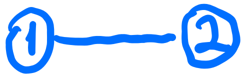
Calculate Eigenvalues and Eigenvectors of Graph Laplacian



$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \end{array}$$

$$L = D - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underbrace{A}_{n \times n \text{ matrix}} \; \underbrace{x}_{\text{non-zero } n \times 1 \text{ vector}} = \underbrace{\lambda}_{\text{scalar}} x$$

any λ that has solution to this equation is an eigenvalue of A (characteristic value)

a vector corresponding to the eigenvalue is called eigenvector

$\rightarrow$ $Ax - \lambda x = 0 \Rightarrow Ax - \lambda I x = 0 \Rightarrow$

$\rightarrow$ $\underbrace{(A - \lambda I)}_{0} \underline{\underline{x}} = 0 \Rightarrow$

$\rightarrow$ $|A - \lambda I| = 0$ (characteristic equation)

n^{th} order polynomial in λ with n roots. Roots are eigenvalues of A. Roots can be distinct or repeated.

For each eigenvalue, there will be an eigenvector for which the equation holds.

Substitute A with L

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \underline{|A - \lambda I|} = \left| \overset{A}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} - \overset{\lambda I}{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}} \right| = 0$$

$$\underbrace{\left| \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} \right|}_{\text{determinant}} = 0 \Rightarrow (1-\lambda)^2 - 1 = 0$$

$$(1-\lambda-1)(1-\lambda+1) = 0$$

$$-\lambda(2-\lambda) = 0$$

$$\underline{\lambda_1 = 0} \qquad \underline{\lambda_2 = 2}$$

Eigenvectors

$$L \cdot x_1 = \lambda_1 x_1 \Rightarrow (L - \lambda_1 I) x_1 = 0$$

$$\rightarrow \underset{L}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \underset{\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}}{x_1} = \underset{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}{0}$$

$$x_{11} - x_{12} = 0 \Rightarrow x_{11} = x_{12}$$

$$-x_{11} + x_{12} = 0$$

$$\underline{x_1 = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Second Eigenvector

$$L \cdot x_2 = \lambda_2 x_2$$

$$(L - \lambda_2 I) x_2 = 0 \qquad \left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) x_2 = 0$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x_2 = 0 \qquad -x_{21} - x_{22} = 0$$

$$-x_{21} - x_{22} = 0 \qquad \underline{x_{21} = -x_{22}}$$

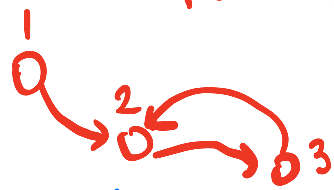
$$x_2 = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Try L = np.array([[1,-1],[-1,1]])

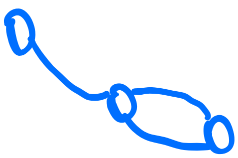
ev, evv = np.linalg.eig(L)

Strong Connectivity
for Directed Graphs

G_1



node 1 is not accessible
from node 2
So the graph G_1 is not
strongly connected
but weakly connected



ignore directions