Deformed Cartan matrices and Generalized preprojective algebras

(joint work with Ryo Fujita (Paris・RIMS))

@ Preprojective algebras and Calabi-Yau algebras
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Aim

Cartan matrices $\xleftarrow[\text{invariant}]{\text{combinatorial}}$ Rep. theory of Dynkin quivers
(root sys. ...) (preprojective algebras (=: PA))

(q,t)-deformation [Frenkel-Reshetikhin] ; grading.

Deformed Cartan matrices (=: DCM) $\xleftarrow{?}$ Rep. theory of graded quivers
(quantum root sys. ...) (graded PA)

- Give rep. theoretical interpretations of DCM from viewpoints of generalized preprojective algebras.
- Prove numerical properties of DCM which are not easily understood directly from its definition.

Setting $\mathfrak{g}$: simple Lie algebra / $\mathbb{C}$ $\xleftrightarrow{1:1}$ Cartan matrix $C = (c_{ij})$

Take $D = \mathrm{diag}(d_1, \cdots, d_n)$ DC : symm.
(s.t. $d_i = 1$ or r)

Def (DCM, E. Frenkel - Reshetikhin)

$$C_{ij}(q,t) := \begin{cases} q_i t^{-1} + q_i^{-1} t & (i = j) \\ [c_{ij}]_q & (i \neq j) \end{cases} \qquad q_i := q^{d_i} \qquad [c_{ij}]_{q_i} = \frac{q^k - q^{-k}}{q - q^{-1}}$$

e.g. $C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$$C(q,t) = \begin{pmatrix} qt^{-1}+q^{-1}t & -(q+q^{-1}) \\ -1 & q^2t^{-1}+q^{-2}t \end{pmatrix}$$

take inverse

$$\rightsquigarrow \quad \widetilde{C}(q,t) = \frac{q^3t^{-2}}{1+q^6t^{-4}} \boxed{\begin{pmatrix} q^2t^{-1}+q^{-2}t & q+q^{-1} \\ 1 & qt^{-1}+q^{-1}t \end{pmatrix}}$$

We can read some properties from the above

$$\widetilde{C}_{ij}(q,t) = \sum_{u,v\in\mathbb{Z}} \widetilde{c}_{ij}(u,v)\, q^u t^v$$

e.g.).
- $\widetilde{c}_{ij}(u+6,-v-4) = -\widetilde{c}_{ij}(u,-v)$ (periodicity)
- invariant under $(q,t) \longleftrightarrow (q^{-1},t^{-1})$ (duality)

<u>Generalized preprojective algebras</u> (Geiss-Leclerc-Schröer)

<u>Def</u> Quiver : $\overline{Q}_0 = I$ (index of CM)

$\overline{Q}_1 = \{\alpha_{ij} : c_{ij}<0\} \amalg \{\varepsilon_i \mid i\in I\}$

$i \longleftarrow j$

$\rightsquigarrow \quad \widetilde{\Pi} := \Bbbk\overline{Q} \Big/ \left(\begin{array}{l} \cdot\ \varepsilon_i^{-c_{ij}}\cdot\alpha_{ij} = \alpha_{ij}\,\varepsilon_j^{-c_{ji}} \\ \cdot\ (\text{preproj. rel.}) \end{array} \right.$

$$\varepsilon := \sum_{i\in I} \varepsilon_i^{r/d_i} \qquad \Pi = \Pi(\ell) := \widetilde{\Pi}/(\varepsilon^\ell)$$

We define a (q,t)-grading on Π s.t.

$q \cdots$ ($\widetilde{\Pi}$-q-gr. f.g. mod. cat) $\simeq$ f.g. module cat. of Jacobian algebra of quiver with potential in Bernard's talk (in [Hernandez-Leclerc])

$t \cdots$ pos. grading counting α

$$\begin{array}{c} d_i c_{ij} \\ \| \end{array}$$

Def $\deg(\alpha_{ij}) := (b_{ij}, 1)$

$\deg(\varepsilon_i) := (b_{ii}, 0) = (2d_i, 0)$

Bigrading

$\Bbbk\text{-mod}_{\mathbb{Z}^2} \ni V = \bigoplus_{x,y\in\mathbb{Z}} V_{x,y}, \quad a(q,t) := \sum_{u,v\in\mathbb{Z}} a(u,v)\, q^u \cdot t^v \quad \in \mathbb{Z}_{\geq 0}[q^{\pm}, t^{\pm}]$

$$a(q,t)\cdot V := \bigoplus_{u,v} \Big(\bigoplus_{x,y} V_{x-u,\, y-v} \Big)^{\oplus a(u,v)}$$

Lem ([Chari], [Bouwknegt - Pilch])

$\bigoplus_{i\in I} \mathbb{Q}(q,t)\cdot\alpha_i \circlearrowleft {}^{\exists}$ Braid group action

$$\alpha_i^\vee \longmapsto \alpha_j^\vee - q_j^{-1}\cdot t\ C_{ji}(q,t)\,\alpha_i^\vee \qquad \Big(\alpha_i^\vee := \frac{q_i^{-1} t}{[d_i]_q}\alpha_i\Big)$$

$\rightsquigarrow$ refl. functors of bigraded GPA are understood in terms of this actions.

In $K_0(\Pi\text{-mod}) \otimes \mathbb{Q}(q,t)$ $\big(=: K_0(\Pi)_{loc}\big)$, $\quad J_i := \Pi(1-e_i)\Pi$.

$$[J_i \otimes \underset{\sim\sim}{E_j}] = [E_j] - q_j^{-1}\cdot t\ C_{ji}(q,t)\,[E_i] \qquad (i\neq j)$$

maximal iterated self-ext of S_i
(called generalized simple)

By using this, we can extract symmetry about (q,t) - Cartan of projective Π-modules because there is a filtration :

$$\Pi \supset J_{i_1} \supset J_{i_2}J_{i_1} \supset \cdots \supset J_{i_\ell}\cdots J_{i_1} = 0$$

for any red. exp. $\mathring{\mathbf{i}} = (i_\ell, \cdots, i_1)$ of w_0.

$$\frac{J_{i_{k-1}}\cdots J_{i_1}}{J_{i_k}\cdots J_{i_1}} \simeq \begin{cases} E'_{i_k} \otimes_\Pi J_{i_{k-1}}\cdots J_{i_1} & \text{(I)} \quad \text{(right)} \\ \underset{\text{(II)}}{E_{i_k}^{\oplus a}} \quad ({}^{\exists} a\in\mathbb{Z}_{\geq 0}[q^{\pm}, t^{\pm}]) & \text{(left)} \end{cases}$$

Compairing q,t-dimension of (I), (II) and using some combinatorics of quantum root system, we obtain

$\mathbb{Q}(q,t)$- bilinear form on $\bigoplus \mathbb{Q}(q,t)\alpha_i$

$(\alpha_i,\alpha_j)_{q,t} = [d_i]_q C_{ij}(q,t)$

$\cdot\ (\tilde{\varpi}_i^\vee, \alpha_j) = \delta_{ij}$

<u>Lem</u> [Fujita-M]

- In $K_0(\Pi)_{loc}$

$$[P_i] = \sum_{k=1}^{\ell} (\tilde{\varpi}_i^\vee, T_{i_1}\cdots T_{i_{k-1}}\alpha_{i_k})_{q,t}[E_{i_k}]$$

(for any red. exp $\mathbb{i}$)

- $$0 \to \underbrace{q^{-rh^\vee} t^h E_{i^*}}_{\text{"soc" of } P_i} \to q_i^{-2} t^2 P_i \to \bigoplus_{j\sim i}(-q_j^{-1} t\, C_{ij}(q,t))P_j \to P_i \to E_i \to 0$$

and $T_{w_0}\alpha_i = -q^{-rh^\vee} t^h \alpha_{i^*}$

This part is an analogue of [GLS]

We consider this Lem. from a viewpoint of Euler-Poincaré principle.

$$\begin{tikzcd}
 & P_i \arrow[r, leftrightarrow, "\text{dual}"] \arrow[d, phantom, "\wr\ \text{Lem}"] & S_i \\
? \arrow[r, leftrightarrow, "\text{dual}"] & E_j &
\end{tikzcd}$$

"generic kernel"

<u>Def</u>

$$\bar{I}_i := D((\tilde{\Pi}/\tilde{\Pi}\varepsilon_i)e_i)$$

$$\left(0 \longrightarrow \bar{I}_i \longrightarrow I_i \longrightarrow q^{-2d_i} I_i \right)$$

$\bar{I}_i$: object in Π-mod.

$\Bbbk[\varepsilon_i]/\langle\varepsilon_i^{\ell r/d_i}\rangle$

$$\underline{\mathrm{Hom}}_\Pi(M,\bar{I}_i) \simeq D(e_i(M/\varepsilon_i M)) \simeq \underline{\mathrm{Hom}}_{H_i}(e_i M,\Bbbk)$$

$$\underline{\mathrm{Ext}}^m_\Pi(E_i,\bar{I}_j) \simeq \begin{cases}\Bbbk & (m=0,\ i=j)\\ 0 & \text{other}\end{cases}$$

$$\left(\underline{\mathrm{Ext}}^k_\Pi(M,N) = \bigoplus_{u,v}\mathrm{Ext}^k_\Pi(q^u t^v M, N) \right)$$

Def $M,N\in\Pi$-mod.

$$\langle M,N\rangle = \sum_{k\geq 0}(-1)^k \dim_{q,t}\underline{\mathrm{Ext}}^k_\Pi(M,N) \in \mathbb{Z}[\![q^\pm,t^\pm]\!]$$

Rem With our (q,t)-grading,

$\forall u,v\in\mathbb{Z}$ $\quad \mathrm{Ext}^m_\Pi(q^u t^v M,N)=0$ $(\exists m\gg 0)$

$\rightsquigarrow$ $\langle -,-\rangle_{q,t}$ only depends on $[M]$ & $[N]$

$$0\to q^{-rh^\vee}t^h E_{i^*}\to q_i^{-2}t^2 P_i \longrightarrow \bigoplus_{j\sim i}(-q_j^{-1}t\, c_{ij}(q,t))P_j \longrightarrow P_i \longrightarrow E_i \longrightarrow 0$$

$$\rightsquigarrow \langle E_i,S_j\rangle_{q,t} = \frac{q_i t^{-1}}{1-(q^{rh^\vee}t^{-h})^2}\left(c_{ij}(q,t) - q^{rh^\vee}t^{-h}c_{i^*j}(q,t)\right)$$

$$\in \mathbb{Z}[q^\pm](\!(t^{-1})\!)$$

(δ_{ij^*})

$$\rightsquigarrow (\langle E_i,S_j\rangle_{q,t})_{i,j\in I} = \frac{q^D t^{-1}(\mathrm{id}-q^{rh^\vee}t^{-h}\nu)}{1-(q^{rh^\vee}t^{-h})^2}\,C(q,t) \quad \cdots ①$$

On the other hand,

$$[P_i] = \sum_{j\in I}\dim_{q^{-1},t^{-1}}\underbrace{\underline{\mathrm{Hom}}_\Pi(P_i,\bar{I}_j)}_{\cong\, e_i\bar{I}_j}[E_i]$$

$$= \sum_{k=1}^{\ell} (\tilde{\varpi}_i^\vee, T_{i_1} \cdots T_{i_{k-1}} \alpha_{i_k})_{q,t} [E_{i_k}]$$

compair

$$\rightsquigarrow \dim_{q,t}^{-1} (e_i \bar{I}_j) = \sum_{i_k = j} (\tilde{\varpi}_i^\vee, T_{i_1} \cdots T_{i_{k-1}} \alpha_j)_{q,t}.$$

By EP principle

$$Id = (\langle P_i, S_j \rangle_{q,t})_{ij}$$

$$= (\dim e_i \bar{I}_j)_{ij} (\langle E_i, S_j \rangle_{q,t})_{i,j} \quad \cdots ②$$

By comparing ① and ②, $C(q,t)$ is invertible and obtain (by a bit of calculation)

<u>Thm</u> [Fujita-M]

$$\begin{cases} \tilde{C}_{ij}(q,t) = \dfrac{q^{d_j} t^{-1}}{1-(q^{rh^\vee} t^{-h})^2} (\dim_{q,t} e_i \bar{I}_j - q^{rh^\vee} t^{-h} \dim_{q,t} e_i \bar{I}_{j^*}) \\ \dim_{q,t} e_i \bar{I}_j = q^{-d_j} t \sum_{u=0}^{rh^\vee} \sum_{v=0}^{h} \tilde{C}_{ij}(u,-v) q^u t^{-v} \end{cases}$$

- periodicity of proj. resl of E_i

$$\rightsquigarrow \tilde{C}_{ij}(u,v) = -\tilde{C}_{ij^*}(u+rh^\vee, v-h)$$

$$(u \geq 0, v \leq 0).$$

- $D(\bar{I}_i) \simeq q^{2d_i - rh^\vee} t^{h-2} \bar{I}_{i^*} \rightsquigarrow \tilde{C}_{ij}(rh^\vee - u, -h-v)$

$$= \tilde{C}_{ij^*}(u,v)$$

$$(0 \leq u \leq rh^\vee \ \&\ -h \leq v \leq 0)$$

- $\dim_{q,t} e_i \bar{I}_j$ is a dim. of module

$$\rightsquigarrow \tilde{C}_{ij}(u,v) \geq 0 \quad (\text{—} \ '' \ \text{—})$$

Cor (cf [Hernandez-Leclerc; ADE])

$$\tilde{C}_{ij}(q,t) = q^{d_j} t^{-1} \sum_{k>0,\, i_k=j} (\tilde{\varpi}_i^\vee, T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1} \alpha_j)_{q,t}.$$

$$(\underbrace{i_1, \cdots, i_\ell}_{\text{any red. word of } w_0}, \cdots) \quad \text{w/} \quad i_{k+\ell} = i_k^* \quad (k \in \mathbb{Z}_{>0})$$