RPP crystal from Nakajima tensor product variety

Based on arxiv.org/abs/2202.02490 joint with Balazs Elek, Calder Morton-Ferguson, and Joel Kamnitzer.

$M \in X(\Phi)$ f_i : $S(i) \to f_i M \to M$

Lowering in RPPs

$R(w,n)$ is (reverse 🗜️ into) a model for the crystal $B(n\lambda)$ as follows.

- Decompose $\Phi \in R(w,n)$ as $\phi_1 \otimes \cdots \otimes \phi_n$ such that $\phi_i \le \phi_{i+1}$ in $J(H(w))$
- Apply the signature rule in $B(\lambda)^{\otimes n}$
- And reassemble into $f_i \Phi \in R(w,n)$

Example. Apply f_2 to an element of $R(s_2 s_1 s_3 s_2, 5)$

$$\begin{matrix} & 1 & \\ 3 & & 4 \\ & 5 & \end{matrix} = \begin{matrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{matrix} \otimes \begin{matrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{matrix} \otimes \begin{matrix} & 0 & \\ 1 & & 1 \\ & 1 & \end{matrix} \otimes \begin{matrix} & {\color{red}0} & \\ 1 & & 1 \\ & 1 & \end{matrix} \otimes \begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix} \xrightarrow{S_2 = \cancel{-}\ \ \cancel{+}{\color{red}+}-} \begin{matrix} & 2 & \\ 3 & & 4 \\ & 5 & \end{matrix}$$

Recall: Nakajima core quiver variety

Let $\lambda \in P_+$ and $\mu \in P$ be such that $\lambda = \sum w_i \omega_i$ and $\lambda - \mu = \sum v_i \alpha_i$ for some $\vec{w} = (w_i), \vec{v} = (v_i) \in \mathbb{N}^I$.

Let V, W be I-graded vector spaces of dimension $\vec{v}, \vec{w}$.

1. $\mathrm{Hom}(\vec{v}, \vec{w}) := \bigoplus \mathrm{Hom}(V_i, V_j) \oplus \bigoplus \mathrm{Hom}(V_i, W_i)$
2. $\prod GL(V_i) \circlearrowright T^*\mathrm{Hom}(\vec{v}, \vec{w}) \xrightarrow{\psi} \prod \mathfrak{gl}(V_i)$
3. $M_\chi(\vec{v}, \vec{w}) := \psi^{-1}(0) /\!/_\chi \prod GL(V_i) \to M_0(\vec{v}, \vec{w})$ $\chi =$ determinantal char
4. the preimage of zero under this projection is the so-called *core* quiver variety

$$L(\vec{w}) := \bigsqcup_{\vec{v}} L(\vec{v}, \vec{w})$$

Recall: Lusztig's isomorphism

Theorem. With $\lambda = \sum w_i \omega_i, \lambda - \mu = \sum v_i \alpha_i$ as above

$$\mathrm{Gr}(T(\lambda)) \cong L(\vec{w})$$

Splitting

Let $\vec{w} = \vec{w}^1 + \cdots + \vec{w}^n$ be a splitting giving $W = W^1 \oplus \cdots \oplus W^n$ associated to a composition $\lambda = \lambda^1 + \cdots + \lambda^n$ and define an action of $\mathbb{C}^\times$ on W by

$$s \cdot (u_1, \ldots, u_n) = (s^{n-1}u_1, \ldots, su_{n-1}, u_n)$$

This yields $\mathbb{C}^\times$ actions on

- $M(\vec{w})$ $= \bigsqcup M(\vec{v}, \vec{w})$
- $L(\vec{n})$
- $T(\lambda)$ and its submodules
- $\mathrm{Gr}(T(\lambda))$

Fixed points

Let $\vec{w} = \vec{w}^1 + \cdots + \vec{w}^n$ be a splitting giving $W = W^1 \oplus \cdots \oplus W^n$ associated to a composition $\lambda = \lambda^1 + \cdots + \lambda^n$ and define an action of $\mathbb{C}^\times$ on W by

$$s \cdot (u_1, \ldots, u_n) = (s^{n-1} u_1, \ldots, s u_{n-1}, u_n)$$

This yields $\mathbb{C}^\times$ actions on so that

- $M(\vec{w})^{\mathbb{C}^\times} \cong M(\vec{w}^1) \times \cdots \times M(\vec{w}^n)$
- $L(\vec{w})^{\mathbb{C}^\times} \cong L(\vec{w}^1) \times \cdots \times L(\vec{w}^n)$
- $T(\lambda)$ and its submodules
- $\mathrm{Gr}(T(\lambda))^{\mathbb{C}^\times} \cong \mathrm{Gr}(T(\lambda^1)) \times \cdots \times \mathrm{Gr}(T(\lambda^n))$

Tensor product variety

Nakajima: Irreducible components of

$$Z(\vec{w}) = \left\{ x \in M(\vec{w}) : \lim_{s \to 0} s \cdot x \in L(\vec{w})^{\mathbb{C}^\times} \right\}$$

are in crystal bijection ⛵ with irreducible components of

$$L(\vec{w})^{\mathbb{C}^\times} \cong L(\vec{w}^1) \times \cdots \times L(\vec{w}^n)$$

The bijection ⛵ is

$$(X_1, \ldots, X_n) \mapsto \overline{\left\{ x \in M(\vec{w}) : \lim_{s \to 0} s \cdot x \in X_1 \times \cdots \times X_n \right\}}$$

Tensor product variety

Fact: $L(\vec{w}) \subset Z(\vec{w})$ because it is a projective variety, so every point has a limit, and this inclusion yields an inclusion on irreducible components. Combined with ⛵ we have

Theorem. The crystal structure on $\mathrm{Irr} L(\vec{w})^{\mathbb{C}^\times}$ extends the crystal structure on $\mathrm{Irr} L(\vec{w})$. In other words, there is an inclusion of crystals.

$$\mathrm{Irr}\mathrm{Gr}(T(\lambda)) = B(\lambda) \to \prod \mathrm{Irr} L(\vec{w}^i) = \bigotimes B(\lambda^i)$$

$$\xrightarrow{\lambda^i = w_i \omega_i} \bigotimes B(\omega_i)^{\otimes w_i}$$

Components and filtrations

$T(\lambda) = \bigoplus T(\lambda^i)$

$\lambda = \lambda^1 + \dots + \lambda^n$

Submodules $M \subset T(\lambda)$ can be filtered by $M^{\leq i} := M \cap (T(\lambda^1) \oplus \cdots \oplus T(\lambda^i))$ and composition factors $M^i = M^{\leq i}/M^{\leq i-1}$ regarded as $M^i \subset T(\lambda^i)$.

This filtration is compatible with respect to the $\mathbb{C}^\times$ action:

$$\lim_{s \to 0} s \cdot M = M^1 \oplus \cdots \oplus M^n \in \mathrm{Gr}(T(\lambda))^{\mathbb{C}^\times} \longleftrightarrow (M^1, \dots, M^n)$$

Theorem. Thus we get another version of the ⛵ crystal isomorphism

$$B(\lambda) \to B(\lambda^1) \otimes \cdots \otimes B(\lambda^n)$$
$$X_1 \otimes \cdots \otimes X_n \mapsto \overline{\{M \in \mathrm{Gr}\, T(\lambda) : M^i \in X_i \text{ for all } i\}}$$

Heaps and composition factors

Reducing again to the special case $\lambda = \overset{n}{\cancel{v}}\omega$ recall that $T(\lambda, w) = \mathbb{C}H(w)$ and irreducible components of the quiver grassmannian of $T(\lambda, w)$ are in bijection with $J(H(w))$.

Theorem. Given $\Phi =$ $(\phi^1, \dots, \phi^n)$

$$Z(\Phi) = \overline{\{M \in \mathbb{C}H(w)^{\oplus \overset{n}{\cancel{v}}} : M^i = \mathbb{C}\phi^i \text{ for all } i\}}$$

is an irreducible component of $\mathrm{Gr}T(\lambda, w)$ and supplies the crystal isomorphism

$$R(w, n) = B_w(n\lambda)$$

Another filtration

Recall that $H(w)_i = \{x_1^i < \cdots < x_q^i\}$ so define $A_i : x_j^i \mapsto x_{j-1}^i$ and $A_i x_1^i = 0$.

← extend this shift operator to a linear transformation of $M \subseteq \mathbb{C}H(w)^{\oplus n}$

Let $\Phi \in R(w, n)$. This yields the Jordan filtrations

$$M_i \supset \mathrm{Ker}\, A_i \supset \mathrm{Ker}\, A_i^2 \supset \cdots \supset \mathrm{Ker}\, A_i^q$$

with composition factors $\mathrm{Ker}\, A_i^j / \mathrm{Ker}\, A_i^{j-1}$ so that

$$X(\Phi) = \overline{\left\{ M : \dim \mathrm{Ker}\, A_i^j - \dim \mathrm{Ker}\, A_i^{j-1} = \Phi(x_j^i) \text{ for all } i \right\}}$$

is an irreducible component of $\mathrm{Irr}\mathrm{Gr}\mathbb{C}H(w)$.

Literature

Stembridge 1999: Minuscule elements of Weyl groups

- Clarify exactly which fully commutative w are λ-minuscule in terms of
  - reduced words
  - heaps
- Extend Proctor's classification of (dominant) λ-minuscule w (equivalently, their heaps) from simply-laced to symmetrizable Kac-Moody Weyl groups

$$\sum_{\rho} q^{|\rho|} = \prod_{u \in \lambda} \frac{1}{1-q^{h(u)}}$$

Literature

Garver, Patrias, Thomas 2018: Minuscule reverse plane partitions via quiver representations kQ.

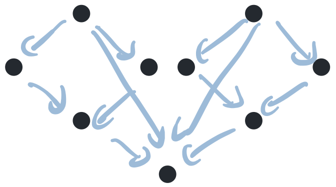
Generalize Hillman-Grassl (1976) bijective *proof* of Stanley's generating function (1971) for RPPs of shape λ, via multisets of rim hooks, to a proof of Proctor's generating function (1984) for RPPs of shape P a minuscule poset *using* isomorphism classes of representations of simply-laced Dynkin quivers.

Connection?

Consider in $R(32413524, 2)$ with $\Gamma = A_5$

$$\Phi = \begin{array}{ccccc} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 2 & & \end{array} = \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \otimes \begin{array}{ccccc} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

Regarding the heaps on the RHS as modules $N^1, N^2 \subset I(3)$ calculate $\operatorname{Ext}(N^1, N^2)$ to check that there is just one non-trivial extension, and verify that its socle does not contain $S(3)$



Beta Testing

We could alternatively view the RPP as a filtration of a $\Pi(Q)$-module by $\mathbb{C}Q$-modules.

$$M \in X(\Phi)\,, \;\; \Phi = \begin{matrix} & & 1 & & 1 & & \\ 1 & & & 1 & & & 1 \\ & & 1 & & 1 & & \\ & & & 2 & & & \end{matrix} \;\Rightarrow\; M\big|_{\mathbb{C}Q} = \frac{2 \to 3 \leftarrow 4}{1 \to 2 \to 33 \leftarrow 4 \leftarrow 5} \; = \tau^{-1}(\quad)$$

The upshot is that we can use the AR quiver to predict e.g. if $S(3)$ is a simple quotient.

Challenge: can we describe irreducible components of $\mathrm{Irr}\,\mathrm{Gr}T(\lambda)$ using the AR quiver?

characterize M in $X(\Phi)$ using $M|_{\mathbb{C}Q}$ or

the fact that Φ indexes nilpotent endomorphosim of an iso class $\mathbb{C}Q$ modules

Literature

Kleshchev, Ram 2008: Homogeneous representations of Khovanov-Lauda algebras

Explicit construction of irreducible graded representations of simply laced KLR algebras which are concentrated in one degree ("homogeneous") from $J(H(w))$.

Open questions

- The MV cycles associated to order ideals $J(H(w))$ are Schubert varieties in partial flag varieties G/P_λ
  - What about MV cycles associated to $R(w, n)$
- The homogeneous coordinate ring of G/P_λ is isomorphic to $\bigoplus V(n\lambda)$ and it carries a cluster algebra structure
  - How does the cluster combinatorics interact with the combinatorics of $\coprod_{n} R(w, n)$

Qu

$M \in \Pi(Q)$-mod $\longleftarrow$ $N \in kQ$-mod

choice of orientation of Q

gen JF$(M|_{kQ})$

minuscule ⊆ simple.

iso classes of kQ reps have "generic Jordan forms" (GPT)

rpp's.

Thank you for listening

Baumann-Kamnitzer: orientations adapted to reduced expression