Recitation 12
25.12.2023

$\boxed{\text{Quiz 5}}$ 1

BLG 210E — Quiz V — December 21, 2023

1. Let A be an $m \times n$ matrix and $\lambda_1, \ldots, \lambda_n$ be eigenvalues of $A^T A$ such that $\lambda_i \neq \lambda_j$.
   (a) (10 pts) Determine the number of eigenspaces of $A^T A$ and their dimensions.
   (b) (20 pts) Let v_1 be an eigenvector of $A^T A$ for λ_1 and v_2 be an eigenvector of $A^T A$ for λ_2. Compute the dot product $v_1.v_2$.
   (c) (30 pts) Find the result of the dot product $Av_1.Av_1$ assuming v_1 has norm 1.
2. (40 pts) Let A be an $m \times n$ matrix. Write down the pseudo code outputs the singular value decomposition of A.

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a) A^TA has n distinct eiguvalues ($\lambda_i \neq \lambda_j$), so it has n eigenspaces. Each eigenspace is formed by an eiguvector and zero vector. Since each eigenspace has 1 linearly indep. vector, the dimension is 1.

b)
- Remember: $\lambda . A = v A$ for a square matrix.
- A^TA is a square matrix. $(A^T_{n\times m} A_{m\times n} = A^TA_{n\times n})$
- A^TA is a symmetric matrix.
  ↳ $(A^TA)^T = A^T A^{TT} = A^TA$
- Let's call $A^TA = B$ for simplicity.
- $v_1 \bullet v_2 = v_1^T v_2$.

$$\lambda_1 v_1^T v_2 = (\lambda_1 v_1)^T v_2 = (B v_1)^T v_2 = v_1^T B^T v_2 = v_1^T B v_2 = v_1^T \lambda_2 v_2$$

$$\lambda_1 v_1^T v_2 = v_1^T \lambda_2 v_2$$

$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0}(v_1^T v_2) = 0 \Rightarrow v_1^T v_2 = 0 \,//$$

(2)

c) $Av_1 \cdot Av_1 = (Av_1)^T Av_1 = v_1^T \underbrace{A^T A v_1} = v_1^T \lambda_1 v_1$

$= \lambda_1 \underbrace{\|v_1\|^2}_{1} = \lambda_1$

(2) function SVD(A):

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    ATA = transpose(A) * A
    ATA_lambdas, ATA_vectors = eig(ATA)
    Sort_decreasing (ATA_lambdas, ATA_vectors)
    singular_values = sqrt(ATA_lambdas)
    Sigma = diagonal_matrix(singular_values)
    VT = transpose(ATA_vectors)
    # Ui = 1/σi A vi
    U = construct_U(singular values, A, ATA_vectors)
    return U, Sigma, VT
```

Solve the following first-order initial value problems:

(a) $y' = t^2$, $y(0) = 1$ (b) $y' = t^2\,y$, $y(0) = 1$ (c) $y' = t^2\,y - \exp\left(\frac{t^3}{3}\right)$, $y(0) = 1$

a) $\frac{dy}{dt} = t^2 \Rightarrow dy = t^2 dt \Rightarrow \int dy = \int t^2 dt \rightarrow y = \frac{t^3}{3} + c$

$y(0)=1$, $y = \frac{t^3}{3} + c \Rightarrow c = 1 \longrightarrow \boxed{y = \frac{t^3}{3} + 1}$

b) $\frac{dy}{dt} = t^2 y \Rightarrow \frac{dy}{y} = t^2 dt \Rightarrow \int \frac{1}{y} dy = \int t^2 dt$

$\Rightarrow \ln|y| = \frac{t^3}{3} + c$

$\left\{ \begin{matrix} y(0) = 1 \\ t=0,\ y=1 \end{matrix} \right\}$ $\ln|1| = 0 + c \Rightarrow c = 0$

$\ln|y| = \frac{t^3}{3} \Rightarrow \boxed{y = e^{t^3/3}}$

c) $\underbrace{\frac{dy}{dt}}_{y'} = t^2 y - \underbrace{\exp\left(\frac{t^3}{3}\right)}$

this part must come from a non-homogeneous part which contains exponential term.

I will make an assumption about the form of y.

$y(t) = A(t) \cdot \exp(t^3/3)$

$\hookrightarrow y'(t) = A'(t) \exp(t^3/3) + A(t) \cdot t^2 \cdot \exp(t^3/3)$

Substitude these to the original equation.

$$y' = t^2 y - \exp\left(t^3/3\right)$$

$$A'(t)\exp\left(t^3/3\right) + \cancel{A(t)\,t^2\exp\left(t^3/3\right)} = \cancel{t^2 A(t)\exp\left(t^3/3\right)} - \exp\left(t^3/3\right)$$

$$A'(t) = -1$$

$$A'(t) = -1 \implies A(t) = -t + c$$

Using the initial assumption about the form of y:

$$y(t) = A(t)\cdot\exp\left(t^3/3\right)$$

$y(t) = (-t+c)\exp\left(t^3/3\right)$ and $y(0) = 1. \Rightarrow c = 1$

$$\boxed{\Rightarrow y(t) = (-t+1)\cdot\exp\left(t^3/3\right)}$$

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Ex: Find explicit particular solutions of the initial value problems:

$$\frac{dy}{dx} = 6e^{2x-y}, \quad y(0)=0$$

$$\frac{dy}{dx} = 6e^{2x} \cdot e^{-y}$$

$$e^{y}\,dy = 6e^{2x}\,dx$$

$$\int e^{y}\,dy = e^{y} + C_1$$

$$\int 6e^{2x}\,dx = 6\int e^{2x}\,dx = 6 \cdot \frac{1}{2} e^{2x} + C_2 = 3e^{2x} + C_2$$

$$e^{y} = 3e^{2x} + C$$

$y(0)=0$:

$$e^{0} = 3e^{0} + C \Rightarrow C = -2$$

$$e^{y} = 3e^{2x} - 2$$

$$\ln e^{y} = \ln(3e^{2x} - 2)$$

$$\underline{y = \ln(3e^{2x} - 2)}$$