**Q1.** (a) I have a start-up where I provide web design, database management, and network design services.

- In January, I had 2 web design, 1 database management, and 3 network design jobs, and I made \$14 thousand.
- In February, I had 1 web design, 2 database management, and 2 network design jobs, and I made \$10 thousand.
- In March, I had 2 web design, and 1 network design jobs, and I made \$7 thousand.
- In April, I had 2 web design, 1 database management, and 1 network design jobs, and I made \$8 thousand.

If possible, find how much I charge for each of the three job types. If not, explain why.

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ d \\ n \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \\ 7 \\ 8 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 2 & 1 & 3 & 14 \\ 1 & 2 & 2 & 10 \\ 2 & 0 & 1 & 7 \\ 2 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 2 & 1 & 3 & 14 \\ 2 & 0 & 1 & 7 \\ 2 & 1 & 1 & 8 \end{array}\right] \xrightarrow[\substack{R_3 \leftarrow R_3 - 2R_1 \\ R_4 \leftarrow R_4 - 2R_1}]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & -3 & -1 & -6 \\ 0 & -4 & -3 & -13 \\ 0 & -3 & -3 & -12 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 - R_2]{R_3 \leftarrow R_3 - \frac{4}{3}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & -3 & -1 & -6 \\ 0 & 0 & -5/3 & -5 \\ 0 & 0 & -2 & -6 \end{array}\right] \xrightarrow{R_4 \leftarrow R_4 - \frac{6}{5}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & -3 & -1 & -6 \\ 0 & 0 & -5/3 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} w \\ d \\ n \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \$1K$$

(b) Is $b = \begin{bmatrix} 14 \\ 10 \\ 7 \\ 8 \end{bmatrix}$ in the column space of $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$?

Yes.

$$\begin{bmatrix} 14 \\ 10 \\ 7 \\ 8 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

**Q2.** (a) What could be the last row of $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ ? & ? & ? \end{bmatrix}$, if $\det A = 12$?

$$\begin{array}{ccc} a & b & c \end{array}$$

cofactor expansion along the first row:

$2(4c+2b) - 10a = 12$

$4c + 2b - 5a = 6$

The last row can be $[0 \;\; 0 \;\; 3/2]$

(b) Fill in the following statement:

Matrix $A$ is $45 \times 28$, and its rank is 24. The dimension of its column space is 24, the dimension of its row space is 24, the dimension of its null space is 4, and the dimension of the null space of $A^\top$ is 21.

(c) A basis for the nullspace of $A$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$, and a basis for the nullspace of $B$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Find bases for the nullspaces of block matrices $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $D = \begin{bmatrix} A & 2A \\ 0 & B \end{bmatrix}$, assuming $A$ and $B$ have appropriate sizes.

$Cx = 0 \rightarrow \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x_U \\ x_D \end{bmatrix} = \begin{bmatrix} Ax_U \\ Bx_D \end{bmatrix} = 0 \rightarrow$

$\rightarrow x_U = k_U \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad x_D = k_{D1} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + k_{D2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\text{Null}(C) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$Dx = 0 \rightarrow \begin{bmatrix} A & 2A \\ 0 & B \end{bmatrix} \begin{bmatrix} x_U \\ x_D \end{bmatrix} = \begin{bmatrix} A(x_U + 2x_D) \\ Bx_D \end{bmatrix} = 0 \rightarrow$

$\rightarrow x_U + 2x_D = k_U \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$x_D = k_{D1} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + k_{D2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_U = k_U \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2k_{D1} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - 2k_{D2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\text{Null}(D) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 2 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

**Q3.** Find the inverse of $A = E_1\,E_2\,E_3\,E_4\,E_5\,C$, where

$$E_1 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\3&0&0&1\end{bmatrix},\quad E_2 = \begin{bmatrix}1/3&0&0&0\\0&1/2&0&0\\0&0&1&0\\0&0&0&-1/3\end{bmatrix},\quad E_3 = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix},$$

$$E_4 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\\0&0&0&1\end{bmatrix},\quad E_5 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&0&0&1\end{bmatrix},\quad C = \begin{bmatrix}3&2&0&0\\2&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}.$$

$A^{-1} = C^{-1}E_5^{-1}E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$

C is block diagonal, so

$$C^{-1} = \begin{bmatrix} \begin{bmatrix}3&2\\2&1\end{bmatrix}^{-1} & \begin{bmatrix}0&0\\0&0\end{bmatrix} \\ \begin{bmatrix}0&0\\0&0\end{bmatrix} & \begin{bmatrix}0&1\\1&0\end{bmatrix}^{-1}\end{bmatrix} = \begin{bmatrix}-1&2&0&0\\2&-3&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

$E_1, E_4, E_5$ are elementary matrices for adding a multiple of a row to another. $E_2$ is diagonal. $E_3$ is a permutation matrix, so $E_3^{-1} = E_3^T$.

$$E_1^{-1} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-3&0&0&1\end{bmatrix},\quad E_2^{-1}E_1^{-1} = \begin{bmatrix}3&0&0&0\\0&2&0&0\\0&0&1&0\\9&0&0&-3\end{bmatrix},\quad E_3^{-1} = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

$$E_3^{-1}E_2^{-1}E_1^{-1} = \begin{bmatrix}3&0&0&0\\9&0&0&-3\\0&2&0&0\\0&0&1&0\end{bmatrix},\quad E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1} = \begin{bmatrix}3&0&0&0\\9&0&0&-3\\0&2&-2&0\\0&0&1&0\end{bmatrix}$$

$$E_5^{-1}E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1} = \begin{bmatrix}3&0&0&0\\9&0&0&-3\\9&2&-2&-3\\0&0&1&0\end{bmatrix},\quad C^{-1}E_5^{-1}E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1} =$$

$$= \begin{bmatrix}-1&2&0&0\\2&-3&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}3&0&0&0\\9&0&0&-3\\9&2&-2&-3\\0&0&1&0\end{bmatrix} = \begin{bmatrix}15&0&0&-6\\-21&0&0&9\\0&0&1&0\\9&2&-2&-3\end{bmatrix}$$

**Q4.** Consider the matrix

$$A = \begin{pmatrix} 67 & -15 & 180 \\ 338 & -108 & 1296 \\ 16 & -5 & 79 \end{pmatrix}$$

and the multiplication

$$\begin{pmatrix} 67 & -15 & 180 \\ 338 & -108 & 1296 \\ 16 & -5 & 79 \end{pmatrix} \begin{pmatrix} 10/19 & 3/19 & 3/19 \\ -4/19 & 33/19 & 14/19 \\ -3/19 & 1/19 & 1/19 \end{pmatrix} = \begin{pmatrix} 10/19 & 3/19 & 3/19 \\ -4/19 & 33/19 & 14/19 \\ -3/19 & 1/19 & 1/19 \end{pmatrix} \begin{pmatrix} 19 & 0 & 0 \\ 0 & -38 & 0 \\ 0 & 0 & 57 \end{pmatrix}$$

(a) Find all eigenvalues of $A$.

(b) Find at least one vector in each eigenspace of $A$, i.e. one eigenvector corresponding to each eigenvalue.

(c) Find det $A$.

a) $A . P = P . \Lambda$

$\lambda_1 = 19, \lambda_2 = -38, \lambda_3 = 57$

b) $\lambda_1 = 19$:

$\vec{v_1} = \begin{bmatrix} 10/19 \\ -4/19 \\ -3/19 \end{bmatrix}$

$\lambda_2 = -38$:

$\vec{v_2} = \begin{bmatrix} 3/19 \\ 33/19 \\ 1/19 \end{bmatrix}$

$\lambda_3 = 57$:

$\vec{v_3} = \begin{bmatrix} 3/19 \\ 14/19 \\ 1/19 \end{bmatrix}$

c) $|A| = \lambda_1 . \lambda_2 . \lambda_3$

$= 19 . -38 . 57 =$

$= -41154$