MAT0122 ÁLGEBRA LINEAR I

FOLHA DE SOLUÇÃO

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Assinatura

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Exercício: E47 **Data: 20/10/22**

SOLUÇÃO

(i) This is a proof by contrapositive. For S to be a base of V, Span(S) must equal V (confirmed by hypothesis) and S must be linearly independent.

Let $S = \{u_1, \ldots, u_n\}$ be linearly dependent. Then, there exist a $u_k \in S$ such that u_k is a linear combination of other vectors of S. In mathematical notation,

$$u_k = \sum_{i \neq k}^{n} \alpha_i u_i$$

The fact Span(S) $= V$ shows that for all $v \in V$, v is a linear combination of vectors of S.

$$\begin{aligned}
v &= \sum_{i}^{n} \alpha_i u_i \\
&= (\sum_{i \neq k} \alpha_i u_i) + \beta_k u_k \\
&= (\sum_{i \neq k} \alpha_i u_i) + \beta_k (\sum_{i \neq k} \alpha_i u_i) \\
&= \sum_{i \neq k} \alpha_i u_i + \sum_{i \neq k} \beta_k \alpha_i u_i \\
&= \sum_{i \neq k} \alpha_i u_i + \beta_k \alpha_i u_i \\
&= \sum_{i \neq k} (\alpha_i + \beta_k \alpha_i) u_i
\end{aligned}$$

Note that v can be written as a linear combination of vectors of a subset of S, where u_k is not included. But by the exercise hypothesis, for all subset of S such that this subset is not S, it cannot generate V ($\forall S' \subset S, S' \neq S \rightarrow \text{Span}(S) \neq V$), a contradiction. Therefore, the assumption of linear dependency of S is wrong. Thus, S is linearly independent, as we wanted.

(ii) For S to be a base of V, $\text{Span}(S)$ must equal V and S must be linearly independent (confirmed by hypothesis).
Assume the set $S' = S \cup T$, such that

- $S' = \{v_1, \ldots, v_N\}$
- $S = \{v_1, \ldots, v_n\}$
- $T = \{v_{n+1}, \ldots, v_N\}$
- $S \cap T = \emptyset$ (S and T do not share vectors)

Since S' is linearly dependent by hypothesis, then the 0 vector can be written as a non-trivial linear combination of vectors of S'. Thus, there exists $[\alpha_1, \ldots, \alpha_N] \neq 0$ such that

$$\begin{aligned} 0 &= [\alpha_1, \ldots, \alpha_N] \cdot [v_1, \ldots, v_N] \\ &= [\alpha_1, \ldots, \alpha_n] \cdot [v_1, \ldots, v_n] + [\alpha_{n+1}, \ldots, \alpha_N] \cdot [v_{n+1}, \ldots, v_N] \end{aligned}$$

Since S is linearly independent by hypothesis, the only way the 0 vector could be written as a linear combination in terms of S is if $[\alpha_1, \ldots, \alpha_n] = 0$. But we stated that $[\alpha_1, \ldots, \alpha_N] \neq 0$ and $[\alpha_1, \ldots, \alpha_n] = 0$, so $[\alpha_{n+1}, \ldots, \alpha_N]$ must have at least one $\alpha_x \neq 0$.

$$\begin{aligned} 0 &= [\alpha_1, \ldots, \alpha_n] \cdot [v_1, \ldots, v_n] + \alpha_x v_x \\ \implies v_x &= -\frac{1}{\alpha_x}[\alpha_1, \ldots, \alpha_n] \cdot [v_1, \ldots, v_n] \\ \implies v_x &= [-\frac{\alpha_1}{\alpha_x}, \ldots, -\frac{\alpha_n}{\alpha_x}] \cdot [v_1, \ldots, v_n] \end{aligned}$$

Since v_x is an arbitrary vector of T such that $T \subset V$, and v_x can indeed be expressed as a linear combination in terms of S, then $\text{Span}(S) = V$.