1 Problem 1: Marble Toy Game

Figure 1 illustrates a marble rolling toy. A toy can be dropped from either slot A or B. The levers x1, x2, and x3 cause the marble to fall either to the left (L) or right (R). The configuration provided in Figure 1 is LLL (x1 = L, x2 = L, x3 = L). Whenever a marble encounters a lever, it cause the lever to reverse **after** the marble passes, so the next marble will take the opposite branch. The marble exits from slots C or D depending on the lever settings.

- Model this toy as a Mealy machine and give the state transition table, where A and B are the inputs and C and D are the outputs.
- Suppose the players play the following game: Players drop marbles in turn and whoever gets his/her marble out from slot D wins the game. Using your transition table from the previous part, find the lever settings/state where the player who makes the second move can always win.

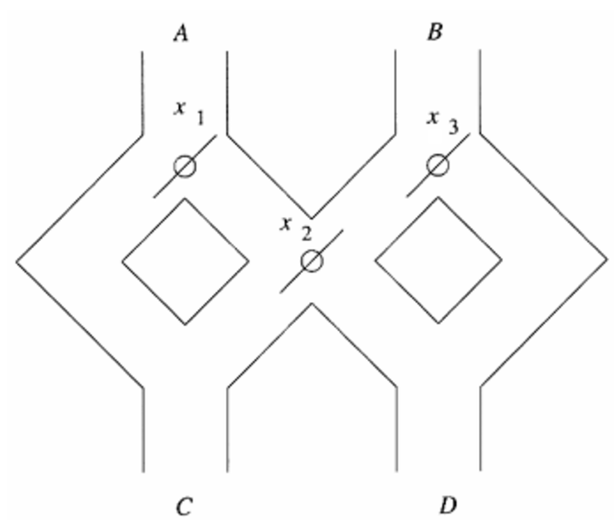

Figure 1: Marble Toy.

3 Problem 3: Proof by Induction

Let $S(n) = 1 + 2 + ... + n$ be the sum of the first n natural numbers and let $C(n) = 1^3+2^3+...+n^3$ be the sum of the first n cubes. First, prove the following two equalities by induction on n. Then, use induction to prove $C(n) = S^2(n)$ for every n.

- $S(n) = \frac{1}{2}n(n + 1)$
- $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n + 1)^2$

Base proof $\rightarrow S(1) = 1 \rightarrow \frac{1}{2} \cdot 1 \cdot 2 = 1$ ✓

Induction step:

Assume that $S(k) = \frac{1}{2}k(k+1)$ is true

$S(k+1) \overset{?}{=} \frac{1}{2} \cdot (k+1)(k+2)$

$S(k+1) = \underbrace{1+2+\dots+k}+k+1$

$S(k+1) = \frac{1}{2} \cdot k \cdot (k+1) + (k+1) = (k+1)\left(\frac{k}{2}+1\right) = \frac{1}{2} \cdot (k+1) \cdot (k+2) =$ ✓

Base proof $\rightarrow C(1) = 1^3 = 1 = \frac{1}{4} \cdot 1 \cdot 4 = 1$ ✓

Induction step:

Assume that $C(k) = \frac{1}{4} \cdot k^2 \cdot (k+1)^2$ is true

$C(k+1) \overset{?}{=} \frac{1}{4}(k+1)^2 \cdot (k+2)^2$

$C(k+1) = C(k) + (k+1)^3$

$C(k+1) = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$C(k+1) = (k+1)^2\left(\frac{1}{4}k^2 + k + 1\right) = \frac{1}{4}(k+1)^2 \cdot (k+2)^2$ ✓

Base $\rightarrow S^2(1) = 1$, $C(1) = 1 \rightarrow S^2(1) = C(1)$

Induction step:

Assume that $S^2(k) = C(k)$ is true.

$S^2(k+1) \overset{?}{=} C(k+1)$

$S^2(k+1) = (S(k)+k+1)^2 = S^2(k) + 2(k+1) \cdot S(k) + (k+1)^2$

$C(k+1) = \underbrace{C(k)} + (k+1)^3 \overset{?}{=} S^2(k) + 2(k+1) \cdot S(k) + (k+1)^2 \rightarrow (k+1)^3 \overset{?}{=} (k+1)(2S(k)+k+1)$

$(k+1)^3 \overset{?}{=} (k+1) \cdot (\cancel{2} \cdot \frac{1}{\cancel{2}} \cdot k \cdot (k+1) + k+1) \rightarrow (k+1)^3 \overset{?}{=} (k+1) \cdot (k+1)(k+1)$ ✓

4 Problem 4: Grammar Formulations for String Structures

Produce the grammars for the following languages. State what type the grammar is in the Chomsky Hierarchy.

- The language composed of strings containing an arbitrary number of substrings AA or BB followed by a single substring BB.
- The language composed by an even number of A's followed by an even number of B's.
- The language over {0, 1} containing strings which have at least one 0 surrounded by 1's.

1) $\Sigma = AA, BB$

$N = n_0, S$

$n_0 \rightarrow XBB$

$X \rightarrow XAA \mid XBB \mid AA \mid BB$

Type - 3

2) $\Sigma = AA, BB$

$N = n_0$

$n_0 \rightarrow n_0BB \mid AAn_0 \mid AA \mid BB$

Type-2

3) $\Sigma = 0, 1$

$N = n_0, S, X$

$n_0 \rightarrow X101X$

$X \rightarrow 1X \mid 0X \mid \lambda$

Type - 2

$\Sigma = h, a$

$N = n_0$

$n_0 \rightarrow w_1 w_2 w_1 w_2 a$

$w_1 \rightarrow ha$

$w_2 \rightarrow w_2 h \mid w_2 a \mid \lambda$

5 Problem 5: The Grammar of Laughter

Long before emojis and keysmashing (random atmak) people would express their amusement by typing sequences of characters resembling "haha".
The rules for producing haha's are:

- There should be at least 2 "ha"s.
- The first "ha" consists of exactly one h followed by one a.
- Any "ha" after the first can have a variable number of h's and a's depending on the intensity of the associated laughter.
- Expressions beginning with a and ending with h are not acceptable.

Some examples of acceptable haha's are: haha, hahaaaa, hahhhhaaaaaa, hahhaaahaa...
Provide the production rules for the grammar described. What is the type of this grammar in the Chomsky Hierarchy?

6 Problem 6:

Consider the following grammar:

$$A \rightarrow AaA$$
$$A \rightarrow b$$

- What is the type of the grammar according to the Chomsky Hierarchy? Why? Type-2. It has only non-terminal item in the left hand side of the exp.
- Design another grammar with a more restrictive type that generates the same language. (e.g. if the grammar is **Type-1**, design a **Type-2** or **Type-3** grammar.) $A \rightarrow Aab$ $A \rightarrow b$ Type-3

7 Problem 7:

Let Δ be any alphabet and let $L_5 \subseteq \Delta^*$ and $L_6 \subseteq \Delta^*$.

a) Suppose that every word in L_5 has an even length, and the empty string λ is not in L_5. Explain why $L_5\Delta^*$ does not contain all the elements of Δ^*.

b) Suppose there exists a word x such that $x \in L_5$ and $x \in L_6$. Using axioms and theorems of formal language theory, show that $(L_5 \cup L_6)^* = \Delta^*$.

8 Problem 8:

Consider the following languages A, B, and C defined over the alphabet $\Sigma = \{x, y\}$:

- $A = xy^+yx$
- $B = x(yy)^+yx$
- $C = x(yy)^*x$

Answer each of the following questions considering the definition above.

a) Give an example string that is accepted by all three languages. None!

b) Give an example string that is accepted by only A. None!

c) Give an example string that is accepted by only A and B. xyyyx

d) Give an example string that is accepted by only A and C. xyyx

e) Give an example string that is accepted by only C. xx

f) Indicate if there is a subset/superset relation between any pair of the three languages.

$B \subset A$

9 Problem 9:

Given a language defined over $\Sigma = \{a, b\}$ that includes words containing at least one instance of "aaa" or "bbb" strings:

a) Provide the production rules for the grammar that this language belongs to.

b) Indicate to which Chomsky class the grammar of this language $L(G)$ belongs.

$\Sigma = a, b$

$N = n_0$

$n_0 \to w_1 w_2 w_1$

$w_1 \to w_1 a \mid w_1 b \mid \lambda$

$w_2 \to aaa \mid bbb$

Type-2

10 Problem 10:

Determine the production rules for the grammars of the languages below and identify which Chomsky class they belong to.

a) $L(G) = \{a^n b^n \mid n \geq 1\}$

b) $L(G) = \{a^n b^{n+m} \mid n \geq 1, m \geq 1\}$

a) $\Sigma = a, b$

$N = n_0$

$\mapsto = \{$

$n_0 \to a n_0 b \mid ab$

$\}$

b) $\Sigma = a, b$

$N = n_0, S$

$\mapsto = \{$

$n_0 \to aSb$

$S \to aSb \mid Sb \mid b$

11 Problem 11:

Let A and B be languages defined over Σ. Show that equation $A^*B^* \cap B^*A^* = A^* \cup B^*$ holds.

$A = \{x\}$

$B = \{y\}$

$A^* = \{\lambda, x, xx, xxx, \ldots\}$

$B^* = \{\lambda, y, yyy, \ldots\}$

We get A^* with concatenation λ with A^*, and B^* with concatenation λ with B^*. There is also $x^n y^n$ in A^*B^* and $y^k x^l$ in B^*A^*. There is only $A^* \cup B^*$ in intersection since the " is not commutative. so

$$A^*B^* \cap B^*A^* = A^* \cup B^*$$

$A^+ = A^1 \cup A^2 \cup \cdots \cup A^n$

$A^+ \cdot A^+ = (A^1 \cup A^2 \cup \cdots \cup A^n) \cdot (A^1 \cup \cdots \cup A^n)$

$= A^2 \cup A^3 \cup \cdots \cup A^{2n \doteq n}$

It doesn't include A^1, so it is not true.

$(AB)^0 \cup (AB)^1 \cup \cdots (AB)^n \stackrel{?}{=} (BA)^0 \cup (BA)^1 \cup \cdots (BA)^n$

The concatenation operation is not commutative so $AB \neq BA$, and $(AB)^* \neq (BA)^*$

12 Problem 12:

Show that following expressions hold. If they do not hold give a counterexample.

a) $A + A+ = A+$

b) $(A^*B^*)^* = (B^*A^*)^*$

c) $(AB)^* = (BA)^*$

13 Problem 13:

Design a regular grammar that generates all strings ending with "01" over the alphabet $\Sigma = \{0, 1\}$.

$\Sigma = 0, 1$

$N = n_0, S$

$n_0 \to S01$

$S \to S0 \mid S1 \mid 0 \mid 1$

14 Problem 14:

Let $L = \{w \mid w \in \{a, b\}^* \wedge |w|$ is odd $\wedge$ the first, middle, and last characters of w are the same$\}$.

a) Give generation rules of a grammar for this language.

b) Identify the Chomsky hierarchy type of your grammar.

a) $\Sigma = a, b$

$N = n_0, S$

$n_0 \to a \mid b \mid aAa \mid bBb$

$A \to a \mid CAC$

$B \to b \mid CBC$

$C \to a \mid b$