Signed Networks

Introduction

Signed networks are graphs where edges can have positive (+) or negative (-) weights, representing relationships such as friendship/enmity, trust/distrust, or agreement/disagreement. These networks provide a richer representation of social and economic interactions than traditional unsigned networks.

Basic Concepts

1. Signed Network Structure

Definition: A signed network is a graph $G = (V, E, \sigma)$ where:

- V is the set of nodes
- E is the set of edges
- $\sigma : E \to \{+1, -1\}$ is the sign function

Edge Types:

- **Positive edges** $(+1)$: Friendship, trust, agreement
- **Negative edges** (-1): Enmity, distrust, disagreement

2. Adjacency Matrix Representation

Signed Adjacency Matrix:

$$A_{ij} = \begin{cases} +1 & \text{if } (i,j) \text{ is a positive edge} \\ -1 & \text{if } (i,j) \text{ is a negative edge} \\ 0 & \text{if no edge exists} \end{cases}$$

Balance Theory

1. Structural Balance

Definition: A signed network is balanced if it can be partitioned into two groups such that all positive edges are within groups and all negative edges are between groups.

Mathematical Definition: A triangle is balanced if the product of its edge signs is positive:

$$\sigma_{ij} \cdot \sigma_{jk} \cdot \sigma_{ki} = +1$$

2. Balance Conditions

Triangle Types:

1. **+++**: All positive edges (balanced) - "Friend of friend is friend"
2. **+--**: One positive, two negative edges (balanced) - "Enemy of enemy is friend"
3. **++-**: Two positive, one negative edge (unbalanced) - "Two friends with common enemy"
4. **---**: All negative edges (unbalanced) - "Three mutual enemies"

3. Balance Index

Definition: Fraction of balanced triangles in the network.

Mathematical Definition:

$$B = \frac{\text{Number of balanced triangles}}{\text{Total number of triangles}}$$

Centrality Measures

Issues with Standard Centrality

Problems:

1. **Shortest Paths**: How to define "shortest" when edges have different signs?
2. **Path Meaning**: A path through enemies is fundamentally different from a path

through friends
3. **Edge Treatment**: Standard algorithms treat all edges equally, ignoring signs
4. **Existence**: Some measures (like eigenvector centrality) may not even exist for signed networks!

1. Signed Degree Centrality

Definition: Net degree considering both positive and negative connections.

Mathematical Definition:

$$d_i^{net} = d_i^+ - d_i^-$$

where:

- d_i^+ is the positive degree of node i
- d_i^- is the negative degree of node i

Interpretation:

- High positive net degree: Well-liked, influential
- High negative net degree: Controversial, many enemies
- Near zero: Balanced relationships

2. Walk-Based Centrality

Walk-based centrality is the most reliable method for signed networks. It properly accounts for how influence propagates along all paths with signs multiplying.

Reference: Liu et al. (2020) "A simple approach for quantifying node centrality in signed and directed social networks"

Key Innovation: Instead of shortest paths, consider ALL walks where effects propagate and signs multiply.

Direct Effect Formula

The direct effect of node i on node j is:

$$a_{ij} = \frac{\sigma_{ij}}{d_j}$$

where:

- σ_{ij} is the sign of edge (i, j) ($+1$ or -1)
- d_j is the degree of node j

Intuition: Effect is stronger when target has fewer connections.

Indirect Effects and Sign Propagation

For a walk $i \to k \to j$, the indirect effect is:

$$\text{effect}(i \to j \text{ via } k) = a_{ik} \times a_{kj}$$

Sign Propagation Rules:

- $(+) \times (+) = +$: Positive influence through positive intermediary
- $(+) \times (-) = -$: Positive becomes negative through negative intermediary
- $(-) \times (-) = +$: Negative through negative (enemy of enemy)

Total Effect (TE)

Definition: Sum of effects along ALL walks up to length n:

$$TE_{ij}^{(n)} = \sum_{l=1}^{n} \sum_{\text{walks of length } l} \text{effect along walk}$$

Total Effect of node i on whole network:

$$TE_i = \sum_{j=1}^{N} TE_{ij}^{(n)}$$

Interpretation: Overall influence magnitude, regardless of sign.

Net Effect (NE)

Definition: Positive effects minus negative effects:

$$NE_{ij}^{(n)} = E_{ij}^{(n)+} - E_{ij}^{(n)-}$$

Net Effect of node i:

$$NE_i = \sum_{j=1}^{N} NE_{ij}^{(n)}$$

Interpretation:

- $NE_i > 0$: Predominantly positive influence
- $NE_i < 0$: Predominantly negative influence
- $NE_i \approx 0$: Balanced positive and negative influence

3. Signed Betweenness Centrality (Approximation Only)

Approach: Structure-Weighted Approximation

```
def signed_betweenness_approximation(G):
    """
    Approximate betweenness for signed networks.

    WARNING: This is an approximation with limitations.
    Consider using walk-based centrality instead.
    """
    # Create unsigned version for path counting
    G_unsigned = nx.Graph()
    G_unsigned.add_nodes_from(G.nodes())
    G_unsigned.add_edges_from(G.edges())

    # Calculate standard betweenness
    betweenness = nx.betweenness_centrality(G_unsigned)

    # Weight by local sign environment
    weighted_betweenness = {}
    for node in G.nodes():
        pos_edges = sum(1 for _, _, d in G.edges(node, data=True)
                       if d.get('sign', 1) == 1)
        neg_edges = sum(1 for _, _, d in G.edges(node, data=True)
                       if d.get('sign', 1) == -1)
        total_edges = pos_edges + neg_edges

        if total_edges > 0:
            sign_ratio = (pos_edges - neg_edges) / total_edges
            weighted_betweenness[node] = betweenness[node] * (1 +
```

```
sign_ratio) / 2
        else:
            weighted_betweenness[node] = 0.0

    return weighted_betweenness
```

---

4. Signed Closeness Centrality

```
def signed_closeness_harmonic(G):
    """
    Harmonic closeness using only positive edges.

    Only considers reachability through friendly relationships.
    """
    # Create subgraph with only positive edges
    G_positive = nx.Graph()
    G_positive.add_nodes_from(G.nodes())
    for u, v, data in G.edges(data=True):
        if data.get('sign', 1) == 1:
            G_positive.add_edge(u, v)

    closeness = {}
    for node in G.nodes():
        harmonic_sum = 0.0
        for target in G.nodes():
            if node != target:
                try:
                    distance = nx.shortest_path_length(G_positive,
node, target)
                    harmonic_sum += 1.0 / distance
                except nx.NetworkXNoPath:
                    pass  # No path through positive edges

        n = len(G.nodes())
        closeness[node] = harmonic_sum / (n - 1) if n > 1 else 0

    return closeness
```

Interpretation:

- High closeness: Well-connected through friendly relationships
- Low closeness: Isolated or only reachable through enemies

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5. Signed Eigenvector Centrality

The Perron-Frobenius theorem does NOT apply when adjacency matrix has negative entries.

When Eigenvector Centrality Fails:

1. No dominant positive eigenvalue
2. Multiple eigenvalues with same magnitude
3. Complex eigenvalues with imaginary components
4. Eigenvector with mixed signs (no clear interpretation)

Mathematical Definition:

$$x_i = \frac{1}{\lambda} \sum_j A_{ij} x_j$$

where A_{ij} can be positive or negative.

Python Implementation with Safety Checks:

```
def signed_eigenvector_centrality(G, tol=1e-6):
    """
    Calculate eigenvector centrality for signed networks.

    WARNING: May not exist! Always check return status.

    Returns
    -------
    centrality : dict or None
        Centrality values if computable, None otherwise
```

```
    status : str
        Status message explaining result
    """
    nodes = list(G.nodes())
    n = len(nodes)

    if n == 0:
        return None, "Empty graph"

    # Build signed adjacency matrix
    A = nx.adjacency_matrix(G, nodelist=nodes,
weight='sign').toarray()

    # Find eigenvalues
    try:
        eigenvalues, eigenvectors = np.linalg.eig(A)
        eigenvalues = eigenvalues.real
    except:
        return None, "Failed to compute eigenvalues"

    # Find largest eigenvalue by magnitude
    max_idx = np.argmax(np.abs(eigenvalues))
    lambda_max = eigenvalues[max_idx]

    # Check 1: Dominant eigenvalue?
    sorted_eigs = sorted(np.abs(eigenvalues), reverse=True)
    if len(sorted_eigs) > 1 and np.abs(sorted_eigs[0] -
sorted_eigs[1]) < tol:
        return None, "No dominant eigenvalue (multiple with same
magnitude)"

    # Check 2: Positive eigenvalue?
    if lambda_max <= 0:
        return None, f"No positive dominant eigenvalue (λ=
{lambda_max:.4f})"

    # Get eigenvector
    eigenvector = eigenvectors[:, max_idx].real

    # Check 3: All same sign?
    if np.all(eigenvector >= 0) or np.all(eigenvector <= 0):
        eigenvector = np.abs(eigenvector)
```

```
        eigenvector = eigenvector / eigenvector.sum()
        centrality = {node: eigenvector[i] for i, node in
enumerate(nodes)}
        return centrality, f"Success (λ={lambda_max:.4f})"
    else:
        return None, "Eigenvector has mixed signs"

# Usage
centrality, status = signed_eigenvector_centrality(G)
if centrality is None:
    print(f"Cannot compute: {status}")
    print("Using walk-based centrality instead")
    centrality = signed_walk_effect(G)
```

Community Detection

1. Signed Modularity

The Problem: Standard Newman modularity treats all edges equally.

Correct Formula: Gómez et al. signed modularity with parameter α:

$$Q_{signed} = \alpha \cdot Q(G^+) + (1 - \alpha) \cdot Q(G^-)$$

Where:

- $Q(G^+)$: Modularity for positive edges (want within communities)
- $Q(G^-)$: Modularity for negative edges (want between communities)
- $\alpha \in [0, 1]$: Balance parameter (typically 0.5)

2. Signed Spectral Clustering

Algorithm: ```python def signed_spectral_clustering(G, num_communities=2): """Spectral clustering for signed networks""" nodes = list(G.nodes()) n = len(nodes)

```
# Create signed adjacency matrix
A = nx.adjacency_matrix(G, weight='sign').toarray()
```

```
# Compute signed Laplacian
D = np.diag(np.sum(np.abs(A), axis=1))
L = D - A

# Find eigenvectors
eigenvalues, eigenvectors = np.linalg.eigh(L)

# Use Fiedler vector (second smallest eigenvalue)
fiedler_vector = eigenvectors[:, 1]

# Partition based on sign
communities = {}
for i, node in enumerate(nodes):
    communities[node] = 0 if fiedler_vector[i] < 0 else 1

return communities
```

```

### 3. Community Quality Metrics

```
def analyze_community_quality(G, communities):
    """Analyze quality of detected communities in signed
network"""
    internal_pos = 0 # Good
    internal_neg = 0 # Bad
    external_pos = 0 # Bad
    external_neg = 0 # Good

    for u, v, data in G.edges(data=True):
        sign = data['sign']
        same_community = (communities[u] == communities[v])

        if same_community:
            if sign == 1:
                internal_pos += 1
            else:
                internal_neg += 1
        else:
            if sign == 1:
                external_pos += 1
```
```

```
            else:
                external_neg += 1

    good_edges = internal_pos + external_neg
    bad_edges = internal_neg + external_pos
    quality = good_edges / (good_edges + bad_edges) if (good_edges
+ bad_edges) > 0 else 0

    return {
        'internal_positive': internal_pos,
        'internal_negative': internal_neg,
        'external_positive': external_pos,
        'external_negative': external_neg,
        'quality_score': quality
    }
```

Link Prediction

Balance-Based Prediction

Principle: Predict signs that maximize structural balance.

Algorithm:

```
def predict_edge_sign(G, node1, node2):
    """
    Predict edge sign using balance theory.

    Returns
    -------
    predicted_sign : int (1 or -1)
        Predicted sign
    confidence : float
        Prediction confidence
    """
    # Find common neighbors
    neighbors1 = set(G.neighbors(node1))
```

```
    neighbors2 = set(G.neighbors(node2))
    common_neighbors = neighbors1.intersection(neighbors2)

    if len(common_neighbors) == 0:
        return None, 0.0

    vote_positive = 0
    vote_negative = 0

    for neighbor in common_neighbors:
        sign1 = G[node1][neighbor]['sign']
        sign2 = G[node2][neighbor]['sign']

        # Balance rule: if signs match → predict positive
        if sign1 * sign2 == 1:
            vote_positive += 1
        else:
            vote_negative += 1

    predicted_sign = 1 if vote_positive > vote_negative else -1
    confidence = max(vote_positive, vote_negative) /
(vote_positive + vote_negative)

    return predicted_sign, confidence
```

Complete Analysis Example

```
import networkx as nx
import numpy as np

# Create signed network
G = nx.Graph()
G.add_edges_from([
    ('A', 'B', {'sign': 1}),
    ('B', 'C', {'sign': 1}),
    ('C', 'A', {'sign': 1}),  # Community 1: all friends
```

```
    ('D', 'E', {'sign': 1}),
    ('E', 'F', {'sign': 1}),  # Community 2: friends
    ('A', 'D', {'sign': -1}),
    ('B', 'E', {'sign': -1}),
    ('C', 'F', {'sign': -1})  # Between communities: enemies
])

# 1. Calculate balance index
balance_idx = calculate_balance_index(G)
print(f"Balance Index: {balance_idx:.3f}")

# 2. Calculate centrality (USE WALK-BASED - RECOMMENDED)
centrality = signed_walk_effect(G, max_steps=3)
print("\nTop 3 by Total Effect:")
sorted_nodes = sorted(centrality.items(),
                     key=lambda x: x[1]['total_effect'],
                     reverse=True)[:3]
for node, metrics in sorted_nodes:
    print(f"  {node}: TE={metrics['total_effect']:.3f}, "
          f"NE={metrics['net_effect']:.3f}")

# 3. Detect communities
communities = signed_spectral_clustering(G, num_communities=2)
print(f"\nCommunities: {communities}")

# 4. Calculate CORRECT signed modularity
Q_signed = signed_modularity(G, communities, alpha=0.5)
print(f"Signed Modularity: {Q_signed:.3f}")

# 5. Evaluate community quality
quality = analyze_community_quality(G, communities)
print(f"Community Quality: {quality['quality_score']:.3f}")

# 6. Link prediction
for node1 in ['A', 'D']:
    for node2 in ['B', 'E']:
        if not G.has_edge(node1, node2):
            pred_sign, conf = predict_edge_sign(G, node1, node2)
            if pred_sign:
                sign_str = '+' if pred_sign == 1 else '-'
                print(f"Predict {node1}-{node2}: {sign_str} "
                      f"(confidence: {conf:.2%})")
```

References

Critical Papers for Signed Networks

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