MAT0122 ÁLGEBRA LINEAR I

FOLHA DE SOLUÇÃO

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Assinatura

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Exercício: E54 **Data: 25/10/2022**

SOLUÇÃO

(i) Let's prove that for all integer $n \geq 0$, it is true

$$\sum_{0 \leq i < n} x^i = \frac{1 - x^n}{1 - x}$$

Proof:

$$\sum_{0 \leq i < n} x^i = x^0 + x^1 + x^2 + \cdots + x^n$$

$$\sum_{0 \leq i < n} x^i - \sum_{0 \leq i < n} x^{i+1} = (x^0 + x^1 + x^2 + \cdots + x^{n-1}) - (x^1 + x^2 + \cdots + x^n)$$

$$\sum_{0 \leq i < n} x^i - x \sum_{0 \leq i < n} x^i = x^0 - x^n$$

$$(1 - x) \sum_{0 \leq i < n} x^i = 1 - x^n$$

$$\sum_{0 \leq i < n} x^i = \frac{1 - x^n}{1 - x}$$

(ii)

$$F_2 = \begin{bmatrix} \omega^{00} & \omega^{01} \\ \omega^{10} & \omega^{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & (e^{2\pi i/2})^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & e^{\pi i} \end{bmatrix}$$

$$F_3 = \begin{bmatrix} \omega^{00} & \omega^{01} & \omega^{02} \\ \omega^{10} & \omega^{11} & \omega^{12} \\ \omega^{20} & \omega^{21} & \omega^{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & (e^{2\pi i/3})^1 & (e^{2\pi i/3})^2 \\ 1 & (e^{2\pi i/3})^2 & (e^{2\pi i/3})^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{8\pi i/3} \end{bmatrix}$$

$$F_4 = \begin{bmatrix} \omega^{00} & \omega^{01} & \omega^{02} & \omega^{03} \\ \omega^{10} & \omega^{11} & \omega^{12} & \omega^{13} \\ \omega^{20} & \omega^{21} & \omega^{22} & \omega^{23} \\ \omega^{30} & \omega^{31} & \omega^{32} & \omega^{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & (e^{2\pi i/4})^1 & (e^{2\pi i/4})^2 & (e^{2\pi i/4})^3 \\ 1 & (e^{2\pi i/4})^2 & (e^{2\pi i/4})^4 & (e^{2\pi i/4})^6 \\ 1 & (e^{2\pi i/4})^3 & (e^{2\pi i/4})^6 & (e^{2\pi i/4})^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{\pi i/2} & e^{\pi i} & e^{3\pi i/2} \\ 1 & e^{\pi i} & e^{2\pi i} & e^{3\pi i} \\ 1 & e^{3\pi i/2} & e^{3\pi i} & e^{9\pi i/2} \end{bmatrix}$$

(iii) Let's show matrix $G_n\,F_n$:

$$G_n\,F_n = \begin{bmatrix} \omega^{-00} & \omega^{-01} & \omega^{-02} & \omega^{-03} & \dots \\ \omega^{-10} & \omega^{-11} & \omega^{-12} & \omega^{-13} & \dots \\ \omega^{-20} & \omega^{-21} & \omega^{-22} & \omega^{-23} & \dots \\ \omega^{-30} & \omega^{-31} & \omega^{-32} & \omega^{-33} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \omega^{00} & \omega^{01} & \omega^{02} & \omega^{03} & \dots \\ \omega^{10} & \omega^{11} & \omega^{12} & \omega^{13} & \dots \\ \omega^{20} & \omega^{21} & \omega^{22} & \omega^{23} & \dots \\ \omega^{30} & \omega^{31} & \omega^{32} & \omega^{33} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Note that if $i \neq j$, the cell with row i and column j in matrix $G_n\,F_n$ is equal to

$$\begin{aligned} (G_n\,F_n)_{i,j} &= \sum_{0 \le k < n} \frac{\omega^{kj}}{\omega^{ik}} \\ &= \sum_{0 \le k < n} \omega^{k(j-i)} \\ &= \frac{1 - (\omega^{(j-i)})^n}{1 - \omega^{(j-i)}} && \text{(using the sum formula in (i))} \\ &= \frac{1-1}{1 - \omega^{j-1}} && \text{(using } \omega^n = (e^{2\pi i/n})^n = e^{2\pi i} = 1\text{)} \\ &= 0 \end{aligned}$$

If $i = j$, then

$$(G_n\,F_n)_{i,j} = \sum_{0 \le k < n} \frac{\omega^{kj}}{\omega^{ik}} = \sum_{0 \le k < n} 1 = n$$

Therefore, matrix $G_n\,F_n = nI_S$.

Now, let's show matrix $F_n\,G_n$:

$$F_n\,G_n = \begin{bmatrix} \omega^{00} & \omega^{01} & \omega^{02} & \omega^{03} & \dots \\ \omega^{10} & \omega^{11} & \omega^{12} & \omega^{13} & \dots \\ \omega^{20} & \omega^{21} & \omega^{22} & \omega^{23} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \omega^{-00} & \omega^{-01} & \omega^{-02} & \omega^{-03} & \dots \\ \omega^{-10} & \omega^{-11} & \omega^{-12} & \omega^{-13} & \dots \\ \omega^{-20} & \omega^{-21} & \omega^{-22} & \omega^{-23} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Similarly, if $i \neq j$, then the cell with row i and column j in matrix $F_n\,G_n$ is equal to

$$\begin{aligned} (F_n\,G_n)_{i,j} &= \sum_{0 \le k < n} \frac{\omega^{ik}}{\omega^{kj}} \\ &= \sum_{0 \le k < n} \omega^{k(i-j)} \\ &= \frac{1 - (\omega^{(i-j)})^n}{1 - \omega^{(i-j)}} && \text{(using the sum formula in (i))} \\ &= \frac{1-1}{1 - \omega^{j-1}} && \text{(using } \omega^n = (e^{2\pi i/n})^n = e^{2\pi i} = 1\text{)} \\ &= 0 \end{aligned}$$

Therefore, matrix $F_n\, G_n = nI_S$ as well.
We have $F_n\, G_n = nI_S = G_n\, F_n$. For that reason, F_n is invertible.

(iv) Lets prove the column space of matrix F_n is a basis for $\mathbb{C}^S$.

The column space of F_n is linearly independent.
Proof: Suppose a vector $\alpha = [\alpha_0, \ldots, \alpha_{n-1}]$, $\alpha \in \mathbb{C}^S$. We must show that

$$F_n \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = 0 \implies \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = 0$$

Or else, show $\mathrm{Null}(F_n) = 0$, such that f_{F_n} is the function associated to the matrix.
We've proven that F_n is an invertible matrix. Because of this, f_{F_n} is injective. Therefore, only the 0 vector maps to 0, and $\mathrm{Null}(F_n) = 0$.

We can easily see that the cardinality of $\{v_0, \ldots, v_{n-1}\}$ is n, which is equal to the cardinality of the standard basis for C^S. Allied to the fact that $\{v_0, \ldots, v_{n-1}\}$ is linearly independent, it implies that $\{v_0, \ldots, v_{n-1}\}$ generates C^S.
Therefore, $\{v_0, \ldots, v_{n-1}\}$ is a basis for C^S.