Constructing the lattice of wide subcategories

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Setting

- Λ: a finite-dimensional algebra.
- $\operatorname{mod}\Lambda$: the category of f.g. right Λ-modules.

Definition

- A subcategory $\mathcal{T}$ of $\operatorname{mod}\Lambda$ is a torsion class
  $:\Leftrightarrow$ closed under extensions and quotients.
  $\operatorname{tors}\Lambda$: the poset of torsion classes.
- A subcategory $\mathcal{W}$ of $\operatorname{mod}\Lambda$ is a wide subcategory
  $:\Leftrightarrow$ closed under extensions, kernels, and cokernels.
  $\operatorname{wide}\Lambda$: the poset of wide subcategories.

$\operatorname{tors}\Lambda$ and $\operatorname{wide}\Lambda$ are *complete lattices* with $\bigwedge = \bigcap$ and $\bigvee$.

Setting

For simplicity, we assume $\operatorname{tors}\Lambda$ is finite.

Assume we are given $\operatorname{tors}\Lambda$ just as an abstract lattice!

$\rightsquigarrow$ I'll construct the poset $\operatorname{wide}\Lambda$.

Example

$\Lambda = k(\bullet \to \bullet)$. The Hasse quiver of $\operatorname{tors}\Lambda$:

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[ddr] & \\
x \arrow[dd] & & \\
 & & z \arrow[ddl] \\
y \arrow[dr] & & \\
 & 0 &
\end{tikzcd}$$

The join-irreducible labeling

Definition

For each Hasse arrow $a \to b$ in $\operatorname{tors}\Lambda$, we label its arrow by

$$\gamma(a \to b) := \min\{x \in \operatorname{tors}\Lambda \mid b \vee x = a\}.$$

This is well-defined and called the join-irreducible label.

$$\begin{tikzcd}
 & 1 \arrow[dl, "z"'] \arrow[ddr, "y"] & \\
x \arrow[dd, "x"'] & & \\
 & & z \arrow[ddl, "z"] \\
y \arrow[dr, "y"'] & & \\
 & 0 &
\end{tikzcd}$$

The extended kappa map $\overline{\kappa}$

Barnard–Todorov–Zhu introduced the map $\overline{\kappa}$ as follows:

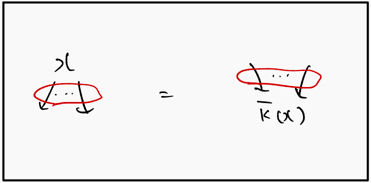
Definition

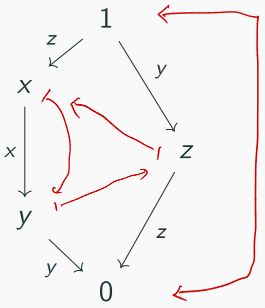
For each $x \in \operatorname{tors}\Lambda$, there's unique $\overline{\kappa}(x) \in \operatorname{tors}\Lambda$ satisfying

$$\begin{aligned} &\{\text{labels of Hasse arrows starting at } x\} \\ =&\{\text{labels of Hasse arrows ending at } \overline{\kappa}(x)\} \end{aligned}$$

The extended kappa map $\overline{\kappa}$

Red: the orbit of $\overline{\kappa}$.





Definition (The kappa order)

For $x, y \in \operatorname{tors}\Lambda$, define

$$x \leq_\kappa y :\iff x \leq y \text{ and } \overline{\kappa}(x) \geq \overline{\kappa}(y).$$

Theorem (E, in today's arxiv!)

$\mathrm{T}\colon \operatorname{wide}\Lambda \to \operatorname{tors}\Lambda$ induces a poset isomorphism:

$$\operatorname{wide}\Lambda \cong (\operatorname{tors}\Lambda, \leq_\kappa)$$

Works with a little modification for any abelian length category.

Example

$\longrightarrow : \overline{\kappa}$

tors Λ

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[ddr] \arrow[dddd, bend left=80, leftrightarrow] & \\
x \arrow[dd] \arrow[dd, bend left] & & \\
 & & z \arrow[ddl] \arrow[ull, bend right] \\
y \arrow[dr] \arrow[urr, bend right] & & \\
 & 0 &
\end{tikzcd}$$

$(\operatorname{tors}\Lambda, \leq_\kappa) \cong \operatorname{wide}\Lambda$

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[d] \arrow[dr] & \\
x \arrow[dr] & y \arrow[d] & z \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

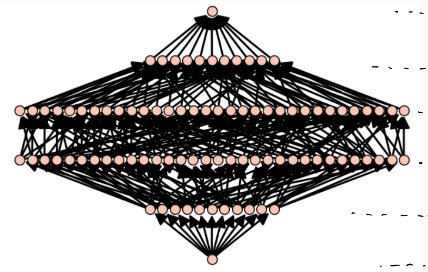
$y \leq x$ but $\underset{\overset{\shortparallel}{z}}{\overline{\kappa}(y)} \not\geq \underset{\overset{\shortparallel}{y}}{\overline{\kappa}(x)}$, hence $y \not\leq_\kappa x$!

Lattice properties of wide Λ

Theorem

The lattice wide Λ *is*

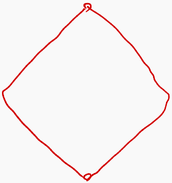
- *ranked (graded) with rank func: $\#\{$simples$\}$,*
- *rank-vector is symmetric and unimodal [Aoki−Higashitani−Iyama−Kase−Mizuno].*



wide Λ

rank	rank-vector
5	1
4	11
3	32
2	32
1	11
0	1

wide Λ looks like

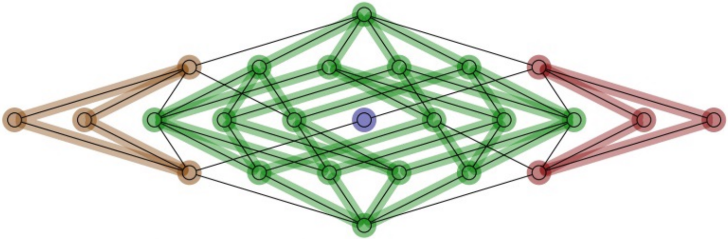


Conjecture for wide Λ

Conjecture

The lattice wide Λ has the following properties:

- (strongly) Sperner.
- The generating function of rank-vector has only real roots.
- Has the symmetric Boolean decomposition.



(d) A symmetric Boolean decomposition into four parts.

Figure from [Mühle, *Symmetric chain decompositions and the strong Sperner property for noncrossing partition lattices*]

SageMath Program

I developed a program

- The Lattice of torsion classes in SageMath

which can compute various things from $\operatorname{tors} \Lambda$.

Youtube Video for how to use this program