MAT0122 ÁLGEBRA LINEAR I

FOLHA DE SOLUÇÃO

Nome: Gabriel Haruo Hanai Takeuchi **Número USP: 13671636**

Assinatura

Gabriel Haruo Hanai Takeuchi

Sua assinatura atesta a autenticidade e originalidade de seu trabalho e que você se compromete a seguir o código de ética da USP em suas atividades acadêmicas, incluindo esta atividade.

Exercício: E77 **Data: 08/12/2022**

SOLUÇÃO

(i) Let $v_1 = [\alpha_1, \alpha_2, \alpha_3], v_2 = [\beta_1, \beta_2, \beta_3]$.
Then

$$M = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} M^\intercal = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

The matrix composition $M^\intercal M$ is

$$M^\intercal M = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \\ \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 & \beta_1^2 + \beta_2^2 + \beta_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} \|v_1\|^2 & \langle a, b\rangle \\ \langle a, b\rangle & \|v_2\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

On the other hand, the matrix composition $MM^\intercal$ is

$$MM^\intercal = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha_1^2 + \beta_1^2 & \alpha_1\alpha_2 + \beta_1\beta_2 & \alpha_1\alpha_3 + \beta_1\beta_3 \\ \alpha_1\alpha_2 + \beta_1\beta_2 & \alpha_2^2 + \beta_2^2 & \alpha_2\alpha_3 + \beta_2\beta_3 \\ \alpha_1\alpha_3 + \beta_1\beta_3 & \alpha_2\alpha_3 + \beta_2\beta_3 & \alpha_3^2 + \beta_3^2 \end{bmatrix}$$

It is fairly easy to notice that $M^\intercal M \neq MM^\intercal$ because in this particular case the first one is a two sided square matrix and the other one is a three sided square matrix.

(ii) The matrix multiplication $M^\intercal M$ has the main diagonal composed of inner products of the same vector

$$\langle v_i, v_i\rangle, 1 \leq i \leq n$$

And the other positions composed by inner products of distinct vectors

$$\langle v_i, v_j\rangle, 1 \leq i \neq j \leq n$$

With the fact that these vectors are orthonormal in pairs, then

$$\langle v_i, v_i \rangle = \|v_i\|^2 = 1, \langle v_i, v_j \rangle = 0$$

And therefore,

$$M^\intercal M = \left[\begin{array}{c} v_1 \\ \hline \dots \\ \hline v_n \end{array}\right] \left[\begin{array}{c|c|c} v_1 & \dots & v_n \end{array}\right] = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_{n-1} \rangle & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_2, v_{n-1} \rangle & \langle v_2, v_n \rangle \\ \vdots & & & & \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \dots & \langle v_n, v_{n-1} \rangle & \langle v_n, v_n \rangle \end{bmatrix} = I_n$$

The matrix MM^T is indeed I_n too.

$$MM^T = \left[\begin{array}{c|c|c} v_1 & \dots & v_n \end{array}\right] \left[\begin{array}{c} v_1 \\ \hline \dots \\ \hline v_n \end{array}\right]$$