MAT0122 ÁLGEBRA LINEAR I

FOLHA DE SOLUÇÃO

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Assinatura

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Exercício: E34 **Data: 06/10/22**

SOLUÇÃO

Caveat: It is already assumed that

$$\text{a function is invertible} \iff \text{a function is bijective (injective and surjective)}$$

(i) The statement is false. A simple counterexample is shown below:
Let $X = \{1\}, Y = \{1, 2\}, Z = \{1\}$. Let $g : X \to Y$ be the function $x \mapsto x$. Let $f : Y \to Z$ be the function $y \mapsto 1$.
Unfortunately, I don't know how to draw set schemes in LaTeX, so follow along in your head :(
It is easy to notice that $f \circ g : X \to Z$ is invertible because $f \circ g$ is bijective – 1 from X is mapped to 1 from Z and vice-versa. But g is not surjective – 2 from Y is not an image of any element in X – and f is not injective both 1 and 2 from Y are mapped to 1 in Z). Therefore, neither f nor g are invertible.

(ii) The statement is false. Here's an counterexample (gathered by PNK, page 246).
Suppose $A \in \mathbb{R}^{2x3}$ and $B \in \mathbb{R}^{3x2}$ such that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.
The product AB is I_2, notably inversive. Let $f_A : \mathbb{R}^3 \to \mathbb{R}^2$ and $f_B : \mathbb{R}^2 \to \mathbb{R}^3$ the associated functions of matrices A and B.
Note that for any $\lambda \in \mathbb{R}$, $A \begin{bmatrix} 0 \\ 0 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, therefore f_A is not injective, and neither A is invertible.

Note that there is no vector $x \in \mathbb{R}^2$ such that $f_B(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. A quick proof is the following:

Suppose a generic vector $x = \begin{bmatrix} m \\ n \end{bmatrix}$. The product $B \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m \\ n \\ 0 \end{bmatrix}$, where 0 is fixed, meaning that the image of f_B will never include $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Therefore, f_B is not surjective, and neither B is invertible.

(iii) The statement is false. Following the same example of exercise (ii), it has been shown that A is not invertible, and consequently does not accept an inverse A^{-1}.

(iv) The statement is false. Following the same example of exercise (ii), BA is indeed $I_C = I_3$, and A is still not invertible, and consequently does not accept an inverse A^{-1}.