401e+05
Low Precision time(s): 9.8890000000e-07
Low Precision relative error: 5.5253439244e-08
Hp * Lp -> Hp time(s): 1.3829000000e-06
Hp * Lp -> Hp relative error: 1.6328092846e-08
Lp * Lp -> Hp time(s): 1.3761000000e-06
Lp * Lp -> Hp relative error: 2.5540873856e-08
Lp * Hp -> Hp time(s): 1.3761000000e-06
Lp * Hp -> Hp relative error: 3.7166469483e-08
```

ginkgo -> examples -> mixed-spmv

https://github.com/ginkgo-project/ginkgo/tree/develop/examples/mixed-spmv

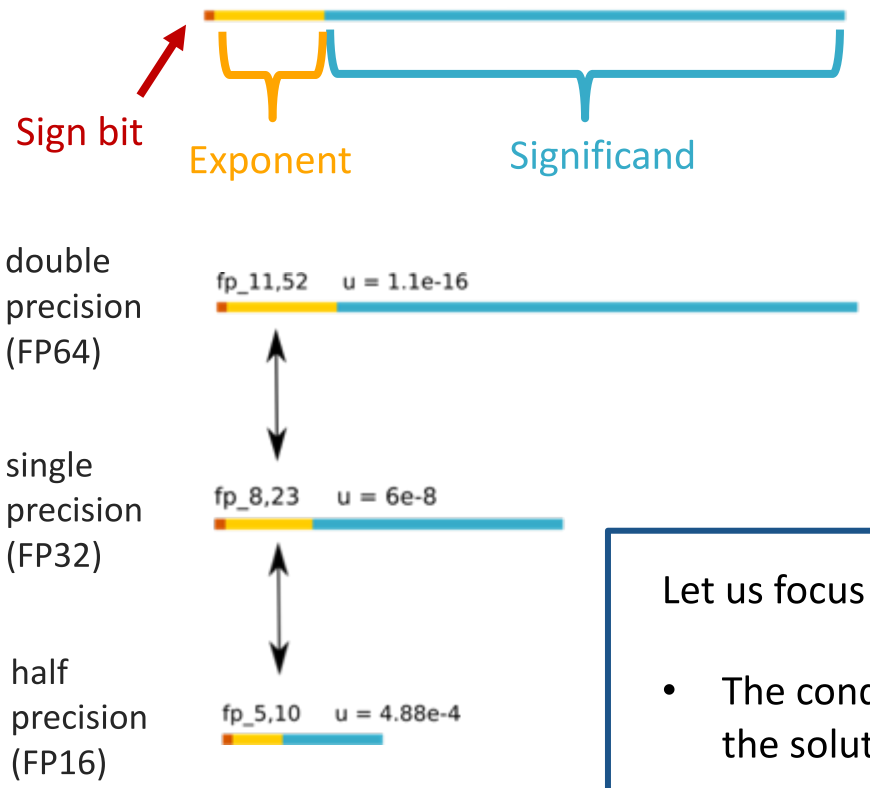
Background: Floating Point Formats, Accuracy, and Performance



Broadly speaking….

- The length of the **exponent** determines the **range** of the values that can be represented;
- The length of the **significand** determines how **accurate** values can be represented;
- **The data access cost linearly depends on the memory volume;**
- **Rounding effects accumulate over a sequence of computations;**

Background: Floating Point Formats, Accuracy, and Performance



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- **The data access cost linearly depends on the memory volume;**
- **Rounding effects accumulate over a sequence of computations;**

Let us focus on linear systems of the form Ax=b.

- The conditioning of a linear system reflects how sensitive the solution x is with regard to changes in the right-hand side b.
- Rule of thumb:

relative residual accuracy = **(unit round-off)** * **(linear system's condition number)**

N. Higham: Accuracy and stability of numerical algorithms. SIAM, 2002.

Running iterative methods in different precision formats

Linear System Ax=b with cond(A) ≈ 10^4

Double Precision

```
Initial residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
111.127
Final residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
5.0775e-10
CG iteration count:     1231
CG execution time [ms]: 140.038
```

Experiments based on the Ginkgo library https://ginkgo-project.github.io/
ginkgo/examples/adaptiveprecision-blockjacobi/adaptiveprecision-blockjacobi.cpp

Running iterative methods in different precision formats

Ginkgo

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Accuracy improvement ~10^{12}

relative residual accuracy = (unit round-off) * (linear system's condition number)

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1 1
5.0775e-10
CG iteration count:     1231
CG execution time [ms]: 140.038
```

Accuracy improvement ~10^{12}

```
…
- ValueType = double; //u=1e-16
+ ValueType = float;  //u=1e-7
…
```

relative residual accuracy = (unit round-off) * (linear system's condition number)

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Running iterative methods in different precision formats

Ginkgo

Linear System Ax=b with cond(A) $\approx 10^4$

Double Precision

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1 1
111.127
Final residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
5.0775e-10
CG iteration count:     1231
CG execution time [ms]: 140.038
```

Accuracy improvement ~10^{12}

Single Precision

```
Initial residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
111.127
Final residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
0.179829
CG iteration count:     1234
CG execution time [ms]: 127.152
```

Accuracy improvement ~10^3

relative residual accuracy = (unit round-off) * (linear system's condition number)

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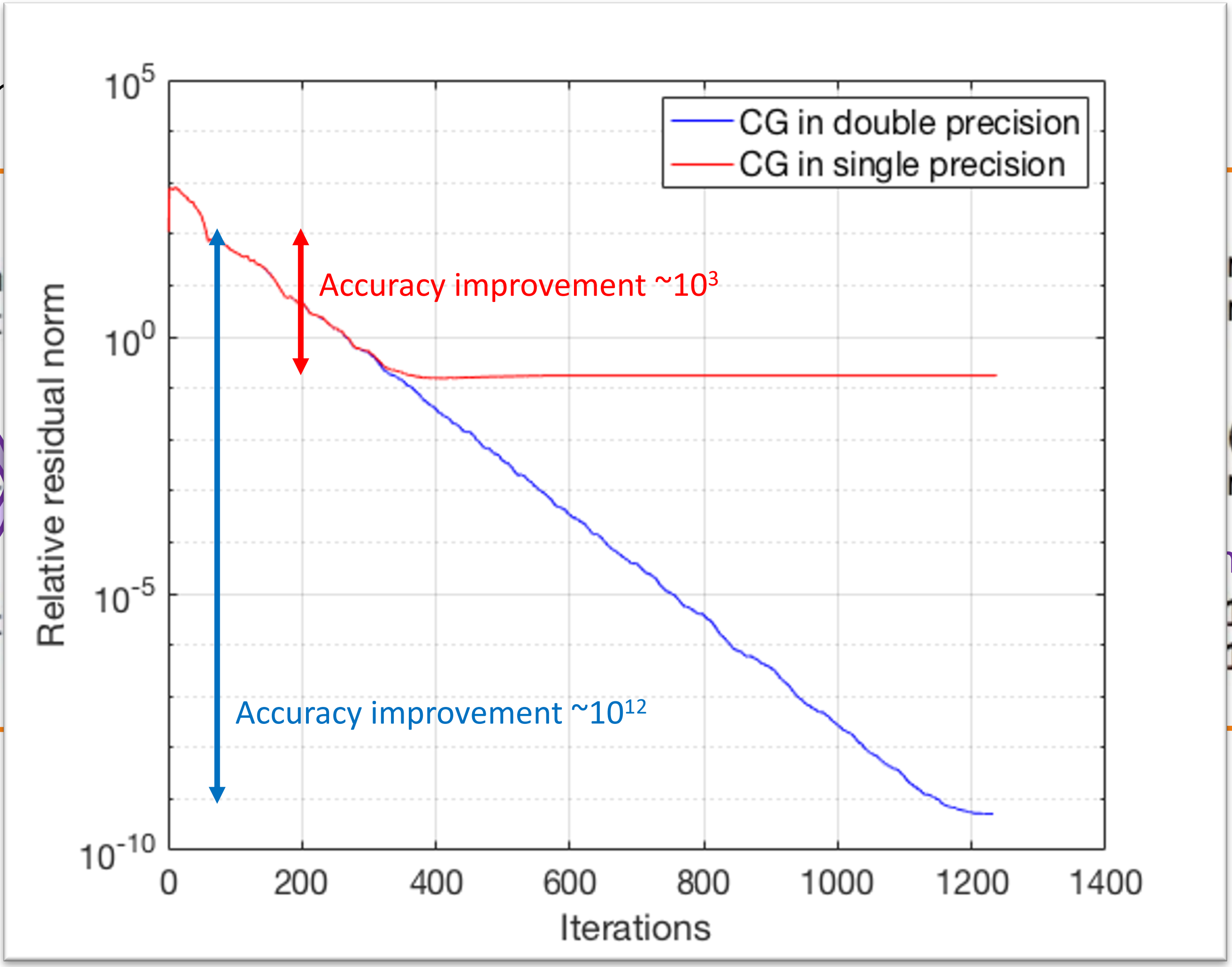
Running iterative methods in different precision formats

Linear System Ax=b with

Double Precision

```
Initial residual n
%%MatrixMarket mat
1 1
111.127
Final residual no
%%MatrixMarket mat
1 1
5.0775e-10
CG iteration count
CG execution time
```




```
rt(r^T r):
ray real general

(r^T r):
ray real general
```

mprovement ~10^3

```
1234
127.152
```

Experiments based on the Ginkgo library https://ginkgo-project.github.io/

ginkgo/examples/adaptiveprecision-blockjacobi/adaptiveprecision-blockjacobi.cpp

Running iterative methods in different precision formats

Ginkgo

Linear System Ax=b with cond(A) ≈ 10^4

Double Precision

```
Initial residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
111.127
Final residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
5.0775e-10
CG iteration count:      1231
CG execution time [ms]:  140.038
```

Single Precision

```
Initial residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
111.127
Final residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
0.179829
CG iteration count:      1234
CG execution time [ms]:  127.152
```

Single Precision is 10% faster!

Experiments based on the Ginkgo library https://ginkgo-project.github.io/
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Running iterative methods in different precision formats

Linear System Ax=b with cond(A) ≈ 10^7 *(apache2 from SuiteSparse)* **NVIDIA A100 GPU**

Double Precision CG + Double Precision Preconditioner

```
Initial residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
1390.67
Final residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
3.97985e-06
CG iteration count:     4797
CG execution time [ms]: 2971.18
```

Accuracy improvement ~10^9

Experiments based on the Ginkgo library https://ginkgo-project.github.io/
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Running iterative methods in different precision formats

Ginkgo

Linear System Ax=b with cond(A) ≈ 10^7 (*a*

Double Precision CG + Double Precision I

```
Initial residual norm sqrt(r^T r)
%%MatrixMarket matrix array real
1 1
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1 1
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CG execution time [ms]: 2971.18
```

Accuracy improvement ~10^9

```
…
   - ValueType = double; //u=1e-16
   + ValueType = float;  //u=1e-7
…
```

Experiments based on the Ginkgo library https://ginkgo-project.github.io/
ginkgo/examples/adaptiveprecision-blockjacobi/adaptiveprecision-blockjacobi.cpp

Running iterative methods in different precision formats

Ginkgo

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Final residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
1588.77
CG iteration count:     8887
CG execution time [ms]: 2972.46
```

No improvement

relative residual accuracy = (unit round-off) * (linear system's condition number)

N. Higham: Accuracy and stability of numerical algorithms. SIAM, 2002.

Experiments based on the Ginkgo library https://ginkgo-project.github.io/
ginkgo/examples/adaptiveprecision-blockjacobi/adaptiveprecision-blockjacobi.cpp

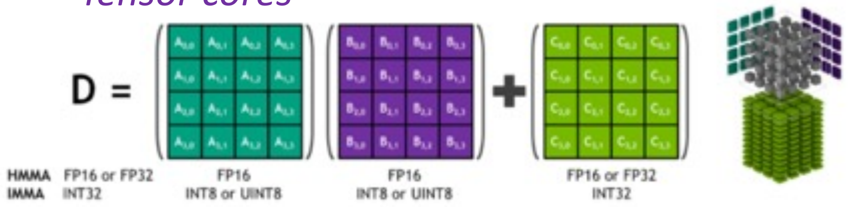
Why are we faster with a single precision CG solver?

double : single : half

2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016 2017 2018 2019 2020 2021

NVIDIA GPU generation	Tesla	Fermi	Kepler	Maxwell	Pascal	Volta	Ampere
Rel. compute performance	1 : 2	1 : 2	1 : 2	1 : 32	1 : 2 : 4	1 : 2 : 16*	1 : 2 : 32 1* : 8* : 32*
Rel. memory performance	1 : 2	1 : 2	1 : 2	1 : 2	1 : 2 : 4	1 : 2 : 4	1 : 2 : 4

**Tensor cores*



For **compute-bound applications**, the performance gains from using lower precision **depend on the architecture**.

Differences between Volta, Maxwell, Tesla...

For **memory-bound applications**, the performance gains from using lower precision are **architecture-independent** and correspond to the floating point format complexity (#bits).

Generally, 2x for FP32, 4x for FP16.

Why are we faster with a single precision CG solver?

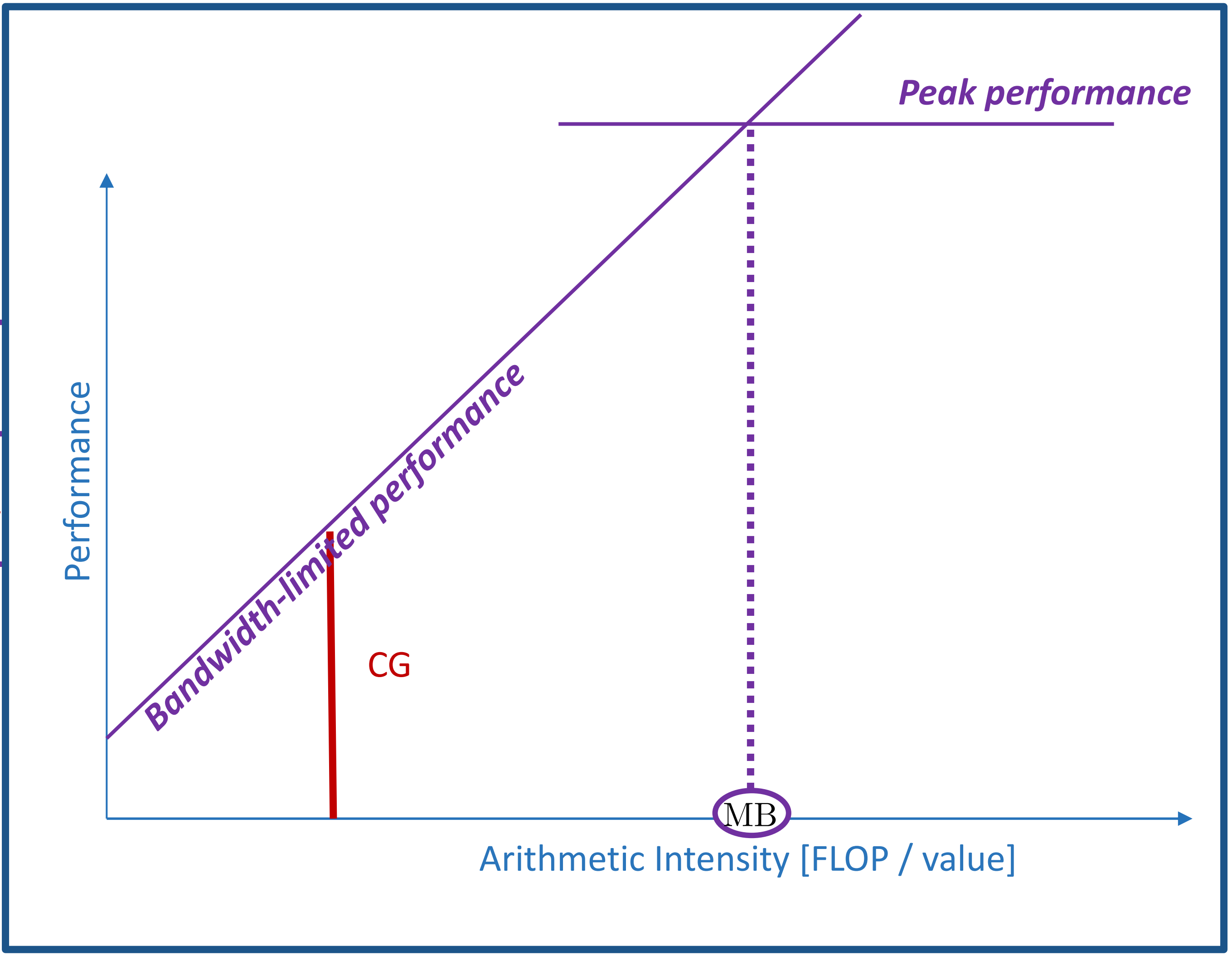
	2006	2007	2008	2009	2010	2011	2012
NVIDIA GPU generation			Tesla	Fermi			Keple
Rel. compute performance			1 : 2	1 : 2			1 : 2
Rel. memory performance			1 : 2	1 : 2			1 : 2

For **compute-bound applications**, the performance gains from using lower precision **depend on the architecture**.

Differences between Volta, Maxwell, Tesla...

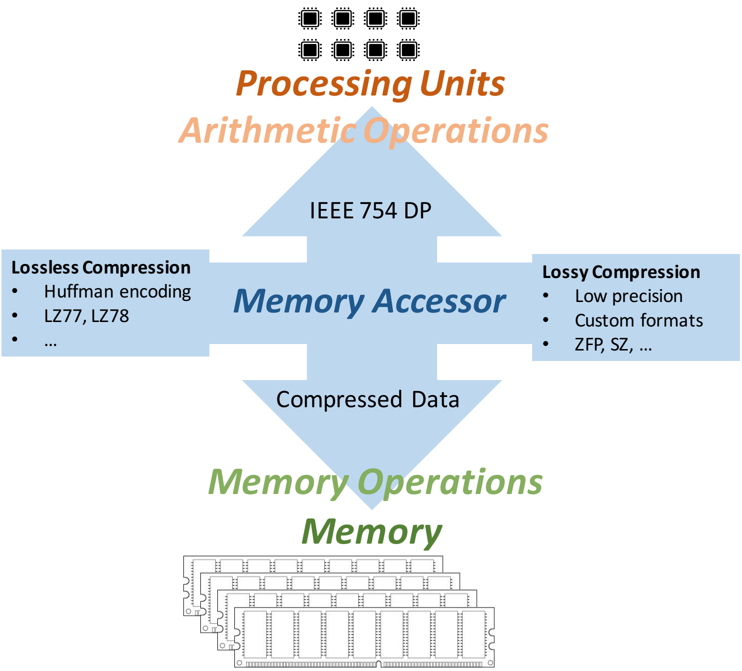
For **memory-bound applications**, the performance gains from using lower precision are **architecture-independent** and correspond to the floating point format complexity (#bits).

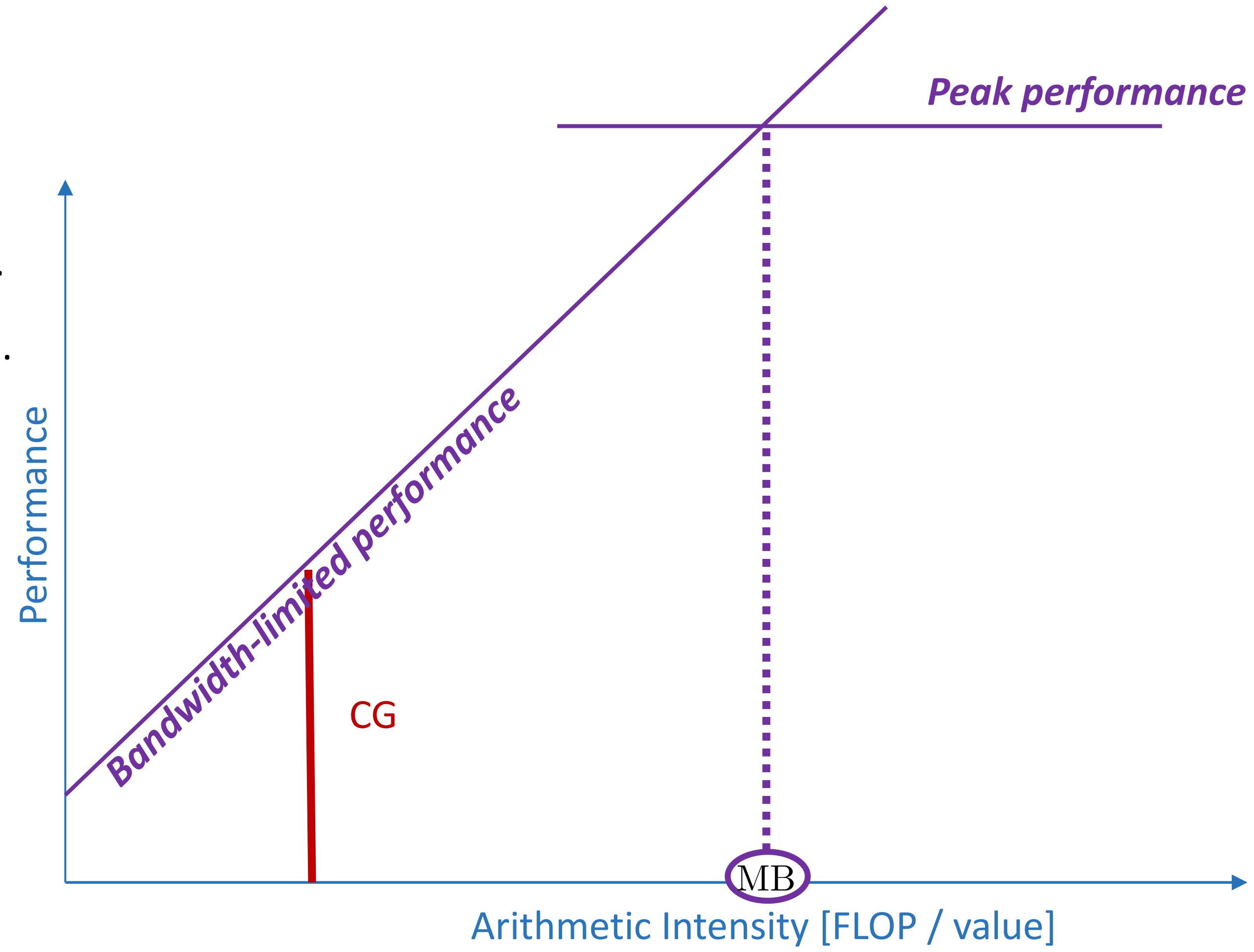
Generally, 2x for FP32, 4x for FP16.



Decoupling the memory precision from the arithmetic precision

- Traditionally, we use a strong coupling between the precision formats used for arithmetic operations and storing data.
- The arithmetic operations are free, use high precision formats.
- **Data access** should be as cheap as possible, **reduced precision**.





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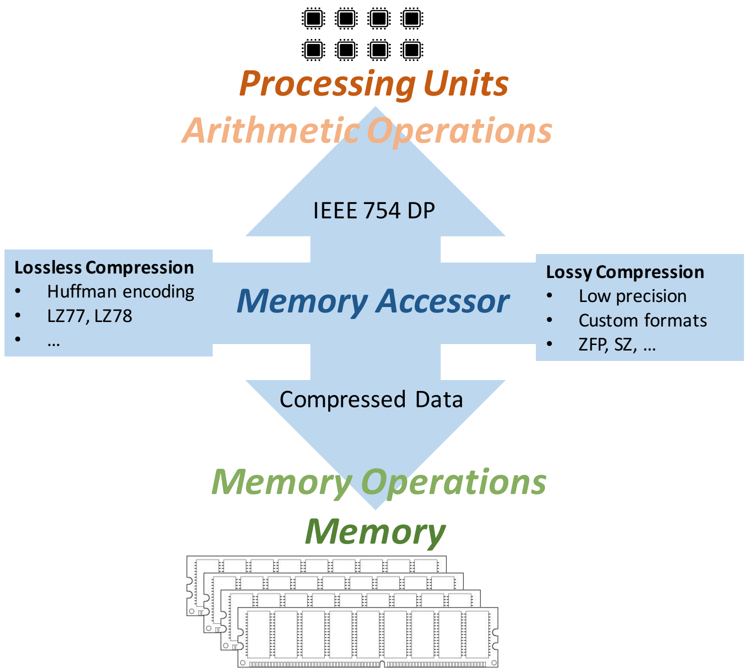
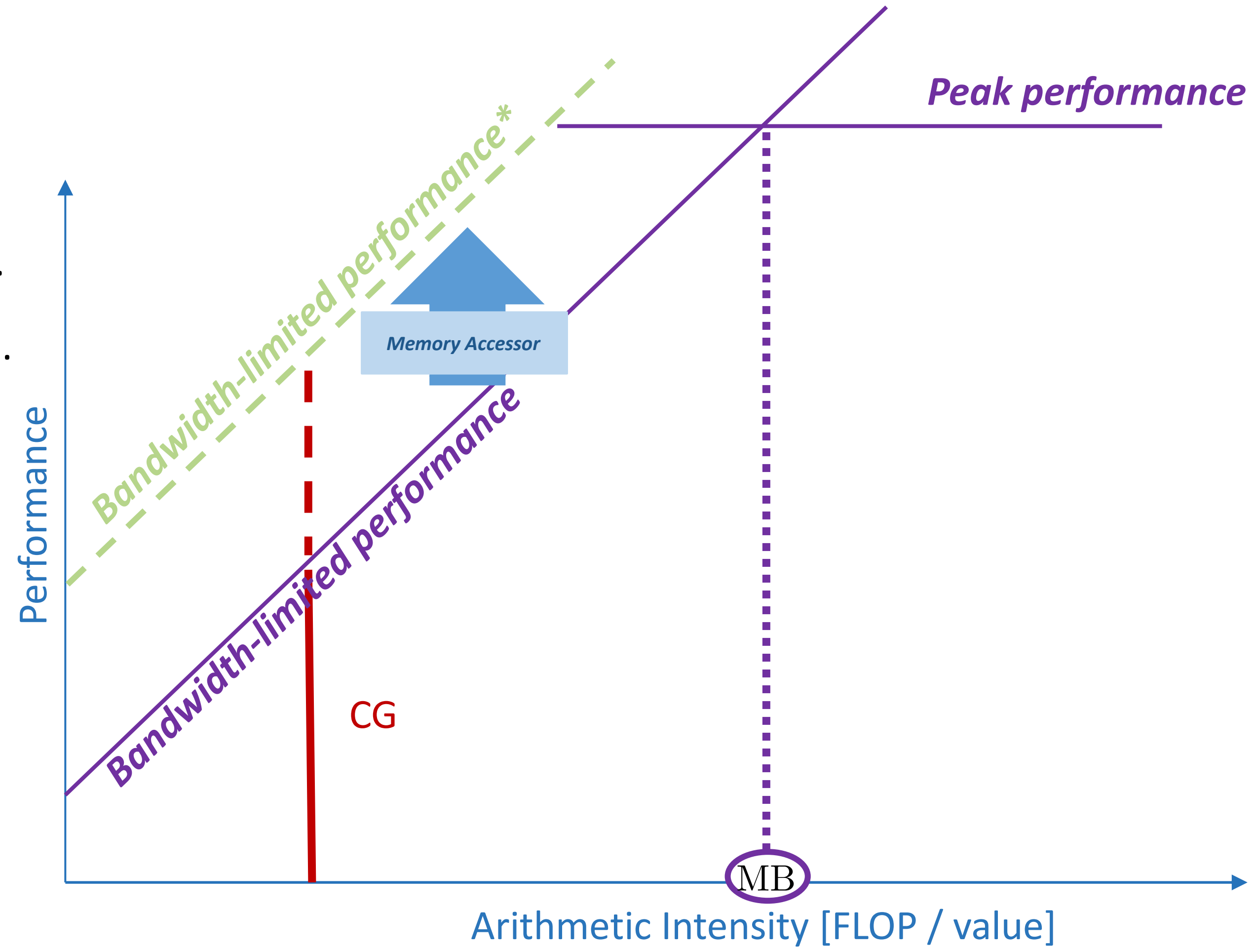
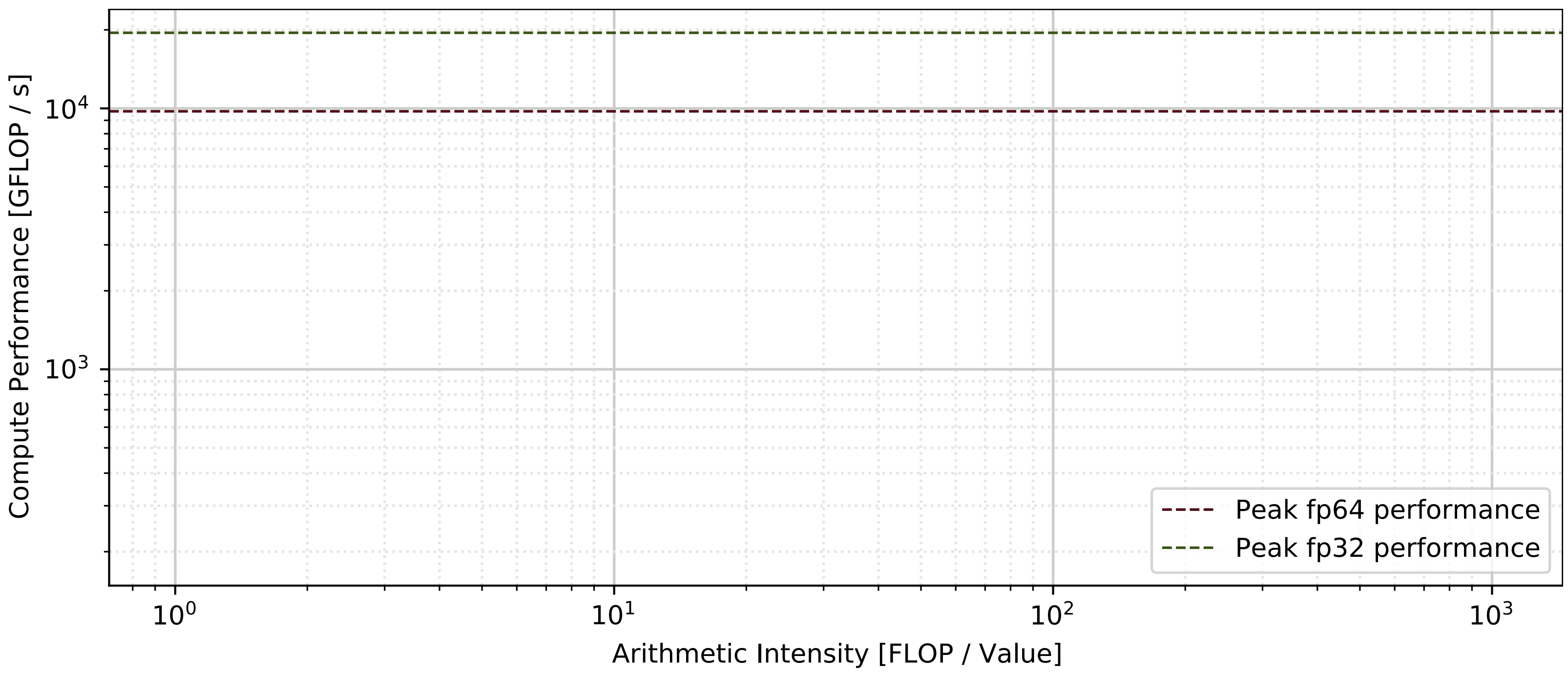




Memory Accessor for NVIDIA A100 GPU



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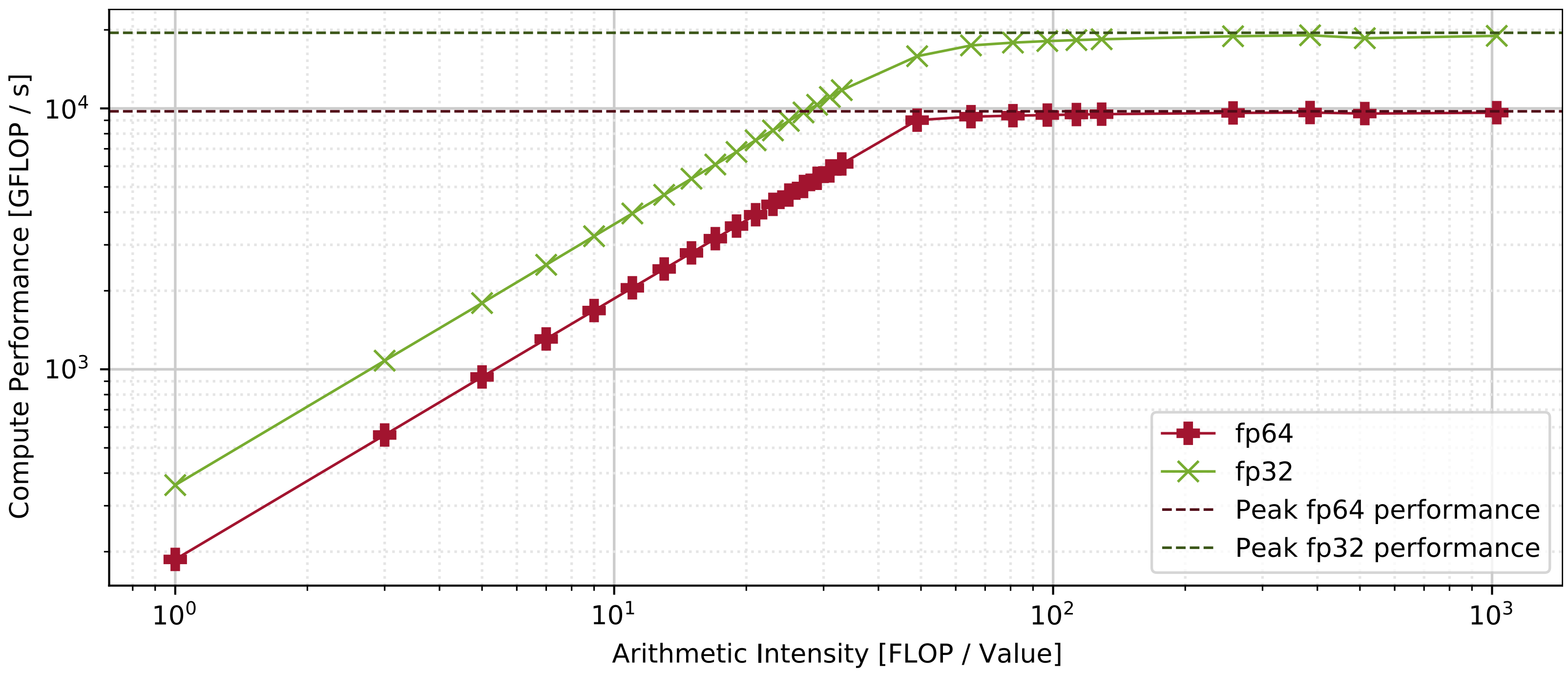

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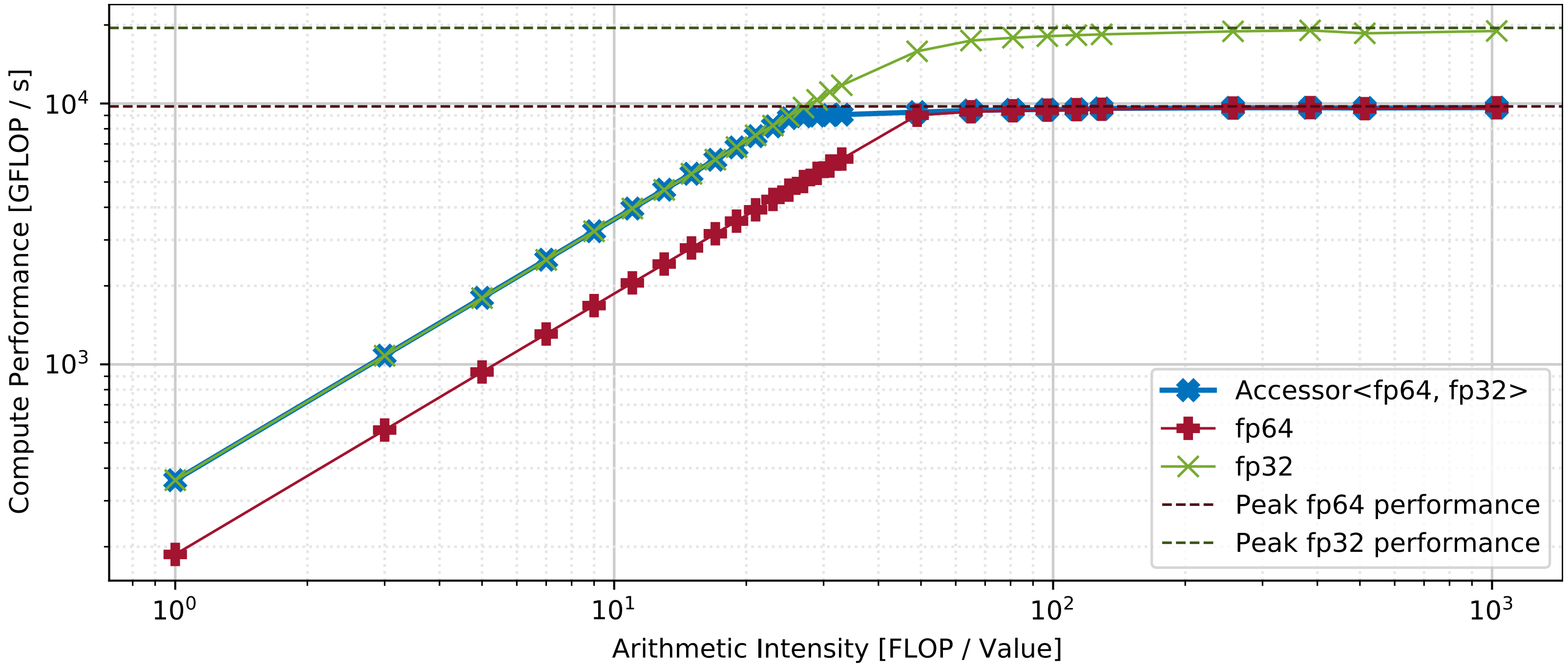

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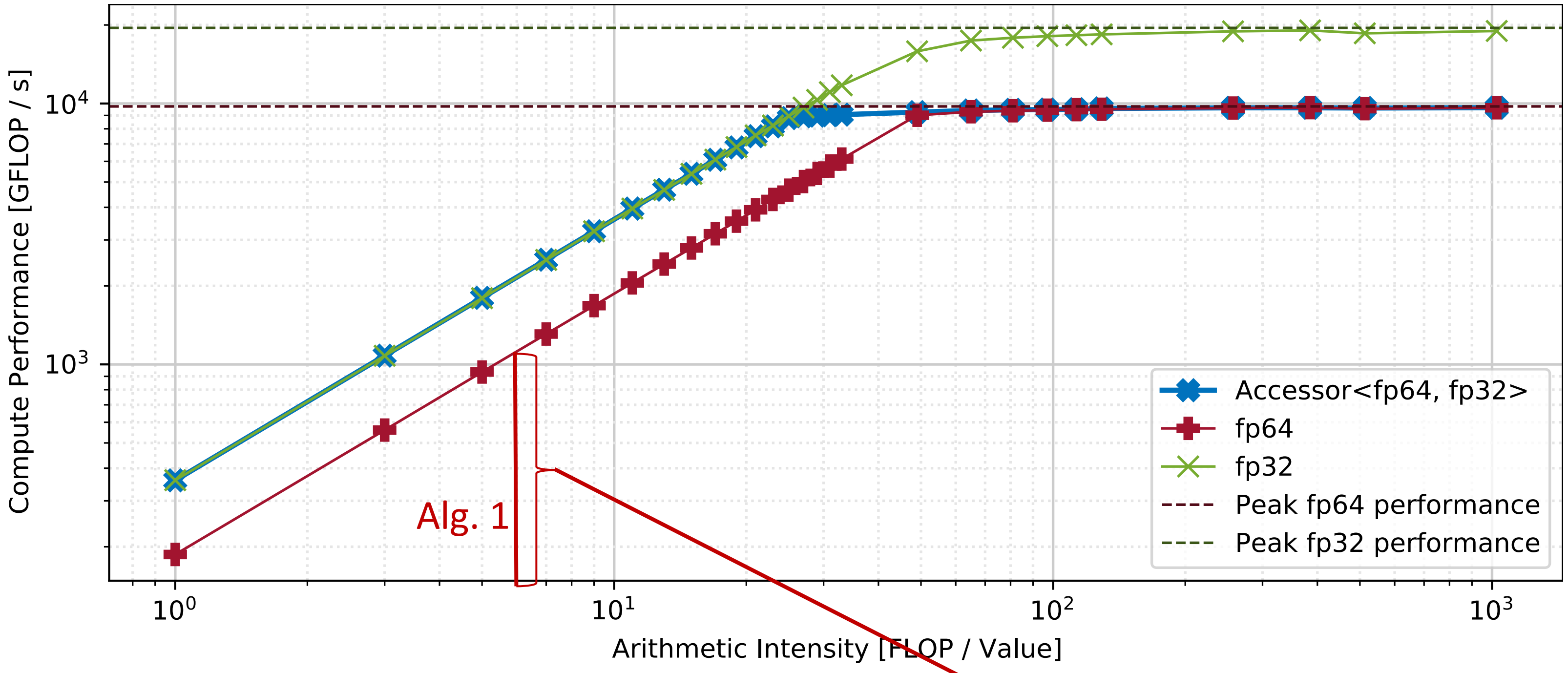
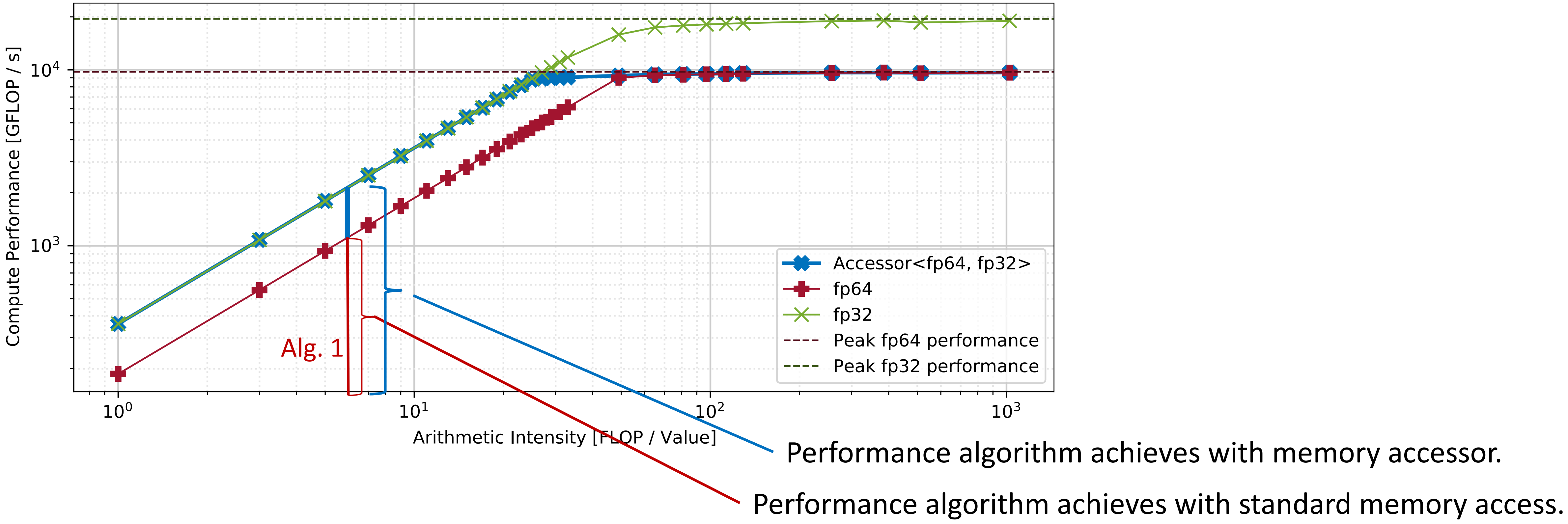


Performance algorithm achieves with standard memory access.

Memory Accessor for NVIDIA A100 GPU



Rethinking Algorithms: Use the memory accessor to increase accuracy

Can we use the memory accessor to reduce the rounding effects?

Design

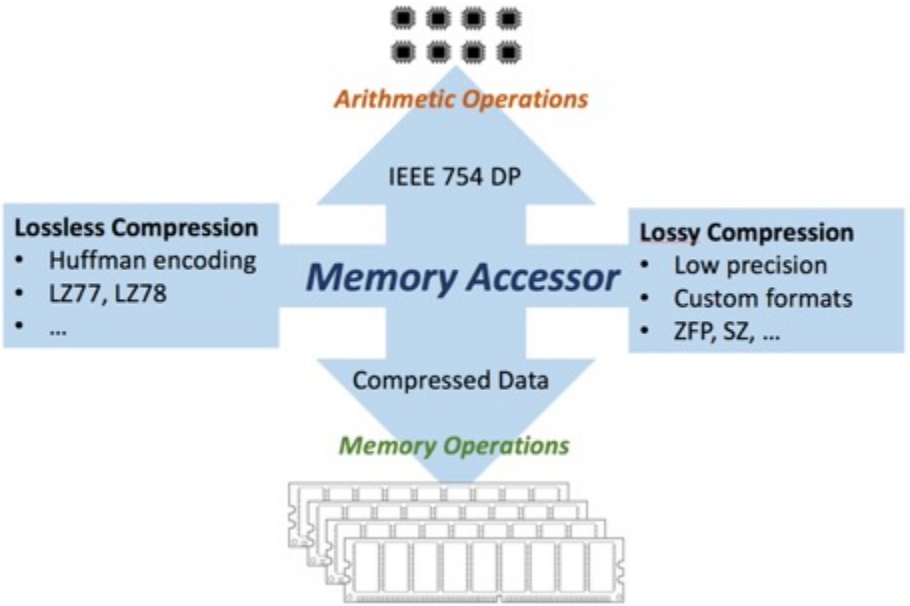
- Memory access in low precision (e.g. fp32);
- Computations in high precision (e.g. fp64);

Characteristics

- Performance of low precision BLAS;
- Higher accuracy of low precision BLAS;

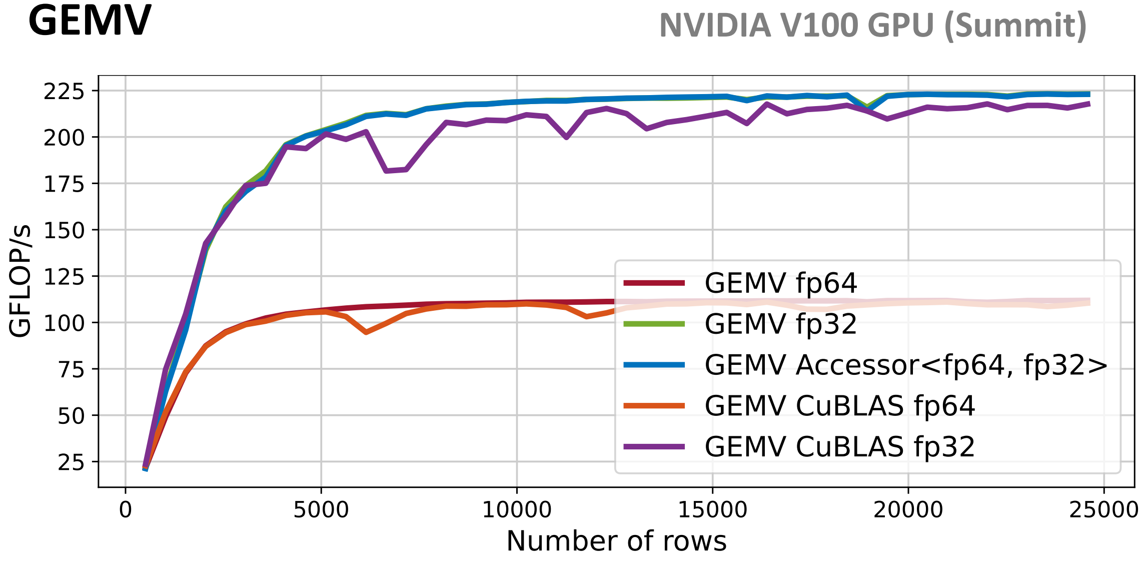
Usage

1. Can replace low precision BLAS to increase accuracy;
2. Can replace high precision BLAS if information loss is acceptable;
   (without having to deal with explicit mixed precision usage)



Accessor-BLAS: Replacing LP BLAS to improve accuracy

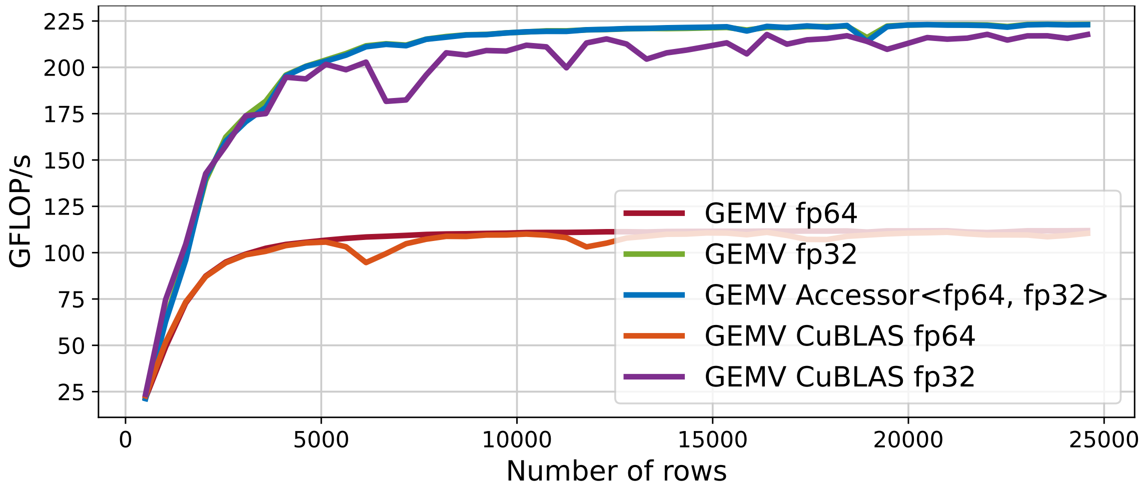
GEMV

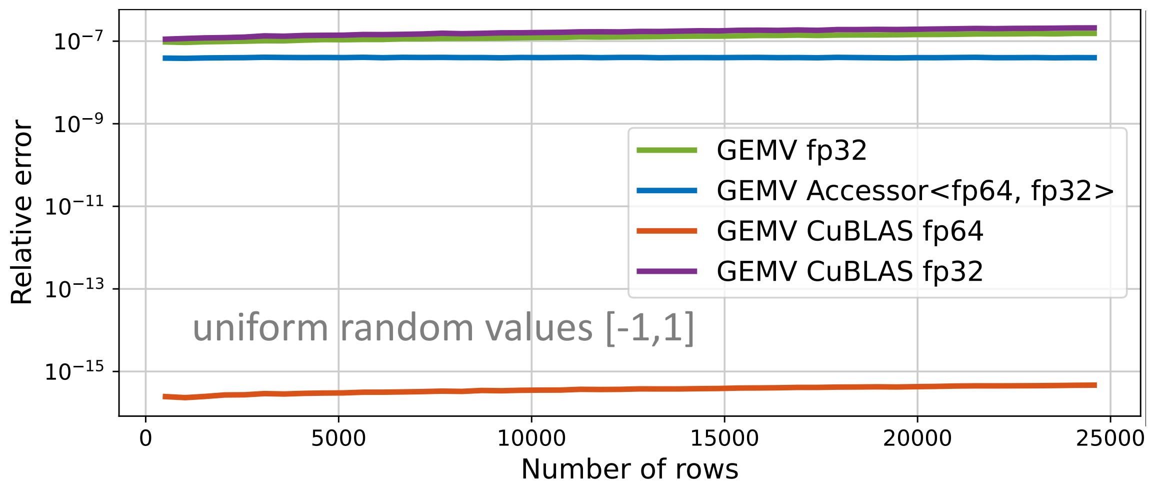


Accessor-BLAS: Replacing LP BLAS to improve accuracy

GEMV

NVIDIA V100 GPU (Summit)





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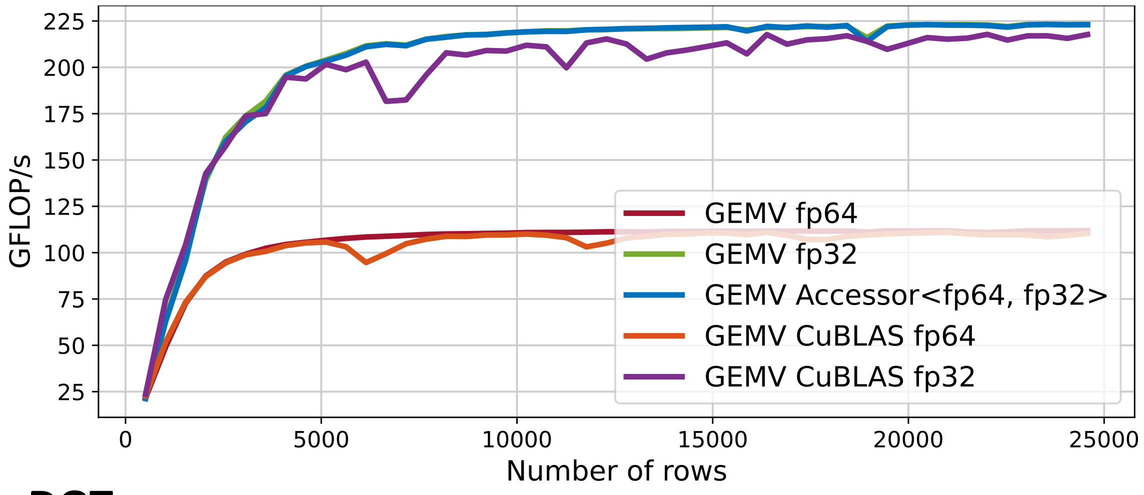
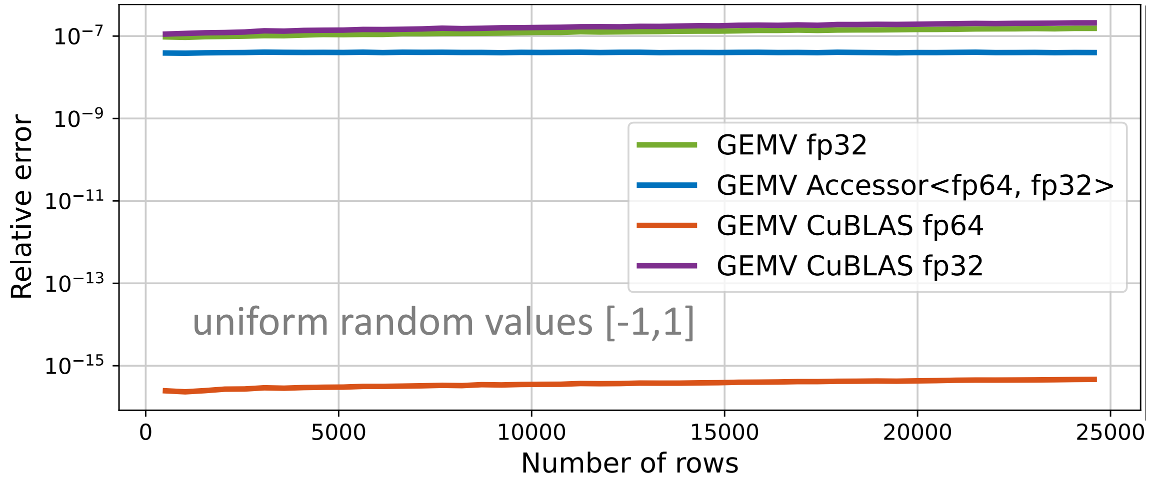
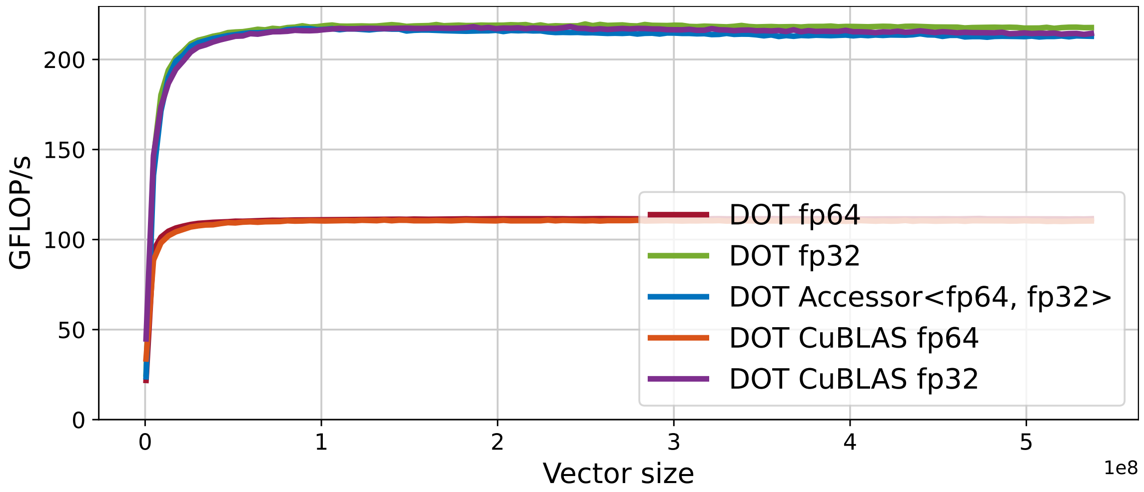
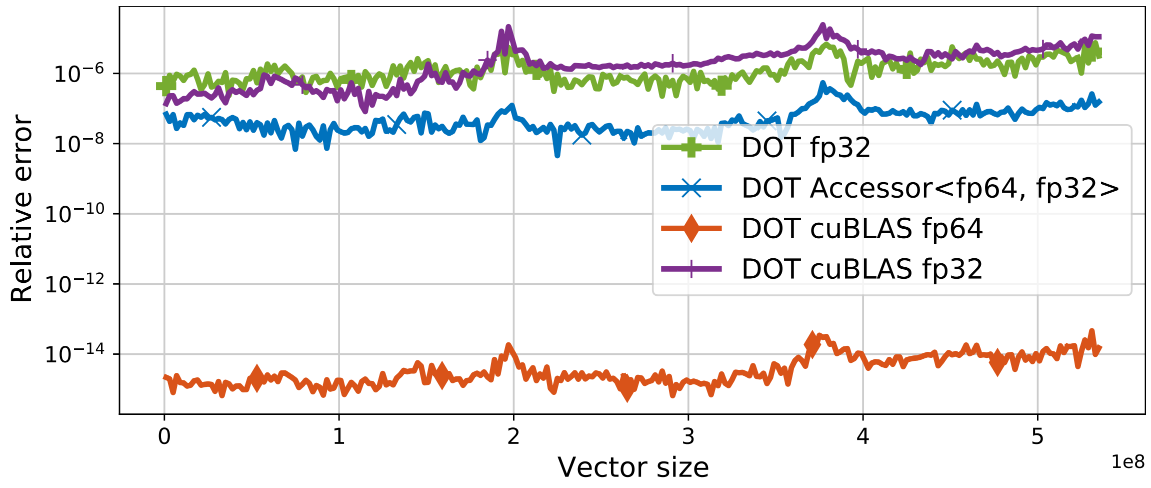




DOT





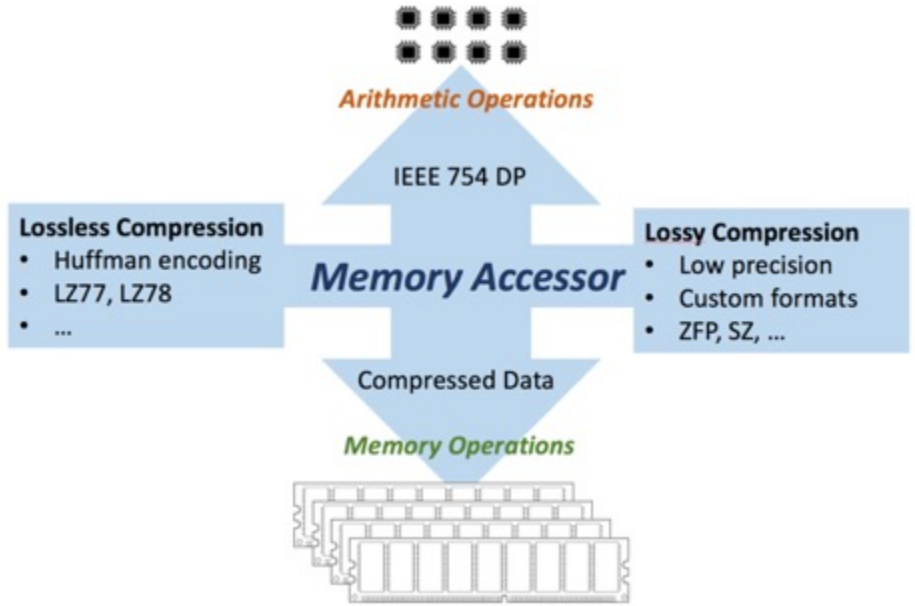
Rethinking Algorithms: Use the memory accessor to boost performance

Can we use the memory accessor to accelerate applications by compressing data without changing the application output?

- No, not in general.
- Yes, in some cases.
- We need to adapt the approach to the application & data.

Two possibilities in the context of solving linear systems:

- *"Self-healing" iterative methods;*
- *Approximate linear operators;*

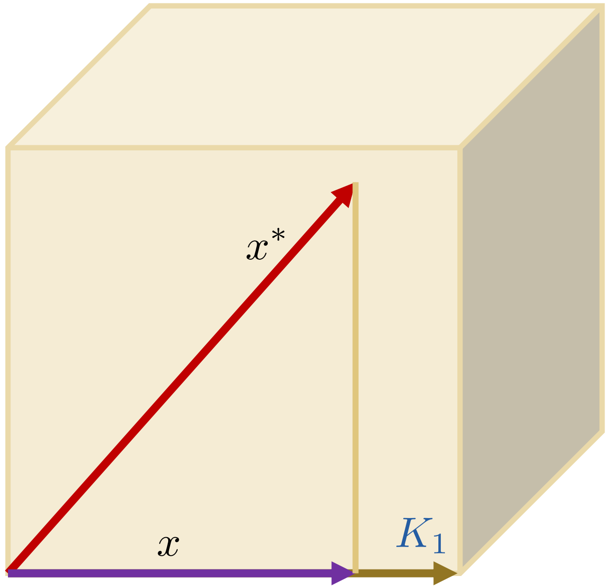


Rethinking Algorithms I: Self-Healing Iterative Methods

- **Krylov iterative solvers**
- Krylov methods aim at approximating the solution to a linear problem in a subspace.
- Over the iterations, a nested sequence of Krylov subspaces is generated, adding one basis vector in each iteration.
- Orthonormalization ensures a orthonormal basis is formed (Classical Gram-Schmidt, Modified Gram Schmidt…).

$$K_0 \subset K_1 \subset K_2 \subset \ldots$$

$$K_i(A, r) = span\{b, Ab, A^2b, \ldots A^{i-1}b\}$$



*GMRES in the modular precision ecosystem

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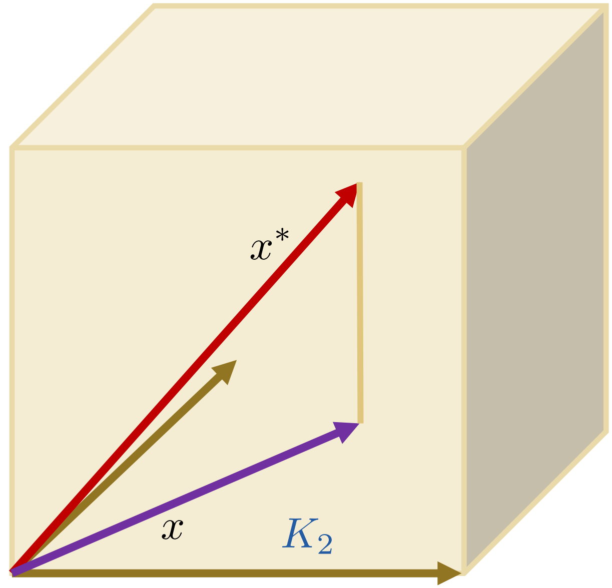

**GMRES in the modular precision ecosystem*

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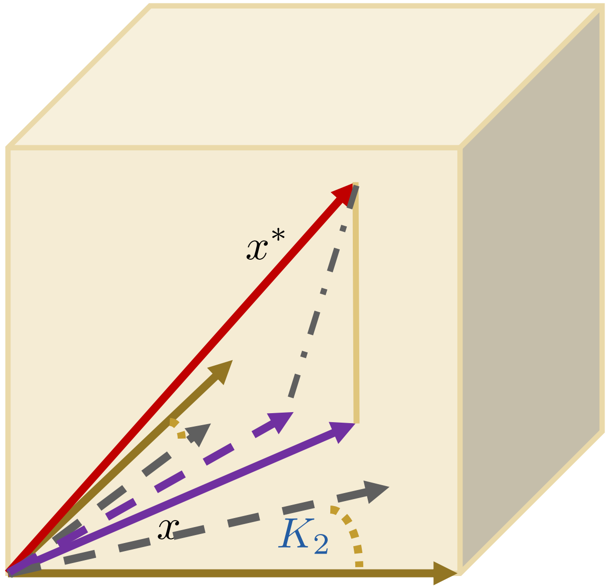

Compressed Basis (CB-) GMRES

- Use double precision in all arithmetic operations;
- Store Krylov basis vectors in lower precision;
    - Search directions are no longer DP-orthogonal;
    - Hessenberg system maps solution to “perturbed” Krylov subspace;
    - Additional iterations may be needed;
    - As long as the loss-of-orthogonality is moderate, we should see moderate convergence degradation;

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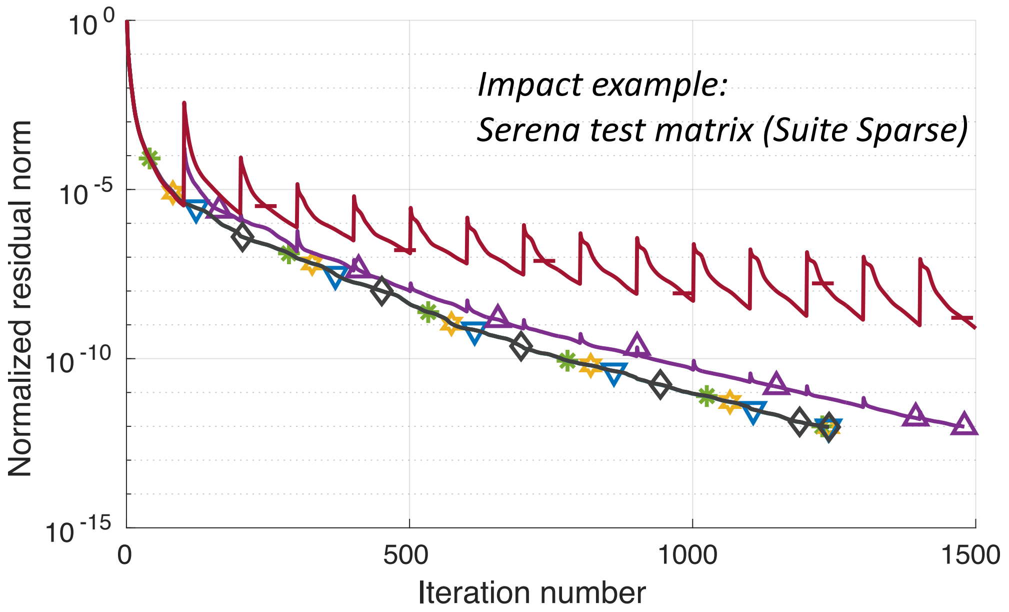
https://ginkgo-project.github.io/

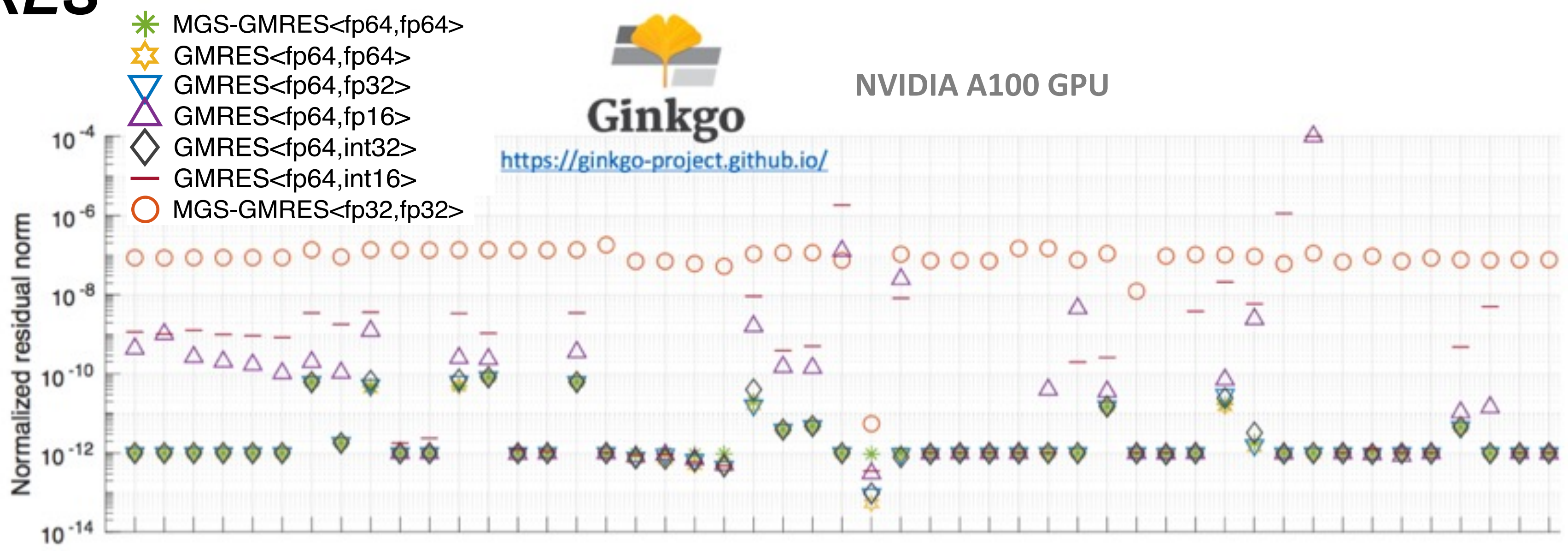
MGS-GMRES<fp64,fp64>
GMRES<fp64,fp64>
GMRES<fp64,fp32>
GMRES<fp64,fp16>
GMRES<fp64,int32>
GMRES<fp64,int16>

arithmetic precision memory precision



Compressed Basis GMRES

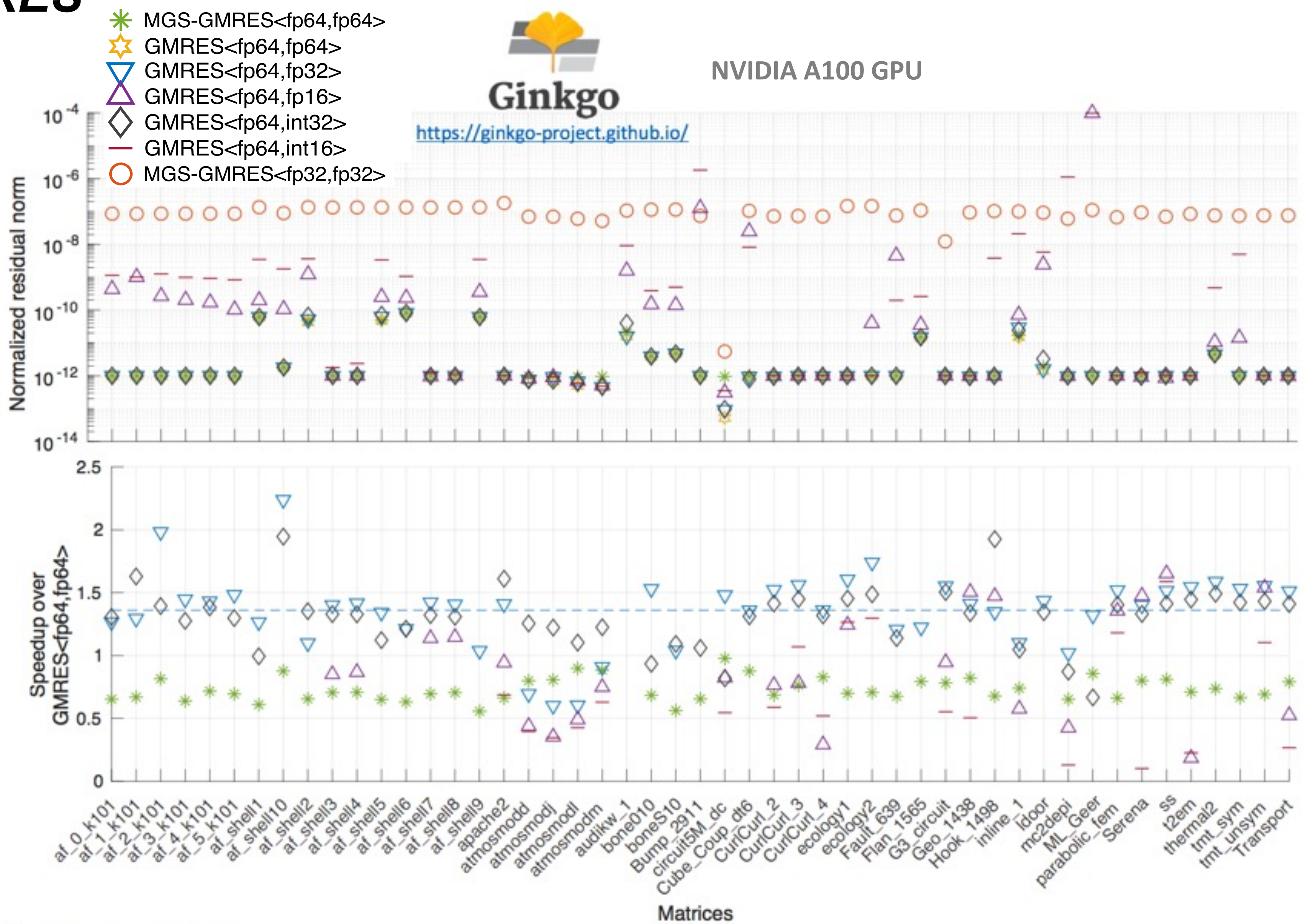
- CB-GMRES using 32-bit storage preserves DP accuracy (SP-GMRES does not)



Compressed Basis GMRES

- CB-GMRES using 32-bit storage preserves DP accuracy (SP-GMRES does not)

- Speedups problem-dependent
- Speedup Ø1.4x (for restart 100)
- 16-bit storage mostly inefficient



Ginkgo
https://ginkgo-project.github.io/

Compressed Basis GMRES on High Performance GPUs

José I. Aliaga, Hartwig Anzt, Thomas Grützmacher, Enrique S. Quintana-Ortí, Andrés E. Tomás

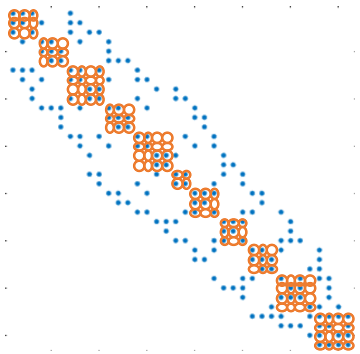
https://arxiv.org/abs/2009.12101

Rethinking Algorithms II: Approximate Linear Operators

Rethinking Algorithms II: Approximate Linear Operators

- **Preconditioning iterative solvers.**
  - Idea: Approximate inverse of system matrix to make the system “easier to solve”: $P^{-1} \approx A^{-1}$
    and solve $Ax = b \quad \Leftrightarrow \quad P^{-1}Ax = P^{-1}b \quad \Leftrightarrow \quad \tilde{A}x = \tilde{b}$.

- **Block-Jacobi preconditioner** is based on **block-diagonal scaling**: $P = diag_B(A)$
  - Each block corresponds to one (small) linear system.
    - *Larger* blocks typically improve convergence.
    - *Larger* blocks make block-Jacobi more expensive.



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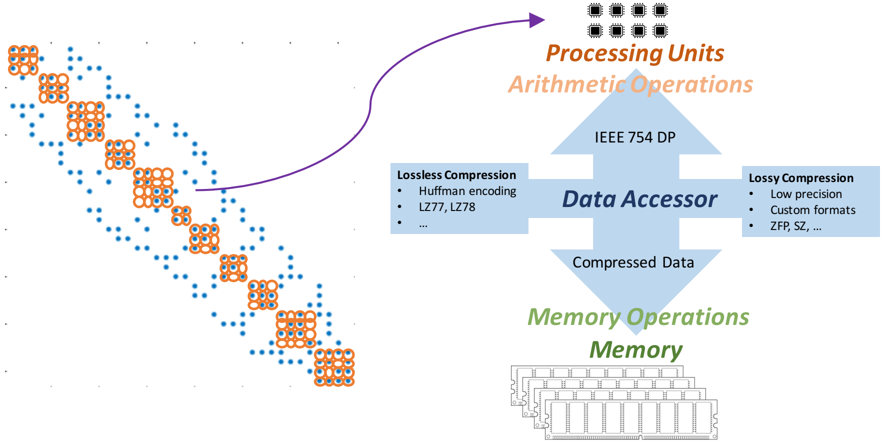
  - Each block corresponds to one (small) linear system.
    - *Larger* blocks typically improve convergence.
    - *Larger* blocks make block-Jacobi more expensive.

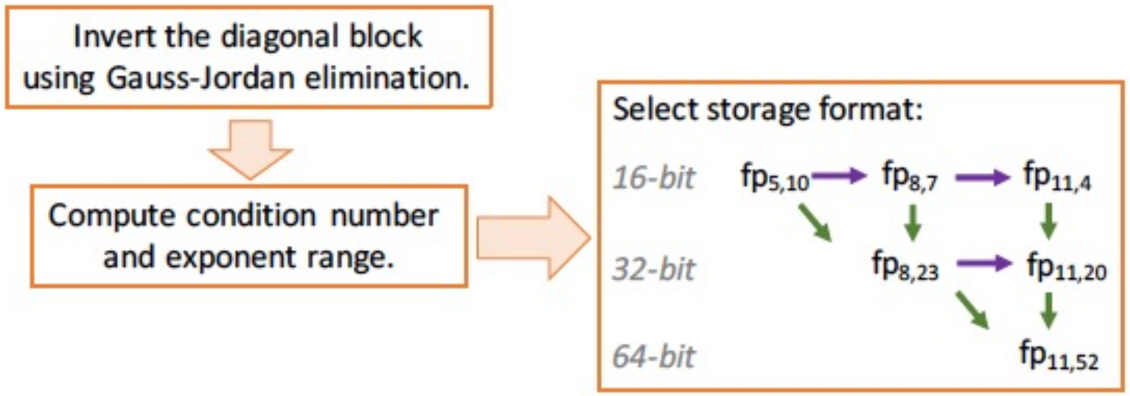
- *Why should we store the preconditioner matrix P^{-1} in full (high) precision?*

- Use the accessor to store the inverted diagonal blocks in lower precision.
  - *Be careful to preserve the regularity of each inverted diagonal block!*



Mixed Precision Preconditioning

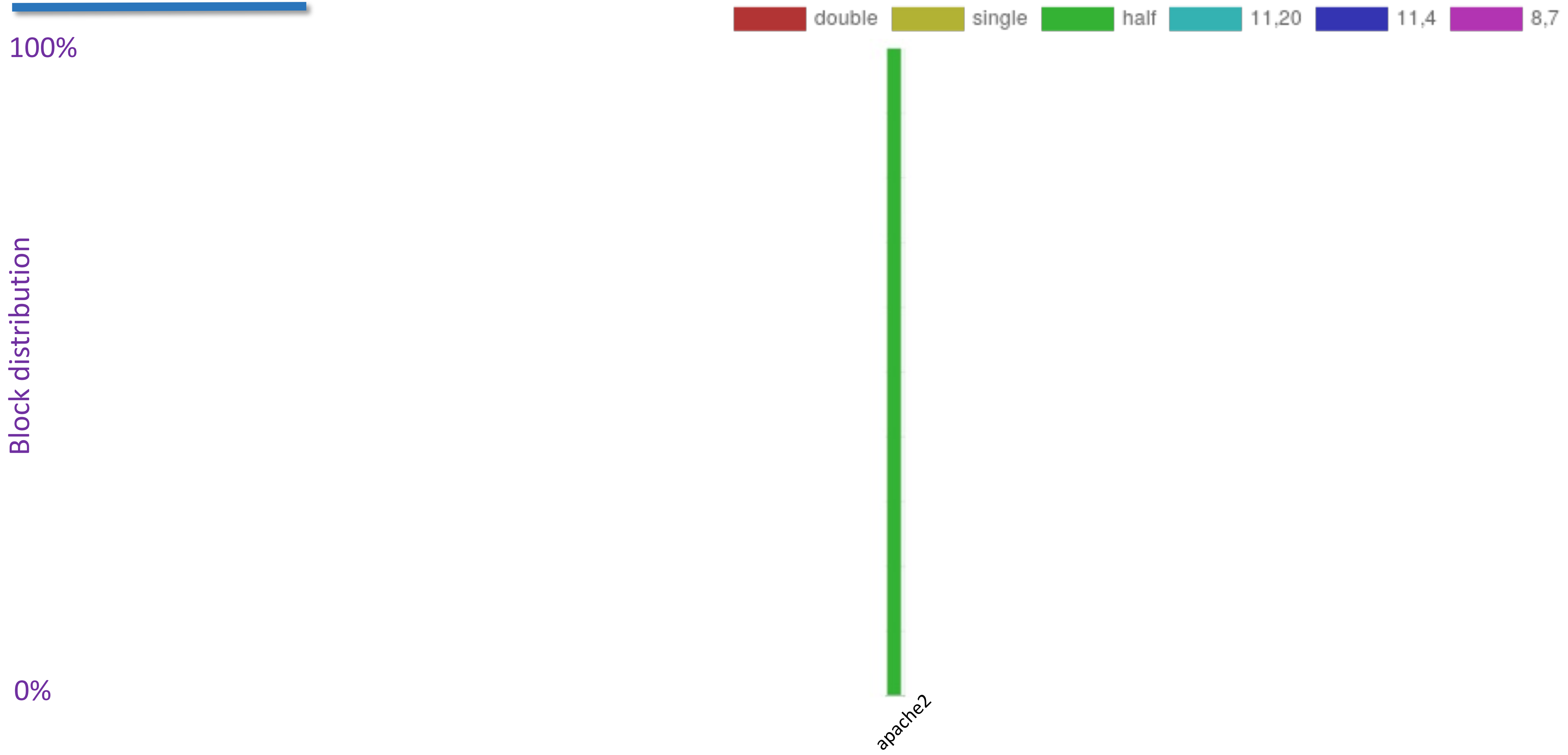
- Choose how much accuracy of the preconditioner should be preserved in the selection of the storage format.
- All computations use double precision, but store blocks in lower precision.



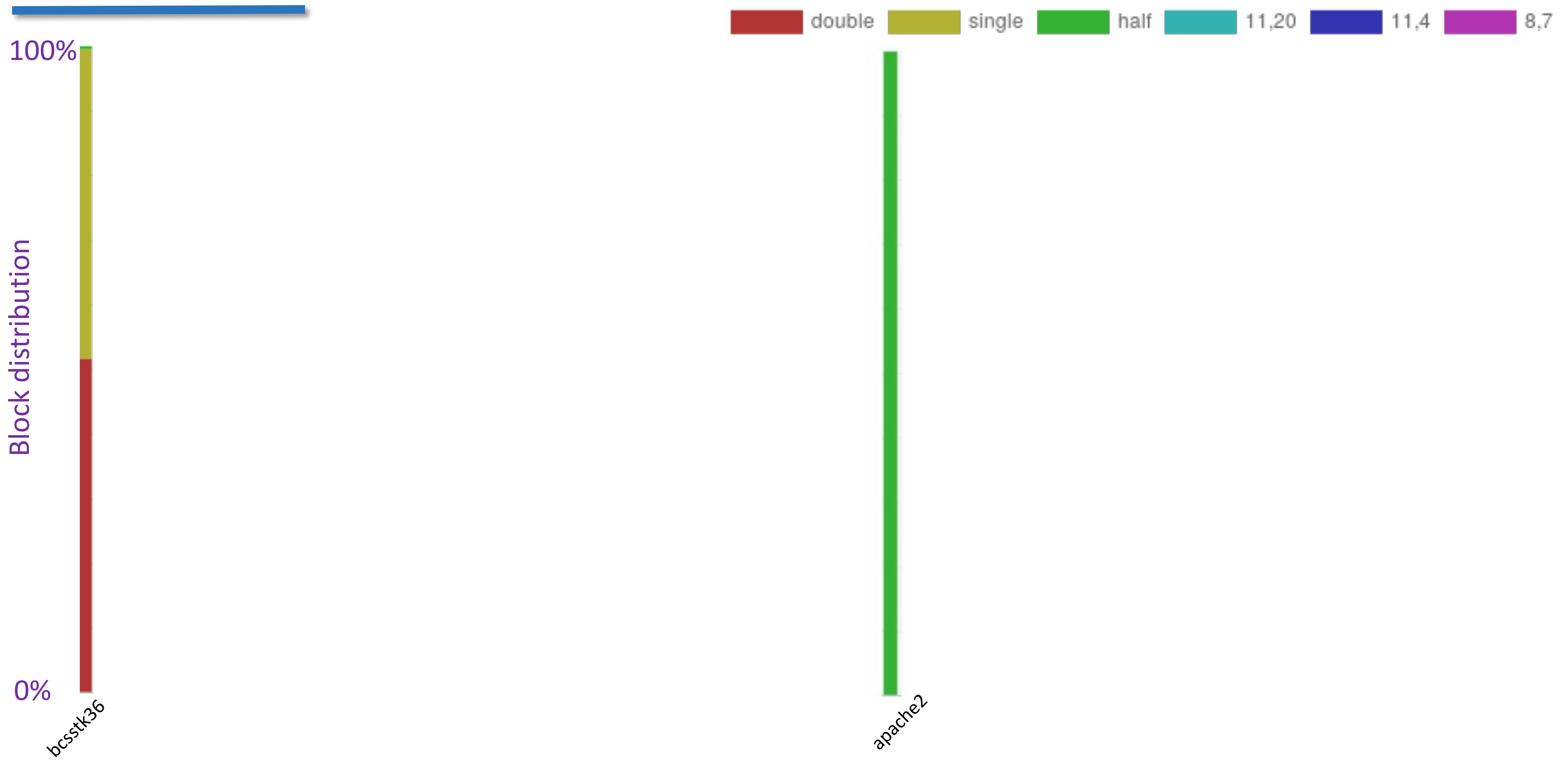
+ **Regularity preserved;**
+ Flexibility in the accuracy;
+ ”Not a low precision preconditioner”
    + Preconditioner is a constant operator;
    + No flexible Krylov solver needed ;

- **Overhead** of the **precision detection** (condition number calculation);
- **Overhead** from storing **precision information** (need to additionally store/retrieve flag);
- Speedups / preconditioner quality **problem-dependent**;

Mixed Precision Preconditioning



Mixed Precision Preconditioning



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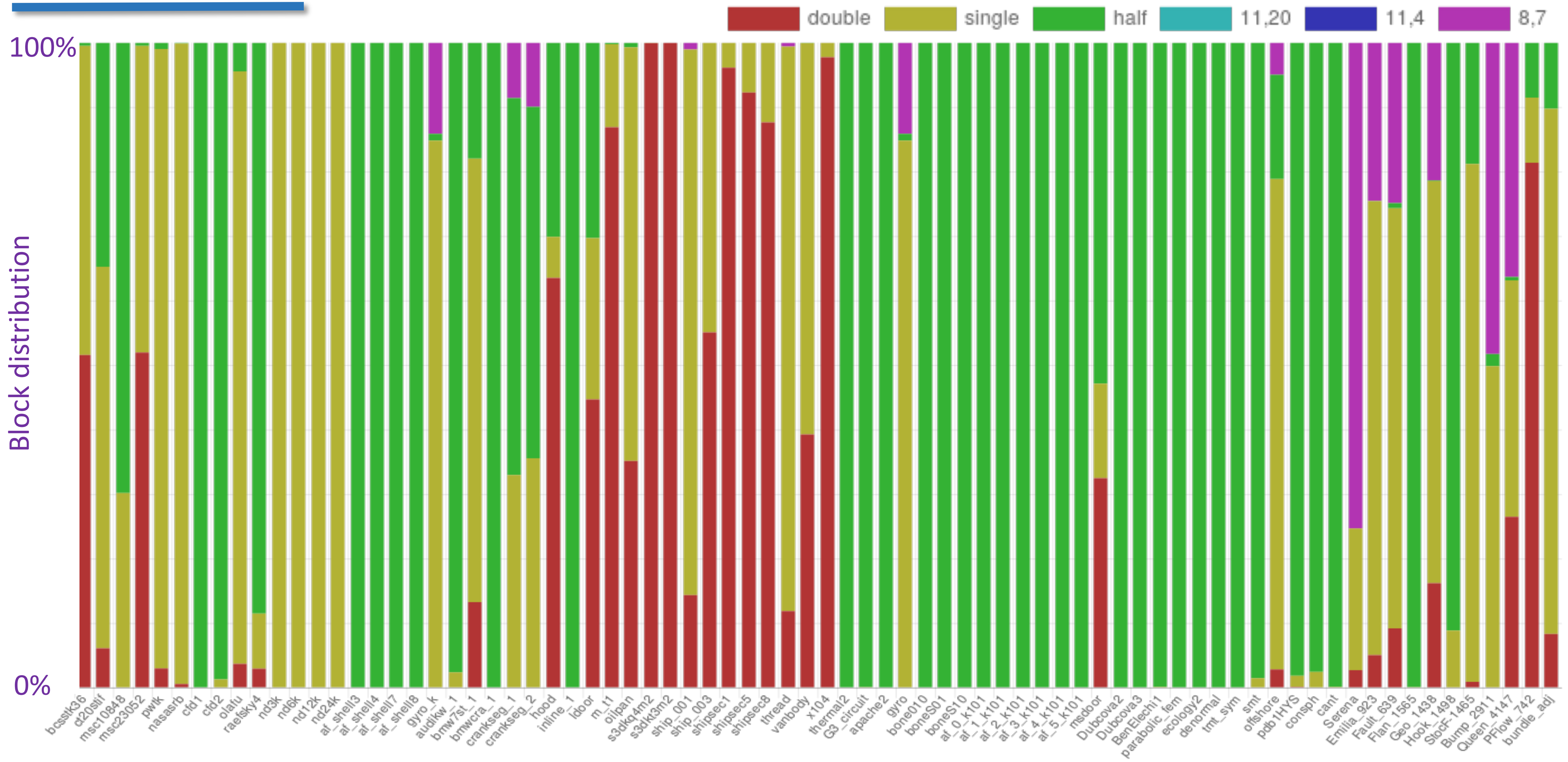

Mixed Precision Preconditioning

Linear System Ax=b with cond(A) ≈ 10^7 *(apache2 from SuiteSparse)* **NVIDIA A100 GPU**

Double Precision CG + Double Precision Preconditioner

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Initial residual norm sqrt(r^T r):
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CG execution time [ms]: 2971.18
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Accuracy improvement ~10^9

Single Precision CG + Single Precision Preconditioner

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No improvement

Experiments based on the Ginkgo library https://ginkgo-project.github.io/
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Mixed Precision Preconditioning

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Double Precision CG + **Mixed Precision Preconditioner**

- *Attainable accuracy of CG unaffected*
- *Preconditioner remains a constant operator*

Experiments based on the Ginkgo library https://ginkgo-project.github.io/
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1390.67
Final residual norm sqrt(r^T r):
%%MatrixMarket matrix array real general
1 1
3.98574e-06
CG iteration count:     4794
CG execution time [ms]: 2568.1
```
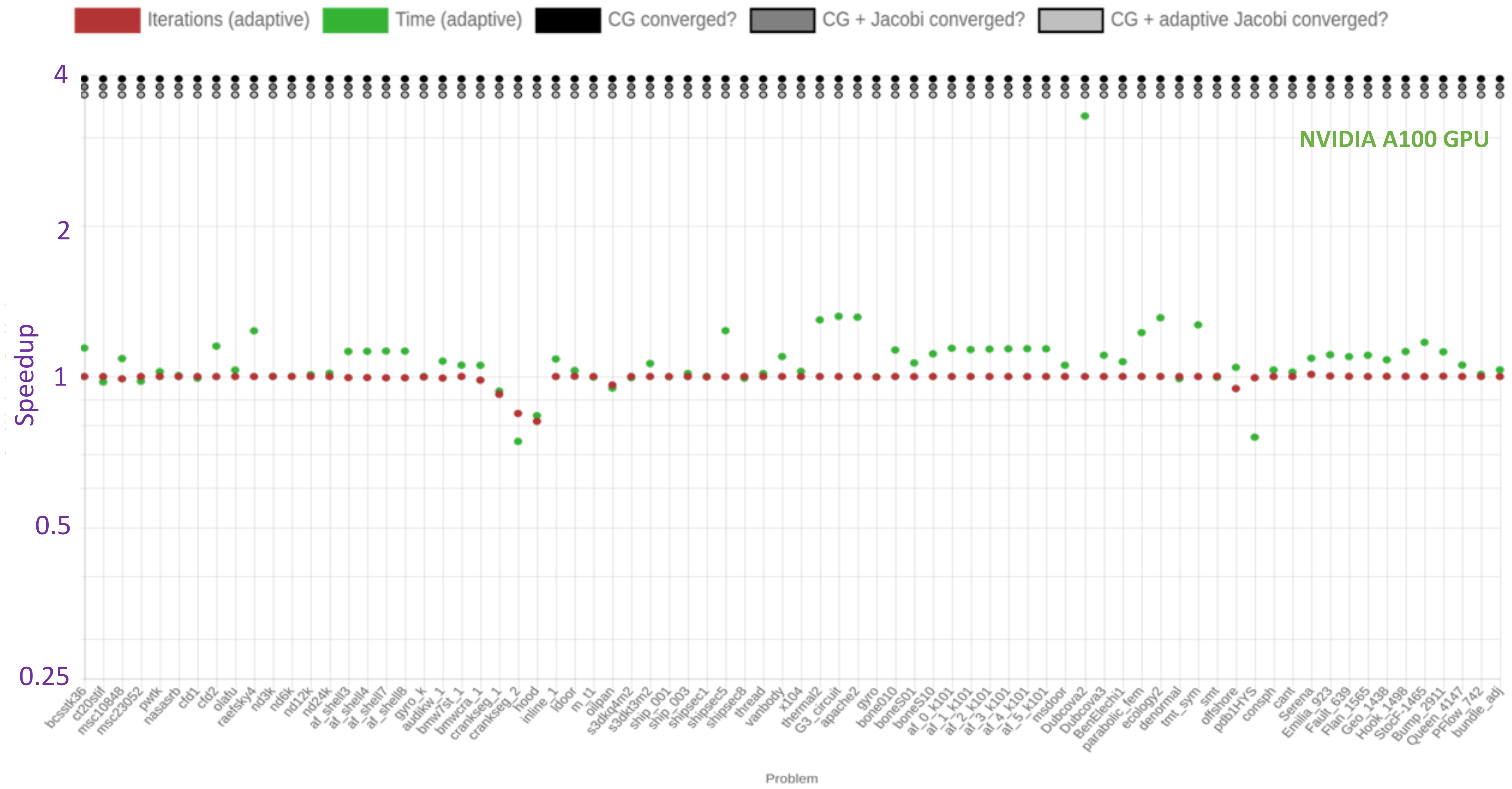
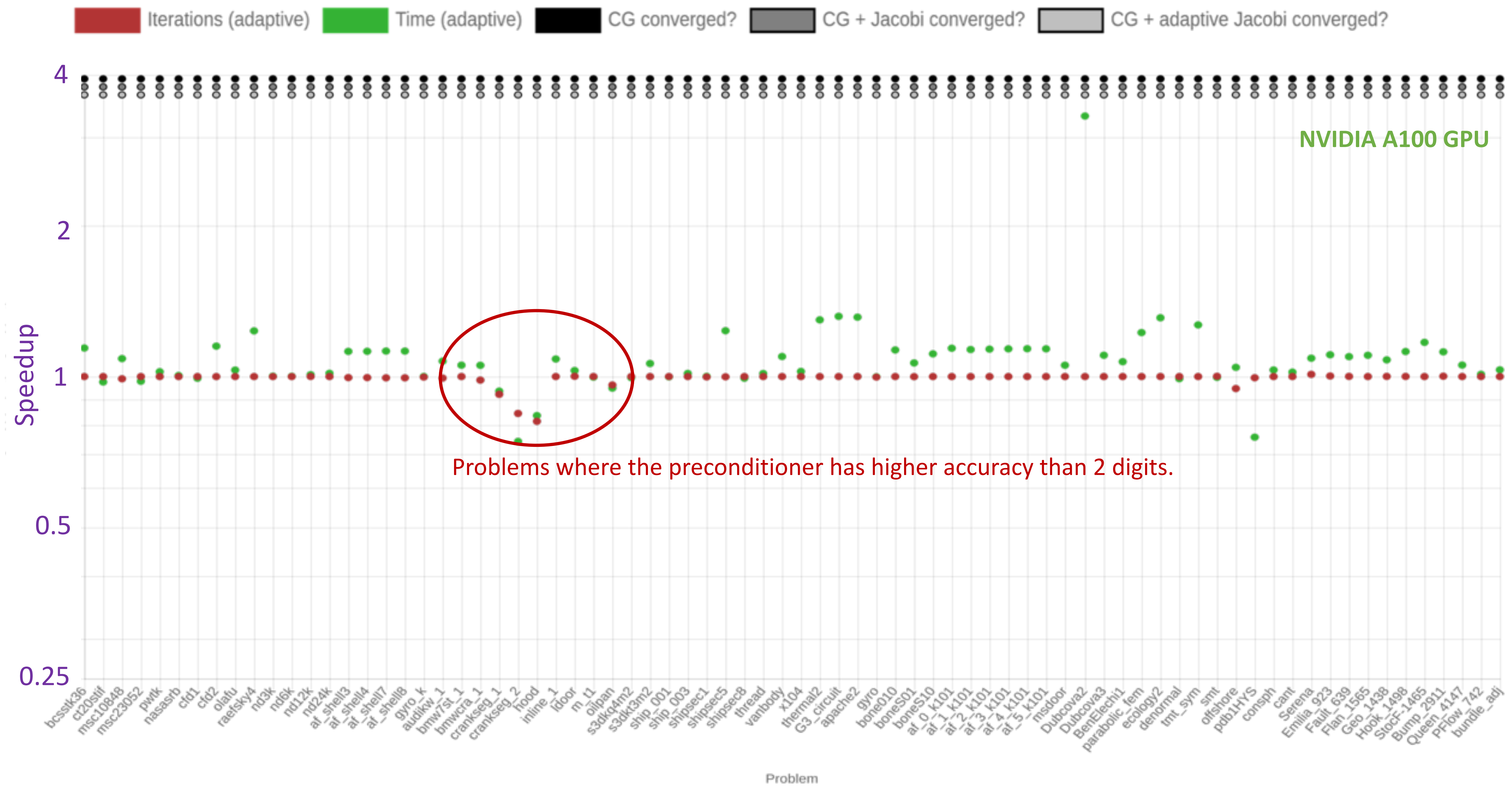
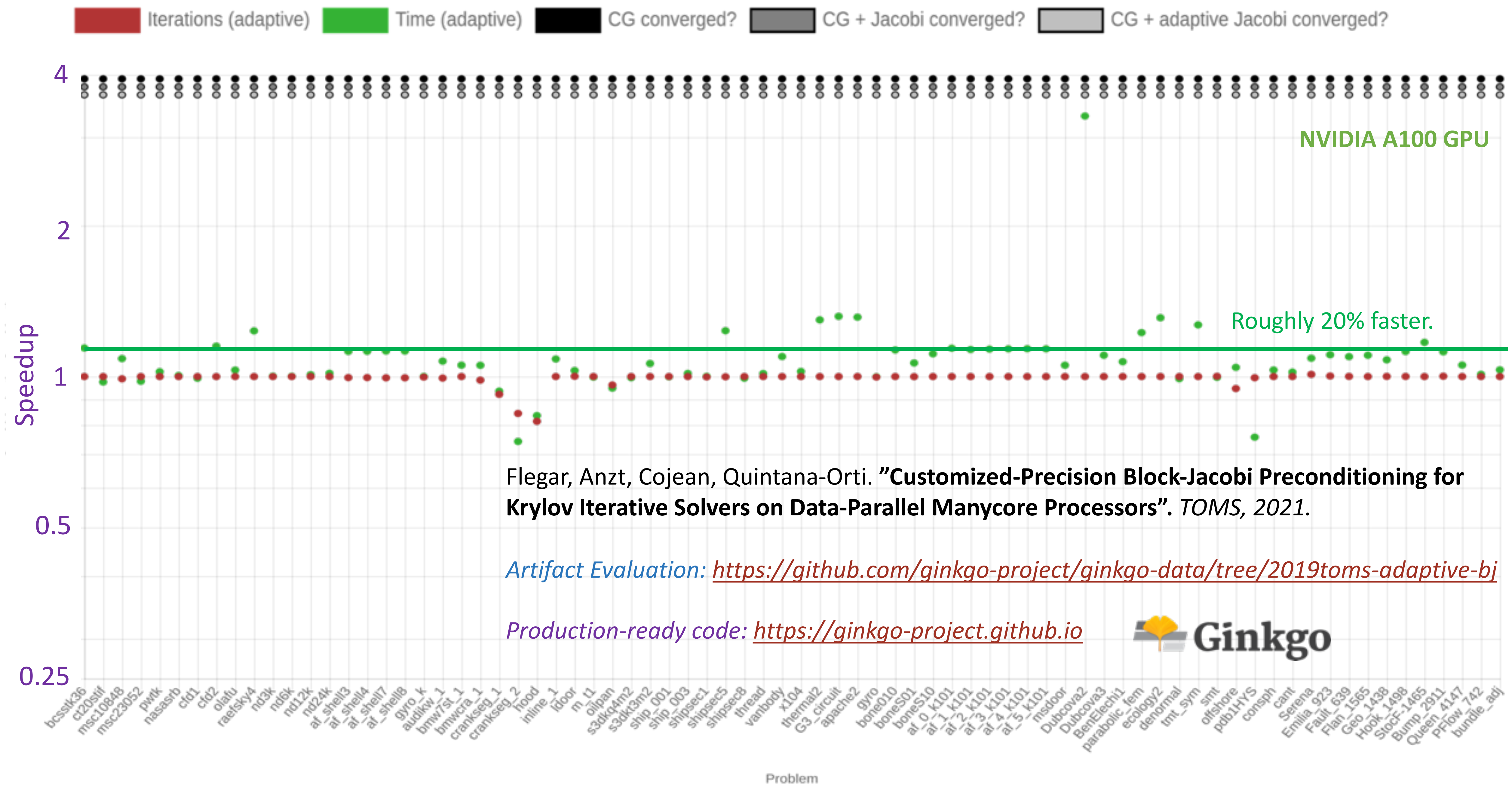

16% runtime improvement

Experiments based on the Ginkgo library https://ginkgo-project.github.io/
ginkgo/examples/adaptiveprecision-blockjacobi/adaptiveprecision-blockjacobi.cpp



Iterations (adaptive)
Time (adaptive)
CG converged?
CG + Jacobi converged?
CG + adaptive Jacobi converged?
NVIDIA A100 GPU
Speedup
4
2
1
0.5
0.25
Problem
bcsstk36
ct20stif
msc10848
msc23052
pwtk
nasasrb
cfd1
cfd2
olafu
raefsky4
nd3k
nd6k
nd12k
nd24k
af_shell3
af_shell4
af_shell7
af_shell8
gyro_k
audikw_1
bmw7st_1
bmwcra_1
crankseg_1
crankseg_2
hood
inline_1
ldoor
m_t1
oilpan
s3dkq4m2
s3dkt3m2
ship_001
ship_003
shipsec1
shipsec5
shipsec8
thread
vanbody
x104
thermal2
G3_circuit
apache2
gyro
bone010
boneS01
boneS10
af_0_k101
af_1_k101
af_2_k101
af_3_k101
af_4_k101
af_5_k101
msdoor
Dubcova2
Dubcova3
BenElechi1
parabolic_fem
ecology2
denormal
tmt_sym
smt
offshore
pdb1HYS
consph
cant
Serena
Emilia_923
Fault_639
Flan_1565
Geo_1438
Hook_1498
StocF-1465
Bump_2911
Queen_4147
PFlow_742
bundle_adj


Iterations (adaptive)
Time (adaptive)
CG converged?
CG + Jacobi converged?
CG + adaptive Jacobi converged?
NVIDIA A100 GPU
Speedup
4
2
1
0.5
0.25
Problems where the preconditioner has higher accuracy than 2 digits.
Problem



Flegar, Anzt, Cojean, Quintana-Orti. **"Customized-Precision Block-Jacobi Preconditioning for Krylov Iterative Solvers on Data-Parallel Manycore Processors".** *TOMS, 2021.*

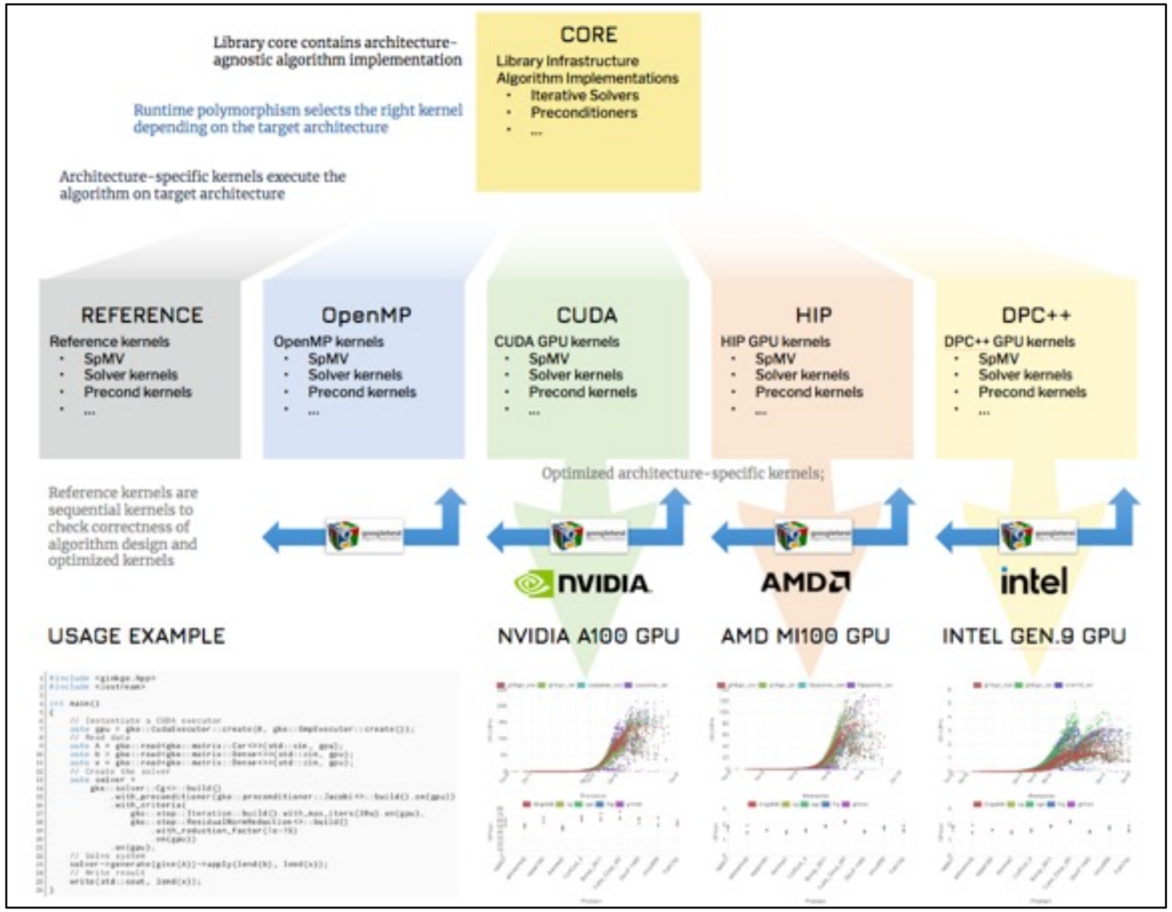
Artifact Evaluation: *https://github.com/ginkgo-project/ginkgo-data/tree/2019toms-adaptive-bj*

Production-ready code: *https://ginkgo-project.github.io*

Ginkgo

Ginkgo

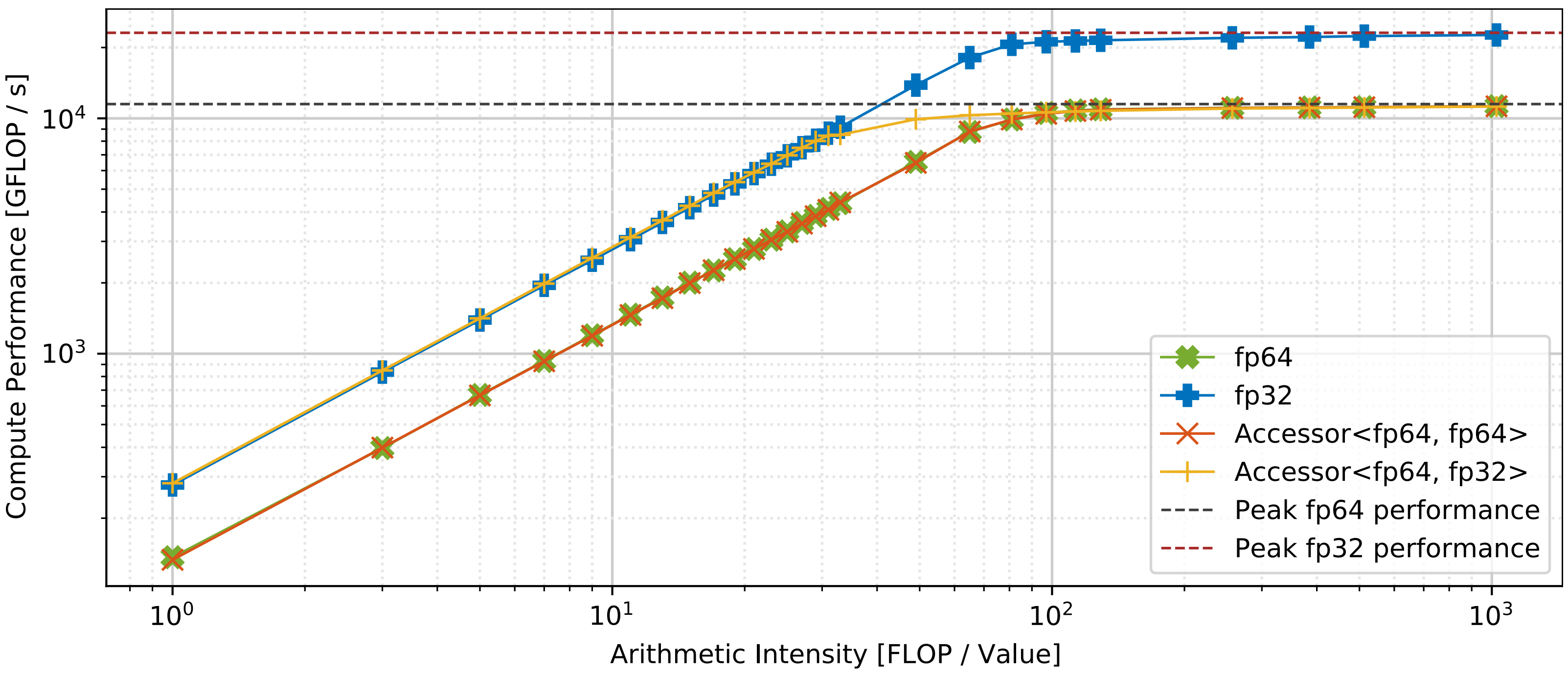
- **High performance sparse linear algebra**
  - Linear algebra building blocks: SpMV, SpMM, SpGEAM,...;
  - Linear solvers: BiCG, BiCGSTAB, CG, CGS, FCG, GMRES, IDR;
  - Advanced preconditioning techniques: ParILU, ParILUT, SAI;
  - Batched iterative solvers;
- **Exascale early systems GPU-readiness**
  - Available: Nvidia GPU (CUDA), AMD GPU (HIP), Intel GPU (DPC++), CPU Multithreading (OpenMP);
  - C++, CMake build;
- **Open source, community-driven**
  - Freely available (BSD License), GitHub, and Spack;
  - Part of the **xSDK and E4S software stack**;
  - Can be used from **deal.II** and **MFEM**;
- **Modular precision ecosystem**
  - Decoupling of arithmetic precision and memory precision;
  - Compressed Basis (CB) Krylov methods;
  - Mixed precision algorithms: adaptive precision Jacobi, FSPAI;



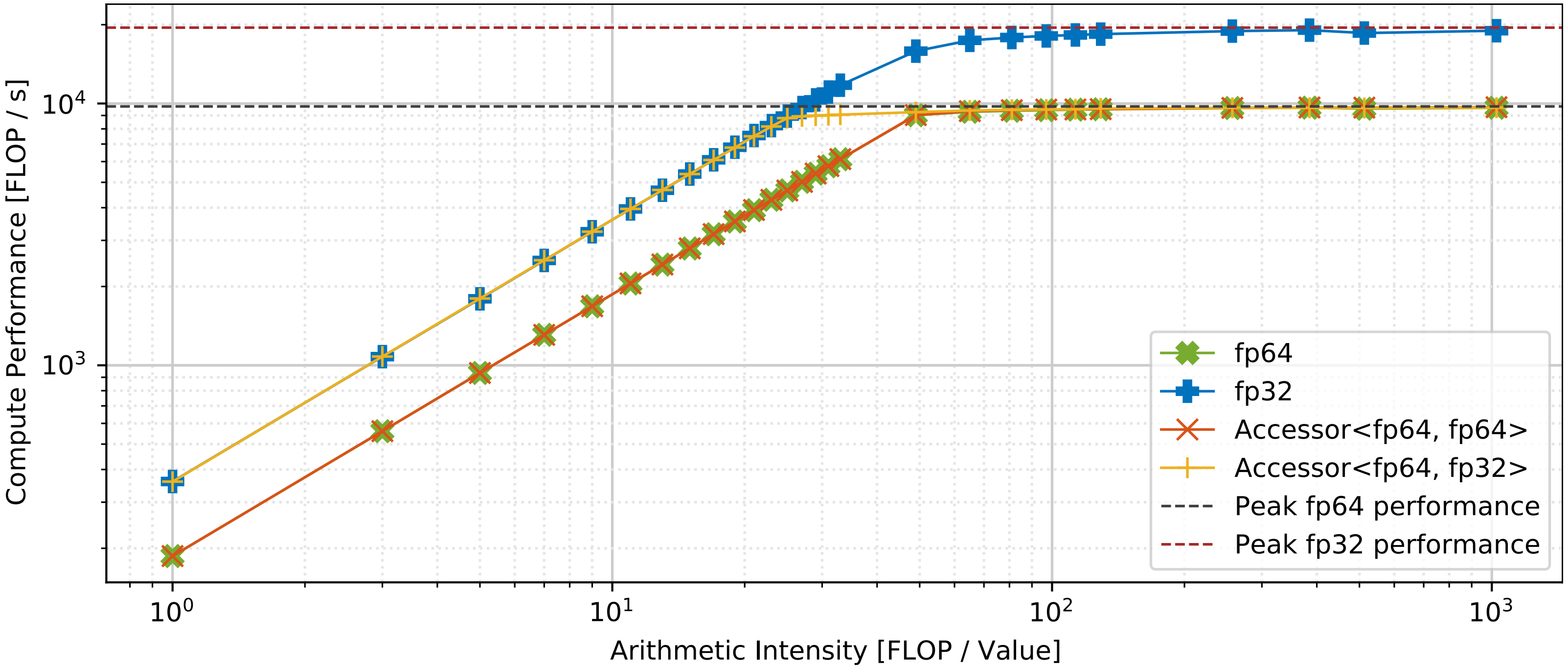
https://ginkgo-project.github.io/

This research was supported by the Exascale Computing Project (17-SC-20-SC), a collaborative effort of the U.S. Department of Energy Office of Science and the National Nuclear Security Administration and the Helmholtz Impuls und Vernetzungsfond VH-NG-1241.

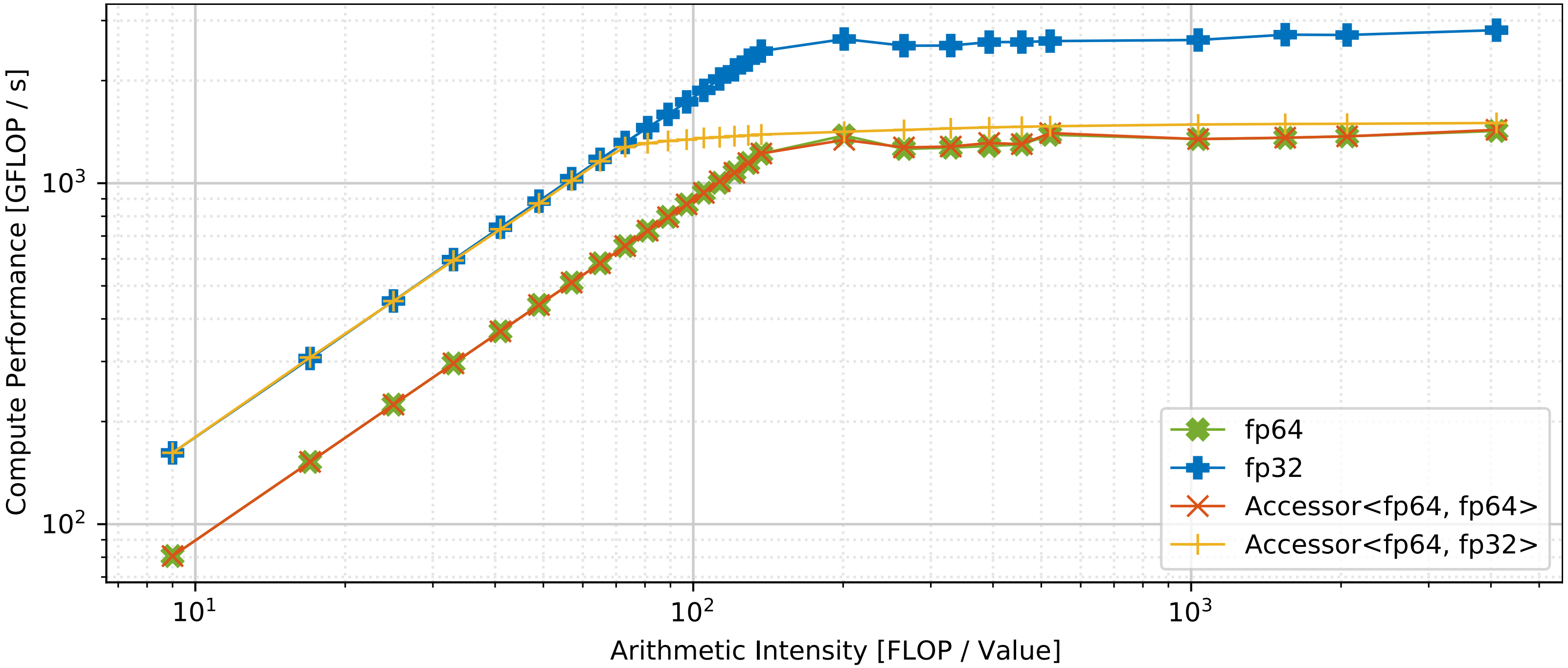
Accessor for AMD MI100 GPU



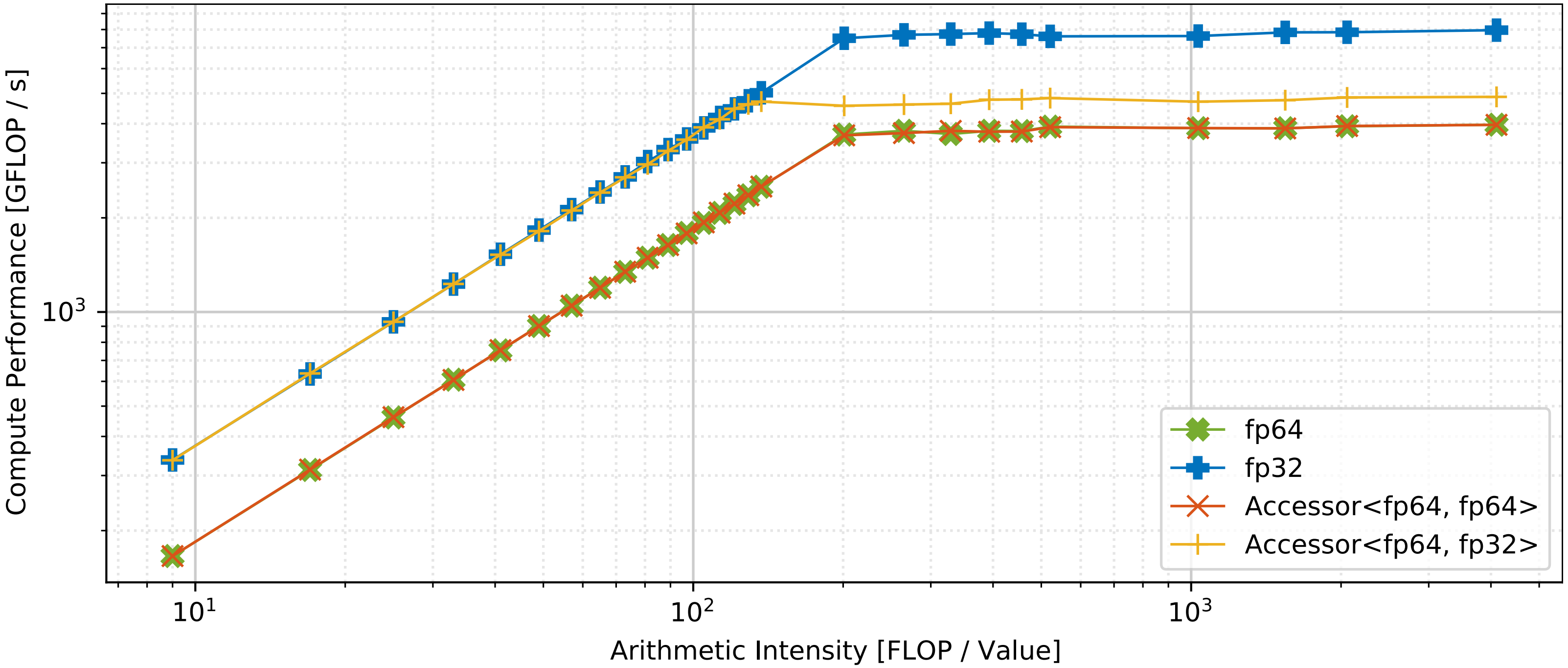
Accessor for NVIDIA A100 GPU



Accessor for Intel Skylake CPU



Accessor for AMD EPYC CPU



Memory accessor for memory-bound routines

Design

- Memory access in low precision (e.g. fp32);
- Computations in high precision (e.g. fp64);

Characteristics

- Performance of low precision routine;
- Higher accuracy of low precision routine;

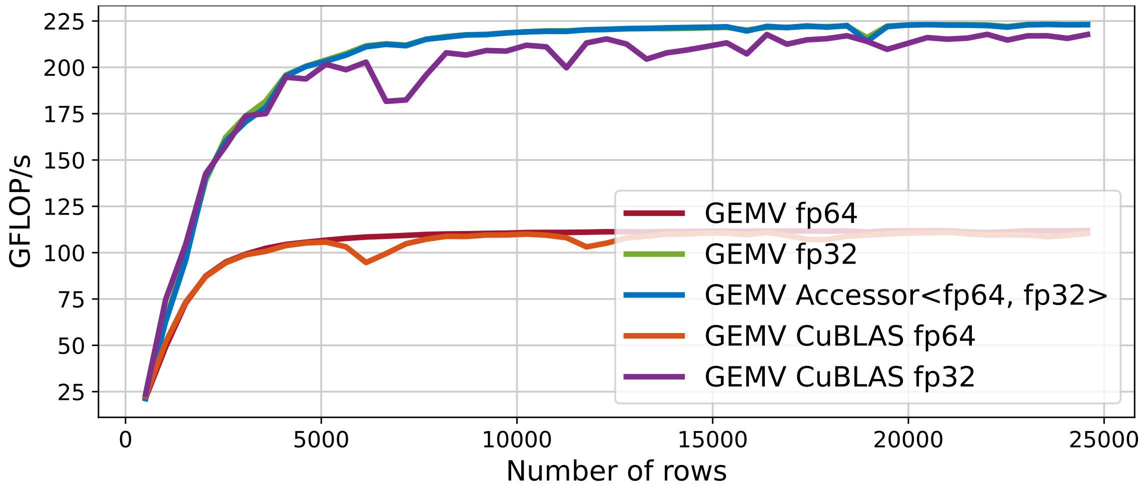
Usage

1. Accessor-BLAS can replace low precision BLAS to increase accuracy;
2. Accessor-BLAS can replace high precision BLAS if information loss is acceptable;
   (without having to deal with explicit mixed precision usage)

Accessor-BLAS: Replacing LP BLAS to improve accuracy

GEMV

NVIDIA V100 GPU (Summit)



Accessor-BLAS: Replacing LP BLAS to improve accuracy

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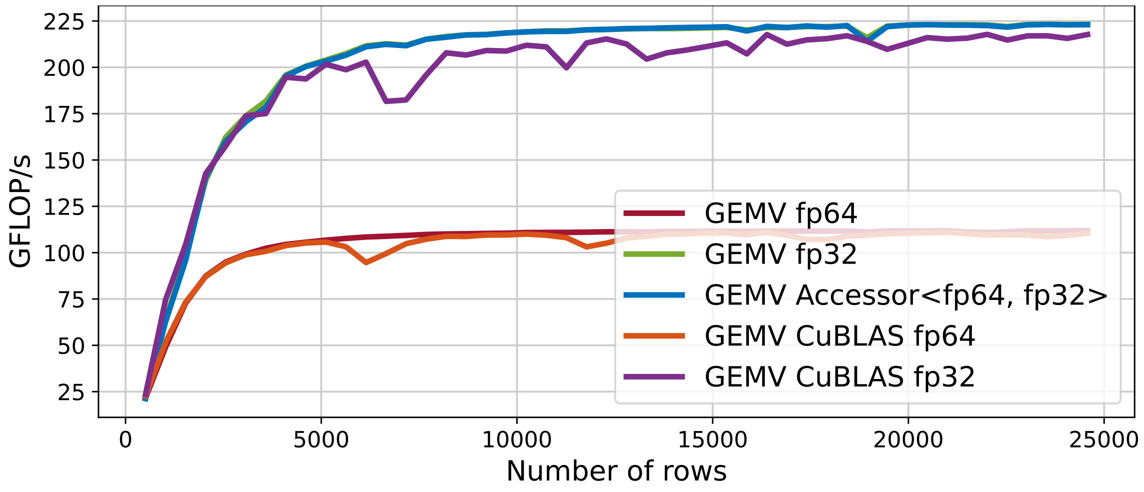

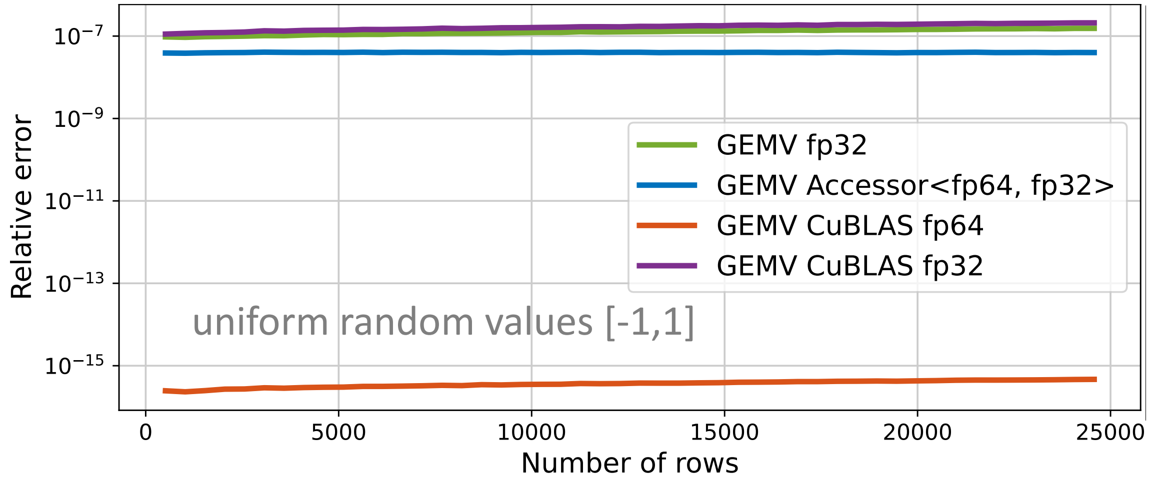

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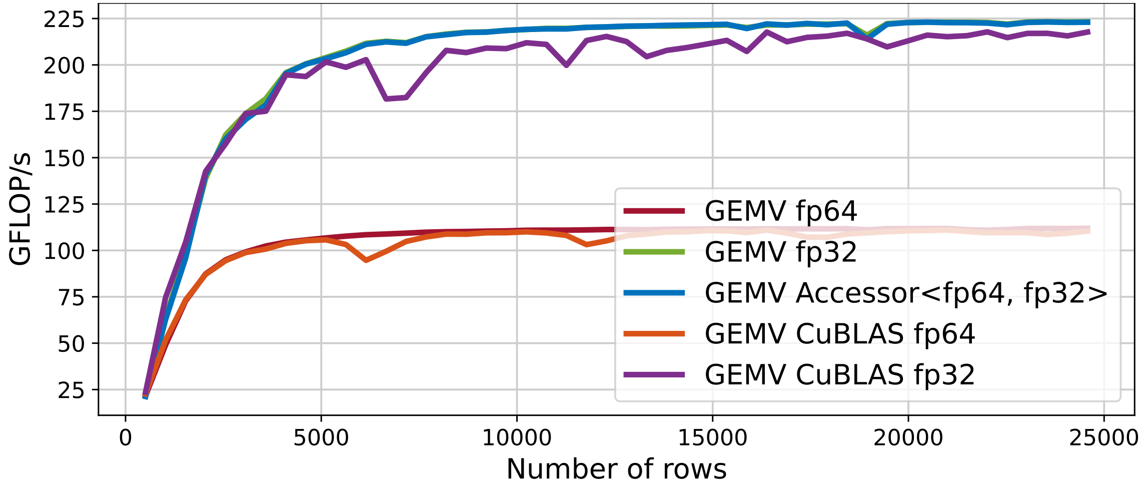
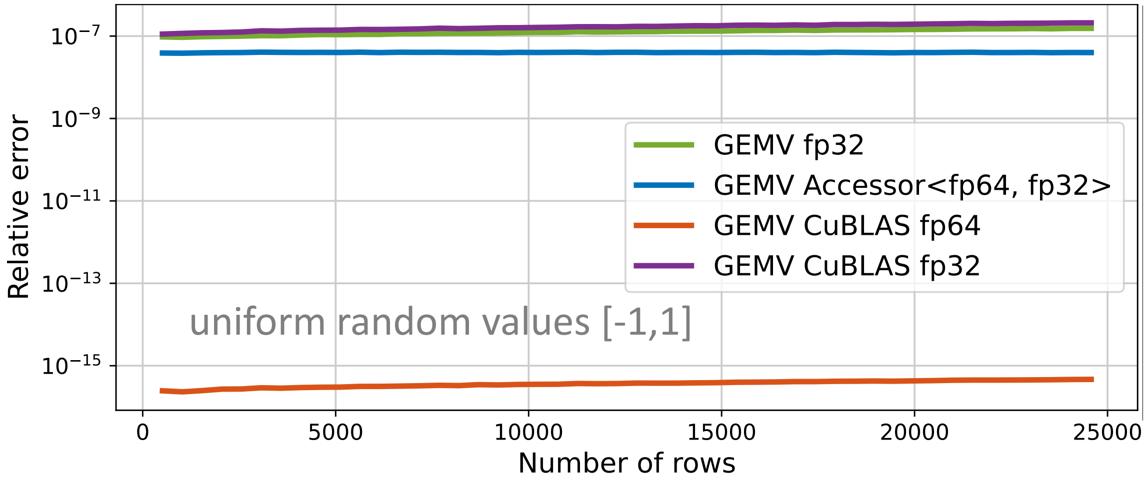
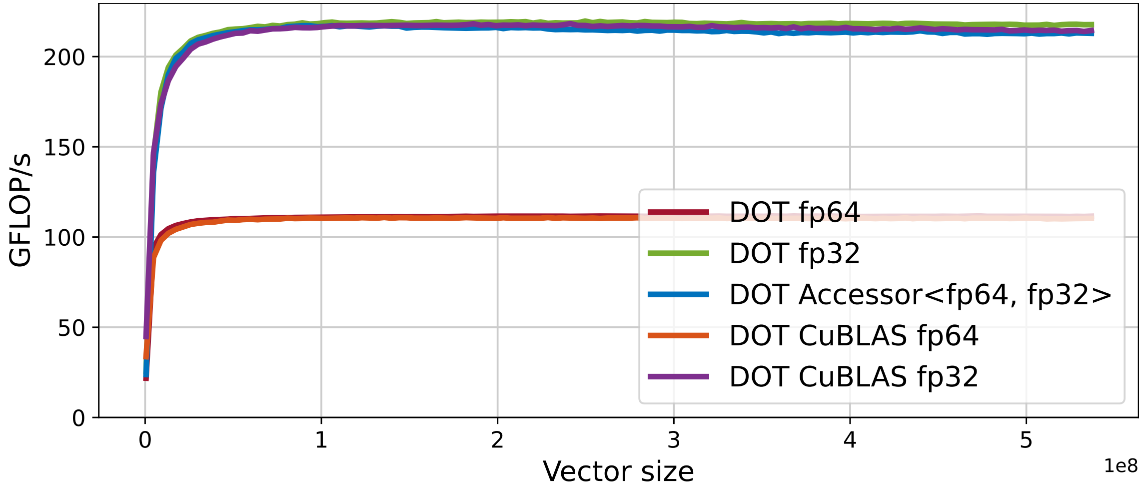
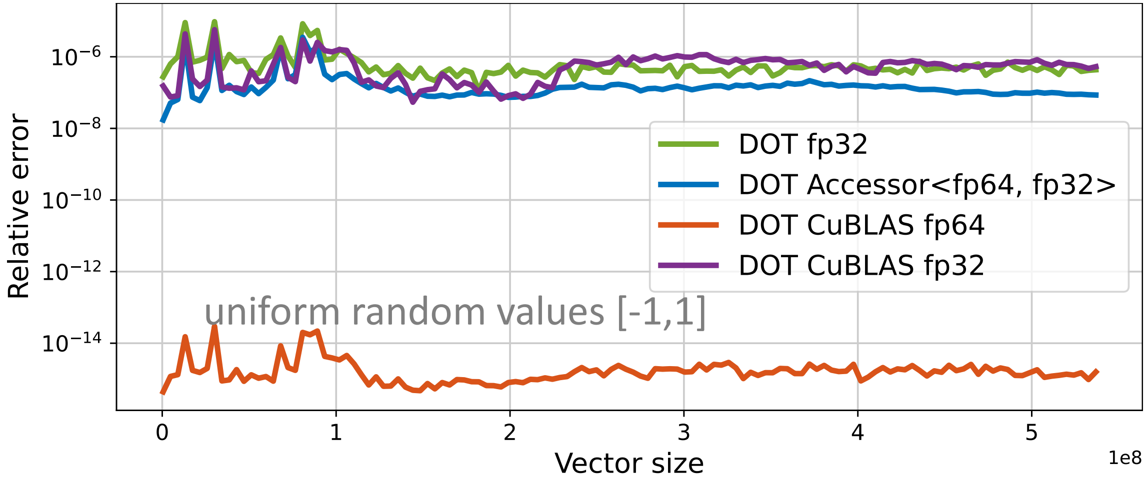




DOT





Accessor-BLAS: Replacing LP BLAS to improve accuracy

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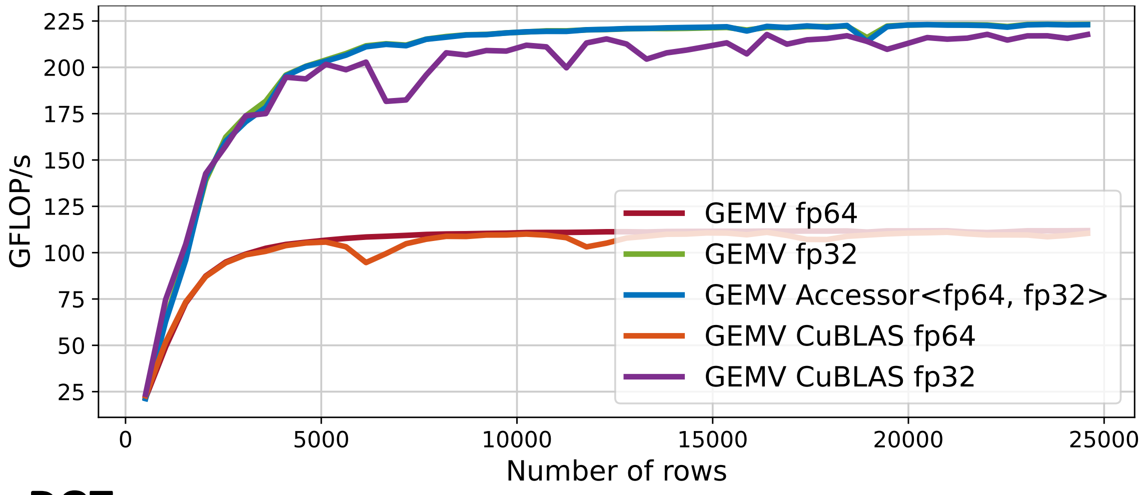

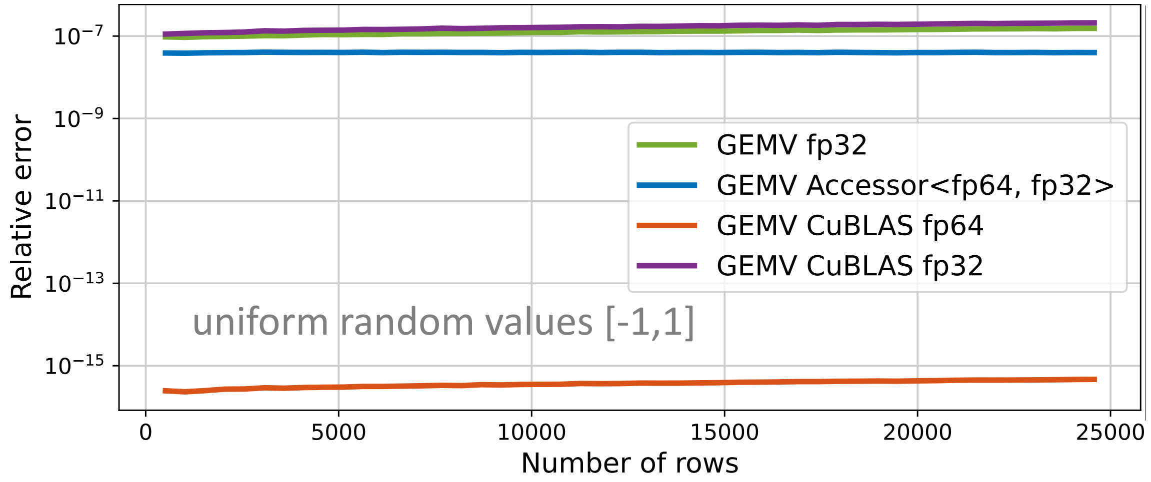

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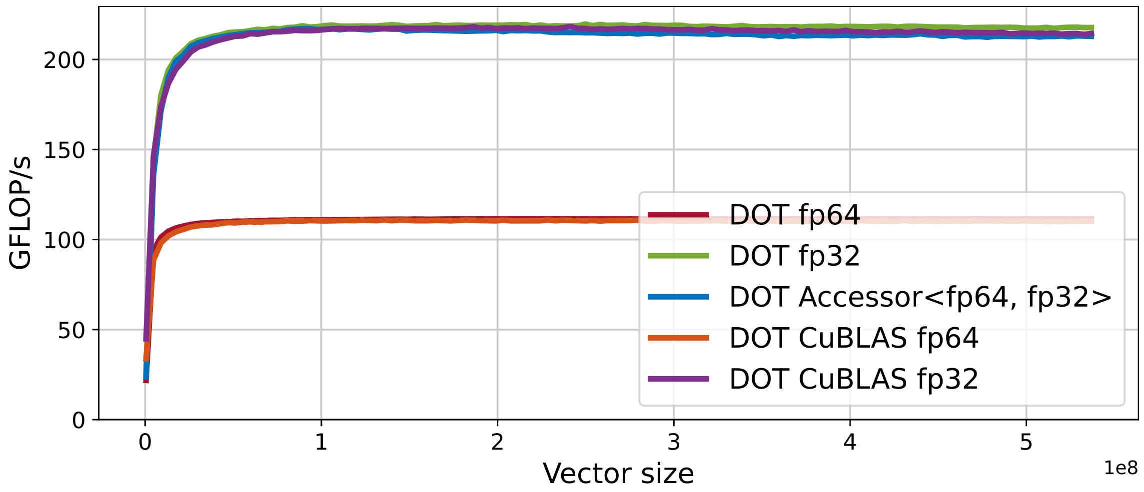
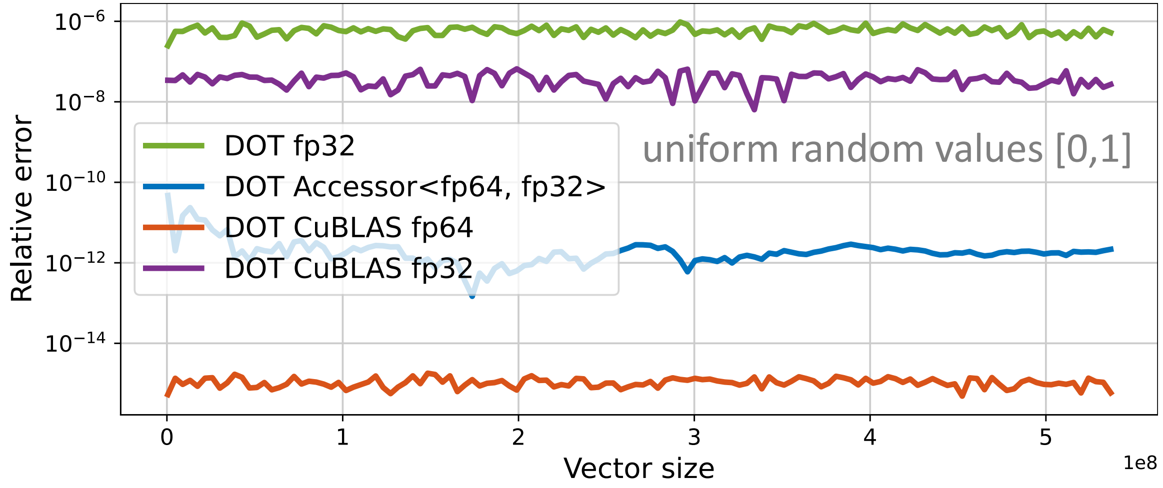




Mixed Precision Sparse Approximate Inverse

- **Preconditioning iterative solvers.**
  - Idea: Approximate inverse of system matrix to make the system "easier to solve": $P^{-1} \approx A^{-1}$

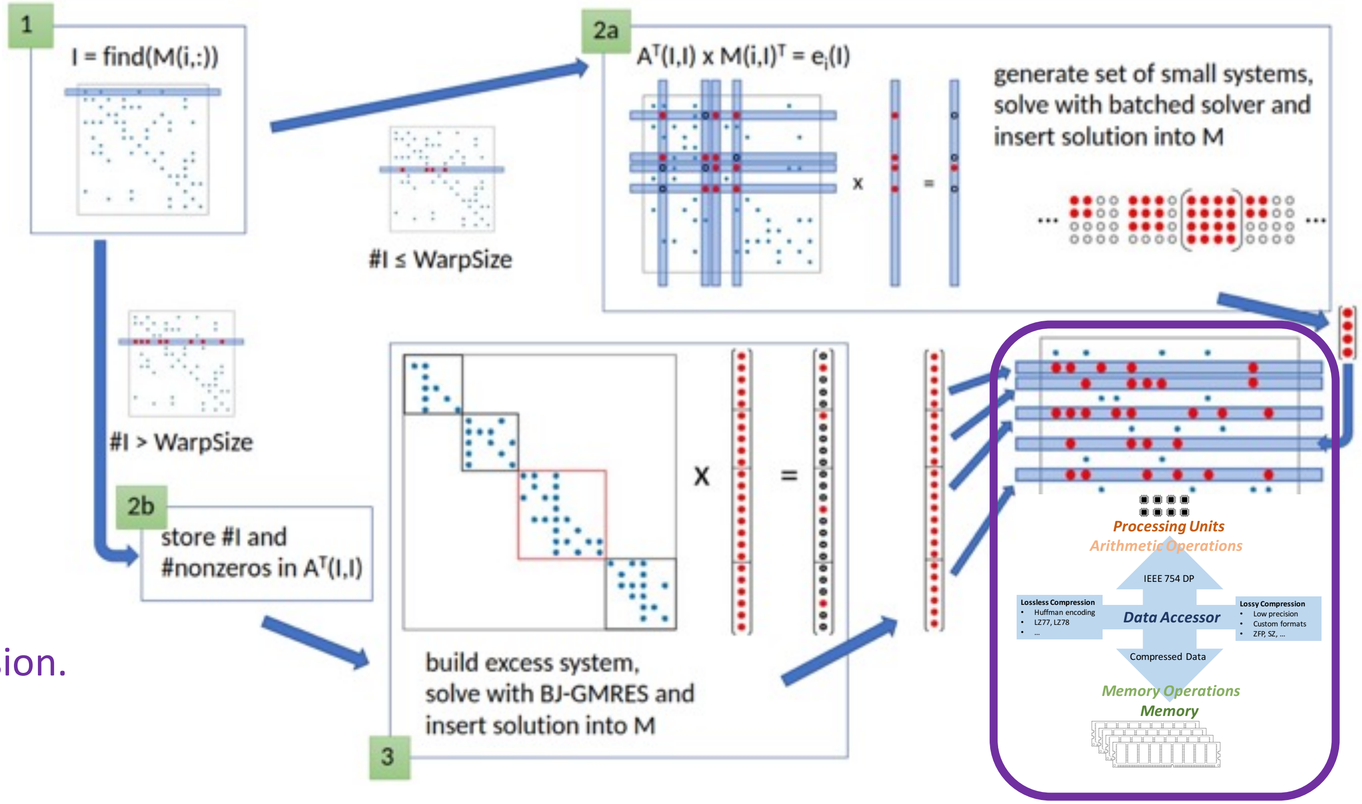
    and solve $Ax = b \quad \Leftrightarrow \quad P^{-1}Ax = P^{-1}b \quad \Leftrightarrow \quad \tilde{A}x = \tilde{b}\,.$

- **Sparse Approximate Inverse Preconditioner**
  - $M \approx A^{-1}$ and sparse
  - Incomplete Sparse Approximate Inverse (ISAI)

    uses sparsity pattern of A;
  - Factorized Sparse Approximate Inverse (FSPAI)

    stores inverse approximation in factorized form;

- Use the accessor to store the preconditioner in lower precision.



Göbel at al: Mixed Precision Incomplete and Factorized Sparse Approximate Inverse Preconditioning on GPUs, EuroPar submitted.

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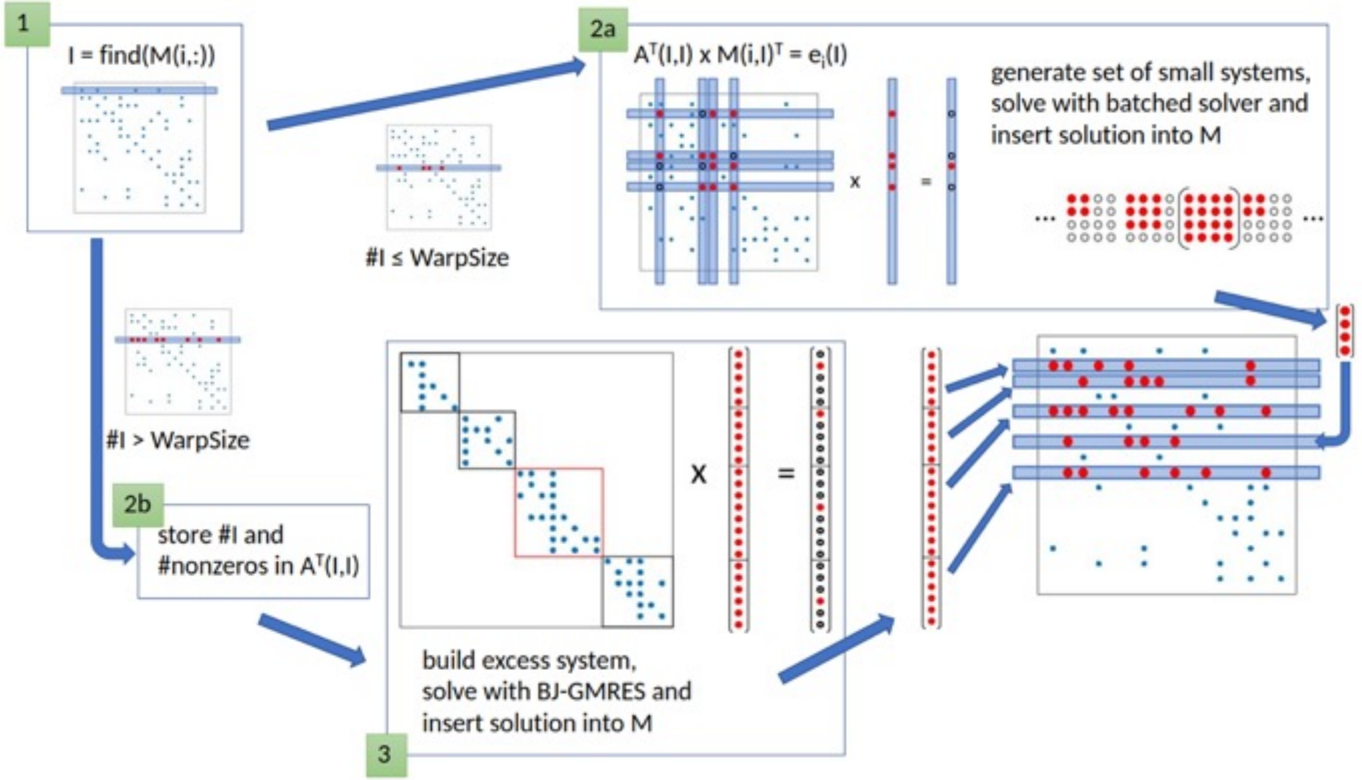

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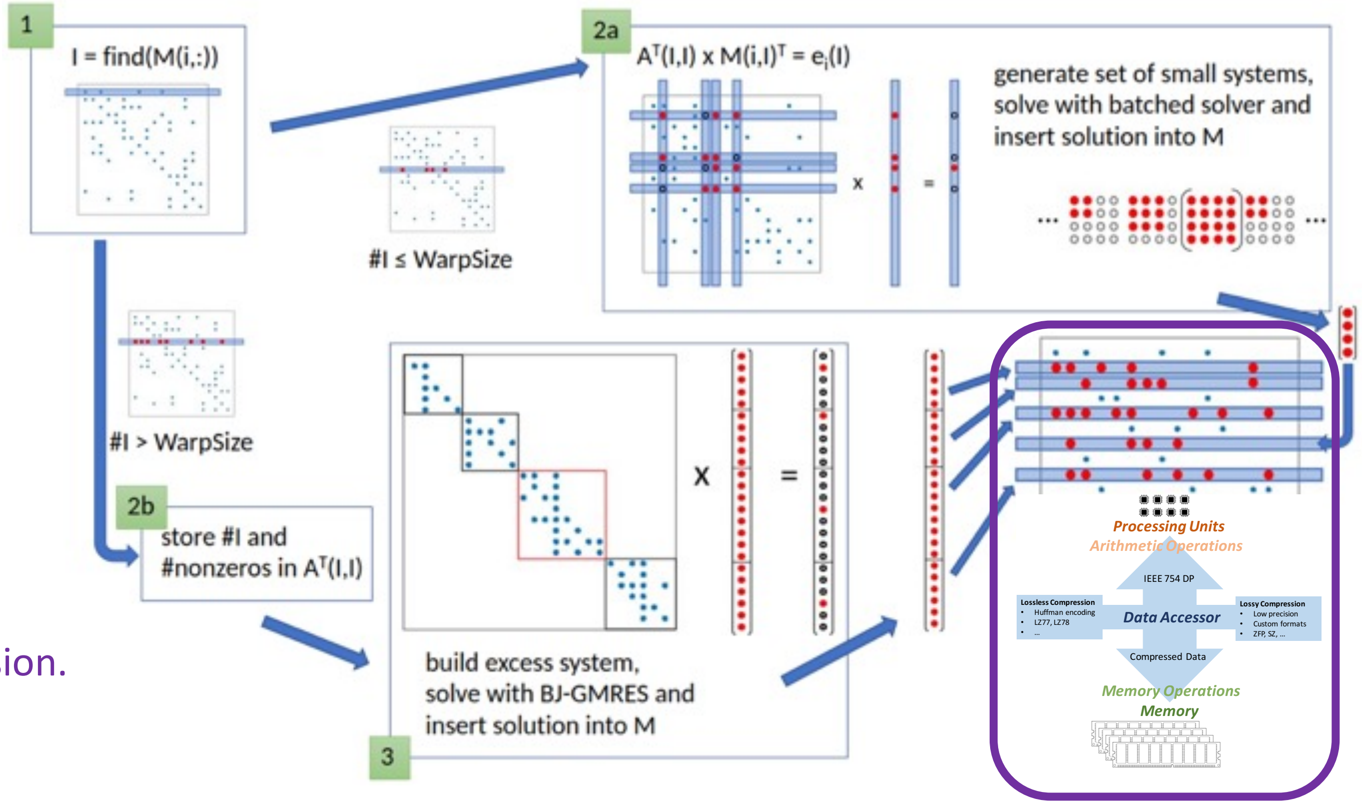

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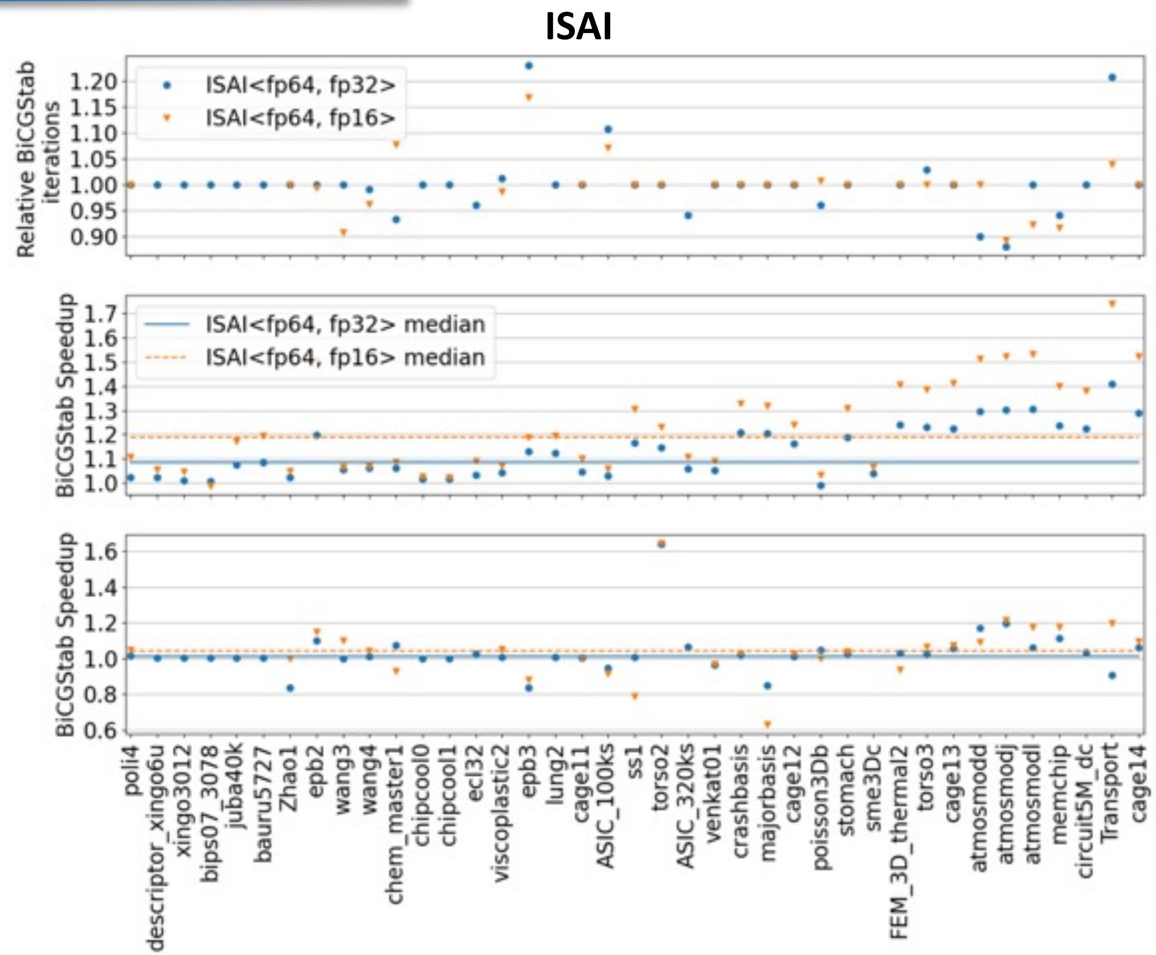
Mixed Precision Sparse Approximate Inverse

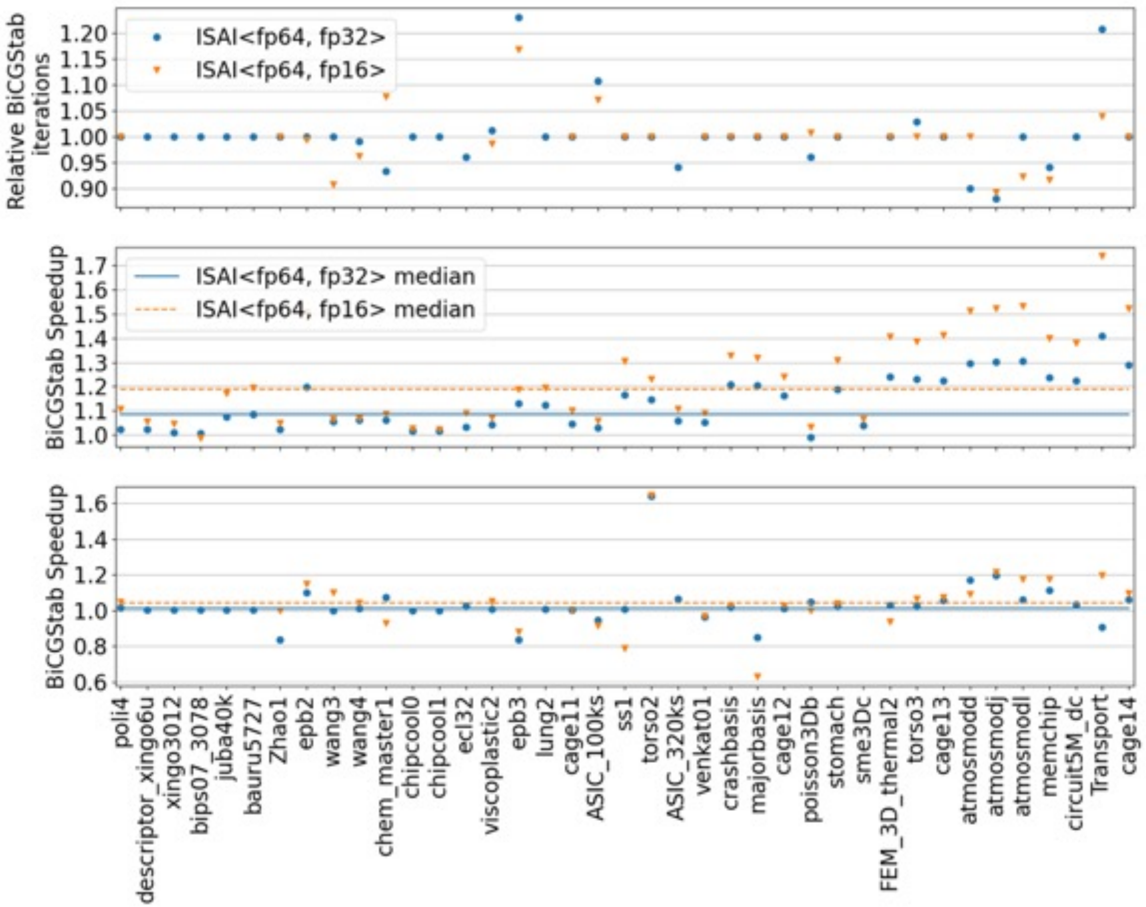


- ~10% speedup for ISAI<fp64,fp32>
- ~20% speedup for ISAI<fp64,fp16> - unstable!

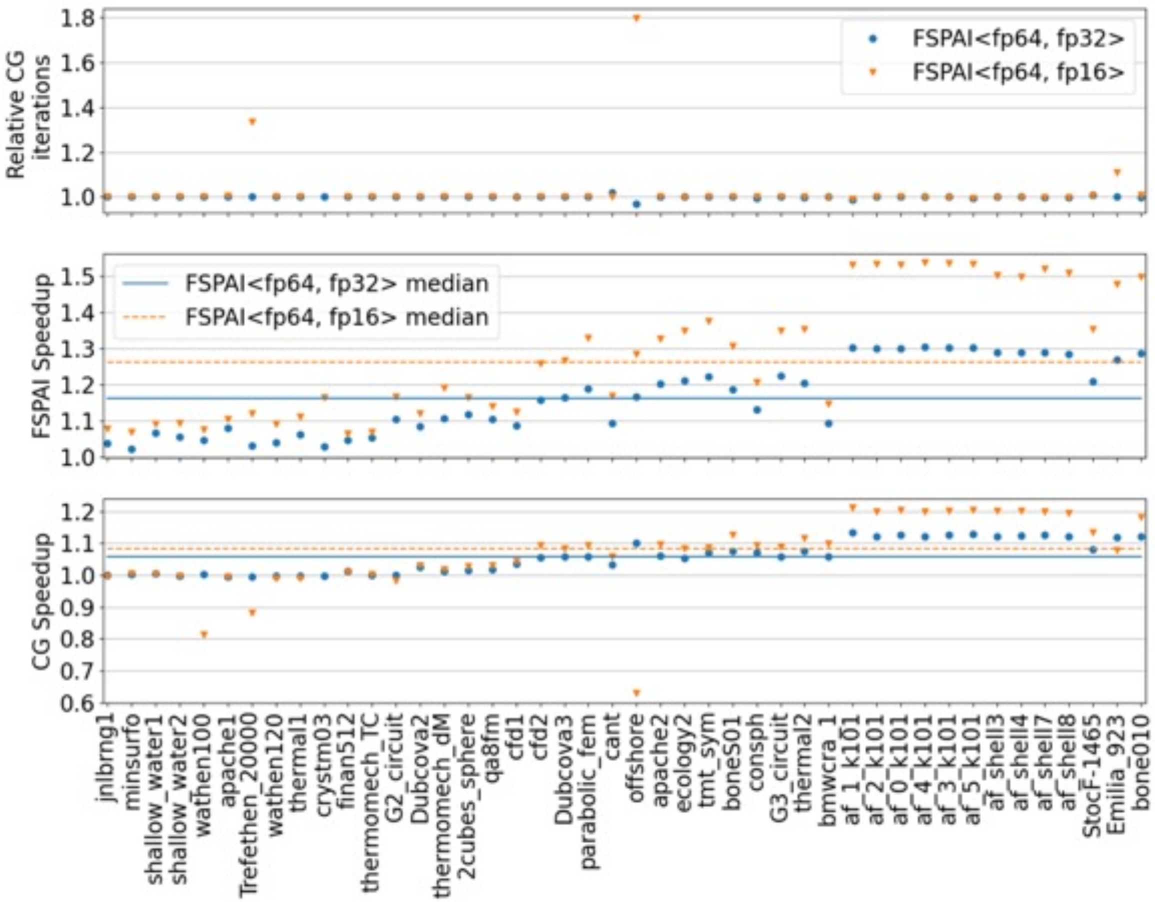
Göbel at al: Mixed Precision Incomplete and Factorized Sparse Approximate Inverse Preconditioning on GPUs, EuroPar submitted.

Mixed Precision Sparse Approximate Inverse

ISAI



FSPAI



- ~10% speedup for ISAI<fp64,fp32>
- ~20% speedup for ISAI<fp64,fp16> - unstable!

- ~15% speedup for FSPAI<fp64,fp32>
- ~25% speedup for FSPAI<fp64,fp16>

Göbel at al: Mixed Precision Incomplete and Factorized Sparse Approximate Inverse Preconditioning on GPUs, EuroPar submitted.