MODS seminar 2020/5/8

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Bruhat inversions in Weyl grp and torsion-free classes over preproj alg.

§0. Intro

§1. Preliminaries on Preprojective alg.

§2. Bruhat inversions, Main Results

§3. Idea of Proof, Brick sequence

§0. Introduction

- Q : Dynkin quiver (ADE)

$\rightsquigarrow$ Π_Q : preprojective alg, W : Weyl grp

For $w \in W$, we can define

$I_w \subseteq \Pi$: 2-sided ideal.

$\rightsquigarrow$ $\mathcal{F}(w) := \mathrm{Sub}\, \Pi / I_w$

: subcats of $\mathrm{mod}\,\Pi$.

- introduced in Buan-Iyama-Reiten-Scott [BIRS]
- is Frobenius and stably 2-CY with cluster-tilting objects

$\rightsquigarrow$ categorifies the cluster alg str.

on the cordinate ring of unipotent cell in Lie grp [Geiss-Lecrec-Schröer]

Today's Aim : study $\mathcal{F}(w)$ via Root System!

§1. Preliminaries

Notation • k: field • module $=$ f.g. right

• Q: Dynkin quiver (A, D, E)

Def
$\Pi := \Pi_Q$: ==preproj alg== is defined by

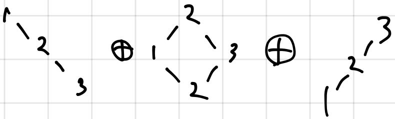
$$kQ̄ \Big/ \left\langle \sum_{a \in Q_1} aa^* - a^*a \right\rangle ,$$

where $\overline{Q}$: double quiver is $Q + i \xleftarrow{\alpha^*} j$ for each $i \xrightarrow{\alpha} j$ $\in Q_1$

Ex $Q: 1 \to 2 \to 3 \rightsquigarrow \overline{Q}: 1 \rightleftarrows 2 \rightleftarrows 3$

$\Pi_Q = k\overline{Q} / \{ \ldots = 0, \ldots \}$

$=$ (over Π-mod)



Fact • Q: Dynkin $\Leftrightarrow$ Π_Q: fin. dim, and

• Π_Q doesn't depend on the orientation of Q

Def
• Φ_Q: the root system of type Q,
$\Phi_Q = \Phi^+ \sqcup \Phi^-$ (positive, negative roots)

• α_i $(i \in Q_0)$: simple root

• $s_i := s_{\alpha_i}$: simple reflection

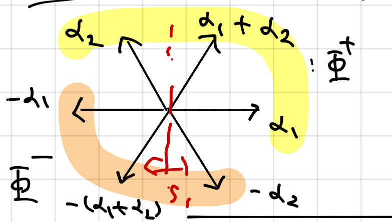
• $W_Q := \langle s_i \mid i \in Q_0 \rangle$: Weyl grp.

Rmk
• If you don't know root system, then

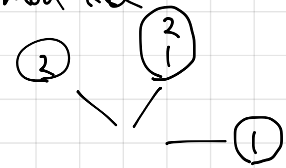
$$\Phi_Q = \underbrace{\{ \underline{\dim} M \mid M: \text{indec } kQ\text{-mod} \}}_{= \Phi^+} \sqcup \underbrace{\{ -\underline{\dim} M \}}_{= \Phi^-}$$

Gabriel $\quad \ni \underline{\dim} S_i = \alpha_i$

Ex $Q: 1 \leftarrow 2$: A_2



mod kQ



$s_1(\alpha_1) = -\alpha_1$ $\quad s_1(\alpha_2) = \alpha_1+\alpha_2, \quad s_1(\alpha_1+\alpha_2) = \alpha_2$

Fact W is presented by gen $= \{s_i\}_{i\in Q_0}$

with relation • $s_i^2 = e$, $s_i s_j s_i = s_j s_i s_j$ (if $i - j$), $s_i s_j = s_j s_i$ ($i \not- j$)

Def $w = s_{u_1} \cdots s_{u_l} \in W$: reduced exp

if l is minimal

e.g. $Q: 1 - 2$ | $s_1 s_2$: reduced

$\underbrace{s_1 s_1}_{= e}$, $s_1 s_2 s_1 s_2$: not reduced

$s_{2122} := s_2 s_1 \underbrace{s_2 s_2}_{= e}$ $= s_2 s_1$

Construction of I_w

• For $i \in Q_0$, $I_i := \langle 1 - e_i \rangle \trianglelefteq \Pi$: ideal

• For $w \in W$, take reduced exp

$$w = s_{u_1} s_{u_2} \cdots s_{u_l}$$

Def $I_w := I_{u_l} \cdots I_{u_2} I_{u_1} \trianglelefteq \Pi$

$\trianglelefteq M$

Ex $M I_i$: delete i in top M !

$Q: 1 - 2 - 3$, $w = s_{123} = s_1 s_2 s_3$

• $\Pi = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix}$

$\cup$

delete 3 in top $\rightsquigarrow$ $\Pi I_3 = I_3 = - \oplus - \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$

$\cup$

delete 2 in top $\rightsquigarrow$ $I_3 I_2 = - \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \oplus 1$

$\overset{1}{\rightsquigarrow}$ $I_w = I_{321} = \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3\\2&\end{smallmatrix} \oplus \phi = 0$

Note $\Pi / I_{321} = 1 \oplus \begin{smallmatrix}&2\\1&\end{smallmatrix} \oplus \begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix}$

Fact: $w \mapsto I_w$ is well-defined.

Def For $w \in W$, $\mathcal{F}(w) = \underbrace{\mathrm{Sub}\ \Pi/I_w}_{\ni e_w} \subset \mathrm{mod}\,\Pi$

($\mathrm{Sub}\,\square := \{$ submods of $\oplus$ of $\square\}$)

Thm [Mizuno] $w \mapsto \mathcal{F}(w)$ is a bij between W and $\mathrm{torf}\,\Pi$, the set of torsion-free classes in $\mathrm{mod}\,\Pi$.

Rmk

All torf in $\mathrm{mod}\ \underset{\text{path alg!}}{KQ}$ are restriction of $\mathcal{F}(w)$

($\mathcal{F} = \mathcal{F}(\overset{\exists w}{w}) \cap \mathrm{mod}\,KQ$ via $\Pi \twoheadrightarrow KQ$)

Def $\mathcal{F}$: torf in $\mathrm{mod}\,\Pi$

- $0 \neq M \in \mathcal{F}$ is Simple in $\mathcal{F}$

$:\Leftrightarrow$ $\nexists$ $0 \to L \to M \to N \to 0$: ex with $0 \neq L, N \in \mathcal{F}$

- $\mathrm{sim}\,\mathcal{F} := \{$ simples in $\mathcal{F}\}/\cong$

Goal Describe $\mathrm{sim}\,\mathcal{F}(w)$!

Rmk
- $\mathcal{F}(w) = \mathrm{Filt}(\mathrm{sim}\,\mathcal{F}(w))$
- $\#\,\mathrm{sim}\,\mathcal{F}(w)$ is important!

(smaller simp $\Rightarrow$ nicer str)

Ex $w = w_0$; longest elem. $\rightsquigarrow I_w = 0$

$\therefore$ $\mathcal{F}(w_0) = \mathrm{mod}\,\Pi$, $\mathrm{Sim}\,\mathcal{F}(w_0) = \{$simple Π-mod$\}$

$\rightsquigarrow$ bij $\underline{\dim} : \mathrm{Sim}\,\mathcal{F}(w_0) \xrightarrow{\sim} \{$simple roots$\} \subset \Phi$

$S_i \mapsto \alpha_i$

Method: extend this bij!

$\mathcal{F} \subset \mathrm{mod}\,\Pi$: torf

$\Leftrightarrow$ $\forall\, 0 \to L \to M \to N \to 0$,

closed under ext (1) $L, N \in \mathcal{F} \Rightarrow M \in \mathcal{F}$

& sub (2) $M \in \mathcal{F} \Rightarrow L \in \mathcal{F}$

§2. Bruhat inv. results

Def For $w\in W$, its inversion set is $\mathrm{inv}(w) := \{\beta\in\Phi^+ \mid w^{-1}(\beta)\in\Phi^-\}$

Fact. If $w = s_{u_1}s_{u_2}\cdots s_{u_\ell}$: red. exp,

$$\mathrm{inv}(w) = \{\alpha_{u_1},\ s_{u_1}(\alpha_{u_2}),\ \dots,\ s_{u_1\cdots u_{\ell-1}}(\alpha_{u_\ell})\}$$

e.g. $w^{-1}s_{u_1}(\alpha_{u_2}) = s_{u_\ell}\cdots s_{u_2}s_{u_1}s_{u_1}(\alpha_{u_2})$
$= \underbrace{s_{u_\ell}\cdots s_{u_2}(\alpha_{u_2})}_{\text{negative}}$

$\#\mathrm{inv}(w)=\ell$

e.g. $Q: A_3$, $w = s_{1321}$

⇝ $\mathrm{inv}(w) = \{\alpha_1,$

$\beta\in (B)\,\mathrm{inv}(w)$
$\iff \ell(t_\beta w) < \ell(w)$
$\ell(t_\beta w) \overset{?}{=} \ell(w)-1$

Obs $\mathrm{inv}(w_0)=\Phi^+$! $\longleftrightarrow$ $\mathcal{F}(w_0) = \mathrm{mod}\,\Pi$

Prop $\mathcal{F}$: torf, $B\in\mathrm{sim}\,\mathcal{F}$

$\Rightarrow$ B : brick, i.e. $\mathrm{End}(B)$: division

(!!) If not, $\exists\, 0\neq f: B\to B$ ⇝ $0\to \underset{\in\mathcal{F}}{\ker f}\to B\to \underset{\in\mathcal{F}}{\mathrm{Im} f}\to 0$ □

Fact $\mathrm{brick}\,\mathcal{F}(w) \xrightarrow{\dim} \mathrm{inv}(w)$

$\mathrm{brick}\,\mathcal{F}(w) = \{B\in\mathcal{F}(w) \mid \text{brick}\}$

e.g. $Q: A_3$

$\begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix}$, $\begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$, $\begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix}$ $\mapsto \alpha_1+\alpha_2+\alpha_3$

bricks

$\begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}$: indec, non-brick, but $\alpha_1+2\alpha_2+\alpha_3$: not root

Def For $w\in W$,

$$\mathrm{Binv}(w) := \{\beta\in\mathrm{inv}(w) \mid \nexists\ \beta \overset{\text{non-trivially}}{=} \textstyle\sum \gamma_i \text{ with } \gamma_i\in\mathrm{inv}(w)\}$$

: the set of Bruhat inversions

simple roots inside $\mathrm{inv}(w)$

e.g. A_3, $w = s_{1321}$

⇝ $\mathrm{inv}(w) = \{1, 3, \underset{\alpha_1+\alpha_2+\alpha_3}{123}, 23\}$

⇝ $\mathrm{Binv}(w) = \{1, 3, \times, 23\}$

Thm [E] For $w \in W$, we have bij

$$\begin{array}{ccc} \underline{\dim} : \operatorname{sim} \mathcal{F}(w) & \xrightarrow{\sim} & \operatorname{Binv}(w) \\ \cap & & \cap \\ \operatorname{brick} \mathcal{F}(w) & \xrightarrow[\rightarrowtail]{\underline{\dim}} & \operatorname{inv}(w) \end{array}$$

$\operatorname{Binv}(w) = ?$

Prop $w = s_{u_1} \cdots s_{u_\ell}$: red. exp.

$\beta_i := s_{u_1} \cdots s_{u_{i-1}} (\alpha_{u_i})$

$(\rightsquigarrow \operatorname{inv}(w) = \{\beta_1, \cdots, \beta_\ell\})$

Then $\beta_i \in \operatorname{Binv}(w) \Leftrightarrow s_{u_1} \cdots \widehat{s_{u_i}} \cdots s_{u_\ell}$ is reduced.

e.g. A_3 $w = s_{21323}$

$\rightsquigarrow \hat{2}1323,\ 21\hat{3}23,\ 2132\hat{3}$: red.

$2\hat{1}323,\ 213\hat{2}3$: not

$\rightsquigarrow$ 3 Binvs.

inv(w)	β_1, β_2, ⋯ β_5
	2, 12, 23, 123, 1

§3. Idea of Proof

Recall $\operatorname{sim} \mathcal{F}(w) \subset \operatorname{brick} \mathcal{F}(w) \xrightarrow{\underline{\dim}} \operatorname{inv}(w)$

Hence, suffices to check which brick is simple.

- If B : brick is non-simple,

then $B =$ "ext. of bricks in $\mathcal{F}(w)$"

$\underset{\underline{\dim}}{\rightsquigarrow}$ $\underline{\dim} B =$ other inversions of w

$\Rightarrow$ $\underline{\dim} B$: not a Bruhat inv.

Thus. $\underline{\dim} B \in \operatorname{Binv}(w) \Rightarrow B$: simple

For converse, we'll use the brick sequence of $\mathcal{F}(w)$,

which categorifies seq $(\beta_1, \beta_2, \cdots, \beta_\ell)$

From now, fix red. exp $w = s_{u_1} \cdots s_{u_\ell}$

Idea: $0 = \mathcal{F}(e) \subset \mathcal{F}(s_{u_1 u_2}) \subset \cdots \subset \mathcal{F}(w)$

: saturated chain of torfs.

• $\mathcal{F}' \subset \mathcal{F}$: cover $\rightsquigarrow \exists B \in \mathcal{F}$: brick

(brick label) [DIRRT] $\quad \mathcal{F} = \mathrm{Filt}(\mathcal{F}' \cup B)$

Def For $1 \le i \le \ell$, (layer)

$$B_i := I_{u_{i-1}} \cdots I_{u_1} \big/ I_{u_i} I_{u_{i-1}} \cdots I_1$$

Lem • $\mathcal{F}(w) = \mathrm{Filt}\{B_1, \dots, B_\ell\}$

Thus $\mathrm{sim}\, \mathcal{F}(w) \subset \{B_1, \dots, B_\ell\}$

(sketch) $e < s_{u_1} < s_{u_1 u_2} < \cdots < s_{u_1 \cdots u_\ell} = w$

in right weak order

$\Rightarrow \quad \mathcal{F}(e) \leftarrow \mathcal{F}(s_{u_1}) \cdots \leftarrow \mathcal{F}(w)$

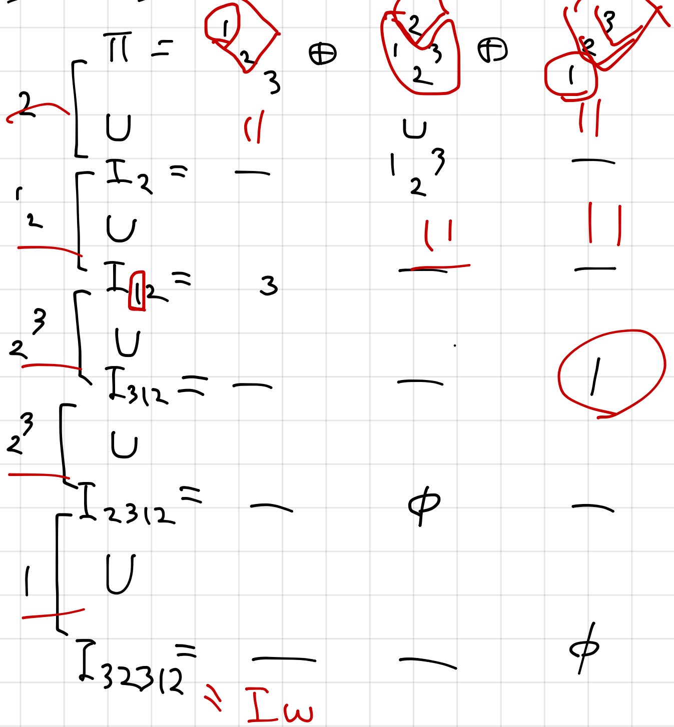
$\mathcal{F}(e) = 0$: chain in torf Π (path in Hasse quiver)

$\rightsquigarrow$ [DIRRT] brick label in torf Π

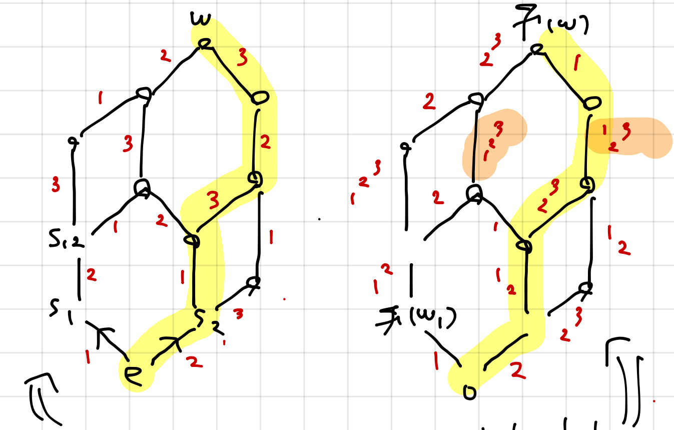
$$0 \xleftarrow{B_1} \circ \xleftarrow{B_2} \circ \cdots \xleftarrow{B_\ell} \mathcal{F}(w) \quad \text{[IRRT]}$$

$\rightsquigarrow$ $\mathcal{F}(w)$ is filtered by them. $\square$

ex (A_3) $\quad w = s_{21323}$



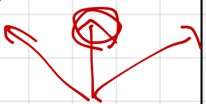
$(W, \leq_R)$ torf Π



$\Uparrow$ all red exps of w

Brick labels below $\mathcal{F}_1(w)$

: path corresp. to $w = s_{21323}$



Prop [AIRT] $\dim B_i = \beta_i \; (= s_{u_1 \cdots u_{i-1}}(\alpha_i))$

$\rightsquigarrow \mathrm{inv}(w) = \{ \underbrace{\dim B_1, \cdots, \dim B_\ell}_{\text{all distinct}} \}$

Recall $\operatorname{sim} \mathcal{F}_1(w) \subset \{B_1, \cdots, B_\ell\}$

$\downarrow$ 1:1 dim

$\mathrm{inv}(w) = \{\beta_1, \cdots, \beta_\ell\}$

Claim B_i : non-simp $\iff \beta_i = \dim B_i$: non-Br. inv

($\Leftarrow$) : OK by taking dim

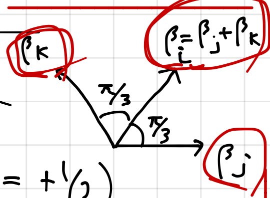
Key Lem in torf

B_i : non-simple $\iff \exists j\,(>i)$, $B_i \xrightarrow{\exists} B_j$: non-zero, non-injection

$\Leftarrow$ holds for any torf with brick seq.

Key Lem in root sys.

β_i : non-Br $\Rightarrow$ Looks like



($\rightsquigarrow$ inner prod. $(\beta, \beta_j) = +1/2$)

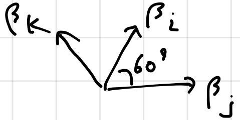
Lem [Crawley-Boevey]

B_i, B_j : bricks, β_i, β_j : dim

$\Rightarrow 2(\beta_i, \beta_j) = \hom(B_i, B_j) + \hom(B_j, B_i) - \mathrm{ext}^1(B_i, B_j)$

In this situ,

value = $+1 > 0$



$i < j$

$\Rightarrow (B_j, B_i) = 0$

Hence β_i : non-Br

$\Rightarrow \exists\ B_i \longrightarrow B_j$, with $\dim B_i < \dim B_j$

non-zero

$\Rightarrow B_i$ is not simple! $\square$

Cor

Q : Dynkin quiver

$\rightsquigarrow \mathrm{sim}(\mathcal{F}(w) \cap \mathrm{mod}\, KQ) \xrightarrow[\sim]{\dim} \mathrm{Binv}(w)$

Cor $\mathcal{F}(w)$ satisfies Jordan-Hölder

[E]

$\Longleftrightarrow \# \mathrm{supp}(w) = \# \mathrm{Binv}(w)$

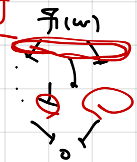
$\#\{\text{proj obj in } \mathcal{F}(w)\}$ $\quad$ $\# \mathrm{sim}\, \mathcal{F}(w)$

$\Longleftrightarrow$ X_w : Schubert var is locally factorial

Conj

Conjecture TFAE? for B : brick

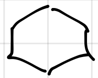
(1) $B \in \mathrm{sim}\, \mathcal{F}(w)$ $\quad$ semibrick



(2) B appears every path

$\Downarrow$

(3) $\nexists$ hexagon



B below $\mathcal{F}(w)$

(4) $\nexists$ s.e.s

$0 \longrightarrow B_j \longrightarrow B \longrightarrow B_k \longrightarrow 0$

with $B_j, B_k \in \mathrm{brick}\, \mathcal{F}(w)$

(True for ADE_6)