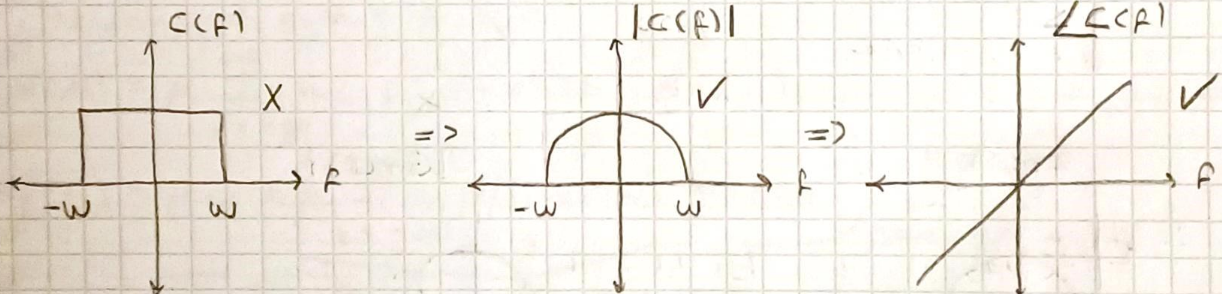
Digital Transmission over Bandlimited AWGN channels

• Due to the bandlimited properties of channel ; Intersymbol interference (ISI) occurs. It distorts the transmitted signal. Design receiver to manage this distortion (distortion means , $P_e \uparrow$)

Bandlimited Channel

• The channel has a frequency components whose amplitude not constant and the phase is linear.

• The channel passes all the frequencies if $|f| < W$, else blocks the signal component. where $W = B_c$



• The channel is around 0 , so it is baseband channel

• Bandlimited channels are modelled by using linear filters. (ex. Telephone wireline channel)

$c(t) \longrightarrow$ linear filter impulse response

$C(f) \longrightarrow \int_{-\infty}^{\infty} c(t) \exp(-j2\pi f t)\, dt$

W : transmitted signal bandwith

• If $B_c = W$, trasmit all freq. components of transmitted signal.

$g_T(t) \Rightarrow$ transmitted pulse (rectangular pulse)

$g_T(t) \longrightarrow \boxed{c(t)} \longrightarrow h(t) = g_T(t) * c(t) \Rightarrow H(f) = G_T(f) \cdot C(f)$

$g_T(t) \longrightarrow \boxed{c(t)} \longrightarrow h(t) \longrightarrow (+) \longrightarrow h(t) + n(t) \longrightarrow$ $\boxed{\text{matched filter}}$

$n(t)$ ↑ (into $+$)

- $G_R(f) = H^*(f)\exp(-j2\pi f t_0)$ (frequency response of the filter)

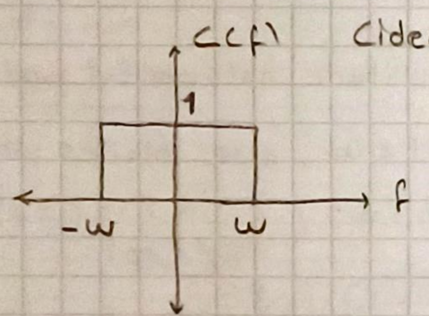
$SNR_0 = \dfrac{2\mathcal{E}_h}{N_0}$, where $\mathcal{E}_h$ is the energy of the pulse and channel, energy of the channel output

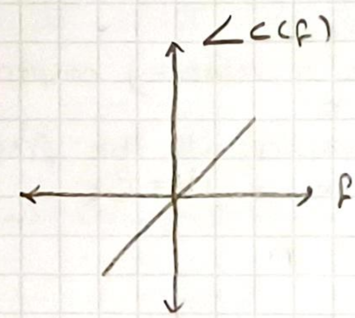
- $\mathcal{E}_h = \int_{-\infty}^{\infty} |H(f)|^2 df$

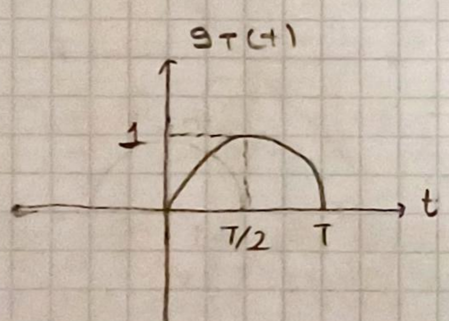
Drawback : assumption is that channel is estimated (not exactly known), the case is channel is known perfectly) önceki sayfadaki $C(f)$ ve $\angle C(f)$'i bilmek

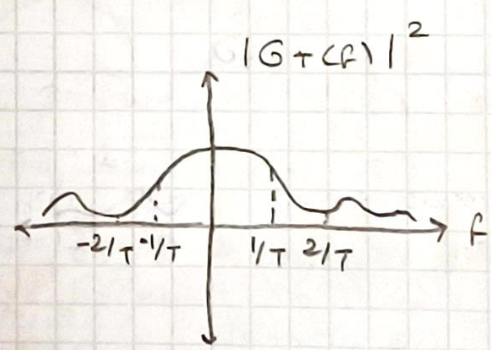
Example : Pulse shape

$$g_T(t) = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi}{T}\left(t - \frac{T}{2}\right)\right)\right] \qquad 0 \le t \le T$$





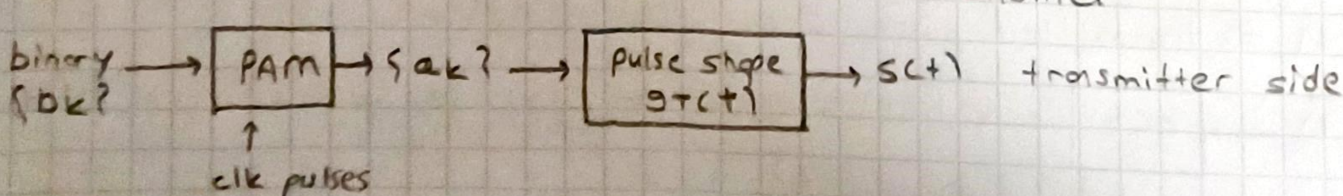


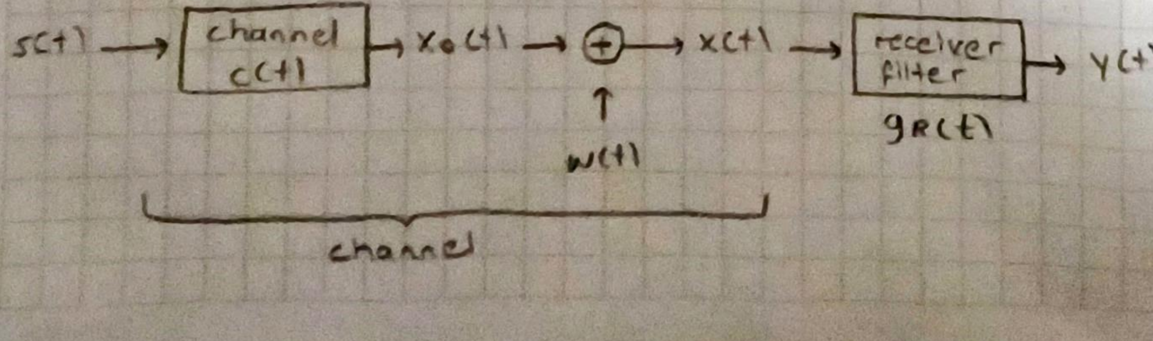


$$G_T(f) = \frac{T}{2}\,\frac{\sin(\pi f T)}{\pi f T(1 - f^2T^2)}\exp(-j\pi f T)$$

$SNR_0 = \dfrac{2\mathcal{E}_h}{N_0}$ $\quad \mathcal{E}_h = \int_{-W}^{W} |G_T(f)|^2 df$, where $W = \dfrac{1}{T}$

Digital PAM Transmission Through Bandlimited Channel





$$y(t) \xrightarrow[t_i = iT_b]{\times} \boxed{\text{detector}} \begin{cases} y(t_i) > 0, & \text{bit } 1 \\ y(t_i) < 0, & \text{bit } 0 \end{cases}$$

- pulse shape, receiver filter will be designed
- Transmitted signal after applying transmitter filter:

$$s(t) = \sum_k a_k\, g_T(t - kT_b)$$

$$y(t) = \mu \sum_k a_k\, p(t - kT_b) + n(t)$$, where μ is scaling factor of channel

$$p(t) = \underbrace{g_T(t)}_{\substack{\text{transmit}\\\text{filter}}} * \underbrace{c(t)}_{\text{channel}} * \underbrace{g_R(t)}_{\substack{\text{receiver}\\\text{filter}}}$$

$$P(f) = G_T(f) \cdot C(f) \cdot G_R(f)$$

The sampled output is

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k\, p((i-k)T_b) + n(t_i)$$

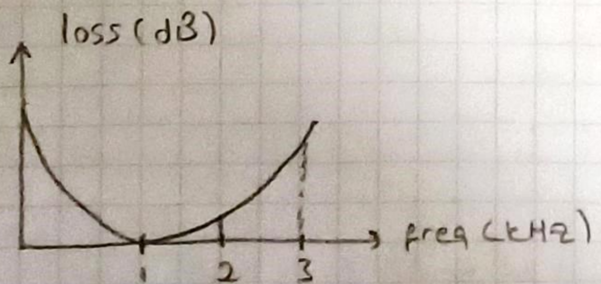
$$y(t_i) = \underbrace{\mu a_i}_{\substack{\text{desired transmitted}\\\text{signal}}} + \underbrace{\mu \sum_{\substack{k=-\infty\\ k \neq i}}^{\infty} a_k\, p((i-k)T_b)}_{\substack{\text{inter-symbol interference}\\\text{(ISI)}}} + \underbrace{n(t_i)}_{\text{sampled noise}}$$

- case no ISI

$$y(t_i) = \mu a_i + n_i$$

- SINR : $\dfrac{\text{signal power}}{\text{interference power + noise power}}$

Example : Telephone Channels



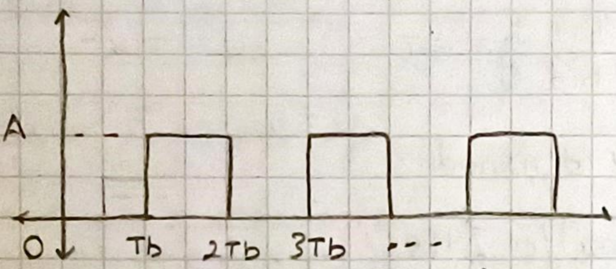
$$\text{Loss} = 10 \log_{10}\left(\frac{P_0}{P_2}\right)$$

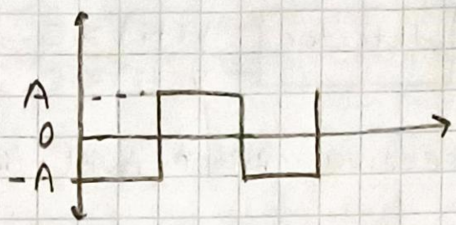
$$B_c = 3.5 \text{ kHz}$$

P_0 : power without channel

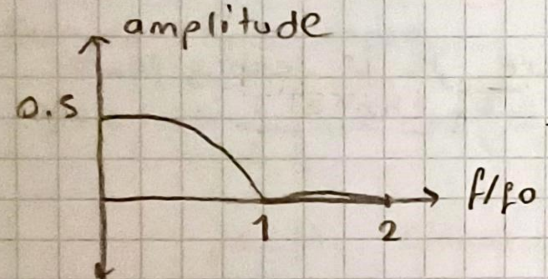
P_2 : power with channel

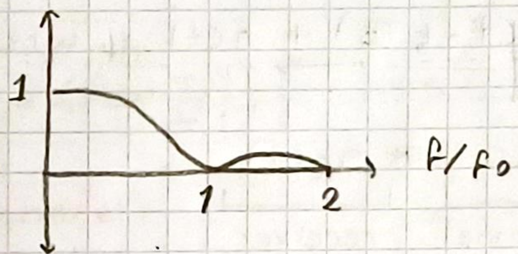
Figure 8.1 (Line Codes)





uni poler NRZ ⇙F ⇙F poler NRZ





f_o [bps] = f_s (sample /s) . $\log_2 L$ (bits /sample) $T_b = \frac{1}{f_o}$

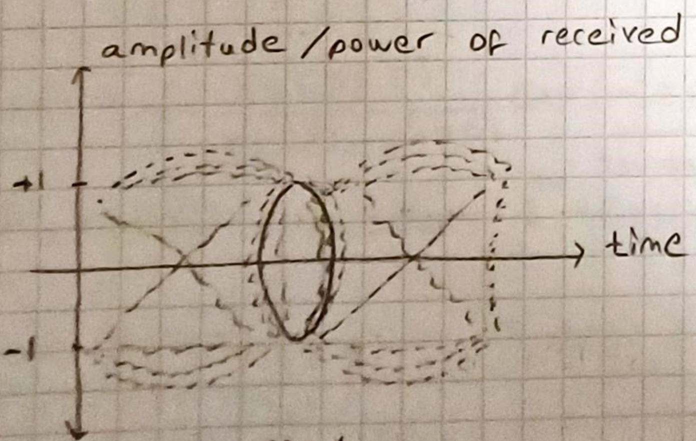
↳ data bit rate

- Trasmission bandwidth is related to line codes that is used.
- Higher data rate requires higher bandwith

Figure 8.11

Eye Diagram

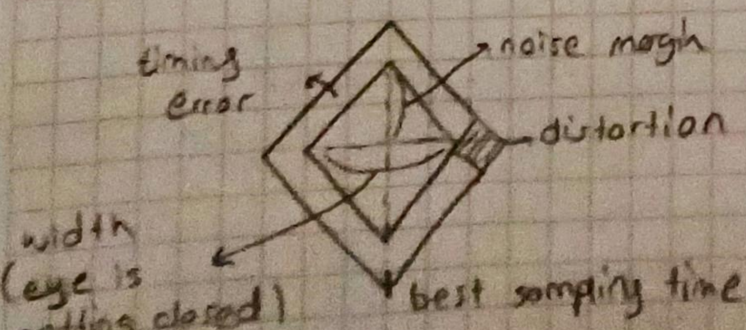
- Operational tool to observe ISI effect
- The eye pattern is defined as the synchronized superposition of all possible realization of signal of interest (observed in a particular signalling interval)



polar NRZ , A=1

figure 8.13

M=1 eyes



we would like to have open eyes

- The width of the eye openning when the effect of ISI is severe
- The height of the eye oppenning at a spesific sampling time defines noise margin
- Sensibility of timing error is determined by the rate of closure of the eye as the sampling time is varied

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ISI occurs due to channel properties => bandlimited channel

Nyquist criterion for distortionless transmission

- ideal bandlimited channel

$$\xrightarrow{} \boxed{g_T(t)} \rightarrow \boxed{c(t)} \rightarrow \oplus \rightarrow \boxed{g_R(t)} \rightarrow \boxed{\text{detector}}$$

symbol amplitude ∓ 1 binary case $\quad b_k = \mp a_k \qquad n(t) \uparrow$

$$p(iT_b - kT_b) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \qquad t = iT_b$$

where $p(0) = 1$
- it is normalized

$$p(t) = g_T(t) * c(t) * g_R(t)$$

$$y(t_i) = \mu a_i + \text{noise} \quad (\text{case no ISI}) \; (\text{assuming no noise})$$

$$P_\delta(f) = R_b \sum_{n=-\infty}^{\infty} P(f - nR_b) \qquad R_b = 1/T_b$$

$\Downarrow$

freq. domain representation of $p((i-k)T_b)$

(FT of infinite periodic sequence of delta function of T_b)

$$P_\delta(f) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [p(mT_b) \cdot \delta(t - mT_b)] e^{-j2\pi f t} dt \qquad m = i - k$$

$m = 0$ if $i = k$

$m \neq 0$ if $i \neq k$

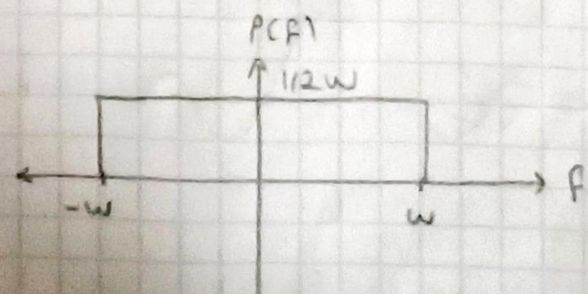
$$P_\delta(f) = \int_{-\infty}^{\infty} p(0)\, \delta(t) \exp(-j2\pi f t)\, dt \qquad \text{where } p(0) = 1$$

$$\sum_{n=-\infty}^{\infty} P(f - nR_b) = T_b$$

condition for no ISI. Nyquist criterion for distortionless transmission

$P(f)$ eliminates the ISI for the samples taken at T_b.

$$P(f) = \begin{cases} 1/2W, & -W < f < W \\ 0, & \text{else} \end{cases}$$



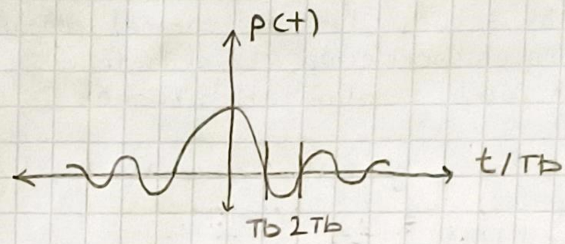
$$P(f) = G_T(f) \cdot C(f) \cdot G_R(f)$$

$$y(t_i) = \mu a_i + \underbrace{\sum_{n=-\infty}^{\infty} a_k \, \delta((i-k)T_b)}_{ISI\ (?)}$$

$$P(f) = \frac{1}{2W} \operatorname{rect}\left(\frac{1}{2W}\right) \qquad (R_b = 2W \text{ Nyquist})$$

$$W = \frac{R_b}{2} = \frac{1}{2T_b} \quad (\text{required BW})$$

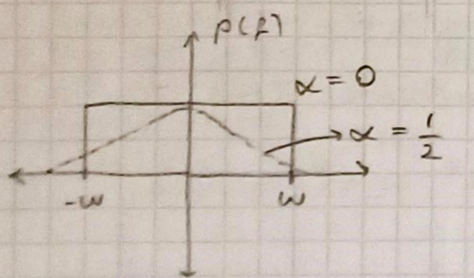
$$p(t) = \frac{\sin(2\pi W t)}{2\pi W t} = \operatorname{sinc}(2Wt)$$



solves the problem of ISI under ideal bandlimited channel

Ideal Nyquist channel achieves min. BW and solves the problem of ISI
1. It requires $P(f)$ flat and very sharp
• practically non-realizable, because there is very sharp transition at $\pm W$.
2. The function $p(t)$ decreases as $1/|t|$ for large $|t|$
• resulting a slow rate of decay, caused discontuinity of $P(f)$ of $\pm W$

Raised Cosine Spectrum (RCS)
• In order to overcome the practical difficulties, which ideal Nyquist channel. (extend BW between W and $2W$ for realizable case.



$B_T = (1+\alpha) W$ $\qquad \alpha$ = roll of factor

$B_T = \frac{3}{2} W$ $\qquad \alpha = 1 - \frac{f_1}{W}$

$$P(f) + p(f-2W) + p(f+2W) = \frac{1}{2W} \qquad -W \le f \le W$$

$$p(t) = \operatorname{sinc}(2Wt)\left(\frac{\cos(2\pi\alpha W t)}{1 - 16\alpha^2 W^2 t^2}\right)$$

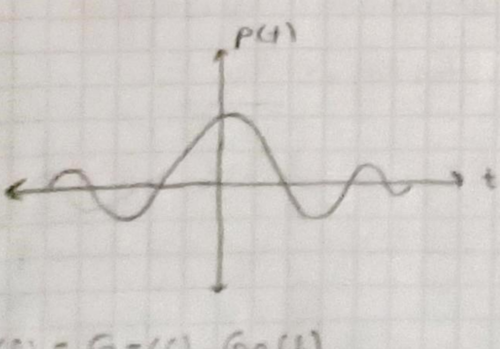


Fig: 8.17

$$P(f) = \begin{cases} 1/2W & 0 \le f \le f_1 \\ 1/4W \left\{1 - \sin\left(\frac{\pi(|f| - W)}{2W - 2f_1}\right)\right\} & f_1 \le |f| < 2W - f_1 \\ 0 & f > 2W - f_1 \end{cases}$$

$$P(f) = \underset{?}{G_T(f)} \cdot \underset{?}{G_R(f)}$$

Ex: 24 independent voice inputs

- 8 bit PCM (2^8 levels)
- $T_b = 0.647\ \mu sn$
- for ideal Nyquist channel $B_T = W = 1/2T_b = 772$ kHZ
- for $\alpha = 1$ RCS $B_T = 1.544$ mHZ $= 1/T_b$

How can design a trasmitter and receiver?

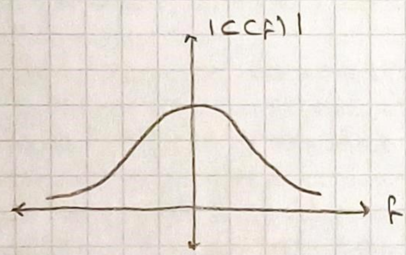
$|C(f)|$ is not constant $|f| < W$
↳ amplitude distortion

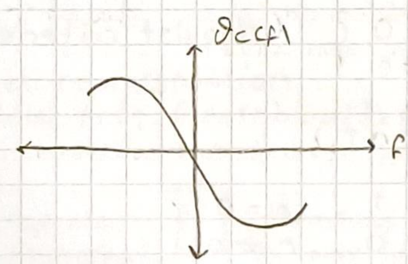
$\theta_C(f)$ is not linear => phase distortion

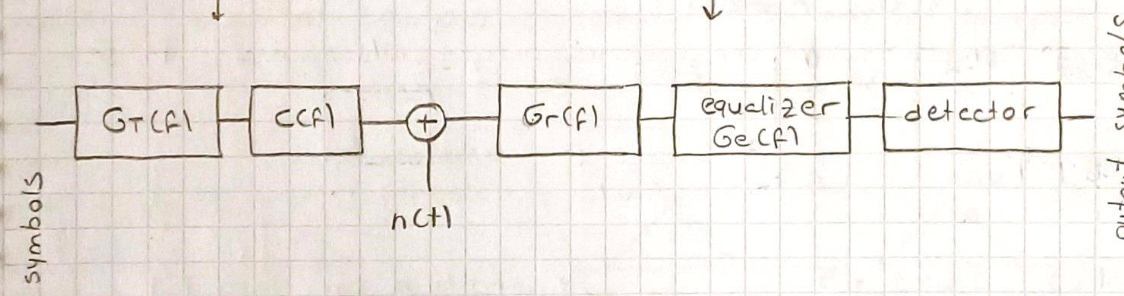
$$\tau(f) = \frac{1}{2\pi} \frac{d\theta_C(f)}{df}$$

↓

delay (different values for different frequencies)

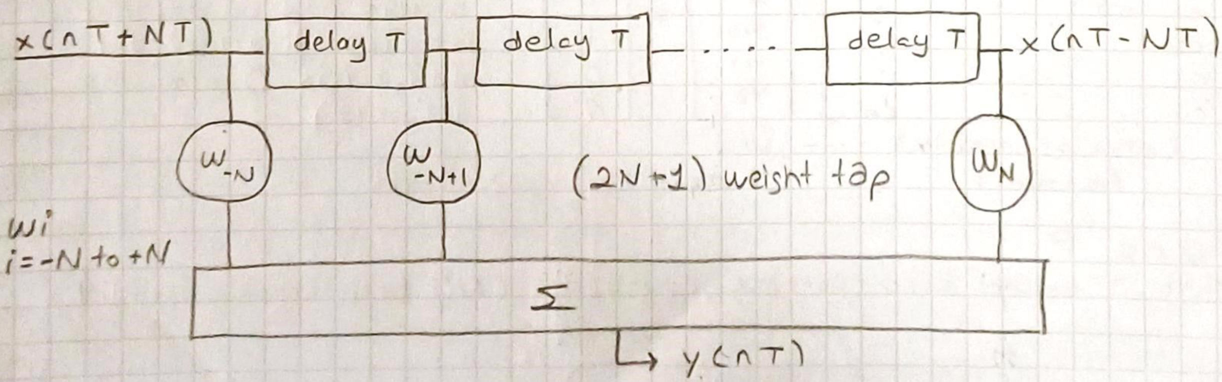






- main purpose of the equalization is inversing the chanel effect

Tapped delay-line filter



- yapılan işlem konvülasyon gibi

Tapped-delay-line equalizer

$$\boxed{c(t) \text{ channel}} - \boxed{\text{equalizer } h(t)} - c(t) * h(t)$$

$$h(t) = \sum_{k=-N}^{N} w_k \, \delta(t-kT)$$

$$c(t) * h(t) = c(t) * \sum_{k=-N}^{N} w_k \, \delta(t-kT)$$

$$p(t) = \sum_{k=-N}^{N} w_k \, c(t-kT) \quad \text{sampled at } t = kT$$

$$p(nT) = \sum_{k=-N}^{N} w_k \, c((n-k)T)$$ ($p(nT)$ is longer than $c(nT)$)

• To eliminate ISI

$$p(nT) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$ (Nyquist criterion)

Notation $c_n = c(nT)$

$$\sum_{k=-N}^{N} w_k \, c_{n-k} = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} c_0 & \cdots & c_{-2N} \\ \vdots & c_0 & \\ & & \ddots \\ \vdots & & \\ c_{2N} & \cdots & c_0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} w_{-N} \\ \vdots \\ \vdots \\ w_N \end{bmatrix}}_{w} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Ex: for $N=2$ $2N+1=5$

→ condition for zero ISI

$$\begin{bmatrix} c_0 & c_{-1} & c_{-2} & c_{-3} & c_{-4} \\ c_1 & c_0 & & & \vdots \\ c_2 & & \ddots & & \vdots \\ c_3 & & & \ddots & \vdots \\ c_4 & \cdots & \cdots & \cdots & c_0 \end{bmatrix} \begin{bmatrix} w_{-2} \\ w_{-1} \\ w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(channel matrix) (known)

w ↳ equalizer coefficients

(örneklediğimiz andaki sinyalin 0'daki değerini istiyoruz diğerleri ISI olduğu için 0 olmasını istiyoruz)

$$w = C^{-1} P$$

• This is called zero-forcing equalizer (N'i belirlemek önemli)

MIDTERM'de
• 3 soru (kolay, orta, tasarım)

PROJECT
• 2G GSM equalizer
• 2G+
• 3G
• 4G
• 5G

Bandpass Transmission model

Difference Between Baseband and Bandpass

- In baseband pulse transmission
  - Transmission directly through channel
  - design pulse shaping + equalization (to avoid ISI)
- In bandpass transmission
  - use carrier to modulate signal (usually sinusoidal)
  - design optimum receiver to minimize the error due to channel
  - satellite channels, cellular channels

The modulation can be on:

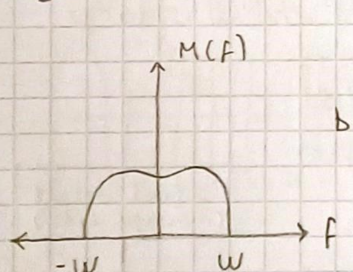
- amplitude (ASK)
- frequency (FSK)
- phase (PSK)
- amplitude + phase (QAM) (better performace, less error)

We'll start with binary, then move on to m-ary SK's.

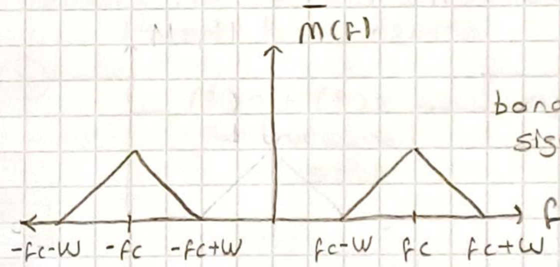
- BASK, BFSK, BPSK

Bandpass or Lowpass signal representation

- Information signal is a low freq. (baseband). Available modulation is at higher freq. Any real narrowband and high freq. signal is called bandpass signal.
- Bandpass signals can be represented in terms of complex low freq. signal. It is called Low-pass Representation of original bandpass signal.



baseband signal



bandpass signal

$BW = 2W$

$$\underbrace{x(t)}_{\text{bandpass representation}} = Re\{\underbrace{x_L(t)}_{\text{lowpass representation}} \exp(j2\pi f_c t)$$

$$X(f) = \frac{1}{2}\{X_L(f-f_c) + X_L^*(-f-f_c)\}$$

$$x_L(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t) + j\hat{x}(t)\cos(2\pi f_c t) - j\hat{x}(t)\sin(2\pi f_c t)$$

$\hat{x}(t) \Rightarrow$ Hermition of $x(t)$

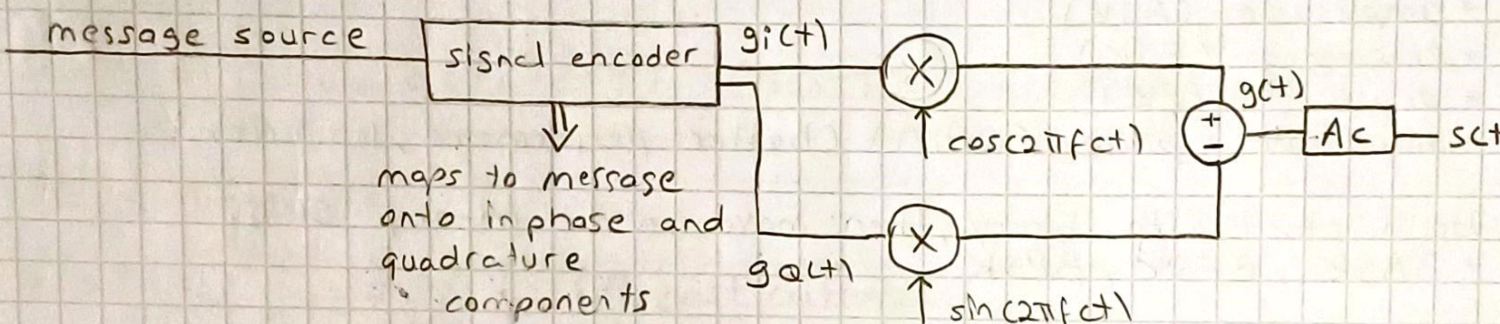
$$x_L(t) = \underbrace{x_I(t)}_{\text{inphase}} + j\underbrace{x_Q(t)}_{\text{quadrature}}$$

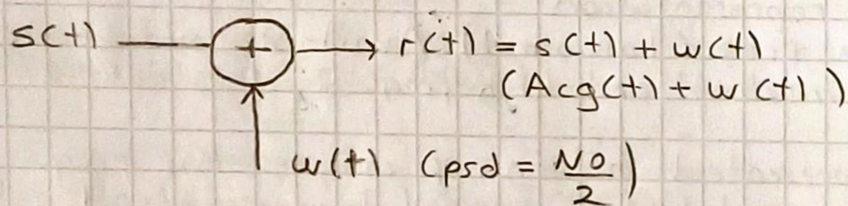
$$x_I(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t)$$
$$x_Q(t) = \hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)$$

$$x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$$

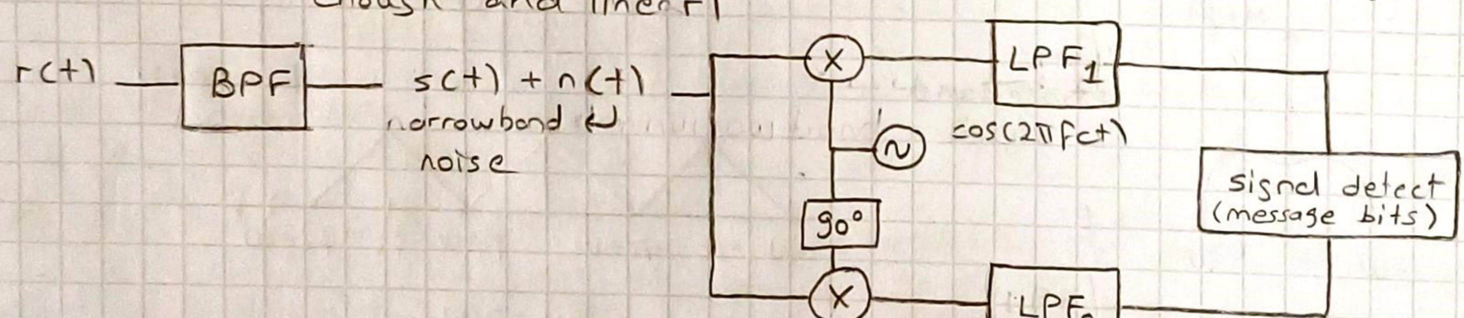
$$x_L(t) = r_x(t)\, e^{j\theta_x(t)}$$
where
$$r_x(t) = \sqrt{|x_I(t)|^2 + |x_Q(t)|^2}$$
$$\theta_x(t) = \arctan\left(\frac{x_Q(t)}{x_I(t)}\right)$$

The signal has inphase and quadrature components





AWGN (the channel has channel impairments ch BW is large enough and linear)



$$LPF_1 = \frac{1}{2}\left[A_c g_I(t) + n_I(t)\right]$$

$$LPF_2 = \frac{1}{2}\left[A_c g_Q(t) + n_Q(t)\right]$$

$n_I(t)$ and $n_Q(t)$ => lowpass complex noise $n(t) = n_I(t) + j n_Q(t)$

$$A_c\Big[\underbrace{g_I(t) + n_I(t)}_{\text{upper branch}} + j\underbrace{(g_Q(t) + n_Q(t)}_{\text{lower branch}}\Big] = x(t)$$

based on $x(t)$, decide 0 or 1 (message bits)
- two class of detection
  - coherent detection (sync. + timing recovery PLL)
  - non-coherent detection (no phase sync. direct recover from bandpass signal)

• PLL çalış hatırlatma için

Transmission of Binary ASK, FSK, PSK

=> transmits binary symbols

bit 0 , $s_0(t)$ $\quad 0 \le t \le T_b$ or

bit 1 , $s_1(t)$ $\quad 0 \le t \le T_s$, $T_b = T_s$

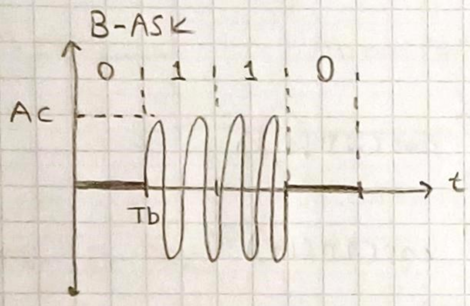
$R_b = \frac{1}{T_b}$ ideal case

For B-ASK

we transmit sinusoidal wave por bit 1, else no wave. (on-off)

$s_1(t) = A_c \cos(2\pi f_c t)$

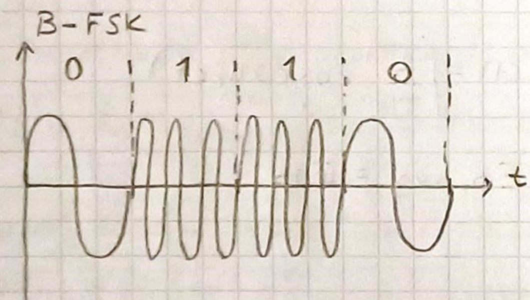
$s_0(t) = 0$



For B-FSK

$s_0(t) = A_c \cos(2\pi f_0 t)$, bit 0 $\qquad f_0 = f_c - \Delta f$

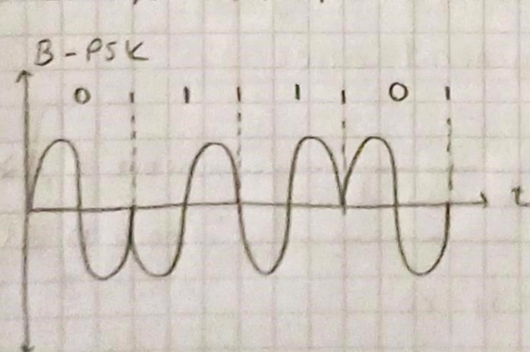
$s_1(t) = A_c \cos(2\pi f_1 t)$, bit 1 $\qquad f_1 = f_c + \Delta f$



For B-PSK

$s_0(t) = A_c \cos(2\pi f_c t)$, bit 0

$s_1(t) = A_c \cos(2\pi f_c t + \pi)$, bit 1



For B-ASK

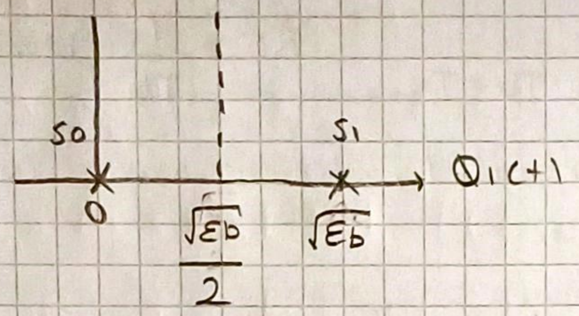
$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$

$E_b = \frac{A_c^2}{2} T_b$

$s_0(t) = \phi(t) \cdot \sqrt{E_b}$

$\|s_1 - s_0\| = \sqrt{E_b}$

$E_{b,avg} = \frac{E_b}{2}$

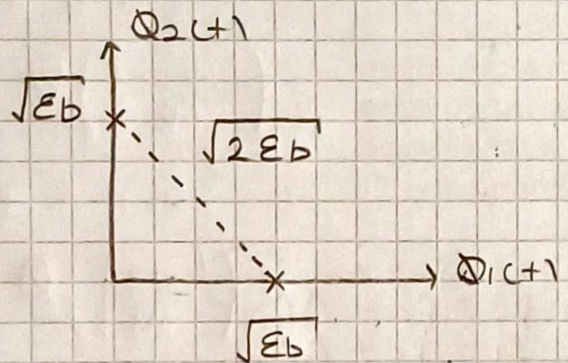


For B-FSK

$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_0 t)$

$\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_1 t)$

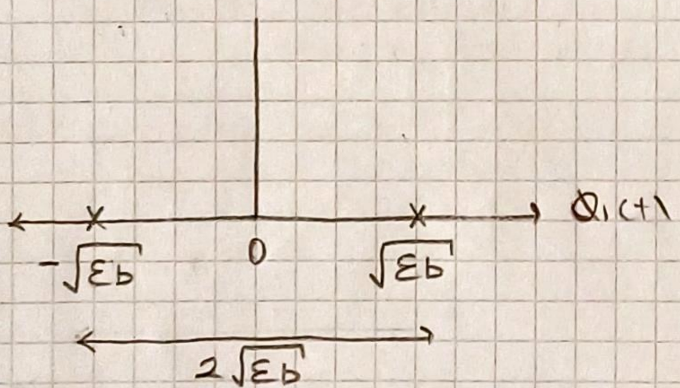
$E_{b,avg} = E_b$



For B-PSK

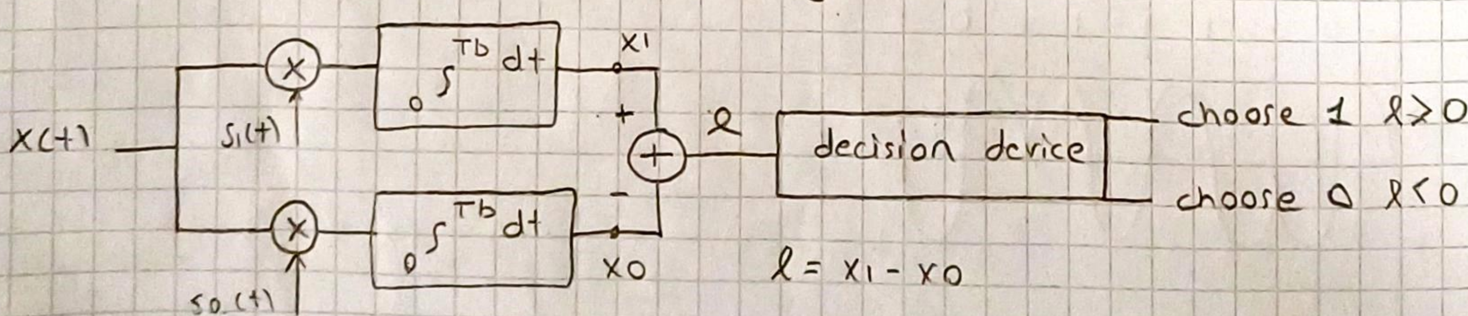
$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$

$E_{b,avg} = E_b$



B-PSK kullanmak erroru düşürmek için daha iyi

Two path correlation receiver for a general case



$l = x_1 - x_0$

$x_1 = \int_0^{T_b} x(t)\, s_1(t)\, dt$

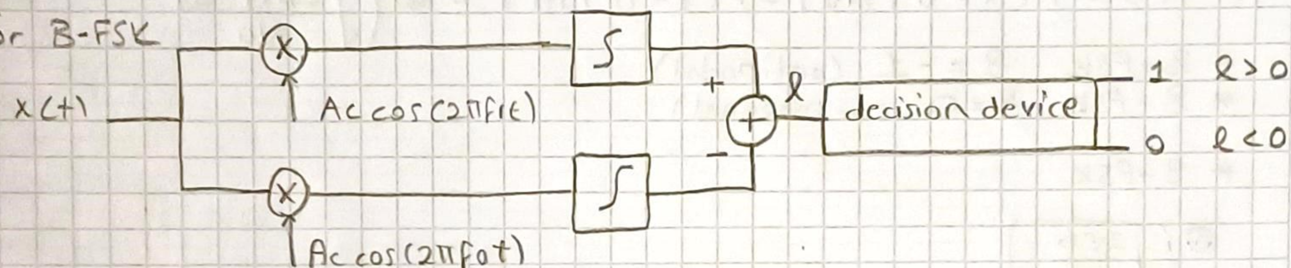
$x_0 = \int_0^{T_b} x(t)\, s_0(t)\, dt$

$H_0 : x(t) = s_0(t) + w(t)$

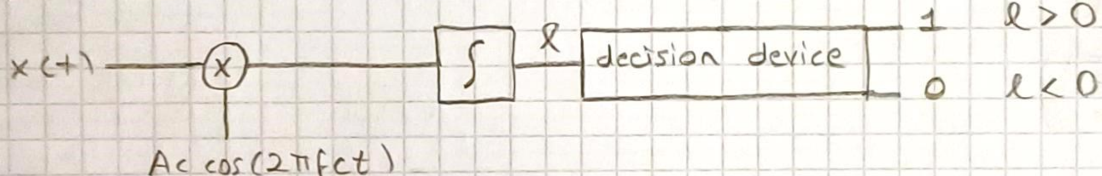
$H_1 : x(t) = s_1(t) + w(t)$

$$\ell = \int_0^{T_b} x(t)\left[s_1(t) - s_0(t)\right] dt$$

For B-FSK



For B-PSK



- These are ALL coherent receivers

$$H_1 : \ell = \int_0^{T_b} s_1(t)\left[s_1(t) - s_0(t)\right] dt + \int_0^{T_b} w(t)\left[s_1(t) - s_0(t)\right] dt$$

$w(t)$: zero mean Gaussian r.v.

$$E\{L \mid H_1\} = \int_0^{T_b} s_1(t)\left[s_1(t) - s_0(t)\right] dt = \underbrace{\int_0^{T_b} s_1(t)^2\, dt}_{\mathcal{E}_b} - \underbrace{\int_0^{T_b} s_1(t) s_0(t)\, dt}_{\mathcal{E}_b \rho}$$

$= E_b(1 - \rho)$

correlation between $s_0(t)$ and $s_1(t)$

$$\rho = \frac{\int_0^{T_b} s_0(t) s_1(t)\, dt}{\left(\int_0^{T_b} s_0(t)^2\, dt \cdot \int_0^{T_b} s_1(t)^2\, dt\right)^{1/2}}$$

$$\rho = \frac{1}{\mathcal{E}_b} \int_0^{T_b} s_0(t) s_1(t)\, dt$$

$E\{L \mid H_0\} = -\mathcal{E}_b(1 - \rho)$

$\mathrm{var}\{L\} = E\{(L - E\{L\})^2\}$

$$= E\left\{\int_0^{T_b}\int_0^{T_b} w(t) w(u)\left[s_1(t) - s_0(t)\right]\left[s_1(u) - s_0(u)\right] dt\, du\right\}$$

$$= \int_0^{T_b}\int_0^{T_b} \left[s_1(t) - s_0(t)\right]\left[s_1(u) - s_0(u)\right] R_w(t,u)\, dt\, du$$

$$R_w(t,u) = \frac{N_0}{2}\delta(t-u)$$

$$\text{var}\{l\} = N_0 E_b(1-\rho)$$

Gaussian Random Variable with mean $\pm E_b(1-\rho)$ and variance $N_0 E_b(1-\rho)$

$$P_e = P_b = P(l>0 \mid H_0) = P(l<0 \mid H_1) = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

* B-PSK $\rho = -1$ (antipodal)
* B-FSK $\rho = 0$ (orthogonal)

* B-PSK

$$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

* B-FSK

$$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$