MAT0122 ÁLGEBRA LINEAR I

FOLHA DE SOLUÇÃO

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Assinatura

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Exercício: E77 **Data: 08/12/2022**

SOLUÇÃO

(i) Let $v_1 = [\alpha_1, \alpha_2, \alpha_3], v_2 = [\beta_1, \beta_2, \beta_3]$.
Then

$$M = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} M^\intercal = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

The matrix composition $M^\intercal M$ is

$$M^\intercal M = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \\ \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 & \beta_1^2 + \beta_2^2 + \beta_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} \|v_1\|^2 & \langle a, b\rangle \\ \langle a, b\rangle & \|v_2\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

On the other hand, the matrix composition $MM^\intercal$ is

$$MM^\intercal = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha_1^2 + \beta_1^2 & \alpha_1\alpha_2 + \beta_1\beta_2 & \alpha_1\alpha_3 + \beta_1\beta_3 \\ \alpha_1\alpha_2 + \beta_1\beta_2 & \alpha_2^2 + \beta_2^2 & \alpha_2\alpha_3 + \beta_2\beta_3 \\ \alpha_1\alpha_3 + \beta_1\beta_3 & \alpha_2\alpha_3 + \beta_2\beta_3 & \alpha_3^2 + \beta_3^2 \end{bmatrix}$$

It is fairly easy to notice that $M^\intercal M \neq MM^\intercal$ because in this particular case the first one is a two sided square matrix and the other one is a three sided square matrix.

(ii) The matrix multiplication $M^\intercal M$ has the main diagonal composed of inner products of the same vector

$$\langle v_i, v_i\rangle, 1 \leq i \leq n$$

And the other positions composed by inner products of distinct vectors

$$\langle v_i, v_j\rangle, 1 \leq i \neq j \leq n$$

With the fact that these vectors are orthonormal in pairs, then

$$\langle v_i, v_i \rangle = \|v_i\|^2 = 1, \langle v_i, v_j \rangle = 0$$

And therefore,

$$M^\intercal M = \begin{bmatrix} v_1 \\ \hline \dots \\ \hline v_n \end{bmatrix} \begin{bmatrix} v_1 \mid \dots \mid v_n \end{bmatrix} = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_{n-1} \rangle & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_2, v_{n-1} \rangle & \langle v_2, v_n \rangle \\ \vdots & & & & \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \dots & \langle v_n, v_{n-1} \rangle & \langle v_n, v_n \rangle \end{bmatrix} = I_n$$

The matrix MM^T is indeed I_n, too.
The argument that will be used consists on definitions and consequences of invertible matrices.
By hypothesis, the vectors are mutually orthogonal and their norms equals 1. By Proposition 9.5.1 of PNK, those vectors are linearly independent. Therefore, the columns of the square matrix M are linearly independent.
By Corollary 6.4.10, if a matrix A is square and their columns are linearly independent, then A is invertible. So M is invertible.
By Corollary 6.4.11, the transpose of an invertible matrix is invertible. Therefore $M^\intercal$ is invertible.
By Corollary 9.7.3, if Q is an orthogonal matrix then its inverse is its transpose $Q^\intercal$. Therefore, $M^\intercal = M^{-1}$.
By Lemma 4.13.11, if a $R \times C$ matrix A has an inverse A^{-1} then AA^{-1} is the $R \times R$ identity. Therefore, $MM^\intercal = I_n$.