$f(x) = \sin x$: Taylor Expansion around "0".

$f(x) = \sin x$
$f'(x) = \cos x$
$f^2(x) = -\sin x$
$f^3(x) = -\cos x$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n \qquad c = 0$$

$$f(x) = f(0) + f'(0)\,x + f^2(0)\frac{x^2}{2!} + f^3(0)\frac{x^3}{3!} + \cdots$$

$$f(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$f(x) = \frac{1}{1-x}$
$\downarrow$
$x \in (-1, 1)$

$$f(x) = \sum_{n=0}^{K} \frac{f^n(c)}{n!}(x-c)^n + E_{n+1}$$

$$E_{n+1} = \frac{f^{n+1}(\epsilon)}{(n+1)!}(x-c)^{n+1} \qquad \epsilon \in (c, x)$$

$c \in (a, b)$

Ex: Let's use Taylor Expansion to approximate $\sqrt{3}$.

why: $f(x) = x^2 - 3$

Consider $g(x) = \sqrt{x} = x^{1/2}$ $\quad (\sqrt{3} = g(3))$

Let's expand $g(x)$ around $c = 4$

$$g(x) = g(4) + g'(4)(x-4) + g^2(4)\frac{(x-4)^2}{2!} + g^3(4)\frac{(x-4)^3}{3!} + \cdots$$

$$g(x) = 2 + \frac{1}{4}(x-4) + \left(\frac{-1}{32}\right)\frac{(x-4)^2}{2} + E_3(x)$$

$g'(x) = \frac{1}{2}x^{-1/2}$
$g'(4) = \frac{1}{2} 4^{-1/2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
$g^2(x) = -\frac{1}{4}x^{-3/2}$
$g^2(4) = -\frac{1}{4} \cdot \frac{1}{8} = -\frac{1}{32}$

$$g(3) = 2 + \frac{1}{4}(-1) - \frac{1}{32} \cdot \frac{1}{2}$$

$$= 2 - \frac{1}{4} - \frac{1}{64} = 2 - \frac{17}{64} = 1.736\ldots$$

$\sqrt{3} = 1.734\ldots$

Bisection Method: Find a root of function $f(x)$:

The Mean Value Theorem: Suppose $f(x)$ is continuous on $[a,b]$. For any y in $f(a) \le y \le f(b)$ there exists $c \in [a,b]$ such that $f(c) = y$.

Ex: $f(x) = x^2 - 3$ we want to find a root of $f(x)$.

Find "a" and "b" so that $f(x)$ is cont on $[a,b]$

and $f(a)f(b) < 0$ or $f(a) < 0 < f(b)$

$a = 1$
$b = 2$ satisfy what we need.

$f(a) = -2$
$f(b) = 1$

there exist $c \in (a,b) = (1,2)$ such that $f(c) = 0$

$f(a) < c < f(b)$

$a \qquad c = \frac{a+b}{2} \qquad b$

$f(a) < 0 \qquad f(b) > 0$

or $f(a)f(b) < 0$

$r \to r$ is the real root

$r - c < \frac{b-a}{2}$

$f(1) < 0 \qquad f\left(\frac{3}{2}\right) < 0 \qquad f(2) > 0$

$$f\left(\tfrac{3}{2}\right) = \frac{9}{4} - 3 = -\frac{3}{4} < 0$$

Then I can say that the root lies in $\left(\frac{3}{2}, 2\right)$

$$\frac{\frac{3}{2} + 2}{2} = \frac{7}{4}$$

$f\left(\frac{3}{2}\right) < 0 \qquad \frac{7}{4} \qquad f(2) > 0$

$f\left(\frac{7}{4}\right) > 0$

$$f\left(\tfrac{7}{4}\right) = \frac{49}{16} - 3 = \frac{1}{16}$$

So the root must lie in $\left(\frac{3}{2}, \frac{7}{4}\right)$

$\frac{3}{2} \qquad \frac{7}{4}$

$a < b$

Steps of Bisection method:

1.) Find a, b such $f(a)f(b) < 0$

2.) Determine which ~~one~~ of $f(a)f\left(\frac{a+b}{2}\right)$ or $f(b)f\left(\frac{b+a}{2}\right)$ is negative.

then assign new a, b so that $f(a)f(b) < 0$

then proceed!

Error Computation:

$a_0 \qquad b_0$
$\wedge \qquad \vee$
$a_1 \qquad b_1$
$\wedge \qquad \vee$
$a_2 \qquad b_2$
$\vdots \qquad \vdots$
$a_n \qquad b_n$

$a_{n-1} \qquad b_{n-1}$
$\downarrow$
$a_n \qquad b_n$

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}$$

$$= \frac{b_{n-2} - a_{n-2}}{4} = \cdots = \frac{b_0 - a_0}{2^n}$$

when I stop at n^{th} step

$$|b_n - a_n| = \left|\frac{b_0 - a_0}{2^n}\right|$$