Final Exam

Upper bound: Gaussian

Gaussian elim

$$L(x,y,z) = \frac{(x+y-z)^2}{2} - \frac{(x-y+z)^2}{2} - \frac{(-x+y+z)^2}{2}$$

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial z} = 0$$

$$\frac{\partial L}{\partial x} = \sum_{i=1}^{n} x_i (x_i + x_j - x_k) = 0$$

$$\frac{\partial L}{\partial y} = \sum_{i=1}^{n} x_i (x_i - x_j + x_k) = 0$$

$$\frac{\partial L}{\partial z} = \sum_{i=1}^{n} x_i (-x_i + x_j + x_k) = 0$$

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$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial z} = 0$$

Hessian

Hessian of a function of the form

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Hessian

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

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Eigenvalues and Eigenvectors

Def: Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue of A if $Ax = \lambda x$ for some $x \neq 0$.

Note: λ is called an eigenvalue of A.

Show: $\det(A - \lambda I) = 0$

Proof: λ is true if and only if $Ax = \lambda x$

$$\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\det(A - \lambda I) = 0$$

Solve for λ (which gives all eigenvalues)

Suppose λ is true that $\vec{0}$ is an eigenvector. Then $\vec{0}$ is not the eigenvector.

Any vector v is $\vec{0}$ $\vec{0}$ is an eigenvector for $\vec{0}$.

$$\vec{0} = 0$$

Ex:

$$A = \begin{bmatrix} 2a & 0 & -a \\ 0 & -a & 0 \\ 2a & 0 & -a \end{bmatrix}$$

Find the eigenvalues of A.

Let $A = \begin{bmatrix} 2a & 0 & -a \\ 0 & -a & 0 \\ 2a & 0 & -a \end{bmatrix}$

$$\det \begin{pmatrix} 2a-\lambda & 0 & -a \\ 0 & -a-\lambda & 0 \\ 2a & 0 & -a-\lambda \end{pmatrix} = (-a-\lambda)(2a-\lambda)(-a-\lambda)$$

$$\det \begin{pmatrix} 2a-\lambda & 0 & -a \\ 0 & -a-\lambda & 0 \\ 2a & 0 & -a-\lambda \end{pmatrix} = (-a-\lambda)(2a-\lambda)(-a-\lambda)$$

So $\lambda = 2a, -a, -a$ eigenvalues of A

Eigenvector: eigenvalue $\lambda = 2$

$$A - 2I = \begin{pmatrix} 0 & 0 & -a \\ 0 & -3a & 0 \\ 2a & 0 & -3a \end{pmatrix} \xrightarrow{\text{Gauss}} \begin{pmatrix} 0 & 0 & -a \\ 0 & -3a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A - 2I)x = 0$$

$$\begin{pmatrix} 0 & 0 & -a \\ 0 & -3a & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} -ax_3 &= 0 \\ -3ax_2 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 \neq 0$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ eigenvector

Properties of Eigenvalues:

1) If $\lambda \neq 0$ is an eigenvalue of A then λ^{-1} is an eigenvalue of A^{-1}

$$Ax = \lambda x$$

$$A^{-1}Ax = A^{-1}\lambda x$$

$$x = \lambda A^{-1}x$$

$$\frac{1}{\lambda}x = A^{-1}x \qquad A^{-1}\left(\frac{1}{\lambda}\right)$$

Eigenvalue

2) If $\lambda \neq 0$ then λ is eigenvalue of A and A^{-1}

3) If $\lambda \neq 0$ is eigenvalue of A then λ^{-1} is eigenvalue of A^{-1}

4) $\det A = \prod_{i=1}^{n} \lambda_i$ where λ_i are eigenvalues

5) Let $\vec{0}$ be a matrix. $\det A = \det A^T$ (transpose)

6) If $\det A = 0$ then A is not an eigenvalue

$$Ax = \lambda x$$

$$Ax^T = \lambda x^T$$

$$x^T A = \lambda x^T$$

$$x^T A^T = \lambda x^T$$

$$(Ax)^T = \lambda x^T$$

Def: For matrices A and B, are called similar matrices if such S, $S^{-1}A$ and SBS matrices. If such that

$$B = S^{-1}AS$$

Ex: Let A be $n \times n$ diagonal $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$A^n = PD^nP^{-1}$$

$$A^n = \begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_n^n \end{pmatrix} = PD^nP^{-1}$$

Markov Processes

"state"

$$\begin{aligned} x_1 &= x_1 + \frac{1}{2}x_2 + x_3 \\ x_2 &= \frac{1}{2}x_2 \\ x_3 &= \frac{1}{2}x_2 + \frac{1}{2}x_3 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\lambda = 1$ is an eigenvalue corresponding to the eigenvector.

Stationary vector could not be equilibrium