MAT0122 ÁLGEBRA LINEAR I

FOLHA DE SOLUÇÃO

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Assinatura

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Exercício: E68 **Data:29/11/2022**

SOLUÇÃO

(i) The statement is true. An example follows below:
Let

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} b = \begin{bmatrix} 29 \\ 31 \\ 41 \end{bmatrix} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Therefore, we have $Ux = b$ as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 29 \\ 31 \\ 41 \end{bmatrix}$$

This can be translated as a linear system of equations

$$\begin{cases} x_1 + x_2 + x_3 + x_4 & = 29 \\ & 0 = 31 \\ & 0 = 41 \end{cases}$$

Where $0 \neq 0$.
Therefore, the solution set is empty.

(ii) The statement is false. By the kernel-image theorem,

$$\begin{aligned} \dim \mathbb{F}^C &= \dim \text{Null } U + \dim \text{Col } U \\ \dim \mathbb{R}^n &= \dim \text{Null } U + \dim \text{Col } U \\ n &= \dim \text{Null } U + \dim \text{Col } U \\ \dim \text{Null } U &= n - \dim \text{Col } U \end{aligned}$$

If this difference equals 0, then the null space of U is empty, therefore the function associated to U is injective, therefore exists U, b such that $\mathrm{Sol}(U, b)$ has exactly one element.
But in this case, $n = 4$, and $\dim \mathrm{Col}\, U$ is at maximum 3. This implies that $\dim \mathrm{Null}\, U > 0$, implying that it is impossible for Null U to be empty.
Therefore, there is no possible way that $\mathrm{Sol}(U, b)$ has exactly one solution.

(iii) The statement is true. An example follows below:
Let

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} b = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Therefore, we have $Ux = b$ as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

This can be translated as a linear system of equations

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 4 \quad (I) \\ \qquad x_2 + x_3 + x_4 = 3 \;\; (II) \\ \qquad\qquad x_3 + x_4 = 2 (III) \end{cases}$$

Where $x_1 = 1$ (subtracting II from I), $x_2 = 1$ (subtracting III from II) and $x_3 = 2 - x_4$ such that x_3 and x_4 are not univocally determined and can be a multiple pair of values.
Therefore, the solution set has infinite elements.