Maximal self-orthogonal modules and a new generalization of tilting modules

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§0. Overview

Setting k : field

Λ : f.d. k-alg.

modules = f.g. right modules.

Def $M \in \operatorname{mod}\Lambda$ is self-orthogonal (SO)

$:\Longleftrightarrow \operatorname{Ext}^{>0}_{\Lambda}(M, M) = 0.$

Hierarchy of SO modules

pd $< \infty$

pd $= \infty$

Today

$\{$tilting modules with pd $\leq 1\}$ $\subseteq$ $\{$tilting modules$\}$ $\subseteq$ $\{$projectively Wakamatsu tilting$\}$ $\supseteq$ $\{$Wakamatsu tilting$\}$

dual

self-dual

$\{$cotilting modules with id $\leq 1\}$ $\subseteq$ $\{$cotilting modules$\}$ $\subseteq$ $\{$injectively Wakamatsu tilting$\}$ $\supseteq$ $\{$Wakamatsu tilting$\}$

§1.

Def Let $T \in \operatorname{mod}\Lambda$.

Define $T^{\perp} \subseteq \operatorname{mod}\Lambda$ by

$$T^{\perp} := \{ X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}_{\Lambda}^{>0}(T, X) = 0 \}$$

$\lrcorner$

Rem T : SO $:\Leftrightarrow$ $T \in T^{\perp}$ $\rfloor$

Def Let $\mathcal{C} \subseteq \operatorname{mod}\Lambda$: subcat closed under extensions & direct summands.

(1) $P \in \mathcal{C}$ is <u>Ext-proj</u> in $\mathcal{C}$

$:\Leftrightarrow$ $\operatorname{Ext}_{\Lambda}^{1}(P, \mathcal{C}) = 0$

(2) $P \in \mathcal{C}$ is an <u>Ext.progenerator</u>

$:\Leftrightarrow$ ∘ $P \in \mathcal{C}$: Ext-proj

∘ $\forall C \in \mathcal{C}$,

$\exists\ 0 \to C_1 \to P_0 \to C \to 0$

: ex with $C_1 \in \mathcal{C}$, $P_0 \in \operatorname{add} P$.

Dually define Ext-inj (cogenerator) $\rfloor$

Ex

(1) $\Lambda^{\perp} = \operatorname{mod}\Lambda$,

Λ : Ext-progen. of $\operatorname{mod}\Lambda$.

(2) T : tilt. mod. with $\operatorname{pd} T \leq 1$.

$\rightsquigarrow$ $T^{\perp} = \operatorname{Fac} T$ ($\operatorname{Gen} T$),

with T : Ext-progen of $T^{\perp}$.

Def $T \in \operatorname{mod}\Lambda$ is projectively Wakamatsu tilting (pW-tilt)

if (1) T : SO

(2) T is an Ext-progen of $T^{\perp}$.

Ex

(1) Λ : pW-tilt

(2) Every tilt. module (with fin. proj. dim) is pW-tilt

[Auslander–Reiten, "Applications of contravariantly ..."]

Relation to (W-tilt)
Wakamatsu tilting modules

Def Let $T \in \operatorname{mod}\Lambda$.
Define $\mathcal{Y}_T \subseteq T^{\perp}$ by

$$Y \in \mathcal{Y}_T :\Longleftrightarrow \exists \cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} Y \to 0$$

: ex with $\operatorname{Im} f_i \in T^{\perp}$ $(\forall i \geq 0)$

($\Leftrightarrow$ T is an Ext-progen. of $\mathcal{Y}_T$) if T: SO

T: SO	$\mathcal{Y}_T$ $\subseteq$	$T^{\perp}$
progen	T	??
inj cogen	??.	$D\Lambda$

This is T
$\Leftrightarrow$ T: pW-tilt.

Def T : W-tilt
$:\Longleftrightarrow D\Lambda \in \mathcal{Y}_T$

($\Leftrightarrow$ $D\Lambda$ is an Ext-inj cogen. of $\mathcal{Y}_T$).
[Auslander-Reiten]

Prop TFAE for $T \in \operatorname{mod}\Lambda$
(1) T: pW-tilt
(2) T : W-tilt s.t. $\mathcal{Y}_T = T^{\perp}$.

Nice subclass
$$\{\text{pW-tilt}\} \subseteq \{\text{W-tilt}\}$$

Rem
- This inclusion may proper.
- $\{\text{W-tilt}\}$ is self-dual

i.e., T : W-tilt in $\operatorname{mod}\Lambda$
$\Leftrightarrow$ DT : W-tilt in $\operatorname{mod}\Lambda^{op}$.

§ Main Results

Def

Let $U \in \mathrm{mod}\,\Lambda$: SO.

The Bongartz completion T of U is a pW-tilt T s.t.

$$\begin{cases} (1) & U \in \mathrm{add}\,T \\ (2) & T^{\perp} = U^{\perp}. \end{cases}$$

(if exists)

$$U : \mathrm{SO} \rightsquigarrow U^{\perp} \underset{\text{(if exists)}}{\overset{\text{progen}}{\rightsquigarrow}} \text{Bon. compl. } T.$$

Rem It's compatible with that of (classical) tilting modules.

Thm 1.

Let $U \in \mathrm{mod}\,\Lambda$: SO. TFAE

(1) U has a Bon. compl.

(2) $U^{\perp}$ has a finite cover

$$\left(:\Longleftrightarrow \begin{array}{l} \exists M \in U^{\perp} \text{ s.t.} \\ \forall X \in U^{\perp}\ \exists\ M^{n} \twoheadrightarrow X : \text{surj} \end{array} \right)$$

Rem

(2) $\Longleftrightarrow$ Λ has left $U^{\perp}$-approx

$\Longleftarrow$ $U^{\perp}$ is covariantly finite.

($\Rightarrow$? I don't know).

Cor

If $U \in \mathrm{mod}\,\Lambda$ is SO and $\# \mathrm{ind}\,U^{\perp} < \infty$

$\{\text{indecs in } U^{\perp}\}/\cong$

then $\exists\ X \in \mathrm{mod}\,\Lambda$ s.t.

$U \oplus X$ is pW-tilt.

Thm 2.

Suppose $T \in \mathrm{mod}\,\Lambda$ satisfies $\#\mathrm{ind}\, T^{\perp} < \infty$.
(e.g. Λ is rep-fin).

Then TFAE

(1) T: pW-tilt

(2) T: W-tilt.

(3) T: SO with $|T| = |\Lambda|$ — number of indec summands.

(4) T: maximal SO

($:\Longleftrightarrow$ • T: SO
• $T \oplus M$: SO $\Rightarrow M \in \mathrm{add}\, T$)

Ex

Λ : $$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[d, "b"] \\ 4 \arrow[u, "d"] & 3 \arrow[l, "c"] \end{tikzcd} \Big/ \langle abc, bcda, cdab \rangle$$

$$T := \underbrace{P(2) \oplus P(3) \oplus P(4)}_{\text{proj-inj}} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \quad \left(\text{or } \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \right)$$

: pW-tilt with $\mathrm{pd}\, T = \mathrm{id}\, T = \infty$.

Key Lemma

Let $\mathcal{E}$: Hom-fin, Krull-Schmidt exact cat with progen P, inj cogen I.

If $\#\mathrm{ind}\,\mathcal{E} < \infty$, then $|P| = |I|$. $\lrcorner$

Rem

In general, (1) – (4) are not equiv. for T : SO

$$\begin{tikzcd} T:\text{pW-tilt} \arrow[r, Rightarrow, shift left] \arrow[d, phantom, "??"] & T:\text{W-tilt} \arrow[l, Rightarrow, shift left, "\times"] \\ |T| = |\Lambda| \arrow[r, phantom, "??"] & T:\text{maximal SO} \arrow[u, Rightarrow, "\times"'] \end{tikzcd}$$

Other result

Thm Let Λ be a ~~rep-fin~~ Iwanaga-Gorenstein alg ($\mathrm{id}\,\Lambda_\Lambda = \mathrm{id}\,{}_\Lambda\Lambda < \infty$).

(1) T: tilting ($\Leftrightarrow$ cotilt)
$\Leftrightarrow$ pW-tilt.
$\Leftrightarrow$ W-tilt.

(2) $U \in \mathrm{mod}\,\Lambda$: SO
$\Longrightarrow$ $\mathrm{pd}\,U < \infty$. $\square$

(!) (2) Take Bongartz comp T of U
$\rightsquigarrow$ T: pW-tilt
$\rightsquigarrow$ T: tilt by (1)
$\rightsquigarrow$ $\mathrm{pd}\,T < \infty$.
$\Longrightarrow$: unknown! $\mathrm{pd}\,U < \infty$ by $U \in \mathrm{add}\,T$ $\square$

Conjectures

We conjecture the following.

Boundedness Conjecture (BC) [Happel]

If $T \in \mathrm{mod}\,\Lambda$: SO,
then $|T| \leq |\Lambda|$.

Maximal SO Conj (MSOC)

Every W-tilt is maximal SO.

Weak maximal SO Conj (WMSOC)

Every pW-tilt is maximal SO.

Proj = Inj Conj (PIC)

T: W-tilt $\Rightarrow$ $|T| = |\Lambda|$.

($\Leftrightarrow$ Key Lem holds w.o. the assumption $\#\mathrm{ind}\,\mathcal{E} < \infty$)

Prop The following implication holds:

$$
\begin{tikzcd}
 & (\mathrm{PIC}) \arrow[dr, Rightarrow] & \\
(\mathrm{BC}) \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & (\mathrm{MSOC}) \arrow[r, Rightarrow] & (\mathrm{wMSOC}) \arrow[d, Leftrightarrow] \\
 & & \text{Auslander-Reiten Conj}
\end{tikzcd}
$$

Auslander-Reiten Conj : Λ is maximal SO.

- These are all true if Λ : rep-fin.

Rem Dually we can consider

dual of $\mathcal{Y}_T \rightsquigarrow \mathcal{X}_T \subseteq {}^{\perp}T$ (T as inj cogen)

(1) $\mathcal{X}_\Lambda = \mathrm{GP}\Lambda$: the cat of Gorenstein-proj modules.

(2) T : injectively W-tilt (iW-tilt)

:$\iff$ T : Ext-inj cogen of ${}^{\perp}T$.

Then Λ_Λ is W-tilt, and

Λ_Λ is iW-tilt

$\iff \mathrm{GP}\Lambda = {}^{\perp}\Lambda$

$\iff$: Λ : weakly Gorenstein [Ringel-Zhang] ⌋

Cor If $\# {}^{\perp}\Lambda < \infty$, then $\mathrm{GP}\Lambda = {}^{\perp}\Lambda$.

Apply dual of Thm 2 to

$\Lambda_\Lambda \rightsquigarrow \Lambda$: iW-tilt □