數值分析 Team5 Homwork3

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[Theoretical problems]

1. (a) Use Gaussian quadrature with $n = 2$, let $f(x) = \frac{2x}{x^2-4}$

$$\begin{aligned}
\int_1^{1.6} \frac{2x}{x^2-4}\,dx &= \frac{1.6-1}{2}\int_{-1}^{1} f\left(\frac{1.6-1}{2}x + \frac{1.6+1}{2}\right)\,dx \\
&= 0.3\int_{-1}^{1} f(0.3x+1.3)\,dx \\
&\approx 0.3\ [\ \alpha_1 f(x_1) + \alpha_2 f(x_2) + \alpha_3 f(x_3)\ ]
\end{aligned}$$

$$\begin{aligned}
\text{where}\quad \alpha_1 &= \frac{5}{9} \approx 0.5556\ ,\ x_1 = -\sqrt{0.6} \approx -0.7746 \\
\alpha_2 &= \frac{8}{9} \approx 0.8889\ ,\ x_2 = 0 \\
\alpha_3 &= \frac{5}{9} \approx 0.5556\ ,\ x_3 = \sqrt{0.6} \approx 0.7746
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_1^{1.6} \frac{2x}{x^2-4}\,dx &\approx 0.3\ [\ 0.5556 * f(\ 0.3(-0.7746)+1.3\ ) + 0.8889 * f(\ 0.3*0+1.3\ ) \\
&\quad +0.5556 * f(\ 0.3(0.7746)+1.3\ )\ ] \\
&= 0.3\ [\ 0.556 * f(1.0676) + 0.8889 * f(1.3) + 0.5556 * f(1.5324)\ ] \\
&= 0.3\ [\ (-0.4427) + (-1.0005) + (-1.0309)\ ] \\
&= -0.7422
\end{aligned}$$

And the exact value is

$$\begin{aligned}
\int_1^{1.6} \frac{2x}{x^2-4}\,dx &= \int_3^{1.44} \frac{dy}{y} \quad (\text{let } y = 4-x^2, dy = -2xdx) \\
&= [\ \ln(y)\ ]_3^{1.44} \\
&= \ln(1.44) - \ln(3) \\
&= -0.734
\end{aligned}$$

(b) Use Gaussian quadrature with $n = 2$, let $f(x) = \frac{x}{\sqrt{x^2-4}}$

$$\begin{aligned}
\int_3^{3.5} \frac{x}{\sqrt{x^2-4}} &= \frac{3.5-3}{2} \int_{-1}^{1} f\left(\frac{3.5-3}{2}x + \frac{3.5+3}{2}\right)\ dx \\
&\approx 0.25\ [\ \alpha_1 f(x_1) + \alpha_2 f(x_2) + \alpha_3 f(x_3)\ ] \\
&= 0.25\ [\ 0.5556 * f(\ 0.25 * (-0.7746) + 3.25\ ) + 0.8889 * f(3.25) \\
&\quad +0.5556 * f(\ 0.25 * (0.7746) + 3.25\ )\ ] \\
&= 0.25\ [\ 0.5556 * f(3.0564) + 0.8889 * f(3.25) + 0.5556 * f(3.4437)\ ] \\
&= 0.25\ [\ 0.7345 + 1.1277 + 0.6825\ ] \\
&= 0.6362
\end{aligned}$$

And the exact value is

$$\begin{aligned}
\int_3^{3.5} \frac{x}{\sqrt{x^2-4}}\ dx &= \int_5^{8.25} \frac{dy}{2\sqrt{y}} \quad (\text{let } y = x^2 - 4, dy = -2xdx) \\
&= [\ \sqrt{y}\ ]_5^{8.25} \\
&= \sqrt{8.25} - \sqrt{5} \\
&= 2.8723 - 2.2361 \\
&= 0.6362
\end{aligned}$$

2. We know that given n point, we can get the order of precision to $2n - 1$

then consider the following function.

$$f(x) = \Pi_{i=1}^{n} (x - x_i)^2$$

then by the lecture note

$$\begin{aligned}
Q_n(x) &= \Sigma_{i=1}^{n} w_i f(x_i) = 0 \\
I(f) &= \int_a^b f(x)\ dx > 0 \\
E_n &= I(f) - Q_n > 0
\end{aligned}$$

By expanding f, we knew that f can be express as

$$f(x) = \Sigma_{i=0}^{2n}\ \alpha_i x^i$$

for each x^i , $0 \leq i \leq 2n - 1$

$$\begin{aligned}
E_n(x^i) &= I(x^i) - Q_n(x^i) = 0 \\
\Rightarrow E_n(\alpha_i x^i) &= I(\alpha_i x^i) - Q_n(\alpha_i x^i) = 0 \\
\text{but } E_n &= I(f) - Q_n > 0 \\
\Rightarrow\ E_n(x^{2n}) &= I(x^{2n}) - Q_n(x^{2n}) > 0 \quad QED
\end{aligned}$$

3. Let $h = \frac{b-a}{3}$, $x_0 = a + h$, $x_1 = b + h$

By Newton formula

$$
\begin{aligned}
f(t) &= f(x_0)\frac{t-x_1}{x_0-x_1} + f(x_1)\frac{t-x_0}{x_1-x_0} \\
\widetilde{\int_a^b} f(t)\ dt &= \int_a^b f(x_0)\frac{t-x_1}{x_0-x_1} + f(x_1)\frac{t-x_0}{x_1-x_0}\ dt \\
&= f(\frac{2a+b}{3})\frac{\frac{b^2-a^2}{2} - \frac{a+2b}{3}(b-a)}{\frac{a-b}{3}} + f(\frac{a+2b}{3})\frac{\frac{b^2-a^2}{2} - \frac{2a+b}{3}(b-a)}{\frac{b-a}{3}} \\
&= \frac{3}{2}(\frac{b-a}{3})\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] \\
&= \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right]
\end{aligned}
$$

(1) If $f(x) = 1$, $\int_a^b 1\ dx = b - a$

$$\widetilde{\int_a^b} 1\ dx = \frac{b-a}{2}[1+1] = b-a$$

(2) If $f(x) = x$, $\int_a^b x\ dx = \frac{b^2-a^2}{2}$

$$\widetilde{\int_a^b} x\ dx = \frac{b-a}{2}(\frac{3a+3b}{3}) = \frac{b^2-a^2}{2}$$

(3) If $f(x) = x^2$, $\int_a^b x^2\ dx = \frac{b^3-a^3}{3}$

$$
\begin{aligned}
\widetilde{\int_a^b} x^2\ dx &= \frac{b-a}{2}\left(\frac{4a^2+4ab+b^2}{9} + \frac{a^2+4ab+4b^2}{9}\right) \\
&= \frac{(b-a)(5a^2+8ab+5b^2)}{18} \\
&\neq \frac{b^3-a^3}{3}
\end{aligned}
$$

[Numerical Problems]

1.

```
for time = 1:6
h=10/10^time;
x1=0;
x2=0;
x3=0;
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```
%Start midpoint
for i = 0:h:1-h
    x1=x1+4/(1+(i+h/2)^2);
end
x1=x1*h;
fprintf('when h=1/%d \tuse midpoint:%f\n',10^time/10,x1);


%Start trapezoid
for i = 0:h:1-h
    x2=x2+4/(1+i^2)+4/(1+(i+h)^2);
end
x2=x2*h/2;
fprintf('when h=1/%d \tuse trapezoid:%f\n',10^time/10,x2);


%Start Simpson;
A=0:h:1;
A=4./(1+A.^2);
x3=h/3*(A(1)+A(end)+4*sum(A(2:2:end-1))+2*sum(A(3:2:end-1)));

fprintf('when h=1/%d \tuse Simpson:%f\n',10^time/10,x3);
end
```