

ICF-closed subcategories of module categories

Haruhisa Enomoto
(joint work with Arashi Sakai)

§0. Intro.

Setting. k : field.

Λ : a f.d. k -alg.

$\text{mod}\Lambda$: the cat of f.g.
right Λ -mods.

Motivation

Want to study subcats
of $\text{mod}\Lambda$.

More precisely
Want to classify certain subcats
of $\text{mod}\Lambda$.

(i) subcats closed under operations.

e.g. taking \oplus , submodules, cokernels,
...

(ii) establish a bij

$$\left\{ \begin{array}{c} \text{certain} \\ \text{subcats} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{certain} \\ \text{modules} \end{array} \right\}$$

Def

• $T \subseteq \text{mod}\Lambda$ is a forsion class
 \Leftrightarrow closed under quotients, extensions.

[Dickson 1966].

$$\begin{array}{c} T_1 \rightarrow T_2 \rightarrow 0 \\ T_1, T_2 \in T \\ \rightarrow E \in T \end{array}$$

• $W \subseteq \text{mod}\Lambda$ is a wide subcat (Hovey 2001)

\Leftrightarrow closed under kernels, coker, ext.
(i.e., ext-closed exact abelian subcat)

Def $\mathcal{C} \subseteq \text{mod } \Lambda$: ICE - closed

subset (ICE)

$\Leftrightarrow \mathcal{C}$ is closed under

(i) Images, (ii) Cokernels, Extensions.

i.e., (i) $\forall c_1, f \rightarrow c_2, c_1, c_2 \in \mathcal{C}$.

$\text{Im } f \in \mathcal{C}$.

(iii)

Coker $f \in \mathcal{C}$

]

Rem

Every tors and wide is ICE.

ICE

many papers.



Aim

classify ICE's of $\text{mod } \Lambda$!

Strategy

Use a progenerator of ICE.

Def $\mathcal{C} \subseteq \text{mod } \Lambda$: ext-closed.

(1) $P \in \mathcal{C}$ is proj in \mathcal{C}

$\Leftrightarrow \text{Ext}_\Lambda^1(P, \mathcal{C}) = 0$.

(2) $P \in \mathcal{C}$ is a progenerator

$\Leftrightarrow \forall X \in \mathcal{C}$

$\exists 0 \rightarrow Y \rightarrow P^2 \rightarrow X \rightarrow 0$

with $Y \in \mathcal{C}$.

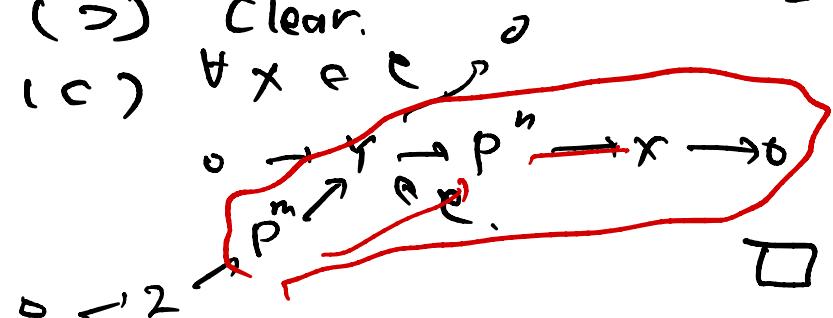
Prop

If an ICE \mathcal{C} has a progeny,
then $\mathcal{C} = \text{cok } P$

$:= \{M \in \text{mod } \Lambda \mid \exists P^m \rightarrow P^n \rightarrow M \rightarrow 0\}$.

$\Leftarrow (\Rightarrow)$ Clear.

(\Leftarrow) $\forall X \in \mathcal{C}$



□

Def $M \in \text{mod } \Lambda$ is rigid

$$\Leftrightarrow \text{Ext}_\Lambda^1(M, M) = 0.$$

We have a map
 of ICEs with progeny $\xrightarrow{P(-)}$ ^{progenerator.}
 { subcats } $\xleftarrow{\text{cok}}$
 s.t. $\text{cok} \circ P = \text{id.}$

§ 1. Path alg case.

$Q:$ Dynkin quiver. e.g. $Q: 1 \leftarrow 2 \rightarrow 3$
 A_3 quiver.

kQ : its path algebra.

(kQ -module = representation of Q)

$\left\{ \begin{array}{l} \circ i \in Q \quad M_i : k\text{-vec. sp} \\ \circ i \rightarrow j \in Q \quad M_i \rightarrow M_j \\ \text{linear map} \end{array} \right.$

Ex $Q: 1 \leftarrow 2 \rightarrow 3.$

$$M: \begin{matrix} k & \xleftarrow{\quad} & k^2 & \xrightarrow{\quad} & 0 \\ & & [1 \ -1] & & \end{matrix}$$

Thm [Gabriel]

We have a bij.

{ indecomp. kQ -module } $\xleftrightarrow{?}$ { positive roots of Q }

$$M \mapsto \underline{\dim} M := \sum_{i \in Q} (\dim M_i) \alpha_i$$

simpk
root.

P.G. $(k \xleftarrow{\quad} k \rightarrow 0) \rightsquigarrow \alpha_1 + \alpha_2$

$$k \leftarrow k \rightarrow k \rightsquigarrow \alpha_1 + \alpha_2 + \alpha_3$$

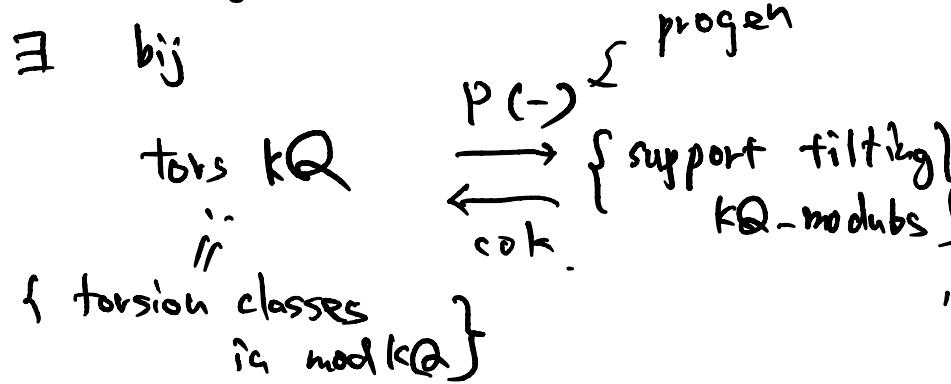
Fact :

Every ICE of mod kQ has
 a progen.

Rigid mods as
ICE-closed ...

Classification of exact structures ...

Thm [Ingalls - Thomas 2009]



where. M is supp. tilt

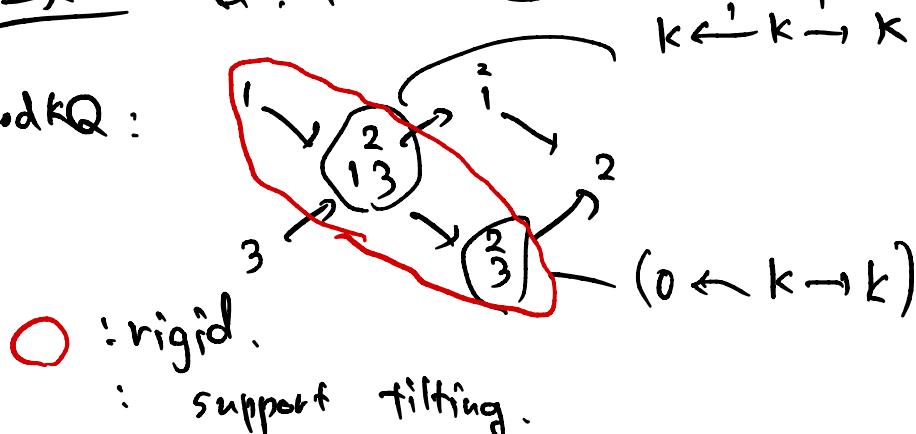
\Leftrightarrow (i) M is rigid. ($\text{Ext}_A^1(A, M) = 0$)

(ii) $|M| = \# \{ \text{comp. factors of } M \}$
 $\# \{ \text{indec. summands of } M \} \leq \dots$ (in general)

Ex

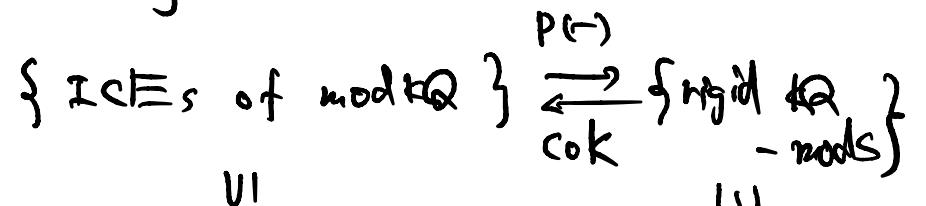
$Q : 1 \leftarrow 2 \rightarrow 3$

$\text{mod } kQ :$

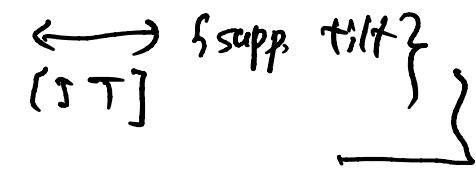


Thm [E]

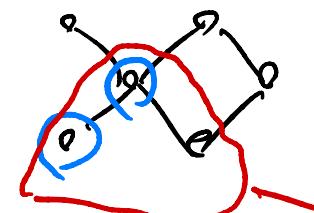
$\exists \text{ bij}$



$\text{tors } kQ$



Ex $Q : 1 \leftarrow 2 \rightarrow 3$



$\circ : M : \text{rigid}$
 $\circ : \text{not supp. tilt}$

$\circ : \text{cok } M$

Rem

$\circ : \text{rigid } kQ\text{-mod}$
 \longleftrightarrow certain faces of
 cluster cpx of Q .

$\leadsto \exists$ formula of this number.

e.g. $A_n : \sum_{i=0}^n \frac{1}{i+1} \binom{n}{i} \binom{n+i}{i}$
 $(\text{Schröder number}).$

Non-trivial part:

cok M is ICE
for a rigid $k\Lambda$ -mod M.

- cok M is a torsion class
in $\langle M \rangle_{\text{wide}}$: the smallest wide, containing M.
- Tors of wide is ICE.

□

§ 2. ICE via torsion class.

It's difficult to characterize
proper of ICE in general.

Instead, we use the poset

$$\text{tors } \Lambda = \{ \text{tors in } \text{mod } \Lambda \}$$

Largest
mod Λ

(complete lattice)

D
smallest.

Def For $u \subseteq T$ in $\text{tors } \Lambda$,

$$H[u, T] := T \cap u^\perp \subseteq \text{mod } \Lambda,$$

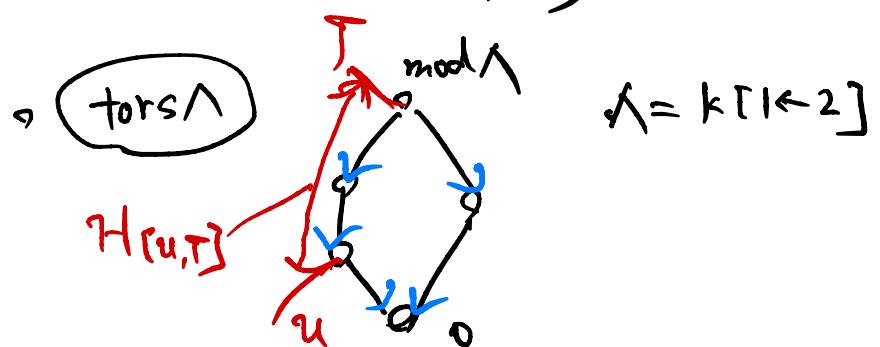
$$(u^\perp := \{x \in \text{mod } \Lambda \mid \text{Hom}_\Lambda(u, x) = 0\})$$

: the heart of an interval $[u, T]$
(like " $T - u$ ")

Ex

$$H[0, T] = T$$

$$("T - 0" = T)$$



Key Prop

$$e \subseteq \text{mod } \Lambda : \text{ICE}$$

$$\Rightarrow \exists u \subseteq T \text{ in } \text{tors } \Lambda$$

$$\text{s.t. } e = H[u, T].$$

Which interval gives ICE?

Def

$\tilde{H}(\text{tors}\Lambda)$: Hasse quiver.

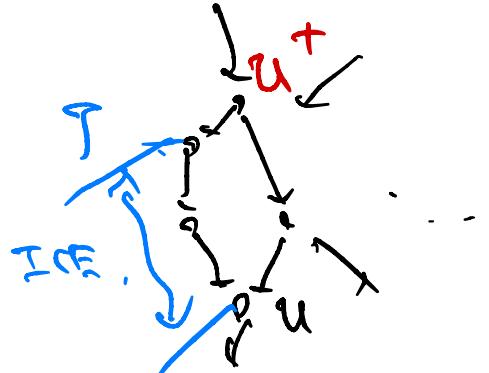
- $u \in T \subseteq \text{tors}\Lambda$
- $T \rightarrow u \Leftrightarrow T \npreceq u,$
 $\nexists v \quad T \npreceq v \npreceq u.$

Thm [E-Sakai]

TFAE for $u \in T$ in $\text{tors}\Lambda$

- (1) $H[u, T]$ is ICE.
- (2) For $u^+ := \bigvee \{u' \mid \exists u' \rightarrow u\}$,
 $(u \leq) T \subseteq u^+$ holds

Ex



We can construct

all ICES if
 $\text{tors}\Lambda$ is given!

Cor For a subset $\mathcal{C} \subseteq \text{mod}\Lambda$,

TFAE

- (1) \mathcal{C} : ICE-closed
- (2) $\exists W \subseteq \text{mod}\Lambda$: wide

s.t. \mathcal{C} : tors in W .

$\therefore (2) \Rightarrow (1)$: Easy.

(1) $\Rightarrow (2)$ \mathcal{C} : ICE

By Key Prop

$$\mathcal{C} = H[u, T]$$

By Thm.

$$u^+ \xrightarrow{(-) \cap u^\perp} H[u, u^+]$$

$$u^- \longrightarrow u^-$$

$$T \longrightarrow H[u, T] = \mathcal{C}$$

$$U \longrightarrow U$$

$$u \longrightarrow 0$$

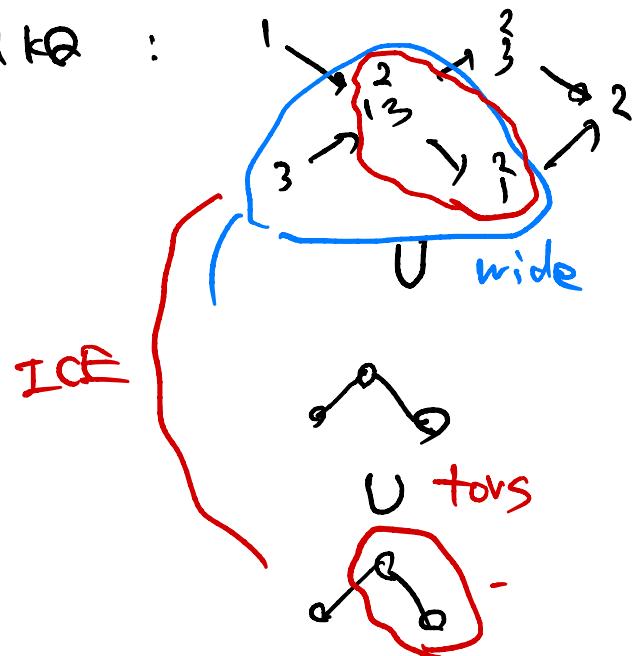
By Asai-Pfeifer 2019.

$H_{\Gamma u, u^+ \gamma}$: wide.

$e \in \text{tors } H_{\Gamma u, u^+ \gamma}.$ \square

Ex $Q : 1 \leftarrow 2 \rightarrow 3$

$\text{mod } kQ :$



What about progeny of ICE?

§ 3. ICES via τ -tilting.

Adachi-Iyama-Reiten 2014.

Def $M \in \text{mod } \Lambda$

(1) M : τ -rigid

\Leftrightarrow If $\nexists M \xrightarrow{\text{surf}} \underline{N}$, then

$$\text{Ext}_\Lambda^1(M, N) = 0.$$

(\Leftrightarrow rigid)
KQ.

(2) M is supp τ -tilt

\Leftrightarrow M is τ -tilt.

- $|M| = \#\{\text{comp. fact. of } M\}$
(\leq in general)

Thm [AIR]

$$\{ \text{tors with progeny} \} \xrightleftharpoons[\text{cok}]{{\mathbf{P}}(\rightarrow)} \{ \text{supp } T\text{-tilt} \}$$

Assume $\{ \text{ICE - closed subsets and wide } T\text{-tilting} \}$
 $\# \text{tors} \wedge < \infty - *$
 (e.g. KQ Q: Dynkin
 preproj alg. of Dynkin)

Fact.

- every ICE has a progeny
- every wide is equiv to $\text{mod } \Gamma \dashv \Gamma$: f.d. alg.

Recall

\mathcal{E} : ICE

$$\Leftrightarrow \mathcal{E} \subset W \subset \text{mod } \Lambda$$

$\begin{matrix} \text{tors} \\ \parallel \\ \text{tors.} \end{matrix}$

$\text{mod } \Gamma$

Def

$M \in \text{mod } \Lambda$: wide $T\text{-tilt}$

$\Leftrightarrow \exists W$: wide

s.t. $M \in \text{supp } T_W\text{-tilt}$.

(under $W \simeq \text{mod } \Gamma$)

Thm [ES]

Assume (*)

\exists bij

$$\{ \text{ICES} \} \xrightleftharpoons[\text{cok}]{{\mathbf{P}}(\rightarrow)} \{ \text{wide } T\text{-tilt} \}$$

$\begin{matrix} \uparrow \\ \# \text{tors} \wedge \\ \text{[AIR]} \end{matrix} \dashv \begin{matrix} \downarrow \\ \{ \text{supp } T\text{-tilt} \} \end{matrix}$

Rem For KQ, wide $T\text{-tilt} = 2\text{-rigid}$
 In general, $\{ \text{tors} \} \subseteq \{ \text{f.g. f-tors} \}$