Recitation 5

- Youtube Videosu!!

https://youtu.be/uQhTuRlWMxw?si=ZXNTd-02qNOu5Pua

- Quiz 2 - Monday
- Quiz 2 - Thursday

- **Questions:**

① Find a basis for the space spanned by the given vectors.

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

Create a matrix:

$R_3 \leftarrow 3R_1 + R_3$
$R_4 \leftarrow -2R_1 + R_4$

$R_3 \leftarrow -2R_2 + R_3$
$R_4 \leftarrow 3R_2 + R_4$

$$\begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 2 & -8 & -5 & 0 \\ 0 & -3 & 12 & 5 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -4 & 8 \end{bmatrix}$$

$R_4 \leftarrow 4R_3 + R_4$

$$\begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns.

- Hence the basis vectors are:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$$

(2) Let $u=(\lambda,1,0)$, $v=(1,\lambda,1)$ and $w=(0,1,\lambda)$. Find all values of λ that make $\{u,v,w\}$ a linearly dependent subset of $\mathbb{R}^3$.

$$\begin{bmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix} \xrightarrow{R_1 \leftarrow 1/\lambda R_1} \begin{bmatrix} 1 & 1/\lambda & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix} \xrightarrow{R_2 \leftarrow -R_1+R_2} \begin{bmatrix} 1 & 1/\lambda & 0 \\ 0 & \lambda-1/\lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{\frac{\lambda^2-1}{\lambda}} R_2$$

$$\rightarrow \begin{bmatrix} 1 & 1/\lambda & 0 \\ 0 & 1 & \lambda/\lambda^2-1 \\ 0 & 1 & \lambda \end{bmatrix}$$

if $\frac{\lambda}{\lambda^2-1} = \lambda$, then they're lin. dependent

$$\frac{\lambda}{\lambda^2-1} = \lambda \Rightarrow \lambda\left(\frac{1}{\lambda^2-1} - 1\right) = 0 \rightarrow \lambda = 0, \quad \lambda = \mp\sqrt{2}$$

(3) What is the dimension of the vector space W of all polynomials of degree less than or equal to n?

A polynomial in the variable x of degree less than or equal to n can be written exactly one way in the form

$$a_n x^n + a_{n-1} x^{n+1} + \cdots + a_1 x + a_0$$

where each a_i is a scalar. That means, this is a linear combination of the $(n+1)$ monomials $x^n, x^{n-1}, \ldots, x, 1$.

$\therefore$ The dimension of the vector space is $n+1$.

Quiz - 2 (MONDAY)

Q1. For each of the following statements, determine whether it is True or False. If True, briefly explain why. If False, briefly explain or give an example as to why not.

(a) If V_1 and V_2 are two vector spaces, and elements from each space can be added (i.e. they have compatible dimensions), the set $S = V_1 + V_2 = \{v_1 + v_2 \mid v_1 \in V_1,\ v_2 \in V_2\}$ containing all vectors obtained by summing a vector from V_1 and another vector from V_2 is also a vector space.

(b) The set of vectors in $\mathbb{R}^3$ whose three components add up to 0 forms a vector space.

(c) The set of 5×5 diagonal matrices forms a vector space.

(d) The set of 5×5 invertible matrices forms a vector space.

Vector Spaces:

- V is a collection of elements that can be like
  1) added together in any combination
  2) multiplied by scalars in any combo
- "Closure"
  - ↳ given $\vec{a} \in V$ and scalar c, then $c\vec{a} \in V$
  - ↳ given $\vec{a} \in V$ and $\vec{b} \in V$, $\vec{a} + \vec{b} \in V$

b) True. $\vec{v}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\quad \vec{v}_1$ and $\vec{v}_2 \in V \Rightarrow a+b+c=0 \wedge d+e+f=0$

$\vec{v}_2 = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ $\quad \vec{v}_1 + \vec{v}_2 = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$

$(a+d)+(b+e)+(c+f)=0$ ✓

$k \cdot \vec{v}_1 = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$ $\quad ka+kb+kc = k(a+b+c) = 0$ ✓

$\therefore T$

c) True. ① $\begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \ddots & \alpha \end{pmatrix} + \begin{pmatrix} \beta & & & \\ & \beta & & \\ & & \beta & \\ & & & \ddots & \beta \end{pmatrix} = \begin{pmatrix} \alpha+\beta & & \\ & \alpha+\beta & \\ & & \ddots & \\ & & & \alpha+\beta \end{pmatrix}$

② $\lambda \cdot \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \ddots & \alpha \end{pmatrix} = \begin{pmatrix} \lambda\alpha & & \\ & \lambda\alpha & \\ & & \ddots & \lambda\alpha \end{pmatrix}$ ✓

③ $\begin{pmatrix} 0 & 0 & 0 & \cdots \\ \vdots & 0 & & \\ & & 0 & \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ is a diagonal matrix.

d) False. Counter Example: $m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (invertible)

$m_2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (invertible)

$m_1 + m_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (not invertible)

a) True.

$v_1 \in V_1 \longrightarrow cv_1 \in V_1$

$v_2 \in V_2 \longrightarrow cv_2 \in V_2$

let $V_1 + V_2 = S$

$v_1 + v_2 = s \in S$

- $cs = c(v_1 + v_2) = \underbrace{cv_1}_{\in V_1} + \underbrace{cv_2}_{\in V_2} \in V_1 + V_2 = S$ ✓

- $v_1 + v_2 = s_1 \in S$ and $v_1' + v_2' = s_2 \in S$

$s_1 + s_2 = (v_1 + v_2) + (v_1' + v_2') = \underbrace{(v_1 + v_1')}_{\in V_1} + \underbrace{(v_2 + v_2')}_{\in V_2}$

$\Rightarrow S_1 + S_2 \in S$ ✓

Q2. Find the determinant of

$$A = \begin{bmatrix} 2 & 1 & 2 & 0 & 97 \\ 0 & 0 & -1 & 0 & -1 \\ -42 & 13 & 1000 & 1 & 71.8 \\ -3 & -1 & -2 & 0 & 78.1 \\ 0 & 0 & 4 & 0 & 2 \end{bmatrix}.$$

Q2) We can row / column reduce the matrix and try to end up with a diagonal matrix whose determinant is the product of its diagonal components.

* Row / column reduction of a matrix doesn't change the determinant.
* Exchanging the rows changes the sign of the determinant.

$$\begin{bmatrix} 2 & 1 & 2 & 0 & 97 \\ 0 & 0 & -1 & 0 & -1 \\ -42 & 13 & 1000 & 1 & 71.8 \\ -3 & -1 & -2 & 0 & 78.1 \\ 0 & 0 & 4 & 0 & 2 \end{bmatrix}$$

Using column 4, we can reduce third row of each column.

$$\begin{bmatrix} 2 & 1 & 2 & 0 & 97 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -3 & -1 & -2 & 0 & 78.1 \\ 0 & 0 & 4 & 0 & 2 \end{bmatrix}$$

Using second row, we can eliminate the fifth row.

$$\begin{bmatrix} 2 & 1 & 2 & 0 & 97 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -3 & -1 & -2 & 0 & 78.1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Using fifth row, we can eliminate the fifth row.

$$\begin{bmatrix} 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -3 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Using second row, we can eliminate the third column.

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Using first row, we can eliminate fourth row.

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Using the fourth row, we can eliminate the first row.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Exchange the following rows:

$R4 - R3 \rightarrow (-)$
$R3 - R2 \rightarrow (+)$
$R2 - R1 \rightarrow (-)$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \Rightarrow$$

$\det = (+)(-1)(1)(-1)(1)(-2).$

$= 2$ //

Quiz 2 - (Thursday)

1. (40 pts) Construct a 3×3 matrix A such that the vector

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

is a solution of $Ax = 0$.

2. (60 pts) Apply Gaussian Elimination method to find the inverse of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 11 \\ 3 & 8 & 17 \end{pmatrix}$$

1) $A . x = 0$

$$\begin{bmatrix} a & b & c \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$a = b + c$

can be anything. No limits.

some solutions. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 2 & 4 \\ -6 & -5 & -1 \\ 3.14 & 1 & 2.14 \end{bmatrix} \ldots$

2)

$R_2 \leftarrow -2R_1 + R_2$
$R_3 \leftarrow -3R_1 + R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 6 & 11 & 0 & 1 & 0 \\ 3 & 8 & 17 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & -2 & 1 & 0 \\ 0 & 2 & 8 & -3 & 0 & 1 \end{array}\right)$$

$R_2 \leftarrow 1/2\, R_2$
$R_3 \leftarrow -2R_2 + R_3$

$R_3 \leftarrow 1/3\, R_3$
$R_2 \leftarrow -5/2\, R_3 + R_2$
$R_1 \leftarrow -3R_3 + R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & -1 & 1/2 & 0 \\ 0 & 0 & 3 & -1 & -1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -1/6 & 4/3 & -5/6 \\ 0 & 0 & 1 & -1/3 & -1/3 & 1/3 \end{array}\right)$$

$R_1 \leftarrow -2R_2 + R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/3 & -5/3 & 4/6 \\ 0 & 1 & 0 & -1/6 & 4/3 & -5/6 \\ 0 & 0 & 1 & -1/3 & -1/3 & 1/3 \end{array}\right)$$