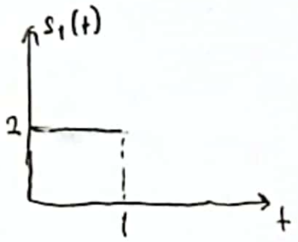
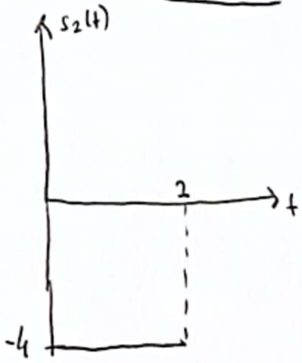
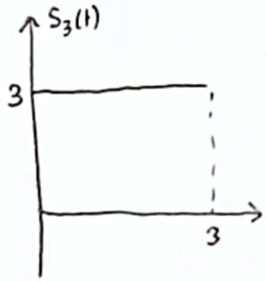
EE451 Recitation - 1

Answers

1.

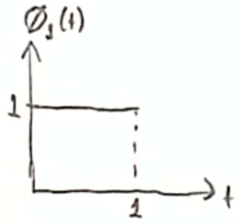






a) Find set of orthonormal basis function

$$\mathcal{E}_1 = \int_{-\infty}^{\infty} (s_1(t))^2 dt = 4 \qquad \varnothing_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}} = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{o.w} \end{cases}$$



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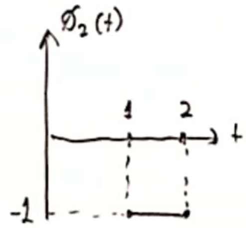
c_{21} : projection of $s_2(t)$ on $\varnothing_1(t)$

$$c_{21} = \langle S_2(t), \varnothing_1(t) \rangle = \int_{-\infty}^{\infty} s_2(t)\, \varnothing_1^*(t)\, dt = -4$$

$y_2(t) \rightarrow$ orthogonal part of $s_2(t)$ (but not orthonormal)

$$y_2(t) = s_2(t) - c_{21}\, \varnothing_1(t) = \begin{cases} -4 & 1 \leq t \leq 2 \\ 0 & \text{o.w} \end{cases}$$

$$\mathcal{E}_2 = \int_{-\infty}^{\infty} (y_2(t))^2 dt = 16 \qquad \varnothing_2(t) = \frac{y_2(t)}{\sqrt{\mathcal{E}_2}} = \begin{cases} -1 & 1 \leq t \leq 2 \\ 0 & \text{o.w} \end{cases}$$

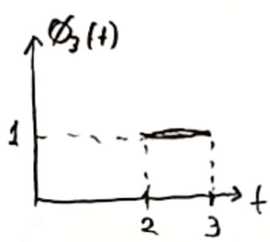


$$c_{31} = \langle S_3(t), \varnothing_1(t) \rangle = \int_{-\infty}^{\infty} s_3(t)\, \varnothing_1(t)\, dt = 3$$

$$c_{32} = \langle S_3(t), \varnothing_2(t) \rangle = \int_{-\infty}^{\infty} S_3(t)\, \varnothing_2(t)\, dt = -3$$

$$y_3(t) = S_3(t) - c_{31}\, \varnothing_1(t) - c_{32}\, \varnothing_2(t) = \begin{cases} 3, & 2 \leq t \leq 3 \\ 0, & \text{o.w} \end{cases}$$

$$\mathcal{E}_3 = \int_{-\infty}^{\infty} (y_3(t))^2 dt = 9 \qquad \varnothing_3(t) = \frac{y_3(t)}{\sqrt{\mathcal{E}_3}} = \begin{cases} 1, & 2 \leq t \leq 3 \\ 0, & \text{o.w} \end{cases}$$



$$S_1 = [\sqrt{\mathcal{E}_1} \;\; 0 \;\; 0] = [2 \;\; 0 \;\; 0] \qquad s_1(t) = 2\varnothing_1(t)$$

$$S_2 = [c_{21} \;\; \sqrt{\mathcal{E}_2} \;\; 0] = [-4 \;\; 4 \;\; 0] \qquad s_2(t) = -4\varnothing_1(t) + 4\varnothing_2(t)$$

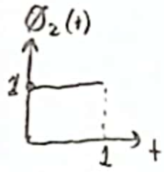
$$S_3 = [c_{31} \;\; c_{32} \;\; \sqrt{\mathcal{E}_3}] = [3 \;\; -3 \;\; 3] \qquad s_3(t) = 3\varnothing_1(t) - 3\varnothing_2(t) + 3\varnothing_3(t)$$

2. You can choose starting signal at any order, but selecting the shortest duration signal generally makes the computation easier.

In the lecture notation confused some of the students so I will give the orthonormal set ids same as the matching signal ids (for $s_2(t)$ it will be $\phi_2(t)$)

We can start from $s_2(t)$:

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}} \qquad E_2 = \int_{-\infty}^{\infty} s_2^2(t)\,dt = 4 \qquad \phi_2(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{o.w} \end{cases}$$



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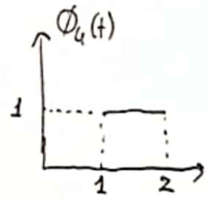
we can select the second signal as $s_4(t)$ for easier computation:

c_{42}: projection of $s_4(t)$ on to $\phi_2(t)$ $\int_{-\infty}^{\infty} s_4(t)\,\phi_2^*(t)\,dt = 2$

$$y_4(t) = s_4(t) - c_{42}\,\phi_2(t) = \begin{cases} 2, & 1 \le t < 2 \\ 0, & \text{o.w} \end{cases}$$

$$E_4 = \int_{-\infty}^{\infty} (s_4(t))^2\,dt = 4 \qquad \phi_4(t) = \begin{cases} 1, & 1 \le t \le 2 \\ 0, & \text{o.w} \end{cases}$$



Select $s_1(t)$

c_{12}: projection of $s_1(t)$ on to $\phi_2(t) = \int_{-\infty}^{\infty} s_1(t)\,\phi_2^*(t)\,dt = 2$

c_{14}: " " " " " $\phi_4(t) = \int_{-\infty}^{\infty} s_1(t)\,\phi_4^*(t)\,dt = 2$

$$y_1(t) = s_1(t) - c_{12}\,\phi_2(t) - c_{14}\,\phi_4(t) = \begin{cases} 2, & 2 \le t \le 3 \\ 0, & \text{o.w} \end{cases}$$

$$E_1 = \int_{-\infty}^{\infty} (s_1(t))^2\,dt = 4 \qquad \phi_1(t) = \frac{y_1(t)}{\sqrt{E_1}} = \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{o.w} \end{cases}$$

$s_3(t)$ can fully be represented by found orthonormal set

Now you may rename $\phi_2(t) \longrightarrow \phi_1(t)$

$\phi_4(t) \longrightarrow \phi_2(t)$

$\phi_1(t) \longrightarrow \phi_3(t)$ to avoid confusion

$s_1 = [\,2 \quad 2 \quad 2\,]$

$s_2 = [\,2 \quad 0 \quad 0\,]$

$s_3 = [\,0 \quad -2 \quad -2\,]$

$s_4 = [\,2 \quad 2 \quad 0\,]$

distance between any vector:

$$d_{mn} = \sqrt{\|s_m - s_n\|^2} \Rightarrow \quad d_{12} = \sqrt{8} \quad d_{23} = 2\sqrt{3}$$

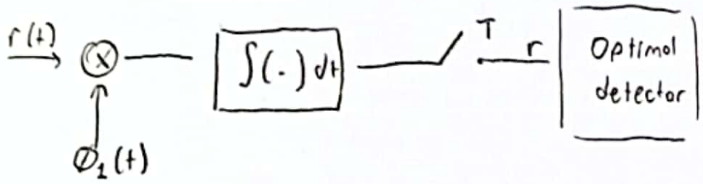
$$d_{13} = 6 \quad d_{24} = 2$$

$$d_{14} = 2 \quad d_{34} = 2\sqrt{6}$$

$$\boxed{d_{min} = \min_{m \ne n}(d_{mn}) = 2}$$

3. $\phi_1(t) = \dfrac{s_1(t)}{\sqrt{\mathcal{E}_1}}$ $\quad \mathcal{E}_1 = A^2 T \quad \phi_1(t) = \begin{cases} \frac{1}{\sqrt{T}}, & 0 \leq t \leq T \\ 0, & \text{o.w} \end{cases}$

since we have only one dimension $\quad s_0 = 0 \quad s_1 = A\sqrt{T}$



$$r = \int_0^T r(t)\,\phi_1(t)\,dt \quad \text{where} \quad r(t) = s_m(t) + n(t)$$

$$= \int_0^T s_m(t)\,\phi_1(t)\,dt + \int_0 n(t)\,\phi_1(t)\,dt$$

$$= s_m + \frac{1}{\sqrt{T}} \int_0^T n(t)\,dt$$

has zero mean and variance of :

$$\sigma_n^2 = E\left[\frac{1}{\sqrt{T}} \int_0^T n(t)\,dt + \frac{1}{\sqrt{T}} n(\tau)\,d\tau \right]$$

$$\sigma_n^2 = \frac{1}{T} \int_0^T \int_0^T E[n(t)\,n(\tau)]\,dt\,d\tau$$

$$= \frac{1}{T} \int_0^T \int_0^T \frac{N_0}{2}\,\delta(t-\tau)\,dt\,d\tau$$

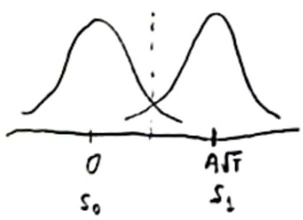
$$= \frac{1}{T} \int_0^T \frac{N_0}{2}\,dt = \frac{N_0}{2}$$

AWGN variance cheat sheete yaz

Gaussian pdf with mean s_m and $\sigma = \sqrt{\frac{N_0}{2}}$ $\qquad f(r|s_m) = \dfrac{1}{\sqrt{\pi N_0}}\, e^{-\frac{(r-s_m)^2}{N_0}}$

$$\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(r-\mu)^2}{2\sigma^2}}$$

Since prior probabilities are equal



$$f(r|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} f(r|s_1)$$

$$\frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{r^2}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{(r-A\sqrt{T})^2}{N_0}} \Rightarrow e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

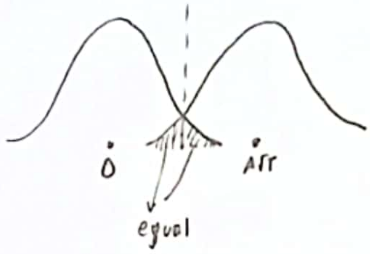
$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2} A\sqrt{T}$$

optimum threshold

optimum threshold is $\frac{1}{2} A\sqrt{T}$

3.b. $P_e = P(e|s_0)P(s_0) + P(e|s_1)P(s_1)$ probability of error

$$= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} f(r|s_0)\,dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} f(r|s_1)\,dr$$



→ Due to symmetricity we can compute one of them and multiply by 2

$$= 2.\frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} f(r|s_0)\,dr = \int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{\frac{-r^2}{N_0}}\,dr$$

→ convert to standard normal

std normal $z = \frac{r-\mu}{\sigma}$

$r = \sqrt{\frac{N_0}{2}}\,z$

$dr = \sqrt{\frac{N_0}{2}}\,dz$

$r = \frac{1}{2}A\sqrt{T} \Rightarrow \sqrt{\frac{N_0}{2}}z = \frac{1}{2}A\sqrt{T}$

$z = \frac{A\sqrt{T}}{\sqrt{2N_0}}$

$$= \int_{\frac{A\sqrt{T}}{\sqrt{2N_0}}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{\frac{-\frac{N_0}{2}z^2}{N_0}} \frac{\sqrt{N_0}}{\sqrt{2}}\,dz$$

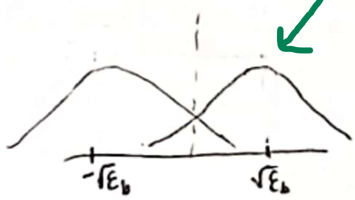
$$P_e = \int_{\frac{A\sqrt{T}}{\sqrt{2N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = Q\left[\frac{A\sqrt{T}}{\sqrt{2N_0}}\right]$$

$\mathcal{E}_b = \mathcal{E}_{avg} = \frac{A^2T}{2} \Rightarrow$

$$P_e = Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right]$$

antipodal demek birisi A ise birisi -A o yüzden bu signal constellation grafii çkyor

4. Maximum likelihood decision rule selects m that maximizes $f(r|s_m)$



$\mathcal{E}_{bavg} = \frac{\mathcal{E}_{avg}}{k} = A^2T$

so $f(r|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} f(r|s_1)$

$$e^{-\frac{(r+\sqrt{E_b})^2}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} e^{-\frac{(r-\sqrt{E_b})^2}{N_0}}$$

$$e^{\frac{(r-\sqrt{E_b})^2-(r+\sqrt{E_b})^2}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

yanlis

$$\frac{-4r\sqrt{E_b}}{N_0} \underset{s_1}{\overset{s_0}{\gtrless}} \cancel{0} \qquad r \underset{s_0}{\overset{s_1}{\gtrless}} \cancel{0}$$

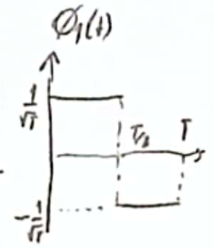
b) $P(e) = P(e|s_1)P(s_1) + P(e|s_0)P(s_0)$

$$P_e = p\int_{-\infty}^{0}\frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{E_b})^2/N_0}\,dr + (1-p)\int_{0}^{\infty}\frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{E_b})^2/N_0}$$

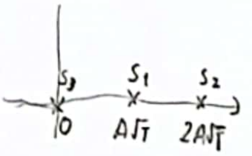
$$P_e = \frac{p}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\sqrt{E_b}}{\sqrt{N_0/2}}} e^{-\frac{z^2}{2}}\,dz + (1-p)\int_{\frac{-\sqrt{E_b}}{\sqrt{N_0/2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = p\,Q\left[\sqrt{\frac{2E_b}{N_0}}\right] + (1-p)\,Q\left[\sqrt{\frac{2E_b}{N_0}}\right] = Q\left[\sqrt{\frac{2E_b}{N_0}}\right]$$

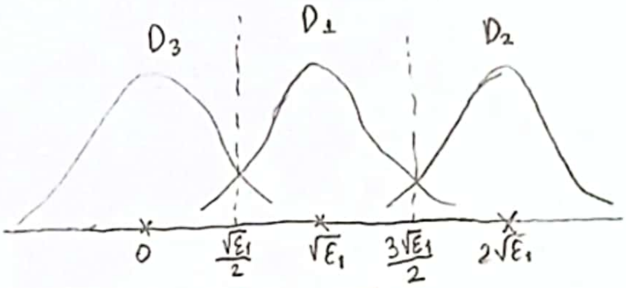
5. a. Select s_1 & normalize

$$\mathcal{E}_1 = \frac{A^2T}{3} + \frac{2A^2T}{3} = A^2T \qquad \phi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}} = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{3} \\ -\frac{1}{\sqrt{T}} & \frac{T}{3} \leq t \leq T \end{cases}$$



$$s_1 = \sqrt{\mathcal{E}_1} = A\sqrt{T}$$
$$s_2 = 2\sqrt{\mathcal{E}_1} = 2A\sqrt{T}$$
$$s_3 = 0$$





b) Since equiprobable

$$D_3 = \{r \in R : f(r|s_3) > f(r|s_1),\ f(r|s_3) > f(r|s_2)\} = \{r \in R : r < \tfrac{\sqrt{\mathcal{E}_1}}{2}\}$$

$$D_1 = \{r \in R : f(r|s_1) > f(r|s_2),\ f(r|s_1) > f(r|s_3)\} = \{r \in R : \tfrac{\sqrt{\mathcal{E}_1}}{2} < r < \tfrac{3\sqrt{\mathcal{E}_1}}{2}\}$$

$$D_2 = \{r \in R : f(r|s_2) > f(r|s_1),\ f(r|s_2) > f(r|s_3)\} = \{r \in R : r > \tfrac{3\sqrt{\mathcal{E}_1}}{2}\}$$

c) $$P_e = \sum_{m=1}^{M} P_m \left(\sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} \int_{D_{m'}} \left(p(r|s_m)\, dr \right) \right)$$

$$P_e = \frac{1}{3} \int_{D_3} p(r|s_1)\, dr + \frac{1}{3} \int_{D_2} p(r|s_1)\, dr + \frac{1}{3}\left(\int_{D_1} p(r|s_3)\, dr + \int_{D_2} p(r|s_3)\, dr \right) + \frac{1}{3}\left(\int_{D_3} p(r|s_2)\, dr + \int_{D_1} p(r|s_2)\, dr \right)$$

$$\frac{1}{3} Q\left[\sqrt{\frac{\mathcal{E}_1}{2N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{\mathcal{E}_1}{2N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{\mathcal{E}_1}{2N_0}}\right] \qquad + \frac{1}{3} Q\left[\sqrt{\frac{\mathcal{E}_1}{2N_0}}\right]$$

$$P_e = \frac{4}{3} Q\left[\sqrt{\frac{\mathcal{E}_1}{2N_0}}\right] \longmapsto$$ we need it interms of SNR:

$$\mathcal{E}_{avg} = \frac{1}{3}\left(0 + A^2T + 4A^2T \right)$$

$$\mathcal{E}_{avg} = \frac{1}{3}\left(0 + \mathcal{E}_1 + 4\mathcal{E}_1 \right) = \frac{5\mathcal{E}_1}{3}$$

$$\mathcal{E}_{bavg} = \frac{\mathcal{E}_{avg}}{\log_2 M} = \frac{5\mathcal{E}_1}{3\log_2 3} \Rightarrow \mathcal{E}_1 = 0.95\,\mathcal{E}_{bavg}$$

$$P_e = \frac{4}{3} Q\left[\frac{0.95\,\mathcal{E}_{bavg}}{2N_0}\right]$$

d) $R = R_s \log_2 M = 3000 \log_2 3$

Q6. a) Structure of optimal receiver

optimum receiver metric to maximize

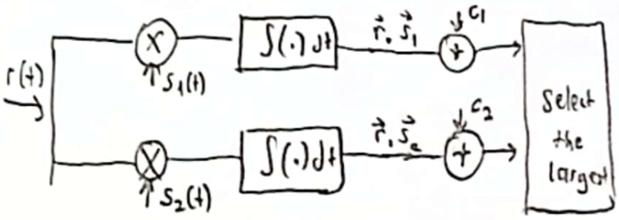
$$f(\vec{r}|\vec{s}_m) = \prod_{k=1}^{N} f(r_k|s_{mk})$$

$$C(r, s_m) = \underbrace{\int_{-\infty}^{\infty} r(t)\, s_m(t)\, dt}_{\text{correlator}} \underbrace{- \frac{1}{2}\int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2}\ln P(s_m)}_{\text{bias term}}$$

Do not write

Vector form:

$$C(\vec{r}, \vec{s}_m) = \vec{r}\cdot\vec{s}_m - \frac{1}{2}|s_m|^2 + \frac{N_0}{2}\ln(P(s_m))$$



Since $s_1(t) = -s_2(t)$ energy of the signals are same

$$\int_{-\infty}^{\infty} r(t)\, s_1(t)\, dt + \frac{N_0}{2}\ln P(s_1) \underset{s_2}{\overset{s_1}{\gtrless}} \int_{-\infty}^{\infty} -r(t)\, s_1(t)\, dt + \frac{N_0}{2}\ln P(s_2)$$

$$\int_{-\infty}^{\infty} r(t)\, s_1(t)\, dt \underset{s_2}{\overset{s_1}{\gtrless}} \frac{N_0}{4}\ln\frac{P_2}{P_1}$$

b) $P_e = ?$

first $P(e|s_1)$

$$\int_{-\infty}^{\infty} (s_1(t) + n(t))\,(s_1(t))\, dt = \int_0^T (s_1(t))^2 dt + \underbrace{\int_0^T n(t)\, s_1(t)\, dt}$$

$\mathcal{E}_s + n \rightarrow$ zero mean gaussian with variance

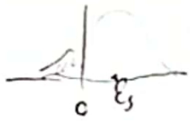
$$\sigma_n^2 = E\left[\int_0^T n(\tau)s_1(\tau)d\tau \int_0^T n(\upsilon)s_1(\upsilon)d\upsilon\right]$$

$$= \int_0^T\int_0^T s_1(\tau)\, s_1(\upsilon)\, E[n(\tau)n(\upsilon)]\, d\tau\, d\upsilon$$

$E[n(\tau)n(\upsilon)] \rightarrow \frac{N_0}{2}\delta(\tau-\upsilon)$

$$\frac{N_0}{2}\int_0^T (s_1(\tau))^2 d\tau$$

$$\sigma_n^2 = \frac{N_0}{2}\mathcal{E}_s$$

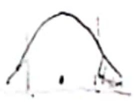


$$P(e|s_1) = \int_{-\infty}^{\frac{N_0}{4}\ln\frac{P_2}{P_1} - \mathcal{E}_s} \frac{1}{\sqrt{\pi N_0 \mathcal{E}_s}}\, e^{-\frac{x^2}{N_0\mathcal{E}_s}}\, dx$$

$$r = \sqrt{\frac{N_0\mathcal{E}_s}{2}}\, z$$

$$z = \left(\frac{N_0}{4}\ln\frac{P_2}{P_1} - \mathcal{E}_s\right) \Big/ \sqrt{\frac{N_0\mathcal{E}_s}{2}}$$

$$Q\left(\frac{\mathcal{E}_s\sqrt{2}}{\sqrt{N_0\mathcal{E}_s}} - \frac{\sqrt{N_0}}{2\sqrt{2}}\ln\frac{P_2}{P_1}\right)$$



$$P(e|s_1) = Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}} - \frac{\sqrt{N_0}}{2\sqrt{2}}\ln\frac{P_2}{P_1}\right)$$

$$P(e) = P_1\, Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}} - \frac{\sqrt{N_0}}{2\sqrt{2}}\ln\frac{P_2}{P_1}\right) + (1-P_1)\, Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}} + \frac{\sqrt{N_0}}{2\sqrt{2}}\ln\frac{P_2}{P_1}\right)$$

Q.7) Equiprobable binary antipodal (as we did in Q4)

a)

$r = n \pm \sqrt{E_b}$

$P(e) = P(e|s_0)P(s_0) + P(e|s_1)P(s_1)$

$$P_e = \frac{1}{2}\int_0^{\infty} f(r|s_0)\,dr + \frac{1}{2}\int_{-\infty}^{0} f(r|s_1)\,dr$$

$$z = \frac{x-\mu}{\sigma}$$

$$x = \sigma z + \mu$$

$$r = \sqrt{\frac{N_0}{2}}z - \sqrt{E_b}$$

this part is solved in Q4.b

$$P_e = \int_{\sqrt{\frac{N_0}{2}}z - \sqrt{E_b} = 0}^{\infty} e^{-\frac{z^2}{2}}\,dz$$

$$P_e = Q\left[\sqrt{\frac{2E_b}{N_0}}\right]$$

b) Optimal Detector does not change for N_0 or N_1 decision region is at 0 regardless

$P_1 = Q\left[\sqrt{\frac{2E_b}{N_1}}\right]$ $\quad P_1 > P_e$

c) same, does not effect the detector so $P_{e_1} = Q\left[\sqrt{\frac{2E_0}{N_1}}\right] = P_1$

d)

Decision region $\quad \frac{P_0}{\sqrt{\pi N_0}} e^{-(r-\sqrt{E_s})^2/N_0} = \frac{P_1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{E_s})^2/N_0}$ $\quad P(s_0)\,p(r|s_0) = P(s_1)\,p(r|s_1)$

$$e^{-(r-\sqrt{E_s})^2/N_0 + (r+\sqrt{E_s})^2/N_0}$$

$$e^{4r\sqrt{E_s}/N_0} = \frac{P_1}{P_0}$$

$$r_{th} = \frac{N_0}{4\sqrt{E_s}} \ln\frac{(1-p)}{p}$$ → threshold depends on N_0

$P_1 > P_{e_1} > P_e$

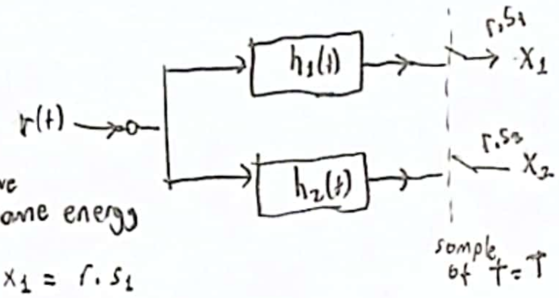
due to sub optimal solution

due to higher noise level

Q.8) a) Assume we use matched filter

$h_1(t) = s_1(T-t)$

$h_2(t) = s_2(T-t)$



$\hat{m} = \arg\max\left[\eta_m + r.s_m\right]$

where $\eta_m = \frac{N_0}{2}\ln P_m - \frac{1}{2}\mathcal{E}_m$

you can discard these terms: if equiprobable, if equal energy

since singnals are equiprobable & have same energy

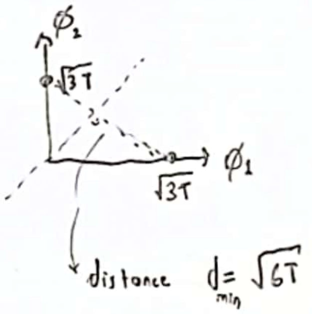
$$x_1 \underset{s_2}{\overset{s_1}{\gtrless}} x_2 \qquad \text{where } x_1 = r.s_1, \quad x_2 = r.s_2$$

b) $P(e) = ?$ if $\frac{E}{N_0} = 4$

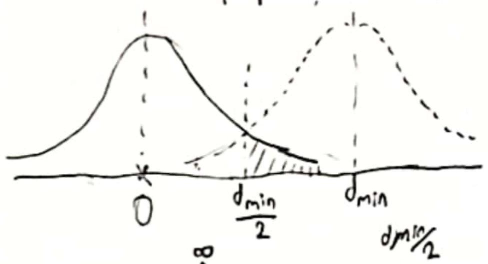
$$E_2 = E_1 = \int_0^T (1)^2 dt + \int_T^{2T} (-1)^2 dt + \int_{2T}^{3T} (1)^2 dt = 3T = E$$ given as signal energy

$\phi_1(t) = \frac{s_1(t)}{\sqrt{3T}}$ $\quad \phi_2(t) = \frac{s_2(t)}{\sqrt{3T}}$

since orthogonal



Since binary & equiprobable: we can regard this as:



$$P_e = \frac{1}{2}\int_{\frac{d_{min}}{2}}^{\infty} f(r|s_1)dr + \frac{1}{2}\int_{-\infty}^{d_{min}/2} f(r|s_2)\,dr$$

same

$$P_e = \int_{\frac{d_{min}}{2}}^{\infty} f(r|s_1)\,dr =$$

$$P_e = \int_{\frac{d_{min}}{2}/\sqrt{\frac{N_0}{2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = Q\left[\frac{d_{min}/2}{\sqrt{N_0/2}}\right] = Q\left[\sqrt{\frac{3T}{N_0}}\right] = Q\left[\sqrt{\frac{E}{N_0}}\right]$$

Note: $Q(x)$ can be denoted as $\frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$ in some solutions.

Q.9 Optimum detection of sinusoidal signal $s(t) = \sin\left(\frac{8\pi t}{T}\right) \quad 0 \le t \le T$

a) Determine correlator output of noiseless input

$$y(T) = \int_0^T r(\tau)\, s(\tau)\, d\tau$$

$$y(T) = \int_0^T s^2(\tau)\, d\tau = \int_0^T \sin^2\left(\frac{8\pi\tau}{2}\right) d\tau = \int_0^T \frac{1}{2}\left[1 - \cancel{\cos\left(\frac{16\pi\tau}{T}\right)}\right] d\tau$$

$$\boxed{\frac{T}{2}}$$

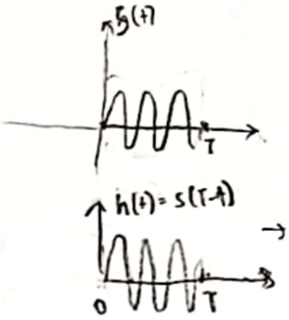
b) Determine the corresponding matched filter

$$h(t) = s(T-t)$$

$$y(t) = \int_{-\infty}^{\infty} r(\lambda)\, h(t-\lambda)\, d\lambda$$

$\sin x \cdot \sin y = -\frac{1}{2}\left[\cos(x+y) - \cos(x-y)\right]$

$$y(t) = \int_{-\infty}^{\infty} s(\lambda)\, s(T-t+\lambda)\, d\lambda = \int_{-\infty}^{\infty} \sin\left(\frac{8\pi\lambda}{T}\right) \sin\left(\frac{8\pi(T-t+\lambda)}{T}\right) d\lambda$$



→ Convolve with this

$$= \frac{1}{2}\int_{-\infty}^{\infty} \cos\left[\frac{8\pi(T-t)}{T}\right] d\lambda - \frac{1}{2}\int_{-\infty}^{\infty} \cos\frac{8\pi(T-t+2\lambda)}{T}\, d\lambda$$

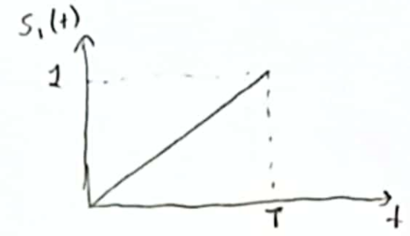
$$= \frac{1}{2}\cos\left[\frac{8\pi(T-t)}{T}\right]\lambda\Big|_0^T - \frac{T}{32\pi}\sin\frac{8\pi(T-t+2\lambda)}{T}\Big|_{\lambda=0}^{\lambda=T}$$

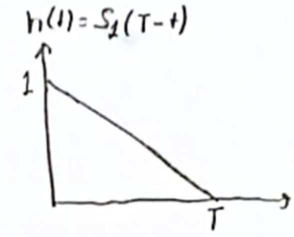
$$= \frac{T}{2}\cos\left[\frac{8\pi(T-t)}{T}\right] - \frac{T}{32\pi}\sin\frac{8\pi(3T-t)}{T} + \frac{T}{32\pi}\sin\frac{8\pi(T-t)}{T}$$

at $t = T$ $\boxed{y(T) = \frac{T}{2}}$

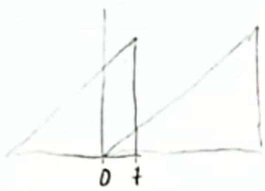
→ Results are the same

Q10.





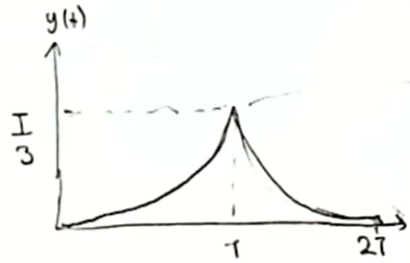
$$s_1(\tau) = \begin{cases} \tau/T & 0<\tau<T \\ 0 & \text{o.w.} \end{cases}$$

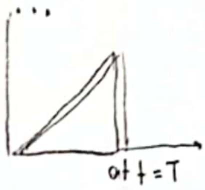


$$y_1(t) = \int_0^\infty s_1(\tau)\, h(t-\tau)\, d\tau$$

$$h(\tau) = \begin{cases} 1 - \frac{\tau}{T} & 0<\tau<T \\ 0 & \text{o.w} \end{cases}$$

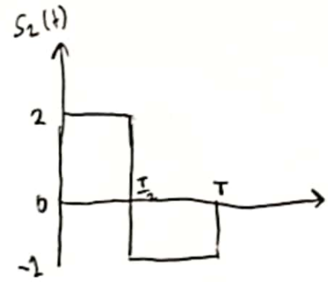
$$y_1(t) = \int_0^\infty \frac{\tau}{T}\left(1 - \frac{(t-\tau)}{T}\right) d\tau \quad \ldots$$

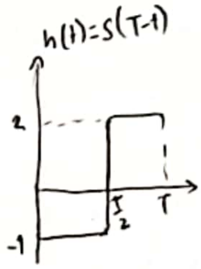


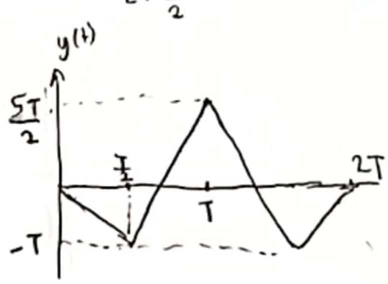


$$\int_0^T \frac{t^2}{T^2}\, dt = \frac{T}{3}$$

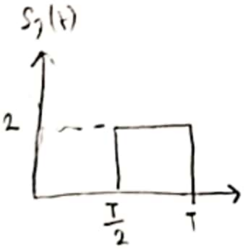
b)

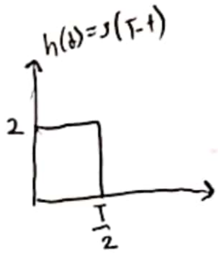


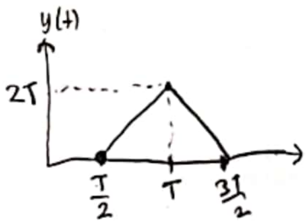




c)

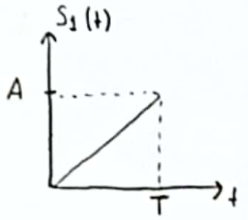


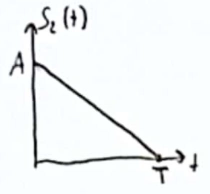




CamScanner ile tarandı

11.





→ Two signals have the same energy

a) Structure of the optimal receiver → equiprobable
→ same energy so no bias term

$$\int_{-\infty}^{\infty} r(t)\, s_1(t)\, dt \underset{s_2}{\overset{s_1}{\gtrless}} \int_{-\infty}^{\infty} r(t)\, s_2(t)\, dt$$

b) $P(e) = ?$ first try to find $P(e|s_1)$

$$\int_{-\infty}^{\infty} s_1(t)\left(s_1(t) - s_2(t)\right) dt + \int_{-\infty}^{\infty} n(t)\left(s_1(t) - s_2(t)\right) dt \underset{s_2}{\overset{s_1}{\gtrless}} 0$$

$$w = \frac{A^2 T}{6} + n$$

$$\sigma_n^2 = E\left[\int_{-\infty}^{\infty} n(\tau)\left(s_1(\tau) - s_2(\tau)\right) d\tau \int_{-\infty}^{\infty} n(\upsilon)\left(s_1(\upsilon) - s_2(\upsilon)\right) d\upsilon\right]$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(s_1(\tau) - s_2(\tau)\right)\left(s_1(\upsilon) - s_2(\upsilon)\right) \frac{N_0}{2}\,\delta(\upsilon - \tau)\, d\upsilon\, d\tau$$

$$\frac{N_0}{2}\int_{-\infty}^{\infty}\left(s_1(\tau) - s_2(\tau)\right)^2 d\tau$$

$$\int_0^T\left(\frac{2A\tau}{T} - A\right)^2 d\tau$$

$$= \frac{N_0 A^2 T}{6}$$

$$P(e|s_1) = \int_{-\infty}^{0} f(w)\, dw$$

, w has $\frac{A^2 T}{6}$ mean and $\sqrt{\frac{N_0 A^2 T}{6}}$ standard deviation

$r = \sigma z + \mu$

$$r = \sqrt{\frac{N_0 A^2 T}{6}}\, \overset{\text{std normal}}{z} + \frac{A^2 T}{6}$$

$$P(e|s_1) = \int_{-\infty}^{-\frac{A^2 T}{6} \Big/ \sqrt{\frac{N_0 A^2 T}{6}}} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz = Q\left[\sqrt{\frac{A^2 T}{6 N_0}}\right] = P(e|s_2)$$

$$P_e = \frac{1}{2} Q\left[\sqrt{\frac{A^2 T}{6 N_0}}\right] + \frac{1}{2} Q\left[\sqrt{\frac{A^2 T}{6 N_0}}\right] = Q\left[\sqrt{\frac{A^2 T}{6 N_0}}\right]$$

12. $r = \pm A + n$

$\pm A$: equally probable $\qquad n$: laplacian

$$p(n) = \frac{1}{\sqrt{2\sigma}} e^{-\frac{|n|\sqrt{2}}{\sqrt{\sigma}}} \qquad \lambda = \frac{\sqrt{2}}{\sqrt{\sigma}}$$

$$f(n) = \frac{\lambda}{2} e^{-\lambda|n|}$$

Optimal Receiver

$$\frac{f(r|A)}{f(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \underset{-A}{\overset{A}{\gtrless}} 1$$

$$-\lambda[|r-A|-|r+A|] \underset{-A}{\overset{A}{\gtrless}} 0 \Rightarrow r \underset{-A}{\overset{A}{\gtrless}} 0$$

$$P(e) = \frac{1}{2} P(e|A) + \frac{1}{2} P(e|-A)$$

$$\frac{1}{2}\int_{-\infty}^{0} f(r|A)\,dr + \frac{1}{2}\int_{0}^{\infty} f(r|-A)\,dr$$

$$= \frac{1}{2}\int_{-\infty}^{0} \frac{\lambda}{2} e^{-\lambda|r-A|}\,dr + \frac{1}{2}\int_{0}^{\infty} \frac{\lambda}{2} e^{-\lambda|r+A|}\,dr$$

symmetric

$$= \frac{\lambda}{2}\int_{A}^{\infty} e^{-\lambda|x|}\,dx$$

$$= -\frac{1}{2} e^{-\lambda x}\Big|_{A}^{\infty} = \frac{1}{2} e^{-\lambda A}$$