數值分析 Team5 Homwork4

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1. By the equation, we can have $-y_{n+3} - ay_{n+2} + y_{n+1} + y_n + hb(f_{n+2} + f_{n+1}) = 0$,

   we have $\begin{cases} a_{-1} & = -1 \\ a_0 & = -a \\ a_1 & = a \\ a_2 & = 1 \end{cases}$ and $\begin{cases} b_{-1} & = 0 \\ b_0 & = b \\ b_1 & = b \\ b_2 & = 0 \end{cases}$ then observe that

$$C_0 = -1 - a + a + 1 = 0$$

$$C_1 = \sum_{i=-1}^{2} (1-i)a_i + \sum_{i=-1}^{2} b_i = -2 - a - 1 + 2b = 0$$

$$\Rightarrow a - 2b = -3$$

$$C_2 = \frac{1}{2!}\sum_{i=-1}^{2} (1-i)^2 a_i + \sum_{i=-1}^{2} (1-i)b_i = \frac{1}{2}(-4 - a + 1) + b = 0$$

$$\Rightarrow a - 2b = -3$$

$$C_3 = \frac{1}{3!}\sum_{i=-1}^{2} (1-i)^3 a_i + \frac{1}{2!}\sum_{i=-1}^{2} (1-i)^2 b_i = \frac{1}{3!}(-8 - a - 1) + \frac{1}{2!}b = 0$$

$$\Rightarrow a - 3b = -9$$

$$C_4 = \frac{1}{4!}\sum_{i=-1}^{2} (1-i)^4 a_i + \frac{1}{3!}\sum_{i=-1}^{2} (1-i)^3 b_i = \frac{1}{4!}(-16 - a + 1) + \frac{1}{3!}b = 0$$

$$\Rightarrow a - 4b = -15$$

   by C_1 and C_3, we have $a = 9$ and $b = 6$

$$\begin{aligned} C_5 &= \frac{1}{5!}\sum_{i=-1}^{2} (1-i)^5 a_i + \frac{1}{4!}\sum_{i=-1}^{2} (1-i)^4 b_i \\ &= \frac{1}{5!}(-32 - a - 1) + \frac{1}{4!}b \\ &= \frac{1}{5!}(-33 - 9 + 5 * 6) \neq 0 \end{aligned}$$

therefore,it has degree of accuracy 4 Plug in $a = 9,\ b = -6$, We get

$$\begin{aligned} P_1 &= \sum_{i=-1}^{2} a_i r^i \\ &= -(r-1)(r^2 + (a+1)r + 1) \\ &= -(r-1)(r^2 + 10r + 1) \end{aligned}$$

$$\Rightarrow r = 1 \vee -5 \pm 2\sqrt{6}$$

$$\text{But } |-5 - 2\sqrt{6}| \ > 1 \ \Rightarrow \text{Not zero stable}$$

2. $f(x) = x^2$, $x \in [-\pi, \pi]$ and period is 2π , then we get

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right) \right]$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\ dx \\ &= \frac{1}{2\pi} \left[\frac{1}{3} x^3\right]_{-\pi}^{\pi} \\ &= \frac{\pi^2}{3} \end{aligned}$$

for $n \geq 1$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos\left(\frac{n\pi x}{\pi}\right)\ dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\ dx \\ &= \frac{1}{\pi} \left\{ \left[x^2 * \frac{1}{n} \sin(nx) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x * \frac{1}{n} \sin(nx)\ dx \right\} \\ &= \frac{1}{\pi} \left[0 - \frac{2}{n} \int_{-\pi}^{\pi} x \sin(nx)\ dx \right] \\ &= \frac{-2}{n\pi} \left\{ \left[x(\frac{-1}{n}) \cos(nx) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (\frac{-1}{n}) \sin(nx)\ dx \right\} \\ &= \frac{-2}{n\pi} \left\{ \frac{-2\pi}{n} \cos(n\pi) - \left[\frac{1}{n^2} \cos(nx) \right]_{-\pi}^{\pi} \right\} \\ &= \frac{4}{n^2} \cos(n\pi) \end{aligned}$$

Since $x^2 \sin(nx)$ is odd function

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(nx)\ dx = 0$$

therefore

$$
\begin{aligned}
f(x) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(n\pi)\cos(nx) \\
&= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)
\end{aligned}
$$

3.

$$
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \text{ and } \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

therefore

$$
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ , where } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

then $\det\begin{pmatrix} 0-x & -1 \\ 1 & 0-x \end{pmatrix} = x^2+1 \Rightarrow$ eigenvalues are i and $-i$

let $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ be two eigenvectors , then

$$
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{2}e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2}e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}
$$

therefore

$$
y_1 = \frac{1}{2}(e^{it} + e^{-it}) \text{ , } y_2 = \frac{i}{2}(e^{it} - e^{-it})
$$

$$
y_1^2 + y_2^2 = \frac{1}{4}[(e^{2it} + 2 + e^{-2it}) - (e^{2it} - 2 + e^{-2it})] = \frac{1}{4} * 4 = 1
$$

MATLAB code for two methods

```matlab
h = 1/10^4;
N = 0:h:10;
L = length(N);

%Start euler
y1 = ones(2,L);
y1(:,1) = [1;0];
for n = 1:L
    y1(:,n+1) = [1,-h;h,1] * y1(:,n);
end
E = sum( y1(:,n) .^ 2);

%Start Runge-Kutta
y2 = ones(2,L);
k1 = ones(2,L);
k2 = ones(2,L);
k3 = ones(2,L);
k4 = ones(2,L);
y2(:,1) = [1;0];
for i = 1:length(N)
    k1(:,i) = [0,-1 ; 1,0] * y2(:,i);
    k2(:,i) = k1(:,i) + h/2 .* k1(:,i);
    k3(:,i) = k1(:,i) + h/2 .* k2(:,i);
    k4(:,i) = k1(:,i) + h .* k3(:,i);
    y2(:,i+1) = y2(:,i) + h/6 .* ( k1(:,i) + k2(:,i) ./ 2 + k3(:,i) ./ 2 + k4(:,i) );
end
R = sum( y2(:,i) .^ 2);

fprintf('By Euler method %d\n',E);
fprintf('By Runge-Kutta method %d\n',R);
```