BLG 210E Engineering Mathematics
2023 Fall, Quiz 4

04.12.2023

Name: ______________________________

Find the orthogonal projection of $\begin{bmatrix}2\\3\\5\end{bmatrix}$ $\vec{v}$ onto

(a) $\begin{bmatrix}1\\0\\0\end{bmatrix}$ (b) $\begin{bmatrix}0\\1\\0\end{bmatrix}$ (c) span$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ (d) $\begin{bmatrix}1\\1\\0\end{bmatrix}$ (e) span$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

a) $\text{proj}_u v = \dfrac{v \cdot u}{u \cdot u} u = \dfrac{\begin{bmatrix}2 & 3 & 5\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}}{1} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}$ //

b) $\dfrac{\begin{bmatrix}2 & 3 & 5\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}}{1} \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\3\\0\end{bmatrix}$ //

c) u_1 and u_2 (Col A) are orthogonal

$\text{proj}_u v = \underbrace{\dfrac{v \cdot u_1}{u_1 \cdot u_1} u_1}_{a} + \underbrace{\dfrac{v \cdot u_2}{u_2 \cdot u_2} u_2}_{b} = \begin{bmatrix}2\\0\\0\end{bmatrix} + \begin{bmatrix}0\\3\\0\end{bmatrix} = \begin{bmatrix}2\\3\\0\end{bmatrix}$ //

$A(A^TA)^{-1}A^Tv$ can also be used.

d) $\dfrac{\begin{bmatrix}2 & 3 & 5\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}}{2} \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}5/2\\5/2\\0\end{bmatrix}$ //

e) u_1 and u_2 are not orthogonal so $\text{proj} = A(A^TA)^{-1}A^Tv$:

$\text{proj} = \begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}2\\3\\5\end{bmatrix} =$

$= \begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}^{-1}\begin{bmatrix}2\\5\end{bmatrix}$

$= \begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}2\\3\\0\end{bmatrix}$ //

Since (c) and (e) have the same subspace; the result also equals to (c).

1. (40 pts) Let S be a set of orthogonal vectors in $\mathbb{R}^6$. Can S contain more than 6 vectors? Justify your answer.

2. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 1 & -1 & 11 \\ 2 & 6 & -2 \end{pmatrix}$$

   (a) (10 pts) Find a basis for Col$A$.

   (b) (20 pts) Find an orthogonal basis for Col$A$.

   (c) (30 pts) Find the projection of $v = (4, 1, 8)$ onto Col$A$.

1) Orthoganal vectors are independent of each other. Therefore, their number cannot exceed n if they are in R^n. (Invertible Matrix Theorem). Orthoganal vectors are in R^6 so S can not contain more than 6 vectors.

2) a) $\begin{pmatrix} 1 & 3 & -1 \\ 1 & -1 & 11 \\ 2 & 6 & -2 \end{pmatrix} \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 3 & -1 \\ 0 & -4 & 12 \\ 0 & 0 & 0 \end{pmatrix}$

$\text{Col } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \right\}$

b) $\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1} \vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} - \dfrac{[3\ -1\ 6]\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -10/3 \\ 4/3 \end{bmatrix}$

c) $\text{proj}_{\text{col}A} \vec{v} = A(A^T A)^{-1} A^T \vec{v} =$

$= \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 2 & 6 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 2 & 6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} =$

$= \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 23/40 & -7/40 \\ -7/40 & 3/40 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ //

$\left(\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} = \dfrac{7}{4} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \dfrac{3}{4} \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \right)$

a) Orthogonally diagonalize $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ given $\lambda = -2, 7$.

b) Construct a spectral decomposition of A.

a) $\lambda = 7$:

$$\begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - R_1/2} \begin{bmatrix} -4 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vec{v_1} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ v_1 and v_2 are not orthogonal, Gram-Schmidt is applied:

$\vec{v_1} = \vec{q_1} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$\vec{q_2} = \vec{v_2} - \text{proj}_{\vec{q_1}} \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{[1\ 0\ 1]\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}{5} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$\vec{q_2} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$ (linear combination of $\vec{v_1}$ and $\vec{v_2}$).

$\lambda = -2$:

$$\begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_3} \begin{bmatrix} 1 & -4 & -1 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - 4R_1]{R_2 \leftarrow R_2 + 2R_1}$$

$$\begin{bmatrix} 1 & -4 & -1 \\ 0 & 0 & 0 \\ 0 & 18 & 9 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{R_3 \leftarrow R_3/9} \begin{bmatrix} 1 & -4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ \vec{v_3} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$U_1 = \dfrac{V_1}{\|v_1\|} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \ U_2 = \begin{bmatrix} 4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}, \ U_3 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$

$$A = P_1 D_1 P^T = \begin{bmatrix} -1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 4/\sqrt{45} & 2/\sqrt{45} & 5/\sqrt{45} \\ -2/3 & -1/3 & 2/3 \end{bmatrix}$$

b) $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T =$

$= 7 u_1 u_1^T + 7 u_2 u_2^T - 2 u_3 u_3^T =$

$$= 7 \begin{bmatrix} 1/5 & -2/5 & 0 \\ -2/5 & 4/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 7 \begin{bmatrix} 16/45 & 8/45 & 20/45 \\ 8/45 & 4/45 & 10/45 \\ 20/45 & 10/45 & 25/45 \end{bmatrix}$$

$$- 2 \begin{bmatrix} 4/9 & 2/9 & -4/9 \\ 2/9 & 1/9 & -2/9 \\ -4/9 & -2/9 & 4/9 \end{bmatrix}$$