數值分析 Team5 Homwork3

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[Theoretical problems] 修改 1.(d)(e)

1. (a)

$$T_0 = \cos(0 * \cos^{-1} x) = 1$$
$$T_1 = \cos(1 * \cos^{-1} x) = x$$

Let $T_{n-1} = \cos((n-1) * \cos^{-1} x)$

$$\begin{aligned} 2x * T_{n-1} &= 2\cos(\frac{2}{2} * \cos^{-1} x) * \cos(\frac{2n-2}{2} * \cos^{-1} x) \\ &= 2\cos(\frac{n-(n-2)}{2} * \cos^{-1} x) * \cos(\frac{n+(n-2)}{2} * \cos^{-1} x) \\ &= \cos(n * \cos^{-1} x) + \cos((n-2) * \cos^{-1} x) \\ &= T_n + T_{n-2} \end{aligned}$$

(b) For $x \in [-1, 1],\ k \in \mathbb{Z}$

$$\begin{aligned} \cos(n * \cos^{-1} x_k) &= 0 \\ n * \cos^{-1} x_k &= \cos^{-1} 0 = \frac{2k-1}{2}\pi \\ \cos^{-1} x_k &= \frac{2k-1}{2n}\pi \\ x_k &= \cos(\frac{2k-1}{2n})\pi \end{aligned}$$

For $k = 1, 2, \cdots, n$, we have n zeros.

(c)

$$\begin{aligned} (T_n)' = \cos(n \cos^{-1} \tilde{x}_k)' &= 0 \\ \sin(n \cos^{-1} \tilde{x}_k)(\frac{n}{\sqrt{1-\tilde{x}_k^2}}) &= 0 \\ \sin(n \cos^{-1} \tilde{x}_k) &= 0 \\ n \cos^{-1} \tilde{x}_k &= k\pi\ ,\quad k \in \mathbb{Z} \\ \cos^{-1} \tilde{x}_k &= \frac{k\pi}{n} \\ \tilde{x}_k &= \cos\frac{k\pi}{n} \end{aligned}$$

For $k = 0, 1, 2, \cdots, n$, we have n+1 zeros.

$$\begin{aligned} T_n(\tilde{x}_k) &= \cos(n\cos^{-1}(\cos\frac{k\pi}{n})) \\ &= \cos(n * \frac{k\pi}{n}) \\ &= \cos(k\pi) \\ &= (-1)^k \end{aligned}$$

(d) Since $\max\left|T_n(x)\right| = 1$, the problem can be reduce as

$$\frac{1}{2^{n-1}} \leq \max_{-1\leq x\leq 1}\left|p(x)\right|$$

By the hint , assume $\exists\ p \in P_n$

$$\max_{x\in[-1,1]}\left|p(x)\right| < \frac{1}{2^{n-1}}$$

Consider function $r = \frac{T_n}{2^{n-1}} - p$, $p \neq \frac{T_n}{2^{n-1}}$, and deg $r \leq n-1$
By Fundamental Theory of Algebra , # roots of r $\leq n-1$
But $\left|p(x)\right| \leq \left\|\frac{T_n}{2^{n-1}}\right\|_\infty$, $\forall x \in [-1, 1]$
of the roots shouldn't be less than n $\Rightarrow$ # roots of r $\geq n \rightarrow\leftarrow$
therefore , $\forall p \in P_n$

$$\max_{x\in[-1,1]}\frac{|T_n|}{2^{n-1}} \leq \max_{-1\leq x\leq 1}\left|p(x)\right|$$

(e) Since $(x-x_0)\cdots(x-x_n) \in P_{n+1}$
Choose $\tilde{x}_k = \cos(\frac{2k+1}{2(n+1)}\pi)$, $k = 0, \cdots, n$
to minimize the maximum of $(x-x_0)\cdots(x-x_n)$
by (d) we get

$$\frac{1}{2^n} = \max_{-1\leq x\leq 1}\left|(x-\tilde{x}_1)\cdots(x-\tilde{x}_n)\right| \leq \max_{-1\leq x\leq 1}\left|(x-x_1)\cdots(x-x_n)\right|$$

Since $f^{(n+1)}$ is continuous, by extreme value theorem, there exist $X = \max|f(x)|$, where $x \in [-1, 1]$. Therefore, the maximal error should be

$$\Longrightarrow \max_{x\in(-1,1)}\left|f(x) - p(x)\right|$$

$$= \max_{x\in(-1,1)}\left|\frac{f(\xi(x))}{(n+1)!}\right| \cdot \left|(x-\tilde{x_1})(x-\tilde{x_2})\cdots(x-\tilde{x_n})\right|$$

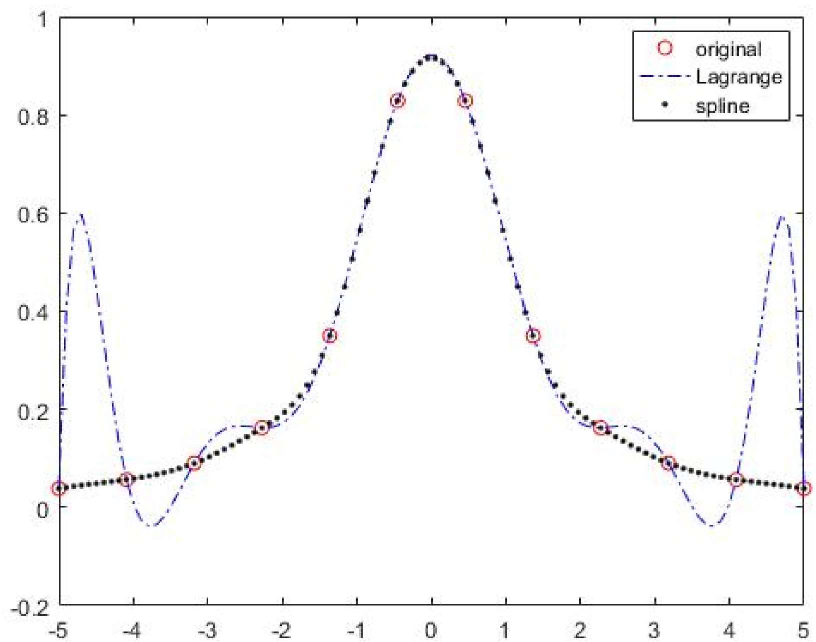
$$\leq \left|\frac{M}{(n+1)!}\right| \max_{x\in(-1,1)}\left|(x-x_1)\cdots(x-x_n)\right|$$

where x_i are randomly picked in (-1,1) satisfy $-1 = x_0 < x_1 < \cdots < x_n = 1,\ \forall\ i$

[Numerical Problems] 修改 1. 和 3.

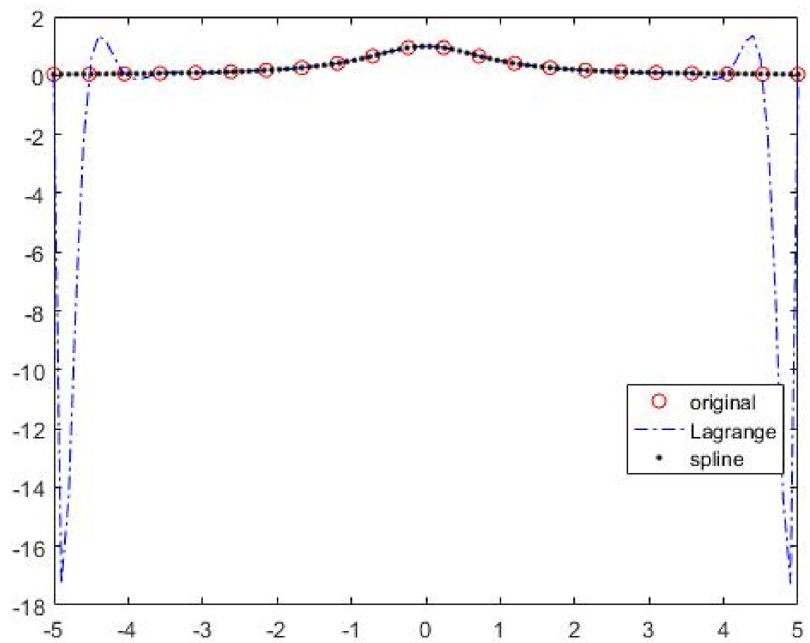
1. (a) **for** $n_1 = 11$

```
n1=11;
X=linspace(-5,5);
x = [-5:10/n1:5];
y = 1./(1+x.^2);
Y = polyval(polyfit(x,y,n1),X);
yy = spline(x,y,X);
plot(x,y,'ro',X,Y,'b-.',X,yy,'k.');
legend('original','Lagrange','spline','Location','best');
```



for $n_2 = 21$

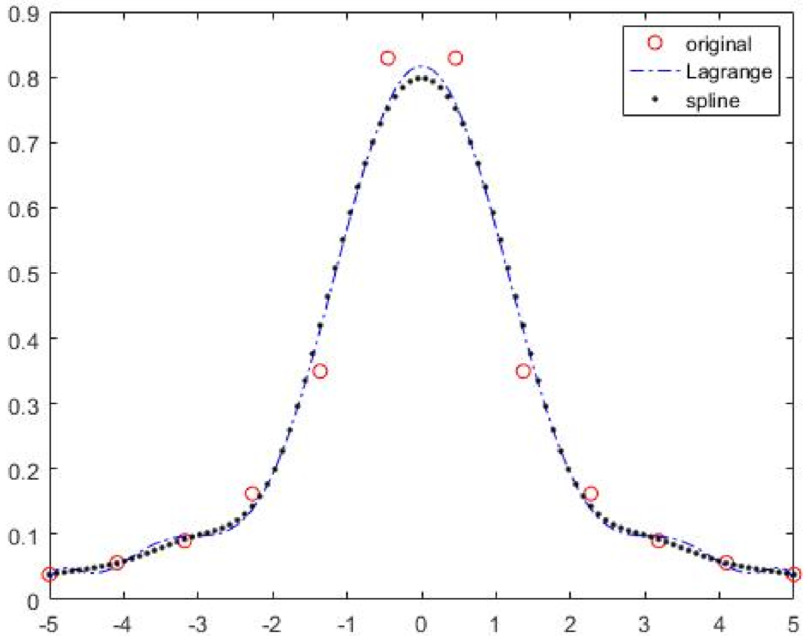
```
n2=21;
X=linspace(-5,5);
x = [-5:10/n2:5];
y = 1./(1+x.^2);
Y = polyval(polyfit(x,y,n2),X);
yy = spline(x,y,X);
plot(x,y,'ro',X,Y,'b-.',X,yy,'k.');
legend('original','Lagrange','spline','Location','best');
```



(b) As $n \to \infty$, the biggest error in the interval will also approximate infinity .

(c) **for** $n_1 = 11$

```
n1=11;
X=linspace(-5,5);
x = [-5:10/n1:5];
y = 1./(1+x.^2);
xc = 1/2*[(5-(-5))*cos((2*(1:n1+1)-1)*pi/(2*n1+2))-5+5];
yc = 1./(1+xc.^2);
Y = polyval(polyfit(xc,yc,n1),X);
yy = spline(xc,yc,X);
plot(x,y,'ro',X,Y,'b-.',X,yy,'k.');
legend('original','Lagrange','spline','Location','best');
```



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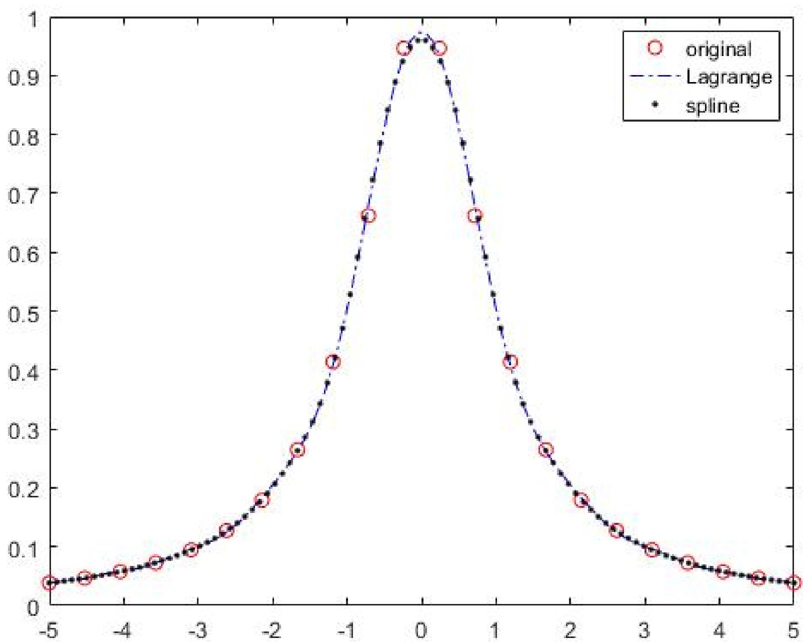
y = 1./(1+x.^2);
xc = 1/2*[(5-(-5))*cos((2*(1:n2+1)-1)*pi/(2*n2+2))-5+5];
yc = 1./(1+xc.^2);
Y = polyval(polyfit(xc,yc,n2),X);
yy = spline(xc,yc,X);
plot(x,y,'ro',X,Y,'b-.',X,yy,'k.');
legend('original','Lagrange','spline','Location','best');
```


original
Lagrange
spline
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5