

Maximal self-orthogonal modules

and a new generalization of
tilting modules

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§0. Overview

Setting

k : field

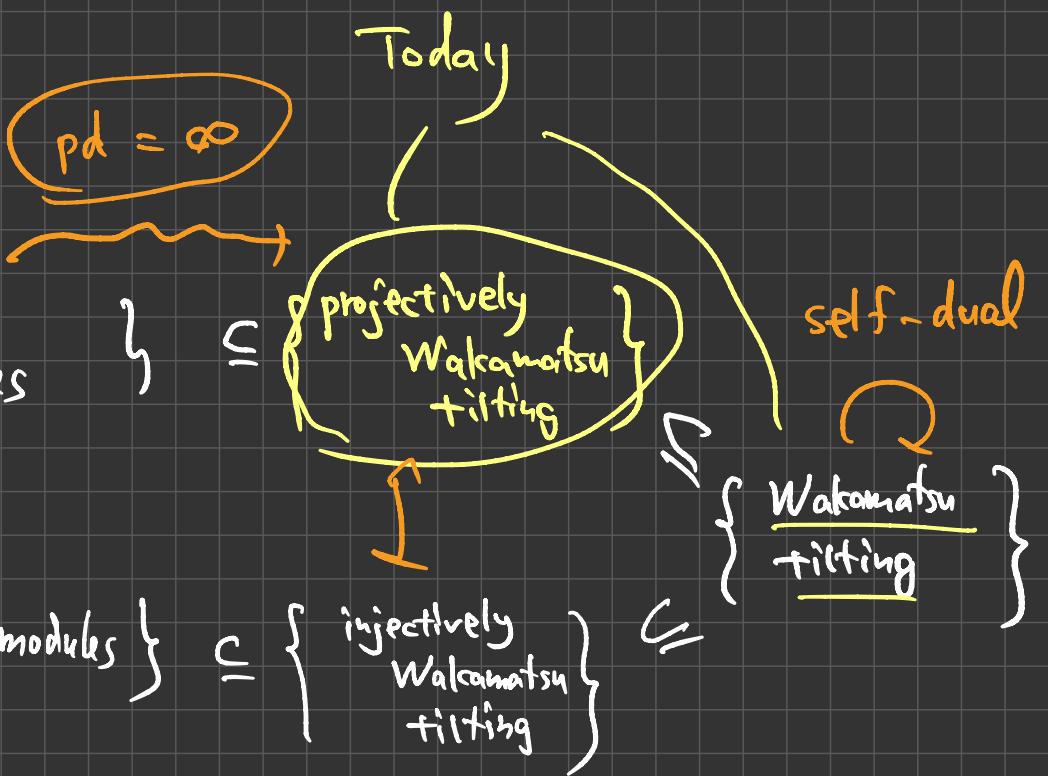
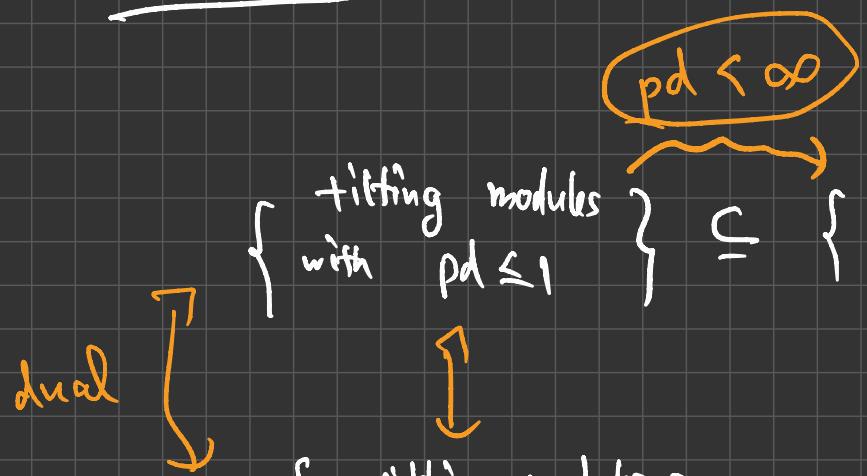
Λ : f.d. k -alg.

modules = f.g. right modules.

Def $M \in \text{mod } \Lambda$ is self-orthogonal
(SO)

$$\Leftrightarrow \text{Ext}_\Lambda^{>0}(M, M) = 0.$$

Hierarchy of SO modules



Today

§1.

Def Let $T \in \text{mod}\Lambda$.

Define $T^\perp \subseteq \text{mod}\Lambda$ by

$$T^\perp := \{ X \in \text{mod}\Lambda \mid \text{Ext}_\Lambda^{>0}(T, X) = 0 \}$$

Rem $T : \text{SO} \iff T \in T^\perp$

Def Let $\mathcal{C} \subseteq \text{mod}\Lambda$: subcat
closed under extensions & direct summands.

(1) $P \in \mathcal{C}$ is Ext-proj in \mathcal{C}

$$\iff \text{Ext}_\Lambda^1(P, \mathcal{C}) = 0$$

(2) $P \in \mathcal{C}$ is an Ext-progenerator

$\iff \exists P \in \mathcal{C} : \text{Ext-proj}$

$$\mathcal{C} \subseteq \mathcal{C},$$

$$0 \rightarrow C_1 \rightarrow P_0 \rightarrow C \rightarrow 0$$

: fx with $C_1 \in \mathcal{C}$, $P_0 \in \text{add } P$.

Duality defines Ext-inj (cogenerator)

Ex

$$(1) \quad \Lambda^\perp = \text{mod}\Lambda,$$

Λ : Ext-progen. of $\text{mod}\Lambda$.

(2) T : tilt. mod. with $\text{pd } T \leq 1$,

$$\sim T^\perp = \text{Fac } T \text{ (Gen } T),$$

with T : Ext-progen of T^\perp .

Def $T \in \text{mod}\Lambda$ is projectively
Wakamatsu filtering (pw-tilt)

if (1) $T : \text{SO}$

(2) T is an Ext-progen of T^\perp .

Ex

(1) Λ : pw-filt

(2) Every tilt. module (with fin. proj. dim)
is pw-filt

[Auslander-Reiten, "Applications of
contravariantly ..."]

Relation to. (W-tilt)

Wakamatsu tilting modules

Def Let $T \in \text{mod } A$.

Define $\mathcal{Y}_T \subseteq T^\perp$ by

$\gamma \in \mathcal{Y}_T : \Leftrightarrow \exists \cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} \gamma \rightarrow 0$
 : ex with $\text{Im } f_i \in T^i$
 $(\forall i \geq 0)$

$\hookrightarrow T$ is an Ext-progen. of \mathcal{Y}_T
 if $T: \text{SO}$

$T: \text{SO}$	$\mathcal{Y}_T \subseteq T^\perp$
progen	T
inj cogen	??

This is T

$\Leftrightarrow T: \text{pw-tilt.}$

Def $T: W\text{-tilt}$

$\Leftrightarrow D\Lambda \in \mathcal{Y}_T$

$(\Leftrightarrow D\Lambda$ is an Ext-irg cogen. of \mathcal{Y}_T)
 [Auslander Reiten]

Prop TFAE for $T \in \text{mod } A$

(1) $T: \text{pw-tilt}$

(2) $T: W\text{-tilt}$ s.t. $\mathcal{Y}_T = T^\perp$

Nice subclass

$$\{ \text{pw-tilt} \} \subseteq \{ W\text{-tilt} \}$$

Rmk

• This inclusion may proper.

• $\{ W\text{-tilt} \}$ is self-dual

i.e., $T: W\text{-tilt}$ in $\text{mod } A$

$\Leftrightarrow DT: W\text{-tilt}$
 in $\text{mod } A^{\text{op}}$.

§ Main Results

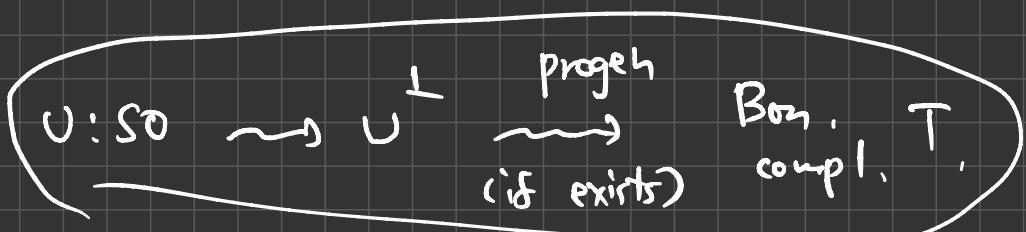
Def

Let $U \in \text{mod}\Lambda : \text{SO}$.

The Bongartz completion T of U
is a pW-tilt T s.t.

- (1) $U \in \text{add } T$
- (2) $T^\perp = U^\perp$.

(if exists)



Rem It's compatible with
that of (classical) tilting modules.

Thm 1.

Let $U \in \text{mod}\Lambda : \text{SO}$. TFAE

(1) U has a Bon. compl.

(2) U^\perp has a finite cover

$\left(:\Leftrightarrow \exists M \in U^\perp \text{ s.t. } \forall X \in U^\perp \exists M^n \xrightarrow{\sim} X : \text{surj} \right)$

Rem

(2) $\Leftrightarrow \Lambda$ has left U^\perp -approx

$\Leftrightarrow U^\perp$ is covariantly finite.

($\Rightarrow ?$. I don't know).

{induces in $U^\perp\} / \cong$

Cor

If $U \in \text{mod}\Lambda$ is SO and $\#\text{ind } U^\perp < \infty$

then $\exists X \in \text{mod}\Lambda$ s.t.

$U \oplus X$ is pW-tilt.

Thm 2.

Suppose $T \in \text{mod } \Lambda$ satisfies $\#^{\text{ind}} T < \infty$.

(e.g. Λ is rep-fin).

Then TFAE

(1) $T: pW\text{-tilt}$

number of
indec submods.

(2) $T: W\text{-tilt}$.

(3) $T: \text{SO}$ with $|T| = |\Lambda|$

(4) $T: \text{maximal SO}$

$\left(\begin{array}{l} \Leftrightarrow \\ \bullet T: \text{SO} \\ \bullet T \oplus M: \text{SO} \Rightarrow M \in \text{add } T \end{array} \right)$

Ex

$$\Lambda: \begin{array}{c} 1 \xrightarrow{a} 2 \\ d \uparrow \quad \downarrow b \\ 4 \leftarrow 3 \\ c \end{array} \quad \langle abc, bcd, cdab \rangle?$$

$$T := \underbrace{P(2) \oplus P(3)}_{\text{proj-inj}} \oplus P(4) \oplus \begin{pmatrix} 1 \\ 2 \\ \text{or} \\ 3 \\ 4 \end{pmatrix}$$

: $pW\text{-tilt}$ with $\text{pd } T = \text{id } T = \infty$.

key lemma

Let $\varepsilon: \text{Hom-fin. Krull-Schmidt exact}$

cat with proj P ,
inj cogen I .

If $\#^{\text{ind}} \varepsilon < \infty$,

then $|P| = |I|$

Rem In general, (1)-(4) are
not equiv. for $T: \text{SO}$

$T: pW\text{-tilt} \implies T: W\text{-tilt}$

? ?

? ?

X

$|T| = |\Lambda|$? ? $T: \text{maximal SO}$

Other result

Thm Let Λ be a ~~rep fin~~

Iwanaga-Gorenstein alg

$$(\text{id}_{\Lambda} = \text{id}_{\Lambda} < \infty).$$

(1) T : tilting (\Leftrightarrow cotilt)

\Leftrightarrow pw-tilt.

\Leftrightarrow W-tilt.

(2) $V \in \text{mod } \Lambda : \text{SO}$

$$\Rightarrow \text{pd } V < \infty. \quad \boxed{\downarrow}$$

(3) Take Bongarts comp T of V

$\rightsquigarrow T$: pw-tilt

$\rightsquigarrow T$: tilt by (1)

$\rightsquigarrow \text{pd } T < \infty.$

\rightsquigarrow : ~~unknown~~ $\text{pd } T < \infty$ by $V \in \text{add } T$



Conjectures

We conjecture the following.

Boundedness Conjecture (BC) [Happel]

If $T \in \text{mod } \Lambda : \text{SO}$,

$$|T| \leq |\Lambda|.$$

Maximal SO Conj (MSOC)

Every W-tilt is maximal SO.

Weak maximal SO Conj (wMSOC)

Every pw-tilt is maximal SO.

Proj = Inj Conj (PIC)

T : W-tilt $\Rightarrow |T| = |\Lambda|$.

\Leftrightarrow Key Lem holds w.o.
the assumption $\# \text{ind } \mathcal{E} < \infty$

Prop The following implication holds:

$$\begin{array}{c} \text{(BC)} \Rightarrow \text{(PI C)} \\ \text{(BC)} \Rightarrow \text{(MSOC)} \Rightarrow \text{(w MSOC)} \end{array}$$

\Updownarrow

Ausland-Reiten Conj
 Λ is maximal SD.

. These are all true if Λ : rep-fin.

Rem Dually we can consider
dual of $\Lambda \hookrightarrow X_T \subseteq {}^\perp T$ (T as rigid)

(1) $X_\Lambda = \text{GP} \Lambda$: the cat of Gorenstein-proj modules.

(2) T : injectively W-tilt (iW-tilt)
 $\Leftrightarrow T$: Ext-rigid of ${}^\perp \Lambda$.

Then Λ_Λ is W-tilt, and

Λ is iW-tilt

$\Leftrightarrow \text{GP} \Lambda = {}^\perp \Lambda$

$\Leftrightarrow \Lambda$: weakly Gorenstein
[Ringel-Zhang] \square

Cor If $\# {}^+ \Lambda < \infty$, then

$\text{GP} \Lambda = {}^\perp \Lambda$.

\because Apply dual of Thm 2 to
 $\Lambda_\Lambda \hookrightarrow \Lambda$: iW-tilt

