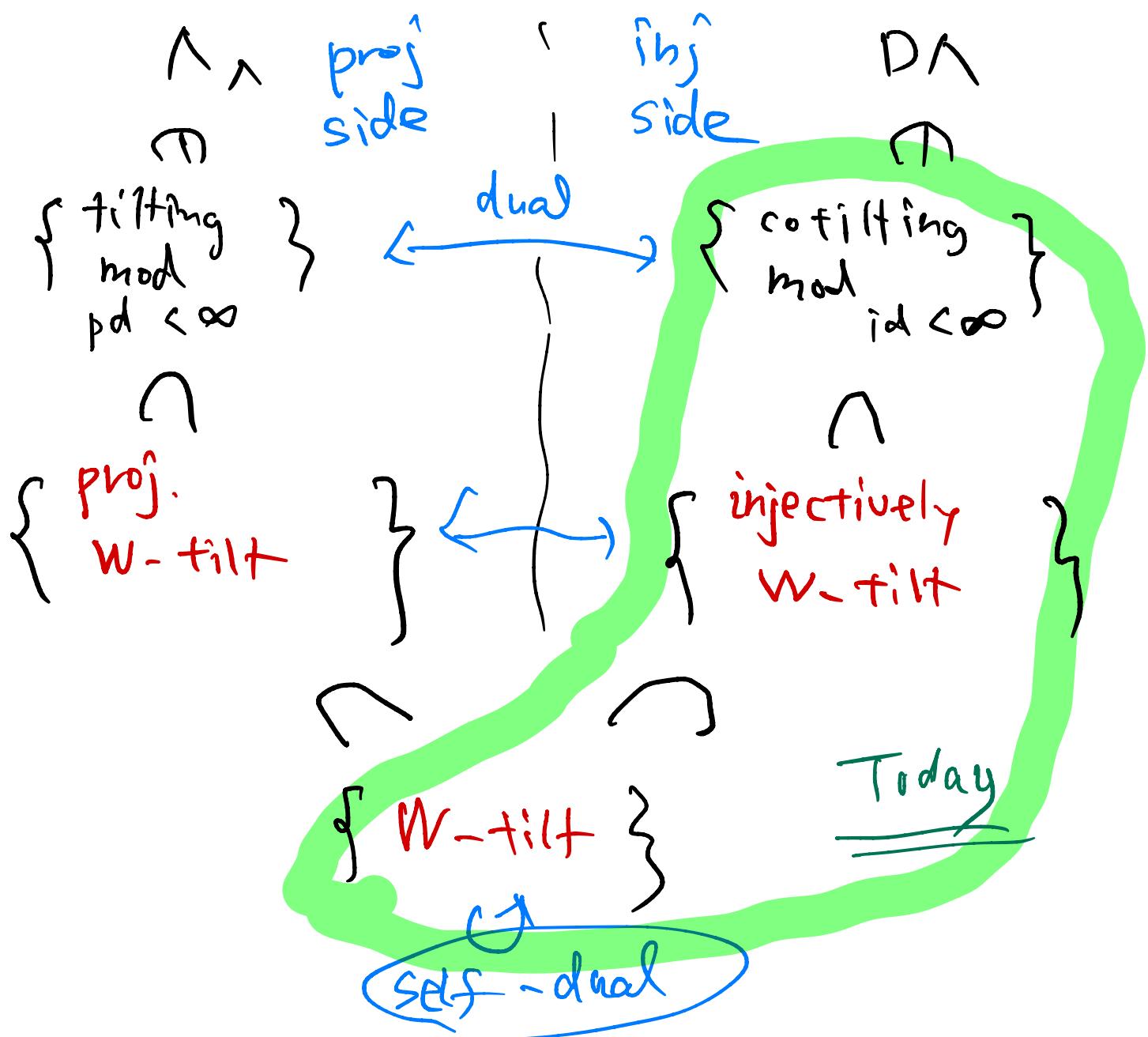


# Orthogonal modules and ~~(projectively)~~ Wakamatsu tilting modules

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~~injectively~~ W-tilt

## Overview

Hierarchy of "full rank" ortho. modules.



## Setting

$\Lambda$ : f.d.  $k$ -alg /  $k$ : field

$\text{mod } \Lambda$ : the cat of f.g. right  $\Lambda$ -modules.

Def Let  $\mathcal{E} \subseteq \text{mod } \Lambda$  be a subcat.

(1)  $P \in \mathcal{E}$  is progenerator of  $\mathcal{E}$ .

$\Leftrightarrow$  (i)  $\text{Ext}_{\Lambda}^1(P, \mathcal{E}) = 0$

(ii)  $\forall X \in \mathcal{E}$ ,  $\exists$  s.e.s.

$$0 \rightarrow X' \rightarrow P_0 \rightarrow X \rightarrow 0$$

s.t.  $X' \in \mathcal{E}$ ,  $P_0 \in \text{add } P$ .

(2) Dually for inj. cogen. of  $\mathcal{E}$ . ]

W-tilting theory.  $\text{Ext}_{\Lambda}^{>0}(T, T) = 0$

Def Let  $T \in \text{mod } \Lambda$  be orthogonal.

Define  $X_T \subseteq {}^{\perp} T \subseteq \text{mod } \Lambda$ .

(1)  $X \in {}^{\perp} T : \Leftrightarrow \text{Ext}_{\Lambda}^{>0}(X, T) = 0$ .

(2)  $X \in X_T$

$\Leftrightarrow$  (i)  $X \in {}^{\perp} T$

(\*)

(ii)  $\exists 0 \rightarrow X \rightarrow T_0 \rightarrow T_1 \rightarrow \dots$   
 s.t.  $T_i \in \text{add } T$   
 and.  $\text{Hom}_\Lambda(X, T)$  is  
 exact. ex. 

Ex  
 $X_{D\Lambda} = {}^\perp D\Lambda = \text{mod } \Lambda.$

$X_\Lambda = \text{GP } \Lambda$  (= Gorenstein-proj).

Thm Let  $T \in \text{mod } \Lambda$  : ortho.

(1)  $\Lambda_T$  is a progen of  ${}^\perp T$ .

(2)  $T$  is an inj cogen. of  $X_T$ .

$T: \text{ortho}$	$X_T \subseteq {}^\perp T$
progen	$\Lambda_T$ iff $T: w\text{-tilt.}$
inj cogen	$T$ iff $T: iW\text{-tilt.}$

Def  $T \in \text{mod } \Lambda$  is **cotilting**

if (1)  $\text{id } T < \infty$

(2)  $T$ : ortho.

(3)  $\exists 0 \rightarrow T_n \rightarrow \dots \rightarrow T_0 \rightarrow D\Lambda \rightarrow 0$

:  $\mathbb{R}$  with  $T_i \in \text{add } T$ .

Thm [Auslander - Reiten '91]

If  $T$  is cotilt, then

$$X_T = {}^\perp T.$$

$\left( \rightarrow {}^\perp T \text{ has progeny} \begin{array}{c} \nearrow \\ \text{inj cogen } \end{array} \begin{array}{c} \nwarrow \\ T \end{array} \right)$

Def  $T \in \text{mod } \Lambda$  is **W-tilt**.

if (1)  $T$ : ortho.

(2)  $\Lambda \in X_T$

( $\iff$   $\Lambda$  is a progeny of  $X_T$ )

Ex

(1)  $\{\text{cotilt}\} \subset \{\text{W-tilt}\}$

(2)  $T_n$  is W-tilt  $\iff \Lambda(DT)$  is W-tilt

(3)  $\{ \text{tilt} \} \subset \{ W\text{-tilt} \}$ .

(4)  $\Lambda_\lambda$  is  $W\text{-tilt}$ .

$$\hookrightarrow X_\lambda = \text{EPA}$$



Rem

Consider a "pair"  $(P, I)$  s.t.

$P$ : progen,  $I$ : inj cogen of some exact cat  $\left( \begin{array}{l} \stackrel{\text{def}}{=} E \subseteq_{\text{mod}} \Gamma \\ \exists \Gamma : \text{f.d. alg.} \end{array} \right)$   
ext-closed.

(1) If  $T_\lambda : W\text{-tilt}$ ,

$(\Lambda, T)$  is such a pair  
( $X_T$ ).

[E] (2) If  $(P, I)$  : pair (+ assump)

$(P, I)$   $\xrightarrow{\text{"equiv"}}$   $(\Lambda, T)$

for some alg  $\Lambda$ ,  $T_\lambda : W\text{-tilt}$ .

$W\text{-tilt}$  = "universal  
inj cogen"

Def  $T \in \text{mod } \Lambda$  is injectively  
W-tilt (iW-tilt) if.

(1)  $T$ : ortho,

(2)  $T$  is an inj cogen of  ${}^+T$ .

( $\Leftarrow$ )  $\forall X \in {}^\perp T, \exists x \hookrightarrow T^n$ )

Prop

$T$ : iW-tilt  $\Leftrightarrow T$ : W-tilt s.t.

$$X_T = {}^\perp T.$$

Ex

(1)  $\{\text{cotilt}\} \subset \{\text{iW-tilt}\}$

(2)  $\Lambda_X$  is iW-tilt

$$\Leftrightarrow X_{\Lambda} = {}^+ \Lambda \\ \Downarrow \\ \text{Gp } \Lambda$$

$\Leftrightarrow: \Lambda$ : weakly Gorenstein.

## Main Results

### Classical Result [Bongartz]

$T \in \text{mod } \Lambda$  :  $i\text{W-tilt}$  with  $\underline{\text{id}} \leq 1$

$$\Leftrightarrow \left\{ \begin{array}{l} \circ \text{id } T \leq 1 \\ \circ T : \text{ortho.} \end{array} \right.$$

$$\circ |T| = |\Lambda|$$

$\cdots 0 \rightarrow T_1 \rightarrow T_0 \rightarrow D\Lambda \rightarrow 0$   $\sim \frac{\# \text{ of indec summands}}{\# \text{ of } T} \geq 1$

This depends "completion"

which works for  $\underline{\text{id}} \leq 1$

At least for  $\Lambda$  : rep-fin?

Thm Let  $\Lambda$  be rep-fin.

(1) For any  $M \in \text{mod } \Lambda$  : ortho,

$\exists M' \in \text{mod } \Lambda$ ,  $M \oplus M'$ : (i)  $i\text{W-tilt}$

(2) TFAE for  $T \in \text{mod } \Lambda$

(i)  $T$  :  $i\text{W-tilt}$

(ii)  $T$  :  $\text{W-tilt}$

(iii)  $T$  : ortho s.t.  $|T| = |\Lambda|$ .

(iv)  $T$ : maximal orthogonal

$\Leftrightarrow$   $\begin{array}{l} \circ T : \text{ortho.} \\ \circ T \oplus M : \text{ortho} \\ \Rightarrow M \subset \text{add } T. \end{array}$

Cor  $\Lambda$  : rep-fin. then.

(1)  $M$  : ortho  $\Rightarrow |M| \leq |\Lambda|$ .

(2)  $T \in \text{mod } \Lambda$  : W-tilt

$$\Rightarrow X_T = {}^\perp T.$$

(3)  $\text{GP } \Lambda = {}^\perp \Lambda$  ( $\Lambda$  : weakly Cor)

Related Conj

(Boundedness Conj [Happel])

$M$  : ortho  $\Rightarrow |M| \leq |\Lambda|$ .

(Proj = Inj Conj)

$T_\Lambda$  : W-tilt  $\Rightarrow |T| = |\Lambda|$

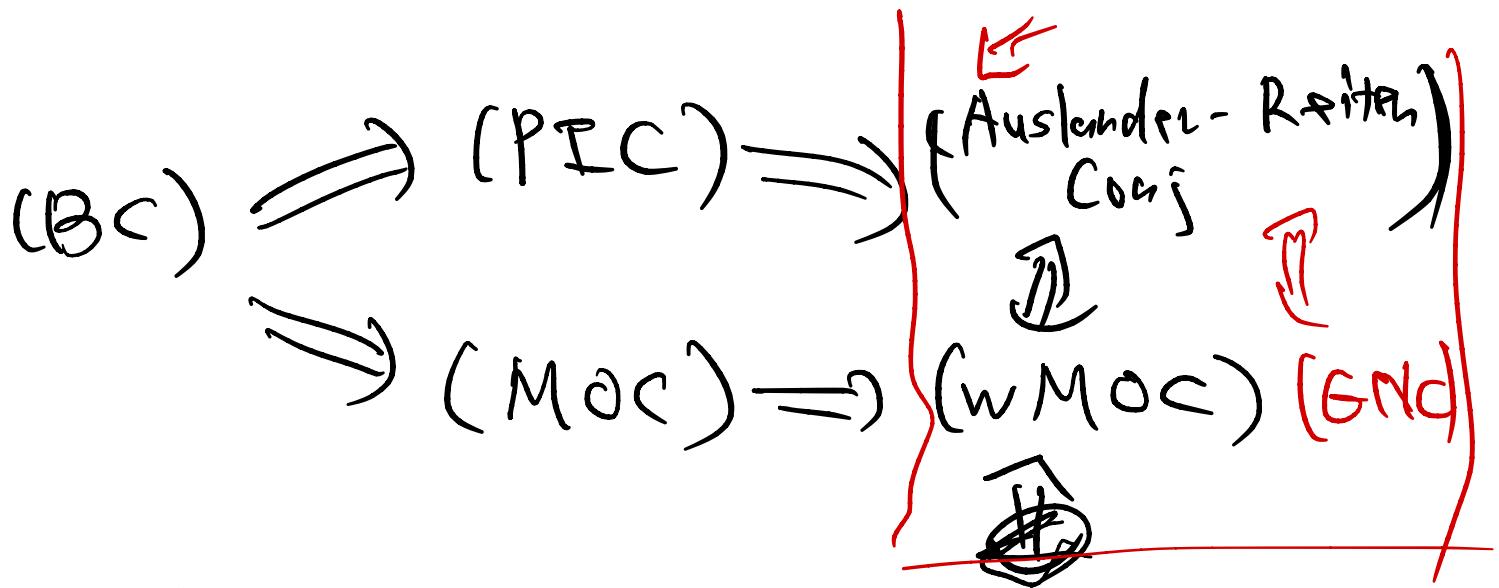
(weak)

(Maximal Ortho. Conj)

$T_\Lambda$  : W-tilt  $\Rightarrow T$  : maximal ortho,

(is W-tilt)

$\Lambda$  is maximal ortho.



These are trap

$T: W\text{-tit}$  if  $\Lambda$  is rep-fin,

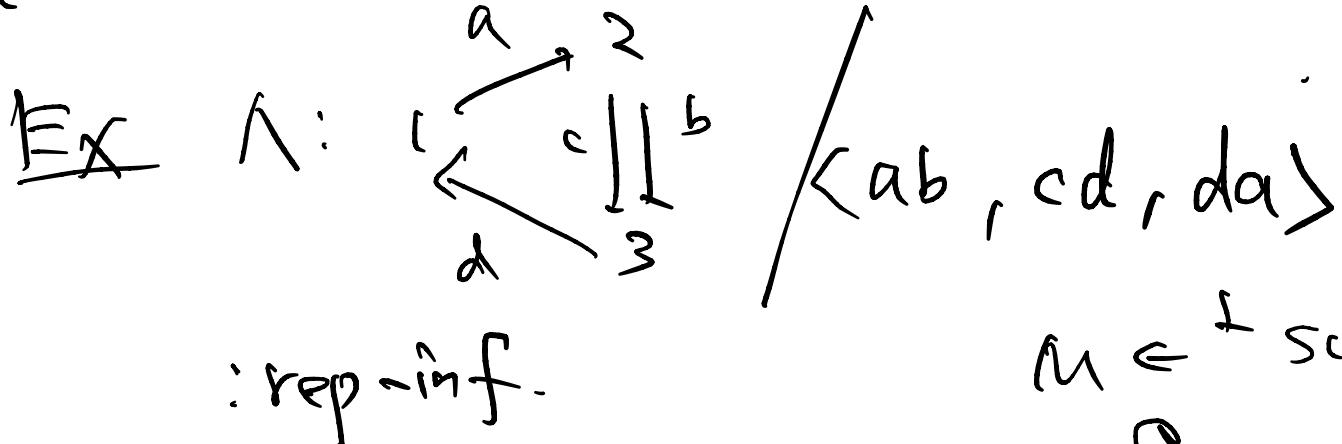
$\vdash T$  has inj cog  
 $T: iW\text{-eif}$   
 $\vdash$  GP $\Lambda = \vdash \Lambda$

$T: iW\text{-eif} \Rightarrow W\text{-eif}$

~~max~~

$$|T| \approx \omega$$

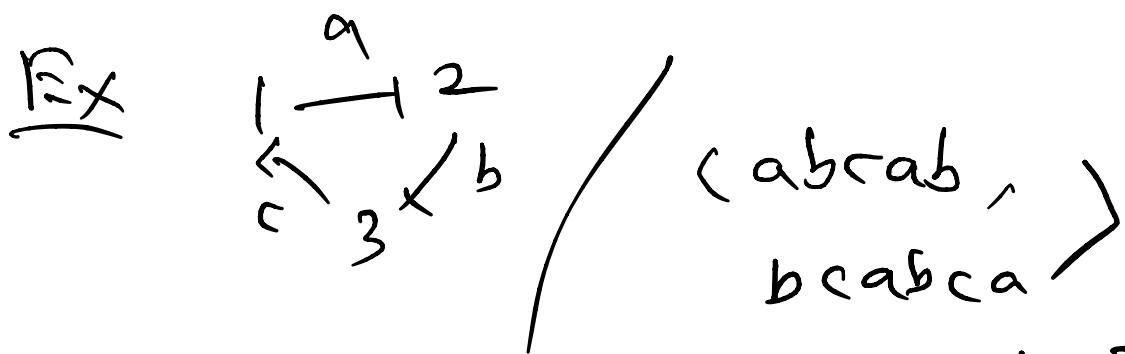
[Rickard-Schofield]



$$\begin{matrix} M \in & {}^{\perp} \text{Scl} \\ & \cap \text{Scl}^{\perp} \end{matrix}$$

$S(\vee)$  : simple at 1

$\Rightarrow S(\wedge)$  : maximal ortho.



$$\underbrace{P(2) \oplus P(3)}_{\text{proj-1-ij}} \oplus X \left( X : \begin{pmatrix} \frac{1}{2}, \frac{2}{3}, \\ \frac{3}{1}, \frac{1}{2}, \frac{2}{3}, -1 \end{pmatrix} \right)$$

$\leadsto$  W-filt,  $\text{pd } M = \text{id} = \infty$

Conj

$\Lambda$ : Iwanaga-Gorenstein

$\Lambda$ : self-inj

$\cap M$ : ortho.

$$\Rightarrow \text{pd } M < \infty$$

(This is true for rep-fin)

$M : W\text{-tilt}$   $\longleftrightarrow$   $M : \text{tilt}$   
 $M : \text{cofilt}$

$\exists \lambda_\lambda$  s.t.  $\neg\lambda \neq \text{GPA}$   
 " "  
 $\lambda_\lambda$   
 (  $\lambda$ : not weakly G or )

$$\begin{aligned} \lambda_{\lambda}: W - \pi^*(t) & , \\ \lambda_{\lambda}: i_{W - \pi^*(t)} & \Leftrightarrow t_{\lambda} = \text{GP}(\lambda). \end{aligned}$$