Calabi-Yau algebras & canonical bundles

$\overset{\shortparallel}{\text{CY}}$

1. Intro, dictionary
2. CY alg is "CY"
3. CY completion vs local CY mfd.

$\shortparallel$ derived preprojective alg

$\shortparallel$ total sp of the canonical bundle

1. Intro

Aim

(1) To explain why CY alg is "CY"

(2) What is the CY completion (derived preproj)

by comparing with geometric settings

dictionary

algebra (rep theory)	geometry

A : algebra (smooth)	X : variety of $\dim = d$ cpx mfd
Θ_A : inverse dualizing complex of A	$K_X^{-1}[-d]$: shifts of the inverse of the canonical bundle K_X
$\mathrm{Hom}(M \otimes_A \Theta_A, N[-i])$ $\cong$	$\mathrm{Ext}^{d-i}(E, F \otimes K_X)$ $\cong \qquad \mathrel{\Vert}$ $\mathrm{Hom}(E \otimes K_X[-d], F[-i])$

$\mathrm{Hom}(N[-i], M)^*$	$\mathrm{Ext}^i(F, E)^*$ $\|$ $\mathrm{Hom}(F[-i], E)$
alg A is CY of $\dim = d$ $\overset{\text{def}}{\iff} \mathbb{H}_A \xrightarrow{\sim} A[-d]$	X is CY mfd of $\dim = d$ $\overset{\text{def}}{\iff} K_X \xrightarrow{\sim} \mathcal{O}_X$ $\iff K_X^{-1}[-d] \xrightarrow{\sim} \mathcal{O}_X[-d]$
$(d+1)$ dim CY completion of A	local CY of X is $Y :=$ total sp of K_X $\curvearrowleft d+1$ dim.

$\Pi_{d+1}(A) := T^\bullet(\Theta_A[d])$

$\sqsubseteq \mathrm{Spec}_X \, \mathrm{Sym}^\bullet K_X^{-1}$

$\qquad \| $

$\mathrm{Spec}_X \, T^\bullet K_X^{-1}$

2. CY alg is "CY"

X: smooth quasi-projective variety $/\mathbb{C}$ (cpx mfd)

$d := \dim_{\mathbb{C}} X$

Def $K_X := \bigwedge^d T^*X \longrightarrow X$

↪ line bundle called the "canonical bundle"

Note

E : vector bundle $\longleftrightarrow$ $\mathcal{E}$: sheaf of hol section of E

K_X $\longleftrightarrow$ $\mathcal{K}_X$: canonical sheaf
||
dualizing sheaf

Def

X is CY mfd $\overset{\text{def}}{\iff}$ $\mathcal{K}_X \cong \mathcal{O}_X$

$(K_X \cong \mathbb{C} \times X$ ← trivial bundle$)$

Remark

This is "loose" definition but covenient in homological alg.

Maybe, proper geometers demand compactness of X

$\pi_1(X) \cong \{1\}$

$\lrcorner$

Examples

$X = \mathbb{C}^d$ ← CY of dim $= d$.

$= \mathbb{E}$: elliptic curve dim $= 1$

$= K3$: $K3$ surface dim $= 2$

= Quintic (= degree 5 hypersurface in $\mathbb{P}^4$) $\dim = 3$

Serre duality

E, F : vector bundles on X with cpt support

$\left(\sum_i \dim H^i(M) < \infty \right)$

$$\boxed{\mathrm{Ext}^{d-i}(E, F \otimes K_X) \simeq \mathrm{Ext}^i(F, E)^*}$$

rewrite in derived cat

$$\mathrm{Hom}(E, F[-i]\otimes K_X[d]) \xrightarrow{\sim} \mathrm{Hom}(F, E[i])^*$$

$$\mathrm{Hom}(E\otimes K_X^{-1}[-d], F[-i]) \xrightarrow{\sim} \mathrm{Hom}(\underbrace{F[-i]}_{= F'}, E)^*$$

$\boxed{\text{if } K_X \xrightarrow{\sim} \mathcal{O}_X}$

$$\mathrm{Hom}(E[-d], F') \xrightarrow{\sim} \mathrm{Hom}(F', E)^*$$

$\boxed{\text{Serre functor} = [d]}$

<u>Inverse dualizing complex</u> ($\doteqdot$ sheaf of the inverse

of the Canonical bundle)

A: (smooth dg) algebra

(ex, A: alg with $\mathrm{gd}\,A < \infty$)

$A^e := A^{op} \otimes A$

$$\left(\begin{array}{l} A\text{-mod} = \text{right } A\text{-modules} \\ A^{op}\text{-mod} = \text{left} \quad '' \\ A^e\text{-mod} = \text{two-sided} \quad '' \end{array}\right)$$

Def

The dualizing complex of A is defined by

$\Theta_A :=$ (projective resolution of) $\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)$

two-sided A-mod

in $\mathrm{Per}(A^e)$

<u>Lemma</u> (Keller)

For $M, N \in \mathcal{D}_{fd}(A)$,

cpt support cond in geometry

$$\mathrm{Hom}(M \otimes_A \Theta_A, N) \xrightarrow{\sim} \mathrm{Hom}(N, M)^*$$

$\boxed{\text{cf}}$

$$\mathrm{Hom}(E \otimes K_X^{-1}[-d], F) \xrightarrow{\sim} \mathrm{Hom}(F, E)^*$$

alg $\qquad\qquad\qquad\qquad$ geometry

$$\Theta_A \longleftrightarrow K_X^{-1}[-d]$$

<u>CY alg</u>

<u>Def</u> A : (smooth dg) alg

A is CY of $\dim = d \overset{\text{def}}{\iff} \Theta_A \xrightarrow{\sim} A[-d]$

$\boxed{\mathrm{CY}}$ $\qquad K_X^{-1}[-d] \xrightarrow{\sim} \mathcal{O}_X[-d]$

(due to Kontsevich)

(explicit correspondence through the dg affinization
in my preprint with Qiu Yu)

3, CY completion (= derived preprojective alg)

$\parallel$

the total space of the canonical bundle.

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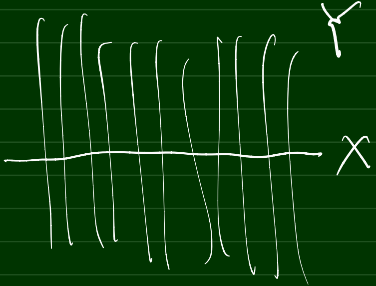
for X : smooth quasi-projective var $/\mathbb{C}$

not necessarily CY

$\boxed{\text{very simple construction of open CY}}$

$$\underset{\text{dim}=d+1}{Y} := \text{total space of } K_X \xrightarrow{\text{line bundle}} \underset{\text{dim}=d}{X}$$



$\boxed{Y \text{ is called local "CY"}}$

Prop

$Y := \mathrm{tot}\, K_X$

$K_Y \xrightarrow{\sim} \mathcal{O}_Y$

ref

Ballard,
Sheaves on local CY
varieties

$\boxed{\text{alg}}$

A : alg not necessarily CY

$\rightsquigarrow$? ? ?

CY completion

$\boxed{\text{geometry}}$

X : variety not necessarily CY

$d := \dim_{\mathbb{C}} X$

$\rightsquigarrow$

$Y = \mathrm{Tot}\, K_X$: CY of dim $= d+1$

the total sp of the canonical

"derived" preprojective, bundle

linear coord functions

Point

- V : vector sp $\rightsquigarrow$ $\mathbb{V} := \operatorname{Spec} \operatorname{Sym}^\bullet \boxed{V^*}$

$V = \mathbb{C}^n \leftarrow$ linear coord $= x_1, x_2, \cdots, x_n$

$\mathbb{V} = \mathbb{C}^n \simeq \operatorname{Spec} \mathbb{C}[x_1, \cdots, x_n]$

- If $\dim V = 1$, $\operatorname{Sym}^\bullet V^* = T^\bullet V^*$

tensor alg.

$E \longrightarrow X$: vector bundle, $\mathcal{E}$: sections.

$$\boxed{\operatorname{tot} E = \operatorname{Spec}_X \operatorname{Sym}^\bullet \mathcal{E}^*}$$

In particular,

(local CY) $Y := \operatorname{tot} K_X$

$$= \operatorname{Spec}_X \operatorname{Sym}^\bullet \mathcal{K}_X^{-1}$$

$$\cong \operatorname{Spec}_X T^\bullet \mathcal{K}_X^{-1}$$

K_X is $\mathrm{rk} = 1$

alg		geometry
A	$\longleftrightarrow$	$\mathcal{O}_X$
$\mathbb{H}_A$	$\longleftrightarrow$	$K_X^{-1}[-d]$
$\mathbb{H}_A[d]$	$\longleftrightarrow$	K_X^{-1}
$T^\bullet(\mathbb{H}_A[d])$	$\longleftrightarrow$	$T^\bullet K_X^{-1} \rightsquigarrow Y = \mathrm{Tot}\, K_X$

CY alg of dim $= d+1$??

Y is CY of dim $d+1$

<u>Def</u> $N \in \mathbb{Z}$,

N-CY completion $\Pi_N(A)$ of A is defined by

$$\Pi_N(A) := T^{\bullet}\left(\Theta_A[N-1]\right)$$

<u>Theorem</u> (Keller, some errors are pointed out by Wai-kit Yeung)

$\Pi_N(A)$ is CY-N alg.

$\boxed{\text{cf}}$ $\qquad \Theta_A[d] \longleftrightarrow K_X^{-1}$

$$\Pi_{d+1}(A) = T^{\circ}(\Theta_A[d]) \longleftrightarrow T^{\circ}(K_X^{-1}) \rightsquigarrow \Upsilon$$

d+1 dim CT alg. $\qquad\qquad$ d+1 dim local CT

<u>Final remark</u> (with Qiu Yu)

$$K_{\mathbb{C}^*}(X) \xrightarrow{\sim} K(X) \otimes \mathbb{Z}[\xi, \xi^{-1}]$$

$$\Pi_{d+1}(A) \longleftrightarrow \Upsilon = \mathrm{Tot}\, K_X$$

extra grading $\longleftrightarrow$ $\mathbb{C}^*$-action for fibers

double graded version of CY completion for A $\rightsquigarrow$ q-deformed arguments

X: smooth quasi-projective.

$\rightsquigarrow$ $\exists G$: generator for $D\,\mathrm{Qcoh}\,X$

Bondal - van den Bergh

$A = \mathbb{R}\mathrm{Hom}(G, G)$: dga

$$\mathrm{RHom}(-, G) : D\,\mathrm{Qcoh}\,X \xrightarrow{\sim} D(A)$$