Derivations of Neutron Transport Equation

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1. General Neutron Transport Equation

Neutron Transport Equation can be written using the principle of neutron conservation within $dVd\Omega dE$. That is, written in differential form.

$$\begin{bmatrix} Time\ rate\ of \\ change\ of\ neutrons \end{bmatrix} = \begin{bmatrix} Rate\ of \\ gain \end{bmatrix} - \begin{bmatrix} Rate\ of \\ loss \end{bmatrix} \tag{1.1}$$

Time rate of change of neutrons is defined as:

$$\frac{dN}{dt} = \frac{1}{v(E)} \frac{\partial \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t)}{\partial t} dVd\Omega dE \tag{1.2a}$$

Rate of loss is defined as:

Rate of loss = collision rate + net leakage rate

Collision rate = $\Sigma_t(\vec{\boldsymbol{x}}, E, t)\psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dVd\Omega dE$ (1.2b)

Collision rate (J_R) (1.2c)

$$J_R = \widehat{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\nabla}}\psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dVd\Omega dE$$

Rate of gain is defined as:

Rate of gain = in-scattering rate + fission rate

In-scattering rate (1.2d)

$$\left[\int_0^\infty \int_{4\pi} \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}' \to \mathrm{E}, t)\psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, dVd\Omega' dE'\right] d\Omega dE$$

Fission rate (prompt) (1.2e)

$$\left[\frac{\chi_p(E)}{4\pi} \int_0^\infty \int_{4\pi} (1-\beta)\nu(E')\Sigma_f(\vec{\boldsymbol{x}}, \mathrm{E}', t)\psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, dVd\Omega' dE'\right] d\Omega dE$$

Fission rate (delayed) (1.2f)

$$\sum_{k=1}^{6} \frac{\chi_{dk}(E)}{4\pi} \lambda_{dk} C(\vec{\boldsymbol{x}}, t)\, dVd\Omega dE$$

Source rate (1.2g)

$$Q(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dVd\Omega dE$$

Therefore, combining all of the above and divide with $dVd\Omega dE$, we obtain:

$$\begin{aligned} &\frac{1}{v(E)} \frac{\partial \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t)}{\partial t} + \widehat{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\nabla}}\psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) + \Sigma_t(\vec{\boldsymbol{x}}, E, t)\psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) \\ &\quad = \int_0^\infty \int_{4\pi} \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}' \to \mathrm{E}, t)\psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, d\Omega' dE' \\ &\quad + \frac{\chi_p(E)}{4\pi} \int_0^\infty \int_{4\pi} (1-\beta)\nu(E')\Sigma_f(\vec{\boldsymbol{x}}, \mathrm{E}', t)\psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, d\Omega' dE' \\ &\quad + \sum_{k=1}^{6} \frac{\chi_{dk}(E)}{4\pi} \lambda_{dk} C(\vec{\boldsymbol{x}}, t) + Q(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) \end{aligned} \tag{1.3a}$$

With the delayed neutron precursor defined as:

$$\frac{\partial C_i}{\partial t} + \lambda_i C_i(\vec{\boldsymbol{x}}, t) = \int_0^\infty \int_{4\pi} \beta_i \nu(E') \Sigma_f(\vec{\boldsymbol{x}}, E', t) \psi\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t\big)\, d\Omega' dE'. \quad ; \quad i = 1, \dots, 6 \tag{1.3b}$$

Boundary Conditions:

$$\begin{gathered} \psi\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t\big) = \psi^b\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t\big) \\ \vec{\boldsymbol{x}} \in \partial\Gamma \\ \widehat{\boldsymbol{\Omega}} \cdot \widehat{\boldsymbol{n}}(\vec{\boldsymbol{x}}) < 0 \end{gathered} \tag{1.3c}$$

Special cases of boundary conditions

- Vacuum Boundary Condition

$$\psi^b\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t\big) = 0 \tag{1.3d}$$

- Reflected Boundary Condition

$$\begin{gathered} \psi^b\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t\big) = \psi^b\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}_{\mathbf{r}}, E, t\big) \\ \widehat{\boldsymbol{\Omega}}_{\mathbf{r}} = \widehat{\boldsymbol{\Omega}} - 2\left(\widehat{\boldsymbol{\Omega}} \cdot \widehat{\boldsymbol{n}}(\vec{\boldsymbol{x}})\right) \widehat{\boldsymbol{n}}(\vec{\boldsymbol{x}}) \end{gathered} \tag{1.3e}$$

Initial Conditions:

$$\psi\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, 0\big) = \psi^i\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E\big) \tag{1.3f}$$

Example: 1-D Transport (z-component)

For 1-D transport, we define:

$$\begin{gathered} \Sigma(\vec{\boldsymbol{x}}, t) = \Sigma(\mathrm{z}, t); Q(\vec{\boldsymbol{x}}) = Q(z) \\ \psi\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}\big) = \psi\big(z, \widehat{\boldsymbol{\Omega}}\big) = \psi(z, \mu, \gamma) \\ \widehat{\boldsymbol{\Omega}} = \left[\mu, \sqrt{1-\mu^2}\cos(\mu), \sqrt{1-\mu^2}\sin(\mu)\right] \\ \widehat{\boldsymbol{\Omega}}' = \left[\mu', \sqrt{1-\mu'^2}\cos(\mu'), \sqrt{1-\mu'^2}\sin(\mu')\right] \\ \widehat{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\nabla}}\psi\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}\big) = \mu \frac{\partial}{\partial z}\psi(z, \mu, \gamma) \end{gathered} \tag{1.4}$$

Assuming steady state, the equation reduces to:

$$\begin{aligned} &\mu \frac{\partial}{\partial z}\psi\big(z, \widehat{\boldsymbol{\Omega}}\big) + \Sigma_t(\vec{\boldsymbol{x}}, E, t)\psi\big(z, \widehat{\boldsymbol{\Omega}}, E\big) \\ &\qquad = \int_0^\infty \int_{4\pi} \Sigma_s\big(z, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}' \to \mathrm{E}, t\big)\psi\big(z, \widehat{\boldsymbol{\Omega}}', E'\big)\, d\Omega' dE' \\ &\qquad + \frac{\chi_p(E)}{4\pi}\int_0^\infty \int_{4\pi} (1-\beta)\nu(E')\Sigma_f(\vec{\boldsymbol{x}}, \mathrm{E}', t)\psi\big(z, \widehat{\boldsymbol{\Omega}}', E'\big)\, d\Omega' dE' \\ &\qquad + \sum_{k=1}^{6} \frac{\chi_{dk}(E)}{4\pi}\lambda_{dk} C(z) + Q\big(z, \widehat{\boldsymbol{\Omega}}, E\big) \end{aligned} \tag{1.5}$$

2. Approximations of Energy Variable

2.1 Multigroup Approximations

In this approximation, we divide the energy in energy bins, such that:

$$\int_{E_g}^{E_{g-1}} \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, dV d\Omega' dE' = \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) \tag{2.1}$$

We do this for each term in neutron transport equation:

- Time rate of change

$$\begin{aligned}\int_{E_g}^{E_{g-1}} \frac{1}{v(E)} \frac{\partial \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t)}{\partial t} dE &= \frac{\partial}{\partial t}\left[\frac{\int_{E_g}^{E_{g-1}} \frac{1}{v(E)} \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dE}{\int_{E_g}^{E_{g-1}} \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dE}\right] \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) \\ &= \frac{1}{v_g} \frac{\partial \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t)}{\partial t}\end{aligned} \tag{2.2a}$$

- Net leakage rate

$$\int_{E_g}^{E_{g-1}} \widehat{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\nabla}} \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dE = \widehat{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\nabla}} \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) \tag{2.2b}$$

- Total reaction rate

$$\begin{aligned}\int_{E_g}^{E_{g-1}} \Sigma_t(\vec{\boldsymbol{x}}, E, t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dE &= \left[\frac{\int_{E_g}^{E_{g-1}} \Sigma_t(\vec{\boldsymbol{x}}, E, t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dE}{\int_{E_g}^{E_{g-1}} \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dE}\right] \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) \\ &= \Sigma_{t,g}(\vec{\boldsymbol{x}}, t) \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t)\end{aligned} \tag{2.2c}$$

- In-scattering rate

$$\begin{aligned}&\int_{E_g}^{E_{g-1}} \left[\int_0^\infty \int_{4\pi} \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}' \to \mathrm{E}, t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, d\Omega' dE'\right] dE \\ &= \sum_{g'=1}^{G} \int_{4\pi} \left[\frac{\int_{E_g}^{E_{g-1}} \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}' \to \mathrm{E}, t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t) dE}{\int_{E_g}^{E_{g-1}} \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t) dE}\right] \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', t) d\Omega' \\ &= \sum_{g'=1}^{G} \int_{4\pi} \Sigma_{s, g' \to g}(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, t)\, \psi_{g'}(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', t) d\Omega'\end{aligned} \tag{2.2d}$$

- Fission rate

$$\begin{aligned}&\int_{E_g}^{E_{g-1}} \left[\frac{\chi(E)}{4\pi} \int_0^\infty \int_{4\pi} \nu(E') \Sigma_f(\vec{\boldsymbol{x}}, \mathrm{E}', t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, d\Omega' dE'\right] dE \\ &= \sum_{g'=1}^{G} \int_0^\infty \frac{\chi(E)}{4\pi} dE \int_{4\pi} \left[\frac{\int_{E_g}^{E_{g-1}} \nu(E') \Sigma_f(\vec{\boldsymbol{x}}, \mathrm{E}', t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t) dE}{\int_{E_g}^{E_{g-1}} \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t) dE}\right] \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', t) d\Omega' \\ &= \frac{\chi_g(E)}{4\pi} \sum_{g'=1}^{G} \int_{4\pi} \nu_{g'}(E') \Sigma_{f g'}(\vec{\boldsymbol{x}}, \mathrm{E}', t)\, \psi_{g'}(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', t) d\Omega'\end{aligned} \tag{2.2e}$$

- Source rate

$$\int_{E_g}^{E_{g-1}} Q(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dE = Q_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) \tag{2.2f}$$

Combining altogether, we got:

$$\begin{aligned} &\frac{1}{v_g} \frac{\partial \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t)}{\partial t} + \widehat{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\nabla}} \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) + \Sigma_{t,g}(\vec{\boldsymbol{x}}, t) \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) \\ &= \sum_{g'=1}^{G} \int_{4\pi} \Sigma_{s,g' \to g}(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, t)\, \psi_{g'}(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', t) d\Omega' \\ &+ \frac{\chi_g(E)}{4\pi} \sum_{g'=1}^{G} \int_{4\pi} \nu_{g'}(E') \Sigma_{f g'}(\vec{\boldsymbol{x}}, \mathrm{E}', t)\, \psi_{g'}(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', t) d\Omega' + Q_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) \end{aligned} \tag{2.3}$$

With boundary conditions:

$$\begin{gathered} \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) = \psi_g^b(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) = \int_{E_g}^{E_{g-1}} \psi^b(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t) dE \\ \vec{\boldsymbol{x}} \in \partial\Gamma\,; 1 < \mathrm{g} < \mathrm{G};\ \widehat{\boldsymbol{\Omega}} \cdot \widehat{\boldsymbol{n}}(\vec{\boldsymbol{x}}) < 0 \end{gathered} \tag{2.4a}$$

And Initial Conditions:

$$\begin{gathered} \psi_g(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, 0) = \psi_g^i(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}) = \int_{E_g}^{E_{g-1}} \psi^i(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, 0) dE \\ \vec{\boldsymbol{x}} \in \partial\Gamma\,; 1 < \mathrm{g} < \mathrm{G};\ \widehat{\boldsymbol{\Omega}} \cdot \widehat{\boldsymbol{n}}(\vec{\boldsymbol{x}}) < 0 \end{gathered} \tag{2.4b}$$

2.2 One Speed Approximations

Assumption:

Assume that the neutron energy does not change in scattering collision.

$$\Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}' \to \mathrm{E}, t) = \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}', t) \delta(E' - E) \tag{2.5}$$

Using this assumption, the scattering term can be simplified into:

$$\begin{aligned} &\int_0^\infty \left[\int_0^\infty \int_{4\pi} \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}' \to \mathrm{E}, t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, d\Omega' dE' \right] dE \\ &= \int_0^\infty \left[\int_0^\infty \int_{4\pi} \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}', t) \delta(E' - E) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, d\Omega' dE' \right] dE \\ &= \int_0^\infty \int_{4\pi} \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}', t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, d\Omega' dE' \\ &= \int_{4\pi} \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, \mathrm{E}', t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, d\Omega' \end{aligned} \tag{2.6}$$

Then the neutron transport equation can be written as:

$$\begin{aligned} &\frac{1}{v} \frac{\partial \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t)}{\partial t} + \widehat{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\nabla}} \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) + \Sigma_t(\vec{\boldsymbol{x}}, t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) \\ &= \int_{4\pi} \Sigma_s(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}' \cdot \widehat{\boldsymbol{\Omega}}, E', t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', E', t)\, d\Omega' \\ &+ \frac{\chi_p}{4\pi} \int_{4\pi} (1 - \beta) \nu \Sigma_f(\vec{\boldsymbol{x}}, t) \psi(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}', t)\, d\Omega' + \sum_{k=1}^{6} \frac{\chi_{dk}}{4\pi} \lambda_{dk} C(\vec{\boldsymbol{x}}, t) + Q(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t) \end{aligned} \tag{2.7a}$$

With boundary conditions:

$$\psi\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t\big) = \psi^{b}\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, t\big) = \int_{0}^{\infty} \psi^{b}\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t\big) dE \tag{2.7b}$$

$$\vec{\boldsymbol{x}} \in \partial\Gamma \,; \widehat{\boldsymbol{\Omega}} \cdot \widehat{\boldsymbol{n}}(\vec{\boldsymbol{x}}) < 0$$

And Initial Conditions:

$$\psi\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, 0\big) = \psi^{i}\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}\big) = \int_{0}^{\infty} \psi^{i}\big(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E\big) dE \tag{2.7c}$$

$$\vec{\boldsymbol{x}} \in \partial\Gamma \;\; \widehat{\boldsymbol{\Omega}} \cdot \widehat{\boldsymbol{n}}(\vec{\boldsymbol{x}}) < 0$$

3. Approximations of Angular Variable

3.1 Continuous P1 Approximations and P1 Closure

Simplifications: Scattering is linearly anisotropic

$$\Sigma_s\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}}'\cdot\widehat{\boldsymbol{\Omega}},E'\to E,t\right)=\frac{1}{4\pi}\left[\Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E,t)+3\left(\widehat{\boldsymbol{\Omega}}'\cdot\widehat{\boldsymbol{\Omega}}\right)\Sigma_{s1}(\vec{\boldsymbol{x}},E'\to E,t)\right] \tag{3.1}$$

Recall:

Scalar flux : $\phi(\vec{\boldsymbol{x}},E,t)=\int_{4\pi}\psi\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,t\right)d\Omega$

Neutron current : $\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)=\int_{4\pi}\widehat{\boldsymbol{\Omega}}\psi\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,t\right)d\Omega$

Integrate Neutron Transport Equation over $\widehat{\boldsymbol{\Omega}}$, $\widehat{\boldsymbol{\Omega}}\in 4\pi$

$$\begin{aligned}\frac{1}{v}\frac{\partial}{\partial t}\left(\int_{4\pi}\psi\,d\Omega\right)+\vec{\boldsymbol{\nabla}}\cdot\left(\int_{4\pi}\widehat{\boldsymbol{\Omega}}\psi\,d\Omega\right)+\Sigma_t\left(\int_{4\pi}\psi\,d\Omega\right)&\\
=\int_0^\infty\int_{4\pi}\left[\int_{4\pi}\frac{1}{4\pi}\Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E,t)\psi\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}}',E',t\right)d\Omega\right.&\\
\left.+\int_{4\pi}\frac{1}{4\pi}3\left(\widehat{\boldsymbol{\Omega}}'\cdot\widehat{\boldsymbol{\Omega}}\right)\Sigma_{s1}(\vec{\boldsymbol{x}},E'\to E,t)\psi\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}}',E',t\right)d\Omega\right]d\Omega'dE&\\
+\frac{\chi(E)}{4\pi}\int_0^\infty\int_{4\pi}\int_{4\pi}\nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}',t)\psi\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}}',E',t\right)d\Omega\,d\Omega'dE'&\\
+\int_0^\infty Q\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,t\right)d\Omega&\end{aligned} \tag{3.2a}$$

$$\begin{aligned}\frac{1}{v(E)}\frac{\partial\phi(\vec{\boldsymbol{x}},E,t)}{\partial t}+\vec{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)+\Sigma_t(\vec{\boldsymbol{x}},E,t)\phi(\vec{\boldsymbol{x}},E,t)&\\
=\int_0^\infty\Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E,t)\phi(\vec{\boldsymbol{x}},E',t)\,dE'&\\
+\frac{\chi(E)}{4\pi}\int_0^\infty\nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}',t)\psi(\vec{\boldsymbol{x}},E',t)\,dE'+Q_0(\vec{\boldsymbol{x}},E,t)&\end{aligned} \tag{3.2b}$$

This is neutron continuity equation.

But we have 4 unknowns $\left(\phi,J_x,J_y,J_z\right)$. We need to relate ϕ to $\boldsymbol{J}$ to “close” the equation.

Try multiple neutron transport equation with $\widehat{\boldsymbol{\Omega}}$ and integrate over $\widehat{\boldsymbol{\Omega}}$, $\widehat{\boldsymbol{\Omega}}\in 4\pi$:

$$\begin{aligned}\frac{1}{v}\frac{\partial}{\partial t}\left(\int_{4\pi}\widehat{\boldsymbol{\Omega}}\psi\,d\Omega\right)+\vec{\boldsymbol{\nabla}}\cdot\left(\int_{4\pi}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Omega}}\psi\,d\Omega\right)+\Sigma_t\left(\int_{4\pi}\widehat{\boldsymbol{\Omega}}\psi\,d\Omega\right)&\\
=\int_0^\infty\int_{4\pi}\left[\int_{4\pi}\frac{\boldsymbol{\Omega}}{4\pi}\Sigma_{s0}(\vec{\boldsymbol{x}},\mathrm{E}'\to\mathrm{E},t)\psi\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}}',E',t\right)d\Omega\right.&\\
\left.+\int_{4\pi}\frac{\widehat{\boldsymbol{\Omega}}}{4\pi}3\left(\widehat{\boldsymbol{\Omega}}'\cdot\widehat{\boldsymbol{\Omega}}\right)\Sigma_{s1}(\vec{\boldsymbol{x}},\mathrm{E}'\to\mathrm{E},t)\psi\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}}',E',t\right)d\Omega\right]d\Omega'dE&\\
+\frac{\chi(E)}{4\pi}\int_0^\infty\int_{4\pi}\int_{4\pi}\widehat{\boldsymbol{\Omega}}\,\nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}',t)\psi\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}}',E',t\right)d\Omega\,d\Omega'dE'&\\
+\int_0^\infty\widehat{\boldsymbol{\Omega}}\,Q\left(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,t\right)d\Omega&\end{aligned} \tag{3.3a}$$

$$\frac{1}{v}\frac{\partial \bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \bar{\bar{\boldsymbol{\Pi}}}(\vec{\boldsymbol{x}},E,t) + \Sigma_t(\vec{\boldsymbol{x}},E,t)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \int_0^\infty \Sigma_{s1}(\vec{\boldsymbol{x}},E' \to E,t)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E',t)\,dE' + Q_1(\vec{\boldsymbol{x}},E,t) \tag{3.3b}$$

This makes 4 equations but unknows. If we continue, it will just add more unknowns. In order to solve this, P1 closure (approximation) is needed.

3.1.1. P1 Closure (Approximation)

Assumption: angular flux is linear in $\widehat{\boldsymbol{\Omega}}$

$$\psi(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,t) = f(\vec{\boldsymbol{x}},E,t) + \widehat{\boldsymbol{\Omega}} \cdot g(\vec{\boldsymbol{x}},E,t) \tag{3.4}$$

Therefore,

$$\begin{aligned} \int_{4\pi} \psi(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,t)\,d\Omega = \phi(\vec{\boldsymbol{x}},E,t) &= \int_{4\pi} f(\vec{\boldsymbol{x}},E,t)\,d\Omega + g(\vec{\boldsymbol{x}},E,t)\int_{4\pi} \widehat{\boldsymbol{\Omega}}\,d\Omega \\ &= 4\pi f(\vec{\boldsymbol{x}},E,t) + \vec{0} \\ f(\vec{\boldsymbol{x}},E,t) &= \frac{\phi(\vec{\boldsymbol{x}},E,t)}{4\pi} \end{aligned} \tag{3.5a}$$

$$\begin{aligned} \int_{4\pi} \widehat{\boldsymbol{\Omega}} \cdot \psi(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,t)\,d\Omega = \bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t) &= \int_{4\pi} \widehat{\boldsymbol{\Omega}} \cdot f(\vec{\boldsymbol{x}},E,t)\,d\Omega + g(\vec{\boldsymbol{x}},E,t)\int_{4\pi} \widehat{\boldsymbol{\Omega}} \cdot \widehat{\boldsymbol{\Omega}}\,d\Omega \\ &= \vec{0} + g(\vec{\boldsymbol{x}},E,t) \cdot \frac{4\pi}{3}\bar{\bar{I}} \\ g(\vec{\boldsymbol{x}},E,t) &= \frac{3}{4\pi}\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t) \end{aligned} \tag{3.5b}$$

Therefore, the P1 Closure is:

$$\psi(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,t) = \frac{1}{4\pi}\left[\phi(\vec{\boldsymbol{x}},E,t) + 3\left(\widehat{\boldsymbol{\Omega}} \cdot \bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)\right)\right] \tag{3.6}$$

Apply the P1 closure to $\bar{\bar{\boldsymbol{\Pi}}}(\vec{\boldsymbol{x}},E,t)$

$$\begin{aligned} \bar{\bar{\boldsymbol{\Pi}}}(\vec{\boldsymbol{x}},E,t) &= \int_{4\pi} \widehat{\boldsymbol{\Omega}} \cdot \widehat{\boldsymbol{\Omega}}\psi(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,t)\,d\Omega \\ &= \frac{1}{4\pi}\left[\int_{4\pi} \widehat{\boldsymbol{\Omega}} \cdot \widehat{\boldsymbol{\Omega}}\phi(\vec{\boldsymbol{x}},E,t)d\Omega + 3\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)\int_{4\pi} \widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Omega}}d\Omega\right] \\ &= \frac{1}{4\pi}\left[\frac{4\pi}{3}\bar{\bar{I}}\phi(\vec{\boldsymbol{x}},E,t) + \vec{0}\right] \\ &= \frac{1}{3}\bar{\bar{I}}\phi(\vec{\boldsymbol{x}},E,t) \\ \vec{\boldsymbol{\nabla}} \cdot \bar{\bar{\boldsymbol{\Pi}}}(\vec{\boldsymbol{x}},E,t) &= \frac{1}{3}\vec{\boldsymbol{\nabla}} \cdot \bar{\bar{I}}\phi(\vec{\boldsymbol{x}},E,t) \\ &= \frac{1}{3}\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},E,t) \end{aligned} \tag{3.7}$$

Apply this to the P1 equation (3.3b) and convert $\vec{\boldsymbol{\nabla}} \cdot \bar{\bar{\boldsymbol{\Pi}}}(\vec{\boldsymbol{x}},E,t)$ term

$$\frac{1}{v}\frac{\partial \bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)}{\partial t} + \frac{1}{3}\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},E,t) + \Sigma_t\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \int_0^\infty \Sigma_{s1}(\vec{\boldsymbol{x}},E' \to E,t)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E',t)\,dE' + Q_1(\vec{\boldsymbol{x}},E,t) \tag{3.8}$$

In conclusion, we have 4 equations:

- Neutron continuity equation

$$\frac{1}{v(E)}\frac{\partial\phi(\vec{\boldsymbol{x}},E,t)}{\partial t}+\vec{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)+\Sigma_t(\vec{\boldsymbol{x}},E,t)\phi(\vec{\boldsymbol{x}},E,t)$$
$$=\int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E,t)\phi(\vec{\boldsymbol{x}},E',t)\,dE'$$
$$+\frac{\chi(E)}{4\pi}\int_0^\infty \nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}',t)\psi(\vec{\boldsymbol{x}},E',t)\,dE'+Q_0(\vec{\boldsymbol{x}},E,t) \tag{3.9a}$$

- P1 Closure

$$\frac{1}{v}\frac{\partial\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)}{\partial t}+\frac{1}{3}\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},E,t)+\Sigma_t(\vec{\boldsymbol{x}},E,t)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)$$
$$=\int_0^\infty \Sigma_{s1}(\vec{\boldsymbol{x}},E'\to E,t)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E',t)\,dE'+Q_1(\vec{\boldsymbol{x}},E,t) \tag{3.9b}$$

With 4 unknowns (ϕ,J_x,J_y,J_z), initial conditions:

$$\phi(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},0)=\phi^i(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},0)=\int_{4\pi}\psi^i(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,0)d\Omega \tag{3.10a}$$

$$\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},0)=\bar{\boldsymbol{J}}^i(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},0)=\int_{4\pi}\widehat{\boldsymbol{\Omega}}_{\boldsymbol{i}}\cdot\psi^i(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},E,0)d\Omega \tag{3.10b}$$

And boundary condition

$$\int_{\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}<0}\left|\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}\right|\psi(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},t)d\Omega\approx\int_{\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}<0}\left|\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}\right|\psi^b(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},t)d\Omega$$
$$=\bar{\boldsymbol{J}}^-(\vec{\boldsymbol{x}},E,t)$$
$$\vec{\boldsymbol{x}}\in\partial\Gamma\,;\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}(\vec{\boldsymbol{x}})<0 \tag{3.11a}$$

$$\int_{\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}<0}\left|\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}\right|\psi(\vec{\boldsymbol{x}},\widehat{\boldsymbol{\Omega}},t)d\Omega\approx-\frac{1}{4}\int_{\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}<0}\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}\left[\phi+3\left(\Omega_xJ_x+\Omega_yJ_y+\Omega_zJ_z\right)\right]dz$$
$$=\frac{1}{4}\phi(\vec{\boldsymbol{x}},E,t)-\frac{1}{2}\left[\mathrm{n_x}J_x+\mathrm{n}_yJ_y+\mathrm{n}_zJ_z\right]$$
$$\vec{\boldsymbol{x}}\in\partial\Gamma\,;\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}(\vec{\boldsymbol{x}})<0 \tag{3.11b}$$

Thus,

$$\phi(\vec{\boldsymbol{x}},E,t)-2\left(\mathrm{n_x}J_x+\mathrm{n}_yJ_y+\mathrm{n}_zJ_z\right)=4\bar{\boldsymbol{J}}^-(\vec{\boldsymbol{x}},E,t) \tag{3.12}$$

This is called Marshak boundary conditions.
Note: Identities

$\int_{4\pi} d\Omega=4\pi$	$\int_{4\pi}\widehat{\boldsymbol{\Omega}}\,d\Omega=\vec{0}$	$\int_{4\pi}\widehat{\boldsymbol{\Omega}}\,\widehat{\boldsymbol{\Omega}}d\Omega=\frac{4\pi}{3}\bar{\bar{I}}$
$\int_{4\pi}\widehat{\boldsymbol{\Omega}}\,\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Omega}}d\Omega=\vec{0}$	$\int_{4\pi}\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}d\Omega=-\pi$	$\int_{\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}<0}\widehat{\boldsymbol{\Omega}}\left(\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{n}}\right)d\Omega=\frac{2\pi}{3}\widehat{\boldsymbol{n}}$

3.2 Diffusion Approximations

We need to employ 3 assumptions:

a. Assume all temporal dependence is in scalar flux. This means the time derivatives of the current vector terms are neglected.

$$\frac{\partial\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}},E,t)}{\partial t}\approx 0 \tag{3.13a}$$

b. Assume source is isotropic.

$$Q_0(\vec{\boldsymbol{x}}, E, t) \approx \frac{Q(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t)}{4\pi} \tag{3.13b}$$

$$Q_1(\vec{\boldsymbol{x}}, E, t) \approx \int_{4\pi} \frac{\widehat{\boldsymbol{\Omega}}}{4\pi} Q(\vec{\boldsymbol{x}}, \widehat{\boldsymbol{\Omega}}, E, t)\, d\Omega = \vec{0}$$

c. Outscatter approximation.

$$\int_0^\infty \Sigma_{s1}(\vec{\boldsymbol{x}}, E' \to E, t)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}}, E', t)\, dE' \approx \int_0^\infty \Sigma_{s1}(\vec{\boldsymbol{x}}, E, t)\delta(E' - E)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}}, E', t)\, dE' \\ = \Sigma_{s1}(\vec{\boldsymbol{x}}, E, t)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}}, E, t) \tag{3.13c}$$

Applying these approximations to the P1 closure equation, yields to:

$$\frac{1}{3}\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}}, E, t) + \Sigma_t(\vec{\boldsymbol{x}}, E, t)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}}, E, t) = \Sigma_{s1}(\vec{\boldsymbol{x}}, E, t)\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}}, E, t) \tag{3.14}$$

Solve for $\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}}, E, t)$ by algebraic manipulation:

$$\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}}, E, t) = -\frac{1}{\Sigma_t(\vec{\boldsymbol{x}}, E, t) - \Sigma_{s1}(\vec{\boldsymbol{x}}, E, t)}\frac{1}{3}\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}}, E, t) \tag{3.15}$$

Using Fick's law, we define:

$$D_0(\vec{\boldsymbol{x}}, E, t) = \frac{1}{3\left(\Sigma_t(\vec{\boldsymbol{x}}, E, t) - \Sigma_{s1}(\vec{\boldsymbol{x}}, E, t)\right)} = \frac{1}{3\Sigma_{tr}(\vec{\boldsymbol{x}}, E, t)} \tag{3.16}$$

Therefore,

$$\bar{\boldsymbol{J}}(\vec{\boldsymbol{x}}, E, t) = -D_0(\vec{\boldsymbol{x}}, E, t)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}}, E, t) \tag{3.17}$$

Apply this to the neutron continuity equation:

$$\frac{1}{v(E)}\frac{\partial\phi(\vec{\boldsymbol{x}}, E, t)}{\partial t} - \vec{\boldsymbol{\nabla}} \cdot D_0(\vec{\boldsymbol{x}}, E, t)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}}, E, t) + \Sigma_t(\vec{\boldsymbol{x}}, E, t)\phi(\vec{\boldsymbol{x}}, E, t) \\ = \int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}}, E' \to E, t)\phi(\vec{\boldsymbol{x}}, E', t)\, dE' + \frac{\chi(E)}{4\pi}\int_0^\infty \nu(E)\Sigma_f(\vec{\boldsymbol{x}}, E', t)\psi(\vec{\boldsymbol{x}}, E', t)\, dE' \\ + Q_0(\vec{\boldsymbol{x}}, E, t) \tag{3.18}$$

With the boundary condition:

$$\frac{\phi(\vec{\boldsymbol{x}}, E, t)}{4} + \frac{D_0(\vec{\boldsymbol{x}}, E, t)}{2}\left(\widehat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}}, E, t)\right) = \bar{\boldsymbol{J}}^-(\vec{\boldsymbol{x}}, E, t) \tag{3.19a}$$

And initial condition

$$\phi(\vec{\boldsymbol{x}}, E, 0) = \phi^i(\vec{\boldsymbol{x}}, E) \tag{3.19b}$$

4. Spatial Approximation: Buckling Approximation

Suppose we have a one-speed neutron diffusion equation derived from continuous energy diffusion equation:

$$\frac{1}{v(E)}\frac{\partial \phi(\vec{\boldsymbol{x}},E,t)}{\partial t} - \vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}},E,t)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},E,t) + \Sigma_t(\vec{\boldsymbol{x}},E,t)\phi(\vec{\boldsymbol{x}},E,t) \\ = \int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E,t)\phi(\vec{\boldsymbol{x}},E',t)\,dE' \\ + \frac{\chi(E)}{4\pi}\int_0^\infty \nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}',t)\phi(\vec{\boldsymbol{x}},E',t)\,dE' + Q_0(\vec{\boldsymbol{x}},E,t) \tag{4.1}$$

We define:

$$\int_0^\infty \phi(\vec{\boldsymbol{x}},E,t)\,dE = \phi(\vec{\boldsymbol{x}},t) \qquad \int_0^\infty \Sigma_x(E,t)dE = \Sigma_x\,(t)$$

$$\int_0^\infty Q_0(\vec{\boldsymbol{x}},E,t)\,dE = Q_0(\vec{\boldsymbol{x}},t) \qquad \Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E,t) = \Sigma_{s0}(\vec{\boldsymbol{x}},t)\delta(E'-E)$$

Thus, the scattering term:

$$\int_0^\infty \left[\int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E,t)\phi(\vec{\boldsymbol{x}},E',t)\,dE'\right]dE = \int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E,t)\phi(\vec{\boldsymbol{x}},E,t)\,dE \\ = \Sigma_{s0}(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) \tag{4.2}$$

Therefore, the one-speed neutron diffusion equation:

$$\frac{1}{v}\frac{\partial \phi(\vec{\boldsymbol{x}},t)}{\partial t} - D_0(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{\nabla}}^2\phi(\vec{\boldsymbol{x}},t) + \Sigma_t(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) \\ = \Sigma_{s0}(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) + \frac{\chi}{4\pi}\nu\Sigma_f(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) + Q_0(\vec{\boldsymbol{x}},t) \tag{4.3}$$

For buckling approximation, we assume:

- Homogeneous system ($\frac{\chi}{4\pi} = 1$)
- No source
- Steady state

Therefore,

$$-D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}^2\phi(\vec{\boldsymbol{x}}) + \Sigma_t(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}}) = \Sigma_{s0}(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}}) + \nu\Sigma_f(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}}) \tag{4.4}$$

Doing some simplifications:

$$\begin{aligned} -D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}^2\phi(\vec{\boldsymbol{x}}) + \left(\Sigma_t(\vec{\boldsymbol{x}}) - \Sigma_{s0}(\vec{\boldsymbol{x}})\right)\phi(\vec{\boldsymbol{x}}) &= \nu\Sigma_f(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}}) \\ -D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}^2\phi(\vec{\boldsymbol{x}}) + \Sigma_a(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}}) &= \nu\Sigma_f(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}}) \\ -D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}^2\phi(\vec{\boldsymbol{x}}) + \left(\Sigma_a(\vec{\boldsymbol{x}}) - \nu\Sigma_f(\vec{\boldsymbol{x}})\right)\phi(\vec{\boldsymbol{x}}) &= 0 \\ \vec{\boldsymbol{\nabla}}^2\phi(\vec{\boldsymbol{x}}) + \frac{\left(\nu\Sigma_f(\vec{\boldsymbol{x}}) - \Sigma_a(\vec{\boldsymbol{x}})\right)}{D_0(\vec{\boldsymbol{x}})}\phi(\vec{\boldsymbol{x}}) &= 0 \\ \vec{\boldsymbol{\nabla}}^2\phi(\vec{\boldsymbol{x}}) + B^2(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}}) &= 0 \end{aligned} \tag{4.5}$$

We define:

$$\Sigma_a(\vec{\boldsymbol{x}}) = \Sigma_t(\vec{\boldsymbol{x}}) - \Sigma_{s0}(\vec{\boldsymbol{x}}) \tag{4.6}$$
$$B^2(\vec{\boldsymbol{x}}) = \frac{\left(\nu\Sigma_f(\vec{\boldsymbol{x}}) - \Sigma_a(\vec{\boldsymbol{x}})\right)}{D_0(\vec{\boldsymbol{x}})} = \frac{k_\infty - 1}{L^2}$$
$$L^2 = \frac{D_0(\vec{\boldsymbol{x}})}{\Sigma_a(\vec{\boldsymbol{x}})}$$

This is an eigenvalue/eigenvector that can be solved analytically.
Example: 1-D Slab

$$\frac{d^2\phi(\vec{\boldsymbol{x}})}{dx^2} + B^2(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}}) = 0 \tag{4.7}$$

General solution:

$$\phi(\vec{\boldsymbol{x}}) = C_1\cos(Bx) + C_2\sin(Bx) \tag{4.8}$$

Here we need to apply boundary conditions.
Note for Buckling Approximation:

- Buckling factor (B) are for bare, unreflected core.
- Solution of the eigenvalue problem is called geometric buckling (B_g) while $\frac{\left(\nu\Sigma_f(\vec{x}) - \Sigma_a(\vec{x})\right)}{D_0}$ is called material buckling (B_m)
- For a homogeneous system:

$$B_g^2 = B_m^2 \rightarrow k = 1 \rightarrow critical$$
$$B_g^2 > B_m^2 \rightarrow k < 1 \rightarrow subcritical$$
$$B_g^2 < B_m^2 \rightarrow k > 1 \rightarrow supercritical$$

- Geometrical buckling (B_g) for various geometries

Geometry:	Dimensions:	Buckling:	Flux Shape:
Rectangular Block	$a \times b \times c$	$B^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2$	$\phi(x,y,z) = A_0\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)\cos\left(\frac{\pi z}{c}\right)$
Sphere	$Radius : R$	$B^2 = \left(\frac{\pi}{R}\right)^2$	$\phi(r) = \frac{A_0}{r}\sin\left(\frac{\pi r}{R}\right)$
Cylinder	$Radius : R$ $Height : H$	$B^2 = \left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$	$\phi(r,z) = A_0 J_0\left(\frac{2.405 r}{R}\right)\cos\left(\frac{\pi z}{H}\right)$

Taken from: J. Lamarsh, "Nuclear Reactor Analysis, p.298

5. Kinetics

5.1 3D Spatial Kinetics

The 3D spatial kinetics equations are basically the 3D multigroup neutron diffusion equation.

$$\frac{1}{v_g(\vec{\boldsymbol{x}})}\frac{\partial \phi_g(\vec{\boldsymbol{x}},t)}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot D_g(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{\nabla}}\phi_g(\vec{\boldsymbol{x}},t) + \Sigma_{t,g}(\vec{\boldsymbol{x}},t)\phi_g(\vec{\boldsymbol{x}},t)$$
$$= \sum_{g'=1}^{G} \Sigma_{s,g'\to g}(\vec{\boldsymbol{x}},t)\,\phi_g(\vec{\boldsymbol{x}},t) + \sum_{k=1}^{6}\frac{\chi_{p,g}}{4\pi}\frac{1}{k_{eff}}\sum_{g'=1}^{G}(1-\beta_k)\nu_{g'}\Sigma_{f\,g'}(\vec{\boldsymbol{x}},\mathrm{t})\phi_g(\vec{\boldsymbol{x}},\mathrm{t})$$
$$+\sum_{k=1}^{6}\frac{\chi_{dk,g}}{4\pi}\lambda_k(\vec{\boldsymbol{x}})C_k(\vec{\boldsymbol{x}},t) \tag{5.1a}$$

And the precursor concentration:

$$\frac{\partial C_k(\vec{\boldsymbol{x}},t)}{\partial t} + \lambda_k(\vec{\boldsymbol{x}})C_k(\vec{\boldsymbol{x}},t) = \beta_k\frac{1}{k_{eff}}\nu_{g'}\Sigma_{f\,g'}(\vec{\boldsymbol{x}},\mathrm{t})\phi_g(\vec{\boldsymbol{x}},\mathrm{t}) \tag{5.1b}$$
$$k = 1,2,3,\dots,6$$

5.2 Adjoint Flux, Exact, and First-order Perturbation Theory

5.2.1 Adjoint Operator

We start form steady state, continuous energy neutron diffusion equation with no external source.

$$-\vec{\boldsymbol{\nabla}} \cdot D_0(\vec{\boldsymbol{x}},E)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},E) + \Sigma_t(\vec{\boldsymbol{x}},E)\phi(\vec{\boldsymbol{x}},E)$$
$$= \int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E)\phi(\vec{\boldsymbol{x}},E')\,dE' + \frac{\chi(E)}{4\pi}\int_0^\infty \nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}')\phi(\vec{\boldsymbol{x}},E')\,dE' \tag{5.2}$$

We define an operator form:

$$M\phi = -\vec{\boldsymbol{\nabla}} \cdot D_0(\vec{\boldsymbol{x}},E)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},E) + \Sigma_t(\vec{\boldsymbol{x}},E)\phi(\vec{\boldsymbol{x}},E) - \int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E)\phi(\vec{\boldsymbol{x}},E')\,dE' \tag{5.3a}$$

$$F\phi = \frac{\chi(E)}{4\pi}\int_0^\infty \nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}')\phi(\vec{\boldsymbol{x}},E')\,dE' \tag{5.3b}$$

Therefore, the adjoint neutronic problem can be written as:

$$M\phi = \lambda F\phi \tag{5.4a}$$

And the corresponding adjoint problem:

$$M^*\phi^* = \lambda F^*\phi^* \tag{5.4b}$$

Where:

$$M^*\phi^* = -\vec{\boldsymbol{\nabla}} \cdot D_0(\vec{\boldsymbol{x}},E)\vec{\boldsymbol{\nabla}}\phi^*(\vec{\boldsymbol{x}},E) + \Sigma_t(\vec{\boldsymbol{x}},E)\phi^*(\vec{\boldsymbol{x}},E) - \int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E\to E')\phi(\vec{\boldsymbol{x}},E')\,dE' \tag{5.5a}$$

$$F^*\phi^* = \frac{\nu(E)\Sigma_f(\vec{\boldsymbol{x}},E)}{4\pi}\int_0^\infty \chi(E')\phi(\vec{\boldsymbol{x}},E')\,dE' \tag{5.5b}$$

5.3 First-order Perturbation Theory

System of equations:

- Base (unperturbed) system (Real system and adjoint system)

$$M_0\phi_0 = \lambda_0F_0\phi_0$$
$$M_0^*\phi_0^* = \lambda_0F_0^*\phi_0^* \tag{5.6a}$$

- Perturbed system

$$\begin{aligned} M &= M_0 + \delta M \\ F &= F_0 + \delta F \\ \phi &= \phi_0 + \delta\phi \\ \lambda &= \lambda_0 + \delta\lambda \\ M\phi &= \lambda F \lambda \end{aligned} \tag{5.6b}$$

Derivation:
From the perturbed system,

$$\begin{gathered} M\phi = \lambda F\phi \\ (M_0 + \delta M)(\phi_0 + \delta\phi) = (\lambda_0 + \delta\lambda)(F_0 + \delta F)(\phi_0 + \delta\phi) \\ M_0\phi_0 + M_0\delta\phi + \delta M\phi_0 + \delta M\delta\phi = (\lambda_0 + \delta\lambda)(F_0\phi_0 + F_0\delta\phi + \delta F\phi_0 + \delta F\delta\phi) \\ M_0\phi_0 + M_0\delta\phi + \delta M\phi_0 = (\lambda_0 + \delta\lambda)(F_0\phi_0 + F_0\delta\phi + \delta F\phi_0) \\ M_0\phi_0 + M_0\delta\phi + \delta M\phi_0 \\ = \lambda_0 F_0\phi_0 + \lambda_0 F_0\delta\phi + \lambda_0\, \delta F\phi_0 + \delta\lambda F_0\phi_0 + \delta\lambda F_0\delta\phi + \delta\lambda\, \delta F\phi_0 \\ M_0\phi_0 + M_0\delta\phi + \delta M\phi_0 = \lambda_0 F_0\phi_0 + \lambda_0 F_0\delta\phi + \lambda_0\, \delta F\phi_0 + \delta\lambda F_0\phi_0 \end{gathered} \tag{5.7}$$

Removing the second order effect:

$$\begin{aligned} (M_0\phi_0 - \lambda_0 F_0\phi_0) + M_0\delta\phi + \delta M\phi_0 &= \lambda_0 F_0\delta\phi + \lambda_0\, \delta F\phi_0 + \delta\lambda F_0\phi_0 \\ 0 + M_0\delta\phi + \delta M\phi_0 &= \lambda_0 F_0\delta\phi + \lambda_0\, \delta F\phi_0 + \delta\lambda F_0\phi_0 \end{aligned} \tag{5.8}$$

Therefore,

$$\begin{gathered} M_0\delta\phi + \delta M\phi_0 = \lambda_0 F_0\delta\phi + \lambda_0\, \delta F\phi_0 + \delta\lambda F_0\phi_0 \\ \delta\lambda F_0\phi_0 = (M_0 - \lambda_0 F_0)\delta\phi + (\delta M - \lambda_0\, \delta F)\phi_0 \end{gathered} \tag{5.9}$$

Multiply by weighting function $\int_V \int_E \phi^w \, dEdV$ and write using scalar product notation:

$$\begin{aligned} \delta\lambda(\phi^w, F_0\phi_0) &= (\phi^w, (M_0 - \lambda_0 F_0)\delta\phi) + (\phi^w, (\delta M - \lambda_0\, \delta F)\phi_0) \\ \delta\lambda &= \frac{(\phi^w, (M_0 - \lambda_0 F_0)\delta\phi) + (\phi^w, (\delta M - \lambda_0\, \delta F)\phi_0)}{(\phi^w, F_0\phi_0)} \\ &= \frac{(\phi^w, (M_0 - \lambda_0 F_0)\delta\phi)}{(\phi^w, F_0\phi_0)} + \frac{(\phi^w, (\delta M - \lambda_0\, \delta F)\phi_0)}{(\phi^w, F_0\phi_0)} \end{aligned} \tag{5.10}$$

If we choose the unperturbed adjoint flux as the weighting function:

$$\delta\lambda = \frac{(\phi_0^*, (M_0 - \lambda_0 F_0)\delta\phi)}{(\phi_0^*, F_0\phi_0)} + \frac{(\phi_0^*, (\delta M - \lambda_0\, \delta F)\phi_0)}{(\phi_0^*, F_0\phi_0)} \tag{5.11}$$

Note that:

$$\begin{aligned} (\phi_0^*, (M_0 - \lambda_0 F_0)\delta\phi) &= (\delta\phi, (M_0^* - \lambda_0 F_0^*)\phi_0^*) \\ &= (\delta\phi, (0)\phi_0^*) \\ &= 0 \end{aligned} \tag{5.12}$$

Therefore,

$$\begin{aligned} \delta\lambda &= 0 + \frac{(\phi_0^*, (\delta M - \lambda_0\, \delta F)\phi_0)}{(\phi_0^*, F_0\phi_0)} \\ &= \frac{(\phi_0^*, (\delta M - \lambda_0\, \delta F)\phi_0)}{(\phi_0^*, F_0\phi_0)} \\ &= \frac{\int_V \int_E \phi_0^*(\vec{\boldsymbol{x}}, E)(\delta M - \lambda_0\, \delta F)\phi_0(\vec{\boldsymbol{x}}, E) dEdV}{\int_V \int_E \phi_0^*(\vec{\boldsymbol{x}}, E) F_0\phi_0(\vec{\boldsymbol{x}}, E) dEdV} \end{aligned} \tag{5.13}$$

5.4 Exact Perturbation Theory

The exact eigenvalue difference can be computed in a form similar to that of the first-order perturbation theory. However, the solution requires the knowledge of the perturbed problem.
System of equations:

- Perturbed real system

$$M\phi = \lambda F\phi \tag{5.14a}$$

- Unperturbed adjoint system

$$M_0^*\phi_0^* = \lambda_0 F_0^*\phi_0^* \tag{5.14b}$$

Doing some manipulations to the system of equations:

- Multiply (5.14a) with ϕ_0^* (unperturbed adjoint flux) and integrate over E and V. Write in terms of scalar products.

$$(\phi_0^*, M\phi) = \lambda(\phi_0^*, F\phi) \tag{5.15a}$$

- Multiply (5.14b) with ϕ (perturbed real flux) and integrate over E and V. Write in terms of scalar products.

$$(\phi, M_0^*\phi_0^*) = \lambda_0(\phi, F_0^*\phi_0^*) \tag{5.15b}$$

- Revolve the operator of equation (5.15b)

$$(\phi_0^*, M_0\phi) = \lambda_0(\phi_0^*, F_0\phi) \tag{5.15c}$$

- Expand operator F_0 in (5.15c)

$$\begin{aligned}(\phi_0^*, M_0\phi) &= \lambda_0(\phi_0^*, (F - \delta F)\phi) \\ (\phi_0^*, M_0\phi) &= \lambda_0(\phi_0^*, F\phi) - \lambda_0(\phi_0^*, \delta F\phi)\end{aligned} \tag{5.15d}$$

- Substract (5.15a) with (5.15d)

$$\begin{aligned}(\phi_0^*, M\phi) - (\phi_0^*, M_0\phi) &= \lambda(\phi_0^*, F\phi) - \{\lambda_0(\phi_0^*, F\phi) - \lambda_0(\phi_0^*, \delta F\phi)\} \\ (\phi_0^*, \delta M\phi) &= \delta\lambda(\phi_0^*, F\phi) + \lambda_0(\phi_0^*, \delta F\phi)\end{aligned} \tag{5.15e}$$

- Manipulate to obtain $\delta\lambda$

$$\begin{aligned}\delta\lambda(\phi_0^*, F\phi) &= \frac{(\phi_0^*, \delta M\phi) - \lambda_0(\phi_0^*, \delta F\phi)}{(\phi_0^*, F\phi)} \\ &= \frac{(\phi_0^*, (\delta M - \lambda_0\delta F)\phi)}{(\phi_0^*, F\phi)} \\ &= \frac{\int_V \int_E \phi_0^*(\vec{x}, E)(\delta M - \lambda_0\, \delta F)\phi(\vec{x}, E)dEdV}{\int_V \int_E \phi_0^*(\vec{x}, E)F\phi(\vec{x}, E)dEdV}\end{aligned} \tag{5.15e}$$

5.5 Exact Point Kinetics

Exact point kinetics for an initially critical system:

$$\begin{aligned}\frac{1}{v}\frac{\partial\phi}{\partial t} &= (F_p - M)\phi + S_d \\ S_d &= \sum_{k=1}^{6} \chi_{dk}\lambda_k C_k\end{aligned} \tag{5.16}$$

Initially critical: $t \leq 0$

- Real (fundamental mode)

$$\begin{aligned}0 &= (F_{p0} - M_0)\phi_0 + S_{d0} \\ 0 &= (F_{p0} - M_0)\phi_0 + F_{d0}\phi_0 \\ 0 &= (F_{p0} + F_{d0})\phi_0 - M_0\phi_0 \\ 0 &= (F_0 - M_0)\phi_0\end{aligned} \tag{5.17a}$$

- Adjoint (fundamental mode)

$$0 = (F_0^* - M_0^*)\phi_0^* \tag{5.17b}$$

Multiply by the real equation by adjoint flux ϕ_0^* and integrate over space and energy:

$$\langle\frac{1}{v}\phi_0^*\frac{\partial\phi}{\partial t}\rangle_{rE} = \langle\phi_0^*,(F-M)\phi\rangle_{rE} - \langle\phi_0^*,F_d\phi\rangle_{rE} + \langle\phi_0^* S_d\rangle_{rE} \tag{5.18}$$

Add/substract $F_d\phi$.
Introduced factorized flux

$$\phi(\vec{\boldsymbol{x}},E,t) = \psi(\vec{\boldsymbol{x}},E,t)p(t) \tag{5.19}$$

Therefore, the left-hand side of (5.18):

$$\begin{aligned}\langle\frac{1}{v}\phi_0^*\frac{\partial\big(\psi(\vec{\boldsymbol{x}},E,t)p(t)\big)}{\partial t}\rangle_{rE} &= \langle\frac{1}{v}\phi_0^*\psi(\vec{\boldsymbol{x}},E,t)\rangle_{rE}\frac{\partial p(t)}{\partial t} + \frac{\partial}{\partial t}\left[\langle\frac{1}{v}\phi_0^*\psi(\vec{\boldsymbol{x}},E,t)\rangle_{rE}\right]p(t) \\ &= \langle\frac{1}{v}\phi_0^*\psi(\vec{\boldsymbol{x}},E,t)\rangle_{rE}\frac{\partial p(t)}{\partial t} + \frac{\partial K_0}{\partial t}p(t)\end{aligned} \tag{5.20}$$

Choose K_0 to be constant in time (which means that we are renormalizing the flux with the power shape in every step of calculation:

$$\frac{\partial K_0}{\partial t} = 0 \tag{5.21}$$

> The second equation required to make the factorization unique can be used to shift the major time dependence into the amplitude function by constraining the time variation of the "magnitude" of the shape function. A convenient way to do this is to hold some integral (over space and energy) of the shape function constant in time. The derivation leads to a specific integral of ψ, which is held constant in time in order to obtain the desired form of the point kinetics equations.

Therefore, the equation changes into:

$$\begin{aligned}\langle\frac{1}{v}\phi_0^*\psi(\vec{\boldsymbol{x}},E,t)\rangle_{rE}\frac{\partial p(t)}{\partial t} &= \langle\phi_0^*,(F-M)\phi\rangle_{rE} - \langle\phi_0^*,F_d\phi\rangle_{rE} + \langle\phi_0^* S_d\rangle_{rE} \\ &= [\langle\phi_0^*,(F-M)\psi\rangle_{rE} - \langle\phi_0^*,F_d\psi\rangle_{rE}]p(t) + \langle\phi_0^* S_d\rangle_{rE}\end{aligned} \tag{5.22}$$

Divide by weighted fission source:

$$F(t) = \langle\phi_0^*,F\psi\rangle_{rE} \tag{5.23}$$

Therefore,

$$\frac{\langle\frac{1}{v}\phi_0^*\psi(\vec{\boldsymbol{x}},E,t)\rangle_{rE}}{\langle\phi_0^*,F\psi\rangle_{rE}}\frac{\partial p(t)}{\partial t} = \left[\frac{\langle\phi_0^*,(F-M)\psi\rangle_{rE}}{\langle\phi_0^*,F\psi\rangle_{rE}} - \frac{\langle\phi_0^*,F_d\psi\rangle_{rE}}{\langle\phi_0^*,F\psi\rangle_{rE}}\right]p(t) + \frac{\langle\phi_0^* S_d\rangle_{rE}}{\langle\phi_0^*,F\psi\rangle_{rE}} \tag{5.24}$$

Then we can define

$$\frac{\langle\frac{1}{v}\phi_0^*\psi(\vec{\boldsymbol{x}},E,t)\rangle_{rE}}{\langle\phi_0^*,F\psi\rangle_{rE}} = \Lambda(t) \qquad \frac{\langle\phi_0^*,(F-M)\psi\rangle_{rE}}{\langle\phi_0^*,F\psi\rangle_{rE}} = \rho(t) \qquad \frac{\langle\phi_0^*,F_d\psi\rangle_{rE}}{\langle\phi_0^*,F\psi\rangle_{rE}} = \beta(t) \qquad \frac{\langle\phi_0^* S_d\rangle_{rE}}{\langle\phi_0^*,F\psi\rangle_{rE}} = S_d(t)$$

Therefore,

$$\Lambda(t)\frac{\partial p(t)}{\partial t} = [\rho(t)-\beta(t)]p(t) + S_d(t) \tag{5.25}$$

$$\frac{\partial p(t)}{\partial t} = \frac{[\rho(t)-\beta(t)]}{\Lambda(t)}p(t) + \frac{S_d(t)}{\Lambda(t)} \tag{5.26}$$

From here, we can simplify the equation by manipulating the delayed neutron source $S_d(t)$:

$$S_d(t) = \frac{\langle \phi_0^* S_d \rangle_{rE}}{\langle \phi_0^*, F\psi \rangle_{rE}} \tag{5.27}$$

$$= \frac{\langle \phi_0^*, F_0\psi_0 \rangle_{rE}}{\langle \phi_0^*, F\psi \rangle_{rE}} \sum_k \lambda_k \frac{\langle \phi_0^*, \gamma_{dk} C_k \rangle_{rE}}{\langle \phi_0^*, F_0\psi_0 \rangle_{rE}}$$

We define:

$$\xi_k(t) = \frac{\langle \phi_0^*, \gamma_{dk} C_k \rangle_{rE}}{\langle \phi_0^*, F_0\psi_0 \rangle_{rE}} \tag{5.28}$$

Thus,

$$S_d(t) = \frac{\langle \phi_0^*, F_0\psi_0 \rangle_{rE}}{\langle \phi_0^*, F\psi \rangle_{rE}} \sum_k \lambda_k \, \xi_k(t) \tag{5.29}$$

$$\frac{\partial p(t)}{\partial t} = \frac{[\rho(t) - \beta(t)]}{\Lambda(t)} p(t) + \frac{\langle \phi_0^*, F\psi \rangle_{rE}}{\langle \frac{1}{v} \phi_0^* \psi \rangle_{rE}} \frac{\langle \phi_0^*, F_0\psi_0 \rangle_{rE}}{\langle \phi_0^*, F\psi \rangle_{rE}} \sum_k \lambda_k \, \xi_k(t) \tag{5.30}$$

$$= \frac{[\rho(t) - \beta(t)]}{\Lambda(t)} p(t) + \frac{\langle \phi_0^*, F_0\psi_0 \rangle_{rE}}{\langle \frac{1}{v} \phi_0^* \psi \rangle_{rE}} \sum_k \lambda_k \, \xi_k(t)$$

We define the prompt neutron lifetime with initial flux:

$$\Lambda_0(t) = \frac{\langle \frac{1}{v} \phi_0^* \psi \rangle_{rE}}{\langle \phi_0^*, F_0\psi_0 \rangle_{rE}} \tag{5.31}$$

Thus,

$$\frac{\partial p(t)}{\partial t} = \frac{[\rho(t) - \beta(t)]}{\Lambda(t)} p(t) + \frac{1}{\Lambda_0(t)} \sum_k \lambda_k \, \xi_k(t) \tag{5.32}$$

For the Delayed Neutron Precursor Equation:

$$\frac{\partial C_k}{\partial t} = -\lambda_k C_k(\vec{\boldsymbol{x}}, t) + \int_0^\infty \nu_{dk} \Sigma_f(\vec{\boldsymbol{x}}, E') \phi(\vec{\boldsymbol{x}}, E', t) \, dE' \tag{5.33}$$
$$k = 1, \dots, 6$$

Multiply by $\phi_0^*(\vec{\boldsymbol{x}}, E)$ and $\chi_{dk}(E)$,

$$\frac{\partial}{\partial t} \chi_{dk} \phi_0^* C_k = -\lambda_k \chi_{dk} \phi_0^* C_k(\vec{\boldsymbol{x}}, t) + \chi_{dk}(E) \phi_0^*(\vec{\boldsymbol{x}}, E) \int_0^\infty \nu_{dk} \Sigma_f(\vec{\boldsymbol{x}}, E') \phi(\vec{\boldsymbol{x}}, E', t) \, dE' \tag{5.34}$$
$$k = 1, \dots, 6$$

Factorize $\phi(\vec{\boldsymbol{x}}, E, t) = \psi(\vec{\boldsymbol{x}}, E, t) p(t)$

$$\begin{aligned} &\frac{\partial}{\partial t} \chi_{dk}(E) \phi_0^*(\vec{\boldsymbol{x}}, E) C_k(\vec{\boldsymbol{x}}, t) \\ &\quad = -\lambda_k \chi_{dk}(E) \phi_0^*(\vec{\boldsymbol{x}}, E) C_k(\vec{\boldsymbol{x}}, t) \\ &\quad + p(t) \chi_{dk}(E) \phi_0^*(\vec{\boldsymbol{x}}, E) \int_0^\infty \nu_{dk} \Sigma_f(\vec{\boldsymbol{x}}, E') \psi(\vec{\boldsymbol{x}}, E', t) \, dE' \end{aligned} \tag{5.35}$$
$$k = 1, \dots, 6$$

and integrate over $dVdE$,

$$\frac{\partial}{\partial t} \langle \chi_{dk}(E) \phi_0^*(\vec{\boldsymbol{x}}, E) C_k(\vec{\boldsymbol{x}}, t) \rangle = -\lambda_k \langle \chi_{dk}(E) \phi_0^*(\vec{\boldsymbol{x}}, E) C_k(\vec{\boldsymbol{x}}, t) \rangle + p(t) \langle \phi_0^*(\vec{\boldsymbol{x}}, E) F_{dk} \psi(\vec{\boldsymbol{x}}, E', t) \rangle \tag{5.36}$$
$$k = 1, \dots, 6$$

Divide by initial adjoint fission source $\langle \phi_0^* F_0 \psi_0 \rangle$

$$\frac{\partial}{\partial t}\left(\frac{\langle \chi_{dk}(E)\phi_0^*(\vec{\boldsymbol{x}},E)C_k(\vec{\boldsymbol{x}},t)\rangle}{\langle \phi_0^*(\vec{\boldsymbol{x}},E)F_0\psi_0(\vec{\boldsymbol{x}},E,t)\rangle}\right) \tag{5.37}$$

$$= -\lambda_k \frac{\langle \chi_{dk}(E)\phi_0^*(\vec{\boldsymbol{x}},E)C_k(\vec{\boldsymbol{x}},t)\rangle}{\langle \phi_0^*(\vec{\boldsymbol{x}},E)F_0\psi_0(\vec{\boldsymbol{x}},E,t)\rangle}$$

$$+ p(t)\frac{\langle \phi_0^*(\vec{\boldsymbol{x}},E)F\psi(\vec{\boldsymbol{x}},E',t)\rangle}{\langle \phi_0^*(\vec{\boldsymbol{x}},E)F_0\psi_0(\vec{\boldsymbol{x}},E,t)\rangle}\frac{\langle \phi_0^*(\vec{\boldsymbol{x}},E)F_{dk}\psi(\vec{\boldsymbol{x}},E',t)\rangle}{\langle \phi_0^*(\vec{\boldsymbol{x}},E)F\psi(\vec{\boldsymbol{x}},E,t)\rangle}$$

$$k = 1, \dots, 6$$

$$\frac{\partial}{\partial t}\big(\xi(t)\big) = -\lambda_k\xi(t) + \frac{\langle \phi_0^*(\vec{\boldsymbol{x}},E)F\psi(\vec{\boldsymbol{x}},E',t)\rangle}{\langle \phi_0^*(\vec{\boldsymbol{x}},E)F_0\psi_0(\vec{\boldsymbol{x}},E,t)\rangle}\beta_k p(t) \tag{5.38}$$

$$k = 1, \dots, 6$$

6. Reactor Noise

6.1 One-group Reactor Noise

Start from neutron diffusion equation (no source, isotropic scattering).

$$\frac{1}{v(E)}\frac{\partial\phi(\vec{\boldsymbol{x}},E,t)}{\partial t} - \vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}},E,t)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},E,t) + \Sigma_t(\vec{\boldsymbol{x}},E,t)\phi(\vec{\boldsymbol{x}},E,t)$$
$$= \int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E' \to E,t)\phi(\vec{\boldsymbol{x}},E',t)\,dE'$$
$$+ \chi_p(E)\int_0^\infty (1-\beta)\nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}',t)\phi(\vec{\boldsymbol{x}},E',t)\,dE' + \chi_{dk}(E)\sum_{i=1}^{6}\lambda_i C_i(\vec{\boldsymbol{x}},t) \tag{6.1}$$

And the precursor equation:

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\vec{\boldsymbol{x}},t) + \int_0^\infty \beta_i \nu(E')\Sigma_f(\vec{\boldsymbol{x}},E')\phi(\vec{\boldsymbol{x}},E',t)\,dE' \qquad i = 1,\ldots,6 \tag{6.2}$$

With boundary condition:

$$\phi_0(r_B) = 0 \tag{6.3}$$

Integrate both equations over E to make it one energy group:

- Time rate of change

$$\int_0^\infty \frac{1}{v(E)}\frac{\partial\phi(\vec{\boldsymbol{x}},E,t)}{\partial t}dE = \frac{1}{v}\frac{\partial\phi}{\partial t} \tag{6.4a}$$

- Net leakage rate

$$\int_0^\infty \vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}},E,t)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},E,t)dE = \vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},t) \tag{6.4b}$$

- Total reaction rate

$$\int_0^\infty \Sigma_t(\vec{\boldsymbol{x}},E,t)\phi(\vec{\boldsymbol{x}},E,t)dE = \Sigma_t(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) = \Sigma_a(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) + \Sigma_{s0}(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) \tag{6.4c}$$

- In-scattering rate

$$\int_0^\infty\left[\int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E'\to E,t)\phi(\vec{\boldsymbol{x}},E',t)\,dE'\right]dE$$
$$= \int_0^\infty\left[\int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E',t)\delta(E'-E)\phi(\vec{\boldsymbol{x}},E',t)\,dE'\right]dE$$
$$= \int_0^\infty \Sigma_{s0}(\vec{\boldsymbol{x}},E',t)\phi(\vec{\boldsymbol{x}},E',t)\,dE'$$
$$= \Sigma_{s0}(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) \tag{6.4d}$$

- Prompt Fission rate

$$\int_0^\infty\left[\chi_p(E)\int_0^\infty (1-\beta)\nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}',t)\phi(\vec{\boldsymbol{x}},E',t)\,dE'\right]dE$$
$$= \int_0^\infty (1-\beta)\nu(E')\Sigma_f(\vec{\boldsymbol{x}},\mathrm{E}',t)\phi(\vec{\boldsymbol{x}},E',t)\,dE' = (1-\beta)\nu\Sigma_f(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}},t) \tag{6.4e}$$

- Delayed Fission rate

$$\int_0^\infty \left[\chi_{dk}(E)\sum_{i=1}^{6}\lambda_i C_i\,(\vec{\boldsymbol{x}},t)\right]dE = \sum_{i=1}^{6}\lambda_i C_i\,(\vec{\boldsymbol{x}},t) \tag{6.4f}$$

- Precursor Equation
  - Left-hand side

$$\int_0^\infty \left[\frac{\partial C_i}{\partial t}\right]dE = \frac{\partial C_i}{\partial t} \tag{6.4g}$$

  - First term right-hand side

$$\int_0^\infty [\lambda_i C_i(\vec{\boldsymbol{x}},t)]dE = \lambda_i C_i(\vec{\boldsymbol{x}},t) \tag{6.4h}$$

  - Second term right-hand side

$$\int_0^\infty \left[\int_0^\infty \beta_i \nu(E')\Sigma_f(\vec{\boldsymbol{x}},E',t)\phi(\vec{\boldsymbol{x}},E',t)\,dE'\right]dE = \beta_i\nu\Sigma_f(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) \tag{6.4i}$$

Therefore, the one-group neutron diffusion equation:

$$\begin{aligned}\frac{1}{v}\frac{\partial\phi}{\partial t} - \vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},t) + \Sigma_a(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) + \Sigma_{s0}(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) \\ = \Sigma_{s0}(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) + (1-\beta)\nu\Sigma_f(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) + \sum_{i=1}^{6}\lambda_i C_i\,(\vec{\boldsymbol{x}},t)\end{aligned} \tag{6.5a}$$

$$\frac{1}{v}\frac{\partial\phi}{\partial t} - \vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},t) + \Sigma_a(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) = (1-\beta)\nu\Sigma_f(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) + \sum_{i=1}^{6}\lambda_i C_i\,(\vec{\boldsymbol{x}},t) \tag{6.5b}$$

And the precursor equation:

$$\begin{gathered}\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\vec{\boldsymbol{x}},t) + \beta_i\nu\Sigma_f(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}},t) \\ i = 1,\dots,6\end{gathered} \tag{6.6}$$

Then, we have some assumptions:

- Temporal fluctuation in diffusion coefficient is neglected.

$$\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},t) \tag{6.7}$$

- Cross-product of fluctuating terms is neglected.
- System is assumed to be critical.
- One-group precursor is used.

Applying the first and fourth assumption:

$$\frac{1}{v}\frac{\partial\phi}{\partial t} - \vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},t) + \Sigma_a(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) = (1-\beta)\nu\Sigma_f(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t) + \lambda C(\vec{\boldsymbol{x}},t) \tag{6.8}$$

And the precursor equation:

$$\frac{\partial C}{\partial t} = -\lambda C(\vec{\boldsymbol{x}},t) + \beta\nu\Sigma_f(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}},t) \tag{6.9}$$

Then, we apply small (relative to its mean value) perturbations to all the fluxes and cross section in the neutron diffusion equation, such that:

$$X(\vec{\boldsymbol{x}},t) = X_0(\vec{\boldsymbol{x}}) + \delta X(\vec{\boldsymbol{x}},t) \tag{6.10}$$

Therefore,
Integrate both equations over E to make it one energy group:

- Time rate of change

$$\frac{1}{v}\frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial t}=\frac{1}{v}\frac{\partial\phi_0(\vec{\boldsymbol{x}})}{\partial t}+\frac{1}{v}\frac{\partial\big(\delta\phi(\vec{\boldsymbol{x}},t)\big)}{\partial t}=\frac{1}{v}\frac{\partial\big(\delta\phi(\vec{\boldsymbol{x}},t)\big)}{\partial t} \tag{6.11a}$$

- Net leakage rate

$$\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\phi_0(\vec{\boldsymbol{x}})+\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\big(\delta\phi(\vec{\boldsymbol{x}},t)\big) \tag{6.11b}$$

- Absorption reaction rate

$$\begin{aligned}&\Sigma_a(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t)\\&=[\Sigma_{a0}(\vec{\boldsymbol{x}})+\delta\Sigma_a(\vec{\boldsymbol{x}},t)][\phi_0(\vec{\boldsymbol{x}})+\delta\phi(\vec{\boldsymbol{x}},t)]\\&=\Sigma_{a0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})+\Sigma_{a0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)+\delta\Sigma_a(\vec{\boldsymbol{x}},t)\phi_0(\vec{\boldsymbol{x}})+\delta\Sigma_a(\vec{\boldsymbol{x}},t)\delta\phi(\vec{\boldsymbol{x}},t)\\&=\Sigma_{a0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})+\Sigma_{a0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)+\delta\Sigma_a(\vec{\boldsymbol{x}},t)\phi_0(\vec{\boldsymbol{x}})\end{aligned} \tag{6.11c}$$

- Fission rate

$$\begin{aligned}&(1-\beta)\nu\Sigma_f(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{x}},t)\\&=(1-\beta)\big[\nu\Sigma_{f0}(\vec{\boldsymbol{x}})+\delta\nu\Sigma_f(\vec{\boldsymbol{x}},t)\big][\phi_0(\vec{\boldsymbol{x}})+\delta\phi(\vec{\boldsymbol{x}},t)]\\&=(1-\beta)\big[\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})+\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)+\delta\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})+\delta\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)\big]\\&=(1-\beta)\big[\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})+\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)+\delta\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})\big]\end{aligned} \tag{6.11d}$$

- Delayed Fission rate

$$\lambda C(\vec{\boldsymbol{x}},t)=\lambda C_0(\vec{\boldsymbol{x}})+\lambda\delta C(\vec{\boldsymbol{x}},t) \tag{6.11e}$$

- Precursor Equation
    - Left-hand side

$$\frac{\partial C}{\partial t}=\frac{\partial C_0(\vec{\boldsymbol{x}})}{\partial t}+\frac{\partial\big(\delta C(\vec{\boldsymbol{x}},t)\big)}{\partial t}=\frac{\partial\big(\delta C(\vec{\boldsymbol{x}},t)\big)}{\partial t} \tag{6.11f}$$

    - First term right-hand side

$$\lambda C(\vec{\boldsymbol{x}},t)=\lambda C_0(\vec{\boldsymbol{x}})+\lambda\delta C(\vec{\boldsymbol{x}},t) \tag{6.11g}$$

    - Second term right-hand side

$$\beta\nu\Sigma_f(\vec{\boldsymbol{x}})\phi(\vec{\boldsymbol{x}},t)=\beta\big[\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})+\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)+\delta\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})\big] \tag{6.11h}$$

Combining them all together:

$$\begin{aligned}\frac{1}{v}\frac{\partial\big(\delta\phi(\vec{\boldsymbol{x}},t)\big)}{\partial t}&-\big\{\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\phi_0(\vec{\boldsymbol{x}})+\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\big(\delta\phi(\vec{\boldsymbol{x}},t)\big)\big\}\\&+\{\Sigma_{a0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})+\Sigma_{a0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)+\delta\Sigma_a(\vec{\boldsymbol{x}},t)\phi_0(\vec{\boldsymbol{x}})\}\\&=\big\{(1-\beta)\big[\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})+\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)+\delta\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})\big]\big\}\\&+\{\lambda C_0(\vec{\boldsymbol{x}})+\lambda\delta C(\vec{\boldsymbol{x}},t)\}\end{aligned} \tag{6.12}$$

And the precursor equation:

$$\begin{aligned}\frac{\partial\big(\delta C(\vec{\boldsymbol{x}},t)\big)}{\partial t}&=-\lambda\{C_0(\vec{\boldsymbol{x}})+\delta C(\vec{\boldsymbol{x}},t)\}\\&+\beta\big\{\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})+\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)+\delta\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})\big\}\end{aligned} \tag{6.13}$$

Subtracting equation (6.12) with the time-independent one-group neutron diffusion at critical condition ($k_{eff}=1$):

$$\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\phi_0(\vec{\boldsymbol{x}})+\left[\frac{\nu\Sigma_{f0}(\vec{\boldsymbol{x}})}{k_{eff}}-\Sigma_{a0}(\vec{\boldsymbol{x}})\right]\phi_0(\vec{\boldsymbol{x}})=0 \tag{6.14a}$$

$$\vec{\nabla}\cdot D_0(\vec{x})\vec{\nabla}\phi_0(\vec{x}) + \left[\nu\Sigma_{f0}(\vec{x}) - \Sigma_{a0}(\vec{x})\right]\phi_0(\vec{x}) = 0 \tag{6.14b}$$

This yields to:

$$\begin{aligned}\frac{1}{v}\frac{\partial\left(\delta\phi(\vec{x},t)\right)}{\partial t} &- \left\{\vec{\nabla}\cdot D_0(\vec{x})\vec{\nabla}\left(\delta\phi(\vec{x},t)\right)\right\} + \{\Sigma_{a0}(\vec{x})\delta\phi(\vec{x},t) + \delta\Sigma_a(\vec{x},t)\phi_0(\vec{x})\} \\ &= \left\{\nu\Sigma_{f0}(\vec{x})\delta\phi(\vec{x},t) + \delta\nu\Sigma_f(\vec{x},t)\phi_0(\vec{x})\right\} \\ &- \ \beta\left\{\nu\Sigma_{f0}(\vec{x})\phi_0(\vec{x}) + \nu\Sigma_{f0}(\vec{x})\delta\phi(\vec{x},t) + \delta\nu\Sigma_f(\vec{x},t)\phi_0(\vec{x})\right\} \\ &+ \{\lambda C_0(\vec{x}) + \lambda\delta C(\vec{x},t)\}\end{aligned} \tag{6.15}$$

And the precursor equation:

$$\begin{aligned}\frac{\partial\left(\delta C(\vec{x},t)\right)}{\partial t} &= -\lambda\{C_0(\vec{x}) + \delta C(\vec{x},t)\} \\ &\quad + \beta\left\{\nu\Sigma_{f0}(\vec{x})\phi_0(\vec{x}) + \nu\Sigma_{f0}(\vec{x})\delta\phi(\vec{x},t) + \delta\nu\Sigma_f(\vec{x},t)\phi_0(\vec{x})\right\}\end{aligned} \tag{6.16}$$

Then, we perform Fourier Transform by multiplying all terms with $\int_{-\infty}^{\infty}\exp(-i\omega t)\,dt$

- Time rate of change

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1}{v}\frac{\partial\left(\delta\phi(\vec{x},t)\right)}{\partial t}\exp(-i\omega t)\,dt \\ &= \frac{\exp(-i\omega t)}{v}\delta\phi(\vec{x},\omega)\bigg|_{-\infty}^{\infty} - \frac{(-i\omega)}{v}\int_{-\infty}^{\infty}\exp(-i\omega t)\,\delta\phi(\vec{x},t)\,dt \\ &= \frac{i\omega}{v}\delta\phi(\vec{x},\omega)\end{aligned} \tag{6.17a}$$

- Net leakage rate

$$\int_{-\infty}^{\infty}\left\{\vec{\nabla}\cdot D_0(\vec{x})\vec{\nabla}\left(\delta\phi(\vec{x},t)\right)\right\}\exp(-i\omega t)\,dt = \vec{\nabla}\cdot D_0(\vec{x})\vec{\nabla}\left(\delta\phi(\vec{x},\omega)\right) \tag{6.17b}$$

- Absorption reaction rate

$$\begin{aligned}&\int_{-\infty}^{\infty}\{\Sigma_{a0}(\vec{x})\delta\phi(\vec{x},t) + \delta\Sigma_a(\vec{x},t)\phi_0(\vec{x})\}\exp(-i\omega t)\,dt \\ &= \int_{-\infty}^{\infty}\Sigma_{a0}(\vec{x})\delta\phi(\vec{x},t)\exp(-i\omega t)\,dt + \int_{-\infty}^{\infty}\delta\Sigma_a(\vec{x},t)\phi_0(\vec{x})\exp(-i\omega t)\,dt \\ &= \Sigma_{a0}(\vec{x})\delta\phi(\vec{x},\omega) + \delta\Sigma_a(\vec{x},\omega)\phi_0(\vec{x})\end{aligned} \tag{6.17c}$$

- Fission rate
  - Part 1

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\nu\Sigma_{f0}(\vec{x})\delta\phi(\vec{x},t) + \delta\nu\Sigma_f(\vec{x},t)\phi_0(\vec{x})\right\}\exp(-i\omega t)\,dt \\ &= \Sigma_{f0}(\vec{x})\delta\phi(\vec{x},\omega) + \delta\nu\Sigma_f(\vec{x},\omega)\phi_0(\vec{x})\end{aligned} \tag{6.17d}$$

  - Part 2

$$\begin{aligned}&\int_{-\infty}^{\infty}\beta\left\{\nu\Sigma_{f0}(\vec{x})\phi_0(\vec{x}) + \nu\Sigma_{f0}(\vec{x})\delta\phi(\vec{x},t) + \delta\nu\Sigma_f(\vec{x},t)\phi_0(\vec{x})\right\}\exp(-i\omega t)\,dt \\ &= \beta\left[0 + \int_{-\infty}^{\infty}\nu\Sigma_{f0}(\vec{x})\delta\phi(\vec{x},t)\exp(-i\omega t)\,dt + \int_{-\infty}^{\infty}\delta\nu\Sigma_f(\vec{x},t)\phi_0(\vec{x})\exp(-i\omega t)\,dt\right] \\ &= \beta\left[\nu\Sigma_{f0}(\vec{x})\delta\phi(\vec{x},\omega) + \delta\nu\Sigma_f(\vec{x},\omega)\phi_0(\vec{x})\right]\end{aligned} \tag{6.17e}$$

- Delayed Fission rate

$$\int_{-\infty}^{\infty}\{\lambda C_0(\vec{\boldsymbol{x}}) + \lambda\delta C(\vec{\boldsymbol{x}},t)\}\exp(-i\omega t)\,dt = 0 + \int_{-\infty}^{\infty}\lambda\delta C(\vec{\boldsymbol{x}},t)\exp(-i\omega t)\,dt = \lambda\delta C(\vec{\boldsymbol{x}},\omega) \quad (6.17f)$$

- Precursor Equation
  - Left-hand side

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\partial\big(\delta C(\vec{\boldsymbol{x}},t)\big)}{\partial t}\exp(-i\omega t)\,dt \\ &= \frac{\exp(-i\omega t)}{v}\delta C(\vec{\boldsymbol{x}},\omega)\bigg|_{-\infty}^{\infty} - (-i\omega)\int_{-\infty}^{\infty}\exp(-i\omega t)\,\delta C(\vec{\boldsymbol{x}},t)dt \\ &= i\omega\;\delta C(\vec{\boldsymbol{x}},\omega)\end{aligned} \quad (6.17g)$$

  - First term right-hand side

$$\begin{aligned}&\int_{-\infty}^{\infty}\lambda\{C_0(\vec{\boldsymbol{x}}) + \delta C(\vec{\boldsymbol{x}},t)\}\exp(-i\omega t)\,dt \\ &= \lambda\left[\int_{-\infty}^{\infty}C_0(\vec{\boldsymbol{x}})\exp(-i\omega t)\,dt + \int_{-\infty}^{\infty}\delta C(\vec{\boldsymbol{x}},t)\exp(-i\omega t)\,dt\right] \\ &= \lambda[0 + \delta C(\vec{\boldsymbol{x}},\omega)] \\ &= \lambda\delta C(\vec{\boldsymbol{x}},\omega)\end{aligned} \quad (6.17h)$$

  - Second term right-hand side

$$\begin{aligned}&\int_{-\infty}^{\infty}\beta\big\{\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}}) + \nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t) + \delta\nu\Sigma_f(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})\big\}\exp(-i\omega t)\,dt \\ &= \beta\left[\int_{-\infty}^{\infty}\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})\exp(-i\omega t)\,dt + \int_{-\infty}^{\infty}\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)\exp(-i\omega t)\,dt\right. \\ &\qquad\qquad \left. + \int_{-\infty}^{\infty}\delta\nu\Sigma_f(\vec{\boldsymbol{x}},t)\phi_0(\vec{\boldsymbol{x}})\exp(-i\omega t)\,dt\right] \\ &= \beta\left[0 + \int_{-\infty}^{\infty}\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},t)\exp(-i\omega t)\,dt + \int_{-\infty}^{\infty}\delta\nu\Sigma_f(\vec{\boldsymbol{x}},t)\phi_0(\vec{\boldsymbol{x}})\exp(-i\omega t)\,dt\right] \\ &= \beta\big[\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega) + \delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}})\big]\end{aligned} \quad (6.17i)$$

Combining them all together:

$$\begin{aligned}&\frac{i\omega}{v}\delta\phi(\vec{\boldsymbol{x}},\omega) - \vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\big(\delta\phi(\vec{\boldsymbol{x}},\omega)\big) + \Sigma_{a0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega) + \delta\Sigma_a(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}}) \\ &\qquad = (1-\beta)\big\{\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega) + \delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}})\big\} + \lambda\delta C(\vec{\boldsymbol{x}},\omega)\end{aligned} \quad (6.18)$$

And the precursor equation:

$$\begin{aligned}i\omega\;\delta C(\vec{\boldsymbol{x}},\omega) + \lambda\delta C(\vec{\boldsymbol{x}},\omega) &= \beta\big[\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega) + \delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}})\big] \\ \delta C(\vec{\boldsymbol{x}},\omega) &= \frac{\beta}{i\omega+\lambda}\big[\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega) + \delta\nu\Sigma_f(\vec{\boldsymbol{x}})\phi_0(\vec{\boldsymbol{x}})\big]\end{aligned} \quad (6.19)$$

Apply (6.19) to the right-hand side of (6.18)

$$\frac{i\omega}{v}\delta\phi(\vec{\boldsymbol{x}},\omega)-\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\big(\delta\phi(\vec{\boldsymbol{x}},\omega)\big)+\Sigma_{a0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega)+\delta\Sigma_a(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}}) \tag{6.20}$$

$$=(1-\beta)\{\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega)+\delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}})\}$$

$$+\frac{\lambda\beta}{i\omega+\lambda}\left[\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega)+\delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}})\right]$$

$$=\left(1-\beta+\frac{\lambda\beta}{i\omega+\lambda}\right)\{\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega)+\delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}})\}$$

$$=\left(1-\frac{i\omega\beta}{i\omega+\lambda}\right)\{\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega)+\delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}})\}$$

Therefore,

$$\frac{i\omega}{v}\delta\phi(\vec{\boldsymbol{x}},\omega)-\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\big(\delta\phi(\vec{\boldsymbol{x}},\omega)\big)+\Sigma_{a0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega)+\delta\Sigma_a(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}})$$

$$=\left(1-\frac{i\omega\beta}{i\omega+\lambda}\right)\{\nu\Sigma_{f0}(\vec{\boldsymbol{x}})\delta\phi(\vec{\boldsymbol{x}},\omega)+\delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\phi_0(\vec{\boldsymbol{x}})\}$$

$$\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\big(\delta\phi(\vec{\boldsymbol{x}},\omega)\big)+\left[\left\{1-\frac{i\omega\beta}{i\omega+\lambda}\right\}\nu\Sigma_{f0}(\vec{\boldsymbol{x}})-\Sigma_{a0}(\vec{\boldsymbol{x}})-\frac{i\omega}{v}\right]\delta\phi(\vec{\boldsymbol{x}},\omega) \tag{6.21}$$

$$=\left[\delta\Sigma_a(\vec{\boldsymbol{x}},\omega)-\left\{1-\frac{i\omega\beta}{i\omega+\lambda}\right\}\delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\right]\phi_0(\vec{\boldsymbol{x}})$$

6.2 Solution to the One-group Reactor Noise using Green's function

Assuming diffusion coefficient is constant, we do some manipulations for the noise equation.

$$\vec{\boldsymbol{\nabla}}\cdot D_0(\vec{\boldsymbol{x}})\vec{\boldsymbol{\nabla}}\big(\delta\phi(\vec{\boldsymbol{x}},\omega)\big)+\left[\left\{1-\frac{i\omega\beta}{i\omega+\lambda}\right\}\nu\Sigma_{f0}(\vec{\boldsymbol{x}})-\Sigma_{a0}(\vec{\boldsymbol{x}})-\frac{i\omega}{v}\right]\delta\phi(\vec{\boldsymbol{x}},\omega) \tag{6.22}$$

$$=\left[\delta\Sigma_a(\vec{\boldsymbol{x}},\omega)-\left\{1-\frac{i\omega\beta}{i\omega+\lambda}\right\}\delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\right]\phi_0(\vec{\boldsymbol{x}})$$

We define:

$$S(\vec{\boldsymbol{x}},\omega)=\left[\delta\Sigma_a(\vec{\boldsymbol{x}},\omega)-\left\{1-\frac{i\omega\beta}{i\omega+\lambda}\right\}\delta\nu\Sigma_f(\vec{\boldsymbol{x}},\omega)\right]\phi_0(\vec{\boldsymbol{x}}) \tag{6.23}$$

Thus,

$$\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\cdot\delta\phi(\vec{\boldsymbol{x}},\omega)+\left[\left\{1-\frac{i\omega\beta}{i\omega+\lambda}\right\}\frac{\nu\Sigma_{f0}}{D_0}-\frac{\Sigma_{a0}}{D_0}\left\{1-\frac{i\omega\beta}{i\omega+\lambda}\right\}-\frac{1}{D_0}\frac{i\omega}{v}\right]\delta\phi(\vec{\boldsymbol{x}},\omega)=\frac{S(\vec{\boldsymbol{x}},\omega)}{D_0} \tag{6.24}$$

Then, we do some manipulations to the bracket which yields to:

$$\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\cdot\delta\phi(\vec{\boldsymbol{x}},\omega)+B^2(\omega)\delta\phi(\vec{\boldsymbol{x}},\omega)=\frac{S(\vec{\boldsymbol{x}},\omega)}{D_0} \tag{6.25}$$

Where,

$$B^2(\omega)=B_0^2\left(1-\frac{1}{\rho_\infty\cdot G_0(\omega)}\right)\qquad \rho_\infty=1-\frac{\Sigma_a}{\nu\Sigma_f}\qquad B_0^2=\frac{\nu\Sigma_f-\Sigma_a}{D_0}$$

$$\Lambda=\frac{1}{v}\frac{1}{\nu\Sigma_f}\qquad G_0(\omega)=\frac{1}{i\omega\left(\Lambda+\frac{\beta}{i\omega+\lambda}\right)}$$

In this form, the solution can be obtained using Green's function technique, where:

$$\boldsymbol{Ly}(\vec{\boldsymbol{x}})=f(\vec{\boldsymbol{x}})$$

Where in this case:

$$\boldsymbol{L}=\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}D_0+B^2(\omega)\qquad f(\vec{\boldsymbol{x}})=S(\vec{\boldsymbol{x}},\omega)$$

Applying Green's function, we got:

$$\boldsymbol{L}G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega) = \delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}') \tag{6.26}$$

$$\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega) + B^2(\omega)G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega) = \delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}')$$

Multiplying both side with $S(\vec{\boldsymbol{x}}',\omega)$, yields:

$$\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega)S(\vec{\boldsymbol{x}}',\omega) + B^2(\omega)G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega)S(\vec{\boldsymbol{x}}',\omega) = \delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}')S(\vec{\boldsymbol{x}}',\omega) \tag{6.27}$$

Integrate over all volume:

$$\int \vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega)S(\vec{\boldsymbol{x}}',\omega)dV + \int B^2(\omega)G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega)S(\vec{\boldsymbol{x}}',\omega)dV = \int \delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}')S(\vec{\boldsymbol{x}}',\omega)dV \tag{6.28}$$

$$\int \vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega)S(\vec{\boldsymbol{x}}',\omega)dV + \int B^2(\omega)G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega)S(\vec{\boldsymbol{x}}',\omega)dV = S(\vec{\boldsymbol{x}},\omega)$$

Therefore, we can write:

$$\delta\phi(\vec{\boldsymbol{x}},\omega) = \int G(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}',\omega)S(\vec{\boldsymbol{x}}',\omega)dV \tag{6.29}$$