CS 181 Spring 2017 Section 4

1 Naive Bayes

A Naive Bayes classifier is a generative classifier, meaning that we are building a model for the joint distribution $p(\mathbf{x}, y)$. What we are actually interested in during classification is $p(y|\mathbf{x})$, which is related to the joint $p(\mathbf{x}, y)$ through Bayes' Rule:

$$p(y|\mathbf{x}) = \frac{p(y)p(\mathbf{x}|y)}{p(\mathbf{x})} \tag{1}$$

$$\propto p(y)p(\mathbf{x}|y) \tag{2}$$

The $p(y)$ is the prior over the classes. In the case of binary classification, $p(y; \theta) \sim \text{Bern}(\theta)$. $p(\mathbf{x})$ is constant and therefore we can disregard it as a normalizing constant, allowing us to go from (1) to (2).

The "naive" of Naive Bayes means that we assume that each feature in $\mathbf{x}$ is independently distributed, conditioned on the class. In the case of Multinomial Naive Bayes, we have $p(\mathbf{x}|y; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1) \propto \prod_{j=1}^{m} \pi_{yj}^{x_j}$.

If we are doing MLE, we then find the values of our parameters $\theta, \boldsymbol{\pi}_0, \boldsymbol{\pi}_1$ that maximize the log-likelihood:

$$\max_{\theta, \boldsymbol{\pi}_0, \boldsymbol{\pi}_1} \sum_{i=1}^{n} \ln p(\mathbf{x}_i, y_i) = \max_{\boldsymbol{\pi}_1, \boldsymbol{\pi}_2} \sum_{i=1}^{n} \ln p(\mathbf{x}_i | y_i; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1) + \max_{\theta} \sum_{i=1}^{n} \ln p(y_i; \theta) \tag{3}$$

Why is it that Naive Bayes is a linear classifier? When we classify a new example $\mathbf{x}$, we choose the class that is more likely by comparing $p(y|\mathbf{x})$ for every possible class value of y. In particular, for a binary classifier, we can compute the value of $h(\mathbf{x})$, which is the difference of the log of $p(y = 1|\mathbf{x})$ and the log of $p(y = 0|\mathbf{x})$:

$$\begin{aligned}
h(\mathbf{x}) &= [\ln p(\mathbf{x}|y = 1) + \ln p(y = 1)] - [\ln p(\mathbf{x}|y = 0) + \ln p(y = 0)] &(4)\\
&= \big[\ln \prod_{j=1}^{m} \pi_{1j}^{x_j} - \ln \prod_{j=1}^{m} \pi_{0j}^{x_j}\big] + \big[\ln \theta - \ln(1 - \theta)\big] &(5)\\
&= \sum_{j=1}^{m} x_j \ln \frac{\pi_{1j}}{\pi_{0j}} + \ln \left(\frac{\theta}{1 - \theta}\right) &(6)\\
&= \mathbf{x}^\top \mathbf{ln}(\frac{\boldsymbol{\pi}_1}{\boldsymbol{\pi}_0}) + \ln \left(\frac{\theta}{1 - \theta}\right), &(7)
\end{aligned}$$

where $\mathbf{ln}(\mathbf{q})$ for vector $\mathbf{q}$ applies ln elementwise, and we write $\boldsymbol{\pi}_1/\boldsymbol{\pi}_0$ to mean elementwise division. The resulting equation is in the same form as linear regression, and in particular the decision boundary $h(\mathbf{x}) = 0$ is linear in $\mathbf{x}$. With different Naive Bayes models, the only difference would be the corresponding weight and bias terms.

1. **Redundant Features in Naive Bayes**

Suppose, as in Bishop 4.2.3, that we use a Naive Bayes classifier to classify binary feature vectors $\mathbf{x}_i \in \mathbb{R}^m$ into two classes. The class conditional distributions will then be of the form

$$p(\mathbf{x} \,|\, y = C_k) = \prod_{i=1}^{m} \pi_{ki}^{x_i} \, (1 - \pi_{ki})^{(1-x_i)}, \qquad \text{(Bishop 4.81)}$$

where $x_i \in \{0, 1\}$, and $\pi_{ki} = p(x_i = 1 \,|\, y = C_k)$. This is a Bernoulli Naive Bayes, where all the features are binary instead of representing count data, as in the Multinomial case. Assume also that the class priors are $p(y = C_1) = p(y = C_2) = \frac{1}{2}$.

a. How is the quantity $\ln(p(y = C_1 \,|\, \mathbf{x})/p(y = C_2 \,|\, \mathbf{x}))$ used for classification of a new example $\mathbf{x}$?

b. If $m = 1$ (i.e., there is only one feature), use the equations above to write out $\ln \frac{p(y=C_1 \,|\, x)}{p(y=C_2 \,|\, x)}$ for a single binary feature x.

c. Now suppose we change our feature representation so that instead of using just a single feature, we use two redundant features (i.e., two features that always have the same value). With this feature representation, instead of x we will use $\mathbf{x} = [x, x]^\top$. What is $\ln \frac{p(y=C_1 \,|\, \mathbf{x})}{p(y=C_2 \,|\, \mathbf{x})}$ in terms of the value for $\ln \frac{p(y=C_1 \,|\, x)}{p(y=C_2 \,|\, x)}$ you calculated in part (a.)?

d. Is this a bug or a feature?

2 Gaussian Probabilistic Classification

Shapes of Decision Boundaries - Part I

Consider now a generative model with $c > 2$ classes, and output label $\mathbf{y}$ encoded as a "one hot" vector of length c.
We adopt class prior $p(\mathbf{y} = C_k; \boldsymbol{\pi}) = \pi_k$ for all $k \in \{1, \ldots, c\}$ (where π_k is a parameter of the prior).
Let $p(\mathbf{x} \mid \mathbf{y} = C_k)$ denote the class-conditional density of features $\mathbf{x}$ (in this case for class C_k). Let the class-conditional probabilities be Gaussian distributions

$$p(\mathbf{x} \mid \mathbf{y} = C_k) = \mathcal{N}(\mathbf{x} \mid \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k), \text{ for } k \in \{1, \ldots, c\} \tag{8}$$

We will predict the class of a new example $\mathbf{x}$ as the class with the highest conditional probability, $p(\mathbf{y} = C_k \mid \mathbf{x})$.
Luckily, a little bird came to the window of your dorm, and claimed that you can classify an example $\mathbf{x}$ by finding the class that maximizes the following function:

$$f_k(\mathbf{x}) = \ln(\pi_k) - \frac{1}{2}\ln(|\boldsymbol{\Sigma}_k|) - \frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_k)^T \boldsymbol{\Sigma}_k^{-1}(\mathbf{x} - \boldsymbol{\mu}_k).$$

Derive this formula by comparing two different classes' conditional probabilities. What can we claim about the shape of the decision boundary given this formula?

Shapes of Decision Boundaries - Part II

Let's say the little bird comes back and now tells you that every class has the same covariance matrix, and so $\boldsymbol{\Sigma}_\ell = \boldsymbol{\Sigma}'_\ell$ for all classes C_ℓ and $C_{\ell'}$. Simplify this formula down further. What can we claim about the shape of the decision boundaries now?

3 Neural Networks

Recall that in the case of binary classification, we can think about neural network as being equivalent to logistic regression with parameterized, adaptive basis functions.

Let's think about a neural network binary classifier with $\mathbf{x} \in \mathbb{R}^2$ (and thus $m = 2$ features) and with two a single hidden layer.

For the activation function, we use the *ReLU* function defined by the following:

$$\text{ReLU}(z) = \begin{cases} z & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Let's consider a function h defined by the following:

$$\begin{aligned} h(\mathbf{x}) &= \mathbf{w}^{\top}\phi(\mathbf{x}) + w_0 && (10) \\ &= \mathbf{w}^{\top}\mathbf{ReLU}\left(\mathbf{W}^1\mathbf{x} + \mathbf{w}_0^1\right) + w_0, && (11) \end{aligned}$$

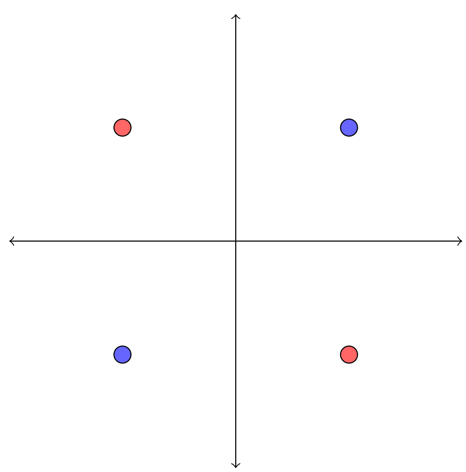
where example $\mathbf{x} \in \mathbb{R}^{2\times 1}$, weights in the hidden layer $\mathbf{W}^1 \in \mathbb{R}^{2\times 2}$, bias in the hidden layer $\mathbf{w}_0^1 \in \mathbb{R}^{2\times 1}$, output weight vector $\mathbf{w} \in \mathbb{R}^{2\times 1}$, and output bias $w_0 \in \mathbb{R}$.

Suppose we want to fit the following data:

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \; y_1 = 1, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \; y_2 = -1$$

$$\mathbf{x}_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \; y_3 = -1, \quad \mathbf{x}_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \; y_4 = 1$$

This looks as follows (blue = 1, red = -1):



Why can't we solve this problem with a linear classifier? What values of parameters $\mathbf{W}^1, \mathbf{w}_0^1, \mathbf{w}$, and w_0 will allow the neural network to solve the problem?

Hint: Think carefully about why we need a nonlinearity in finding a basis function for the examples. What can the ReLU do for us? What does it do to various kinds of vectors?