MAT0122 ÁLGEBRA LINEAR I

FOLHA DE SOLUÇÃO

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Assinatura

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Exercício: E35 **Data: 06/10/22**

SOLUÇÃO

The objective is to prove $\text{Null}H = \text{Span}\{G_{*1}, \ldots, G_{*4}\}$.

Firstly, let's prove $\text{Null}H \supset \text{Span}\{G_{*1}, \ldots, G_{*4}\}$.
Let $c = [G_{*1}|\ldots|G_{*4}][\alpha_1, \alpha_2, \alpha_3, \alpha_4] \in \text{Span}\{G_{*1}, \ldots, G_{*4}\}$ be a codeword. We have to show $Hc = 0$.

$$c = \begin{bmatrix} \alpha_1 + \alpha_3 + \alpha_4 \\ \alpha_1 + \alpha_2 + \alpha_4 \\ \alpha_4 \\ \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \end{bmatrix}$$

$$Hc = \begin{bmatrix} 0 + 0 + 0 + (\alpha_1 + \alpha_2 + \alpha_3) + \alpha_3 + \alpha_2 + \alpha_1 \\ 0 + (\alpha_1 + \alpha_2 + \alpha_4) + \alpha_4 + 0 + 0 + \alpha_2 + \alpha 1 \\ (\alpha_1 + \alpha_3 + \alpha_4) + 0 + \alpha_4 + 0 + \alpha_3 + 0 + \alpha_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the sum of two equal elements in GF(2) is 0, then $Hc = 0$, thus $\text{Null}H \supset \text{Span}\{G_{*1}, \ldots, G_{*4}\}$.

Now, let's prove $\text{Null}H \subset \text{Span}\{G_{*1}, \ldots, G_{*4}\}$.
Let $c = [c_1, \ldots, c_7] \in \text{Null}H$ be a codeword. We have to find $[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ such that $c = G[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$.

Below is the matrix-vector product.

$$G\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = \begin{bmatrix}\alpha_1+\alpha_2+\alpha_4\\ \alpha_1+\alpha_3+\alpha_4\\ \alpha_1\\ \alpha_2+\alpha_3+\alpha_4\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix}$$

Let $c = G[\alpha_1 \ldots \alpha_4]$.

$$c = \begin{bmatrix}c_1\\ c_2\\ c_3\\ c_4\\ c_5\\ c_6\\ c_7\end{bmatrix} = G\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = \begin{bmatrix}\alpha_1+\alpha_2+\alpha_4\\ \alpha_1+\alpha_3+\alpha_4\\ \alpha_1\\ \alpha_2+\alpha_3+\alpha_4\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix}$$

We've found candidates to the alphas: $\alpha_1 = c_3$, $\alpha_2 = c_5$, $\alpha_3 = c_6$, $\alpha_4 = c_7$. The matrix-vector product can be rewritten as:

$$\begin{bmatrix}\alpha_1+\alpha_2+\alpha_4\\ \alpha_1+\alpha_3+\alpha_4\\ \alpha_1\\ \alpha_2+\alpha_3+\alpha_4\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = \begin{bmatrix}c_3+c_5+c_7\\ c_3+c_6+c_7\\ c_3\\ c_5+c_6+c_7\\ c_5\\ c_6\\ c_7\end{bmatrix}$$

Note below that $c = G[\alpha_1 \ldots \alpha_4]$ is indeed in Null$H$.

$$H\begin{bmatrix}\alpha_1+\alpha_2+\alpha_4\\ \alpha_1+\alpha_3+\alpha_4\\ \alpha_1\\ \alpha_2+\alpha_3+\alpha_4\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = H\begin{bmatrix}c_3+c_5+c_7\\ c_3+c_6+c_7\\ c_3\\ c_5+c_6+c_7\\ c_5\\ c_6\\ c_7\end{bmatrix} = \begin{bmatrix}(c_5+c_6+c_7)+c_5+c_6+c_7\\ (c_3+c_6+c_7)+c_3+c_6+c_7\\ (c_3+c_5+c_7)+c_3+c_5+c_7\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Thus, if $\alpha_1 = c_3$, $\alpha_2 = c_5$, $\alpha_3 = c_6$, $\alpha_4 = c_7$, then $c \in \text{Span}\{G_{*1}, \ldots, G_{*4}\}$, and consequently Null$H \subset \text{Span}\{G_{*1}, \ldots, G_{*4}\}$.