

Maximal self-orthogonal modules

and a new generalization of

tilting modules

Haruhisa Enomoto

(Osaka Metropolitan Univ)

§ 1. Intro

Setting

Throughout this talk, k : field.

A : a finite-dimensional k -algebra

Modules = finitely generated A -modules

Thm (Krull - Schmidt)

Every A -module X is uniquely written as

$$X = X_1 \oplus \dots \oplus X_m$$

for X_i : indecomposable A -module \oplus_i

\rightsquigarrow Considering A -module

is equiv. to Considering **subsets** of

$\text{ind } A := \{ \text{indecomposable } A\text{-modules} \} / \cong$.

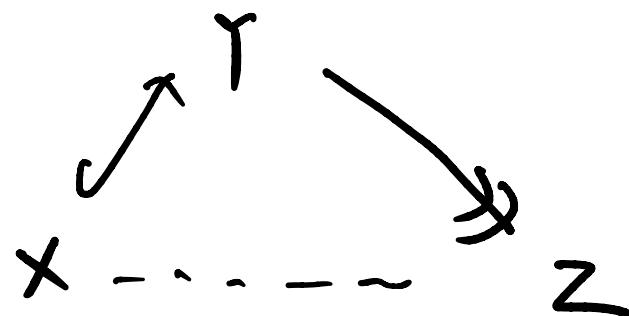
Example

$$A = K[1 \rightarrow 2]$$

(other: demo of
FD Applet)

$\rightsquigarrow \text{ind } A :$

$(\text{mod } A)$



Def

(SO)

$M : A\text{-module}$ is self-orthogonal

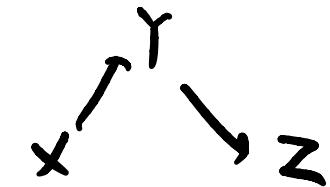
$\Leftrightarrow \operatorname{Ext}_A^i(M, M) = 0 \text{ for } \forall i > 0.$

$\left(\Leftrightarrow \operatorname{Ext}_A^{>0}(M_i, M_j) = 0 \text{ for } \forall i, j \text{ in } M = M_1 \oplus \dots \oplus M_d : \text{index decomp} \right)$

Example

$$A = K[1 \rightarrow 2]$$

ind A :



- $A = X \oplus Y$ is SO (A is always SO)
- $X, Y, Z : \text{SO}$
- $X \oplus Z : \text{NOT SO}$ by $\operatorname{Ext}_A^1(Z, X) \neq 0$.

SO modules are elementary, but

Open Conjectures

(1) [Happel, Boundedness Conj]

If $M : SO$ A -module,

then

$$|M| \leq |A|$$

rank of algebra
(# vertices)

(2) [Auslander - Reiten Conj]

A is maximal SO module,

that is, $\{$

- A is SO
- For $M : \text{indec}$, if $A \oplus M : SO$,
then M is a direct summand of A

Today

Give affirmative answer for these conjs

if A is representation-finite,

$$\left(:\Leftrightarrow \# \text{ind } A < \infty \right)$$

using the theory of

(projectively) Wakamatsu tilting modules.

若松

§ 1.

Recall

A-module P is a **progenerator**

\Leftrightarrow (1) $\forall X: A\text{-module}, \text{Ext}_A^{>0}(P, X) = 0$

proj

(2) $\forall X: A\text{-mod},$

\exists surjection $P^{\oplus n} \rightarrow X$

generator

Rem

(a) $P : \text{proj} \Rightarrow P : \text{so} \quad (\text{Ext}_A^{>0}(P, P) = 0)$

(b) $P : \text{progen} \Rightarrow P \cong A$ up to direct summands
(so there's only one progen)

Def

$T : A\text{-module}$ is projectively Wakamatsu tilting
 (pW-tilt)

\Leftrightarrow (1) $T : \text{SO}$ ($\text{Ext}_A^{>0}(T, T) = 0$)

(2) If $X : A\text{-module}$ satisfies

$\text{Ext}_A^{>0}(T, X) = 0$, then

\exists surj $T^{\oplus n} \longrightarrow X$.

Rem

- $T : \text{pW-tilt} \iff T$ is a progen of
 exact cat $T^\perp := \{X : A\text{-mod} \mid \text{Ext}_A^{>0}(T, X) = 0\}$,
- Progenerators are pW-tilt.

Def

$T : A\text{-module}$ is projectively Wakamatsu tilting
(pW-tilt)

\Leftrightarrow (1) $T : \text{SO}$ ($\text{Ext}_A^{>0}(T, T) = 0$)

(2) If $X : A\text{-module}$ satisfies

$\text{Ext}_A^{>0}(T, X) = 0$, then

\exists surj $T^{\oplus n} \longrightarrow X$.

Rem.

$|T|$: smaller \rightsquigarrow bigger

Above

(1) likely not likely

(2) not likely likely

Def

$T : A\text{-module}$ is tilting

: \iff (0) $\boxed{\operatorname{pd} T < \infty}$

(1) $T : \mathcal{S}\mathcal{O}$

(2) $\exists 0 \rightarrow A \rightarrow T_0 \xrightarrow{\pi_0} \dots \rightarrow T_d \xrightarrow{\pi_d} 0$

: exact seq

Fact • $A \in \{ \text{tilting} \} \subset \{ \text{pW-tilt} \}$

[Auslander - Reiten]

• $T : \text{tilting} \left(\Rightarrow A \text{ and } \operatorname{End}_A(T) \text{ are derived equiv} \right)$
 $\Rightarrow |T| = |A|.$

§ Results

Main Thm

Suppose A is rep-fin.

(1) $\forall M : \text{SO } A\text{-mod}, \exists X \text{ s.t. } M \oplus X : \text{pW-tilt.}$

(2) TFAE for $T : A\text{-module}$

Easy to compute

(i) $T : \text{pW-tilt}$

(ii) $T : \text{SO}$ with $|T| = |A|$

(iii) $T : \text{maximal SO}$

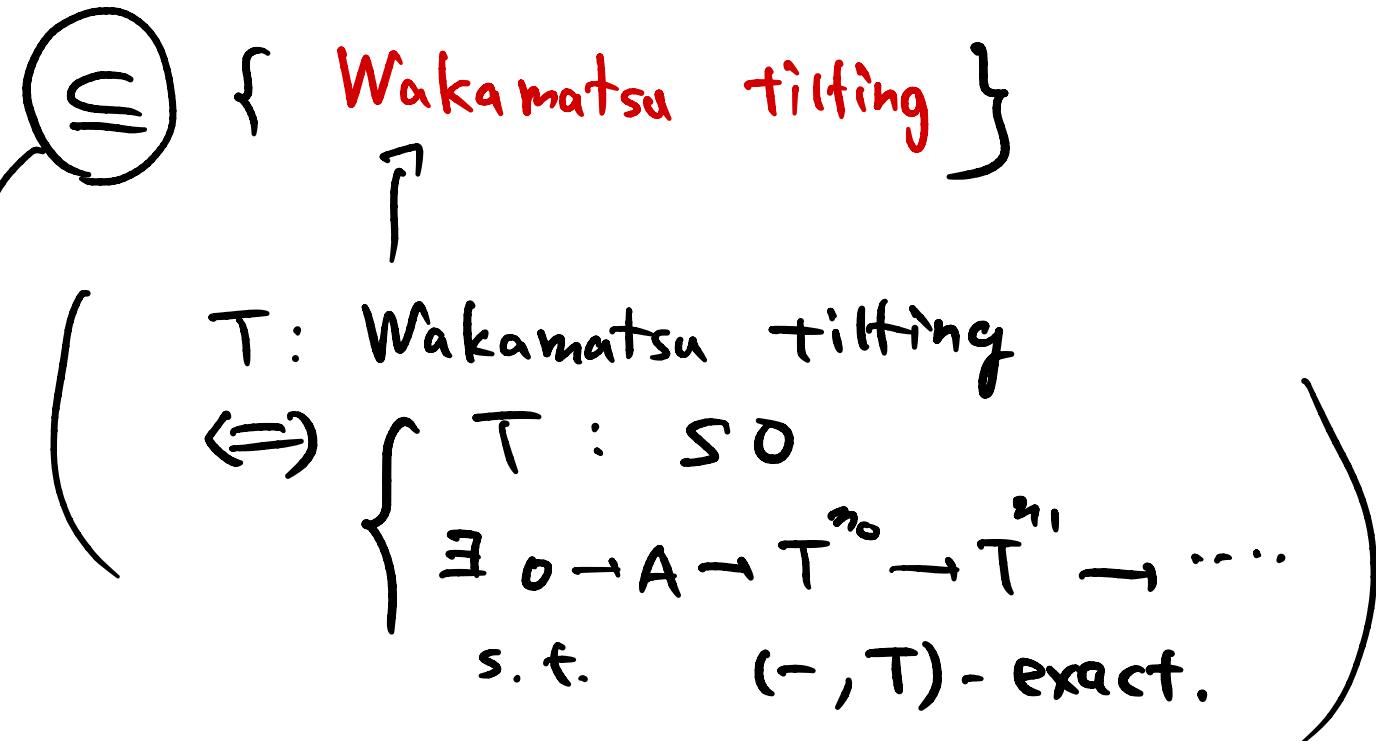
($M : \text{indec, } T \oplus M : \text{SO} \Rightarrow M \triangleleft T$)

(i) $T : \text{Wakamatsu tilting}$

\rightsquigarrow If A is rep-fin, $\begin{cases} \text{if } M : \text{SO} \Rightarrow |M| \leq |A| \text{ (Boundedness)} \\ \text{if } A \text{ is maximal SO. (AR conj)} \end{cases}$

Wakamatsu tilting modules

- $\{ \text{pw-tilt} \}$ coincide if rep-fin



Naturally arise when considering exact category

$$\begin{array}{ccc}
 A & (P, I) & \xrightarrow{\sim} (A, T) \\
 \uparrow & \uparrow & \uparrow \\
 \text{progen} & \text{inj cogen} & \text{alg W-tilt } A\text{-mod.}
 \end{array}$$

of exact cat

If A : not rep-fin, then for $T: \text{SO}$,

$T: \text{pW-tilt}$



$T: \text{W-tilt}$



$|T| = |A|$



$T: \text{maximal SO}$

ALL other implications are open,

and is related "Homological Conjectures" (AR conj...)

Summary figure

Proj.dim

0 A "full rank?" ; "partial"

}

finite { tilting module }

}

(possibly) ∞ { projectively Wakamatsu
tilting }

$T: SO$
 $pd < \infty$

∩

completion

∩

II

$T: SO$

{ Wakamatsu tilting
modules }

• : rep-fin