ALGORITHMS AND DATA STRUCTURES II

<u>Lecture 2</u>

Heaps,
Heapsort Algorithm,
Priority Queues

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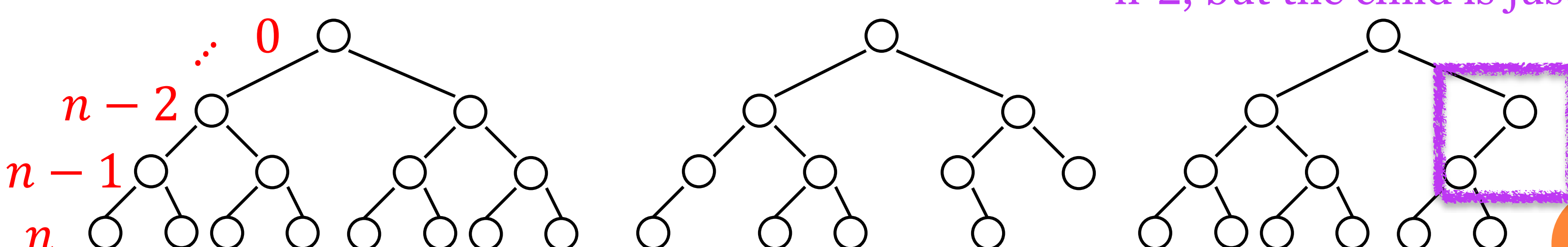
HEAPS

- Binary trees.
  - The depth of a node is its distance from the root.
  - The depth of a tree is the depth of the deepest node.

- A binary tree of depth n is balanced if all the nodes at depths 0 through $n-2$ have two children.

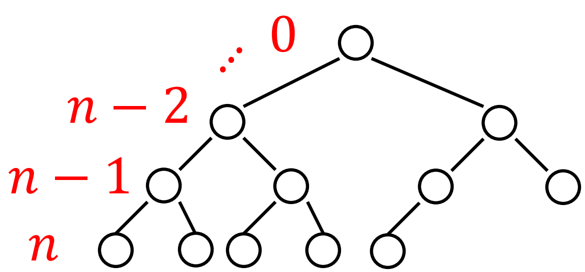
n-2, but the child is just 1



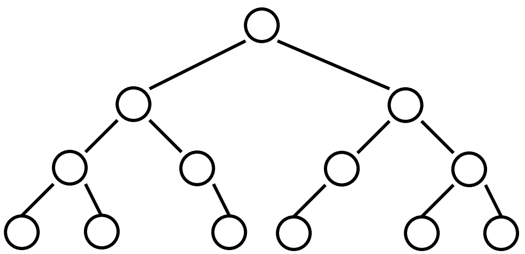
Balanced Balanced **Not** balanced

HEAPS

- A balanced binary tree is left-justified if:
  - all the leaves are at the same depth, or
  - all the leaves at depth n are to the left of all the nodes at depth $n-1$.



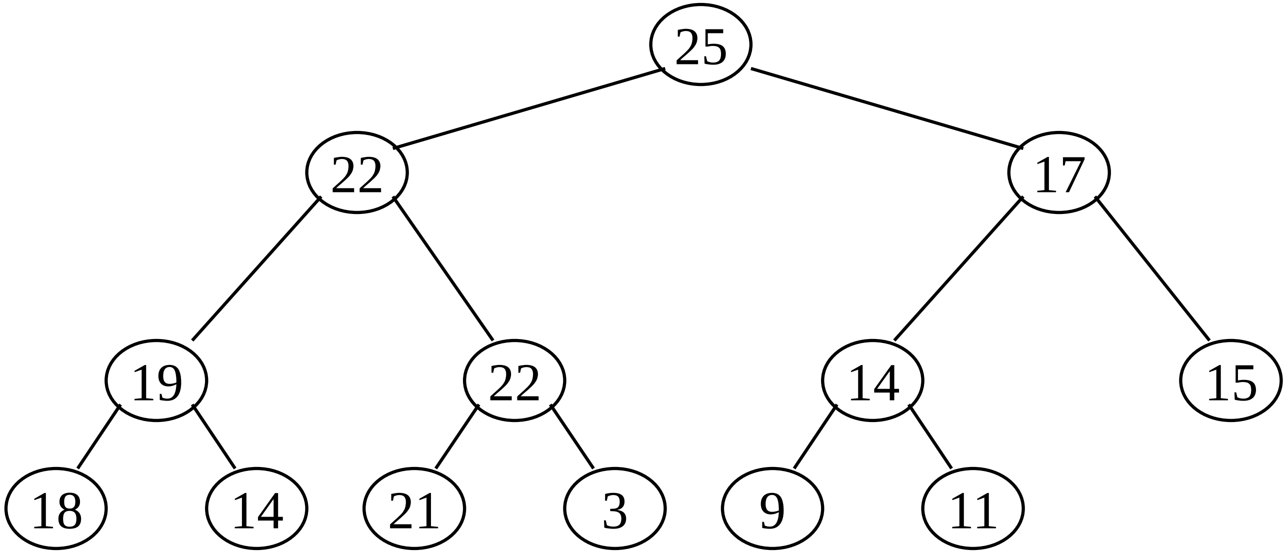
Left-justified



Not left-justified

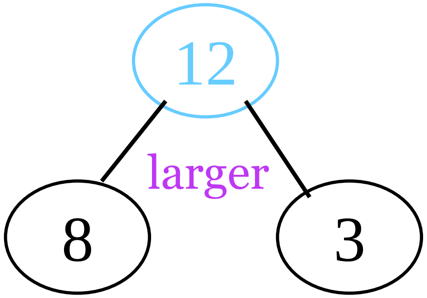
HEAPS

- What is a heap?
  - *A balanced, left-justified binary tree in which no node has a value greater than the value in its parent.*

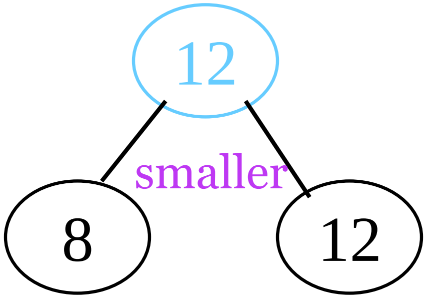


HEAPS

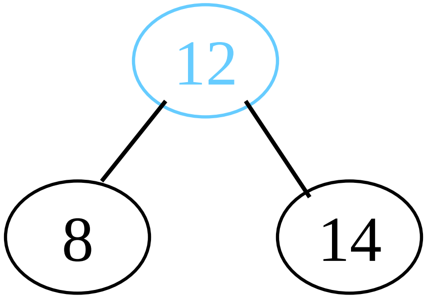
- Heap property.
  - A node has the heap property if the value in the node is as large as or larger than the values in its children.



Blue node has heap property



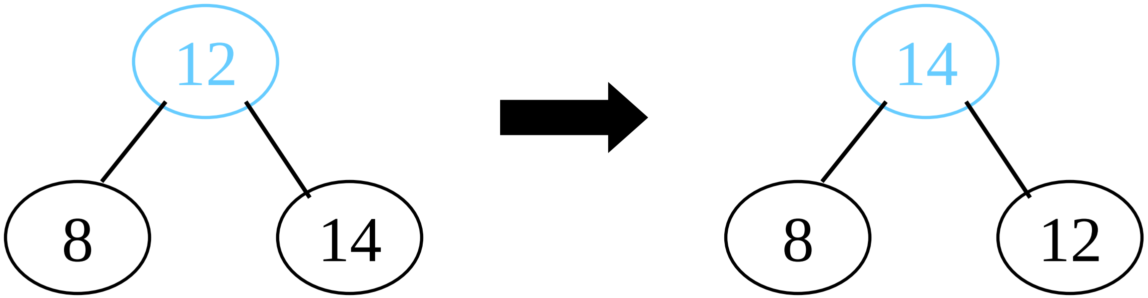
Blue node has heap property



Blue node does **not** have heap property

HEAPS

- "Heapify".
  - Given a node that does not have the heap property, you can give it the heap property by exchanging its value with the value of the larger child.

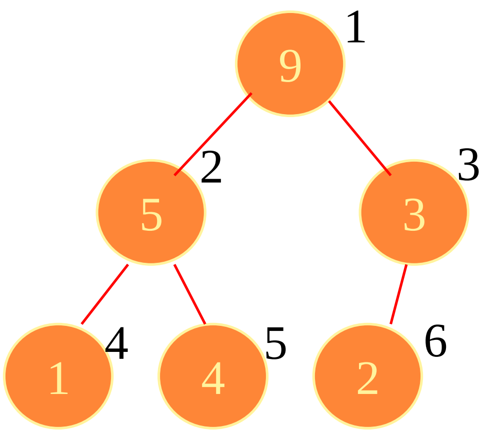


Blue node does **not** have heap property

Blue node has heap property

HEAPS

- Array representation.
  - In practice, it is easier and more efficient to implement a heap using an array.



1	2	3	4	5	6
9	5	3	1	4	2

  - Relationships between indexes of parents and children

PARENT(i):
 {return $\lfloor i/2 \rfloor$}

LEFT(i):
 {return 2i}

RIGHT(i):
 {return 2i+1}

HEAPS

- Maintaining heap property.
  - **Input**: array $\boldsymbol{A}$ and index $\boldsymbol{i}$.
  - **Output**: sub-tree rooted at $\boldsymbol{i}$ with heap property.

search the largest num.

```
def MaxHeapify (A, i)
    l = LEFT(i)
    r = RIGHT(i)
    if l ≤ A.heapsize and A[l] > A[i]:          // if L child exists and is > A[i]
          largest = l
   else:
          largest = i     (i=parent)
    if r ≤ A.heapsize and A[r] > A[largest]:
          largest = r
    if i ≠ largest:
        swap(A[i], A[largest])
        MaxHeapify (A, largest)                  // heapify the subtree
```

HEAPS

- Constructing a heap.
  - Bottom-up: Put everything in an array and then heapify the trees in a bottom-up way.
  - Given a heap of n nodes, what's the index of the last parent? ($\lfloor n/2 \rfloor$)

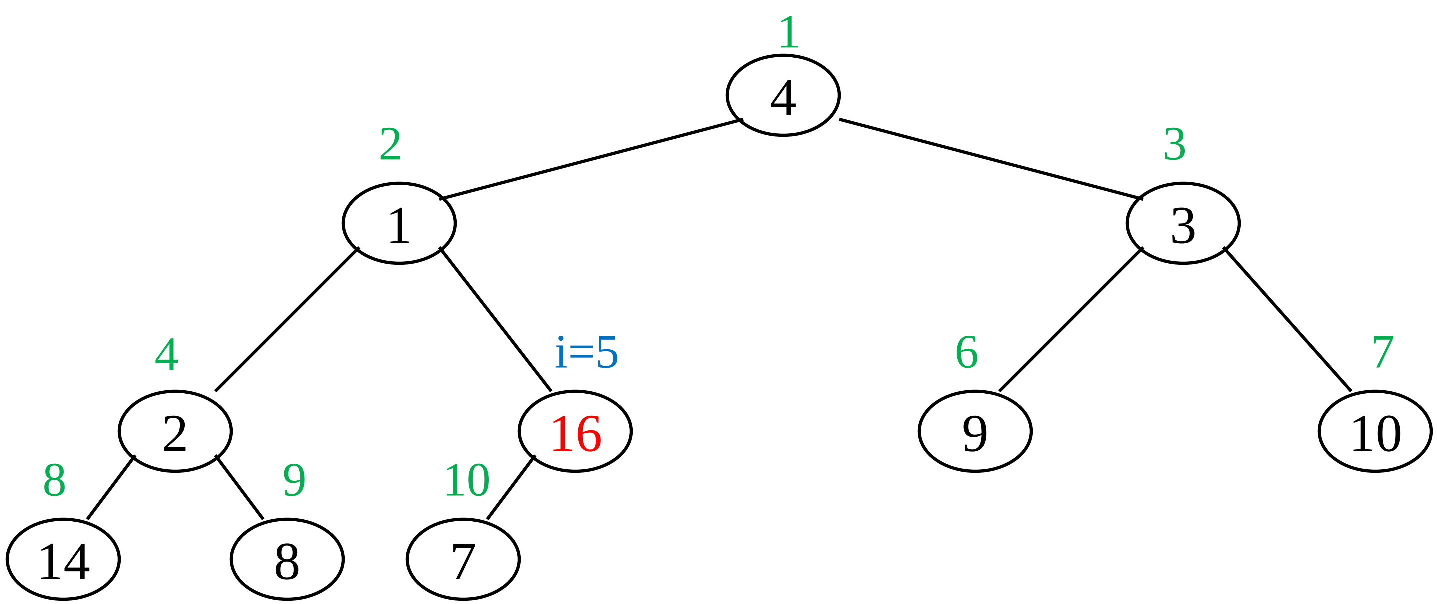
```
def HeapBottomUp (A)
    //Constructs a heap from the elements
    //of a given array by the bottom-up algorithm
    //Input: An array A[1..n] of orderable items
    //Output: A heap A[1..n]
    A.heapsize = A.length
    for i = ⌊A.length/2⌋ to 1
        MaxHeapify (A, i)
```

HEAPS

- Constructing a heap.

A

4	1	3	2	16	9	10	14	8	7

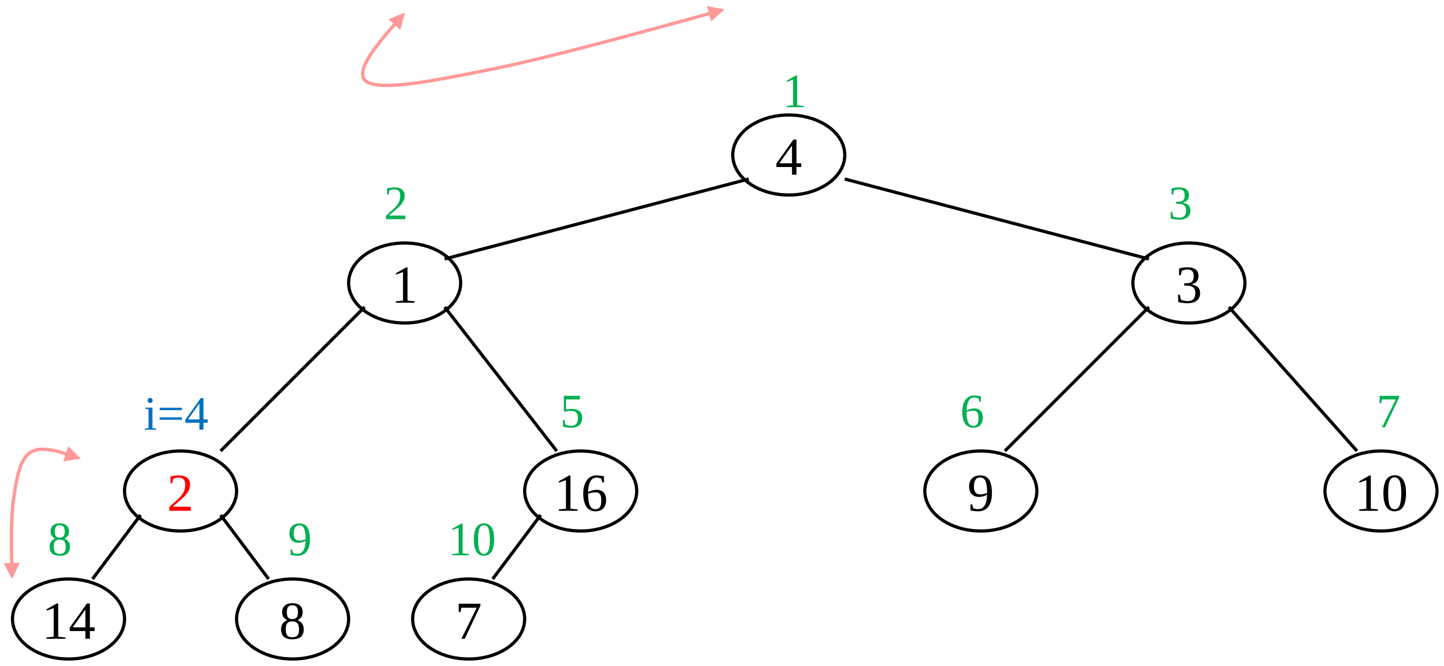


Heaps

- Constructing a heap.

A

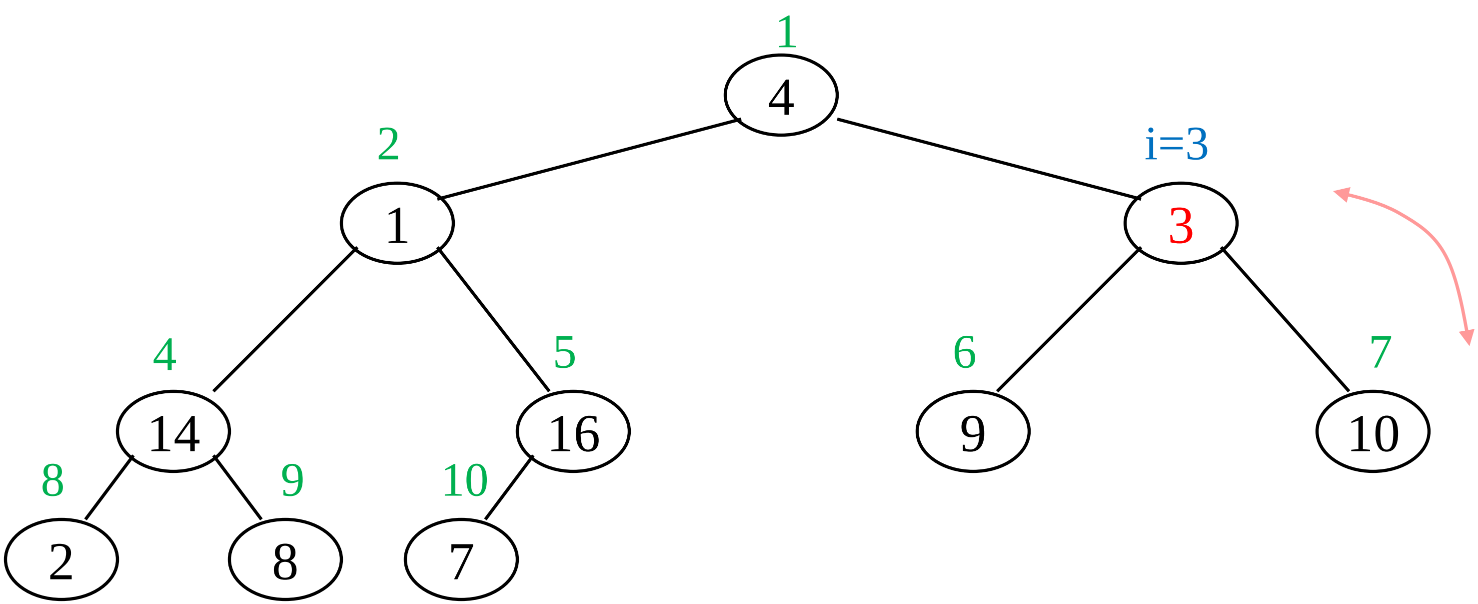
4	1	3	**2**	16	9	10	**14**	8	7



Heaps

- Constructing a heap.

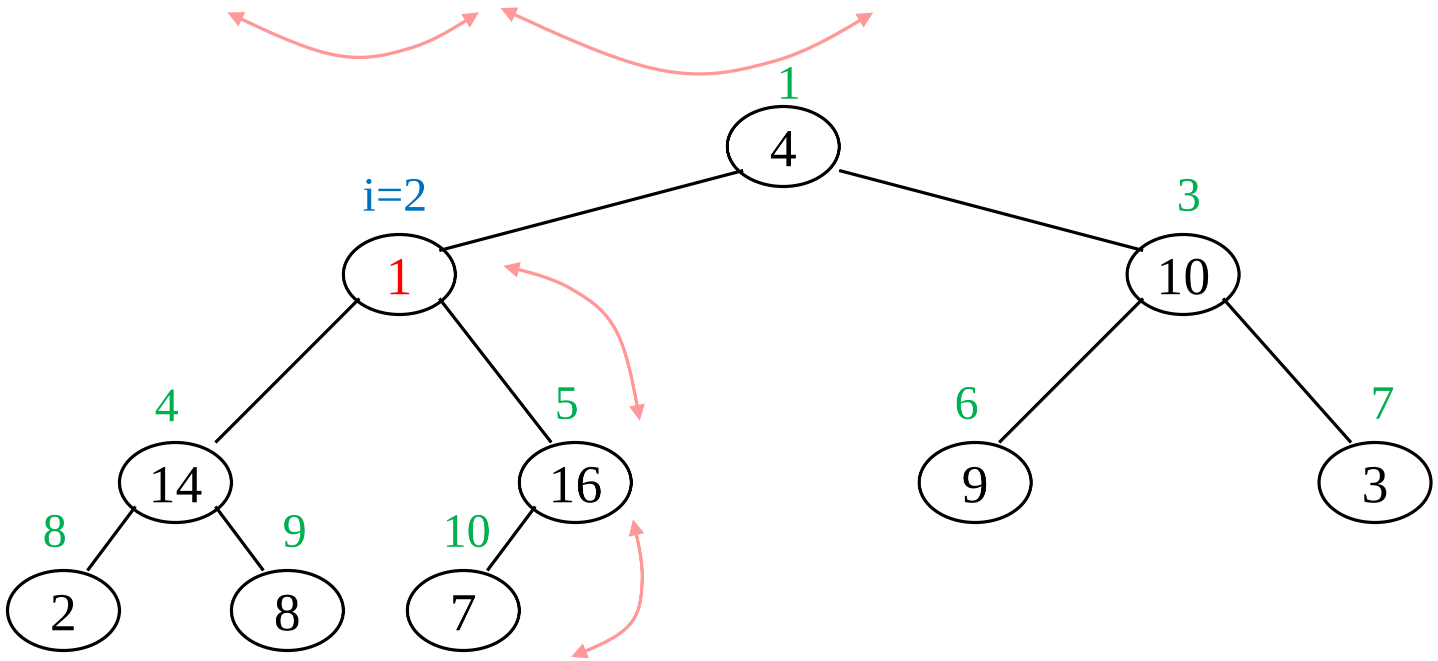
A	4	1	**3**	14	16	9	**10**	2	8	7



HEAPS

- Constructing a heap.

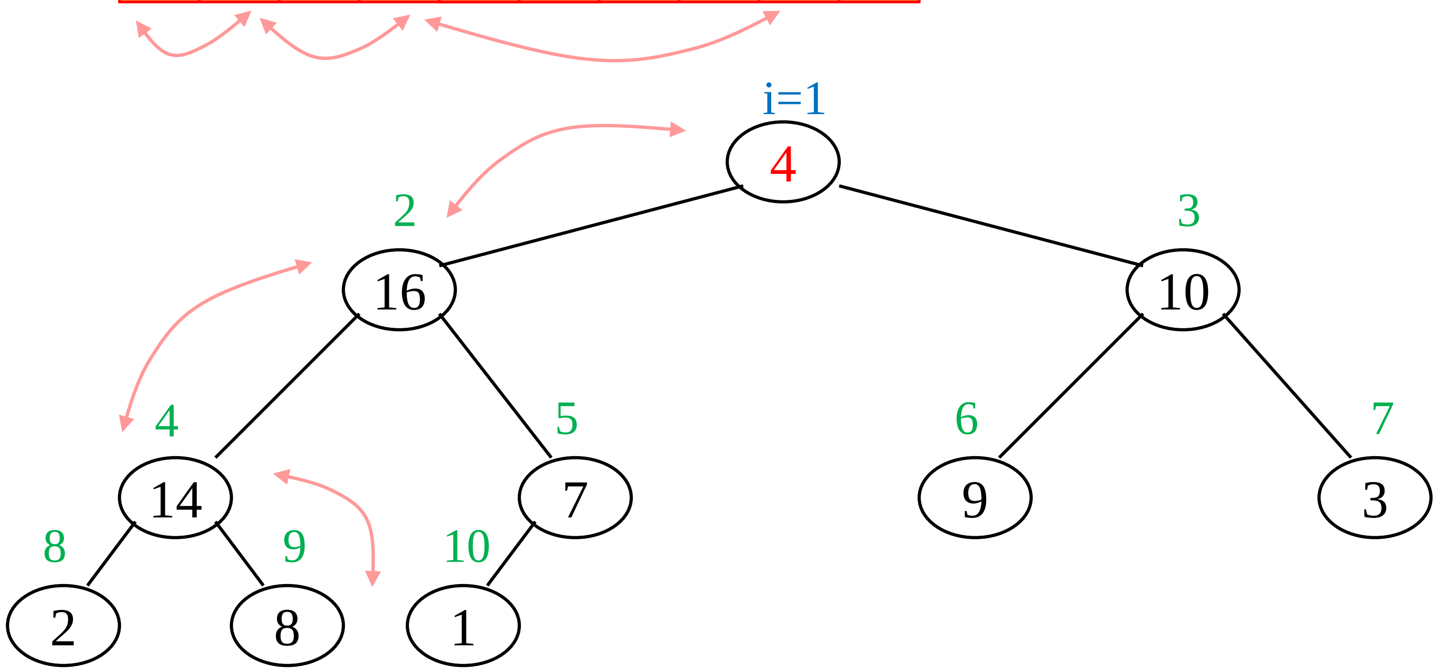
A	4	**1**	10	14	**16**	9	3	2	8	**7**



HEAPS

- Constructing a heap.

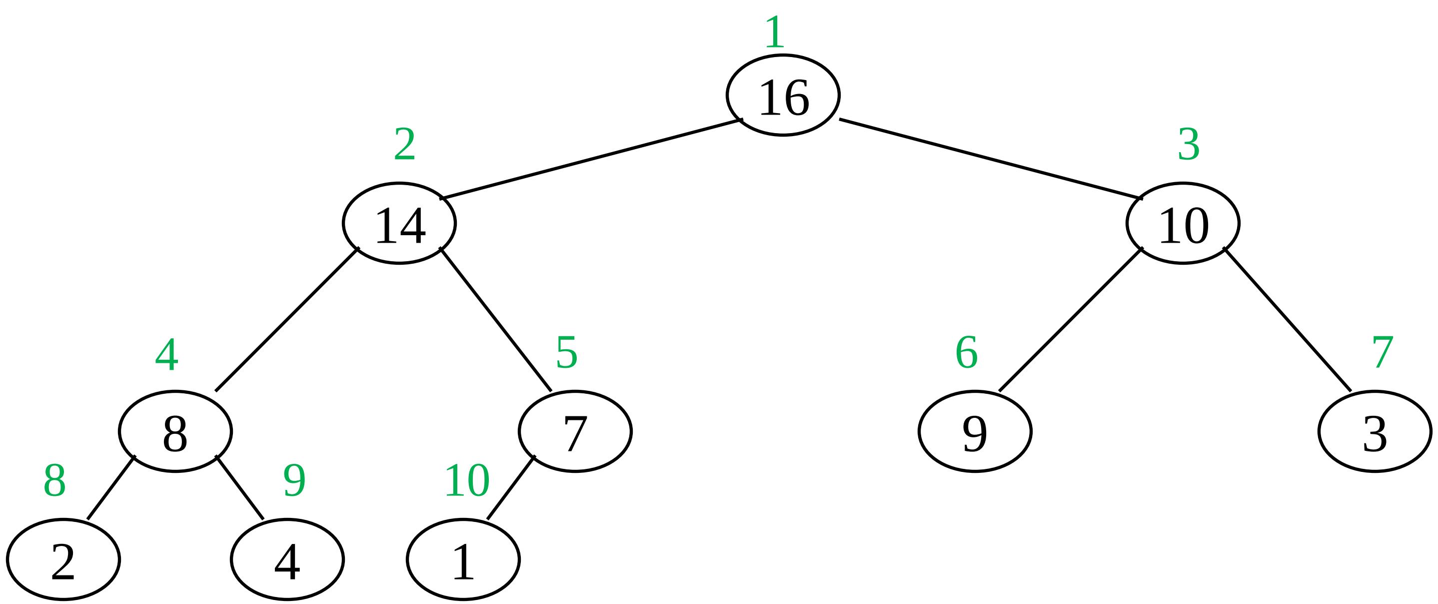
A									
4	**16**	10	**14**	7	9	3	2	**8**	1



Heaps

- Constructing a heap.

A	16	14	10	8	7	9	3	2	4	1



HEAPS

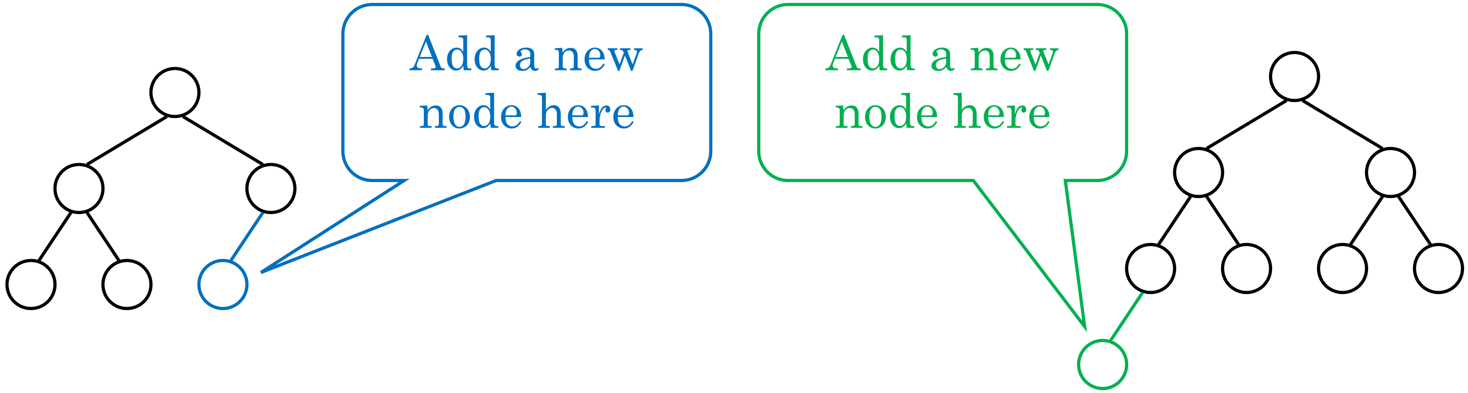
- Time complexity of heap construction.
  - An $\boldsymbol{n}$ node heap has height $\boldsymbol{h} = \lfloor \mathbf{log}\, \boldsymbol{n} \rfloor$.
  - There are at most $\lceil \boldsymbol{n}/\mathbf{2}^{\boldsymbol{h}+\mathbf{1}} \rceil$ nodes at any height $\boldsymbol{h}$.
  - **MaxHeapify** complexity is $\boldsymbol{O}(\mathbf{log}\, \boldsymbol{n}) = \boldsymbol{O}(\boldsymbol{h})$.
  - The time complexity of **HeapBottomUp** is then bounded from above by:

$$\sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) = O(n)$$

since $\displaystyle\sum_{h=0}^{\infty} \frac{h}{2^h} = 2$

HEAPS

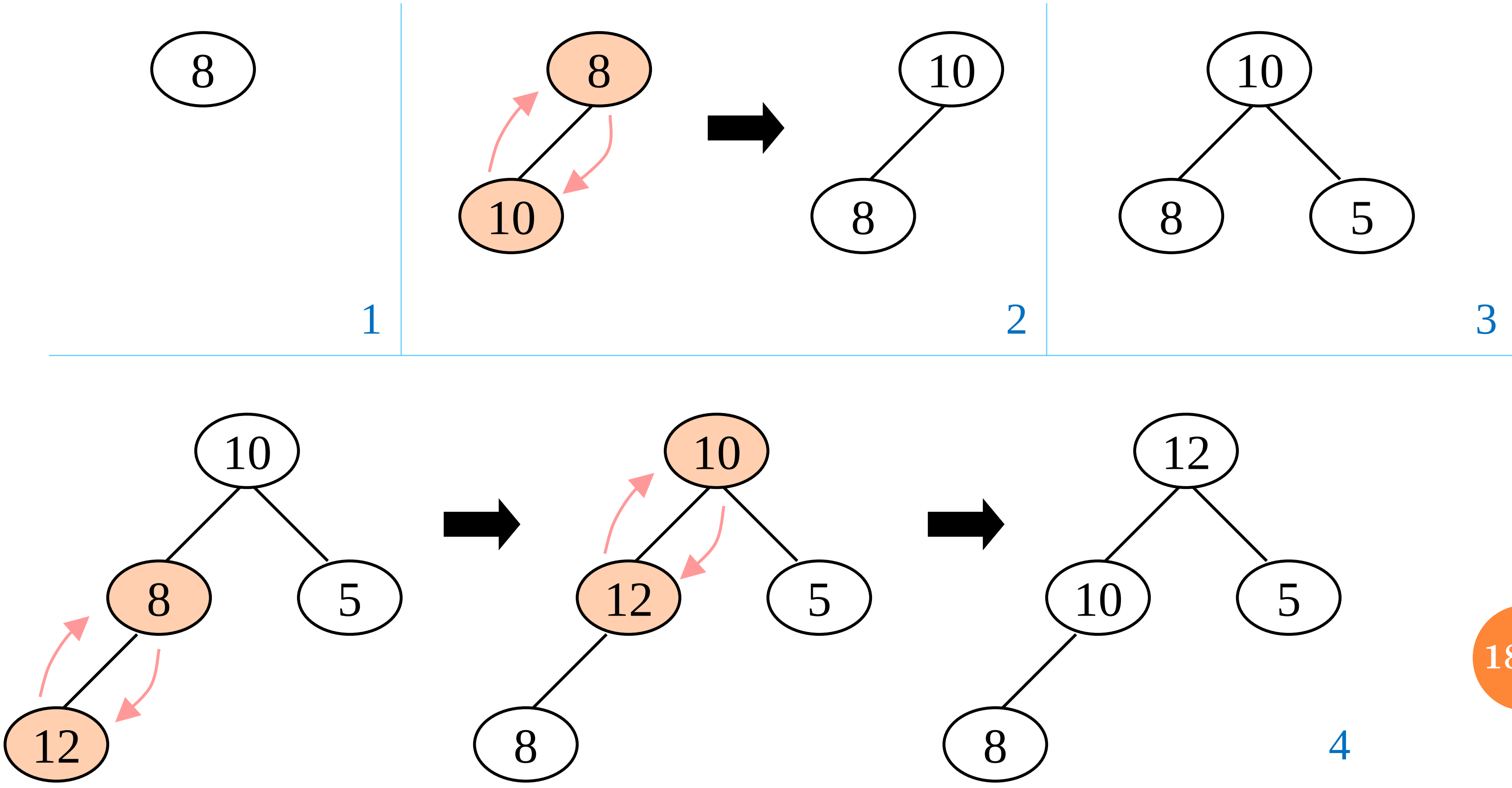
- New element insertion.
  - Insert element at the last position in heap.
    - Add the node just to the right of the rightmost node in the deepest level
    - If the deepest level is full, start a new level.



  - Compare with its parent, and exchange them if it violates the heap property.
  - Continue comparing the new element with nodes up the tree until the heap property condition is satisfied.

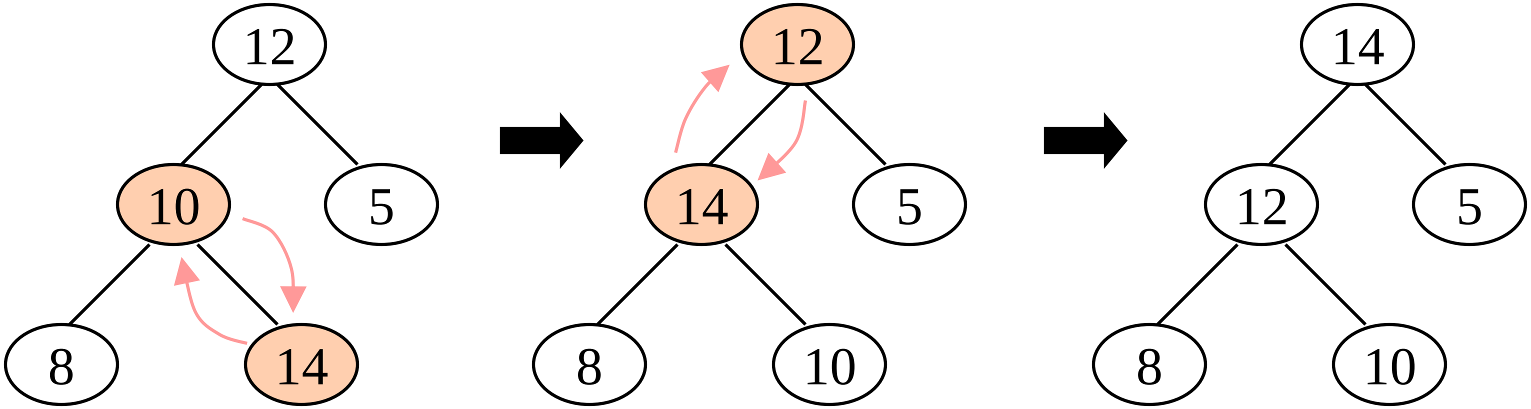
HEAPS

- Top down heap construction.
  - Start with empty heap.
  - Insert all nodes one at a time.



Heaps

- Top down heap construction.
  - Start with empty heap.
  - Insert all nodes one at a time.



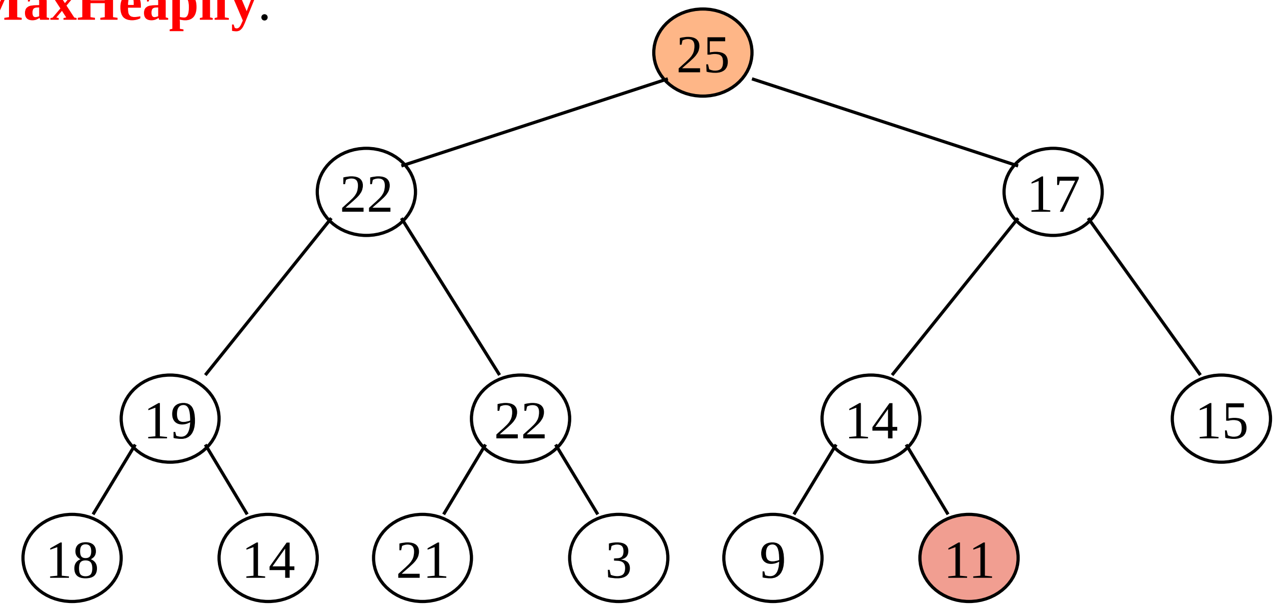
HEAPS

- Root deletion.
  - The root of a heap can be deleted and the heap fixed up as follows:

    - Exchange the root with the last leaf .

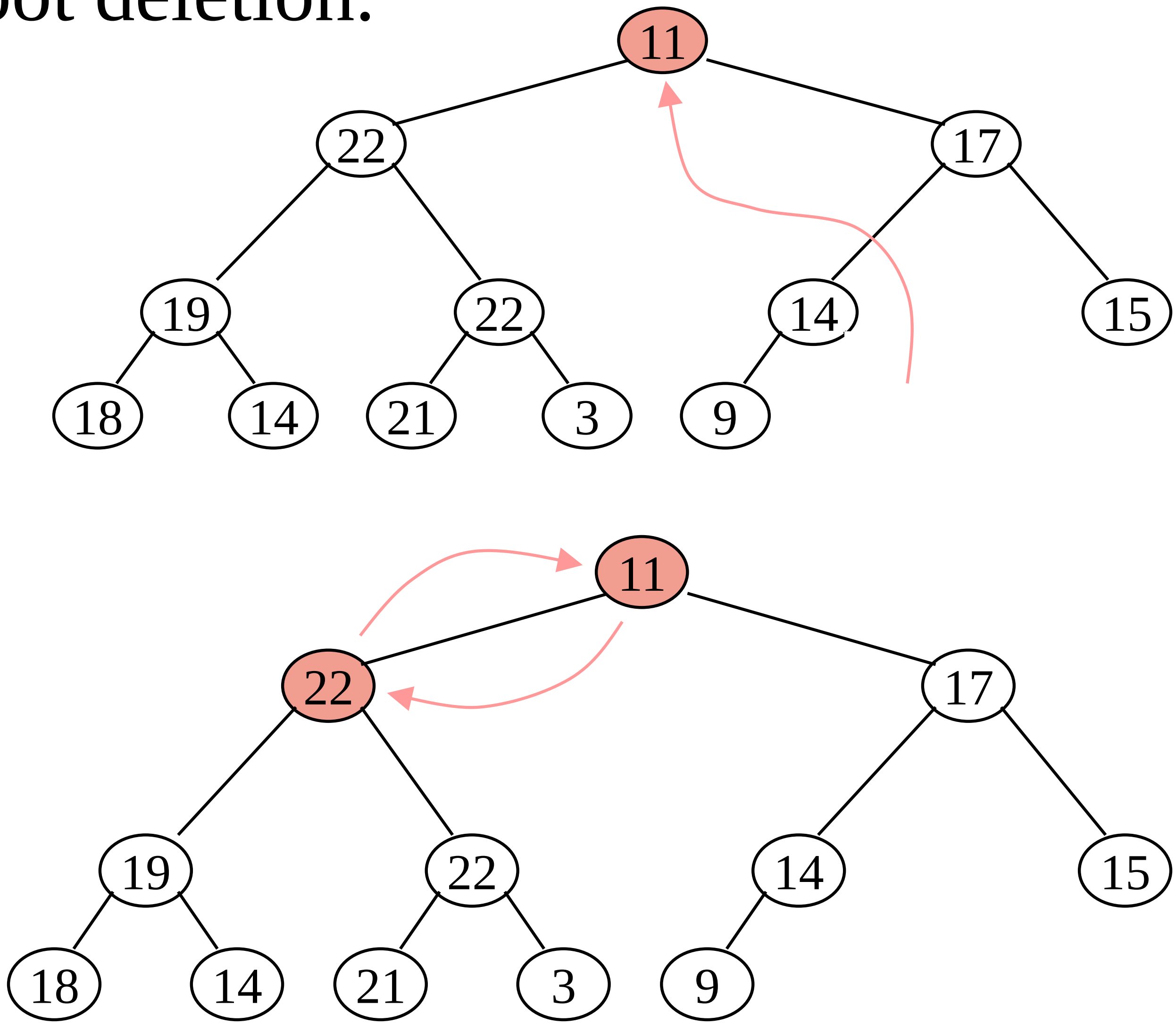
    - Decrease the heap’s size by 1.

    - Heapify the smaller tree in exactly the same way we did it in **MaxHeapify**.



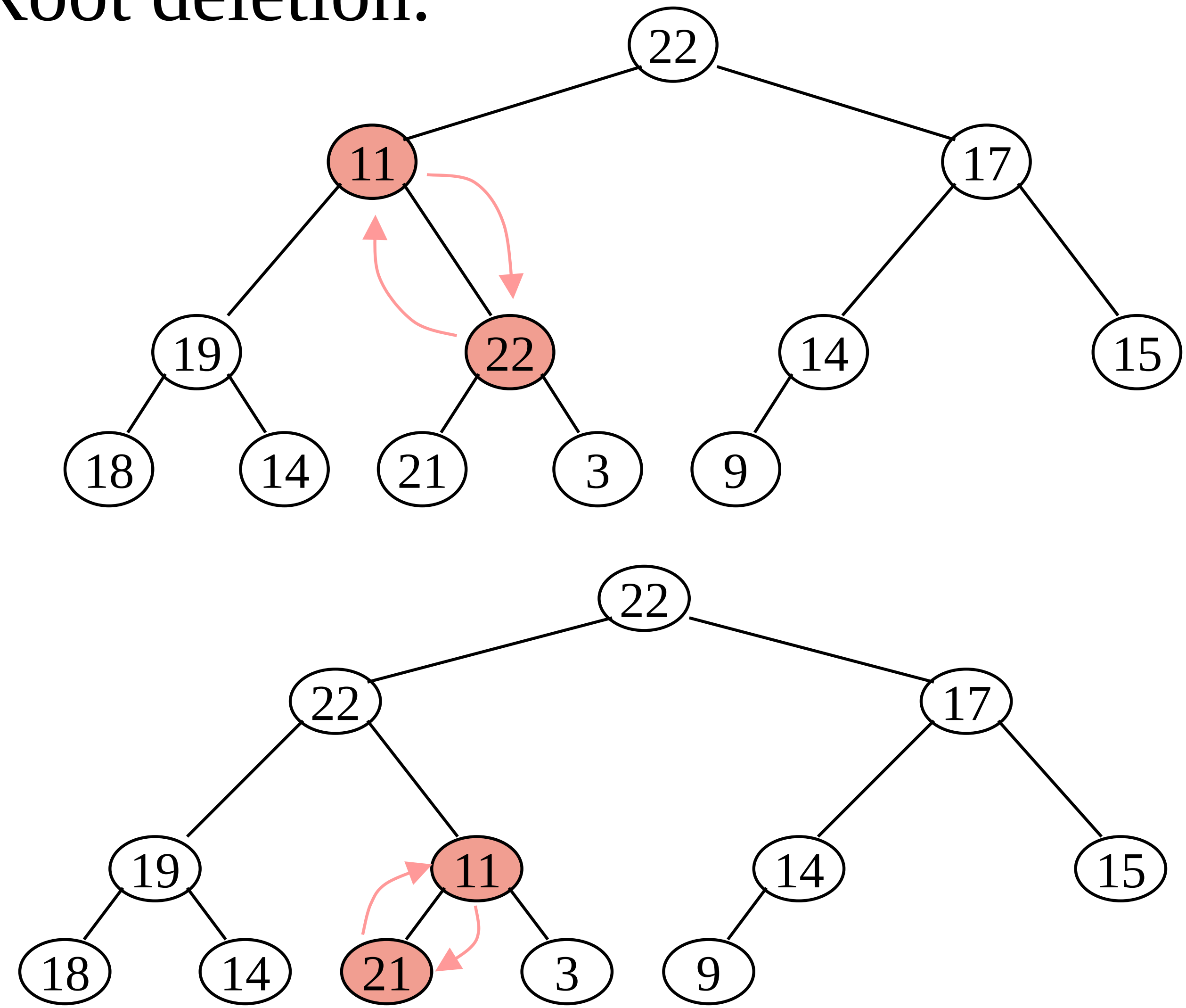
HEAPS

- Root deletion.



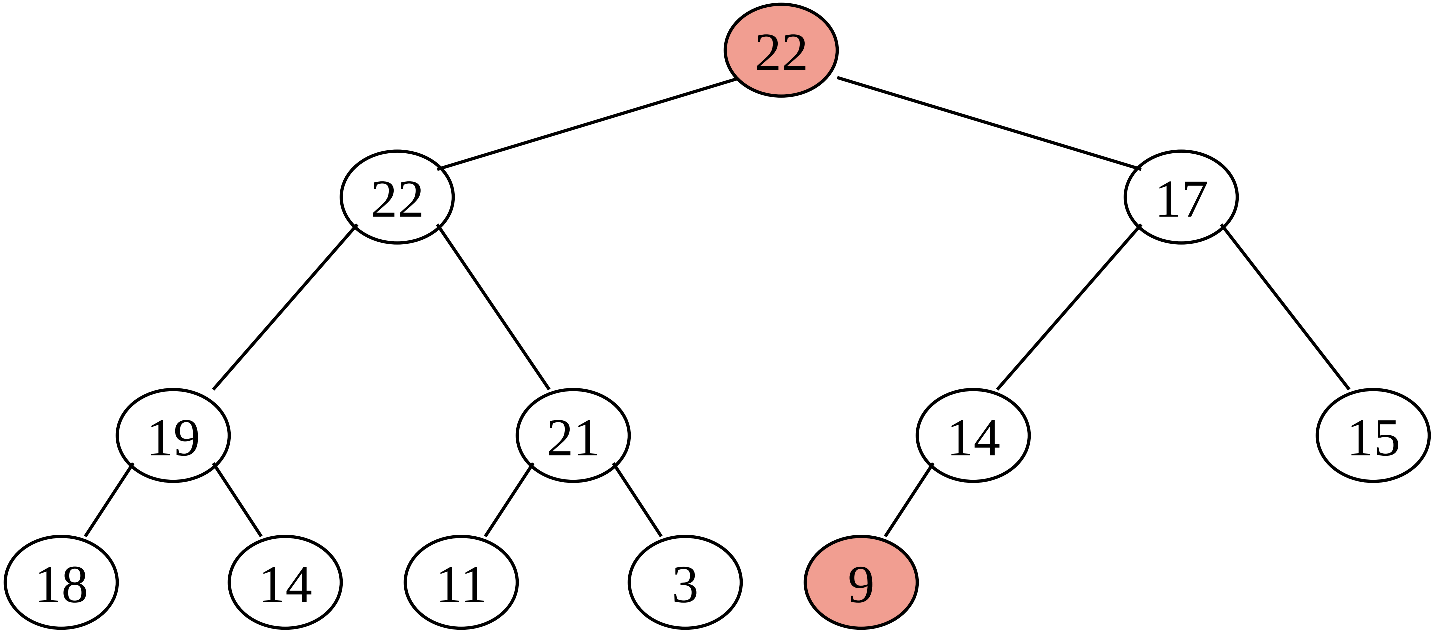
Heaps

- Root deletion.



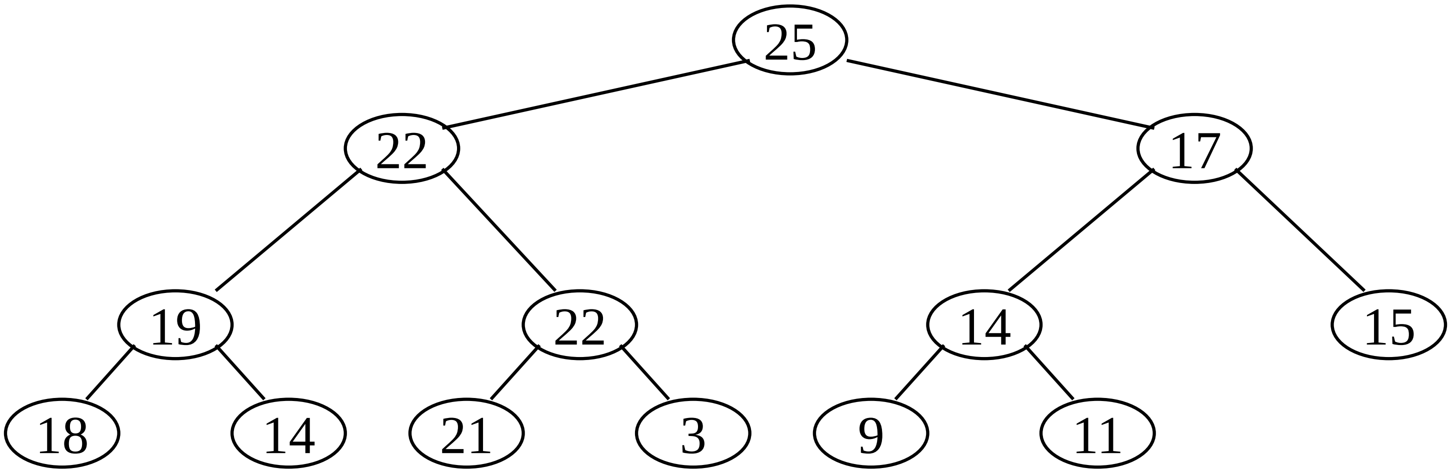
HEAPS

- Root deletion.
  - Removing next root.



Heapsort

- Heapsort algorithm.
  - (Heap construction) Build heap for a given array (either bottom-up or top-down).
  - (Maximum deletion) Apply the root-deletion operation n-1 times to the remaining heap until heap contains just one node.



0	1	2	3	4	5	6	7	8	9	10	11	12
25	22	17	19	22	14	15	18	14	21	3	9	11

HEAPSORT

- The “root” is the first element in the array.
- The “rightmost node at the deepest level” is the last element.
- Swap them...

0	1	2	3	4	5	6	7	8	9	10	11	12
25	22	17	19	22	14	15	18	14	21	3	9	11

0	1	2	3	4	5	6	7	8	9	10	11	12
11	22	17	19	22	14	15	18	14	21	3	9	25

- ...And pretend that the last element in the array no longer exists—that is, the “last index” is 11 (9).

HEAPSORT

- Reheap the root node (index 0, containing 11).

0	1	2	3	4	5	6	7	8	9	10	11	12
11	22	17	19	22	14	15	18	14	21	3	9	25

0	1	2	3	4	5	6	7	8	9	10	11	12
22	22	17	19	21	14	15	18	14	11	3	9	25

0	1	2	3	4	5	6	7	8	9	10	11	12
9	22	17	19	22	14	15	18	14	21	3	22	25

- ...And again, remove and replace the root node.
- Repeat until the last becomes first, and the array is sorted!

HEAPSORT

- Animated example.

6 5 3 1 8 7 2 4

HEAPSORT

- Pseudo-code of the Heapsort algorithm.

```
def Heapsort (A)
    //Constructs a heap from the elements
    //Sorts elements of the array.
    //Input: An array A[1..n] of orderable items
    //Output: A sorted A[1..n]
    HeapBottomUp (A)
    for i = A.length to 2:   // n-1 times
        swap(A[1], A[i])
        A.heapsize = A.heapsize − 1
        MaxHeapify (A, 1)
```

HEAPSORT

- Analysis of the Heapsort.
- Heapsort consists of two parts:
  - Heap construction which has $\boldsymbol{O(n)}$ time complexity.
  - For loop (repeated $\boldsymbol{n-1}$ times) which has time complexity $\boldsymbol{O(\log n)}$.

Total:

$$O(n) + (n-1)O(\log n) = O(n \log n)$$

Priority Queue

- Priority queue is an abstract data type which is like a regular queue or stack data structure, but additionally, each element is associated with a "priority".
  - **stack** — elements are pulled in last-in first-out-order (e.g. a stack of papers).
  - **queue** — elements are pulled in first-in first-out-order (e.g. a line in a cafeteria).
  - **priority queue** — elements are pulled highest-priority-first (e.g. cutting in line, or VIP service).

PRIORITY QUEUE

- Operations on priority queues.
  - ***Insert (S, x)*** – insert element ***x*** into queue ***S***.
  - ***Maximum (S)*** – return the element of ***S*** with the largest key.
  - ***Extract-Max (S)*** – removes and returns the element of ***S*** with the largest key.
  - ***Increase-Key (S, x, k)*** – increases the value of ***x***'s key to the new value ***k*** which should be at least as large as ***x***'s current key value.

PRIORITY QUEUE

- Priority queue is implemented as **heap**.
- Operations on priority queues.
  - ***Extract-Max (S)*** operation:

def **Extract-Max** (A)
    // Input: heap A[1..n]
    // Removes and returns the root element
    $\text{max} = A[1]$
    $A[1] = A[A.\text{heapsize}]$
    $A.\text{heapsize} = A.\text{heapzise} - 1$
    MaxHeapify ($A, 1$)
    return max

  - Time complexity: $\boldsymbol{O}(\mathbf{log}\,\boldsymbol{n})$

PRIORITY QUEUE

- Operations on priority queues.
  - ***Increase-Key (S)*** operation:

```
def Heap-Increase-Key (A, i, k)
    // Input: heap A[1..n], element index i, and its new key k.
    // Output: heap A[1..n] conforming to heap property.
    A[i] = k
    while i > 1 and A[Parent(i)] < A[i]:
        swap(A[Parent(i)], A[i])
        i = Parent(i)
```

  - Time complexity: $\boldsymbol{O(\log n)}$

Priority Queue

- Operations on priority queues.
  - ***Insert (S, x)*** operation:

def **Max-Heap-Insert** $(\boldsymbol{A}, \boldsymbol{k})$

    // Input: heap A[1..n] and new key k.
    // Output: heap A[1..n+1].
    $A.\text{heapsize} = A.\text{heapsize} + 1$
    $A[A.\text{heapsize}] = -\infty$
    Heap-Increase-Key $(\boldsymbol{A}, \boldsymbol{A}.\textbf{heapsize}, \boldsymbol{k})$

  - Time complexity: $\boldsymbol{O}(\mathbf{log}\,\boldsymbol{n})$

That's All for Today!