Recitation 7

- Solutions to Quiz 3
- Questions for the last week
- Review.

- Quiz 3 (Monday)

Find the eigenvalues and the eigenvectors of the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 8 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 8 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -5 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 16 & 2 \end{bmatrix}.$$

- $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - 5 = 0 \Rightarrow \lambda^2 - 4 - 5 = 0 \to \lambda_1 = 3,\ \lambda_2 = -3$

$\lambda = 3 \to (A - 3I)x_1 = 0 \Rightarrow \begin{bmatrix} 2-3 & 5 \\ 1 & -2-3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 1 & -5 \end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$\lambda = -3 \to (A + 3I)x_2 = 0 \Rightarrow \begin{bmatrix} 2+3 & 5 \\ 1 & -2+3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

- $|B - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 8 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) - 8 = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0 \to \lambda_1 = 5,\ \lambda_2 = -1$

$\lambda = 5 \to (B - 5I)x_1 = 0 \to \begin{bmatrix} 3-5 & 1 \\ 8 & 1-5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -1 \to (B + I)x_2 = 0 \to \begin{bmatrix} 3+1 & 1 \\ 8 & 1+1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

- Realize that $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ ... "block diagonal"

$\lambda = \{\lambda_{A1}, \lambda_{A2}, \lambda_{B1}, \lambda_{B2}\} = 3, -3, 5, -1$

$v = \{[v_{A1}\ 0]^T, [v_{A2}\ 0]^T, [v_{B1}\ 0]^T, [v_{B2}\ 0]^T\}$

$= \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$

• Realize that $D = \begin{bmatrix} B & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & 2B \end{bmatrix}$

$\lambda = \{\lambda_{B1}, \lambda_{B2}, -\lambda_{A1}, -\lambda_{A2}, 2\lambda_{B1}, 2\lambda_{B2}\}$
$= \{5, -1, -3, 3, 10, -2\}$

$v = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} \right\}$

Quiz 3 - Thursday

(2)

1. (40 pts)Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?

• The set of three vetors in $\mathbb{R}^4$ form a matrix of size 4x3. The matrix can have maximum of 3 pivot elements. Hence, three vectors can not span $\mathbb{R}^4$.

• As a general rule, if $n < m$, n vectors in $\mathbb{R}^m$ cannot span $\mathbb{R}^m$!

2. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 11 \\ 3 & 6 & 9 \end{pmatrix}$$

(a) (20 pts) Find det A.
(b) (10 pts) Determine if A is invertible.
(c) (15 pts) Find Null$A$.
(d) (10 pts) Find Col$A$.
(e) (5 pts) Find the rank of A.

a) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 6 & 11 \\ 3 & 6 & 9 \end{vmatrix} \xrightarrow[-3R_1+R_3]{\text{row reduce}} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 6 & 11 \\ 0 & 0 & 0 \end{vmatrix} \rightarrow \det A = 0.$

b) Since $\det A = 0$, it is not invertible.

c) Null A : consists of all vectors x such that $Ax=0$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 6 & 11 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 6 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_2 + 5x_3 = 0.$$

$$x_2 = t$$
$$x_3 = -2/5\, t$$
$$x_1 = -2x_2 - 3x_3$$

ex) $t=1 \rightarrow \begin{bmatrix} -2 \\ 1 \\ -2/5 \end{bmatrix}$

d) Col A: The set of pivot colums:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 11 \\ 3 & 6 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Col}\,A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix} \right\}$$

e) rank A = size of Col A $= 2$.

This week's Questions

1) Consider the following vectors and calculate their inner product.

$$v = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

$$\langle v, w \rangle = (2 \cdot 1) + (-1 \cdot 4) + (3 \cdot -2) = 2 - 4 - 6 = -8$$

2) Consider the following vectors and determine the angle (in degrees) between them.

$$u = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|} = \frac{(1 \cdot 3) + (2 \cdot -1) + (-1 \cdot 4)}{\sqrt{1^2 + 2^2 + (-1)^2} \cdot \sqrt{3^2 + (-1)^2 + 4^2}} = \frac{3 - 2 - 4}{\sqrt{6} \cdot \sqrt{26}} = \frac{-3}{2\sqrt{39}}$$

$$\theta = \cos^{-1}\left(\frac{-3}{2\sqrt{39}}\right)$$

3) Let $v = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$. Find a unit vector "$u$" in the same direction as v.

$$u = \frac{1}{\|v\|} \cdot v$$

$$\|v\| = \sqrt{1^2 + (-2)^2 + (2)^2 + 0^2} = \sqrt{9} = 3.$$

$$\|v\| = \sqrt{1^2 + (-2)^2 + (2)^2 + 0^2} = \sqrt{9} = 3.$$

$$\Rightarrow u = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

... Sanity check: if $\|u\| = 1$.

4) let $c = \begin{bmatrix} 4/3 \\ -1 \\ 2/3 \end{bmatrix}$ and $d = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$

a) Find a unit vector in the direction of c.

b) Show that d is orthogonal to c.

c) Use the results of (a) and (b) to explain why d must be orthogonal to the unit vector u.

a) Scale c to ease the calculations.

$c = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$, $\|c\| = \sqrt{(4^2) + (-3)^2 + 2^2} = \sqrt{29}$

$u = \frac{1}{\sqrt{29}} \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{29} \\ -3/\sqrt{29} \\ 2/\sqrt{29} \end{bmatrix}$

b) if $\langle d, c \rangle = 0$, then d is orthogonal to c.

$\begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4/3 \\ -1 \\ 2/3 \end{bmatrix} = 20/3 - 6 - 2/3 = 0.$ $\therefore$ they're orthogonal.

c) $u = k \cdot c$ where k is a scalar.

if $\langle d, u \rangle = 0$, then d is orthogonal to u.

$d \cdot u = d \cdot (kc) = k(dc) = k \cdot 0 = 0$ $\therefore$ they're orthogonal.

5) Let $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$. Determine if

u, v, and w form an orthogonal set.

— Orthogonal set if $\langle u,v \rangle = \langle u,w \rangle, \langle v,w \rangle = 0$.

$$\langle u,v \rangle = (1 \cdot (-2)) + (1 \cdot 1) + (1 \cdot 1) = 0$$

$$\langle u,w \rangle = 1 \cdot 3 + (-2) \cdot 1 + (-1) \cdot 1 = 0$$

$$\langle v,w \rangle = (-2) \cdot 3 + (1)(-2) + (1)(-1) = -6 - 2 - 1 = -9 \quad \times$$

6) Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and find the orthogonal projection of y onto u.

$$\boxed{\hat{y} = \text{Proj}_L u = \frac{\langle y,u \rangle}{\langle u,u \rangle} u}$$

$$\left.\begin{aligned} \langle y,u \rangle &= 28 + 12 = 40 \\ \langle u,u \rangle &= 16 + 4 = 20 \end{aligned}\right\} \quad \hat{y} = \frac{40}{20} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

7) Find q_1, q_2, q_3 (orthonormal) from a, b, c (columns of A).
(Gram–Schmidt Process)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

- $q_1 = \dfrac{a}{\|a\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$q_2' = b - \frac{\langle b, q_1 \rangle \cdot q_1}{\|q_1\|} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{1} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

- $\left.\begin{aligned} q_2' &= b - \text{proj}_b\, q_1 \\ q_2 &= \frac{q_2'}{\|q_2'\|} \end{aligned}\right\} \quad q_2 = \dfrac{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}{3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- $q_3' = c - (\text{proj}_c\, q_1) - (\text{proj}_c\, q_2)$

$$= c - \left(\frac{\langle c, q_1\rangle q_1}{\|q_1\|}\right) - \left(\frac{\langle c, q_2\rangle q_2}{\|q_2\|}\right)$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \left(\frac{4 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{1}\right) - \left(\frac{6 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{1}\right) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$q_3 = \frac{q_3'}{\|q_3'\|} = \frac{\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}}{5} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

8) Find QR decomposition of A using orthonormal vectors of A.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

$$A = QR \Rightarrow \underset{A}{\begin{array}{ccc} a_1 & a_2 & a_3 \\ \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}} = \underset{Q}{\begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$$\left.\begin{aligned} r_{11} &= \langle a_1, q_1\rangle = 1 \\ r_{12} &= \langle a_2, q_1\rangle = 2 \\ r_{13} &= \langle a_3, q_1\rangle = 4 \\ r_{22} &= \langle a_2, q_2\rangle = 3 \\ r_{23} &= \langle a_3, q_2\rangle = 6 \\ r_{33} &= \langle a_3, q_3\rangle = 5 \end{aligned}\right\} \quad R = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

General Review

1) LU Decomposition:

$$A=\begin{bmatrix}1&2&1\\2&-1&3\\3&2&4\end{bmatrix}, \quad b=\begin{bmatrix}1\\2\\1\end{bmatrix} \qquad L=\begin{bmatrix}1&0&0\\2&1&0\\3&4/5&1\end{bmatrix}$$

$$\begin{bmatrix}1&2&1\\2&-1&3\\3&2&4\end{bmatrix}\xrightarrow[R_3\leftarrow -3R_1+R_3]{R_2\leftarrow -2R_1+R_2}\begin{bmatrix}1&2&1\\0&-5&1\\0&-4&1\end{bmatrix}\xrightarrow{R_3\leftarrow -4/5R_2+R_3}\begin{bmatrix}1&2&1\\0&-5&1\\0&0&1/5\end{bmatrix}=U$$

$Ax=b \quad , \quad A=LU$

$LUx=b \Rightarrow L(Ux)=b \quad , \quad Ux=y \Rightarrow Ly=b.$

- $Ly=b \Rightarrow \left[\begin{array}{ccc|c}1&0&0&1\\2&1&0&2\\3&4/5&1&1\end{array}\right] \xrightarrow[R_3\leftarrow -3R_1+R_3]{R_2\leftarrow -2R_1+R_2} \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&0\\0&4/5&1&-2\end{array}\right] \xrightarrow{R_3\leftarrow -4/5R_2+R_3}$

$$\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&0\\0&0&1&-2\end{array}\right] \Rightarrow y=\begin{bmatrix}1\\0\\-2\end{bmatrix}$$

- $Ux=y \Rightarrow \left[\begin{array}{ccc|c}1&2&1&1\\0&-5&1&0\\0&0&1/5&-2\end{array}\right] \xrightarrow[R_3\leftarrow 5R_3]{R_2\leftarrow -1/5R_2} \left[\begin{array}{ccc|c}1&2&1&1\\0&1&-1/5&0\\0&0&1&-10\end{array}\right]$

$R_2\leftarrow \frac{1}{5}R_3+R_2$
$R_1\leftarrow -R_3+R_1$
$R_1\leftarrow -2R_2+R_1$

$$\left[\begin{array}{ccc|c}1&0&0&15\\0&1&0&-2\\0&0&1&-10\end{array}\right] \Rightarrow x=[15 \quad -2 \quad -10]^T$$

2) Find a matrix A such that $W = \text{Col} A$.

$$W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} \right\} \quad \text{where } a, b \in \mathbb{R}$$

— Write W as a set of linear combinations:

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \to \text{Span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

— A matrix whose columns are W is $\begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix} = A$.

3) Consider the linear transformation T defined by:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ 3x-y \end{bmatrix}$$

a) Determine the image of the vector $v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ under the lin. trm. T.

$$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 2-2 \\ 3+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

b) Verify if T is a linear transformation.

- $T(a+b) = T(a) + T(b) \to T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1+x_2)+(y_1+y_2) \\ 3(x_1+x_2)-(y_1+y_2) \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2+y_1+y_2 \\ 3x_1+3x_2-y_1-y_2 \end{bmatrix}$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1+y_1 \\ 3x_1-y_1 \end{bmatrix} + \begin{bmatrix} 2x_2+y_2 \\ 3x_2-y_2 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2+y_1+y_2 \\ 3x_1+3x_2-y_1-y_2 \end{bmatrix}$$

They're equal.

- $T(ca) = c\,T(a) \to T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} 2cx+cy \\ 3cx-cy \end{bmatrix}$

$$c \cdot T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c \cdot \begin{bmatrix} 2x+y \\ 3x-y \end{bmatrix} = \begin{bmatrix} 2cx+cy \\ 3cx-cy \end{bmatrix}$$

They're equal.

c) Find the standard matrix A that represents trm T.

- A is obtained by applying T to the standard basis vectors.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2+0\\3+0\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0+1\\0-1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$$

$$A=\begin{bmatrix}2&1\\3&-1\end{bmatrix}$$

4) Consider the following vectors: $u=\begin{bmatrix}1\\2\\-1\end{bmatrix}$, $v=\begin{bmatrix}-1\\0\\2\end{bmatrix}$, $w=\begin{bmatrix}2\\4\\-2\end{bmatrix}$

a) Determine whether these vectors span $\mathbb{R}^3$.

$$\begin{bmatrix}1&-1&2\\2&0&4\\-1&2&-2\end{bmatrix}\sim\begin{bmatrix}1&-1&2\\0&2&0\\0&0&0\end{bmatrix}$$

two pivots.
∴ They don't span $\mathbb{R}^3$.

b) Find a vector x such that u, v, w, and x span $\mathbb{R}^3$.

↳ a vector that is lin. indep from the others. ex: $\begin{bmatrix}1\\1\\1\end{bmatrix}$