Taylor Expansion

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)\cdot(x-a)^2}{2!} + \frac{f^{(3)}(a)\cdot(x-a)^3}{3!} + \cdots + \frac{f^{(n)}(a)\cdot(x-a)^n}{n!} + \overset{\text{ERROR}}{R_n(x)}$$

$$R_n(x) = \frac{f^{(n+1)}(\varepsilon)\cdot(x-a)^{n+1}}{(n+1)!} \qquad \varepsilon \in (a-\delta, a+\delta)$$

- Approximation of $\sqrt{10}$ using taylor expansion. $f(x) = \sqrt{x}$, approximate around $a = 9$.

$$f(x) = f(9) + f'(9)\cdot(x-9) + \frac{f''(9)\cdot(x-9)^2}{2!} + \frac{f^{(3)}\cdot(x-9)^3}{3!} + \cdots$$

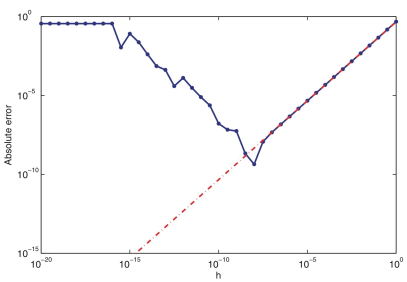
$f(x) = x^{1/2}$ $\quad f'(x) = \frac{1}{2}\cdot x^{-1/2}$

$f''(x) = \frac{-1}{4} x^{-3/2}$ $\quad f^{(3)}(x) = \frac{3}{8} x^{-5/2}$

$$f(x) = 3 + \frac{1}{6}\cdot(x-9) + \left(\frac{-1}{108}\right)\cdot\frac{(x-9)^2}{2} + \frac{1}{8\cdot 3^4}\cdot\frac{(x-9)^3}{6} + R_3$$

$$f(10) = \sqrt{10} \cong 3 + \frac{1}{6} - \frac{1}{108} + \frac{1}{16\cdot 3^5} \cong 3{,}158$$

$$R_3 = f^4(\varepsilon)\cdot\frac{(10-9)^4}{4!} \rightarrow A.E = \frac{f^4(\varepsilon)}{24} = \left|\frac{-15}{16}\cdot\frac{1}{24}\cdot\frac{1}{\sqrt{\varepsilon^7}}\right| < \overbrace{\left|\frac{-15}{16}\cdot\frac{1}{24}\cdot\frac{1}{3^7}\right|}^{\text{absolute err.}}$$



* $$f'(x) = \frac{f(x_0+h) - f(x_0)}{h}$$

h 10^{-8} degerine yaklaştıkça $f'(x_0)$ asıl değeri ve yaklaşık değeri arasındaki hata oranı azalırken, 10^{-8} değerinden sonra hata oranı artmaya başlar.

IEEE - 754 Floating Point Representation

1-bit | 8-bit | 23-bit

Sign | Exponent | Mantissa

32-bit Number

1-bit | 11-bit | 52-bit

Sign | Exponent | Mantissa

64-bit Number

Rounding Unit: $\eta = \frac{1}{2}\beta^{1-t}$ $\quad \beta$: base, t: digit

A.E. → $\beta^{1-t}\cdot\beta^e$ for chopping
→ $\frac{1}{2}\beta^{1-t}\cdot\beta^e$ for rounding

R.E → β^{1-t} for chopping
→ $\frac{1}{2}\beta^{1-t}$ for rounding

- Represent 13,76 in IEEE-754 standards.

$$13.76 = 1101.\overline{11000010100011110101} \rightarrow 1.101\overline{11000010100011110101}\cdot 2^3$$

0.76 → 1.52 (MSB)	0.64 → 1.28	0.96 → 1.92	0.44 → 0.88 (LSB)
0.52 → 1.04	0.28 → 0.56	0.92 → 1.84	0.88 → 1.76
0.04 → 0.08	0.56 → 1.12	0.84 → 1.68	⋮
0.08 → 0.16	0.12 → 0.24	0.68 → 1.36	
0.16 → 0.32	0.24 → 0.48	0.36 → 0.72	
0.32 → 0.64	0.48 → 0.96	0.72 → 1.44	

sign → 0

exp → 3 + 127 = 130

man

0	10000010	10111000010100011110101

Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Error $\rightarrow |r - x_{n+1}| \leq C \cdot \underbrace{|r - x_n|}_{\text{the error of previous step}}{}^2$

r: root , this theorem satisfies if r is simple zero.

Bisection Method

General error: $\left| \frac{x_{n+1} - x_n}{x_{n+1}} \right|$

$f(x)$ fonksiyonunun $x \in [a,b]$ aralığındaki kökünü bul.

1) $f(a) \cdot f(b) < 0$

2) $x_n = \frac{a+b}{2}$

$f(x_n) \cdot f(a) < 0$ ise $\rightarrow$ $b = x_n$ kök $\in [a, x_n] = [a,b]$

$f(x_n) \cdot f(b) < 0$ ise $\rightarrow$ $a = x_n$ kök $\in [x_n, b] = [a,b]$

2. adımı tekrar et

the error in nth step is $\leq \frac{b_0 - a_0}{2^n}$

$$n = \left\lceil \log_2 \left(\frac{b-a}{2\,atol} \right) \right\rceil$$

Secant Method

$$f'(x) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} \rightarrow x_{n+1} = x_n - \frac{f(x_n) \cdot (x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Least Squares Method

$$V(a,b) = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$$

$$\frac{\partial V}{\partial a} = \sum_{i=1}^{n} -2 \, [y_i - (ax_i + b)] \cdot x_i = 0 \rightarrow \sum_{i=1}^{n} y_i x_i = a \cdot \sum_{i=1}^{n} x_i^2 + b \cdot \sum_{i=1}^{n} x_i$$

$$\frac{\partial V}{\partial b} = \sum_{i=1}^{n} -2 \, [y_i - (ax_i + b)] = 0 \rightarrow \sum_{i=1}^{n} y_i = a \cdot \sum_{i=1}^{n} x_i + n \cdot b$$